Lesson 7: Estimation of Autocorrelation and Partial Autocorrelation Function

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Estimation of Autocorrelation Function

We observe that the autocovariance function depends on the unit of measurement of the random variables. Thus it is very difficult to evaluate the dependence of random variables of a stochastic process by using autocovariances.

A more useful measure of dependence, since it is dimensionless, is the autocorrelation function, defined as follows:

Estimation of Autocorrelation Function

Let $\{x_t; t \in \mathbb{Z}\}$ be a weakly stationary process.
The function

$$\rho_x : \mathbb{Z} \to \mathbb{R}$$

defined by

$$\rho_x(k) = \frac{\gamma_x(k)}{\gamma_x(0)} \ \ \forall k \in \mathbb{Z}$$

is called autocorrelation function of the weakly stationary process $\{x_t; t \in \mathbb{Z}\}$.

Estimation of Autocorrelation Function

We see that the autocorrelation function is the autocovariance function normalized so that $\rho_x(0) = 1$.

We note that $\rho_x(k)$ measures the correlation between x_t and x_{t-k}

In fact, we have

$$\rho_x(k) = \frac{\gamma_x(k)}{\gamma_x(0)} = \frac{\gamma_x(k)}{\sqrt{\gamma_x(0)}\sqrt{\gamma_x(0)}} = \frac{\mathrm{Cov}(x_t, x_{t-k})}{\sqrt{\mathrm{Var}(x_t)}\sqrt{\mathrm{Var}(x_{t-k})}}$$

and hence $-1 \leq \rho_x(k) \leq 1$.

We have seen that a consistent estimator for the autocovariance function is given by

$$\hat{\gamma}(k) = \frac{1}{T}\sum_{t=k+1}^{T}(x_t - \bar{x})(x_{t-k} - \bar{x}) \text{ for } k = 0, 1, \ldots, T-1.$$

Estimation of Autocorrelation Function

Since the autocorrelation $\rho(k)$ is related to the autocovariance $\gamma(k)$, by

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} \ \ \forall k \in \mathbb{Z}$$

a natural estimate of $\rho(k)$ is

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)}$$

Estimation of Autocorrelation Function

When we calculate the sample autocorrelation, $\hat{\rho}(k)$, of any given series with a fixed sample size T, we cannot put too much confidence in the values of $\hat{\rho}(k)$ for large k's, since fewer pairs of (x_{t-k}, x_t) will be available for computing $\hat{\rho}(k)$ when k is large. Only one if $k = T - 1$.

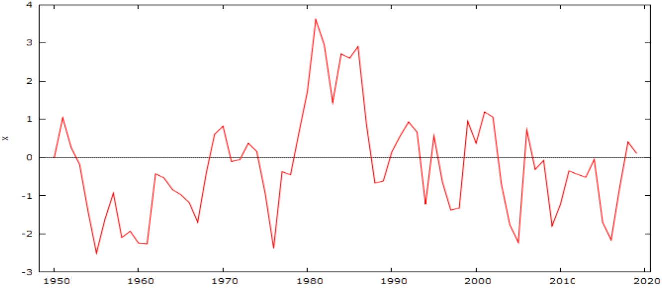
Estimation of Autocorrelation Function

One rule of thumb is not to evaluate $\hat{\rho}(k)$ for $k > T/3$.

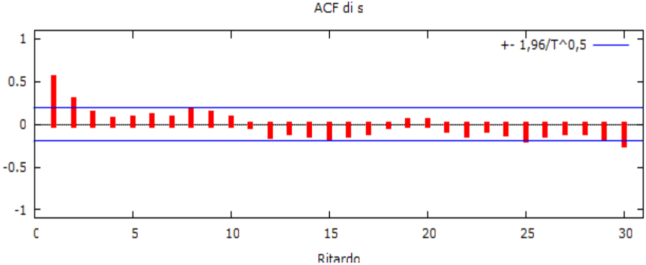
The correlogram

The plot of $\hat{\rho}(k)$ vs. k, is called the correlogram.

Consider the time series represented in the following figure.



The correlogram for the data of this example is

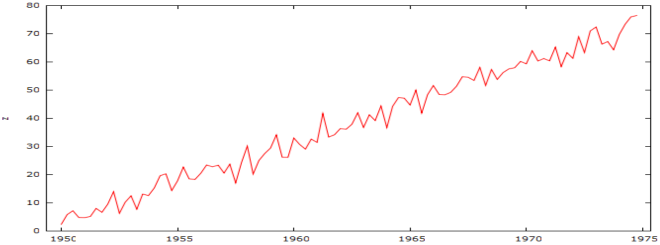


The correlogram

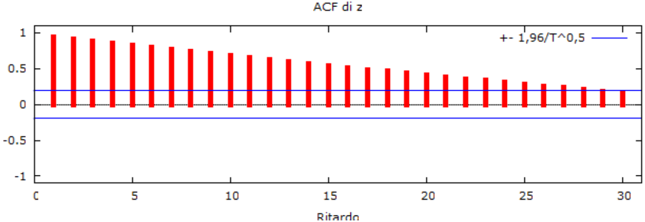
The sample autocorrelation function, $\hat{\rho}(k)$, can be computed for any given time series and are not restricted to observations from a stationary stochastic process.

The correlogram

As an example consider the time series plotted in the following figure.

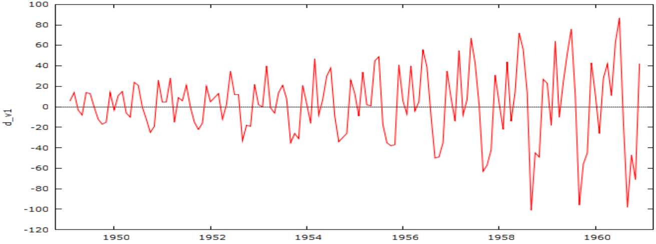


The correlogram for the data of this example is

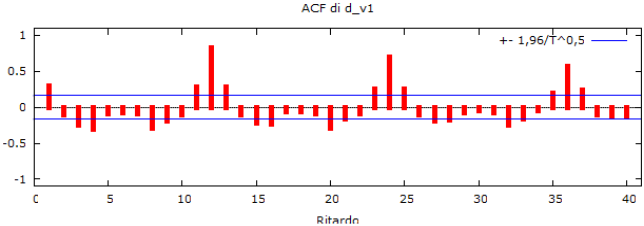


The correlogram

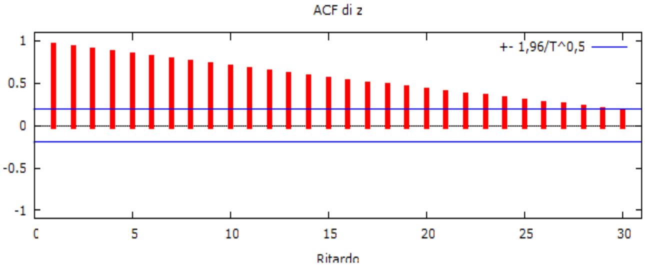
Let us consider the time series plotted in the following figure.



The correlogram for the data of this example is

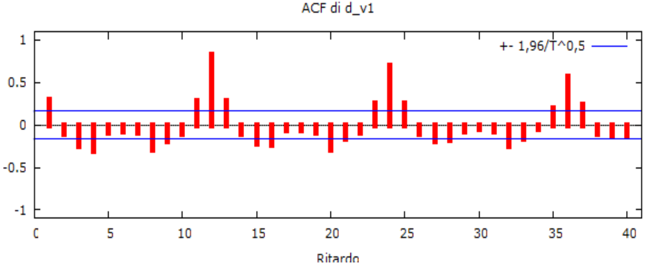


For the time series containing a trend, the sample autocorrelation function, $\hat{\rho}(k)$, exhibits slow decay as k increases.



The correlogram

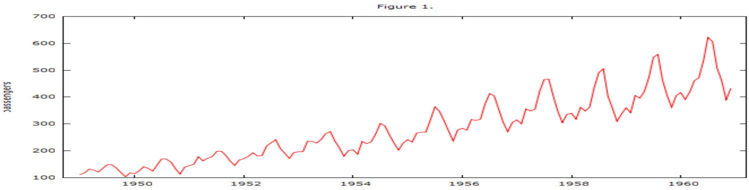
For the time series containing a periodic component (like seasonality), the sample autocorrelation function, $\hat{\rho}_k$, exhibits a behavior with the same periodicity.



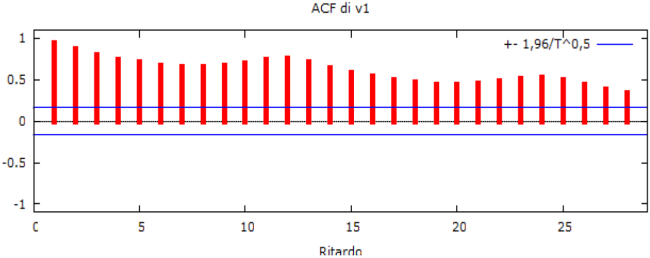
Thus $\hat{\rho}(k)$ can be useful as an indicator of nonstationarity.

The correlogram

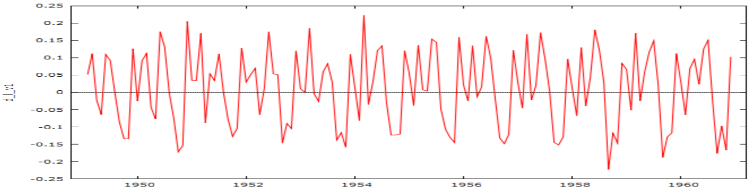
Consider again the number of airline passengers



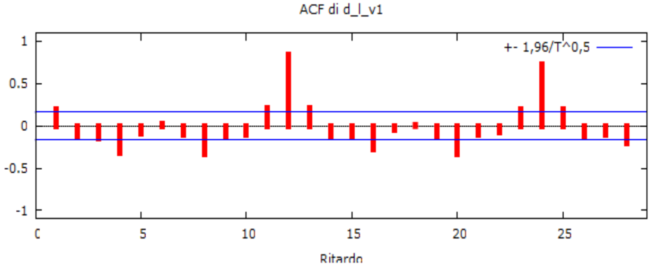
The correlogram for the data of this series is



Now, the first difference of the log tranformation



The correlogram for this series is



Finally, we consider the correlogramm of the series

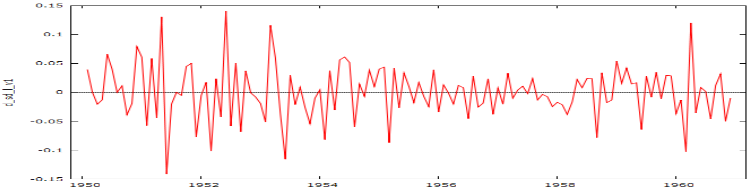
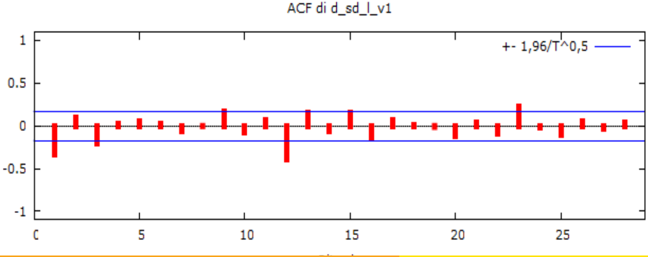


Figure : $(1 - L)(1 - L^{12})$ Log international airline passengers



The Partial Autocorrelation Function

Definition. The function $\pi : \mathbb{Z} \to \mathbb{R}$ defined by the equations

$$\pi_x(0) = 1$$

$$\pi_x(k) = \phi_{kk} \quad k \geq 1$$

where ϕ_{kk}, is the last component of

$$\boldsymbol{\phi}_k = \mathbf{R}_k^{-1}\boldsymbol{\rho}_k$$

with

$$\mathbf{R}_k = \begin{bmatrix} \rho(0) & \rho(1) & \cdots & \rho(k-1) \\ \rho(1) & \rho(0) & \cdots & \rho(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(0) \end{bmatrix}$$

and $\boldsymbol{\rho}_k = (\rho(1), ..., \rho(k))'$ is called partial autocorrelation function of the stationary process $\{x_t; t \in \mathbb{Z}\}$.

It is possible to show that

$$\pi_x(k) = \phi_{kk} \quad k \geq 1$$

is equal to the coefficient of correlation between

$$x_t - E(x_t | x_{t-1}, ..., x_{t-k+1})$$

and

$$x_{t-k} - E(x_{t-k} | x_{t-1}, ..., x_{t-k+1})$$

Thus $\pi_x(k)$ measures the linear link between x_t and x_{t-k} once the influence of $x_{t-1}, ..., x_{t-k+1}$ has been removed.

Estimation of Partial Autocorrelation Function

The sample partial autocorrelation function (SPACF) of a stationary processes x_t is given by the sequence

$$\hat{\pi}_x(0) = 1$$

$$\hat{\pi}_x(k) \;\; k \geq 1$$

where $\hat{\pi}_x(k)$, is the last component of

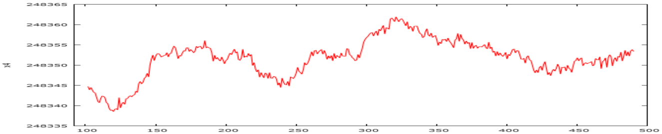
$$\hat{\boldsymbol{\phi}}_k = \hat{\Gamma}_k^{-1}\hat{\boldsymbol{\gamma}}_k$$

Estimation of Partial Autocorrelation Function

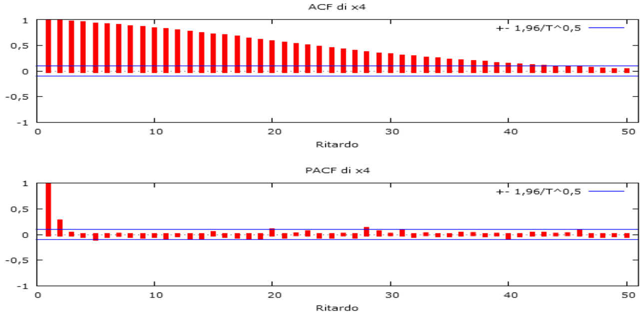
The plot of $\hat{\pi}_x(k)$ vs. k is called the **partial correlogram**.

Correlogram and Partial Correlogram

Let us consider the time series plotted in the following figure.



The correlogram and the partial correlogram for the data of this example are



Correlogram and Partial Correlogram

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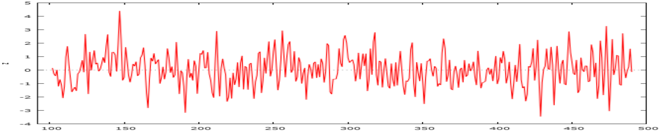
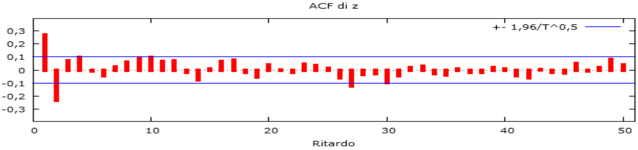
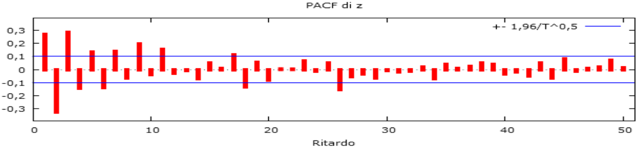


The correlogram and the partial correlogram for the data of this example are





Correlogram and Partial Correlogram

As we will see the correlogram and the partial correlogram will be used in the model identification stage for Box-Jenkins autoregressive moving average time series models.