5.1 Eigenvalues and Eigenvectors

If A is $n \times n$ matrix, then a non-zero vector $\boldsymbol{x}$ in R^n is called an eigenvector of A if $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$ that is

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

$$\Rightarrow A\boldsymbol{x} - \lambda\boldsymbol{x} = 0$$

$$\Rightarrow (A - \lambda I)\boldsymbol{x} = 0$$

For some scalar λ. The scalar λ is called an eigenvalue of A. $\boldsymbol{x}$ is said to be an eigenvector corresponding to λ.

Example1: Eigenvector of a 2 × 2 Matrix

The vector $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 3$, since

$$A\boldsymbol{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\boldsymbol{x}.$$

THEOREM 5.1.1 If A is $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$det(A - \lambda I) = 0 \qquad (1)$$

This is called the **charactristic equation** of A.

Note: In Example 1 we observed that $\lambda = 3$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

but we did not explain how we found it. Use the characteristic equation to find all eigenvalues of this matrix.

Example 2: Find eigenvalues of the matrix A

$$\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Solution: The eigenvalues of A are the solution of the equation $det(A - \lambda I) = 0$. As

$$A - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix}$$

$$\Rightarrow det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$$

Therefore, the characteristic equation $det(A - \lambda I) = 0$ is

$$\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3-\lambda)(-1-\lambda) - 8(0) = 0$$

$$\Rightarrow -(3-\lambda)(1+\lambda) = 0$$

$$\lambda = 3, \; \lambda = -1$$

So, eigenvalues are $\lambda = 3$ & $\lambda = -1.$

Example 3: Find eigenvalues of the matrix A

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution: The characteristic polynomial of A is

$$det(A - \lambda I) = \det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{bmatrix} = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -17 & 8-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 4 & 8-\lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & -\lambda \\ 4 & -17 \end{vmatrix}$$

$$= -\lambda[-\lambda(8-\lambda) + 17] - 1[0(8-\lambda) - 4(1)] + 0$$

$$= -\lambda[8\lambda + \lambda^2 + 17] - 1[-4]$$

$$= -\lambda^3 + 8\lambda^2 - 17\lambda + 4$$

To find eigenvalue put $det(A - \lambda I) = 0$,

$$-\lambda^3 + 8\lambda^2 - 17\lambda + 4 = 0 \quad \Rightarrow \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$\boldsymbol{\lambda = 4}$ is the one solution so by synthetic division,

$$\begin{array}{r|rrrr} 4 & 1 & -8 & 17 & -4 \\ & & 4 & -16 & 4 \\ \hline & 1 & -4 & 1 & 0 \end{array}$$

$$\lambda^2 - 4\lambda + 1 = 0$$

from quadratic equation,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4}}{2} \quad => \lambda = \frac{4 \pm \sqrt{12}}{2}$$

$$\lambda = 2 \pm \sqrt{3}$$

So the eigenvalues of A are

$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \lambda = 2 - \sqrt{3}$

Exercise: Find the eigenvalues of the following matrices.

a) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

$\lambda = 4, 4$

$-\lambda^3 - 4\lambda^2 - 2\lambda = 0$

$-\lambda(\lambda^2 + 4\lambda + 2) = 0$

$\lambda = 0, \quad \lambda^2 + 4\lambda + 2 = 0$

d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

f) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} 3 & 0 & -5 \\ 1/5 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

h) $\begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$

i) $\begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$

Example 4: Find the eigenvalues and the corresponding eigenvectors of the following matrix.

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Solution: The eigenvalue/s of A are the solution of the equation

$$det(A - \lambda I) = 0$$

$$det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 0 \\ 8 & -1 - \lambda \end{vmatrix}$$

$$-(3 - \lambda)(1 + \lambda) = 0$$

$$\lambda = 3, \qquad \lambda = -1$$

For $\lambda = 3$, eigenvector is

$$A\vec{x} = \lambda\vec{x}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 3 - \lambda & 0 \\ 8 & -1 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using $\lambda = 3$,

$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0x + 0y = 0$$

$$8x - 4y = 0$$

$$8x = 4y \quad \Rightarrow \; x = \frac{1}{2}y$$

Let $y = t$,

$$x = \frac{1}{2}t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 3$.

Now for $\lambda = -1$, eigenvector is

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 3 - \lambda & 0 \\ 8 & -1 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using $\lambda = -1$,

$$\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x + 0y = 0 \qquad \ldots (1)$$

$$8x - 0y = 0 \qquad \ldots (2)$$

$$\Rightarrow \quad x = 0 \;\; from\ both\ equations\ (1) \text{ and } (2)$$

As y is free variable so put $y = t$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = -1$.

Example 5: Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution: The eigenvalues of A are the solution of the equation

$$det(A - \lambda I) = 0$$

$$det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(2-\lambda) - 2] - 2[2-\lambda-1] + 1(2-3+\lambda) = 0$$

$$(2-\lambda)(6 - 3\lambda - 2\lambda + \lambda^2 - \lambda) - 2[1-\lambda] + (-1+\lambda) = 0$$

$$(2-\lambda)(\lambda^2 - 5\lambda + 4) - 2(1-\lambda) + (\lambda - 1) = 0$$

$$2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 2 + 2\lambda + \lambda - 1 = 0$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\lambda = 1$ is one solution of above equation because

$$1 - 7 + 11 - 5 = 0$$

$$0 = 0$$

So by using synthetic division we find

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & 0 \end{array}$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\lambda^2 - 5\lambda - \lambda + 5 = 0$$

$$\lambda(\lambda-1)-5(\lambda-1)=0$$

$$(\lambda-1)(\lambda-5)=0$$

$$\lambda=1\,,\ \lambda=5$$

So eigenvalues of A are

$$\lambda=1\,,\lambda=1\,,\ \lambda=5$$

For $\lambda=1$, eigenvector is

$$(A-\lambda I)\vec{x}=\vec{0}$$

$$\begin{bmatrix}2-\lambda & 2 & 1\\ 1 & 3-\lambda & 1\\ 1 & 2 & 2-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Put $\lambda=1$,

$$\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 1\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x+2y+z=0$$

$$x+2y+z=0$$

$$x+2y+z=0$$

Actually, there is only one equation

$$x+2y+z=0$$

$$\text{So}\quad x=-2y-z$$

Since y & z are free variables, So put $y=s,z=t$

$$\Rightarrow\qquad x=-2s-t$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-2s-t\\ s\\ t\end{bmatrix}=s\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

So $\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$ are eigenvectors for $\lambda=1$.

Find the eigenvectors for $\lambda = 5$.

Exercise: Find the eigenvectors for the eigenvalues of the following matrices.

a) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$

b) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 0 & -5 \\ 1/5 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

d) $\begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$

Exercise Set 5.1

In Exercises 1–2, confirm by multiplication that $\mathbf{x}$ is an eigenvector of A, and find the corresponding eigenvalue.

1. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Answer:

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2. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3. Find the characteristic equations of the following matrices:

(a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$