Linear Independence and Spanning

Linear Dependence and Independence

A set of vectors is said to be **linearly dependent** if at least one of the vectors in the set can be defined as a linear combination of the others; if no vector in the set can be written in this way, then the vectors are said to be **linearly independent**.

Definition

If $S = \{v_1, v_2, v_3, \dots, v_n\}$ is a non-empty set of vectors in a vector space V, then the vector equation

$$k_1v_1 + k_2v_2 + k_3v_3 + \cdots + k_nv_n = 0$$

has at least one solution, namely,

$$k_1 = 0, k_2 = 0, \dots, k_n = 0.$$

We call this the trivial solution. If this is the only solution, then S is said to be linearly independent set. If there are solutions in addition to the trivial solution, then S is called as linearly dependent set.

Example 1: Determine whether the set consists of vectors $v_1 = (1, -2,3), v_2 = (5,6, -1), v_3 = (3,2,1)$ are linearly dependent or independent.

Solution: Let $k_1v_1 + k_2v_2 + k_3v_3 = 0$ be the linear combination. Then,

$$k_1(1, -2,3) + k_2(5,6, -1) + k_3(3,2,1) = 0$$

$$\left(k_1 + 5k_2 + 3k_3, -2k_1 + 6k_2 + 2k_{3,} 3k_1 - k_2 + k_3\right) = (0,0,0)$$

$$k_1 + 5k_2 + 3k_3 = 0$$

$$-2k_1 + 6k_2 + 2k_3 = 0$$

$$3k_1 - k_2 + k_3 = 0$$

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{array}\right) \quad R_2 + 2R_1, R_3 - 3R_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \quad \frac{R_2}{8}, \frac{R_3}{-8}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_3 - R_2$$

$$\begin{cases} k_1 + 5k_2 + 3k_3 = 0 \quad \ldots\ldots(1) \\ 2k_2 + k_3 = 0 \quad \ldots\ldots\ldots(2) \\ 0 = 0 \end{cases}$$

From equation (2), we get

$$k_3 = -2k_2$$

Putting value of k_3 in equation (1), we get:

$$k_1 + 5k_2 + 3(-2k_2) = 0$$

$$k_1 - 11k_2 = 0$$

$$k_1 = 11k_2$$

Let $k_2 = t,$ therefore

$$k_1 = 11t$$

$$k_3 = 2t$$

As,

$$k_1 = k_2 = k_3 \neq 0$$

So, the vectors v_1, v_2, v_3 are linearly dependent.

Example 2: Determine whether the set consists of vectors $v_1 = (1,0,1,2), v_2 = (0,1,1,2), v_3 = (1,1,1,3)$ are linearly dependent or independent.

Solution:

Let $$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$k_1(1,0,1,2) + k_2(0,1,1,2) + k_3(1,1,1,3) = (0,0,0,0)$$

$$(k_1, 0, k_1, 2k_1) + (0, k_2, k_2, 2k_2) + (k_3, k_3, k_3, 3k_3) = (0,0,0,0)$$

$$\left(k_1 + k_3, \quad k_2 + k_3, \quad k_1 + k_2 + k_3, \; 2k_1 + 2k_2 + 3k_3\right) = (0,0,0,0)$$

$$\begin{cases} k_1 + k_3 = 0 & \dots\dots\dots (1) \\ k_2 + k_3 = 0 & \dots\dots\dots (2) \\ k_1 + k_2 + k_3 = 0 & \dots\dots\dots (3) \\ 2k_1 + 2k_2 + 3k_3 = 0 & \dots\dots\dots (4) \end{cases}$$

From (1), we get, $k_1 = -k_3$

From (2), we get, $k_2 = -k_3$

Put values of k_1 and k_2 in equation (3), we get:

$$k_1 + k_2 + k_3 = 0$$

$$-k_3 - k_3 + k_3 = 0$$

$$k_3 = 0$$

$$k_3 = 0 \Rightarrow k_1 = 0, k_2 = 0$$

$$k_1 = k_2 = k_3 = 0$$

$\Rightarrow$ Vectors v_1, v_2, v_3 are linearly independent.

Exercise: Determine whether vectors $v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)$ are linearly dependent or independent in R^3.

Exercise: Does $S = \{(1,2,3), (0,1,2), (3,0,-1)\}$ form a linearly independent set of vectors in R^3?

Example 3: Are the vectors

$$v_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

in M_{22} are linearly independent?

Solution: Let $$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$k_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2k_1 + k_2 & k_1 + 2k_2 - 3k_3 \\ k_2 - 2k_3 & k_1 + k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} 2k_1 + k_2 = 0 \\ k_1 + 2k_2 - 3k_3 = 0 \\ k_2 - 2k_3 = 0 \\ k_1 + k_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \quad R_2 - R_1, R_3 - R_1, R_4 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \quad R_3 + R_2, R_4 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \frac{R_3}{-5}, \frac{R_4}{-2}$$

$\Rightarrow k_3 = 0$

$\Rightarrow k_2 - 3k_3 = 0 \quad \Rightarrow k_2 = 0 \quad \Rightarrow k_1 + k_2 = 0 \Rightarrow k_1 = 0$

Hence the vectors are linearly independent.

Example 4: Are the vectors $P = t^2 + t + 2,\ Q = 2t^2 + t$ and $R = 3t^2 + 2t + 2$ in P_2 are linearly independent?

Solution: Let

$$\boldsymbol{k_1 v_1 + k_2 v_2 + k_3 v_3 = 0}$$

$$k_1(t^2 + t + 2) + k_2(2t^2 + t) + k_3(3t^2 + 2t + 2) = 0t^2 + 0\mathrm{t} + 0$$

$$k_1 t^2 + k_1 t + 2k_1 + 2k_2 t^2 + k_2 t + 3k_3 t^2 + 2k_3 t + 2k_3 = 0t^2 + 0\mathrm{t} + 0$$

$$t^2(k_1 + 2k_2 + 3k_3) + t(k_1 + k_2 + 2k_3) + (2k_1 + 2k_3) = 0t^2 + 0t + 0$$

$$\begin{cases} k_1 + 2k_2 + 3k_3 = 0 \\ k_1 + k_2 + 2k_3 = 0 \\ 2k_1 + 2k_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right)$$

By using Gaussian elimination, it is a trivial exercise to show that these vectors are linearly independent.

Spanning Set

Let S be the set of some vectors of vector space V. If every vector in V is a linear combination of vectors in S, then the set S is said to span V or the V is spanned by the set S.

DEFINITION 3

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the ***span of S***, and we say that the vectors in ***S span*** that subspace. If $S = \{w_1, w_2, w_3, \ldots, w_r\}$, then we denote the span of S by

$$span\ \{w_1, w_2, w_3, \ldots, w_r\} \text{ or } span(S).$$

Example1: Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the set of standard unit vectors in R^3. Prove that S spans R^3.

Solution:

S spans R^3 means every vector of R^3 can be written as linear combination of these three vectors i.e.

$$(a, b, c) = k_1\overrightarrow{v_1} + k_2\overrightarrow{v_2} + k_3\overrightarrow{v_3}$$

$$(a, b, c) = k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1)$$

$$(a, b, c) = (k_1, 0,0) + (0, k_2, 0) + (0,0, k_3)$$

$$(a, b, c) = (k_1, k_2, k_3)$$

$$a = k_1\ ,\ b = k_2\ ,\ c = k_3$$

$$=> (a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$$\text{e.g. } (1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1)$$

Note: The Standard Unit Vectors Span R^n. How?

Recall that the standard unit vectors in R^n are

$$e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots\ , e_n = (0, 0, 0, \dots, 1).$$

These vectors span R^n since every vector $v = (v_1, v_2, \dots, v_n)$ in R^n can be expressed as

$$v = v_1e_1 +\ v_2e_2 + \dots + v_ne_n$$

which is a linear combination of $e_1, e_2, \dots, e_n$. Thus, as in example 1, the vectors

$$\boldsymbol{i} = (1,0,0), \boldsymbol{j} = (0,1,0), \boldsymbol{k} = (0,0,1)$$

span R^3 since every vector in this space can be expressed as linear combination of these three vectors (also called basis).

Can there be vectors other than standard unit vectors in R^n which can span R^n?

Example2: Let V be the vector space R^3 and $\overrightarrow{v_1} = (1,2,1), \overrightarrow{v_2} = (1,0,2), \overrightarrow{v_3} = (1,1,0)$. Check $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ span R^3.

Solution: To check whether $\overrightarrow{v_1}, \overrightarrow{v_2},\ \overrightarrow{v_3}$ span R^3, we pick any vector $\vec{v} = (a, b, c)$ in $V = R^3$ and determine whether there are constants k_1, k_2, k_3 such that

$$\vec{v} = k_1\overrightarrow{v_1} + k_2\overrightarrow{v_2} + k_3\overrightarrow{v_3}$$

$$(a, b, c) = k_1(1,2,1) + k_2(1,0,2) + k_3(1,1,0)$$

$$(a, b, c) = (k_1, 2k_1, k_1) + (k_2, 0, 2k_2) + (k_3, k_3, 0)$$

$$k_1 + k_2 + k_3 = a \quad \dots (1)$$

$$2k_1 + 0 + k_3 = b \quad \dots (2)$$

$$k_1 + 2k_2 + 0 = c \quad \dots (3)$$

From equation (2)

$$k_3 = b - 2k_1 \quad \dots (4)$$

From equation (3)

$$k_2 = \frac{c - k_1}{2} \quad \dots (5)$$

Put the values in (1)

$$k_1 + \frac{c - k_1}{2} + b - 2k_1 = a$$

$$-k_1 + \frac{c - k_1}{2} = a - b$$

$$-2k_1 + c - k_1 = 2a - 2b$$

$$-3k_1 = 2a - 2b - c$$

$$k_1 = \frac{-2a + 2b + c}{3}$$

Put in (5)

$$k_2 = \frac{c - \frac{-2a + 2b + c}{3}}{2}$$

$$k_2 = \frac{c}{2} - \frac{1}{2}\left(\frac{-2a + 2b + c}{3}\right)$$

$$k_2 = \frac{c}{2} - \frac{1}{6}(-2a + 2b + c)$$

$$k_2 = \frac{c}{2} + \frac{2a}{6} - \frac{2b}{6} - \frac{c}{6}$$

$$k_2 = \frac{a-b+c}{3}$$

Put $k_1 = \frac{-2a+2b+c}{3}$ in (4)

$$k_3 = b - 2\left(\frac{-2a+2b+c}{3}\right)$$

$$k_3 = \frac{3b+4a-4b-2c}{3}$$

$$k_3 = \frac{4a-b-2c}{3}$$

Thus $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ span V, i.e.

$$Span\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\} = R^3.$$

Example3: Determine whether $\overrightarrow{v_1} = (1,1,2), \overrightarrow{v_2} = (1,0,1), \overrightarrow{v_3} = (2,1,3)$ span R^3.

Solution: Let $\vec{v} = k_1\overrightarrow{v_1} + k_2\overrightarrow{v_2} + k_3\overrightarrow{v_3}$

$$(a,b,c) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$$

$$(a,b,c) = (k_1, k_1, 2k_1) + (k_2, 0, k_2) + (2k_3, k_3, 3k_3)$$

$$(a,b,c) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

$$k_1 + k_2 + 2k_3 = a \quad \ldots(1)$$

$$k_1 + k_3 = b \quad \ldots(2)$$

$$2k_1 + k_2 + 3k_3 = c \quad \ldots(3)$$

From equation (2)

$$k_1 = b - k_3 \quad \ldots(4)$$

Subtract (1) and (3)

$$k_1 + k_2 + 2k_3 = a$$

$$\pm 2k_1 \pm k_2 \pm 3k_3 = \pm c$$

$$-k_1 - k_3 = a - c$$

Put $\quad k_1 = b - k_3$ in above equation,

$$-b + k_3 - k_3 = a - c$$

$$b = c - a$$

Not possible, No solution.

So $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ do not span R^3.

Another Method: For the matrix of coefficients

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$detA = 1\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} - 1\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= 1(0-1) - 1(3-2) + 2(1-0)$$

$$detA = -1 - 1 + 2 = 0$$

So the system has no solution.

System is consistent if & only if its coefficient matrix has non-zero determinant i.e. $detA \neq 0$.

Example 4: Show that the set

$$\boldsymbol{S = \{t^2 + 1, t - 1, 2t + 2\}}$$

span the vector space P_2.

Solution: For spanning, let

$$(a, b, c) = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$at^2 + bt + c = k_1(t^2 + 1) + k_2(t - 1) + k_3(2t + 2)$$

or $\quad at^2 + bt + c = k_1 t^2 + (k_2 + 2k_3)t + (k_1 - k_2 + 2k_3)$

$$\begin{cases} a = k_1 \dots\dots\dots (1) \\ b = k_2 + 2k_3 \dots\dots\dots (2) \\ c = k_1 - k_2 + 2k_3 \dots\dots\dots (3) \end{cases}$$

Put $k_1 = a$ in equation (3)

$$c = a - k_2 + 2k_3$$

$$-k_2 + 2k_3 = c - a \dots\dots\dots (4)$$

Add (2) and (4) $\qquad k_2 + 2k_3 = b$

$$-k_2 + 2k_3 = c - a$$

$$4k_3 = b + c - a$$

$$k_3 = \frac{b + c - a}{4}$$

And by subtracting both implies $\qquad k_2 = \frac{b-c+a}{2}$

So, $k_1 = a, k_2 = \frac{b-c+a}{2}, k_3 = \frac{b+c-a}{4}$.

It means S spans V.

THEOREM 4.2.5

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ and $S' = \{w_1, w_2, \dots, w_k\}$ are nonempty sets of vectors in a vector space V, then

$$span\{v_1, v_2, \dots, v_r\} = span\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

THEOREM 4.3.3

Let $S = \{v_1, v_2, v_3, \dots, v_r\}$, be a set of vectors in R^n. If , $r > n$ then S is linearly dependent.

EXAMPLE 5. An Important Linearly Independent Set in P_n.
Show that the polynomials

$$1, x, x^2, \dots, x^n$$

form a linearly independent set in P_n.

Exercise 4.2

Q11: In each part determine whether the given vector span R^3.

a) $\overrightarrow{v_1} = (2,2,2),\ \overrightarrow{v_2} = (0,0,3),\ \overrightarrow{v_3} = (0,1,1)$
b) $\overrightarrow{v_1} = (2,-1,3), \overrightarrow{v_2} = (4,1,2), \overrightarrow{v_3} = (8,-1,8)$
c) $\overrightarrow{v_1} = (3,1,4), \overrightarrow{v_2} = (2,-3,5), \overrightarrow{v_3} = (5,-2,9), \overrightarrow{v_4} = (1,4,-1)$
d) $\overrightarrow{v_1} = (1,2,6), \overrightarrow{v_2} = (3,4,1), \overrightarrow{v_3} = (4,3,1), \overrightarrow{v_4} = (3,3,1)$

Answer: (a) and (d) span R^3.

Q2: Let V be the vector space P_2. Let $\overrightarrow{v_1} = t^2 + 2t + 1, \overrightarrow{v_2} = t^2 + 2$.

Does $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ span V?

Q3: In R^3 let $\overrightarrow{v_1} = (2,1,1), \overrightarrow{v_2} = (1,-1,3)$ and Determine whether the vector

$\boldsymbol{\vec{v}} = (\mathbf{1}, \mathbf{5}, -\mathbf{7})$ Belong to the span $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$.

Exercise 4.3

1. Explain why the following are linearly dependent sets of vectors. (Solve this problem by inspection.)

(a) $\mathbf{u}_1 = (-1, 2, 4)$ and $\mathbf{u}_2 = (5, -10, -20)$ in R^3

(b) $\mathbf{u}_1 = (3, -1), \mathbf{u}_2 = (4, 5), \mathbf{u}_3 = (-4, 7)$ in R^2

(c) $\mathbf{p}_1 = 3 - 2x + x^2$ and $\mathbf{p}_2 = 6 - 4x + 2x^2$ in P_2

(d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

Answer:

(a) $\mathbf{u}_2$ is a scalar multiple of $\mathbf{u}_1$.

(b) The vectors are linearly dependent by Theorem 4.3.3.

(c) $\mathbf{p}_2$ is a scalar multiple of $\mathbf{p}_1$.

(d) B is a scalar multiple of A.

2. Which of the following sets of vectors in R^3 are linearly dependent?

(a) $(4, -1, 2)$, $(-4, 10, 2)$

(b) $(-3, 0, 4)$, $(5, -1, 2)$, $(1, 1, 3)$

(c) $(8, -1, 3)$, $(4, 0, 1)$

(d) $(-2, 0, 1)$, $(3, 2, 5)$, $(6, -1, 1)$, $(7, 0, -2)$

3. Which of the following sets of vectors in R^4 are linearly dependent?

(a) $(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (1, 4, 0, 3)$

(b) $(0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, -1)$

(c) $(0, 3, -3, -6), (-2, 0, 0, -6), (0, -4, -2, -2), (0, -8, 4, -4)$

(d) $(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)$

Answer:

None

4. Which of the following sets of vectors in P_2 are linearly dependent?

(a) $2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2$

(b) $3 + x + x^2, 2 - x + 5x^2, 4 - 3x^2$

(c) $6 - x^2$

(d) $1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2$

10. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors, then so are $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1\}$, $\{\mathbf{v}_2\}$, and $\{\mathbf{v}_3\}$.

11. Show that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

12. Show that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in a vector space V, and $\mathbf{v}_4$ is any vector in V that is not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.