38) $X' = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} X$

$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$K, P, Q =$ eigen vectors

$\lambda_1, \lambda_2, \lambda_3 =$ eigen values

$X' = AX$

λ:

$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix}$

$= (4-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 \\ 0 & 4-\lambda \end{vmatrix}$

$= (4-\lambda)^3$

$\lambda_1 = \lambda_2 = \lambda_3 = 4$

K:

$(A - \lambda_1 I) K = 0$

$(A - 4I) K = 0$

$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$k_2 = k_3 = 0$

$k_1 = 1$

$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$X = K e^{\lambda_1 t}$

$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{4t}$

$$\left[\begin{array}{ccc:c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_2 = k_3 = 0$

$k_1 = 1$

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

P:

$(A - \lambda_2 I)P = 0$

$(A - 2I)P = 0$

$$\left[\begin{array}{ccc:c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc:c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-P_1 = 0$

$P_1 = 0$

$P_2 + P_3 = 0$

$P_2 = -P_3$

let $P_3 = \gamma$

$P_2 = -\gamma$

if $\gamma = 1$

$P_2 = -1$

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$X_2 = Pe^{\lambda_2 t}$

$$X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

Q:

$(A - \lambda_3 I)Q = P$

$(A - 2I)Q = P$

$$\left[\begin{array}{ccc:c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

$R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc:c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-q_1 = 0$

$q_1 = 0$

$q_2 + q_3 = -1$

$q_2 = -1 - q_3$

let $q_3 = \gamma$

$q_2 = -1 - \gamma$

if $\gamma = 0$

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 2 & -8 & 2 & 0 \\ -5 & 2 & 4 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$\frac{1}{2}R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ -5 & 2 & 4 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$R_2 + 5R_1 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & -18 & 9 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$R_3 - 4R_1 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & 18 & -9 & 0 \end{array}\right]$$

$R_3 + R_2 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-\frac{1}{18}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 + 4R_2 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_3 = 0$

$k_1 = k_3$

$k_2 - \frac{1}{2}k_3$

$k_2 = \frac{1}{2}k_3$

Let $k_3 = r$

$k_1 = r$

$k_2 = r/2$

if $r = 2$

$k_1 = 2$

$k_2 = 1$

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$X_1 = Ke^{\lambda_1 t}$

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{8t}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_2 - k_3 = 0$

$k_1 = k_2 + k_3$

Let $k_2 = r$, $k_3 = s$

$k_1 = r + s$

if $r = 0$, $s = 1$, $\Rightarrow P_1 = 1$

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} r+s \\ r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$X_2 = Pe^{\lambda_2 t} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Q: $\lambda = \lambda_3 = 2$

$(A - \lambda_3 I) Q = P$

$(A - 2I) Q = P$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$(\lambda-2)\left[\lambda^2-3\lambda+2\right]=0$$

$$(\lambda-2)\left[\lambda^2-2\lambda-\lambda+2\right]=0$$

$$(\lambda-2)\left[\lambda(\lambda-2)-1(\lambda-2)\right]=0$$

$$(\lambda-2)\left[(\lambda-1)(\lambda-2)\right]=0$$

$$\lambda_1=1\ ,\quad \lambda_2=\lambda_3=2$$

K: picking $\lambda=\lambda_1=1$

$$(A-\lambda_1 I)K=0$$

$$(A-I)K=0$$

$$\left[\begin{array}{ccc:c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \quad R_1-R_2\to R_1$$

$$\left[\begin{array}{ccc:c} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \quad R_2-R_3\to R_2$$

$$\left[\begin{array}{ccc:c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \quad R_3-R_1\to R_3$$

$$\left[\begin{array}{ccc:c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1+R_2\to R_1$$

31 $\left[\begin{array}{cc|c} 1 & -1 & -1/5 \\ 0 & 0 & 0 \end{array}\right]$

$P_1 - P_2 = -1/5$

$P_1 = P_2 - \frac{1}{5}$

$P_1 = \frac{5P_2 - 1}{5}$

n Let $P_2 = \gamma$

$P_1 = \frac{5\gamma - 1}{5}$

if $\gamma = \frac{1}{5}$

$P_1 = 0,\ P_2 = 1/5$

$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$

$X_2 = tKe^{\lambda_1 t} + Pe^{\lambda_2 t}$

$X_2 = t\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} e^{-t}$

$X = c_1 X_1 + c_2 X_2$

$= c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + c_2 \left[t\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 1/5 \end{pmatrix} e^{-t} \right]$

30) $\frac{dx}{dt} = -6x + 5y$, $\frac{dy}{dt} = -5x + 4y$

$A = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

let $X = Ke^{\lambda t}$ $\quad$ $\because \lambda_1, \lambda_2, \ldots \lambda_n$ = eigen values, K = eigen vectors

λ:
$(A - \lambda I) = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A - \lambda I| = \begin{bmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{bmatrix}$

$(-6-\lambda)(4-\lambda) - (-5)(5) = 0$

$-24 + 6\lambda - 4\lambda + \lambda^2 + 25 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda^2 + (1+1)\lambda + 1 = 0$

$\lambda^2 + \lambda + \lambda + 1 = 0$

$\lambda(\lambda + 1) + 1(\lambda + 1) = 0$

$(\lambda + 1)(\lambda + 1) = 0$

$\lambda_1 = \lambda_2 = -1$

K: Taking $\lambda_1 = -1$

$(A - \lambda_1 I)K = 0$

$(A + I)K = 0$

$= \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix}$

P: Taking $\lambda = \lambda_2$

$(A - \lambda_2 I) P = K$

$(A - 0) P = K$

$$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & +3 \end{array}\right]$$

$\frac{1}{3} R_1 \to R_1$

$$\left[\begin{array}{cc|c} 1 & -1/3 & 1/3 \\ 9 & -3 & 3 \end{array}\right]$$

$R_2 - 9R_1 \to R_2$

$$\left[\begin{array}{cc|c} 1 & -1/3 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

$P_1 = \frac{1}{3} P_2 = \frac{1}{3}$

$P_1 = \frac{1}{3} + \frac{1}{3} P_2 \Rightarrow \frac{P_2 + 1}{3}$

Let $P_2 = \gamma$

$P_1 = \frac{\gamma + 1}{3}$

if $\gamma = 2$

$P_1 = 1$

$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$X_2 = tKe^{\lambda_1 t} + Pe^{\lambda_2 t} = t\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$X = c_1 X_1 + c_2 X_2$

$= c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]$

Equations:

Assignment: 03

Section: A

Name - AOUN-HAIDER

ID: FA21-BSE-133

Task : Ex 8.6 Q(29-38)

29) $\frac{dx}{dt} = 3x - y$

$\frac{dy}{dt} = 9x - 3y$

$X' = AX \quad -(i)$

$A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$

Let $X = Ke^{\lambda t}$ $\quad \lambda_1, \lambda_2, \ldots \lambda_n =$ eigen values

$K =$ eigen vectors

λ: $|A - \lambda I| = 0$

$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix}$

$= (3-\lambda)(-3-\lambda) - (-1)(9)$

$= -9 - \cancel{3\lambda} + \cancel{3\lambda} + \lambda^2 + 9$

$\lambda^2 = 0$

$\boxed{\lambda_1 = \lambda_2 = 0}$

K: picking $\lambda = \lambda_1$

$(A - \lambda_1 I)K = 0$

$(A - 0)K = 0$

$$\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 9 & -3 & 0 \end{array}\right]$$

$\frac{1}{3}R_1 \to R_1$

$$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 9 & -3 & 0 \end{array}\right]$$

$R_2 - 9R_1 \to R_2$

$$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 - \frac{1}{3}k_2 = 0$

$k_1 = \frac{1}{3}k_2$

let $k_2 = r$

if $r = 3$ $\quad$ ∴ take any none zero value

$k_1 = 1$

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$X_1 = Ke^{\lambda t} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{0t} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -5 & 5 & 0 \\ -5 & 5 & 0 \end{array}\right]$$

$-\frac{1}{5}R_1 \to R_1$

$R_2 + 5R_1 \to R_2$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_2 = 0$

let $k_2 = r$

then $k_1 = r$

if $r = 1$

$k_1 = k_2 = 1$

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1 = K e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

P: picking $\lambda_2 = -1$

$(A - \lambda_2 I) P = K$

$(A + I) P = K$

$$\left[\begin{array}{cc|c} -5 & 5 & 1 \\ -5 & 5 & 1 \end{array}\right]$$

$-\frac{1}{5}R_1 \to R_1$

$$\left[\begin{array}{cc|c} +1 & -1 & -1/5 \\ -5 & 5 & 1 \end{array}\right] \qquad R_2 + 5R_1 \to R_2$$

$9 \quad 1 \quad 2 \quad 1 \quad 1.1.P \to R$ $\qquad 11$

$\cancel{\text{Let } r = \frac{1}{3}}$

$\cancel{P_1 = 0, \; P_2 = \frac{1}{3}}$

$\cancel{P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}}$

Let $r = 0$ $\qquad 9$

$P_2 = 0, \; P_1 = -\frac{1}{3}$

$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$

$X_2 = tKe^{\lambda_1 t} + Pe^{\lambda_2 t}$

$t\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} e^{2t}$

$X = c_1 X_1 + c_2 X_2$

$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} e^{2t} \right]$

32) $\frac{dx}{dt} = 12x - 9y \;, \; \frac{dy}{dt} = 4x$

$A = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$

λ:

$X = Ke^{\lambda t}$

$|A - \lambda I| = \begin{vmatrix} 12-\lambda & -9 \\ 4 & 0-\lambda \end{vmatrix}$

$(12-\lambda)(-\lambda) - (-9)(4) = 0$

$-12\lambda + \lambda^2 + 36 = 0$

$\lambda^2 - 12\lambda + 36 = 0$

$\lambda^2 - 6\lambda - 6\lambda + 36 = 0$

$\lambda(\lambda - 6) - 6(\lambda - 6) = 0$

$(\lambda - 6)(\lambda - 6) = 0$

$\lambda_1 = \lambda_2 = 6$

$R_1 - R_2 = 0$

$R_1 = R_2$

let $R_2 = \gamma$

$R_1 = \gamma$

if $\gamma = 1$

$R_1 = R_2 = 1$

$$K = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1 = K e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

P:

picking $\lambda_2 = 2$

$(A - \lambda I) P = K$

$(A - 2I) P = K$

$$\left[\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right] \quad -\tfrac{1}{3} R_1 \to R_1$$

$$\left[\begin{array}{cc|c} +1 & -1 & -1/3 \\ -3 & 3 & 1 \end{array}\right] \quad R_2 + 3R_1 \to R_2$$

$$\left[\begin{array}{cc|c} +1 & -1 & -1/3 \\ 0 & 0 & 0 \end{array}\right]$$

$P_1 - P_2 = -\frac{1}{3}$

$P_1 = P_2 - \frac{1}{3} \Rightarrow \frac{3P_2 - 1}{3}$

let $P_2 = \gamma$

31) $\frac{dx}{dt} = -x + 3y$, $\frac{dy}{dt} = -3x + 5y$

$A = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$X = Ke^{\lambda t}$

λ:

$|A - \lambda I| = \left| \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$

$= \begin{vmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{vmatrix}$

$(-1-\lambda)(5-\lambda) - (-3)(3)$

$-5 + \lambda - 5\lambda + \lambda^2 + 9 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda^2 - 2\lambda - 2\lambda + 4 = 0$

$\lambda(\lambda - 2) - 2(\lambda - 2) = 0$

$(\lambda - 2)(\lambda - 2) = 0$

$\lambda_1 = \lambda_2 = +2$

K: picking $\lambda_1 = 2$

$(A - 2I)K = 0$

$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$-\frac{1}{3}R_1 \to R_1$

$R_2 + 3R_1 \to R_2$

K:

Picking $\lambda_1 = 6$

$(A - \lambda_1 I)K = 0$

$(A - 6I)K = 0$

$$\left[\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right] \quad -\tfrac{1}{3}R_1 \to R_1,\ \tfrac{1}{2}R_2 \to R_2$$

$R_2 + R_1 \to R_2$

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right] \quad R_2 + R_1 \to R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-2k_1 + 3k_2 = 0$

$-2k_1 = -3k_2$

$k_1 = \frac{-3}{-2}k_2$

$k_1 = \frac{3}{2}k_2$

Let $k_2 = \gamma$

if $\gamma = 2$

$k_1 = 3$

so => $k_1 = 3$, $k_2 = 2$

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

P: picking $\lambda = \lambda_2 = 6$

$(A - \lambda_2 I)P = K$

$(A - 6I)P = K$

$$\left[\begin{array}{cc|c} 6 & -9 & 3 \\ 4 & -6 & 2 \end{array}\right] \quad \tfrac{1}{6}R_1 \to R_1$$

$$\left[\begin{array}{cc|c} 1 & -3/2 & 1/2 \\ 4 & -6 & 2 \end{array}\right] \quad R_2 - 4R_1 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & -3/2 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$$

$$P_1 - \frac{3}{2} P_2 = \frac{1}{2}$$

$$P_1 = \frac{1}{2} + \frac{3}{2} P_2$$

let $P_2 = r$

if $r = 0$

$P_1 = 1/2 \quad , P_2 = 0$

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$X_2 = tKe^{\lambda_1 t} + Pe^{\lambda_2 t}$$

$$= t\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{6t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{6t}$$

$$X = c_1 X_1 + c_2 X_2$$

$$X = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{6t} + c_2 \left[t\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{6t} \right]$$

33)

$$\frac{dx}{dt} = 3x - y - z, \quad \frac{dy}{dt} = x + y - z$$

$$\frac{dz}{dt} = x - y + z$$

$$X' = AX$$

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$X = Ke^{\lambda t}$$

λ:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1-\lambda \\ 1 & -1 \end{vmatrix}$$

$$= (3-\lambda)\left[(1-\lambda)(1-\lambda) + 1\right] + 1(1-\lambda+1) - 1(-1-1+\lambda)$$

$$(3-\lambda)\left[\lambda^2 - 2\lambda + \cancel{1} - \cancel{1}\right] + 1(2-\lambda) - 1(-2+\lambda) = 0$$

$$(3-\lambda)(\lambda^2 - 2\lambda) + 2 - \lambda + 2 - \lambda = 0$$

$$(3-\lambda)(\lambda^2 - 2\lambda) + (4 - 2\lambda) = 0$$

$$(3-\lambda)\left[(\lambda^2 - 2\lambda)\right] + 2(2-\lambda) = 0$$

$$(3-\lambda)\left[\lambda(\lambda - 2)\right] - 2(-2+\lambda) = 0$$

$$(\lambda - 2)\left[\lambda(3-\lambda) - 2\right] = 0$$

$$(\lambda - 2)(3\lambda - \lambda^2 - 2) = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_3 = 0$

$k_2 - k_3 = 0$

$k_1 = k_3$

$k_2 = k_3$

let $k_3 = r$

$k_1 = r, \; k_2 = r$

if $r = 1$

$k_1 = k_2 = 1$

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X_1 = K e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t}$$

P: $(A - \lambda_2 I)P = K$

$(A - 2I)P = K$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \quad R_2 - R_1 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \quad R_3 - R_1 \to R_3$$

$$q_1 - q_2 - q_3 = 1$$

$$q_1 = q_2 + q_3 + 1$$

let $q_2 = r$, $q_3 = s$

$$q_1 = r + s + 1$$

if $r = 0, s = 1 \Rightarrow q_1 = 2$

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$X_3 = tPe^{\lambda_3 t} + Qe^{\lambda_3 t}$$

$$= t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \left[t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right]$$

34) $\frac{dx}{dt} = 3x + 2y + 4z$, $\frac{dy}{dt} = 2x + 2z$, $\frac{dz}{dt} = 4x + 2y + 3z$

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

λ:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix}$$

$$= (3-\lambda)\left[-\lambda(3-\lambda)-4\right] - 2\left[6-2\lambda-8\right] + 4\left[4+4\lambda\right]$$

$$= (3-\lambda)\left[-3\lambda+\lambda^2-4\right] - 2(-2\lambda-2) + 4(4\lambda+4)$$

$$= (3-\lambda)\left[\lambda^2-3\lambda-4\right] + 4\lambda + 4 + 16\lambda + 16$$

$$= (3-\lambda)\left[\lambda^2-4\lambda+\lambda-4\right] + 20\lambda + 20$$

$$= (3-\lambda)\left[\lambda(\lambda-4)+1(\lambda-4)\right] + 20\lambda + 20$$

$$= (3-\lambda)(\lambda+1)(\lambda-4) + 20(\lambda+1)$$

$$= (\lambda+1)\left[(3-\lambda)(\lambda-4)+20\right]$$

$$= (\lambda+1)(3\lambda-12-\lambda^2+4\lambda+20)$$

$$= (\lambda+1)(-\lambda^2+7\lambda+8)$$

$$= (\lambda+1)(\lambda^2-7\lambda-8)$$

$$= (\lambda+1)\left[\lambda^2-8\lambda+\lambda-8\right]$$

$$= (\lambda+1)\left[\lambda(\lambda-8)+1(\lambda-8)\right]$$

$$= (\lambda+1)(\lambda+1)(\lambda-8)$$

$$\lambda_1 = 8, \quad \lambda_2 = \lambda_3 = -1$$

K:

$$\lambda = \lambda_1 = 8$$

$$(A-\lambda_1 I)K = 0$$

$$(A-8I)K = 0$$

P:

$\lambda = \lambda_2 = -1$

$(A - \lambda_2 I)P = 0$

$(A + I)P = 0$

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

$2R_2 - R_1 \to R_2$

$R_3 - R_1 \to R_3$

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{4}R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$P_1 + P_2/2 + P_3 = 0$

$P_1 = -P_2/2 - P_3$

Let $P_2 = r$, $P_3 = s$

$P_1 = -\frac{r}{2} - s$

if $r = -2$, $s = 1$

$P_1 = 0$

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$X_2 = Pe^{\lambda_2 t} = Pe^{-t}$

$$= \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{-t}$$

K:

$$(A - \lambda_1 I)K = 0$$

$$(A - 0)K = 0$$

$$\left[\begin{array}{ccc|c} 5 & -4 & 0 & 0 \\ 1 & -4 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -4/5 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right]$$

$\frac{1}{5}R_1 \to R_1$ $\qquad$ $R_2 - R_1 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -4/5 & 0 & 0 \\ 0 & 4/5 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & -4/5 & 0 \\ 0 & 1 & 5/2 \\ 0 & 2 & 5 \end{bmatrix}$$

$5/4 R_2 \to R_2$ $\qquad$ $R_1 + \frac{4}{5}R_2 \to R_1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/2 \\ 0 & 2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 - 2R_2 \to R_3$

$k_1 + 2k_2 = 0$

$k_1 = -2k_2$

$k_2 + 5/2\, k_3 = 0$

$k_2 = -5/2\, k_3$

let $k_3 = r$

$k_1 = -2r, \quad k_2 = -5/2\, r$

if $r = 2$

$k_1 = -4, \quad k_2 = -5$

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

$$X_1 = Ke^{\lambda_1 t} = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix} e^0$$

$$= \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

Q:

$\lambda = \lambda_3 = -1$

$(A - \lambda_3 I) Q = P$

$(A + I) Q = P$

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & -2 \\ 4 & 2 & 4 & 1 \end{array}\right]$$

$2R_2 - R_1 \to R_2$

$R_3 - R_1 \to R_3$

$\frac{1}{4} R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$q_1 + q_2/2 + q_3 = 0$

$q_1 = -q_2/2 - q_3$

let $q_2 = r$, $q_3 = s$

$q_1 = -\frac{r}{2} - s$

if $r = -2$; $s = 0$

$q_1 = -\frac{2}{-2} - 0 = 1$

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$X_3 = tPe^{\lambda_2 t} + Qe^{\lambda_3 t}$

$$X_3 = t\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} e^{-t}$$

$X = c_1 X_1 + c_2 X_2 + c_3 X_3$

$$X = c_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{t}$$

$$+ c_3 \left[t\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t} \right]$$

35) $X' = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} X$

$X' = AX$

$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

λ: $(A - \lambda I) = \begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix}$

$= (5-\lambda)\begin{vmatrix} -\lambda & 0 \\ 2 & 5-\lambda \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 0 & 5-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & -\lambda \\ 0 & 2 \end{vmatrix}$

$= (5-\lambda)\left[-\lambda(5-\lambda) - 4\right] + 4(5-\lambda) + 0$

$(5-\lambda)\left[-\lambda(5-\lambda) \cancel{-4} + 4\right] = 0$

$-\lambda(5-\lambda)^2 = 0$

$\lambda_1 = 0$

$\lambda_2 = \lambda_3 = 5$

P:

$$(A-\lambda_2 I)P = 0$$

$$(A-5I)P = 0$$

$$\left[\begin{array}{ccc|c} 0 & -4 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -5 & 2 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_1$ $\qquad -\frac{1}{4}R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & -5 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

$R_1 + 5R_2 \rightarrow R_1$ $\qquad R_3 - R_2 \times 2 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$P_1 + 2P_3 = 0$

$P_2 = 0$

$P_1 = -2P_3$

let $P_3 = r$

$P_1 = -2r$

if $P_3 = 1$

$P_1 = -2$

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$X_2 = Pe^{\lambda_2 t}$

$$X_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{5t}$$

Q:-

$(A-\lambda_3 I)Q = P$

$(A-5I)Q = P$

$$\left[\begin{array}{ccc:c} 0 & -4 & 0 & -2 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & -5 & 2 & 0 \\ 0 & -4 & 0 & -2 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_1$ $\qquad -\frac{1}{4}R_2 \to R_2$

$$\left[\begin{array}{ccc:c} 1 & -5 & 2 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 2 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & 2 & 5/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

$R_1+5R_2 \to R_1$ $\qquad R_3-2R_2 \to R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & 2 & 5/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$q_1 + 2q_3 = 5/2$

$q_2 = 1/2$

$q_1 = 5/2 - 2q_3$

Let $q_3 = r$

$q_1 = 5/2 - 2r$

If $r = 0$

$q_1 = 5/2,\ q_3 = 0$

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$X_3 = tPe^{\lambda_2 t} + Qe^{\lambda_3 t}$

$$= t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} 5/2 \\ 1/2 \\ 0 \end{bmatrix} e^{5t}$$

$X = c_1X_1 + c_2X_2 + c_3X_3$

$$\Rightarrow c_1\begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{5t} +$$

$$c_3\left[\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} te^{5t} + \begin{pmatrix} 5/2 \\ 1/2 \\ 0 \end{pmatrix} e^{5t}\right]$$

3b)

$$X' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} X$$

$$X' = AX$$

λ:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} \Rightarrow (1-\lambda)(3-\lambda)(1-\lambda)+1$$

$$= (1-\lambda)\left[(3-\lambda)(1-\lambda)+1\right] = (1-\lambda)\left[3-3\lambda-\lambda-\lambda^2+1\right]$$

$$= (1-\lambda)\left[\lambda^2-4\lambda+4\right] = (1-\lambda)\left[\lambda^2-2\lambda-2\lambda+4\right]$$

$$= (1-\lambda)(\lambda-2)^2$$

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 2$$

K:

$$(A - \lambda_1 I)K = 0$$

$$(A - I)K = 0$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{} \Rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \underset{R_1 + 2R_3 \rightarrow R_3}{}$$

$$\underset{R_2 \leftrightarrow R_3}{\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]} \Rightarrow \underset{\frac{1}{2}R_1 \rightarrow R_1}{\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \Rightarrow \underset{R_1 - \frac{1}{2}R_2 \rightarrow R_1}{\left[\begin{array}{ccc|c} 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

P:

$(A - \lambda_2 I)P = 0$

$(A - I)P = K$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ $\qquad$ $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-1 \times R_2 \to R_2$ $\qquad$ $R_1 + R_2 \to R_1$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{2} R_1 \to R_1$

$-P_2 + P_3 = -1$

$P_1 = 0$

$-P_2 = -1 - P_3$

$P_2 = P_3 + 1$

let $P_3 = r$ $\qquad P_2 = r + 1$

if $r = 0$

$P_2 = 1$

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$X_2 = tKe^{\lambda_1 t} + Pe^{\lambda_2 t}$

$$= t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t}$$

2-2

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ $\quad$ $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-1 \times R_2 \to R_2$ $\quad$ $R_1 + R_2 \to R_1$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{2} R_1 \to R_1$

$k_1 = 0, \quad -k_2 + k_3 = 0$

$-k_2 = -k_3$

$k_2 = k_3$

let $k_3 = r$

if $r = 1$

$k_2 = k_3 = 1$

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$X_1 = K e^{\lambda_1 t}$

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t}$$

$$X = c_1X_1 + c_2X_2 + c_3X_3$$

$$= c_1X_1 + c_2X_2 + c_3X_3$$

$$X = c_1\begin{bmatrix}1\\0\\0\end{bmatrix}e^{t} + c_2\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{2t} + c_3\left[\begin{pmatrix}0\\-1\\1\end{pmatrix}te^{2t} + \begin{pmatrix}0\\-1\\0\end{pmatrix}e^{2t}\right]$$

37) $X' = \begin{pmatrix}1 & 0 & 0\\2 & 2 & -1\\0 & 1 & 0\end{pmatrix}X$

$$A = \begin{bmatrix}1 & 0 & 0\\2 & 2 & -1\\0 & 1 & 0\end{bmatrix}, \quad X = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$X' = AX$$

λ:

$$|A - \lambda I| = \begin{vmatrix}1-\lambda & 0 & 0\\2 & 2-\lambda & -1\\0 & 1 & -\lambda\end{vmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)(-\lambda)+1\right]$$

$$= (1-\lambda)\left[-2\lambda+\lambda^2+1\right] \Rightarrow (1-\lambda)\left[\lambda^2-2\lambda+1\right]$$

$$= (1-\lambda)(\lambda^2-\lambda-\lambda+1) \Rightarrow (1-\lambda)\left[\lambda(\lambda-1)-1(\lambda-1)\right]$$

$$= (1-\lambda)(\lambda-1)^2$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1$$

K:

$$(A-\lambda_1 I)K = 0$$

$$(A-I)K = 0$$

Q :

$(A - r_3 I) Q = P$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ $\qquad$ $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-1 \times R_2 \to R_2$ $\qquad$ $R_1 + R_2 \to R_1$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$1/2\, R_1 \to R_1$

$q_1 = 1/2$

$-q_2 + q_3 = 0$

$q_2 = q_3$

let $q_3 = r$

$q_2 = r$

if $r = 0$

$q_2 = q_3 = 0$

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \frac{t^2}{2} K e^{r_1 t} + t P e^{r_2 t} + Q e^{r_3 t}$$

$$= \frac{t^2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} e^t$$

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$= c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t \right]$$

$$+ c_3 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{t^2}{2} e^t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t e^t + \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} e^t \right]$$