The N-Queens Problem as a Digraph

The N-Queens problem is a rather simple one: if you have a $N \times N$ chess board (for fixed $N \in \mathbb{N}$), can you place N queens such that none of them are checking each other, or in other words, so that none of them are able to take each other. A more detailed description is given at https://en.wikipedia.org/wiki/N_queens. We can visualise this as an $N \times N$ matrix, where 1 represents a queen being in a specific position, and 0 represents an empty tile. For example, the matrix

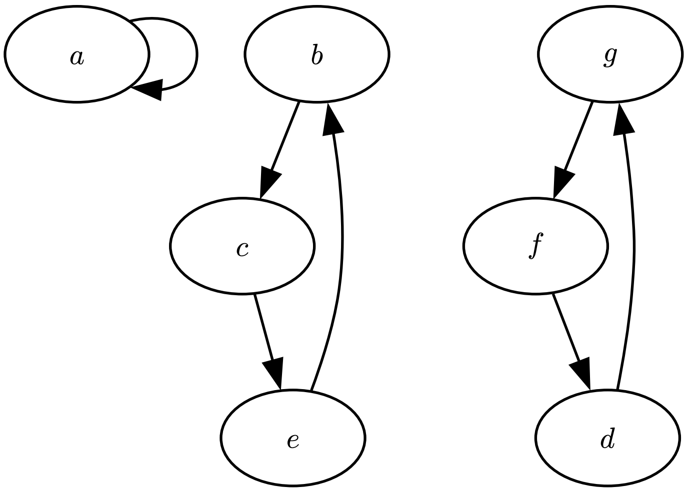
$$M := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is a valid solution, as none of the 1s can "attack" each other on a diagonal, horizontal, or vertical line (the queen's valid moves). Notice that this could form a digraph, where the top of the matrix is labelled $a_0, a_1, ..., a_N$, and the side $b_0, b_1, ..., b_N$, and an edge from a_n to b_m (with $n, m \in [0, N] \cap \mathbb{N}_0$) existing iff $M_{m,n} = 1$ (where M is the matrix in question). Thus, our matrix above would form the digraph

$$\begin{array}{c} a \\ \downarrow \\ c \\ \downarrow \\ d \\ \downarrow \\ b \end{array}$$

Notice the cycle in the digraph. When we checked the first 8 solutions computationally, we found that *all* of the corresponding digraphs had similar cycles (the code is available in the Github repository https://github.com/AowynB/NQueens, along with the source for this document, and a copy of the PDF). Some of them – for example, at $N = 7$ – had multiple small cycles, the matrix and digraph for which is shown below.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



We seek to show that this pattern holds in general iff the matrix exists, since there are cases where no solution is possible, namely $N = 3$ and $N = 4$.