Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

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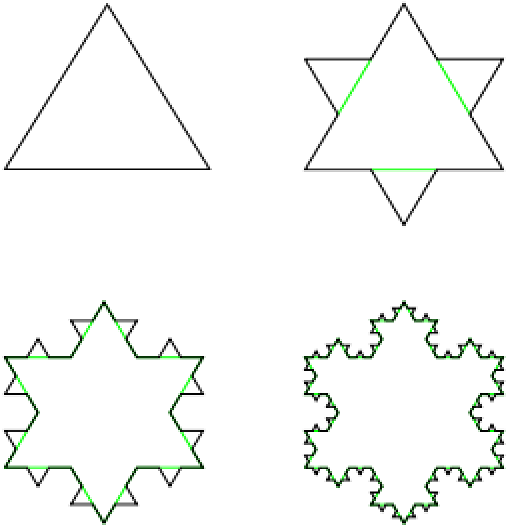
Lecture Outline

- Recursion
  - Recursive definition of function
  - Recursive definition of sequence
  - Recursive definition of sets

Recursive Objects

- **Recursion** is the process of repeating items in a self-similar way.

- Sometimes it is difficult to define an object explicitly, but it is easier to define it in terms of itself.



Factorial Recursive Function

```
int factorial(int n)
{
    if(n == 0)
        return 1;
    else
        return n * factorial(n - 1);
}
```

Basis Step

Recursive Step

Recursive Functions

- We can define a function recursively by specifying:
  - Basis: the value of the function at the smallest element of the domain.
    - e.g. : $f(0) = 1$
  - Recursive step: A rule for finding the value of the function at an integer from its values at smaller integers.
    - e.g. : $f(n+1) = 2 * f(n)$
- Many common functions can be defined recursively.

Example

- Can you write n! as a recursive function
  - $F(0) = 1$
  - $F(n) = n * F(n-1)$
  - What is the value of F(5)?

- $F(5) = 5F(4)$

$$
\begin{aligned}
&= 5 \,.\, 4F(3)\\
&= 5 \,.\, 4 \,.\, 3F(2)\\
&= 5 \,.\, 4 \,.\, 3 \,.\, 2F(1)\\
&= 5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1F(0)\\
&= 5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1 \,.\, 1 = 120
\end{aligned}
$$

Example

- Can you write n! as a recursive function
  - F(0) = 1
  - F(n) = n * F(n-1)
  - What is the value of F(5)?

$$F(5) = 5F(4)$$
$$F(4) = 4F(3)$$
$$F(3) = 3F(2)$$
$$F(2) = 2F(1)$$
$$F(1) = 1F(0)$$
$$F(0) = 1$$

$$F(1) = 1*1 = 1$$
$$F(2) = 2*1 = 2$$
$$F(3) = 3*2 = 6$$
$$F(4) = 4*6 = 24$$
$$F(5) = 5*24 = 120$$

Example

Suppose that f is defined recursively by

- $f(0) = 3$,
- $f(n+1) = 2f(n)+3$. for $n \geq 1$

Find $f(3)$.

Solution:

$f(3) = 2f(2) + 3 = 2(21) + 3 = 45$

$f(2) = 2f(1) + 3 = 2(9) + 3 = 21$

$f(1) = 2f(0) + 3 = 2(3) + 3 = 9$

Example

Suppose that f is defined recursively by

- $f(0) = -1$, $f(1) = 2$
- $f(n+1) = f(n)+3f(n-1)$, for $n \geq 2$

Find $f(4)$.

Solution:

$f(4) = f(3) + 3f(2) = 5 + 3(-1) = 2$

$f(3) = f(2) + 3f(1) = -1 + 3(2) = 5$

$f(2) = f(1) + 3f(0) = 2 + 3(-1) = -1$

Example

- Fibonacci Numbers
  - $F(0) = 0$, $F(1) = 1$
  - $F(n) = F(n-1) + F(n-2)$ for $n = 2, 3, 4, \ldots\ldots$
- Find the Fibonacci number $F(4)$.

Solution:

$F(4) = F(3) + F(2) = 2 + 1 = 3$

$F(3) = F(2) + F(1) =1 + 1 = 2$

$F(2) = F(1) + F(0) = 1 + 0 = 1$

Example

- Let a and b denote positive integers. Suppose a function Q is defined recursively as follows:
- Find the value of Q(4,5) and Q(14,3).
- Find Q(55, 7).

$$Q(a,b) = \begin{cases} 0 & if \;\; a < b \\ Q(a-b,b)+1 & if \;\; a \geq b \end{cases}$$

$$Q(4,5) = 0 \qquad if \;\; a < b$$

Example

$$Q(a,b)=\begin{cases}0 & if\ \ a<b\\ Q(a-b,b)+1 & if\ \ a\geq b\end{cases}$$

$$Q(14,3)=Q(14-3,3)+1=Q(11,3)+1$$

$$Q(11,3)=Q(8,3)+1$$

$$Q(8,3)=Q(5,3)+1$$

$$Q(5,3)=Q(2,3)+1$$

$$Q(2,3)=0$$

$$Q(5,3)=0+1=1$$

$$Q(8,3)=1+1=2$$

$$Q(11,3)=2+1=3$$

$$Q(14,3)=3+1=4$$

Recursive Algorithms

- An algorithm is called recursive if it solves a problem by reducing it to a smaller instance of the same problem.
- Computing n!

```
int factorial(int n)
{ If (n==0; return 1;
  else return  n*factorial(n-1);   }
```

- Computing GCD

```
gcd(a,b)
/* assumption a < b */
  If a=0, then return b
  Else return gcd(b mod a, a)
```

Example

$$GCD(a,b) = \begin{cases} b & \text{if } a = 0 \\ GCD(b\%a, a) & \text{if } a < b \end{cases}$$

Find $GCD(30,108)$.

$$GCD(30,108) = GCD(108\%30, 30)$$
$$= GCD(18, 30) = GCD(30\%18, 18)$$
$$= GCD(12,18) = GCD(18\%12, 12)$$
$$= GCD(6,12) = GCD(12\%6, 6)$$
$$= GCD(0, 6) = 6$$

Sequence

Find the explicit formula and recursive formula of following sequence:

2,4,8,16,32,…

Explicit Formula:

$$a = 2$$
$$r = 2$$
$$a_n = ar^{n-1}$$
$$a_n = 2(2)^{n-1}$$
$$a_n = 2^n$$

Recursive Formula:

$$Basis\ Step:$$
$$a_1 = 2$$
$$Recursive\ Step:$$
$$a_{n+1} = 2a_n$$

$$a_2 = 2a_1$$
$$a_3 = 2a_2$$
$$a_4 = 2a_3$$
$$\vdots$$
$$a_{n+1} = 2a_n$$

After giving the first term, each term of the sequence can be defined from the previous term.

Sequence

Find the explicit formula and recursive formula of following sequence:

5,10,15,20,25,…

Explicit Formula:

$$a = 5$$
$$d = 5$$
$$a_n = a + (n-1)d$$
$$a_n = 5 + (n-1)5$$
$$a_n = 5n$$

Recursive Formula:

$$\mathit{Basis\ Step}:$$
$$a_1 = 5$$
$$\mathit{Recursive\ Step}:$$
$$a_{n+1} = a_n + 5$$

$$a_2 = a_1 + 5$$
$$a_3 = a_2 + 5$$
$$a_4 = a_3 + 5$$
$$\vdots$$
$$a_{n+1} = a_n + 5$$

After giving the first term, each term of the sequence can be defined from the previous term.

Recursively Defined Sets

- Assume S is a set.
- We use two steps to define the elements of S.
- **Basis step:**
  - Specify an initial collection of elements.
- **Recursive step:**
  - Give a rule for forming new elements from those already known to be in S.

Recursively Defined Sets

- Consider $S \subseteq \mathbf{Z}$ defined by
- **Basis step:** (Specify initial elements.)
  - $0 \in S$
- **Recursive step:** (Give a rule using existing elements)
  - If $x \in S$, then $2x + 2 \in S$.

- # of elements in set S after applying 3 time recursion
- 0
- 0, 2 (1^{st})
- 0, 2, 6 (2^{nd})
- 0, 2, 6, 14 (3^{rd})

Recursively Defined Sets

- Consider $S \subseteq \mathbf{Z}$ defined by
- **Basis step:** (Specify initial elements.)
  - $3 \in S$
- **Recursive step:** (Give a rule using existing elements)
  - If $x \in S$ and $y \in S$, then $x + y \in S$.
- # of elements in set S after applying 3 time recursion
- 3
- 3, 6 (1st)
- 3, 6, 9, 12 (2nd)
- 3, 6, 9, 12, 15, 18, 21, 24 (3rd)
- This is the set of all positive multiples of 3.

Exercise Questions

Chapter # 5

Topic # 5.3

Q. 1, 2, 3, 4, 7, 27-a, 48 (Ackermann's Function), 51