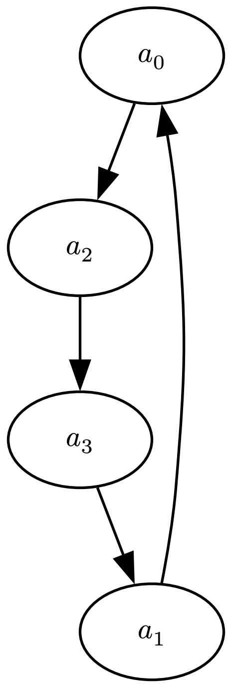
The N-Queens Problem as a Digraph

The N-Queens problem is a rather simple one: if you have a $N \times N$ chess board (for fixed $N \in \mathbb{N}$), can you place N queens such that none of them are checking each other, or in other words, so that none of them are able to take each other. A more detailed description is given at https://en.wikipedia.org/wiki/N_queens. We can visualise this as an $N \times N$ matrix, where 1 represents a queen being in a specific position, and 0 represents an empty tile. For example, the matrix

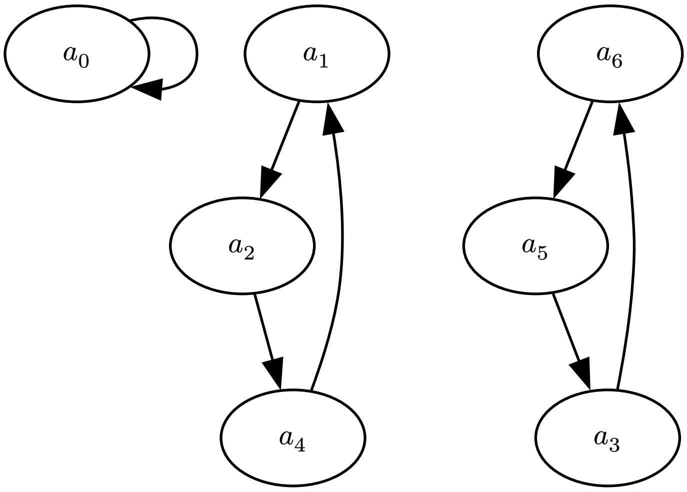
$$M := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is a valid solution, as none of the 1s can "attack" each other on a diagonal, horizontal, or vertical line (the queen's valid moves). Notice that this could form a digraph, where the top of the matrix is labelled $a_0, a_1, ..., a_N$, and the side $b_0, b_1, ..., b_N$, and an edge from a_n to b_m (with $n, m \in [0, N] \cap \mathbb{N}_0$) existing iff $M_{m,n} = 1$ (where M is the matrix in question). Thus, our matrix above would form the digraph



Notice the cycle in the digraph. It is also important to note that there may exist multiple solutions to a given N. When we checked the first 8 solutions computationally, we found that *all* of the corresponding digraphs had similar cycles (the code is available in the Github repository https://github.com/AowynB/NQueens, along with the source for this document, and a copy of the PDF). Some of them – for example, at $N = 7$ – had multiple small cycles, the matrix and digraph for which is shown below.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



We seek to show that this pattern of cyclic digraphs holds in general iff the matrix exists. Note there are cases where no solution is possible, namely $N = 3$ and $N = 4$. First, we need to understand what the digraph is representing about the board.