SubSpaces

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Subspace Criteria:

If W is the subset of a vector space V, then W is called subspace if and only if the following conditions hold:

a) (Closed under addition) If $\vec{u}$ and $\vec{v}$ are vectors in W, then $\vec{u}+\vec{v}$ is in W.
b) (Closed under scalar multiplication) If k is any scalar and $\vec{u}$ is any vector in W then $k\vec{u}$ is in W.

Or in other words, W is subspace of V if and only if it is closed under addition and scalar multiplication.

Example 1: If V is any vector space and if $W=\{\vec{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

$$\vec{0}+\vec{0}=\vec{0}$$

$$k\vec{0}=\vec{0}, \text{ for any scalar k}$$

We call W the zero subspace of V.

Example 2:

$$W=\{(x,y); x\geq 0, y\geq 0\}$$

Prove and disprove W is subspace of R^2.

Solution:

1. W is closed under addition, because

Let, $\vec{u}=(x,y), \vec{v}=(x',y')\in W \qquad (x,y\geq 0 \; and \; x',y'\geq 0)$

$\vec{u}+\vec{v}=(x+x',y+y')\in W \qquad (x+x'\geq 0 \ \ and\ y+y'\geq 0)$

2. W is not closed under scalar multiplication, because if

$$\vec{u}=(1,1)\in W$$

But $-\vec{u}=(-1,-1)\notin W$
So W is not subspace of V.

Example 3:

Which of the following subset of R^2 with usual operations of addition and scalar multiplication are subspaces?

a) $W_1=\{(x,y);x\geq 0\}$
b) $W_2=\{(x,y);x\geq 0,y\geq 0\}$
c) $W_3=\{(0,y);y\in R\}$

Solution:

a) W_1 is closed under addition but not closed under scalar multiplication, because

$$Let\ \vec{u}=(2,3)\in W$$

$$and\ \ (-2)\vec{u}=-2(2,3)=(-4,-6)\notin W$$

b) Done in example 2.
c) $W_3=\{(0,y);y\in R\}$
As $W_3=y-axis$ in R^2. To see whether W_3 is subspace , Let

$$\vec{u}=(0,u_1),\vec{v}=(0,v_1)\in W_3$$

$$\vec{u}+\vec{v}=(0,u_1+v_1)\in W_3,$$

W_3 is closed under addition.
Let $\vec{u}=(0,u_1)$, k is any scalar then

$$k\vec{u}=(0,ku_1)\in W_3$$

W_3 is closed under scalar multiplication.
W_3 is subspace of R^2.

Example 4:

Which of the following subset of R^3 with usual operations of addition and scalar multiplication are subspaces?

a) $W_1 =$ All vectors of the form $(a, 0, 0)$
b) $W_2 =$ All vectors of the form $(a, 1, 1)$
c) $W_3 =$ All vectors of the form (a, b, c) where $b = a + c$.
d) $W_4 =$ All vectors of the form (a, b, c) where $b = a + c + 1$.
e) $W_5 =$ All vectors of the form $(a, b, 0)$.

Solution:

a) $W_1 = \{(a, 0, 0); a \in R\}$

1. W_1 is closed under addition and scalar multiplication as:

$$\vec{u} = (a_1, 0, 0) \in W_1$$
$$\vec{v} = (a_2, 0, 0) \in W_1$$
$$\vec{u} + \vec{v} = (a_1 + a_2, 0, 0) \in W_1$$

2. Let $\vec{u} = (a_1, 0, 0) \in W_1$ and k is any scalar, then

$$k\vec{u} = (ka_1, 0, 0) \in W_1$$

W_1 is subspace of R^3.

b) $W_2 = \{(a, 1, 1); a \in R\}$

Let's check the first property, i.e. W_2 is closed under addition.
Let

$$\vec{u} = (a_1, 1, 1) \in W_2$$
$$\vec{v} = (a_2, 1, 1) \in W_2$$
$$\vec{u} + \vec{v} = (a_1 + a_2, 2,2) \notin W_2$$

W_2 is not subspace of R^3.

c) $W_3 = \{(a, b, c); b = a + c; a, b, c \in R\}$

1. W_1 is closed under addition and scalar multiplication as:

$$\vec{u} = (a_1, b_1, c_1) = (a_1, a_1 + c_1, c_1) \in W_3$$
$$\vec{v} = (a_2, b_2, c_2) = (a_2, a_2 + c_2, c_2) \in W_3$$

Then $\vec{u} + \vec{v} = (a_1 + a_2, (a_1 + c_1) + (a_2 + c_2), c_1 + c_2)$

$$\vec{u} + \vec{v} = (a_1 + a_2, a_1 + a_2 + c_1 + c_2, c_1 + c_2) \in W_3$$

So W_3 is closed under addition.

2. Let $\vec{u} = (a_1, a_1 + c_1, c_1) \in W_3$ and k is any scalar, then

$$k\vec{u} = k(a_1, a_1 + c_1, c_1) = (ka_1, k(a_1 + c_1), kc_1)$$
$$= (ka_1, ka_1 + kc_1, kc_1) \in W_3$$

W_3 is closed under scalar multiplication.

So, W_3 is subspace.

d) W_4 = All vectors of the form (a, b, c) where $b = a + c + 1$.

$$\vec{u} = (a_1, b_1, c_1) = (a_1, a_1 + c_1 + 1, c_1) \in W_4$$
$$\vec{v} = (a_2, b_2, c_2) = (a_2, a_2 + c_2 + 1, c_2) \in W_4$$

Then $\vec{u} + \vec{v} = (a_1 + a_2, (a_1 + c_1) + (a_2 + c_2) + 2, c_1 + c_2)$

$$\vec{u} + \vec{v} = (a_1 + a_2, a_1 + a_2 + c_1 + c_2 + 2, c_1 + c_2) \notin W_4$$

So W_4 is not closed under addition.

So W_4 is not subspace of R^2.

e) W_5 = All vectors of the form $(a, b, 0)$.(Do it yourself)

Example 5.

Consider the unit square shown in the accompanying figure. Let W be the set of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $0 \le x \le 1, 0 \le y \le 1$. That is W is the set of all vectors whose tail is at origin and whose head is a point inside or on the square. Is W a subspace of R^2? Explain.

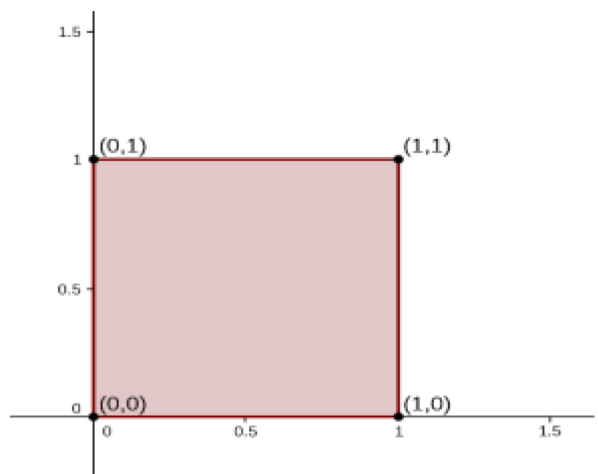
Solution:

Let $\vec{u} = (0,1), \vec{v} = (1,1) \in W$.

Then $\vec{u} + \vec{v} = (1,2) \notin W$.

W is not closed under addition.

W is not subspace of R^2.



Example 6.

Which of the given subsets of the vector space M_{23} of all 2×3 matrices are subspace? The set of all matrices of the form?

a) $W_1 = \left\{ \begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix} ; where\ b = a + c \right\}$

b) $W_2 = \left\{ \begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix} ; where\ \ c > 0 \right\}$

c) $W_3 = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}; where\ a = -2c, f = 2c + d \right\}$

Solution:

Sol (a)

① Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & 0 & 0 \end{bmatrix} \in W_1$

$A = \begin{bmatrix} a_1 & a_1+c_1 & c_1 \\ d_1 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} a_2 & a_2+c_2 & c_2 \\ d_2 & 0 & 0 \end{bmatrix}$

$A+B = \begin{bmatrix} a_1+a_2 & a_1+c_1+a_2+c_2 & c_1+c_2 \\ d_1+d_2 & 0 & 0 \end{bmatrix} \in W_1$

$\Rightarrow W_1$ is closed under '+'.

② $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & 0 & 0 \end{bmatrix} \in W_1$

$kA = \begin{bmatrix} ka_1 & k(a_1+c_1) & kc_1 \\ kd_1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} ka_1 & ka_1+kc_1 & kc_1 \\ kd_1 & 0 & 0 \end{bmatrix} \in W_1$

Closed under scale multiplication.

$\Rightarrow W_1$ is subspace of $M_{2\times3}$.

C: do it your self