Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

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Lecture Outline

- Mathematical Induction
  - Proof using Mathematical Induction
    - Sequence formulas
    - Inequality
    - Divisibility

Mathematical Induction

- Mathematical induction is an extremely important proof technique.

- Mathematical induction can be used to prove
  - results about complexity of algorithms
  - correctness of certain types of computer programs
  - theorem about graphs and trees
  - …

What is Mathematical Induction?

- How to prove "P(n), a mathematical statement, for all positive integer n".
- It is a method of proof.
- It does not generate answers: it only can prove them.

Mathematical Induction

- Assume P(n) is a propositional function.

- **Principle of mathematical induction:**
  To prove that P(n) is true for all positive integers n, we complete two steps.

- Basis Step: P(1)
- Inductive Step: $\forall k(P(k) \rightarrow P(k+1))$
- Result: $\forall$ n P(n) domain: positive integers

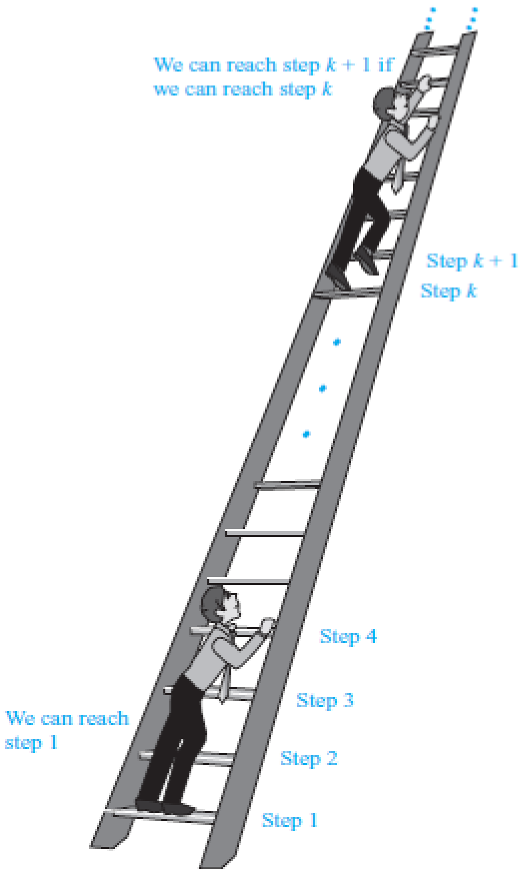
- How to show P(1) is true?
  - P(1): n is replaced by 1 in P(n)
  - Then, show P(1) is true.
- How to show $\forall k\ (P(k) \rightarrow P(k+1))$?
  - Direct proof can be used
  - Assume P(k) is true for some arbitrary k.
  - Then, show P(k+1) is true.

Example

Suppose that we have an infinite ladder

1. We can reach the first step of the ladder.
2. If we can reach a particular step of the ladder, then we can reach the next step.

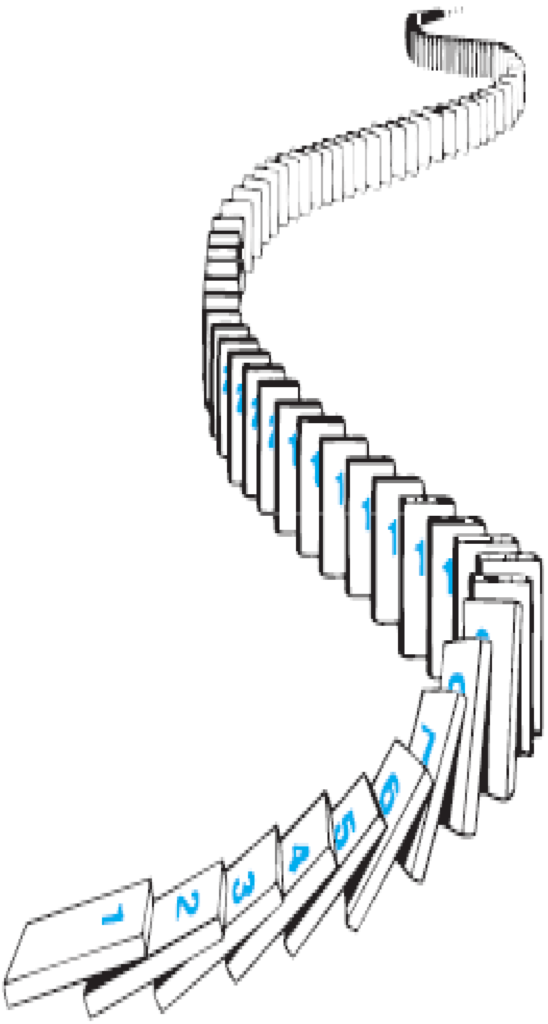
Then, we can conclude that we are able to reach every step of this infinite ladder.



Example

- An infinite row of dominoes, labeled 1, 2, 3, ..., n
- P(n): Domino *n* is knocked over
- P(1): The first domino is knocked over
- P(k): The k^{th} domino is knocked over
- The fact that
  - The first domino is knocked over
  - And whenever the k^{th} domino is knocked over, it also knocks the $(k+1)$st domino over
- Implies that all the dominoes are knocked over

$$[\,P(1) \wedge \forall k\,(\,P(k) \rightarrow P(k+1)\,)\,] \rightarrow \forall n\, P(n)$$



Example

- Show that $1 + 2 + 3 + \ldots + n = n(n+1) / 2$, where n is a positive integer.

- Proof:
  - First define P(n)

    $P(n)$ is $1 + 2 + 3 + \ldots + n = n(n+1) / 2$

  - Basis Step: (Show P(1) is true.)

    $1 = 1(2)/2$

    So, P(1) is true.

Example

Inductive Step: (Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.)

- Assume $P(k)$ is true.

$$1 + 2 + 3 + \ldots + k = k(k+1) / 2$$

- Show $P(k+1)$ is true.

$$P(k+1) : 1 + 2 + 3 + \ldots\ (k+1) = (k+1)(k+2) / 2$$

$$\text{L.H.S of } P(k+1) = 1 + 2 + \ldots + k + k+1$$

$$= (1 + 2 + \ldots + k) + (k+1)$$

$$= k(k+1)/2 + (k+1)$$

$$= [k(k+1) + 2(k+1)]/2$$

$$= (k+1)(k+2)/2 = \text{R.H.S of } P(k+1)$$

- We showed that $P(k+1)$ is true under assumption that $P(k)$ is true. So, by mathematical induction $1+2+\ldots+n = n(n+1)/2$.

What did we show

- Base case: P(1)
- If P(k) was true, then P(k+1) is true
  - i.e., $P(k) \rightarrow P(k+1)$
- We know it's true for P(1)
- Because of $P(k) \rightarrow P(k+1)$, if it's true for P(1), then it's true for P(2)
- Because of $P(k) \rightarrow P(k+1)$, if it's true for P(2), then it's true for P(3)
- Because of $P(k) \rightarrow P(k+1)$, if it's true for P(3), then it's true for P(4)
- Because of $P(k) \rightarrow P(k+1)$, if it's true for P(4), then it's true for P(5)
- And onwards to infinity
- Thus, it is true for all possible values of n
- In other words, we showed that:
  - $[P(1) \wedge \forall k\ (P(k) \rightarrow P(k+1))] \rightarrow \forall n\ P(n)$

Example

- Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

- Proof:
  - First define P(n)

    P(n) is $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

  - Basis step: (Show P(0) is true.)

    $1 = 2^1 - 1$ So, P(0) is true.

Example

Inductive Step: (Show ∀k (P(k) → P(k+1)) is true

- Assume P(k) is true.

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

- Show P(k+1) is true.

$$P(k+1) : 1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{k+2} - 1$$

$$L.H.S\ of\ P(k+1): 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1}$$

$$= 2.2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

$$= R.H.S\ of\ P(k+1)$$

- We showed that P(k+1) is true under assumption that P(k) is true. So, by mathematical induction that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

Example

- Prove by mathematical induction
  $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad for\ all\ integers\ n \geq 1.$

Example

- Use mathematical induction to prove the formula for the sum of a finite number of terms of a geometric progression.
- $\sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n = {(ar^{n+1}-a)}/{(r-1)}$

  $where\ r \neq 1.$
- Proof:
  - First define P(n)

    P(n) is $a + ar + ar^2 + \cdots + ar^n = {(ar^{n+1}-a)}/{(r-1)}$

  - Basis step: (Show P(0) is true.)

    $a = {(ar-a)}/{(r-1)} = a$ So, P(0) is true.

Example

- Inductive Step: (Show ∀k (P(k) → P(k+1)) is true.)
  - Assume P(k) is true. $a + ar + ar^2 + \cdots + ar^k = {(ar^{k+1}-a)}/{(r-1)}$
  - Show P(k+1) is true.

P(k+1) : $a + ar + ar^2 + \cdots + ar^{k+1} = {(ar^{k+2}-a)}/{(r-1)}$

L.H.S of P(k+1): $a + ar + ar^2 + \cdots + ar^k + ar^{k+1}$

$$= \left(ar^{k+1} - a\right)\Big/{(r-1)} + ar^{k+1}$$

$$= \left(ar^{k+1} - a\right)\Big/{(r-1)} + ar^{k+1}\,(r-1)\Big/{(r-1)}$$

$$= \left(ar^{k+1} - a + ar^{k+2} - ar^{k+1}\right)\Big/{(r-1)}$$

$$= \left(ar^{k+2} - a\right)\Big/{(r-1)} = \mathrm{R.H.S\ of\ P(k+1)}$$

- We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction $a + ar + ar^2 + \cdots + ar^n = {(ar^{n+1}-a)}/{(r-1)}$

Proving Divisibility Results

- $Use\ mathematical\ induction\ to\ prove\ that\ n^3 - n\ \ is\ divisible\ by\ 3\ whenever\ n\ is\ a\ positive\ integer.$

Proof:

- First define P(n)

  P(n) is "$n^3 - n\ is\ divisible\ by\ 3$" .

- Basis step: (Show P(1) is true.)

  $1^3 - 1 = 0\ is\ divisible\ by\ 3.$

  So, P(1) is true.

Proving Divisibility Results

- Inductive Step: (Show ∀k (P(k) → P(k+1)) is true.)
  - Assume P(k) is true.
    $k^3 - k \textit{ is divisible by } 3$.
  - Show P(k+1) is true.
    P(k+1) is $(k+1)^3 - (k+1) \textit{ is divisible by } 3$.

$$
\begin{aligned}
(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\
&= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= k^3 + 3k^2 + 3k - k \\
&= (k^3 - k) + 3(k^2 + k)
\end{aligned}
$$

- We showed that P(k+1) is true under assumption that P(k) is true. So given statement is true by mathematical induction.

Proving Divisibility Results

- $Use\ mathematical\ induction\ to\ prove\ that\ 2^{2n} - 1\ \ is\ divisible\ by\ 3\ whenever\ n\ is\ a\ positive\ integer.$

Proof:

- First define P(n)

  P(n) is $"2^{2n}-1\ is\ divisible\ by\ 3"$.

- Basis step: (Show P(1) is true.)

  $2^2 - 1 = 3\ is\ divisible\ by\ 3.$

  So, P(1) is true.

Proving Divisibility Results

- Inductive Step: (Show ∀k (P(k) → P(k+1)) is true.)
  - Assume P(k) is true.
    $2^{2k} - 1$ *is divisible by* 3.
  - Show P(k+1) is true.
    P(k+1) is $2^{2k+2} - 1$ *is divisible by* 3.

$$2^{2k+2} - 1 = 2^{2k}.2^2 - 1 = 2^{2k}.4 - 1$$
$$= 2^{2k}.(3+1) - 1 = 3.2^{2k} + (2^{2k} - 1)$$

- We showed that P(k+1) is true under assumption that P(k) is true. So given statement is true by mathematical induction.

Proving Inequalities Example

- $Use\ mathematical\ induction\ to\ prove\ the\ inequality$
  $2^n < n!\ for\ all\ positive\ integers\ n\ and\ n \geq 4$.

Proof:

- First define P(n)

  P(n) is $2^n < n!$.
- Basis step: (Show P(4) is true.)

  $2^4 < 4!$

  $16 \ < 24$

  So, P(4) is true.

Proving Inequalities Example

- Inductive Step: (Show ∀k (P(k) → P(k+1)) is true.)
- Assume P(k) is true for $k \geq 4$

  $2^k < k!$
- Show P(k+1) is true.

  P(k+1) is $2^{k+1} < (k+1)!$

Proving Inequalities Example

- Inductive Step: (Show ∀k (P(k) → P(k+1)) is true.)
- Assume P(k) is true for $k \geq 4$

  $2^k < k!$
- Show P(k+1) is true.

  P(k+1) is $2^{k+1} < (k+1)!$

$$
\begin{aligned}
2^{k+1} &= 2.2^k && \textit{by definition of exponent} \\
&< 2.k! && \textit{by the induction hypothesis} \\
&< (k+1).k! && \textit{because } 2 < k+1 \\
&= (k+1)! && \textit{by definition of factorial function.}
\end{aligned}
$$

- We showed that P(k+1) is true under assumption that P(k) is true. So, by mathematical induction$2^n < n!$ for all positive integers n and $\mathrm{n} \geq 4$.

Proving Inequalities Example

Show that $n! < n^n$ for all $n > 1$.

Proof:

- First define P(n)
  P(n) is $n! < n^n$
- Basis Step: (Show P(2) is true.)
  $2! < 2^2$
  $2 < 4$
  So, P(2) is true.

Proving Inequalities Example

- Inductive Step: (Show ∀k (P(k) → P(k+1)) is true.)
  - Assume P(k) is true $k > 1$.
    $k! < k^k$
  - Show P(k+1) is true.
    P(k+1) is $(k+1)! < (k+1)^{k+1}$

Proving Inequalities Example

- Inductive Step: (Show ∀k (P(k) → P(k+1)) is true.)
  - Assume P(k) is true $k > 1$.

    $k! < k^k$
  - Show P(k+1) is true.

    P(k+1) is $(k+1)! < (k+1)^{k+1}$

    $(k+1)! = (k+1) \cdot k!$

    $(k+1) \cdot k! < (k+1).k^k$

    $< (k+1)(k+1)^k \; as \quad k^k < (k+1)^k$

    $= (k+1)^{k+1}$
- We showed that P(k+1) is true under assumption that P(k) is true.

Proving Inequalities Example

- $Use\ mathematical\ induction\ to\ prove\ the\ inequality$
  $n < 2^n\ for\ all\ positive\ integers\ n$.

Proof:

- First define P(n)

  P(n) is $n < 2^n$
- Basis step: (Show P(1) is true.)

  1$< 2^1 = 2$

  So, P(1) is true.

Proving Inequalities Example

- Inductive Step: (Show ∀k (P(k) → P(k+1)) is true.)
- Assume P(k) is true $k \geq 1.$

  $k < 2^k$
- Show P(k+1) is true.

  P(k+1) is $k + 1 < 2^{k+1}$

  $k + 1 < 2^k + 1$ $\quad$ $using\ induction\ hypothesis\ \ k < 2^k$

  $< 2^k + 2^k = 2.2^k = 2^{k+1}$

- We showed that P(k+1) is true under assumption that P(k) is true. So, by mathematical induction $n < 2^n$ for all positive integers n.

Chapter Exercise

Chapter # 5

Topic # 5.1

Q 3, 4, 5, 7, 8, 18, 20, 21, 31, 32, 33, 34