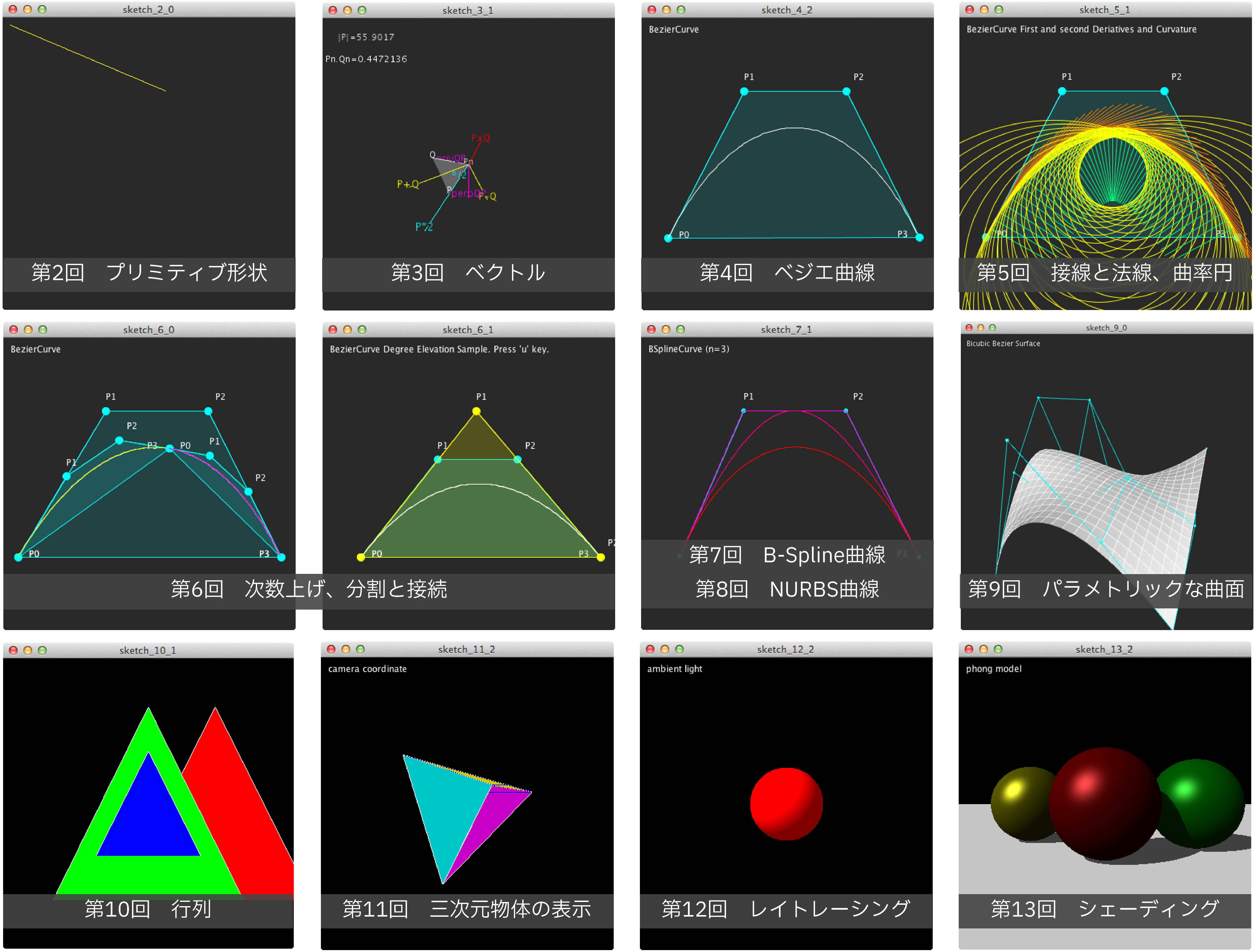
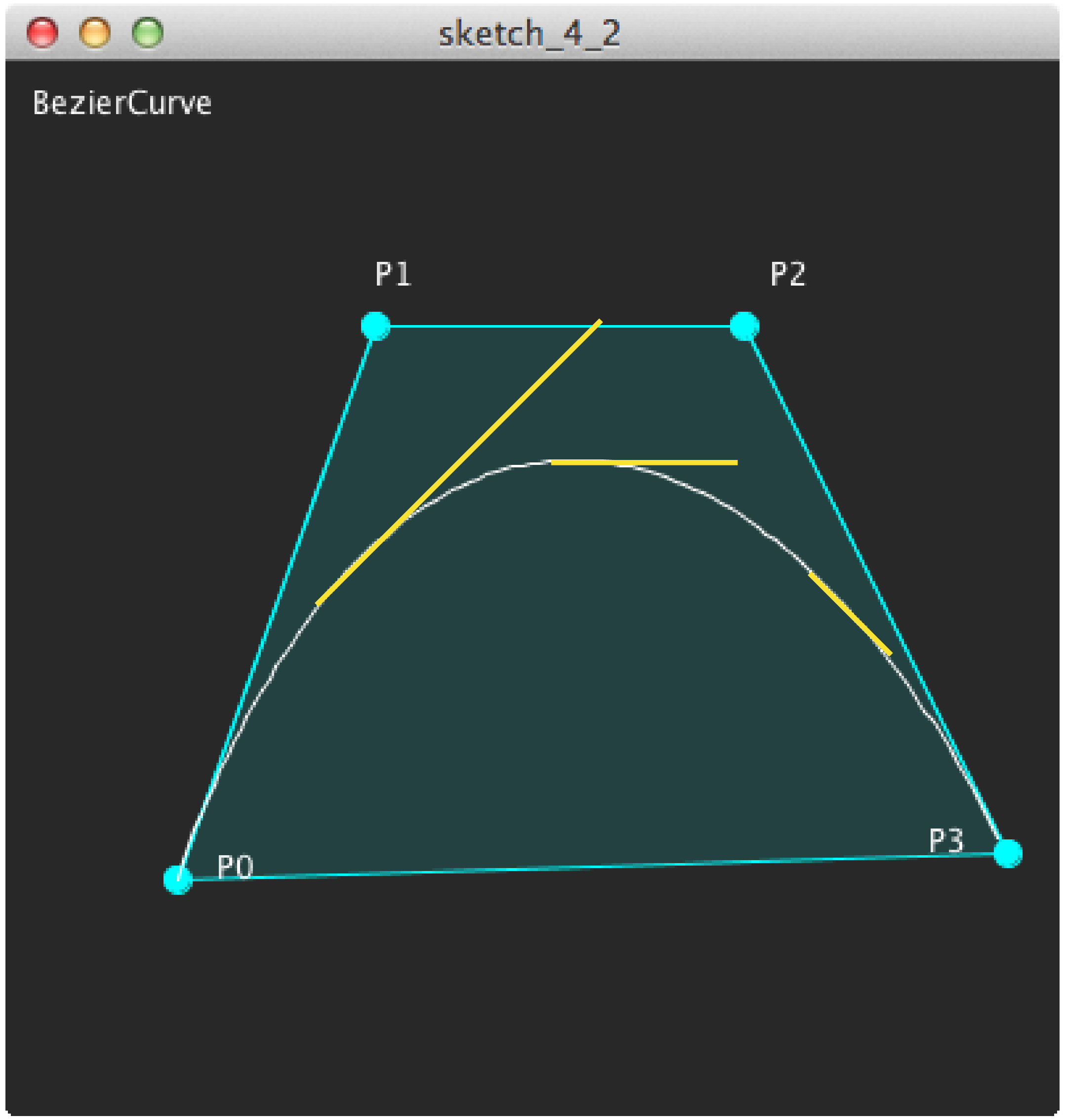
CGとCADの数理

GEOMETRIC MODELING AND COMPUTER GRAPHICS

第05回 接線と法線、曲率円


sketch_2_0
第2回　プリミティブ形状
sketch_3_1
|P|=55.9017
Pn.Qn=0.4472136
第3回　ベクトル
sketch_4_2
BezierCurve
P0
P1
P2
P3
第4回　ベジエ曲線
sketch_5_1
BezierCurve First and second Deriatives and Curvature
第5回　接線と法線、曲率円
sketch_6_0
BezierCurve
sketch_6_1
BezierCurve Degree Elevation Sample. Press 'u' key.
第6回　次数上げ、分割と接続
sketch_7_1
BSplineCurve (n=3)
第7回　B-Spline曲線
第8回　NURBS曲線
sketch_9_0
Bicubic Bezier Surface
第9回　パラメトリックな曲面
sketch_10_1
第10回　行列
sketch_11_2
camera coordinate
第11回　三次元物体の表示
sketch_12_2
ambient light
第12回　レイトレーシング
sketch_13_2
phong model
第13回　シェーディング

ベジエ曲線
Bezier Curve



前回の講義では、ベルンシュタイン基底関数を紹介し、3次ベジエ曲線の描画に挑戦しました。本日の講義では、3次ベジエ曲線の式を1階、2階微分して接線と法線を描画し、最後に曲率の計算までを紹介致します。

$$R(t) = (1-t)^3 P_0 + 3(1-t)^2 t\, P_1 + 3(1-t)\, t^2 P_2 + t^3 P_3$$

t で微分　t で微分　t で微分　t で微分

$$-3(1-t)^2 \qquad 9t^2 - 12t + 3 \qquad 6t - 9t^2 \qquad 3t^2$$

$$\begin{aligned} 9t^2 - 12t + 3 &= 3(3t^2 - 4t + 1) \\ &= 3(t^2 + 2t^2 - 2t - 2t + 1) \\ &= 3(1 - 2t + t^2 + 2t^2 - 2t) \\ &= 3\{(1-t)^2 - 2t(1-t)\} \end{aligned}$$

$$\begin{aligned} 6t - 9t^2 &= 3(2t - 3t^2) \\ &= 3(2t - 2t^2 - t^2) \\ &= 3\{2t(1-t) - t^2\} \end{aligned}$$

$$\begin{aligned} \frac{dR(t)}{dt} &= -3(1-t)^2 P_0 + 3\{(1-t)^2 - 2t(1-t)\} P_1 + 3\{(2t(1-t) - t^2)\} P_2 + 3t^2 P_3 \\ &= -3(1-t)^2 P_0 + 3(1-t)^2 P_1 - 3 \cdot 2t(1-t) P_1 + 3 \cdot 2t(1-t) P_2 - 3t^2 P_2 + 3t^2 P_3 \\ &= (1-t)^2\, 3(P_1 - P_0) + 2t(1-t)\, 3(P_2 - P_1) + t^2\, 3(P_3 - P_2) \end{aligned}$$

$$B_i^n = \binom{n}{i}(1-t)^{n-i}\, t^i$$

$i = 0, n = 2$ の場合　$B_0^2 = \binom{2}{0}(1-t)^{2-0}\, t^0 = \boxed{(1-t)^2}$

$i = 1, n = 2$ の場合　$B_1^2 = \binom{2}{1}(1-t)^{2-1}\, t^1 = \boxed{2t(1-t)}$

$i = 2, n = 2$ の場合　$B_2^2 = \binom{2}{2}(1-t)^{2-2}\, t^2 = \boxed{t^2}$

$$R(t) = (1-t)^3 P_0 + 3(1-t)^2 t\, P_1 + 3(1-t)\, t^2\, P_2 + t^3\, P_3$$

$\boldsymbol{n} = 3$

$$R'(t) = \boxed{(1-t)^2}\,\underbrace{\boxed{3}(P_1 - P_0)}_{=Q_0} + \boxed{2t(1-t)}\,\underbrace{\boxed{3}(P_2 - P_1)}_{=Q_1} + \boxed{t^2}\,\underbrace{\boxed{3}(P_3 - P_2)}_{=Q_2}$$

$$= \boxed{B_0^2}\, Q_0 + \boxed{B_1^2}\, Q_1 + \boxed{B_2^2}\, Q_2$$

$$R(t) = \sum_{i=0}^{\boxed{3}\to n} B_i^{\boxed{3}\to n}(t)\, P_i \qquad R(t) = \sum_{i=0}^{n} B_i^{n}(t) P_i$$

$$R(t) = (1-t)^3 P_0 + 3(1-t)^2 t\, P_1 + 3(1-t)t^2 P_2 + t^3 P_3$$

$$R(t) = B_0^{\boxed{3}}(t) P_0 + B_1^{\boxed{3}}(t) P_1 + B_2^{\boxed{3}}(t) P_2 + B_3^{\boxed{3}}(t) P_3$$

2の次数を*n*で表してみる

$$R'(t) = \sum_{i=0}^{\boxed{2}\to n\text{-}1} B_i^{\boxed{2}\to n\text{-}1}(t) Q_i \qquad \boxed{R'(t) = \sum_{i=0}^{n-1} B_i^{n-1}(t) Q_i}$$

$$R'(t) = (1-t)^2 \underbrace{Q_0}_{\boxed{3}(P_1 - P_0)} + 2t(1-t)\underbrace{Q_1}_{\boxed{3}(P_2 - P_1)} + t^2 \underbrace{Q_2}_{\boxed{3}(P_3 - P_2)}$$

$\boxed{3} \to n$

$$R'(t) = B_0^{\boxed{2}}(t) Q_0 + B_1^{\boxed{2}}(t) Q_1 + B_2^{\boxed{2}}(t) Q_2$$

n 次

$$R(t)=\sum_{i=0}^{3} B_i^{3}(t)\,P_i \qquad R(t)=\sum_{i=0}^{n} B_i^{n}(t)P_i$$

$$R(t)=B_0^{3}(t)P_0+B_1^{3}(t)P_1+B_2^{3}(t)P_2+B_3^{3}(t)P_3$$

$$R(t)=(1-t)^3P_0+3(1-t)^2t\,P_1+3(1-t)t^2P_2+t^3P_3$$

1 階微分

$$R'(t)=\sum_{i=0}^{2} B_i^{2}(t)Q_i \qquad R'(t)=\sum_{i=0}^{n-1} B_i^{n-1}(t)Q_i$$

$$R'(t)=B_0^{2}(t)Q_0+B_1^{2}(t)Q_1+B_2^{2}(t)Q_2$$

$$R'(t)=(1-t)^2Q_0+2t(1-t)Q_1+t^2Q_2$$

$$3(P_1-P_0) \qquad 3(P_2-P_1) \qquad 3(P_3-P_2)$$

2 階微分

$$R''(t)=\sum_{i=0}^{1} B_i^{1}(t)W_i \qquad R''(t)=\sum_{i=0}^{n-2} B_i^{n-2}(t)W_i$$

$$R''(t)=B_0^{1}(t)W_0+B_1^{1}(t)W_1$$

$$R''(t)=(1-t)W_0+tW_1$$

$$2(Q_1-Q_0) \qquad 2(Q_2-Q_1)$$

sketch_5_0.pde をダウンロードして下さい

1 階微分

$$R'(t) = \sum_{i=0}^{\boxed{2}} B_i^{\boxed{2}}(t)Q_i \qquad R'(t) = \sum_{i=0}^{n-1} B_i^{n-1}(t)Q_i$$

$$R'(t) = B_0^{\boxed{2}}(t)Q_0 + B_1^{\boxed{2}}(t)Q_1 + B_2^{\boxed{2}}(t)Q_2$$

$$R'(t) = (1-t)^2 Q_0 + 2t(1-t)Q_1 + t^2 Q_2$$

$$\boxed{3}(P_1 - P_0) \qquad \boxed{3}(P_2 - P_1) \qquad \boxed{3}(P_3 - P_2)$$

```
Q0.x=3*(P1.x-P0.x);
Q0.y=3*(P1.y-P0.y);
```

```
Q1.x=3*(P2.x-P1.x);
Q1.y=3*(P2.y-P1.y);
```

```
Q2.x=3*(P3.x-P2.x);
Q2.y=3*(P3.y-P2.y);
```

2 階微分

$$R''(t) = \sum_{i=0}^{1} B_i^{1}(t)W_i \qquad R''(t) = \sum_{i=0}^{n-2} B_i^{n-2}(t)W_i$$

$$R''(t) = B_0^{\boxed{1}}(t)W_0 + B_1^{\boxed{1}}(t)W_1$$

$$R''(t) = (1-t)W_0 + tW_1$$

$$\boxed{2}(Q_1 - Q_0) \qquad \boxed{2}(Q_2 - Q_1)$$

```
W0.x=2*(Q1.x-Q0.x);
W0.y=2*(Q1.y-Q0.y);
```

```
W1.x=2*(Q2.x-Q1.x);
W1.y=2*(Q2.y-Q1.y);
```

```
PVector Q0, Q1, Q2;
Q0=new PVector();
Q0.x=3*(P1.x-P0.x); Q0.y=3*(P1.y-P0.y);

Q1=new PVector();
Q1.x=3*(P2.x-P1.x); Q1.y=3*(P2.y-P1.y);

Q2=new PVector();
Q2.x=3*(P3.x-P2.x); Q2.y=3*(P3.y-P2.y);

PVector W0, W1;
W0=new PVector();
W0.x=2*(Q1.x-Q0.x); W0.y=2*(Q1.y-Q0.y);

W1=new PVector();
W1.x=2*(Q2.x-Q1.x); W1.y=2*(Q2.y-Q1.y);
```

1階微分

$$R'(t) = \sum_{i=0}^{\boxed{2}} B_i^{\boxed{2}}(t)Q_i$$

$$R'(t) = \sum_{i=0}^{n-1} B_i^{n-1}(t)Q_i$$

$$R'(t) = B_0^2(t)Q_0 + B_1^2(t)Q_1 + B_2^2(t)Q_2$$

$$R'(t) = (1-t)^2 Q_0 + 2t(1-t)Q_1 + t^2 Q_2$$

```
B20t = (1-t)*(1-t);
B21t = 2*(1-t)*t;
B22t = t*t;
```

2階微分

$$R''(t) = \sum_{i=0}^{1} B_i^{1}(t)W_i$$

$$R''(t) = \sum_{i=0}^{n-2} B_i^{n-2}(t)W_i$$

$$R''(t) = B_0^1(t)W_0 + B_1^1(t)W_1$$

$$R''(t) = (1-t)W_0 + tW_1$$

```
B10t = 1-t;
B11t = t;
```

```
B20t =(1-t)*(1-t);
B21t = 2 * (1-t)*t;
B22t = t*t;

B10t = 1-t;
B11t = t;
```

$$R'(t) = \sum_{i=0}^{\boxed{2}} B_i^{\boxed{2}}(t) Q_i \qquad R'(t) = \sum_{i=0}^{n-1} B_i^{n-1}(t) Q_i$$

1階微分

$$R'(t) = \boxed{(1-t)^2}\, Q_0 + \boxed{2t(1-t)}\, Q_1 + \boxed{t^2}\, Q_2$$

```
B20t = (1-t)*(1-t);
Q0.x=3*(P1.x-P0.x);
Q0.y=3*(P1.y-P0.y);

B21t = 2*(1-t)*t;
Q1.x=3*(P2.x-P1.x);
Q1.y=3*(P2.y-P1.y);

B22t = t*t;
Q2.x=3*(P3.x-P2.x);
Q2.y=3*(P3.y-P2.y);
```

D1[tt]

```
D1[tt].x=B20t*Q0.x + B21t*Q1.x + B22t*Q2.x;
D1[tt].y=B20t*Q0.y + B21t*Q1.y + B22t*Q2.y;
```

$$R''(t) = \sum_{i=0}^{1} B_i^{1}(t) W_i \qquad R''(t) = \sum_{i=0}^{n-2} B_i^{n-2}(t) W_i$$

2階微分

```
B10t = 1-t;
```

$$R''(t) = \boxed{(1-t)}\,\boxed{W_0}$$

```
W0.x=2*(Q1.x-Q0.x);
W0.y=2*(Q1.y-Q0.y);
```

D2[tt]

```
B11t = t;
```

$$+\ \boxed{t}\,\boxed{W_1}$$

```
W1.x=2*(Q2.x-Q1.x);
W1.y=2*(Q2.y-Q1.y);
```

```
D2[tt].x=B10t*W0.x + B11t*W1.x;
D2[tt].y=B10t*W0.y + B11t*W1.y;
```

曲率

Curvature

正則な曲線がパラメータ t で表されている時、曲率 $K(t)$ は、以下の式で求められます。

```
D1[tt].x*D2[tt].y-D2[tt].x*D1[tt].y
```

$$K(t) = \frac{R'(t).x \cdot R''(t).y - R''(t).x \cdot R'(t).y}{(R'(t).x \cdot R'(t).x + R'(t).y \cdot R'(t).y)^{3/2}}$$

$$(R'(t).x \cdot R'(t).x + R'(t).y \cdot R'(t).y) \cdot (R'(t).x \cdot R'(t).x + R'(t).y \cdot R'(t).y)^{1/2}$$

$$= (R'(t).x \cdot R'(t).x + R'(t).y \cdot R'(t).y) \cdot \sqrt{R'(t).x \cdot R'(t).x + R'(t).y \cdot R'(t).y}$$

$$= (R'(t).x \cdot R'(t).x + R'(t).y \cdot R'(t).y) \cdot sqrt(R'(t).x \cdot R'(t).x + R'(t).y \cdot R'(t).y)$$

```
D1[tt].x* D1[tt].x + D1[tt].y * D1[tt].y *sqrt(D1[tt].x* D1[tt].x + D1[tt].y * D1[tt].y);
```

```
K[tt] =
(D1[tt].x*D2[tt].y - D2[tt].x*D1[tt].y)/
(
      (D1[tt].x*D1[tt].x + D1[tt].y*D1[tt].y)*
  sqrt(D1[tt].x*D1[tt].x + D1[tt].y*D1[tt].y)
);
```

```
Q0=new PVector(); Q0.x=3*(P1.x-P0.x); Q0.y=3*(P1.y-P0.y);
Q1=new PVector(); Q1.x=3*(P2.x-P1.x); Q1.y=3*(P2.y-P1.y);
Q2=new PVector(); Q2.x=3*(P3.x-P2.x); Q2.y=3*(P3.y-P2.y);

W0=new PVector(); W0.x=(3-1)*(Q1.x-Q0.x); W0.y=(3-1)*(Q1.y-Q0.y);
W1=new PVector(); W1.x=(3-1)*(Q2.x-Q1.x); W1.y=(3-1)*(Q2.y-Q1.y);

B20t =     (1-t)*(1-t)    ;
B21t = 2 *       (1-t)  *t;
B22t =                 t*t;

B10t = 1-t;
B11t =   t;


D1[tt].x=B20t*Q0.x + B21t*Q1.x + B22t*Q2.x;
D1[tt].y=B20t*Q0.y + B21t*Q1.y + B22t*Q2.y;

D2[tt].x=B10t*W0.x + B11t*W1.x;
D2[tt].y=B10t*W0.y + B11t*W1.y;


K[tt] = (D1[tt].x*D2[tt].y -
D2[tt].x*D1[tt].y)/(sqrt(D1[tt].x*D1[tt].x +
D1[tt].y*D1[tt].y)*(D1[tt].x*D1[tt].x + D1[tt].y*D1[tt].y));
```