Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

Mahwish Waqas

Course Outline

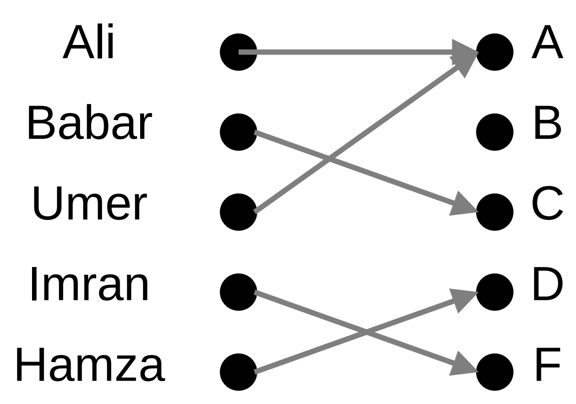
- Functions
  - What is a function?
  - Function Arithmetic
  - One-to-one, Onto and Bijective Functions
  - Identity and Inverse Function
  - Composition of Functions

Application of Functions

- Define discrete structures such as sequences and strings
- Represent the time that a computer takes to solve problems of a given size
- Represent the complexity of algorithms
- …

Functions

- In many examples we assign to each element of a set, a particular element of a second set (which may be the same as the first).
- For example, suppose that each student in a discrete structures class is assigned a letter grade from the set {A, B, C, D, F}.
- This assignment is an example of a function.



Functions

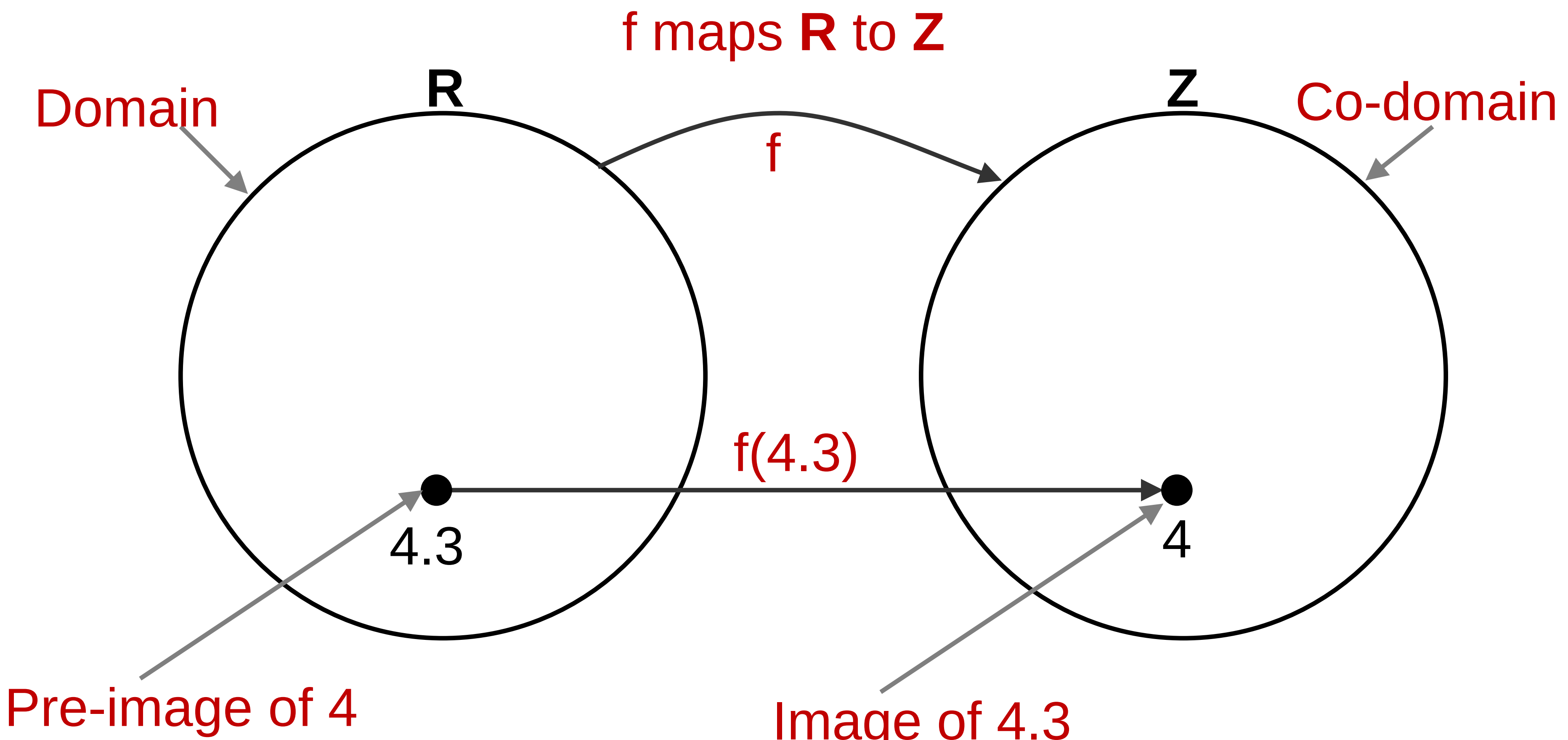
- Let **A** and **B** be nonempty sets.
- A **function** *f* from **A** to **B** is an assignment of exactly one element of **B** to each element of **A**.
- If **b** is the unique element of **B** assigned by the function ***f*** to the element **a** of **A**, we write ***f*(a) = b**.
- If ***f*** is a function from **A** to **B**, we write ***f*: A** → **B**.

Functions

- If ***f*** is a function from **A** to **B**, we say that **A** is the domain of ***f*** and **B** is the codomain of ***f***.
- if ***f*****(a) = b**, we say that **b** is the image of **a** and **a** is the pre-image of **b**.
- The range of ***f*** is the set of all images of elements of **A**.
- Also, if ***f*** is function from **A** to **B**, we say that ***f*** maps **A** to **B**.

Functions

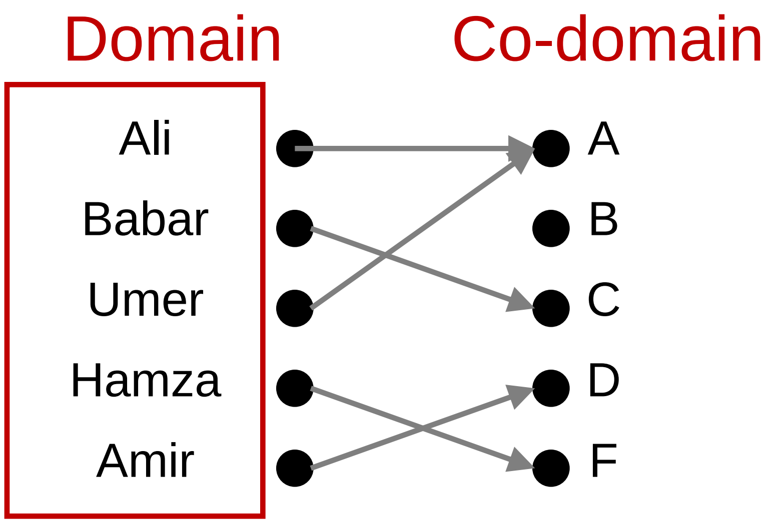
- A function takes an element from a set and maps it to a UNIQUE element in another set.



Arrow Diagram of Functions

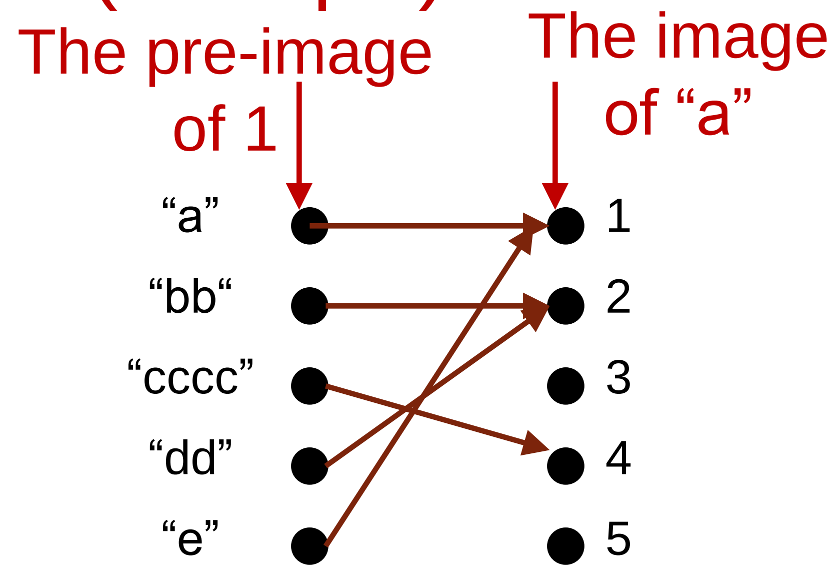
- The definition of a function implies that the arrow diagram for a function ***f*** has the following two properties:
- Every element of **A** has an arrow coming out of it.
- No elements of **A** has two arrows coming out of it that point to two different elements of **B**.

Arrow Diagram of Functions(example)



A class grade function

g(Ali) = A
g(Babar) = C
g(Umer) = A
…

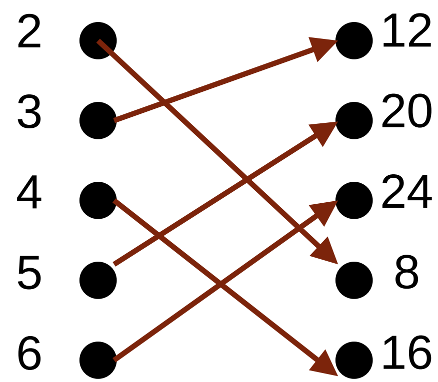


A string length function

f(x) = length x

f("a") = 1
f("bb") = 2
…

Arrow Diagram of Functions(example)



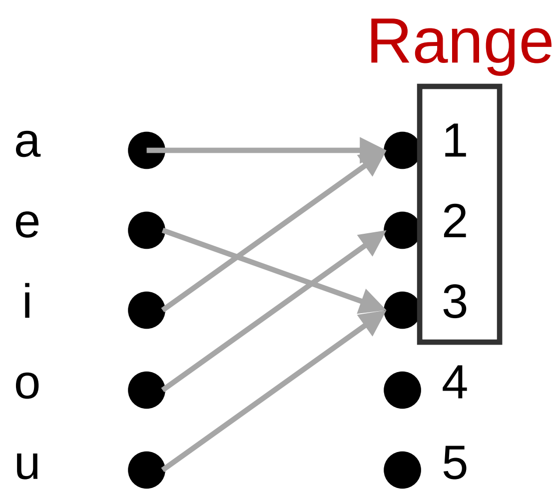
$$f(x) = 4x$$

$$f(2) = 8$$
$$f(3) = 12$$

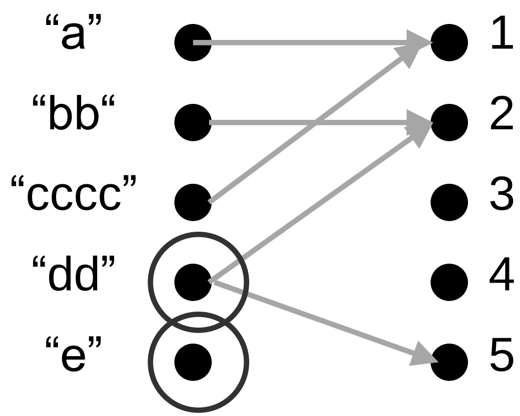
…

Functions

The ***range*** of *f* is the set of all images of elements of *A*.



Functions

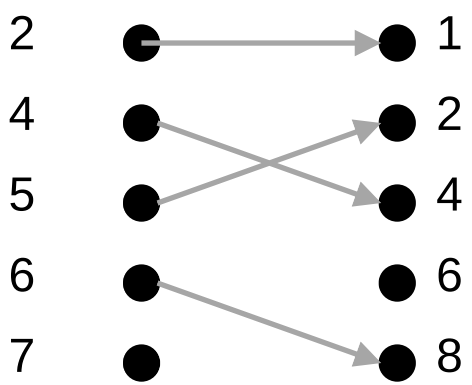


Not a valid function

Functions and Non-Functions

- Which of the arrow diagrams define functions from
- A = {2,4,5,6,7} to B = {1,2,4,6,8}.

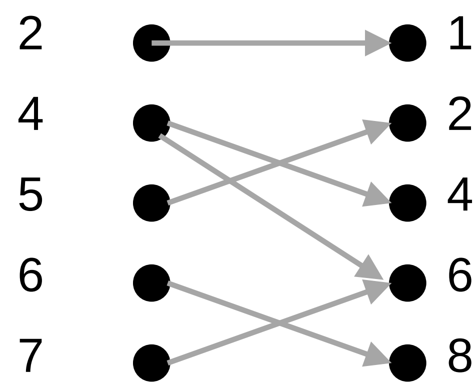
Domain Co-domain



A B

Not a Function

Domain Co-domain



A B

Not a Function

Example

- Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$

  assign the square of an integer to this integer
- What is $f(x)$ =?
  - $f(x) = x^2$
- What is domain of f ?
  - Set of all integers
- What is codomain of f ?
  - Set of all integers
- What is the range of f ?
  - $\{0, 1, 4, 9, \ldots\}$. All integers that are perfect squares

Function Arithmetic

- Just as we are able to **add (+), subtract (-), multiply (*), and divide (÷)** two or more numbers, we are able to **+ , - , * ,** and **÷** two or more functions.

- Let *f* and *g* be functions from ***A*** to **R**. Then $f + g$, $f - g$, $f * g$ and f/g are also functions from ***A*** to **R** defined for all $\boldsymbol{x} \in \boldsymbol{A}$ by:
- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(f * g)(x) = f(x) * g(x)$
- $(f/g)(x) = f(x)/g(x)$ given that $g(x) \neq 0$

Function Arithmetic

Let $\mathbf{f_1}$ and $\mathbf{f_2}$ be functions from **R** to **R** such that:

- $\mathbf{f_1(x) = 2x}$
- $\mathbf{f_2(x) = x^2}$
- Find $f_1 + f_2$ and $f_1 * f_2$.
- $f_1+f_2 = (f_1+f_2)(x) = f_1(x)+f_2(x) = 2x+x^2$
- $f_1*f_2 = (f_1*f_2)(x) = f_1(x)*f_2(x) = 2x*x^2 = 2x^3$

Function Arithmetic

- Let **f** and **g** be functions from **R** to **R** such that:
- $\mathbf{f(x) = 3x+2}$ $\qquad$ $\mathbf{g(x) = -2x + 1}$
- What is the function $f * g$?

Function Arithmetic

- Let **f** and **g** be functions from **R** to **R** such that:
- **f (x) = 3x+2**   **g (x) = -2x + 1**
- What is the function $f * g$?
- $f*g = (f*g)(x) = f(x)*g(x) = (3x+2)*(-2x+1) = -6x^2 - x + 2$

Function Arithmetic

- Let **f** and **g** be functions from **R** to **R** such that:
- **f (x) = 3x+2** **g (x) = -2x + 1**
- What is the function $f * g$?
- f*g = (f*g) (x) = f (x)*g(x) = (**3x**+2)*(-2x+1) = $-6x^2 - x + 2$

Let x = -1, What is f(-1)*g(-1) and (f*g)(-1)?

Function Arithmetic

- Let **f** and **g** be functions from **R** to **R** such that:
- **f (x) = 3x+2** **g (x) = -2x + 1**
- What is the function $f * g$?
- $f*g = (f*g)(x) = f(x)*g(x) = (\mathbf{3x}+2)*(-2x+1) = -6x^2 - x + 2$

Let $x = -1$, What is $f(-1)*g(-1)$ and $(f*g)(-1)$?

- $f(-1) = 3(-1) + 2 = -1$
- $g(-1) = -2(-1) + 1 = 3$
- $f(-1)*g(-1) = -1 \times 3 = -3$
- $(f*g)(-1) = -6(-1)2 - (-1) + 2 = -6+1+2 = -3$

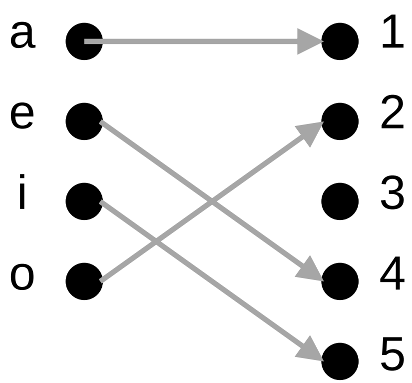
Function Arithmetic

- let $f(x)=5x+2$ and $g(x)=x^2-1$, At $x=4$, $f(4)=22$ and $g(4)=15$.

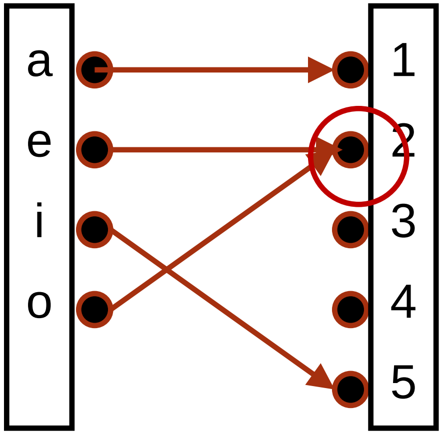
Arithmetic Expression	Combine evaluate	Evaluate, then combine
$(f+g)(x)$ $(5x+2) + (x^2-1)$ $=x^2+5x+1$	$(f+g)(4)$ $4^2+5(4)+1 =16+20+1$ $=37$	$f(4)+g(4)$ $22+15$ $=37$
$(f-g)(x)$ $(5x+2) - (x^2-1)$ $=-x^2+5x+3$	$(f-g)(4)$ $-4^2+5(4)+3$ $=-16+20+3 =7$	$f(4)-g(4)$ $22-15$ $=7$
$(f*g)(x)$ $(5x+2)*(x^2-1)$ $=5x^3+2x^2-5x-2$	$(f*g)(4)$ $5(4^3)+2(4^2)-5(4)-2$ $=5(64)+2(16)-20-2 =330$	$f(4)*g(4)$ $22*15$ $=330$
$(f/g)(x)$ $(5x+2)/(x^2-1)$	$(f/g)(4)$ $[5(4)+2]/[4^2-1]$ $=22/15$	$f(4)/g(4)$ $22/15$

One-to-One Function

- A function is one-to-one if each element in the co-domain has a unique pre-image
- Formal definition: A function ***f*** is one-to-one if f(a) = f(b) implies a = b for all a and b in the domain of ***f***.



A one-to-one function



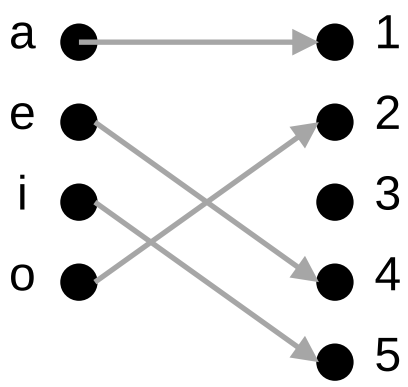
A function that is not one-to-one

One-to-One Function

- f is one-to-one using quantifiers as
- $\forall a \forall b (f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a \forall b \big(a \neq b \rightarrow f(a) \neq f(b)\big)$
- Where a and b in domain of f.

More on One-to-One Functions

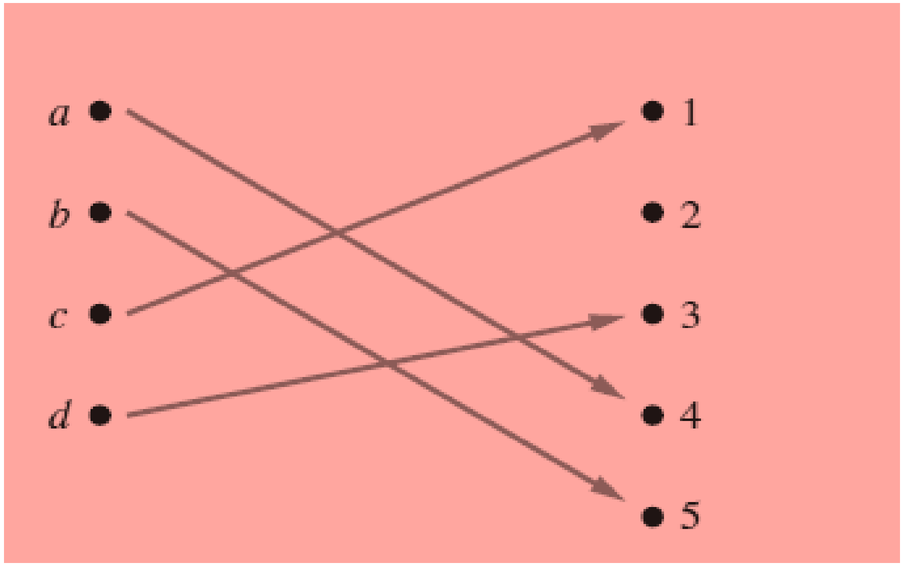
- Injective is synonymous with one-to-one
  - “A function is injective”
- A function is an injection if it is one-to-one
- Note that there can be un-used elements in the co-domain



A one-to-one function

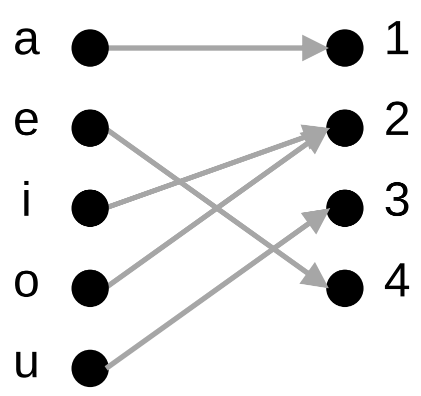
Example one-to-one function

- Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4, f(b) = 5, f(c) = 1, and$ $f(d) = 3$ is one-to-one.

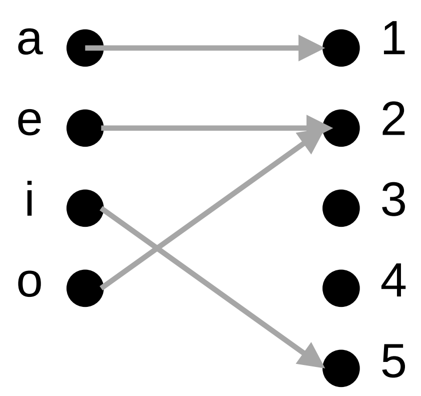


Onto Functions

- A function is onto if each element in the co-domain is an image of some pre-image
- Formal definition: A function f is onto if for all b $\in$ B, there exists a $\in$ A such that f(a) = b.



An onto function



A function that is not onto

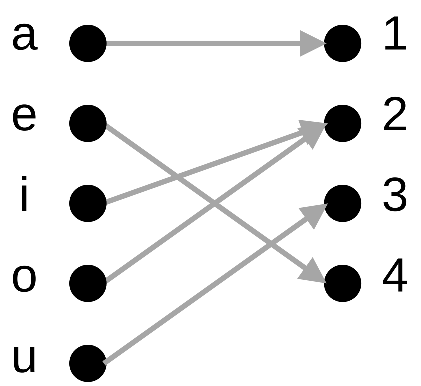
Onto functions

- A function f is onto if $\forall b \exists a (f(a) = b)$, where the domain for a is the domain of the function and the domain for b is the codomain of the function.

More on Onto Functions

- Surjective is synonymous with onto
  - "A function is surjective"
- A function is a surjection if it is onto

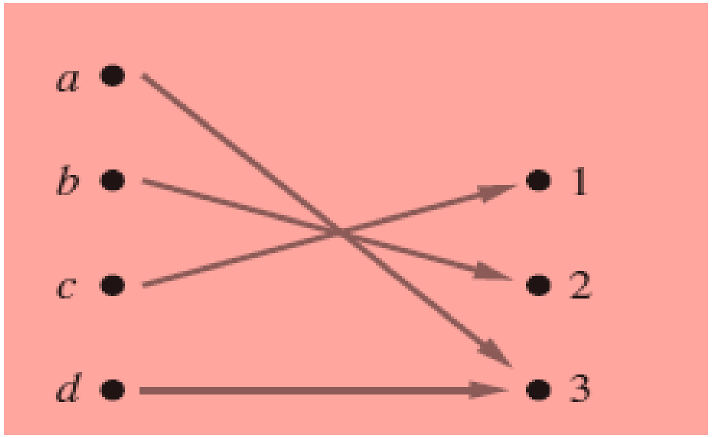
- Note that there can be multiple used elements in the co-domain



An onto function

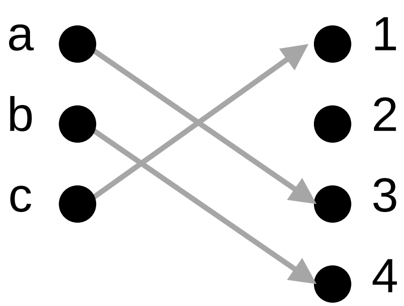
Example onto function

- Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1, and$ $f(d) = 3$. Is f an onto function?

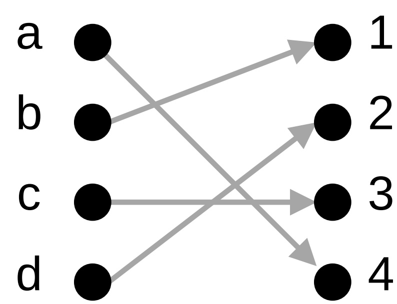


Onto vs One-to-One

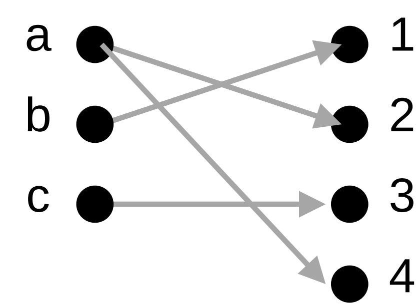
- Are the following functions onto, one-to-one, both, or neither?



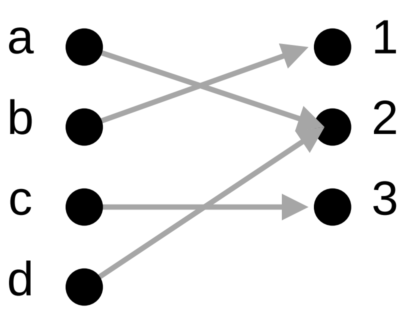
1-to-1, not onto



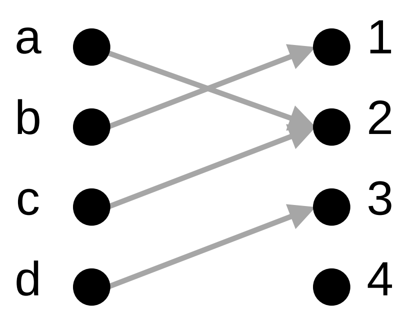
Both 1-to-1 and onto



Not a valid function



Onto, not 1-to-1



Neither 1-to-1 nor onto

Example

- Determine whether the function *f (x)* = x^2 from the set of integers to the set of integers is one-to-one.

$$f: Z \to Z;\ f(x) = x^2$$

To show f is one to one, let $x_{1,} x_2 \in$ Z and suppose

$$f(x_1) = f(x_2)$$

$$x_1^2 = x_2^2$$

$$x_1^2 - x_2^2 = 0$$

$$(x_1 + x_2)(x_1 - x_2) = 0$$

$$x_1 + x_2 = 0,\quad x_1 - x_2 = 0$$

$$x_1 = -x_2,\qquad x_1 = x_2$$

$$x_{1=} \pm x_2$$

Hence f is not one to one.

Example

- Determine whether the function *f (x)* = $x + 1$ from the set of integers to the set of integers is onto.

$$f: Z \to Z;\ f(x) = x + 1$$

Let y $\in$ Z. We look for an x $\in$ Z such that f(x) = y

$f(x) = y$

$x + 1 = y$ By definition of f

$x = y - 1$

Thus for each $y \in Z$, there exists $x = y - 1 \in Z$

such that $f(x) = f(y - 1)$

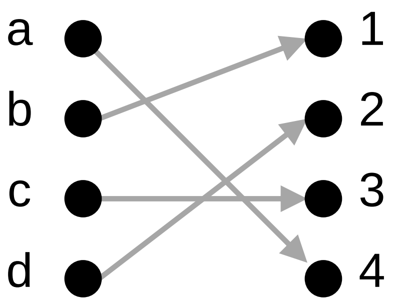
$= (y - 1) + 1 = y$

Hence f is onto.

Bijections

- Consider a function that is both one-to-one and onto:

- Such a function is a one-to-one correspondence, or a bijection.



Example

- Determine whether the following functions are bijective or not?

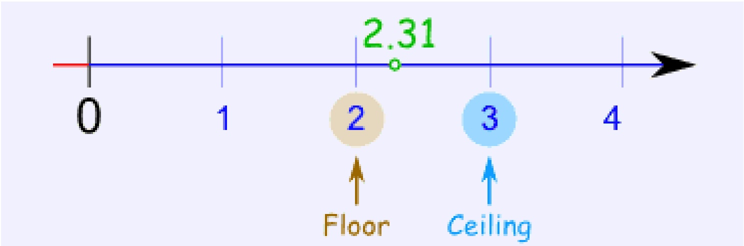
$$f: R \to R;\ f(x) = x^3$$

$$f: R - \{0\} \to R;\ f(x) = \frac{x+1}{x}$$

Floor and Ceiling

- floor(*x*) = ⌊X⌋ is the largest integer that is less than or equal to **x** and ceiling(*x*) = ⌈X⌉ is the smallest integer that is greater than or equal to **x**
- The floor and ceiling functions give you the **nearest integer** up or down.

Sample value *x*	**Floor $\lfloor x \rfloor$**	**Ceiling $\lceil x \rceil$**
12/5 = 2.4	2	3
2.7	2	3
−2.7	−3	−2
−2	−2	−2



Identity Functions

- A function such that the image and the pre-image are ALWAYS equal

- f(x) = 1*x
- f(x) = x + 0

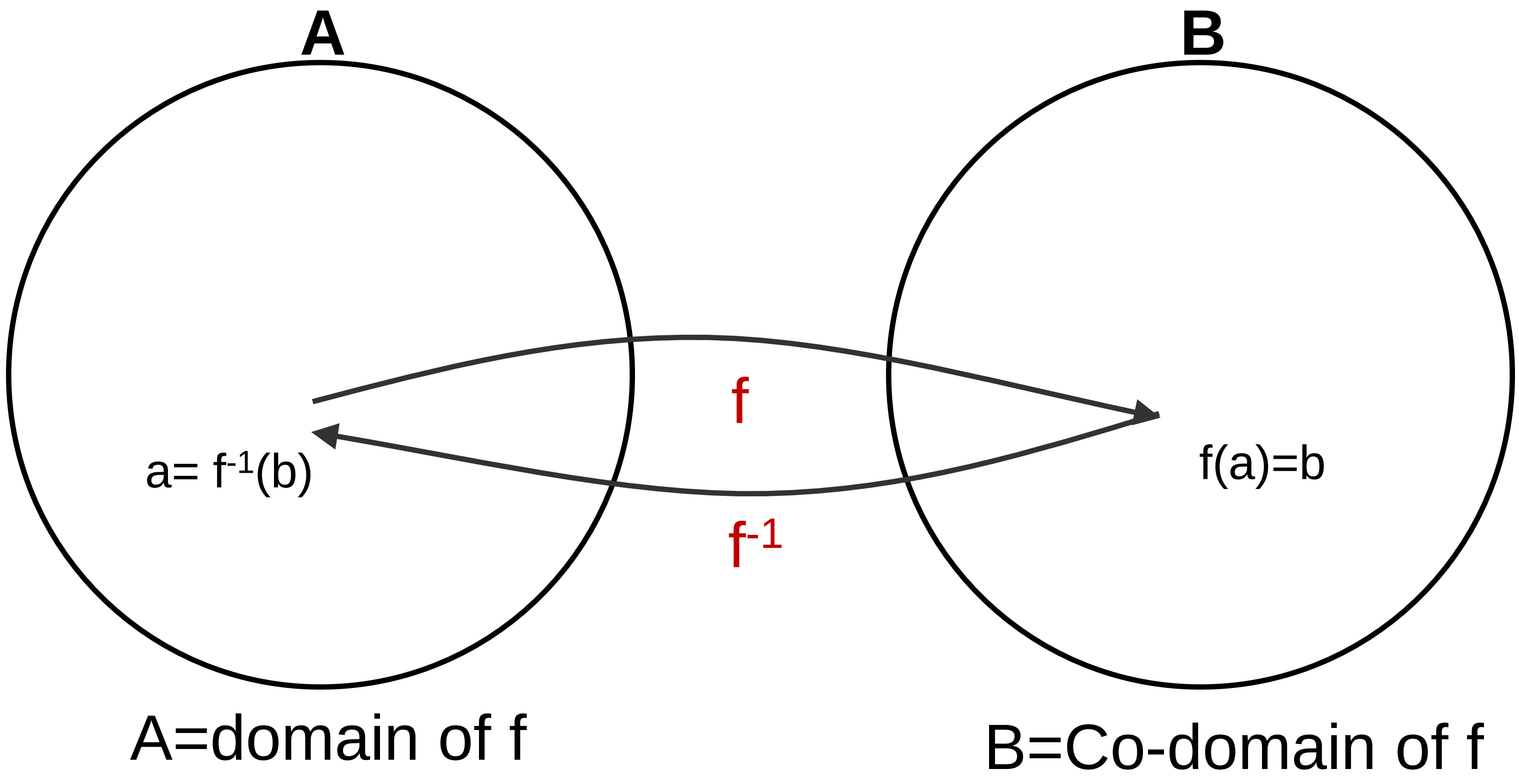
- The domain and the co-domain must be the same set.

Inverse Function

- Let ***f*** be a one-to-one correspondence from the set **A** to the set **B**.
- The inverse function of ***f*** is the function that assigns to an element in **b** belonging to **B** the unique element **a** in **A** such that ***f*****(a) = b**.
- The inverse function of ***f*** is denoted by $\boldsymbol{f^{-1}}$.
- Hence , $\boldsymbol{f^{-1}}$**(b) = a** when ***f*****(a) = b**.

Inverse Functions

If $f(a) = b$, then $f^{-1}(b) = a$

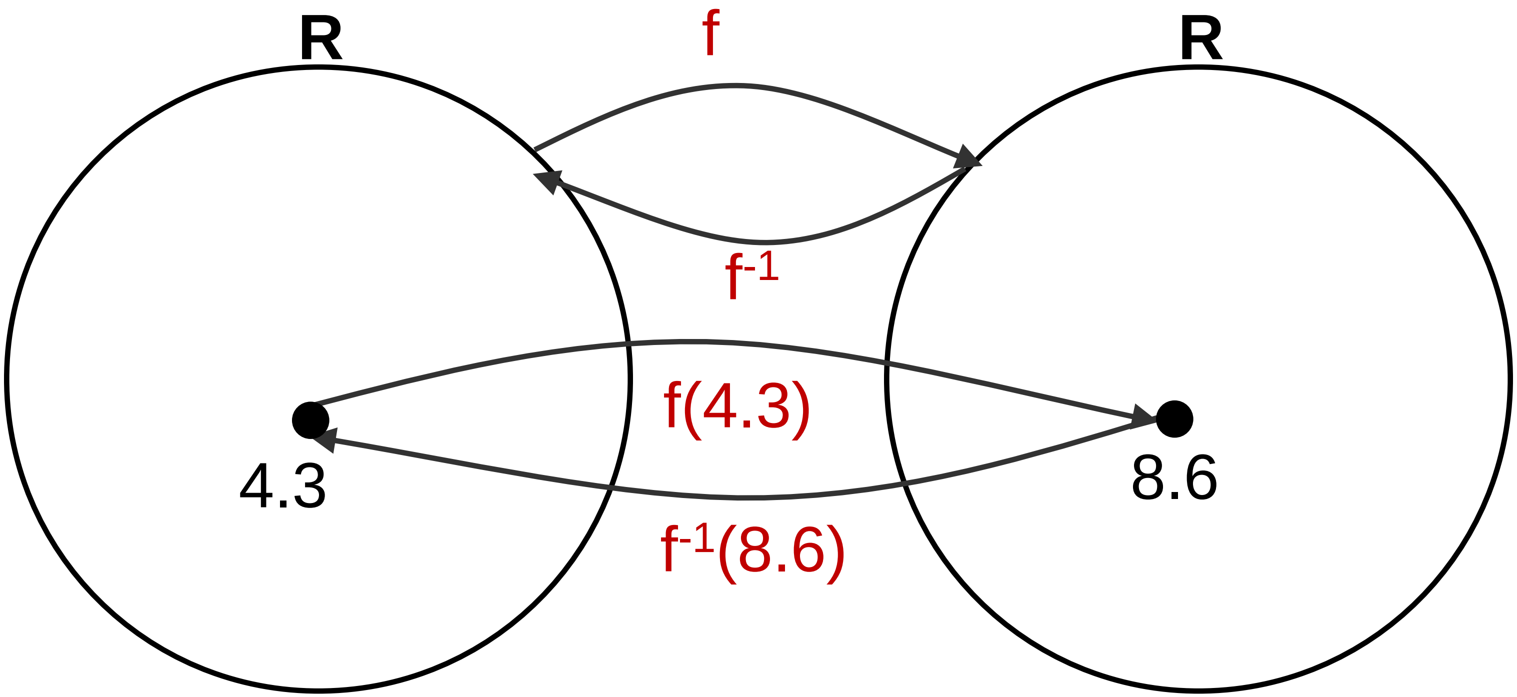


Inverse Functions

If $f(x) = y$, then $f^{-1}(y) = x$

Let $f(x) = 2*x$

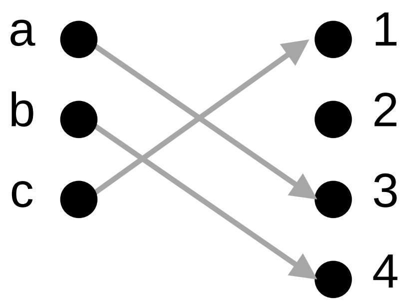
$f(x)=y$



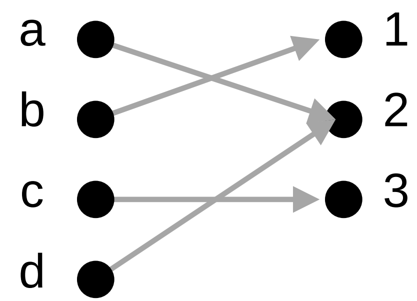
Then $f^{-1}(y) = y/2$

More on Inverse Functions

- Can we define the inverse of the following functions?



What is $f^{-1}(2)$?
Not onto!



What is $f^{-1}(2)$?
Not 1-to-1!

- An inverse function can ONLY be done defined on a bijection

More on Inverse Functions

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Example

- Let ***f*** be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.
- Is ***f*** invertible, and if it is, what is its inverse?

Example

- Let ***g*** be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ such that
  $g(a) = 3$, $g(b) = 4$, $g(c) = 1$, and $g(d) = 3$.
- Is ***g*** invertible, and if it is, what is its inverse?

Working Rule to Find Inverse Function

- Let f: X →Y be a one-to-one correspondence defined by the formula f(x) = y.

1. Solve the equation f(x) = y for x in terms of y.
2. $\mathbf{f}^{-1}$ (y) equals the right hand side of the equation found in step 1.

Example

Let f : Z → Z be such that f (x) = x + 1. Is f invertible, and if it is, what is its inverse?

The given function f is defined by rule

$$f(x) = y$$

$$x + 1 = y$$

$$x = y - 1$$

Hence $f^{-1}(y) = y - 1$

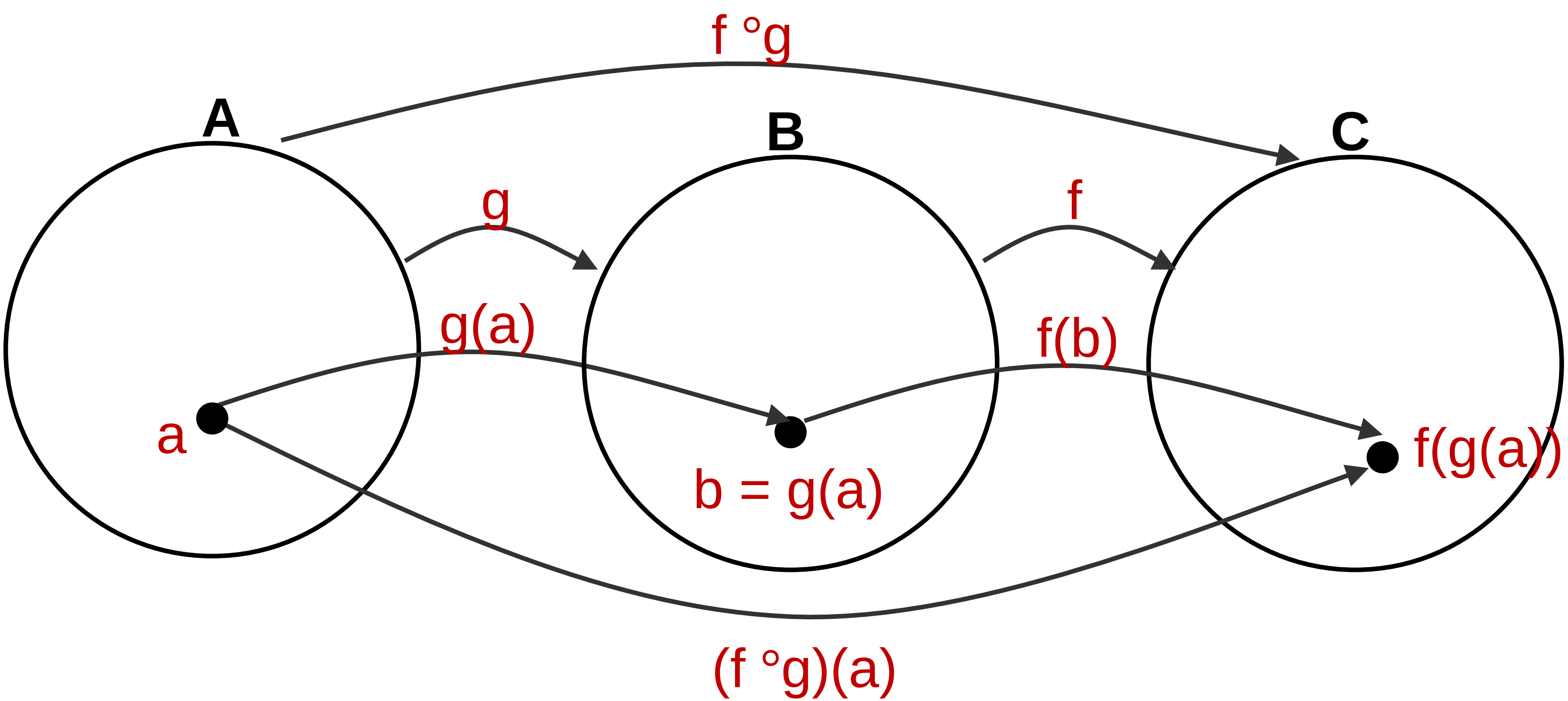
Compositions of Functions

- Let ***g*** be a function from the set **A** to the set **B** and let ***f*** be a function from the set **B** to the set **C**. the compositions of the functions f and g, denoted by $f \circ g$, is defined by

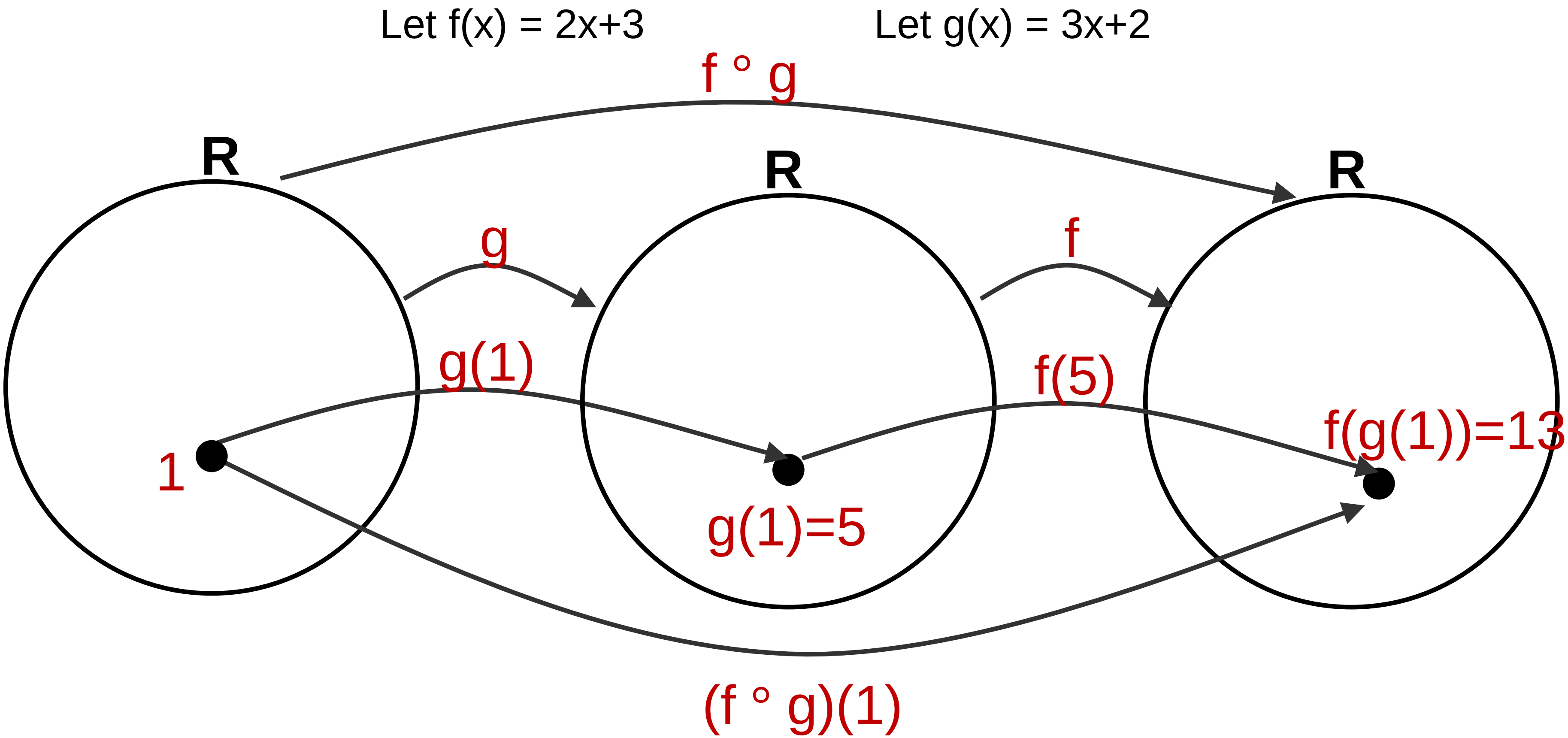
$$(f \circ g)(a) = f(g(a))$$

Compositions of Functions

$$(f \circ g)(a) = f(g(a))$$



Compositions of Functions



f(g(x)) = 2(3x+2)+3 = 6x+7

Compositions of Functions

Does $f(g(x)) = g(f(x))$?

Let $f(x) = 2x+3$

Let $g(x) = 3x+2$

$f(g(x)) = 2(3x+2)+3 = 6x+7$

$g(f(x)) = 3(2x+3)+2 = 6x+11$

Not equal!

Function composition is not commutative!

Compositions of Functions

- Let A = {1,2,3,4,5}

  f : A → A and g : A → A

  $f(1) = 3$, $f(2) = 5$, $f(3) = 3$, $f(4) = 1$, $f(5) = 2$

  $g(1) = 4$, $g(2) = 1$, $g(3) = 1$, $g(4) = 2$, $g(5) = 3$

  Find the composition functions $f \circ g$ and $g \circ f$.

$f \circ g$	$g \circ f$
$(f \circ g)(1) = f(g(1)) = f(4) = 1$	$(g \circ f)(1) = g(f(1)) = g(3) = 1$
$(f \circ g)(2) = ?$	$(g \circ f)(2) = ?$
$(f \circ g)(3) = ?$	$(g \circ f)(3) = ?$
$(f \circ g)(4) = ?$	$(g \circ f)(4) = ?$
$(f \circ g)(5) = ?$	$(g \circ f)(5) = ?$

Compositions of Functions

- Let A = {1,2,3,4,5}

  $f : A \rightarrow A$ and $g : A \rightarrow A$

  $f(1) = 3$, $f(2) = 5$, $f(3) = 3$, $f(4) = 1$, $f(5) = 2$

  $g(1) = 4$, $g(2) = 1$, $g(3) = 1$, $g(4) = 2$, $g(5) = 3$

  Find the composition functions $f \circ g$ and $g \circ f$.

$f \circ g$	$g \circ f$
$(f \circ g)(1) = f(g(1)) = f(4) = 1$	$(g \circ f)(1) = g(f(1)) = g(3) = 1$
$(f \circ g)(2) = 3$	$(g \circ f)(2) = 3$
$(f \circ g)(3) = 3$	$(g \circ f)(3) = 1$
$(f \circ g)(4) = 5$	$(g \circ f)(4) = 4$
$(f \circ g)(5) = 3$	$(g \circ f)(5) = 1$

Compositions of Functions

Let $g : A \rightarrow A$ be the function, Set A = {a, b, c} such that $g(a) = b$, $g(b) = c$, and $g(c) = a$.

Let $f :$ A $\rightarrow B$ be the function, Set A = {a, b, c} to the set B = {1, 2, 3} such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f?

Compositions of Functions

Let $g : A \to A$ be the function, Set A = {a, b, c} such that $g(a) = b$, $g(b) = c$, and $g(c) = a$.

Let $f : A \to B$ be the function, Set A = {a, b, c} to the set B = {1, 2, 3} such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f?

Solution:

The composition $f \circ g$ is defined by

$(f \circ g)(a) = f(g(a)) = f(b) = 2$,

$(f \circ g)(b) = f(g(b)) = f(c) = 1$,

$(f \circ g)(c) = f(g(c)) = f(a) = 3$.

$g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Exercise Questions

Chapter # 2

Topic # 2.3

Questions 1, 2, 8, 9,10,11,12, 22, 23, 36, 37