Q2a) $xy'' + y' = x$ Ex # 6.1

Sol:

$y = y_c + y_p$

let $y = x^m$

$y' = mx^{m-1}$; $y'' = m(m-1)x^{m-2}$

$xy'' + y' = 0$

$m(m-1) + m = 0$

$m^2 - m + m = 0$

$m^2 = 0$

$\boxed{m_1 = m_1 = 0}$

$\boxed{y_c = c_1 + c_2 \ln x}$

$y_p = ?$

$xy'' + y' = x$

Divide by 'x'

$y'' + \frac{1}{x} y' = 1$

So, $f(x) = 1$

$y_p = u_1 y_1 + u_2 y_2$

$w = y_1 y_2' - y_2 y_1'$

$y_1 = 1, \ y_2 = \ln x$

$\boxed{w = (1)(\frac{1}{x}) - (\ln x)(0) = 1/x}$

$\boxed{w_1 = -y_2 f(x) = -\ln x (1) = -\ln x}$

$\boxed{w_2 = y_1 f(x) = 1(1) = 1}$

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{-\ln x}{1/x}\,dx$$

$$= -\int x \ln x \, dx$$

$$= \ln x \frac{x^2}{2} - \int \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2}$$

$$u_1 = \frac{x^2}{2}\ln x - \frac{x^2}{4}$$

$$u_2 = \int \frac{W_2}{W}\,dx$$

$$= \int \frac{1}{1/x}\,dx = \int x\,dx$$

$$u_2 = \frac{x^2}{2}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)(1) + \frac{x^2}{2}\ln x$$

$$y = y_c + y_p$$

$$\boxed{y = c_1 + c_2 \ln x + \frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{x^2}{2}\ln x}$$

30) $xy'' - 4y' = x^4$

Sol: $y = y_c + y_p$

let $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$m(m-1) - 4m = 0$

$m^2 - m - 4m = 0$

$m^2 - 5m = 0$

$m(m-5) = 0$

$\boxed{m_1 = 0,\ m_2 = 5}$

$y_c = c_1x^0 + c_2x^5 = c_1 + c_2x^5$

$y_p = ?$

$xy'' - 4y' = x^4$

Divide by 'x'

$y'' - \frac{4}{x}y' = x^3$

$y_p = u_1y_1 + u_2y_2$, $f(x) = x^3$

$\boxed{y_1 = 1,\ y_2 = x^5}$

$W = y_1y_2' - y_2y_1'$

$\boxed{W = (1)(5x^4) - x^5(0) = 5x^4}$

$W_1 = -y_2 f(x)$

$\boxed{W_1 = -x^5 \cdot x^3 = -x^8}$

$W_2 = y_1 f(x)$

$\boxed{W_2 = 1(x^3) = x^3}$

$$u_1 = \int \frac{w_1}{w}\,dx = \int \frac{x^8}{5x^4}\,dx$$

$$= \frac{1}{5}\int x^4\,dx$$

$$\boxed{u_1 = \frac{1}{5}\,\frac{x^5}{5} = \frac{1}{25}x^5}$$

$$u_2 = \int \frac{w_2}{w}\,dx = \int \frac{x^3}{5x^4}\,dx$$

$$= \frac{1}{5}\int \frac{1}{x}\,dx$$

$$\boxed{u_2 = \frac{1}{5}\ln x}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{1}{25}x^5\right)(1) + \left(\frac{1}{5}\ln x\right)x^5$$

$$y = y_c + y_p$$

$$\boxed{y = c_1 + c_2 x^5 + \frac{1}{25}x^5 + \frac{1}{5}x^5\ln x}$$

31) $2x^2y'' + 5xy' + y = x^2 - x$

Sol: $y = y_c + y_p$

let $y = x^m$, $y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$

$2m(m-1) + 5m + 1 = 0$

$2m^2 - 2m + 5m + 1 = 0$

$2m^2 + 3m + 1 = 0$

$2m^2 + 2m + m + 1 = 0$

$2m(m+\cancel{2}) + 1(m+1) = 0$

$(2m+1)(m+1) = 0$

$\boxed{m_1 = -1/2 \quad , m_2 = -1}$

$y_c = c_1 x^{-1/2} + c_2 x^{-1}$

$y_p = u_1 y_1 + u_2 y_2$

$2x^2 y'' + 5xy' + y = x^2 - x$

Divide by 'x^2' on both sides we got

$$2y'' + \frac{5y'}{x} + \frac{y}{x^2} = 1 - \frac{1}{x}$$

$\boxed{f(x) = 1 - 1/x}$

$W = y_1 y_2' - y_2 y_1'$

$\boxed{y_1 = x^{-1/2} \quad , y_2 = x^{-1}}$

$W = x^{-1/2}(-x^{-2}) - (x^{-1})(-\frac{1}{2}x^{-3/2})$

$$W = -x^{-5/2} + \frac{1}{2}x^{-5/2}$$

$$\boxed{W = \frac{-2x^{-5/2} + x^{-5/2}}{2} = -\frac{1}{2}x^{-5/2}}$$

$w_1 = -y_2 f(x)$

$= -x^{-1}\left(1-\frac{1}{x}\right)$

$= -x^{-1} + x^{-1-1} = -x^{-1} + x^{-2}$

$$\boxed{w_1 = -\frac{1}{x} + \frac{1}{x^2}}$$

$w_2 = y_1 f(x)$

$= x^{-1/2}\left(1 - \frac{1}{x}\right)$

$= x^{-\frac{1}{2}} - x^{-\frac{1}{2}-1}$

$$\boxed{w_2 = x^{-\frac{1}{2}} - x^{-3/2}}$$

$u_1 = \int \frac{w_1}{w} dx = \int \frac{-1/x + 1/x^2}{-1/2\, x^{-5/2}} dx$

$= \int \frac{-x^{-1} + x^{-2}}{-1/2\, x^{-5/2}} dx$

$= -2\int \left(-\frac{x^{-1}}{x^{-5/2}} + \frac{x^{-2}}{x^{-5/2}} \right) dx$

$= -2\int \left(-x^{-1+5/2} + x^{-2+5/2} \right) dx$

$= -2\int \left(-x^{3/2} + x^{1/2} \right) dx$

$= -2\left(-\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right)$

$$= -2\left(\frac{-x^{5/2}}{5/2} + \frac{x^{3/2}}{3/2}\right)$$

$$= -2\left(\frac{-2}{5}x^{5/2} + \frac{2}{3}x^{3/2}\right)$$

$$\boxed{u_1 = \frac{4}{5}x^{5/2} - \frac{4}{3}x^{3/2}}$$

$$u_2 = \int \frac{w_2}{w}\,dx = \int \frac{x^{-1/2} - x^{-3/2}}{-1/2\, x^{-5/2}}\,dx$$

$$= -2\int \left(x^{-\frac{1}{2}+5/2} - x^{-\frac{3}{2}+5/2}\right)dx$$

$$= -2\int (x^2 - x^1)\,dx$$

$$= -2\left(\frac{x^3}{3} - \frac{x^2}{2}\right)$$

$$u_2 = \frac{-2x^3}{3} + x^2$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{4}{5}x^{5/2} - \frac{4}{3}x^{3/2}\right)x^{-\frac{1}{2}} + \left(\frac{-2x^3}{3} + x^2\right)x^{-1}$$

$$= \frac{4}{5}x^2 - \frac{4}{3}x^1 - \frac{2}{3}x^2 + x^1$$

$$= \left(\frac{4}{5} - \frac{2}{3}\right)x^2 + \left(\frac{-4}{3} + 1\right)x$$

$$= \frac{2}{15}x^2 - \frac{1}{3}x$$

$$y = y_c + y_p$$

$$= c_1 x^{-1/2} + c_2 x^{-1} + \frac{2}{15}x^2 - \frac{1}{3}x$$

32) $x^2y'' - 2xy' + 2y = x^4e^x$

Sol: Let $y = x^m$, $y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$

$$m(m-1) - 2m + 2 = 0$$

$$m^2 - m - 2m + 2 = 0$$

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$(m-2)(m-1) = 0$$

$$\boxed{m_1 = 2, \quad m_2 = 1}$$

$$\boxed{y_c = c_1x^2 + c_2x}$$

$$\boxed{y_1 = x^2, \quad y_2 = x}$$

$$y_p = u_1y_1 + u_2y_2$$

$$x^2y'' - 2xy' + 2y = x^4e^x$$

Divide by x^2

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x^2e^x$$

$$f(x) = x^2 e^x$$

$$W = y_1 y_2' - y_2 y_1'$$
$$= x^2(1) - x(2x)$$
$$\boxed{W = x^2 - 2x^2 = -x^2}$$

$$W_1 = -y_2 f(x) = -x \cdot x^2 e^x$$
$$\boxed{W_1 = -x^3 e^x}$$

$$W_2 = y_1 f(x)$$
$$\boxed{W_2 = x^2 \cdot x^2 e^x = x^4 e^x}$$

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{\cancel{-}x^{\cancel{3}} e^x}{\cancel{-x^2}}\,dx$$

$$= \int x e^x\,dx$$

$$= x \cdot e^x - \int e^x\,dx$$

$$\boxed{u_1 = xe^x - e^x}$$

$$u_2 = \int \frac{W_2}{W}\,dx = -\int \frac{x^{\cancel{4}^{2}} e^x}{\cancel{x^2}}\,dx$$

$$= -\left(x^2 e^x - \int 2x e^x\,dx\right)$$

$$= -x^2e^x + 2\int xe^x\,dx$$

$$= -x^2e^x + 2\left(xe^x - \int e^x\,dx\right)$$

$$\boxed{u_2 = -x^2e^x + 2xe^x - 2e^x}$$

$$y_p = u_1y_1 + u_2y_2$$

$$= (xe^x - e^x)x^2 + (-x^2e^x + 2xe^x - 2e^x)\,x$$

$$= \cancel{x^3e^x} - x^2e^x - \cancel{x^3e^x} + 2x^2e^x - 2xe^x$$

$$= x^2e^x - 2xe^x$$

$$y = y_c + y_p$$

$$= c_1x^2 + c_2x + x^2e^x - 2xe^x$$

33) $x^2y'' - xy' + y = 2x$

Sol: let $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$$m(m-1) - m + 1 = 0$$

$$m^2 - m - m + 1 = 0$$

$$m^2 - 2m + 1 = 0$$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$(m-1)(m-1) = 0$$

$$\boxed{m_1 = m_2 = 1}$$

$$y_c = c_1x^1 + c_2x^1\ln x$$

$$\boxed{y_c = c_1 x + c_2 x \ln x}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$x^2 y'' - x y' + y = 2x$$

Divide by 'x^2' on both sides

$$y'' - \frac{1}{x} y' + \frac{y}{x^2} = \frac{2}{x}$$

$$f(x) = \frac{2}{x}$$

$$\boxed{y_1 = x \, , \; y_2 = x \ln x}$$

$$W = y_1 y_2' - y_2 y_1'$$

$$= x\left((1)\ln x + x \cdot \frac{1}{x}\right) - x \ln x (1)$$

$$= \cancel{x \ln x} + x - \cancel{x \ln x}$$

$$\boxed{W = x}$$

$$W_1 = -y_2 f(x) = x \ln x \cdot \frac{2}{\cancel{x}} = 2 \ln x$$

$$\boxed{W_2 = y_1 f(x) = \cancel{x} \cdot \frac{2}{\cancel{x}} = 2}$$

$$u_1 = \int \frac{w_1}{w}\, dx = \int \frac{2\ln x}{x}\, dx$$

$$= 2\int \ln x \cdot \left(\frac{1}{x}\right) dx$$

$$= 2\int x^{-1} \ln x\, dx \qquad \frac{\cancel{2}(\ln x)^2}{\cancel{2}}$$

$$= 2\left(x^{-1} \cdot \frac{1}{x} - \int \frac{1}{x}(-1)x^{-2}\right)$$

$$= 2\left(x^{-2} + \int x^{-3}\right) dx$$

$$= 2\left(x^{-2} + \left(-\frac{x^{-2}}{2}\right)\right)$$

$$= 2x^{-2} - x^{-2}$$

$$u_1 = (\ln x)^2$$

$$u_2 = \int \frac{w_2}{w}\, dx$$

$$u_2 = \int \frac{2}{x}\, dx = 2\ln x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \cancel{2\ln x(x\ln x)}$$

$$= x(\ln x)^2 + 2\ln x(x\ln x)$$

$$= x(\ln x)^2 + 2x(\ln x)^2$$
$$= 3x(\ln x)^2$$

$$y = y_c + y_p$$
$$\boxed{y = c_1 x + c_2 x \ln x + 3x(\ln x)^2}$$

2) $x^2y'' - 2xy' + 2y = x^3 \ln x$

Sol: Cauch-Euler:

$x^2y'' - 2xy' + 2y = 0$

$y = x^m, \quad y' = mx^{m-1}, \quad y'' = m(m-1)x^{m-2}$

$m(m-1) - 2m + 2 = 0$

$m^2 - m - 2m + 2 = 0$

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0$

$m_1 = 1, \quad m_2 = 2$

$y_c = c_1x^1 + c_2x^2$

$\boxed{y_c = c_1x + c_2x^2}$

$y_p = \cancel{(x^3 \ln x)}$

$y_p = u_1 y_1 + u_2 y_2$

$W = y_1 y_2' - y_2 y_1'$

$y_1 = x, \quad y_2 = x^2$

$y_1' = 1, \quad y_2' = 2x$

$= x(2x) - (x^2)(1)$

$W = 2x^2 - x^2 = x^2$

$W_1 = -y_2 f(x)$

$$\frac{x^2 y''}{x^2} - \frac{2xy'}{x^2} + \frac{2y}{x^2} = \frac{x^3 \ln x}{x^2}$$

$$y'' - 2y' + \frac{2y}{x^2} = x \ln x$$

$f(x) = x \ln x$

$W_1 = -x^2 \cdot x \ln x = -x^3 \ln x$

$W_2 = y_1 f(x)$

$= x \cdot x \ln x = x^2 \ln x$

$$u_1 = \int \frac{W_1}{W}\, dx = -\int \frac{x^3 \ln x}{x^2}\, dx$$

$$= \int x \ln x\, dx$$

let $u = \ln x$, $du = 1/x \, dx$

$dv = x\,dx$, $v = \frac{x^2}{2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln x \cdot x\,dx = \ln x \cdot \frac{x^2}{2} - \int \frac{\cancel{x^2}}{2}\cdot\frac{1}{\cancel{x}}\,dx$$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$

$$u_1 = \frac{x^2}{2}\ln x - \frac{x^2}{4} = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$$

$$u_2 = \int \frac{w_2}{w}\,dx = \int \frac{\cancel{x^2}\ln x}{\cancel{x^2}}\,dx$$

$$= \int \ln x\,dx$$

$$= x\ln x - x$$

$y_p = u_1 y_1 + u_2 y_2$

$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)x + (x\ln x - x)x^2$$

$$= \frac{x^3}{2}\left(\ln x - \frac{1}{2}\right) + x^3\ln x - x^2$$

$$= \frac{x^3}{2}\ln x - \frac{x^3}{2} + x^3\ln x - x^2$$

$$\cancel{Y_p = x^3\left(\frac{1}{2}\ln x - \frac{1}{2} + \ln x\right) - x^2}$$

$$\boxed{\cancel{y = c_1 x + c_2 x^2 + x^3\left(\frac{1}{2}\ln x - \frac{1}{2} + \ln x\right) - x^2}}$$

$$Y_p = \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)x + x^3\ln x - x^3$$

$$= \frac{x^3}{2}\ln x - \frac{x^3}{4} + x^3\ln x - x^3$$

$$= \frac{x^3\ln x}{2} + x^3\ln x - \frac{x^3 - 4x^3}{4}$$

$$= \frac{x^3\ln x}{2} + x^3\ln x - \frac{5x^3}{4}$$

$$= \frac{x^3\ln x}{2} + \frac{2}{2}x^3\ln x - \frac{5}{4}x^3$$

$$= \frac{x^3\ln x}{2} + \frac{2x^3\ln x}{2} - \frac{5}{4}x^3$$

$$= \frac{x^3\ln x + 2x^3\ln x}{2} - \frac{5}{4}x^3$$

$$= \frac{3x^3\ln x}{2} - \frac{5}{4}x^3$$

$$\boxed{Y_p = \frac{3x^3\ln x}{2} - \frac{5}{4}x^3}$$

$$y = y_c + y_p = c_1 x + c_2 x^2 + \frac{3}{2} x^3 \ln x - \frac{5}{4} x^3$$