Chapter 4. Vector Space

Vector spaces with real scalars are called ***real vector spaces*** and those with complex scalars are called ***complex vector spaces***. For now, we will be concerned exclusively with real vector spaces.

4.1 Real Vector Spaces

Let V be a nonempty set of objects, on which two operations are defined:

a) Addition
b) Multiplication by scalars

With the following properties:

1. If $\vec{u}$ and $\vec{v}$ are elements in V, then $\vec{u}+\vec{v}$ is in V. (V is closed under addition)
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$, for all $\vec{u}$, $\vec{v}$ in V. (holds Commutative Law)
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ (holds Associative Law)
4. There is an object $\vec{0}$ in V, called the zero vector for V such that $O+\vec{u}=\vec{u}+O=\vec{u}$, for each $\vec{u}$ in V. (have Additive Identity)
5. For each $\vec{u}$ in V, there is an object $-\vec{u}$ in V, called a negative of $\vec{u}$, such that $\vec{u}+(-\vec{u})=-\vec{u}+\vec{u}=O$. (have Additive Inverse)
6. If k is any scalar and $\vec{u}$ is any object in V, then $k\vec{u}$ is in V. (Closed under Scalar Multiplication).
7. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
8. $(k+m)\vec{u}=k\vec{u}+m\vec{u}$
9. $k(m\vec{u})=(km)\vec{u}$
10. $1\vec{u}=\vec{u}$ (have Multiplicative Identity)

then V is called a vector space and the objects in *V are* ***vectors***.

Example 1: Let $V=R^2=\{(x,y);x,y\in R\}$, prove that V is a vector space under the usual operations of addition and scalar multiplication defined by:

$$\vec{u}+\vec{v}=(u_1,u_2)+(v_1,v_2)=(u_1+v_1,\vec{u}_2+\vec{v}_2)$$

$$k\vec{u}=k(\vec{u}_1,\vec{u}_2)=(k\vec{u}_1,k\vec{u}_2)$$

Solution:

1. V is closed under addition. (as defined)
2. Let $\vec{u} = (\vec{u}_1, \vec{u}_2), \vec{v} = (\vec{v}_1, \vec{v}_2)$

$$\vec{u} + \vec{v} = (\vec{u}_1, \vec{u}_2) + (\vec{v}_1, \vec{v}_2)$$

$$= (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2)$$

$$= (\vec{v}_1 + \vec{u}_1, \vec{v}_2 + \vec{u}_2)$$

$$= (\vec{v}_1, \vec{v}_2) + (\vec{u}_1, \vec{u}_2)$$

$$= \vec{v} + \vec{u}$$

3. Let $\vec{u} = (\vec{u}_1, \vec{u}_2), \vec{v} = (\vec{v}_1, \vec{v}_2), \vec{w} = (\vec{w}_1, \vec{w}_2)$

$$\begin{aligned}(\vec{u} + \vec{v}) + \vec{w} &= (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2) + (\vec{w}_1, \vec{w}_2) \\ &= (\vec{u}_1 + \vec{v}_1 + \vec{w}_1, \vec{u}_2 + \vec{v}_2 + \vec{w}_2) \\ &= (\vec{u}_1 + (\vec{v}_1 + \vec{w}_1), \vec{u}_2 + (\vec{v}_2 + \vec{w}_2)) \\ &= (\vec{u}_1, \vec{u}_2) + (\vec{v}_1 + \vec{w}_1, \vec{v}_2 + \vec{w}_2) \\ &= \vec{u} + (\vec{v} + \vec{w})\end{aligned}$$

4. Let $\vec{u} = (\vec{u}_1, \vec{u}_2), \vec{0} = (0,0)$

$$\vec{u} + \vec{0} = (\vec{u}_1, \vec{u}_2) + (0, 0) = (\vec{u}_1, \vec{u}_2) = \vec{u}$$

5. Let $\vec{u} = (\vec{u}_1, \vec{u}_2)$, then there exist $-\vec{u} = (-\vec{u}_1, -\vec{u}_2)$,

$$\vec{u} + (-\vec{u}) = \left(\vec{u}_1 + (-\vec{u}_1), \vec{u}_2 + (-\vec{u}_2)\right) = (\vec{u}_1 - \vec{u}_1, \vec{u}_2 - \vec{u}_2) = (0, 0) = \vec{0}$$

6. V is closed under scalar multiplication. (as defined).
7.
$$\begin{aligned}k(\vec{u} + \vec{v}) &= k\left((\vec{u}_1, \vec{u}_2) + (\vec{v}_1, \vec{v}_2)\right) \\ &= k(\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2) \\ &= (k\vec{u}_1 + k\vec{v}_1, k\vec{u}_2 + k\vec{v}_2) \\ &= (k\vec{u}_1, k\vec{u}_2) + (k\vec{v}_1, k\vec{v}_2) \\ &= k(\vec{u}_1, \vec{u}_2) + k(\vec{v}_1, \vec{v}_2) \\ &= k\vec{u} + k\vec{v}\end{aligned}$$
8.
$$\begin{aligned}(k + m)\vec{u} &= (k + m)(\vec{u}_1, \vec{u}_2) \\ &= (k\vec{u}_1 + m\vec{u}_1, k\vec{u}_2 + m\vec{u}_2) \\ &= (k\vec{u}_1, k\vec{u}_2) + (m\vec{u}_1, m\vec{u}_2) \\ &= k(\vec{u}_1, \vec{u}_2) + m(\vec{u}_1, \vec{u}_2) \\ &= k\vec{u} + m\vec{u}\end{aligned}$$

9. $k(m\vec{u}) = k\big(m(\vec{u}_1, \vec{u}_2)\big) = (km\vec{u}_1, km\vec{u}_2)$
$= km(\vec{u}_1, \vec{u}_2) = km(\vec{u})$
10. $1\vec{u} = 1(\vec{u}_1, \vec{u}_2) = (\vec{u}_1, \vec{u}_2) = \vec{u}$

As the set V satisfies all the properties, so V is vector space.

Example 2: Let $V = R^3$, prove that V is a vector space under the usual operations of addition and scalar multiplication defined by:

$$\vec{u} + \vec{v} = (\vec{u}_1, \vec{u}_2, \vec{u}_3) + (\vec{v}_1, \vec{v}_2, \vec{v}_3) = (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2, \vec{u}_3 + \vec{v}_3)$$

$$k\vec{u} = k(\vec{u}_1, \vec{u}_2, \vec{u}_3) = (k\vec{u}_1, k\vec{u}_2, k\vec{u}_3)$$

Example 3: Let $V = R^2$, under the usual operations of addition defined by:

$$\vec{u} + \vec{v} = (\vec{u}_1, \vec{u}_2) + (\vec{v}_1, \vec{v}_2) = (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2)$$

And if k is any scalar number, then define

$$k\vec{u} = k(\vec{u}_1, \vec{u}_2) = (k\vec{u}_1, 0)$$

The addition operation is standard one on R^2, but the scalar multiplication is not.

Check whether V is vector space or not?

Solution:

All properties of addition are satisfied. (Check it by yourself)

Let's check the properties of scalar multiplication.

6. Let $\vec{u} = (u_1, u_2)$ in V, then $k\vec{u} = k(u_1, u_2) = (ku_1, 0) \in V$.
7. Let $\vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2)$

$$\begin{aligned} k(\vec{u} + \vec{v}) &= k\big((u_1, u_2) + (v_1, v_2)\big) \\ &= k(u_1 + v_1, u_2 + v_2) \\ &= (ku_1 + kv_1, 0) \\ &= (k\vec{u}_1, 0) + (k\vec{v}_1, 0) \\ &= k(\vec{u}_1, 0) + k(\vec{v}_1, 0) \\ &\neq k\vec{u} + k\vec{v} \end{aligned}$$

As the 7th property does not satisfied So it's not a vector space.

Example 4:

Check whether V is vector space or not?

V =The set of all pairs of real numbers of the form $(x, 0)$. i.e. $\{(x, 0); x \in R\}$

with the standard operations on R^2.

$$\vec{u} + \vec{v} = (\vec{u}_1, \vec{u}_2) + (\vec{v}_1, \vec{v}_2) = (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2)$$

$$k\vec{u} = k(\vec{u}_1, \vec{u}_2) = (k\vec{u}_1, k\vec{u}_2)$$

Solution:

1.
$$\vec{u} = (\vec{u}_1, 0), \vec{v} = (\vec{v}_1, 0) \in V$$
$$(\vec{u} + \vec{v}) = (\vec{u}_1 + \vec{v}_1, 0) \in V$$

V is closed under addition.

2.
$$\begin{aligned}(\vec{u} + \vec{v}) &= (\vec{u}_1, 0) + (\vec{v}_1, 0)\\ &= (\vec{u}_1 + \vec{v}_1, 0)\\ &= (\vec{v}_1 + \vec{u}_1, 0)\\ &= (\vec{v}_1, 0) + (\vec{u}_1, 0)\\ &= \vec{v} + \vec{u}\end{aligned}$$

3.
$$\begin{aligned}\vec{u} + (\vec{v} + \vec{w}) &= (\vec{u}_1, 0) + ((\vec{v}_1, 0) + (\vec{w}_1, 0))\\ &= (\vec{u}_1, 0) + (\vec{v}_1 + \vec{w}_1, 0)\\ &= (\vec{u}_1 + \vec{v}_1 + \vec{w}_1, 0)\\ &= (\vec{u}_1 + \vec{v}_1, 0) + (\vec{w}_1, 0)\\ &= (\vec{u} + \vec{v}) + \vec{w}\end{aligned}$$

4. $\vec{u} + \vec{0} = (\vec{u}_1, 0) + (0,0) = (\vec{u}_1, 0) = \vec{u}$
5. $\vec{u} + (-\vec{u}) = (\vec{u}_1, 0) + (-\vec{u}_1, 0)$

$$= (\vec{u}_1 - \vec{u}_1, 0) = (0,0) = \vec{0}$$

6. $\vec{u} = (\vec{u}_1, 0) \in V$

Then $k\vec{u} = (k\vec{u}_1, k0) = (ku_1, 0) \in V$

7. $k(\vec{u} + \vec{v}) = k\big((u_1, 0) + (v_1, 0)\big) = k(\vec{u}_1 + \vec{v}_1, 0) = (k\vec{u}_1 + k\vec{v}_1, 0)$

$$= (k\vec{u}_1, 0) + (k\vec{v}_1, 0) = k(\vec{u}_1, 0) + k(\vec{v}_1, 0)$$

$$= (k\vec{u} + k\vec{v})$$

8. $(k + m)\vec{u} = (k + m)(\vec{u}_1, 0)$

$$= \big((k + m)\vec{u}_1, 0\big) = (k\vec{u}_1 + m\vec{u}_1, 0)$$

$$= (k\vec{u}_1, 0) + (m\vec{u}_1, 0)$$

$$= k(\vec{u}_1, 0) + m(\vec{u}_1, 0) \ = k\vec{u} + m\vec{u}$$

9. $k(m\vec{u}) = k(m\vec{u}_1, 0) = (km\vec{u}_1, 0)$

$$= km(\vec{u}_1, 0) = (km)\vec{u}$$

10. $1\vec{u} = 1(\vec{u}_1, 0) = (\vec{u}_1, 0) = \vec{u}$

So V is a vector space.

<u>**Example 5:**</u> Check whether V is a vector space or not.

V = set of all pairs of real numbers of the form (x, y), where $x \geq 0$, i.e.

$$V = \{(x, y); x \geq 0, y \in R\}$$

With standard operations on R^2.

<u>**Solution:**</u>

As

$$V = \{(x, y); x \geq 0, y \in R\}$$

1. Let $\vec{u} = (\vec{u}_1, \vec{u}_2), \quad \vec{v} = (\vec{v}_1, \vec{v}_2) \in V$

$$(\vec{u} + \vec{v}) = (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2) \in V$$

Because $\vec{u}_1 + \vec{v}_1 \geq 0$. So, V is closed under addition.

2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Easy to verify)

3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w})$ (Easy to verify)
4. Let $\vec{u}=(\vec{u}_1,\vec{u}_2)$, $\vec{u}+\vec{0}=(\vec{u}_1,\vec{u}_2)+(0,0)=(\vec{u}_1,\vec{u}_2)=\vec{u}$
5. Let $\vec{u}=(\vec{u}_1,\vec{u}_2)$, Then there doesn't exist $-\vec{u}=(-\vec{u}_1,-\vec{u}_2)$ because $\vec{u}_1$should be positive.
   5th property fails, So V is not vector space.

Example 6: Show that the set of all pairs of real numbers of the form $(x, 1)$ with the
operations

$(x, 1)+(x', 1)=(x+x', 1)$ & $k(x, 1)=(k^2x, 1)$ is not a vector space.

Example 7: Determine whether the set of all triples of real numbers with standard vector addition but with scalar multiplication defined by

$$k(x, y, z)=(k^2x, k^2y, k^2z)$$

is a vector space or not.

Axiom 8 fails.

Example 8: Determine whether the set of all pairs of real numbers of the form (1, x) with the operations

$$(1, y)+(1, y')=(1, y+y')$$

$$k(1, y)=(1, ky)$$

is a vector space or not.

Example 9: Determine whether V is a vector space or not.

V= the set of all triples of the form (x, y, z) with the operations

$$(x, y, z)+(x', y', z')=(x+x', y+y', z+z')$$

& $k(x, y, z)=(kx, y, z)$

Example 10: Determine whether V is a vector space or not.

Let V be the set of all 2×2 matrices with real entries and take the vector space operations on V to be usual operations of matrix addition and scalar multiplication i.e.

$$\vec{u} + \vec{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$k\vec{u} = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

Solution:

1. V is closed under addition.
2. $$\begin{aligned} \vec{u} + \vec{v} &= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \\ &= \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \\ &= \begin{bmatrix} v_{11} + u_{11} & v_{12} + u_{12} \\ v_{21} + u_{21} & v_{22} + u_{22} \end{bmatrix} \\ &= \vec{v} + \vec{u} \end{aligned}$$
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w})$
4. $\vec{u} + 0 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \vec{u}$
5. $$\begin{aligned} \vec{u} + (-\vec{u}) &= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} \\ &= \begin{bmatrix} u_{11} + (-u_{11}) & u_{12} + (-u_{12}) \\ u_{21} + (-u_{21}) & u_{22} + (-u_{22}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0} \end{aligned}$$

Similarly, you can prove all the properties of scalar multiplication. (Prove it by yourself).

So, V is a vector space.

Example 11: Let $V = R^n$ and define operations on V to be the usual operations of addition and scalar multiplication.

$$\bar{u}^{\rightarrow} + \vec{v} = (u_1, u_2, u_3, \dots, u_n) + (v_1, v_2, v_3, \dots, v_n)$$

$$= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$\& \quad k\bar{u}^{\rightarrow} = (ku_1, ku_2, ku_3, \dots, ku_n)$$

Then V is vector space.

Example 12: Let V be the set of polynomials of the form

$$P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

Determine whether V is a vector space or not under the usual operations of addition and scalar multiplication?