Gamma Distribution

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Recap

Exponential Distribution:

"Infers the probability of the **waiting time between events in a Poisson Process**"

What is it?

The Exponential Distribution only infers the probability of the waiting time for the first event.

However, the **Gamma Distribution** gives us the probability of the waiting time until the ***nth* event.**

Derivation

We start by saying we want to wait time ***T***, for the ***nth (n)*** event to occur. For the Gamma Distribution ***T*** is the random variable. This means we need ***n-1*** events to occur in time ***t***:

$$P(T \leq t) = 1 - P(T > t) = 1 - P(0 \; or \; 1 \; or \; .... \; n-1 \; events \; in \; t)$$

Equation generated in LaTeX by author.

Derivation

$$P(T \leq t) = 1 - P(T > t) = 1 - P(0\ or\ 1\ or\ \ldots.\ n-1\ events\ in\ t)$$

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Now we need to sum up the probabilities of having ***0*** to ***n-1*** events occuring in the time period ***t*** using the Poisson Distribution Probability Mass Function (PMF):

$$P(T \leq t) = 1 - P(T > t) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

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Here, ***n*** is the **number of events** occurring in the time period ***t*** with the ***rate (Poisson) parameter λ***.

Derivation

$$P(T \leq t) = 1 - P(T > t) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

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The above formula is the **Cumulative Distribution Function (CDF)**. To extract the Probability Density Function (PDF) we differentiate the CDF with respect to ***t***.

$$PDF = \frac{d}{dt}\left(1 - P(T > t) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}\right)$$

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Derivation

Now this derivative is quite exhaustive, so I have omitted the full process here. However, the interested reader can find it here.

The final version of the PDF is:

Expected rate of events

$$PDF = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

Number of events your waiting for

Equation generated in LaTeX by author.

Why Called Gamma?

Contains the Gamma Function

$$\Gamma(n) = (n-1)!$$

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Where Γ is the Greek symbol gamma. Therefore, re-writing our PDF:

$$PDF = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{\Gamma(n)}$$

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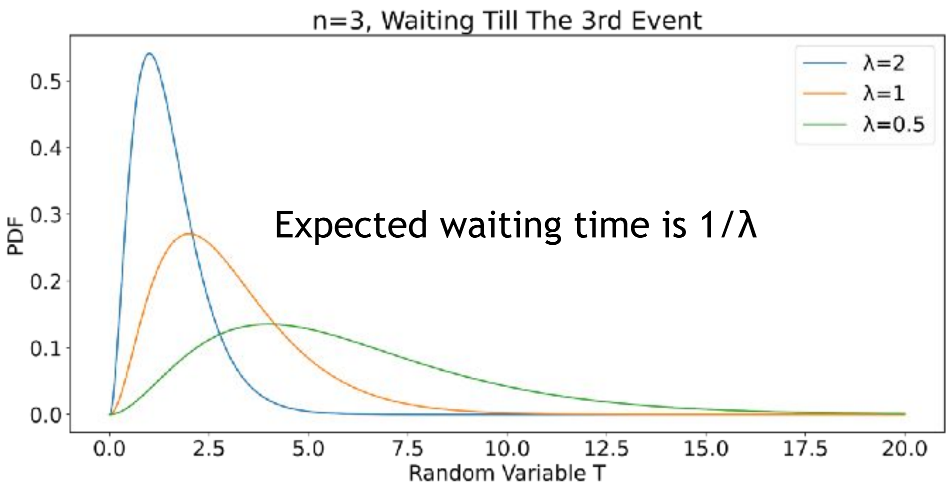
Example Plot




```
# import packages
import numpy as np
import scipy.stats as stats
import matplotlib.pyplot as plt

# get distributions
x = np.linspace(0, 20, 1000)
y1 = stats.gamma.pdf(x, a=3, scale=0.5)
y2 = stats.gamma.pdf(x, a=3, scale=1)
y3 = stats.gamma.pdf(x, a=3, scale=2)

# plot
plt.figure(figsize=(15,7))
plt.plot(x, y1, label='λ=2')
plt.plot(x, y2, label='λ=1')
plt.plot(x, y3, label='λ=0.5')
plt.title('n=3, Waiting Till The 3rd Event', fontsize=24)
plt.xticks(fontsize=20)
plt.yticks(fontsize=20)
plt.legend(fontsize=20)
plt.ylabel('PDF', fontsize=20)
plt.xlabel('Random Variable T', fontsize=20)
plt.show()
```

As λ gets smaller the expected time between events increases. This is observed in the above plots where we see the mean waiting time for the 3rd events increasing as λ gets smaller.

Thanks

✦ Member-only story

Gamma Distribution Simply Explained

An explanation of the Gamma Distribution and its origins

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