Department Of Computer Science, CUI Lahore Campus

CSC102 - Discrete Structures

By

Mahwish Waqas

Lecture Outline

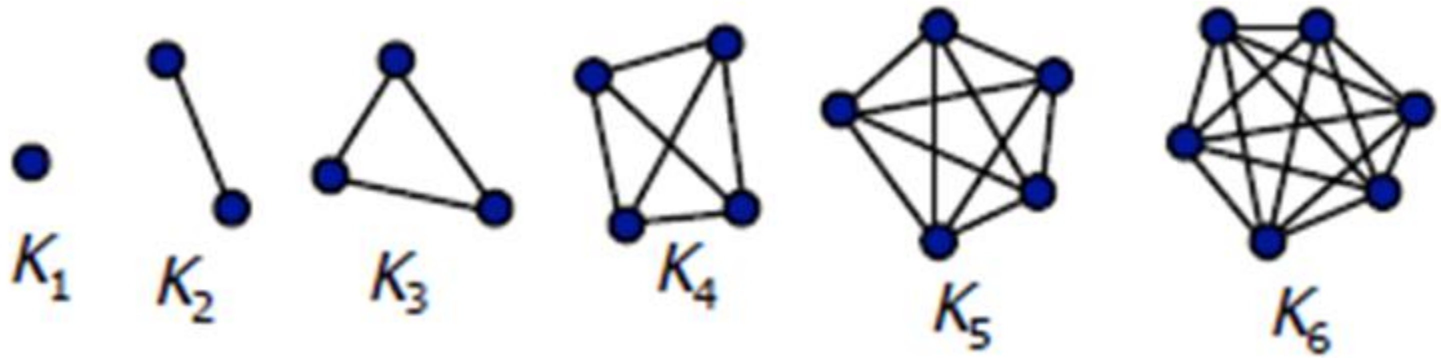
- Graphs
  - Special Graph Structures
  - Sub Graph
  - Adjacency List and Adjacency Matrix
  - Incidence Matrix
  - Path and Distance Matrix

Special Graph Structures

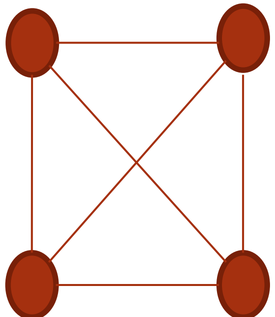
- Special cases of undirected graph structures:
  - Complete graphs K_n
  - Cycles C_n
  - Wheels W_n
  - Bipartite graphs

Complete Graphs

- For any $n \in N$, a complete graph on *n* vertices, denoted by K_n, is a simple graph with *n* nodes in which every node is adjacent to every other node. $\forall_{u,v \in V} : u \neq v \leftrightarrow \{u, v\} \in E$.

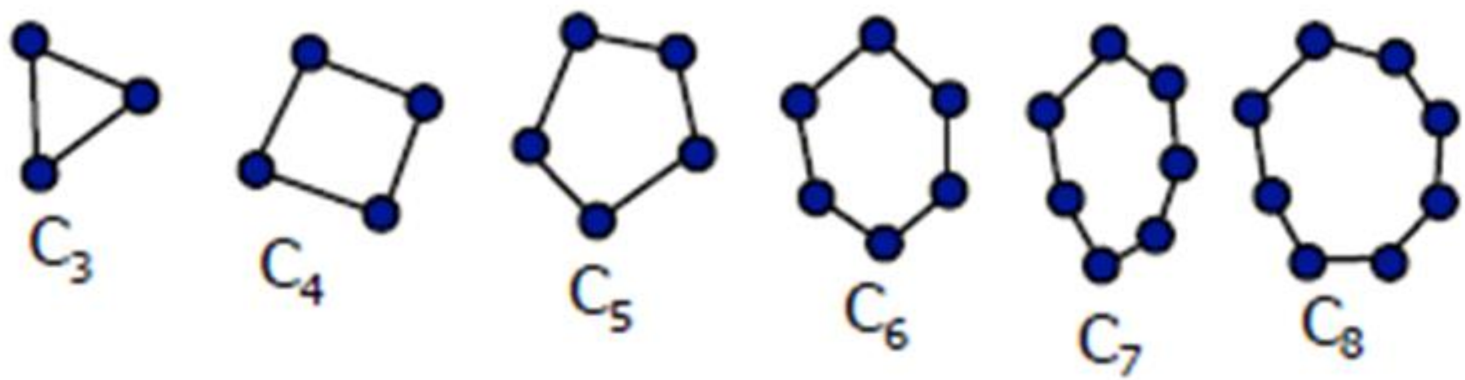


- A simple graph for which there is at least one pair of distinct vertex not connect with edge is called non complete.
- Note that K_n has $n(n-1)/2$ edges.

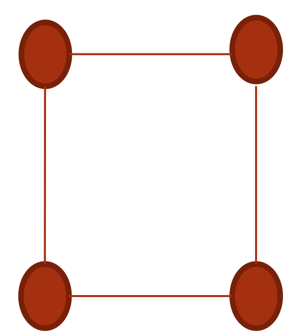


Cycles

- For any $n \geq 3$, a cycle on *n* vertices, denoted by C_n, is a simple graph where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$.



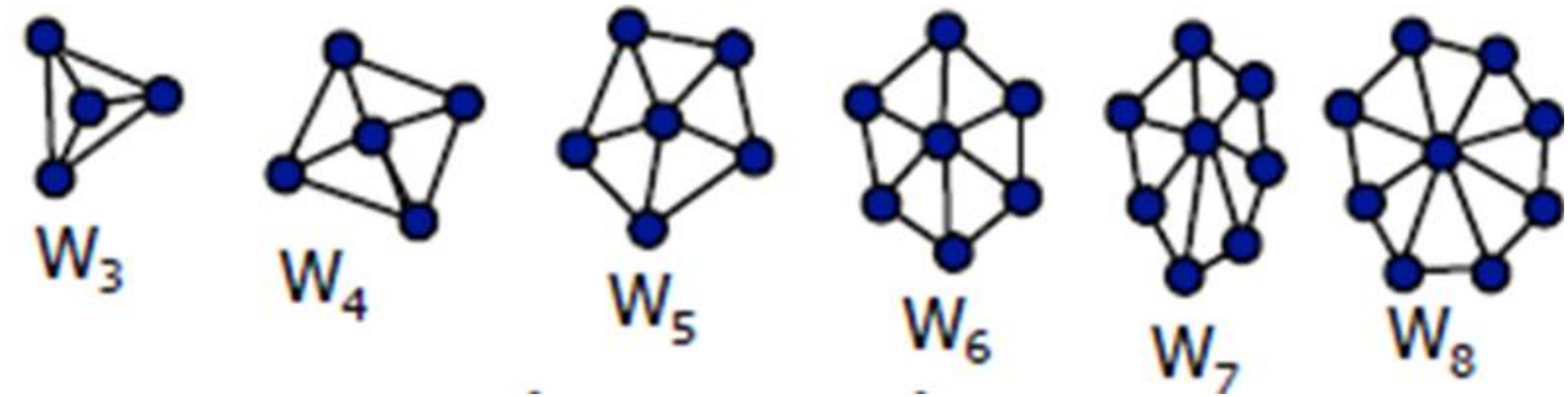
- How many edges are there in $C_n = n$.



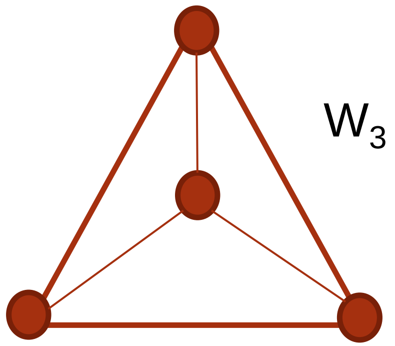
C_4

Wheels

- For any $n \geq 3$, a wheel W_n, is a simple graph obtained by taking the cycle C_n and adding an extra vertex V_{hub} and n extra edges where $\{\{v_{hub}, v_1\}, \{v_{hub}, v_2\}, \dots, \{v_{hub}, v_n\}\}$.

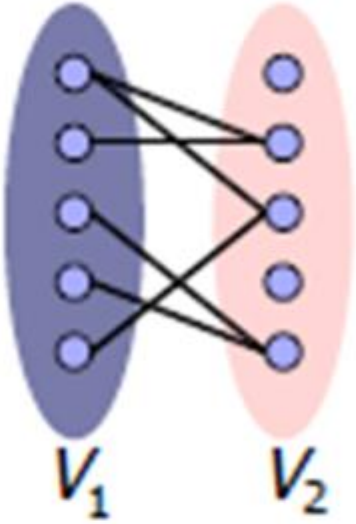


- How many edges are there in $W_n = 2n$.

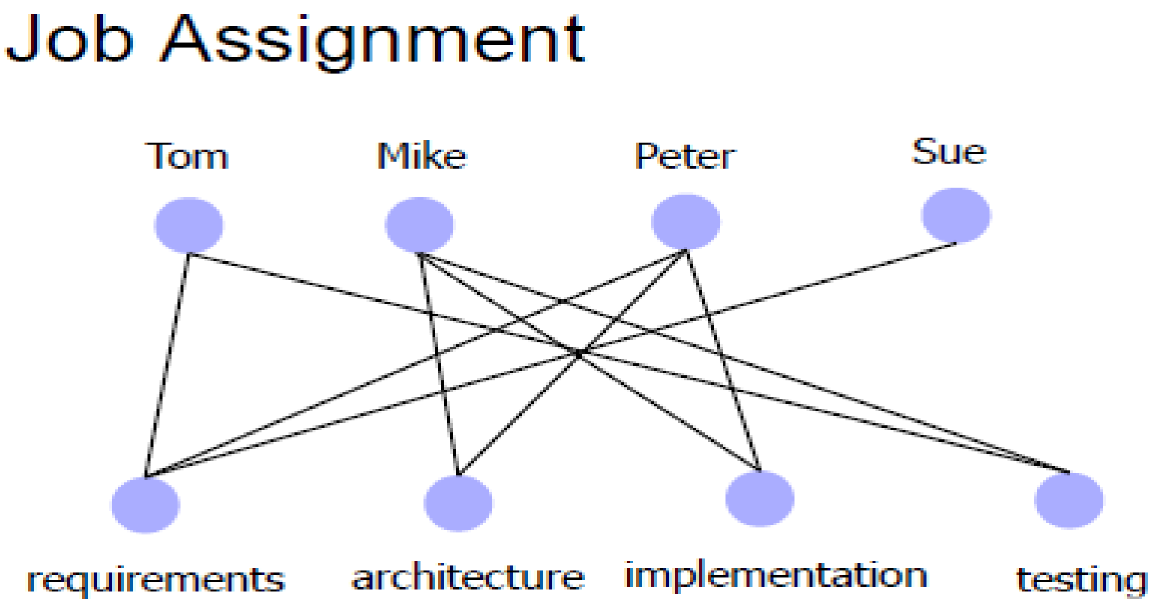


Bipartite Graph

- A graph G = {V, E} is bipartite graph (two parts) if and only if $V = V_1 U\ V_2\ where\ V_1 \cap V_2 = \emptyset$ and $\forall_{e \in E}$: $\exists v_1 \in\ V_1, v_2 \in V_2 : e = \{v_1, v_2\}$
- The graph vertices can be divided into two parts in such a way that all edges go between the two parts. (No edge in G connects either two vertices in V1 or two vertices in V2).



Example

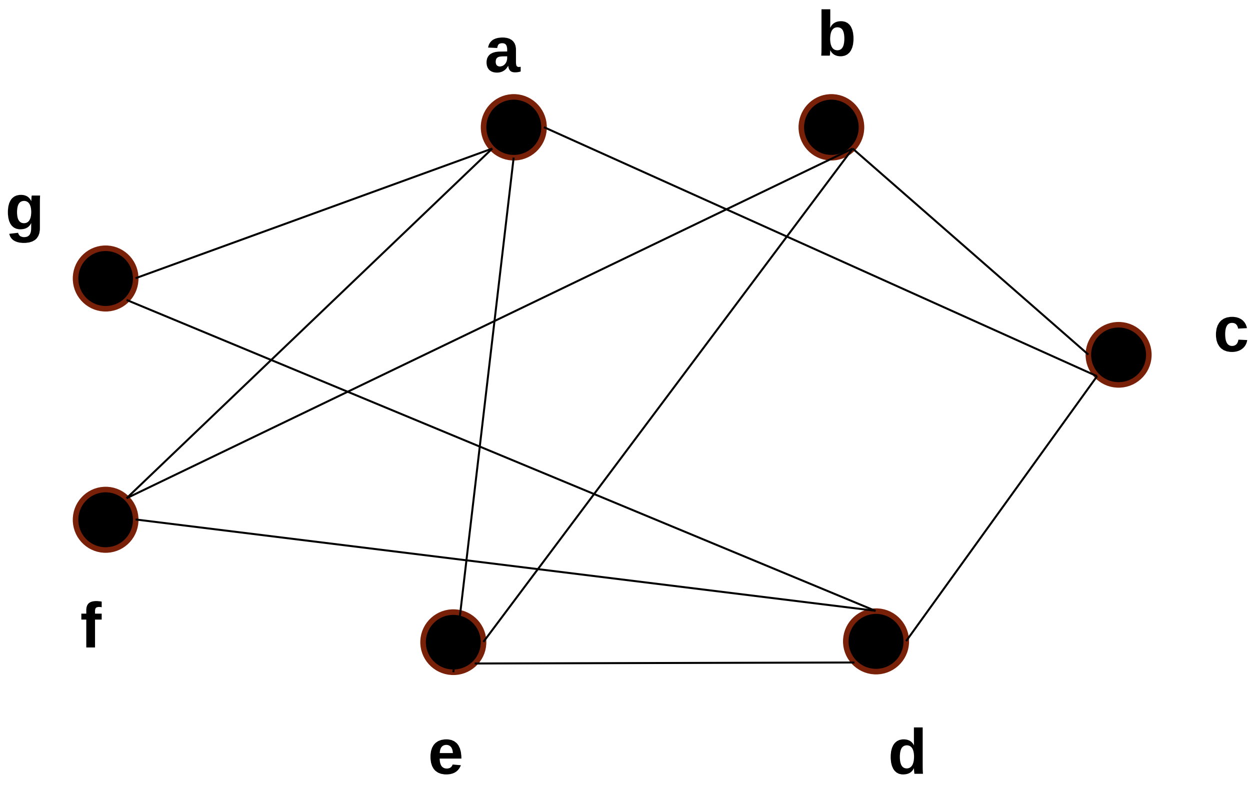

Job Assignment
Tom
Mike
Peter
Sue
requirements
architecture
implementation
testing

Theorem

- A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example

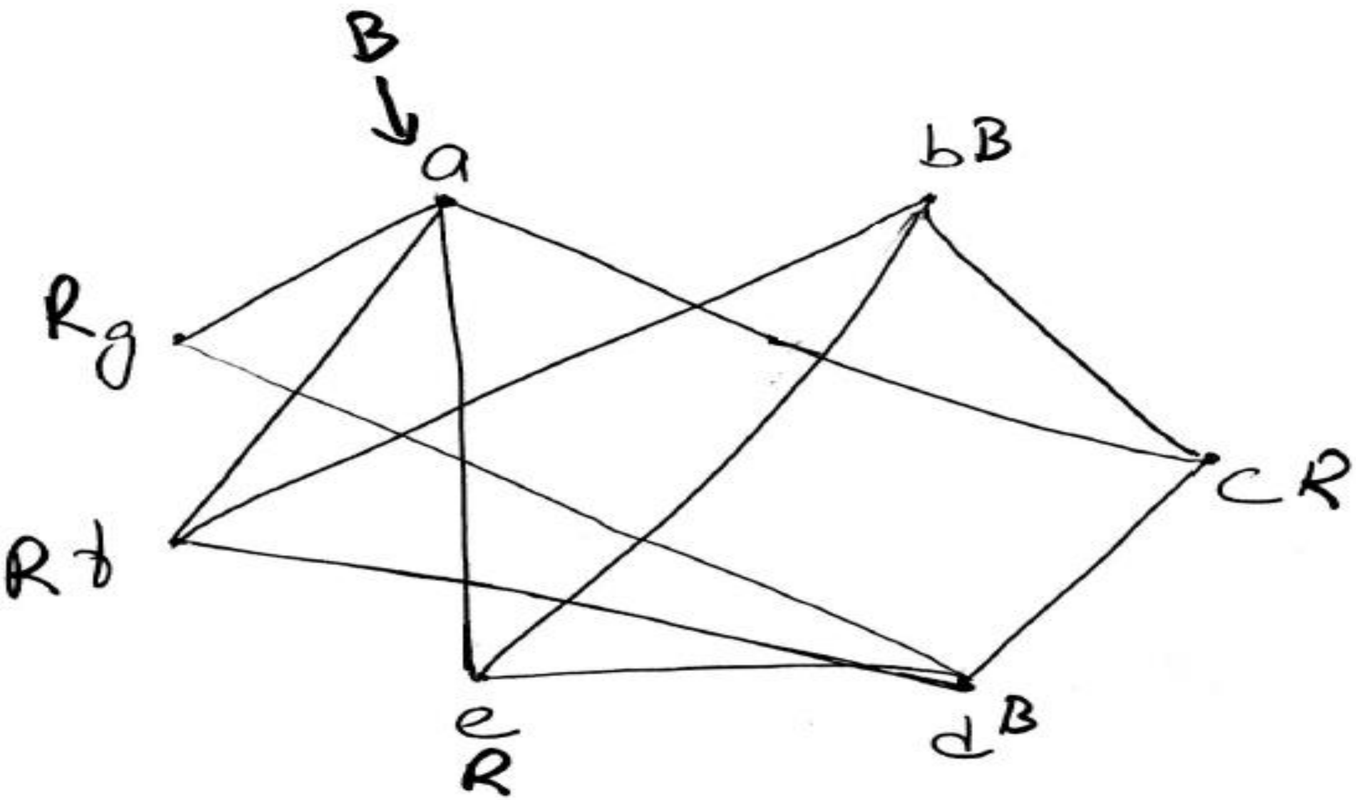
- Is this **bipartite**?



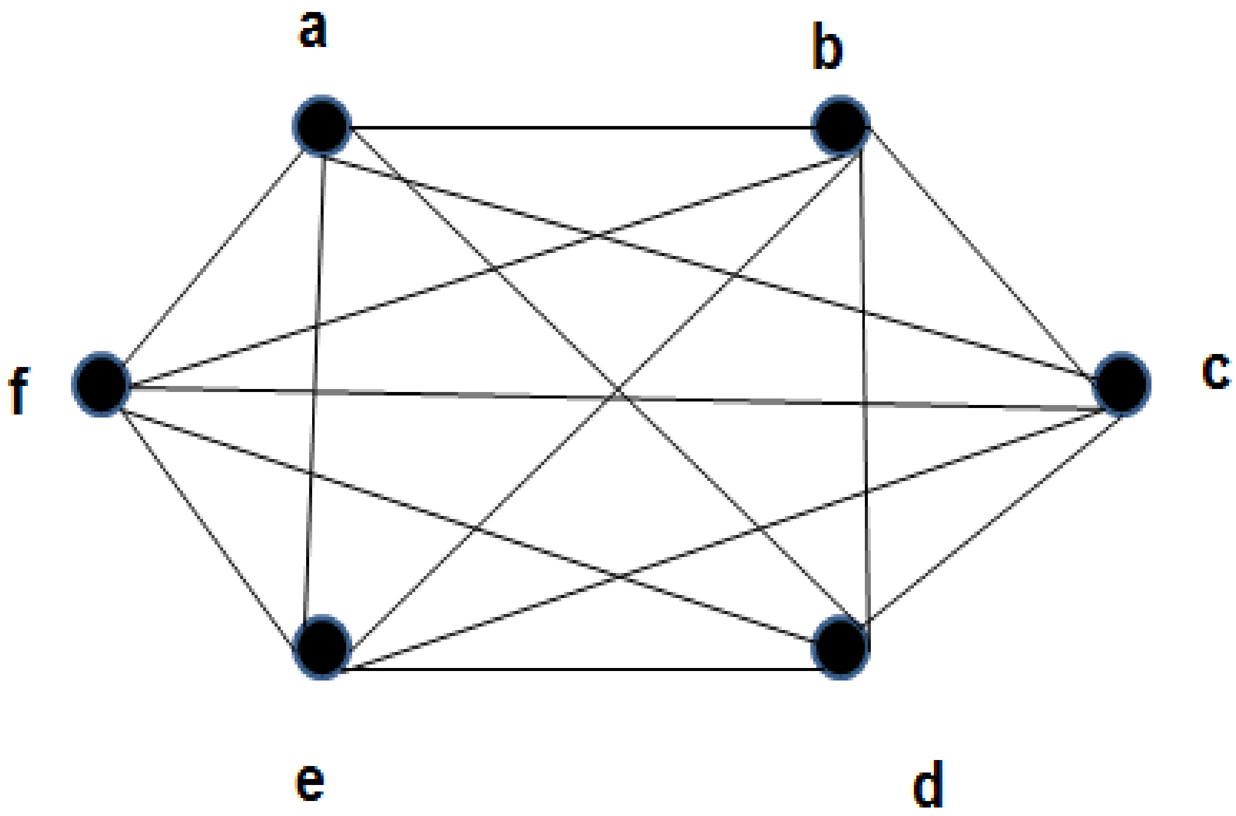
Example

- **Bipartite Graph**

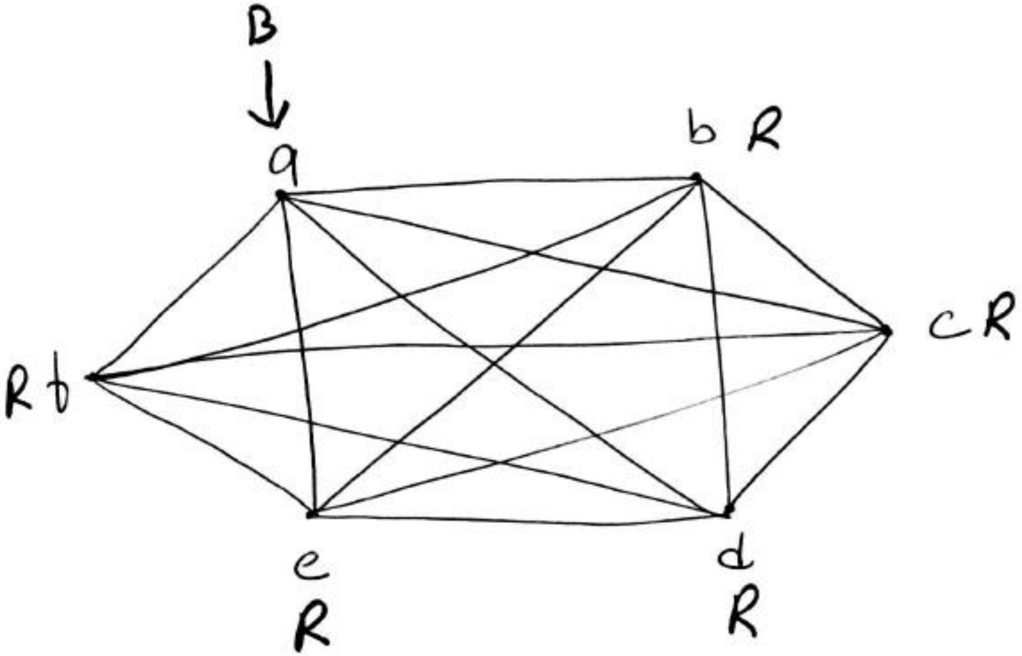
$$R = \{c, e, f, g\} = V_1$$
$$B = \{a, b, d\} = V_2$$



Example



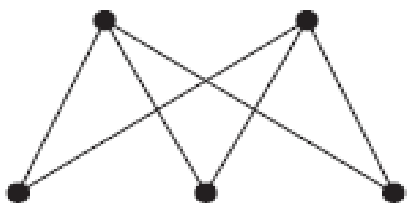
Example



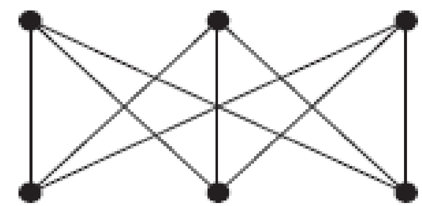
Not Bipartite graph.
Because adjacent vertices
have same color.

Complete Bipartite Graphs

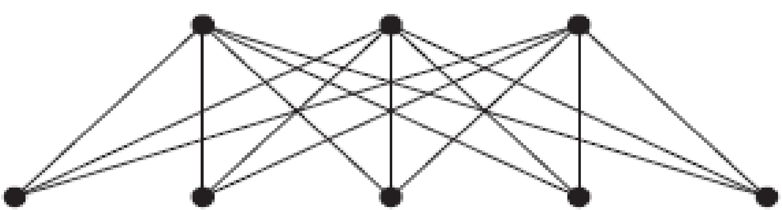
A **complete bipartite graph** $\boldsymbol{K_{m,n}}$ is a graph that has its vertex set partitioned into two subsets of *m* and *n* vertices, respectively with an edge between two vertices iff one vertex is in the first subset and the other vertex is in the second subset.



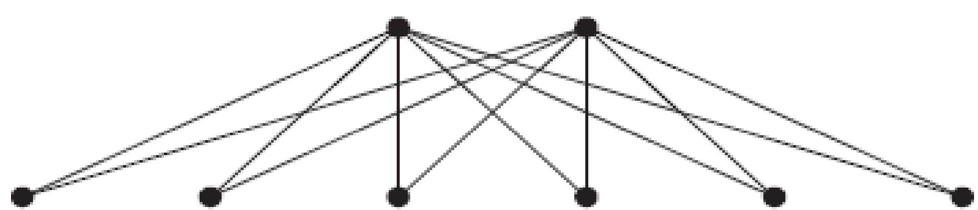
$K_{2,3}$



$K_{3,3}$



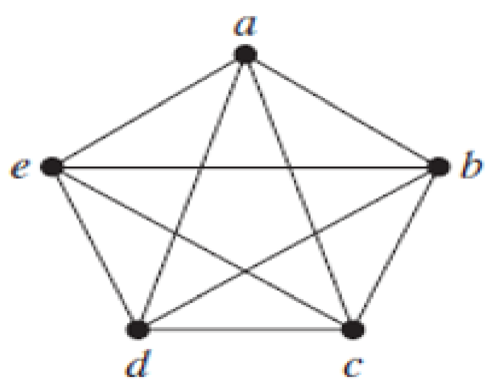
$K_{3,5}$



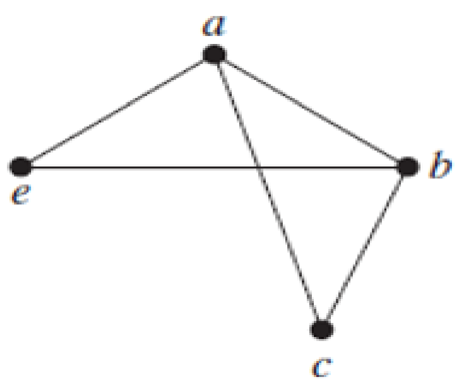
$K_{2,6}$

Sub-graphs

- A *sub-graph of a graph G = (V ,E)* is a graph *H = (W, F)*, where *W* ⊆ *V* and *F* ⊆ *E*.
- A sub-graph *H* of *G* is a *proper sub-graph* of G if $H \neq G$.



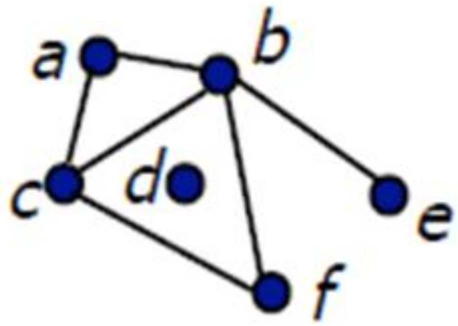
G



H

How to Represent a Graph?

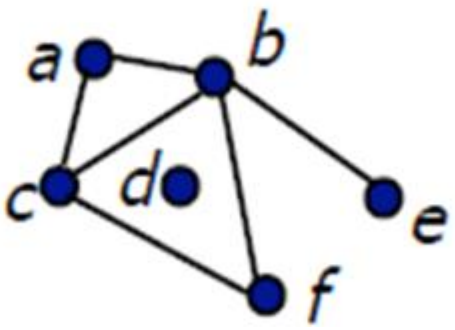
- Mathematically, a graph G is represented by two sets: vertex set V and edge Set E.
- G = (V, E), a tuple of two sets.
- How to represent a graph by computer?



- V = { a, b, c, d, e, f }
- E = {{a,b},{a,c},{b,c}, {b,e},{b,f},{c,f}}

Adjacency List

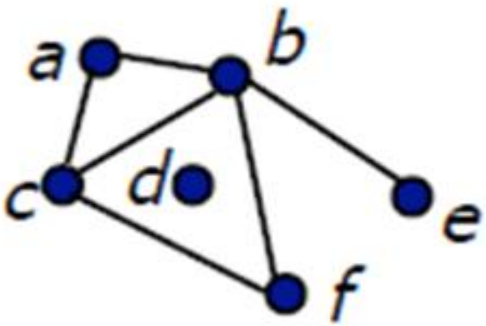
- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	Adjacent Vertices
a	b, c
b	a, c, e, f
c	a, b, f
d	
e	b
f	c, b

Adjacency Matrix

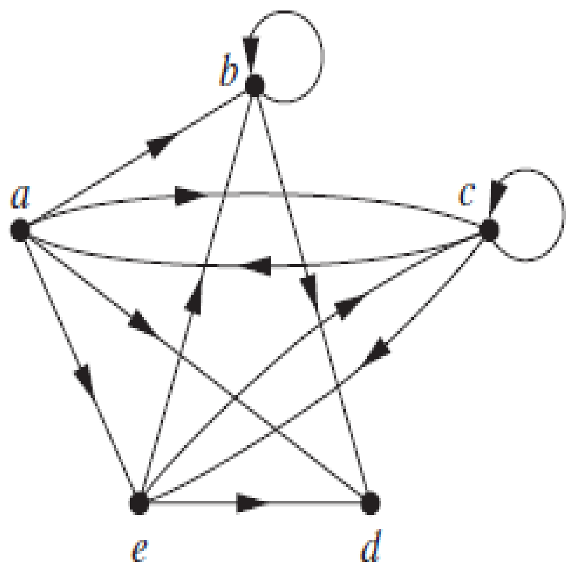
- Matrix $A = [a_{ij}]$, where a_{ij} is 1 if $\{v_i, v_j\}$ is an edge of G, and is 0 otherwise.



Vertex	Adjacent Vertices
a	b, c
b	a, c, e, f
c	a, b, f
d	
e	b
f	c, b

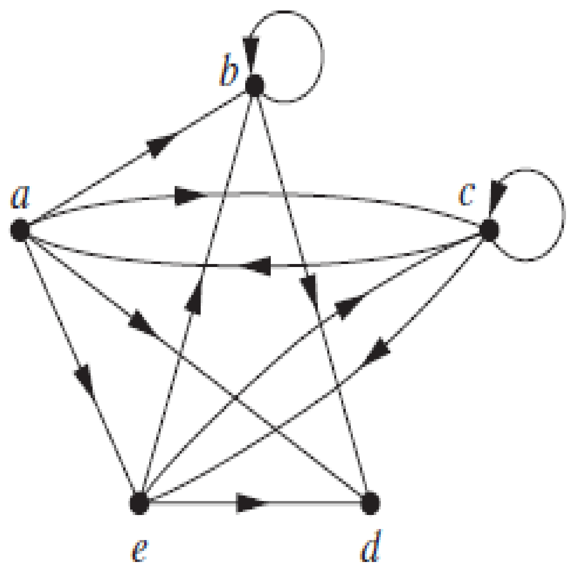
	a	b	c	d	e	f
a	0	1	1	0	0	0
b	1	0	1	0	1	1
c	1	1	0	0	0	1
d	0	0	0	0	0	0
e	0	1	0	0	0	0
f	0	1	1	0	0	0

Adjacency List for Directed Graph



Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrix for Directed Graph



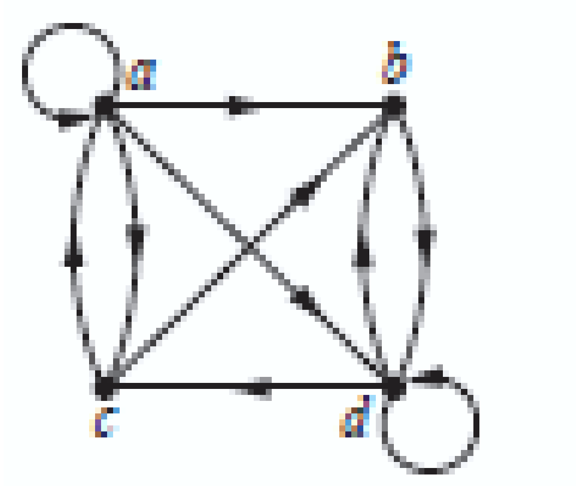
Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Terminal Vertex

initial Vertex

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix for Directed Graph



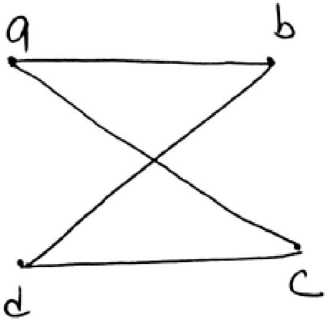
$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Example

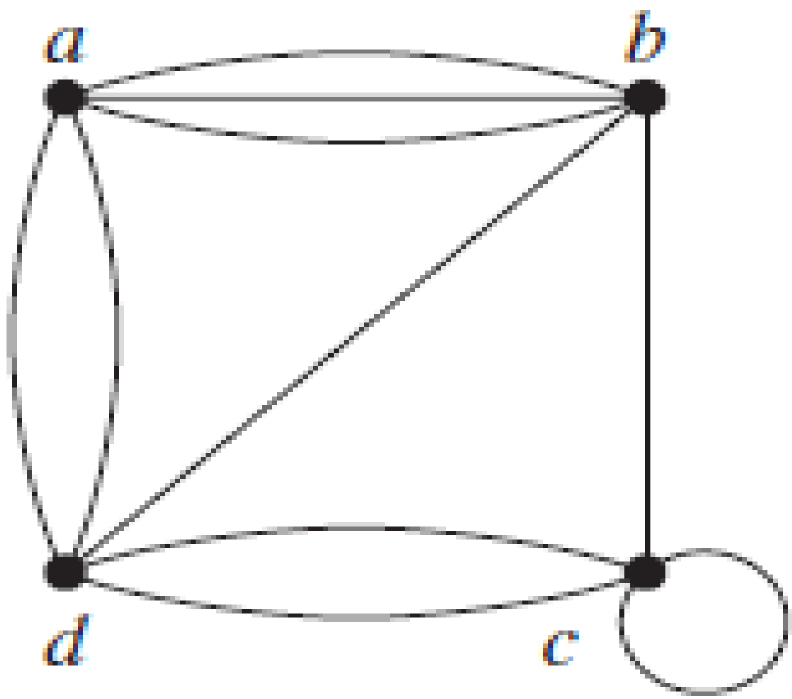
Draw a undirected graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.



Example

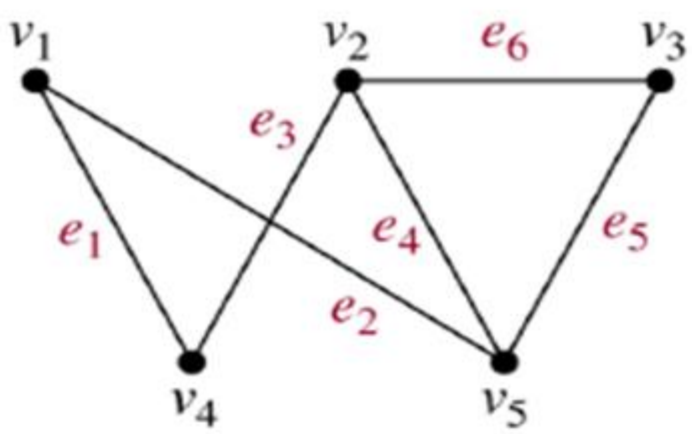


$$
\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{cccc} a & b & c & d \end{array} \\
\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}
\end{array}
$$

Find the adjacency matrix of pseudograph?

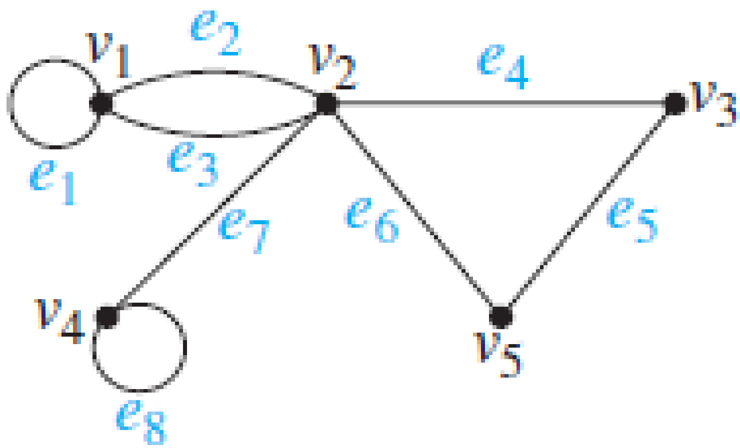
Incidence Matrix

- Let $G = (V, E)$ be an undirected graph. Suppose that $1, 2, \dots, n$ are the vertices and $e_1, e_2, \dots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is $n \times m$ matrix M $= [m_{ij}]$, where
- $m_{ij} = \begin{cases} 1 & \textit{when edge } e_j \textit{ is incident on } i. \\ 0 & \textit{otherwise} \end{cases}$



	e1	e2	e3	e4	e5	e6
v1	1	1	0	0	0	0
v2	0	0	1	1	0	1
v3	0	0	0	0	1	1
v4	1	0	1	0	0	0
v5	0	1	0	1	1	0

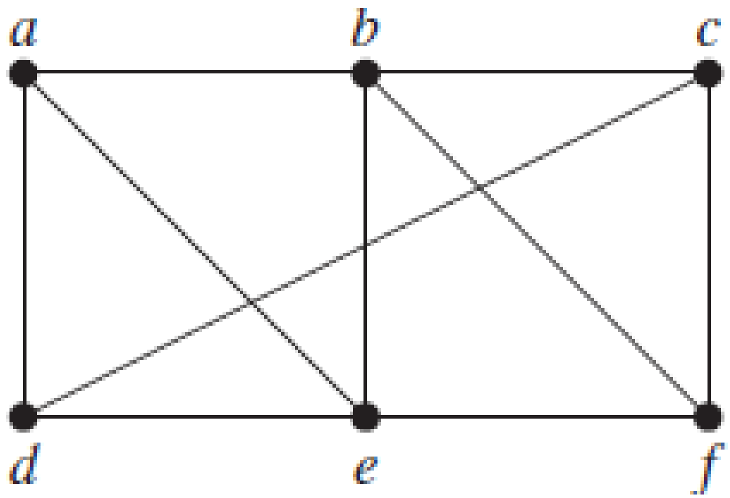
Example



$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}.
$$

Path

- A **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.



a – e length = 1

a – b – e length = 2

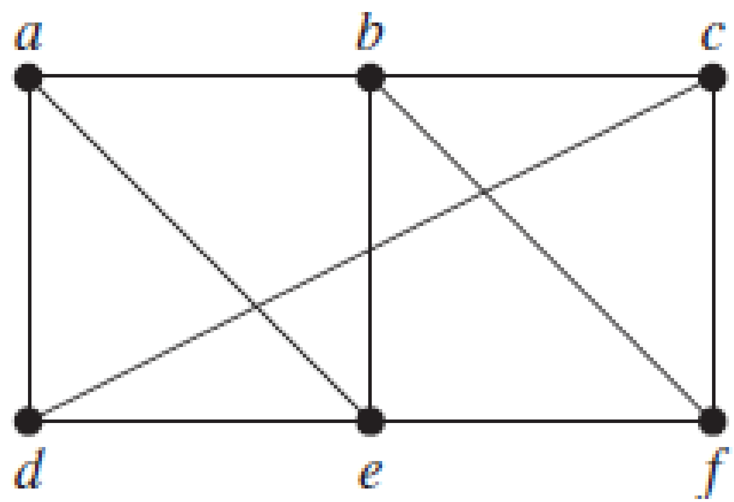
a – b – f – e length = 3

a – b – c – f – e length = 4

- Find a path between ***a*** and ***e***.

Path

Find a path between ***a*** and ***e***.



- ***a*, *d*, *c*, *f* , *e*** is a simple path of length 4, because {*a, d*}, {*d, c*}, {*c, f* }, and {*f, e*} are all edges.
- ***a*, *e*** is a simple path of length 1, because {*a, e*} is the edge connecting both vertices.
- ***a*, *b*, *f* , *e*** is a simple path of length 3, because {*a, b*}, {*b, f*}, and {*f, e*} are all edges.

Graph Distance

- The graph distance between two vertices in a graph is the number of edges in a shortest path connecting them.
- Also known as the **geodesic distance**.
- There may be more than one shortest path between two vertices.
- If we compute the distance between every pair of vertices of graph **G**, we can construct a **distance matrix** ***D***.

Eccentricity

- The eccentricity $\boldsymbol{e(v)}$ of a vertex $\boldsymbol{v}$ in a connected graph $\boldsymbol{G(V,E)}$ is max $\boldsymbol{d(u,v)}$, for all $\boldsymbol{u} \in \boldsymbol{V}$. In other words, a vertex’s eccentricity is equal to the distance from itself to the vertex farthest away.

- The minimum eccentricity of all vertices in a graph is called the radius of the graph,

- The maximum eccentricity of all vertices in a graph is the diameter of the graph.

Example

Draw a undirected graph with the adjacency matrix

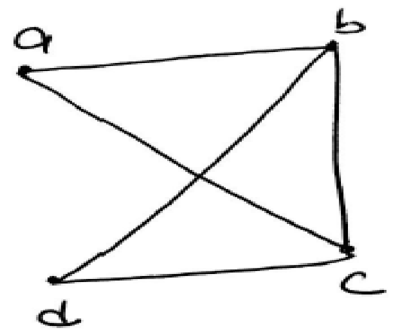
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

Find distance matrix of graph, eccentricity of each vertex, diameter and radius of graph.

Example (Solution)

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 1 & 1 & 0 \\
b & 1 & 0 & 1 & 1 \\
c & 1 & 1 & 0 & 1 \\
d & 0 & 1 & 1 & 0
\end{array}
$$



Un-directed Graph

$$
D_M = \begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 1 & 1 & 2 \\
b & 1 & 0 & 1 & 1 \\
c & 1 & 1 & 0 & 1 \\
d & 2 & 1 & 1 & 0
\end{array}
$$

Distance Matrix

$E(a) = 2$

$E(b) = 1$

$E(c) = 1$

$E(d) = 2$

Eccentricity of each vertex

Radius = 1

Diameter = 2

Radius and Diameter of Graph