Spring-2023

Q1) U.C

$$y'' + 2y' = 2x + 5 - e^{-2x}$$

Sol:- $y'' + 2y' = 0$

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$m^2 + 2m = 0$

$m(m+2) = 0$

$\boxed{m_1 = 0, \quad m_2 = -2}$

$y_c = c_1 e^0 + c_2 e^{-2x}$

$\boxed{y_c = c_1 + c_2 e^{-2x}}$

$y_p = (Ax + B + Ce^{-2x})x$ $\left(\because \text{Multiple by 'x' because } y_c = \boxed{e^{-2x}} \Rightarrow y_p = C\boxed{e^{-2x}}\right)$

$y_p = Ax^2 + Bx + Cxe^{-2x}$

$y_p' = 2Ax + B + Cxe^{-2x}(-2) + Ce^{-2x}$

$y_p' = 2Ax + B - 2Cxe^{-2x} + Ce^{-2x}$

$y_p'' = 2A - 2Cxe^{-2x}(-2) - 2Ce^{-2x} - 2Ce^{-2x} + e^{-2x}(0)$

$y_p'' = 2A + 4Cxe^{-2x} - 4Ce^{-2x}$

$2A + 4Cxe^{-2x} - 4Ce^{-2x} + 4Ax + 2B - 4Cxe^{-2x} + 2Ce^{2x} = 2x + 5 - e^{-2x}$

$2A + 2B = 5$ — (i)

$4A = 2$

$\boxed{A = 2/4 = 1/2}$

$\cancel{2}\left(\frac{1}{\cancel{2}}\right) + 2B = 5$

$2B = 4$

$\boxed{B = 2}$

$2A + 2(2) = 5$

$2A + 4 = 5$

$2A = 5 - 4$

$\boxed{A = 1/2}$

$-4C + 2C = -1$

$-2C = -1$

$\boxed{C = 1/2}$

$y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$

$$\boxed{y = c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

2) $x^2y'' - 2xy' + 2y = x^3 \ln x$

Sol: Cauch-Euler:

$x^2y'' - 2xy' + 2y = 0$

$y = x^m, \; y' = mx^{m-1}, \; y'' = m(m-1)x^{m-2}$

$m(m-1) - 2m + 2 = 0$

$m^2 - m - 2m + 2 = 0$

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0$

$m_1 = 1, \; m_2 = 2$

$y_c = c_1x^1 + c_2x^2$

$\boxed{y_c = c_1x + c_2x^2}$

$y_p = (x^3 \cancel{\ln x})$

$y_p = u_1 y_1 + u_2 y_2$

$W = y_1 y_2' - y_2 y_1'$

$y_1 = x, \; y_2 = x^2$

$y_1' = 1, \; y_2' = 2x$

$= x(2x) - (x^2)(1)$

$W = 2x^2 - x^2 = x^2$

$W_1 = -y_2 f(x)$

$\frac{x^2 y''}{x^2} - \frac{2xy'}{x^2} + \frac{2y}{x^2} = \frac{x^3 \ln x}{x^2}$

$y'' - 2y' + \frac{2y}{x^2} = x \ln x$

$f(x) = x \ln x$

$W_1 = -x^2 \cdot x \ln x = -x^3 \ln x$

$W_2 = y_1 f(x)$

$= x \cdot x \ln x = x^2 \ln x$

$u_1 = \int \frac{W_1}{W} dx = -\int \frac{x^3 \ln x}{x^2} dx$

$= \int x \ln x \, dx$

~~$u_1 = x \cdot \frac{1}{x} - \int \frac{x}{x} dx = 1 - \ln x$~~

$u_1 = \cancel{1 - \ln x}$

Let $u = \ln x$, $du = 1/x \, dx$

$dv = x\,dx$, $v = \frac{x^2}{2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln x \cdot x\,dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$

$$u_1 = \frac{x^2}{2}\ln x - \frac{x^2}{4} = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$$

$$u_2 = \int \frac{w_2}{w}\,dx = \int \frac{x^2 \ln x}{x^2}\,dx$$

$$= \int \ln x\,dx$$

$$= x\ln x - x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)x + (x\ln x - x)x^2$$

$$= \frac{x^3}{2}\left(\ln x - \frac{1}{2}\right) + x^3\ln x - x^2$$

$$= \frac{x^3}{2}\ln x - \frac{x^3}{2} + x^3\ln x - x^2$$

$$\cancel{Y_p = x^3\left(\frac{1}{2}\ln x - \frac{1}{2} + \ln x\right) - x^2}$$

$$\boxed{\cancel{y = c_1 x + c_2 x^2 + x^3\left(\frac{1}{2}\ln x - \frac{1}{2} + \ln x\right) - x^2}}$$

$$Y_p = \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)x + x^3 \ln x - x^3$$

$$= \frac{x^3}{2}\ln x - \frac{x^3}{4} + x^3 \ln x - x^3$$

$$= \frac{x^3 \ln x}{2} + x^3 \ln x - \frac{x^3 - 4x^3}{4}$$

$$= \frac{x^3 \ln x}{2} + x^3 \ln x - \frac{5x^3}{4}$$

$$= \frac{x^3 \ln x}{2} + \frac{2}{2}x^3 \ln x - \frac{5}{4}x^3$$

$$= \frac{x^3 \ln x}{2} + \frac{2x^3 \ln x}{2} - \frac{5}{4}x^3$$

$$= \frac{x^3 \ln x + 2x^3 \ln x}{2} - \frac{5}{4}x^3$$

$$= \frac{3x^3 \ln x}{2} - \frac{5}{4}x^3$$

$$\boxed{Y_p = \frac{3x^3}{2}\ln x - \frac{5}{4}x^3}$$

$$y = y_c + y_p = c_1 x + c_2 x^2 + \frac{3}{2} x^3 \ln x - \frac{5}{4} x^3$$

Q3) $(x^2-1)y'' + xy' - y = 0$

Sol: Let $y = \sum_{n=0}^{\infty} C_n x^n$

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1} \; ; \; y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

$$(x^2-1)\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + x \sum_{n=1}^{\infty} n C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=1}^{\infty} n C_n x^{n-1+1} - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2+2} - \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^n - \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$= \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

Put $K = n-2$, $n = K+2$

$$= \sum_{K=0}^{\infty} (K+2)(K+1) C_{K+2} x^K \quad - (i)$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^n + \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} C_n x^n = 0$$

Put $K = n$

$$\sum_{K=2}^{\infty} K(K-1) C_K x^K + \sum_{K=1}^{\infty} K C_K x^K - \sum_{K=0}^{\infty} C_K x^K = 0 \quad - (ii)$$

Combine (i) and (ii) again.

$$\sum_{K=2}^{\infty} K(K-1) C_K x^K - \sum_{K=0}^{\infty} (K+2)(K+1) C_{K+2} x^K + \sum_{K=1}^{\infty} K C_K x^K - \sum_{K=0}^{\infty} C_K x^K = 0$$

$$\sum_{K=2}^{\infty} K(K-1)\, C_K x^K - (2)(1)C_2 x^0 - (3)(2)C_3 x^1$$

$$- \sum_{K=2}^{\infty} (K+2)(K+1)\, C_{K+2} x^K + (1)C_1 x^1 + \sum_{K=2}^{\infty} K C_K x^K$$

$$- C_0 x^0 - C_1 x^1 - \sum_{K=2}^{\infty} C_K x^K = 0$$

$$-2C_2 - 6C_3 x - C_0 - C_1 x + \sum_{K=2}^{\infty} \left(K(K-1)C_K - (K+2)(K+1)\, C_{K+2} + KC_K - C_K\right) x^K = 0$$

$-2C_2 = 0$ $\qquad$ $\boxed{-2C_2 - C_0 = 0}$

$\boxed{C_2 = 0}$ $\qquad$ $C_0 = 2C_2$

$-6C_3 = 0$ $\qquad$ $-6C_3 = 0$

$\boxed{C_3 = 0}$

$-C_0 = 0$

$\boxed{C_0 = 0}$

$-C_1 = 0$

$\boxed{C_1 = 0}$

$$K(K-1)C_K - (K+2)(K+1)\, C_{K+2} + KC_K - C_K = 0$$

$$K(K-1)C_K + KC_K - C_K = (K+2)(K+1)\, C_{K+2}$$

$$\boxed{C_{K+2} = \frac{K(K-1)C_K + KC_K - C_K}{(K+2)(K+1)}}$$

$K = 2, 3, 4, \ldots$

$$C_{k+2} = \frac{k(k-1)C_k + kC_k - C_k}{(k+2)(k+1)} \quad \boxed{k=2}$$

$$C_4 = \frac{2(1)C_2 + 2C_2 - C_2}{4(3)}$$

$$C_5 = \frac{3(2)C_3 + 3C_3 - C_3}{5(4)}$$

$$C_6 = \frac{4(3)C_4 + 4C_4 - C_4}{6(5)}$$

let $\boxed{C_0 = 1, C_1 = 0}$

$-2C_2 - C_0 = 0$

$-2C_2 - 1 = 0$

$-2C_2 = 1$

$\boxed{C_2 = -1/2}$

$\boxed{C_3 = 0}$

$$C_4 = \frac{\cancel{2}(-1/\cancel{2}) + \cancel{2}(-1/\cancel{2}) - 1/2}{12}$$

$$= \frac{-1 - 1 - \frac{1}{2}}{12} = \frac{-2 + 1/2}{12}$$

$$C_4 = \frac{-3/2}{12} = -\frac{3}{24} = -\frac{1}{8}$$

$$\boxed{C_4 = -\frac{1}{8}}$$

$$\boxed{y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \cdots}$$

let $\boxed{c_0 = 0, c_1 = 1}$

$-2c_2 - c_0 = 0$

$-2c_2 - 0 = 0$

$-2c_2 = 0$

$\boxed{c_2 = 0}$

$\boxed{c_3 = 0}$

$$\boxed{\begin{aligned} c_4 &= 0 \\ c_5 &= 0 \\ c_6 &= 0 \end{aligned}}$$

$$\boxed{y_2 = x}$$

Q4) $\dfrac{dx}{dt} = 3x + 2y + 4z$

$\dfrac{dy}{dt} = 2x + 2z$

$\dfrac{dz}{dt} = 4x + 2y + 3z$

Sol: $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$

$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} = 0$

$(3-\lambda)((-\lambda)(3-\lambda) - 4) - 2(2(3-\lambda) - 8) + 4(4 + 4\lambda) = 0$

$(3-\lambda)(-3\lambda + \lambda^2 - 4) - 2(6 - 2\lambda - 8) + 16 + 16\lambda = 0$

$-9\lambda + 3\lambda^2 - 12 + 3\lambda^2 - \lambda^3 + 4\lambda + 4\lambda + 4 + 16 + 16\lambda = 0$

$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$

$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$

Put $\lambda = -1$

$(-1)^3 - 6(-1)^2 - 15(-1) - 8 = 0$

$\cancel{-1} \cancel{-6} + \cancel{15} - \cancel{8} = 0$

$0 = 0$

So, $\boxed{\lambda_1 = -1}$

$$\begin{array}{r|rrrr} -1 & 1 & -6 & -15 & -8 \\ & & -1 & 7 & 8 \\ \hline & 1 & -7 & -8 & \vert\ 0 \end{array}$$

$\lambda^2 - 7\lambda - 8 = 0$

$\lambda^2 - 8\lambda + \lambda - 8 = 0$

$\lambda(\lambda - 8) + 1(\lambda - 8) = 0$

$(\lambda + 1)(\lambda - 8) = 0$

$\boxed{\lambda_2 = -1}$, $\boxed{\lambda_3 = 8}$

We know that

$X = x_1 c_1 + x_2 c_2 + x_3 c_3$

$x_1 = K e^{\lambda_1 t}$

$(A - \lambda_1 I)K = 0$

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} K = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

$R_1 - R_3 \approx R_3$

$$= \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 - 2R_2 \approx R_2$

$$= \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 / 4 \approx R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$K_1 + \frac{1}{2}K_2 + K_3 = 0$$

$$K_1 = -\frac{1}{2}K_2 - K_3$$

$$\frac{1}{2}K_2 = -K_1 - K_3$$

$$\boxed{K_2 = -2K_1 - 2K_3}$$

$K_1 = r$

$K_3 = s$

$$= \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} r \\ -2r - 2s \\ s \end{bmatrix} = \begin{bmatrix} r \\ -2r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2s \\ 1 \end{bmatrix}$$

let $r = 1, s = 1$

$$= \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

So, $x_1 = Ke^{\lambda_1 t}$

$$= \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} e^{-t}$$

$x_2 = Pe^{\lambda_1 t}$

$$= \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{-t}$$

$$\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right) P = 0$$

$$= \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$R_2/2 \to R_2$

$$= \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 1 & -4 & 1 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$R_1 \approx R_2$

$$= \left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ -5 & 2 & 4 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$5R_1 + R_2 \approx R_2 \ ; 4R_1 - R_3 \approx R_3$

$$= \left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & -18 & 9 & 0 \end{array}\right]$$

$R_2 - R_3 \approx R_3$

$$= \left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 / -18 \approx R_2$

$$= \left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$4R_2 + R_1 \approx R_1$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$P_1 - P_3 = 0 \quad , P_1 = P_3$

$P_2 - P_3/2 = 0$

$P_2 = \frac{P_3}{2}$

$P_3 = 2P_2$

let $P_2 = r$

$P_3 = 2r$

$P_1 = 2r$

$$= \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 2r \\ r \\ 2r \end{bmatrix} \text{ let } r = 1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$X_3 = Pe^{\lambda_3 t} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{8t}$$

$$X = X_1 C_1 + X_2 C_2 + X_3 C_3$$

$$\boxed{X = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} e^{-t} C_1 + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{-t} C_2 + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{8t} C_3}$$

Heat Equation:

$$K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Sol: we know that

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial u}{\partial t} = XT'$$

let $u(x,t) = XT$ — (i)

where u is temperature, x is the position and t is time.

Put $\frac{\partial^2 u}{\partial x^2} = X''T$, and $\frac{\partial u}{\partial t} = XT'$

in Heat equation

$$KX''T = XT'$$

Separation of variable we got

$$\frac{X''}{X} = \frac{T'}{KT} = -\lambda^2$$

$$\frac{X''}{X} = -\lambda^2 \quad , \quad T' = -KT\lambda^2$$

$$X'' = -\lambda^2 X \quad , \quad T' + KT\lambda^2 = 0$$

$$X'' + \lambda^2 X = 0 \quad , \quad \text{let } T = e^{mt}$$

let $X = e^{mx}$

$X' = me^{mx}$, $T' = me^{mt}$

$X'' = m^2 e^{mx}$, $T'' = m^2 e^{mt}$

$m^2 + \lambda^2 = 0$, $m + K\lambda^2 = 0$

$m^2 = -\lambda^2$ $\quad m = -K\lambda^2$

$m = \pm \lambda i$

$X = (c_1 \cos\lambda x + c_2 \sin\lambda x) e^0$; $T = c_3 e^{-K\lambda^2 t}$

Put X and T values in eq.(i)

$= (c_1 \cos\lambda x + c_2 \sin\lambda x) c_3 e^{-K\lambda^2 t}$

$= (c_1 c_3 \cos\lambda x + c_2 c_3 \sin\lambda x) e^{-K\lambda^2 t}$

$c_1 c_3 = A$

$c_2 c_3 = B$

$= (A \cos\lambda x + B \sin\lambda x) e^{-K\lambda^2 t}$ —(ii)

(ii) eq. is the general equation of Heat, know put given condition to get the particular equation for Heat.

i) $u(0,t) = 0$

$u(x,t) = (A \cos\lambda x + B \sin\lambda x) e^{-K\lambda^2 t}$

Apply (i) condition

$u(0,t) = 0 = (A \cos(0) + B \sin(0)) e^{-K\lambda^2 t}$

$0 = (A \cancel{\cos}(1) + B(0)) e^{-K\lambda^2 t}$

$0 = A e^{-K\lambda^2 t}$

$e^{-K\lambda^2 t} \neq 0$ ($\because$ t is a variable)

$A = 0$

So, $u(x,t) = (B \sin \lambda x) e^{-K\lambda^2 t}$ — (iii)

(ii) $u(L,t) = 0$

Apply (ii) condition on (iii) eq

$u(L,t) = 0 = (B \sin \lambda L) e^{-K\lambda^2 t}$

$B \sin \lambda L e^{-K\lambda^2 t} = 0$

$e^{-K\lambda^2 t} \neq 0$ ($\because$ t is a variable)

$B \sin \lambda L = 0$

$B \neq 0$ ($\because$ A is already '0' so if B is '0' then $u(x,t) = 0$)

$\sin \lambda L = 0$

we know that $\sin(0), \sin(\pi), \sin(2\pi), \cdots, \infty = 0$

$\lambda L = n\pi$ $(n = 0, 1, 2, \cdots, \infty)$

$\lambda = \frac{n\pi}{L}$

Put $\lambda = \frac{n\pi}{L}$ in (iii) eq

$$u(x,t) = B \sin \frac{n\pi}{L} e^{-K \frac{n^2\pi^2}{L^2} t}$$

$$u(x,t) = B \sin \frac{n\pi}{L} e^{-K (n\pi/L)^2 t}$$

$U_n(x,t) = U_0 C_0 + U_1 C_1 + U_2 C_2 + \cdots + U_\infty C_\infty$

$U_0 C_0 = 0$ which is trivel solution

So;

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} e^{-K\left(\frac{n\pi}{L}\right)^2 t} \quad -(iv)$$

(iii) $U(x,0) = f(x)$

Apply (iii) condition on (iv) eq.

$$U(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} e^{-K\left(\frac{n\pi}{L}\right)^2 t}$$

By furier sin series we got

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Put B_n in eq. (iv)

$$U(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx\right) \sin \frac{n\pi}{L} \cdot e^{K\left(\frac{n\pi}{L}\right)^2 t} \quad -(v)$$

So, it's the particular equation for Heat. If we know the value of x, and t just put into (v) eq we got the temperature.