Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

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Course Outline

- Nested Quantifiers

Nested Quantifiers

- Two quantifiers are nested if one is within scope of other, such as

$$\forall x \exists y (x + y = 0).$$

- Everything within the scope of a quantifier can be thought of as a propositional function.
- For example,

  $\forall x \exists y (x + y = 0)$

  is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$.

Nested Quantifiers

- $\forall x \exists y\ P(x, y)$
  - "For all x, there exists a y such that $P(x, y)$".
  - Example:
  - $\forall x \exists y\ (x + y\ =\ 0)$ where x and y are integers

- $\exists x \forall y\ P(x, y)$
  - There exists an x such that for all y, $P(x, y)$ is true"
  - Example: $\exists x \forall y\ (x\ \times\ y\ = 0)$

- $\forall x \forall y \forall z (x\ +\ (y\ +\ z) = (x\ +\ y) + z)$

- **THINK QUANTIFICATION AS LOOPS**

Nested Quantifiers Example

- Let Domain of x is the students in this class

  Doamin of y is the courses in software engineering

  $Q(x, y)$ = "x takes course y", true when x takes course y, otherwise false.

Translate the following logical expression:

- $\forall x \forall y Q(x, y)$
- $\exists x \exists y Q(x, y)$
- $\forall x \exists y Q(x, y)$
- $\exists x \forall y Q(x, y)$

Meaning of Multiple Quantifiers

Suppose $P(x, y)$ = "x likes y."

Domain of x: {St1, St2}; Domain of y: {Cricket, Hockey}

- $\boldsymbol{\forall x \forall y\, P(x, y)}$
  - $P(x, y)$ true for all x, y pairs.
- $\boldsymbol{\exists x \exists y\, P(x, y)}$
  - $P(x, y)$ true for at least one x, y pair.
- $\boldsymbol{\forall x \exists y\, P(x, y)}$
  - For every value of x we can find a (possibly different) y so that P(x,y) is true.
- $\boldsymbol{\exists x \forall y\, P(x, y)}$
  - There is at least one x for which P(x,y) is always true.

Predicates - the meaning of multiple quantifiers

Statement	***When True?***	***When False?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Example

- Let $Q(x, y)$: "$x + y = 0$"

What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x,y)$, where the domain for all variables consists of all real numbers?

Example

- Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

- **Solution:**

The quantification $\exists y \forall x Q(x,y)$ denotes the proposition

"There is a real number y such that for every real number x, $Q(x, y)$."

- No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is false.

Example

- The quantification $\forall x \exists y Q(x, y)$ denotes the proposition

  "For every real number x there is a real number y such that $Q(x, y)$."

- Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$.
- Hence, the statement $\forall x \exists y Q(x, y)$ is true.

Order of Quantifiers

- $\exists x \forall y$ and $\forall x \exists y$ are not equivalent!
- $\exists x \forall y\ P(x,y)$
  - P(x,y) = (x+y == 0) is false
- $\forall x \exists y\ P(x,y)$
  - P(x,y) = (x+y == 0) is true

Example

$Q\ (x\ , y, z)$: x + y = z

Domain: Real numbers

- $\forall x \forall y \exists z\, Q(x, y, z)$ True/False???
- For all real numbers x and for all real numbers y there is a real number z such that $x + y = z$.
- **True**
- $\exists z \forall x \forall y\, Q(x, y, z)$ True/False???
- There is a real number z such that for all real numbers x and for all real numbers y that $x + y = z$.
- **False**

Translating between English and Quantifiers

- Translate the statement “The sum of two positive integers is always positive” into a logical expression.

Translating between English and Quantifiers

- Translate the statement "The sum of two positive integers is always positive" into a logical expression.
- **Solution:**
- First rewrite it so that the implied quantifiers and a domain are shown: "For every two integers, if these integers are both positive, then the sum of these integers is positive."
- Next, introduce the variables *x* and *y* to obtain "For all positive integers *x* and *y*, $x + y$ is positive."
- Statement is $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$,
- where the domain for both variables consists positive integers.

Translating between English and Quantifiers

- Translate the following statement into English

$$\forall x(C(x) \lor \exists y(C(y) \land F(x, y)))$$

where

$C(x)$: "x has a computer,"

$F(x, y)$: "x and y are friends,"

The domain for both x and y consists of all students in your school.

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- **Solution:**
- The statement says that for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.
- In other words, every student in your school has a computer or has a friend who has a computer.

Negating Multiple Quantifiers

- Recall negation rules for single quantifiers:
  - $\neg\forall x\, P(x) = \exists x\, \neg P(x)$
  - $\neg\exists x\, P(x) = \forall x\, \neg P(x)$
  - Essentially, you change the quantifier(s), and negate what it's quantifying
- Examples:
  - $\neg(\forall x \exists y\, P(x,y)) = \exists x\, \neg\exists y\, P(x,y) = \exists x \forall y\, \neg P(x,y)$
  - $\neg(\forall x \exists y \forall z\, P(x,y,z)) = \exists x \neg\exists y \forall z\, P(x,y,z)$

    $= \exists x \forall y\, \neg\, \forall z\, P(x,y,z) = \exists x \forall y \exists z\, \neg P(x,y,z)$

Negating Multiple Quantifiers

- Consider $\neg(\forall x \exists y\ P(x,y)) \equiv \exists x \forall y\ \neg P(x,y)$
  - The left side is saying "for all x, there exists a y such that P is true"
  - To negate it, you need to show that "there exists an x such that for all y, P is false"
- Consider $\neg(\exists x \forall y\ P(x,y)) \equiv \forall x \exists y\ \neg P(x,y)$
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Chapter Reading

- ***Chapter 1***, Kenneth H. Rosen, Discrete Mathematics and Its Applications, Section 1.5

Chapter Exercise (For Practice)

- Question # 1, 2, 3, 4, 8, 23, 24, 25, 26, 27, 30, 31, 39, 41