Linear Combinations

If $\vec{w}$ is the vector in vector space V, then $\vec{w}$ is said to be a linear combination of the vectors $\vec{v}_1,\ \vec{v}_2, \dots\ \vec{v}_r$ in V, if $\vec{w}$ can be expressed in the form $\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r$, where $k_1, k_2, \dots, k_r$ are scalars. These scalars are called the coefficients of linear combinations.

Example 1:

Let $\vec{v}_1 = (1, 0, 0), \vec{v}_2 = (0, 1, 0), \vec{v}_3 = (0, 0, 1)$ in R^3. Then $\vec{w} = (2, 3, 4)$ can be written as linear combinations of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Solution:

$$\vec{w} = k_1\vec{v}_1 + \mathrm{k}_2\vec{v}_2 + k_3\vec{v}_3$$

$$(2, 3, 4) = k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1)$$

$$(2, 3, 4) = 2(1, 0, 0) + 3(0, 1, 0) + 4(0, 0, 1)$$

$$\vec{w} = 2\vec{v}_1 + 3\vec{v}_2 + 4\vec{v}_3$$

Example 2:

Consider the vectors $\vec{u} = (1, 2, -1)$ and $\vec{v} = (6,4,2)$ in R^3. Show that $\vec{w} = (9, 2,7)$ is linear combination of $\vec{u}$ and $\vec{v}$.

Also prove that $\vec{w}' = (4, -1, 8)$ is not linear combination of $\vec{u}$ and $\vec{v}$.

Solution:

In order to form linear combination of $\vec{u}$ and $\vec{v}$, there exist k_1 and k_2 such that

$$\vec{w} = k_1\vec{u} + \mathrm{k}_2\vec{v}$$

$$(9,2,7) = k_1(1, 2, -1) + k_2(6,4,2)$$

$$(9,2,7) = (k_1, 2k_1, -k_1) + (6k_2, 4k_2, 2k_2)$$

$$(9,2,7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives:

$$k_1 + 6k_2 = 9$$

$$2k_1 + 4k_2 = 2$$

$$-k_1 + 2k_2 = 7$$

Solving this system by using Gauss Elimination:

$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right] \quad R_2 - 2R_1 \quad and \quad R_3 + R_1$$

$$\begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \qquad -\frac{R_2}{8} \quad and \quad \frac{R_3}{8}$$

$$\begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_3 - R_2$$

$$\begin{cases} k_1 + 6k_2 = 9 \quad ----(1) \\ k_2 = 2 \\ 0 = 0 \end{cases}$$

Put value of $k_2 = 2$ in (1), we get:

$$k_1 + 6(2) = 9$$

$$k_1 = 9 - 12$$

$$k_1 = -3$$

So

$$\vec{w} = -3\vec{u} + 2\vec{v}$$

That is

$$(9,2,7) = -3(1,2,-1) + 2(6,4,2)$$

So $\vec{w}$ is the linear combination of $\vec{u}$ *and* $\vec{v}$.

Similarly, we have to check whether $\vec{w}'$ is linear combination of $\vec{u}$and $\vec{v}$.

If $\vec{w}'$ is linear combination of $\vec{u}$and $\vec{v}$, then there must exist k_1 and k_2, such that:

$$\overrightarrow{w'} = k_1\vec{u} + k_2\vec{v}$$

$$(4,-1,8) = k_1(2,2,-1) + k_2(6,4,2)$$

$$(4,-1,8) = (2k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$\begin{cases} 4 = k_1 + 6k_2 \\ -1 = 2k_1 + 4k_2 \\ 8 = -k_1 + 2k_2 \end{cases}$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{bmatrix} \quad R_2 - 2R_1 \quad and \quad R_3 + R_1$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 2 & 3 \end{bmatrix} \quad -\frac{1}{8}R_2 \quad and \quad \frac{1}{4}R_3$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \quad R_3 - 2R_2$$

$$\begin{cases} k_1 + 6k_2 = 4 \\ k_2 = \frac{9}{8} \\ 0 = \frac{3}{2} \end{cases}$$

This system has no solution. So, no such k_1 and k_2 exist. So $\vec{w}'$ is not the linear combination of $\vec{u}$ *and* $\vec{v}$.

Example 3:

Which of the following are linear combination of $\vec{u} = (0,-2,2)$ and$\vec{v} = (1,3,-1)$.

a) (2, 2, 2)
b) (3,1,5)
c) (0,4,5)

d) (0,0,0)

Example 4:

Express the following as linear combination of $\vec{u} = (2,1,4), \vec{v} = (1,-1,3), \vec{w} = (3,2,5)$

a) (-9, -7,-15)
b) (66,11,6)
c) (0,0,0)
d) (7,8,9)

Example 5:

Which of the following is the linear combination of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

a) $M = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$
b) $N = \begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$
c) $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
d) $P = \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

Solution:

$$M = k_1 A + k_2 B + K_3 C$$

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = k_1 \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + k_2 \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \begin{bmatrix} 4k_1 & 0 \\ -2k_1 & -2k_1 \end{bmatrix} + \begin{bmatrix} k_2 & -k_2 \\ 2k_2 & 3k_2 \end{bmatrix} + \begin{bmatrix} 0 & 2k_3 \\ k_2 & 4k_3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \begin{bmatrix} 4k_2 + k_2 & -k_2 + 2k_3 \\ -2k_1 + 2k_2 + k_3 & -2k_1 + 3k_2 + 4k_3 \end{bmatrix}$$

$$6 = 4k_1 + k_2 - - - - - (1)$$

$$-8 = -k_2 + 2k_3 - - - - - (2)$$

$$-1 = -2k_1 + 2k_2 + k_3 - - - - - (3)$$

$$-8 = -2k_1 + 3k_2 + 4k_3 - - - - - (4)$$

Add (1) and (2):

$$6 = 4k_1 + k_2$$

$$-8 = -k_2 + 2k_3$$

$$-1 = 2k_1 + k_3$$

$k_3 = -1 - 2k_1$, put in (3)

$$-1 = -2k_1 + 2k_2 + k_3$$

$$-1 = -2k_1 + 2k_2 + (-1 - 2k_1)$$

$$-1 = -2k_1 + 2k_2 - 1 - 2k_1$$

$$-1 + 1 = -4k_1 + 2k_2$$

$$-4k_1 + 2k_2 = 0 - - - - - - (5)$$

Add (1) and (5)

$$4k_1 + k_2 = 6$$

$$-4k_1 + 2k_2 = 0$$

$$3k_2 = 6$$

$k_2 = 2,$ Put in (1)

$$4k_1 + 2 = 6$$

$$4k_1 = 4$$

$$k_1 = 1$$

$k = 2$, put in (2)

$$-8 = -2 + 2k_3$$

$$-8 + 2 = 2k_3$$

$$k_3 = -3$$

Put $k_1 = 1, k_2 = 2, k_3 = -3$ in (4)

$$-8 = -2(1) + 3(2) + 4(-3)$$

$$-8 = -2 + 6 - 12$$

$$-8 = -8$$

$$M = 1A + 2B - 3C$$

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

So $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is linear combination of A, B and C.

Polynomials of degree n:

$$P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0; \qquad a_n, a_{n-1}, a_n, \ldots, a_1, a_0 \in R \text{ and } a_n \neq 0$$

Let P_n is the set of all polynomials of degree n, form vector space under addition and scalar multiplication defined by:

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

$$q(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0$$

$$p(t) + q(t) = (a_n + b_n) t^n + (a_{n-1} + b_{n-1}) t^{n-1} + \cdots + (a_1 + b_1) t + (a_0 + b_0)$$

If k is any scalar then

$$kp(t) = k(a_n) t^n + k(a_{n-1}) t^{n-1} + \cdots + k(a_1) t + k(a_0)$$

Then P_n is vector space.

Example 6:

In each part the vectors are as linear combination of $P_1 = 2 + x + 4x^2$, $P_2 = 1 - x + 3x^2$ and $P_3 = 3 + 2x + 5x^2$

a) $-9 - 7x - 15x^2$
b) $6 + 11x + 6x^2$
c) 0
d) $7 + 8x + 9x^2$

Solution:

$$-9-7x-15x^2 = k_1P_1 + k_2P_2 + k_3P_3$$

$$-9-7x-15x^2 = k_1(2+x+4x^2) + \mathrm{k}_2(1-x+3x^2) + k_3(3+2x-5x^2)$$

$$-9-7x-15x^2 = 2k_1 + k_1x + 4k_1x^2 + k_2 - k_2x + 3k_2x^2 + 3k_3 + 2k_3x - 5k_3x^2$$

$$= (2k_1 + k_2 + 3k_3) + (k_1x - k_2x + 2k_3x) + (4k_1x^2 + 3k_2\,x^2 - 5k_3x^2)$$

$$-9-7x-15x^2 = (2k_1 + k_2 + 3k_3) + (k_1 - k_2 + 2k_3)x + (4k_1 + 3k_2 - 5k_3)x^2$$

Comparing equations on both sides, we get:

$$\begin{cases} -9 = 2k_1 + k_2 + 3k_3 ----(1) \\ -7 = k_1 - k_2 + 2k_3 ------(2) \\ -15 = 4k_1 + 3k_2 - 5k_3 ----(3) \end{cases}$$

Row Operation 1:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & -5 & -15 \end{array}\right]$$ multiply the 1st row by **1/2** $$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 1 & -1 & 2 & -7 \\ 4 & 3 & -5 & -15 \end{array}\right]$$

Row Operation 2:

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 1 & -1 & 2 & -7 \\ 4 & 3 & -5 & -15 \end{array}\right]$$ add **-1** times the 1st row to the 2nd row $$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & \frac{-3}{2} & \frac{1}{2} & \frac{-5}{2} \\ 4 & 3 & -5 & -15 \end{array}\right]$$

Row Operation 3:

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & \frac{-3}{2} & \frac{1}{2} & \frac{-5}{2} \\ 4 & 3 & -5 & -15 \end{array}\right]$$ add **-4** times the 1st row to the 3rd row $$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & \frac{-3}{2} & \frac{1}{2} & \frac{-5}{2} \\ 0 & 1 & -11 & 3 \end{array}\right]$$

Row Operation 4:	$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & \frac{-3}{2} & \frac{1}{2} & \frac{-5}{2} \\ 0 & 1 & -11 & 3 \end{array}\right]$	multiply the 2nd row by **-2/3**	$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & 1 & \frac{-1}{3} & \frac{5}{3} \\ 0 & 1 & -11 & 3 \end{array}\right]$
Row Operation 5:	$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & 1 & \frac{-1}{3} & \frac{5}{3} \\ 0 & 1 & -11 & 3 \end{array}\right]$	add **-1** times the 2nd row to the 3rd row	$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & 1 & \frac{-1}{3} & \frac{5}{3} \\ 0 & 0 & \frac{-32}{3} & \frac{4}{3} \end{array}\right]$
Row Operation 6:	$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & 1 & \frac{-1}{3} & \frac{5}{3} \\ 0 & 0 & \frac{-32}{3} & \frac{4}{3} \end{array}\right]$	multiply the 3rd row by **-3/32**	$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & 1 & \frac{-1}{3} & \frac{5}{3} \\ 0 & 0 & 1 & \frac{-1}{8} \end{array}\right]$

Row Operation 7:

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & 1 & \frac{-1}{3} & \frac{5}{3} \\ 0 & 0 & 1 & \frac{-1}{8} \end{array}\right] \quad \text{add } \mathbf{1/3} \text{ times the 3rd row to the 2nd row} \quad \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & 1 & 0 & \frac{13}{8} \\ 0 & 0 & 1 & \frac{-1}{8} \end{array}\right]$$

Row Operation 8:

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-9}{2} \\ 0 & 1 & 0 & \frac{13}{8} \\ 0 & 0 & 1 & \frac{-1}{8} \end{array}\right] \quad \text{add } \mathbf{-3/2} \text{ times the 3rd row to the 1st row} \quad \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{-69}{16} \\ 0 & 1 & 0 & \frac{13}{8} \\ 0 & 0 & 1 & \frac{-1}{8} \end{array}\right]$$

Row Operation 9:

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{-69}{16} \\ 0 & 1 & 0 & \frac{13}{8} \\ 0 & 0 & 1 & \frac{-1}{8} \end{array}\right] \quad \text{add } \mathbf{-1/2} \text{ times the 2nd row to the 1st row} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-41}{8} \\ 0 & 1 & 0 & \frac{13}{8} \\ 0 & 0 & 1 & \frac{-1}{8} \end{array}\right]$$

As $k_1 = -\frac{41}{8}, k_2 = \frac{13}{8}, k_3 = -\frac{1}{8}$

$$\boldsymbol{-9 - 7x - 15x^2 = -\frac{41}{8}(2 + x + 4x^2) + \frac{13}{8}(1 - x + 3x^2) - \frac{1}{8}(3 + 2x - 5x^2)}$$

So $-9 - 7x - 15x^2$ is linear combination of P_1, P_2, P_3.