1 Introduction

INTRODUCCION (VER HCP-GROVER-V1 PARA UNA PRIMERA VERSION)

2 Background

PONER ALGO DE CAMINO HAMILTONIANO, DISTINTOS APROACHES TOMADOS, LOS MEJORES ALGORITMOS. EXPLICAR GROVER. (VER HCP-GROVER-V1 PARA UNA PRIMERA VERSION)

3 Proposed Algorithm for Hamiltonian Cycle Problem

In order to solve an NP-problem with Grover's Algorithm it is necessary to translate a decision problem into a search problem. First, we must encode a way to choose the cycle with a binary code. Let G = (V, E) be the input graph with n vertexes and e edges. Vertexes are numbered as $\{0, 1, 2, ..., N\}$. For every vertex, $n = log_2 N$ bits are used to indicate the position it occupies in the cycle.

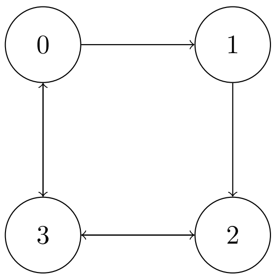


Figure 1: Example Graph

For example, the bitstring 11000110 would be translated to $3 - 0 - 1 - 2$. This means the vertex 0 is on the position 3, the vertex 1 is on the position 0 and so on. The resulting cycle is $1 - 2 - 3 - 0$

The search space consists of every possible permutation of the positions of the vertexes in the hamiltonian cycle. Therefore, its size is $2^{(N*n)}$. We will be searching the space to find solutions that follow this two conditions:

1. No two vertices can have the same position.
2. Vertices with consecutive positions have to be connected by an edge. (The last and first positions are considered consecutive)

INSERTAR TEOREMA QUE DICE QUE ESTO ES RESOLVER EL PROBLEMA DEL CICLO HAMILTONIANO

We present an algorithm to solve HCP of a directed graph with $N = 2^n$ vertexes. Later, a possible adaptation of this algorithm to an arbitrary graph

is proposed.
The Grover operator G is divided into the oracle U_ω, which marks the strings that satisfy the two conditions, and the diffuser D.
The quantum circuit block diagram of the oracle U_ω for the HCP problem is shown in Fig. 3). The construction of U_ω is divided into two blocks: comparator, which checks the first condition; missing edge detector, which checks the second condition.

3.1 Circuit definition

The main register $|\psi\rangle$, which covers the search space, has $m = N \times n$ qubits. Applying U_ω requires the addition of three registers. An ancilla register $|\theta\rangle$ of $k = \binom{N}{2}$ qubits, and a single qubit register $|\zeta\rangle$.

3.2 Initialization

In the initialization, the main register is set as a superposition of all the states in the search space. This is done by applying the Hadamard Gate to each qubit in the main register. The qubits in the ancilla and the output qubit are set to $|1\rangle$ with the NOT gate. The resulting state is expressed as:

$$|\psi_0\rangle \otimes |\theta_0\rangle \otimes |\zeta_0\rangle$$

where:

$$|\psi_0\rangle = \frac{1}{\sqrt{2^m}} \sum_{i=0}^{2^m-1} |i\rangle = |+\rangle^{\otimes m}$$

$$|\theta_0\rangle = |1\rangle^{\otimes k}$$

$$|\zeta_0\rangle = |0\rangle$$

3.3 Oracle U_ω

The oracle has two main building blocks: Positional Exclusivity Block and Missing Edge Detector Block.
The Comparator circuit, described at [**?**], is a fundamental component of these blocks. The circuit, composed of NOT, CNOT, and Toffoli/MCT gates, is applied to two registers of the same size, a and b, and stores the result in the f qubit.

$$\text{Comparator}(a, b, f) = \begin{cases} f = f, & \text{if } a \neq b \\ f = f \oplus 1, & \text{if } a = b. \end{cases}$$

POSIBILIDAD, PONER LA DESCRIPCION DEL BLOQUE, A PESAR DE QUE SE DA LA CITA.

The Plus One circuit, which takes a state $|i\rangle$ to the state $|(i+1) \bmod 2^p\rangle$.

AGREGAR DESCRIPCION DEL PLUS ONE Y EL MINUS ONE.

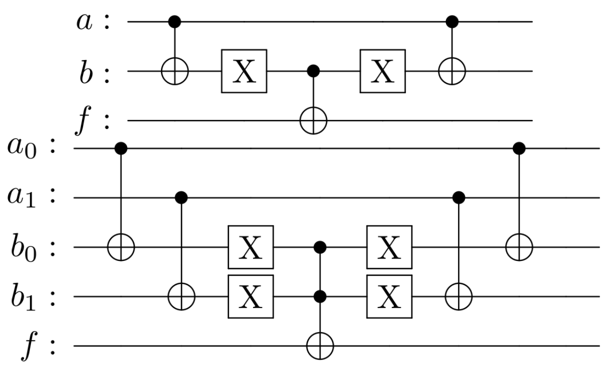
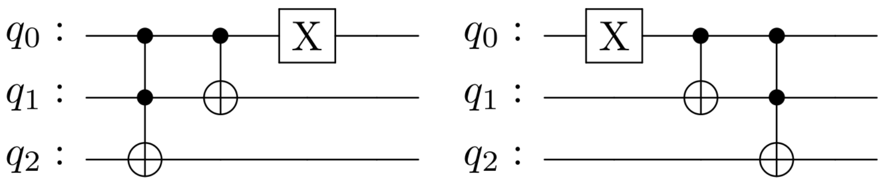


Figure 2: Comparator circuits of one and two qubits



(a) PlusOne Circuit

(b) MinusOne Circuit

Figure 3

3.3.1 Positional Exclusivity Block

To ensure that each position is assigned to exactly one vertex, each vertex is compared with all the other vertices. The comparator circuit is applied to the $\binom{n}{2}$ combinations of vertexes, and the results are stored in $|\theta\rangle$. If any two vertices are in the same position, at least one qubit in $|\theta\rangle$ will be in state $|0\rangle$.

3.3.2 Missing Edge Detector Block

We will refer as missing edges to the set of edges of the complementary graph. Formally, let $E(G)$ be the set of edges in G, and E_{missing} denote the set of missing edges. Then, E_{missing} can be defined as follows:

$$E_{\text{missing}} = \{(u,v) \mid u,v \text{ are vertices in } G, (u,v) \notin E(G)\}$$

In the example, the set of missing edges is $\{(0,2),(1,0),(1,3),(2,0),(2,1),(3,1)\}$. For each vertex, it is necessary to ensure that the vertex that occupies the next position does not form a missing edge.

The Plus One circuit is used to select the next position of the vertex, and then the Comparator circuit is used on the vertexes of the missing edge and an ancilla qubit. Then, it is returned to its previous position with the Minus One circuit.
If the vertex that is assigned the next position in the cycle is not connected through an edge, the Comparator circuit will flip an ancilla qubit.

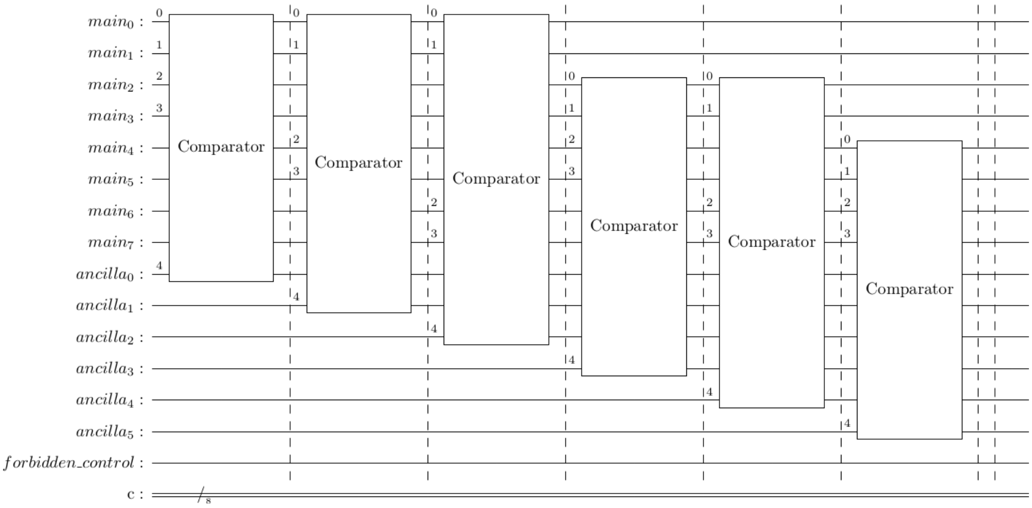


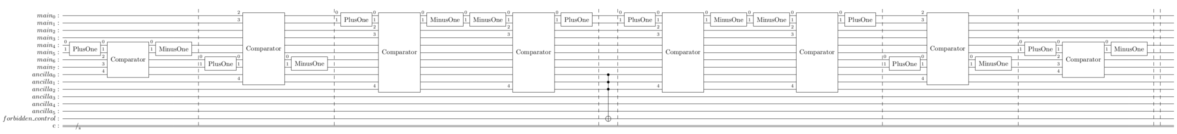
Figure 4: Diagram of the PE Block

ESTE PARRAFO NO SE ENTIENDE TANTO.

The maximum amount of edges an undirected graph can have is $\frac{N(N-1)}{2}$, so the information of which edges violate the constraint can be stored in the k qubits of the ancillary register.
In a directed graph, both (u, v) and (v, u) edges could be missing. The same ancilla qubit can store the result of the Comparator. This is because the same vertex can not have the previous and the next position in any cycle of more than two elements.

We give an example of the Missing Edge Detector Block of a graph with the following missing edge set $\{(0, 1), (1, 0), (2, 3), (3, 0)\}$.

Figure 5



3.3.3 Final Assembly

Figure 5 shows the diagram of the complete oracle U_ω. We will analyze the first generic run of the U_ω oracle. To understand the example, we will name the A states that represent a valid hamiltonian cycle α, the B states that only satisfy the missing edge constraint β, the C states that satisfy the positional exclusivity constraint γ, and the rest of the D states δ.

$$|\rho_0\rangle = \frac{1}{2^m}\left(\sum_{i=1}^{A}|\alpha_i\rangle + \sum_{i=1}^{B}|\beta_i\rangle + \sum_{i=1}^{C}|\gamma_i\rangle + \sum_{i=1}^{D}|\delta_i\rangle\right) \otimes |1\rangle^{\otimes k} \otimes |0\rangle$$

In $|\rho_1\rangle$, the ancilla of the states C and D are have at least one flipped qubit. The string of k qubits that are not all in state $|1\rangle$ will be represented with a $|l\rangle$.

$$|\rho_1\rangle = \frac{1}{2^m}\left(\sum_{i=1}^{A}|\alpha_i\rangle|1\rangle^{\otimes k} + \sum_{i=1}^{B}|\beta_i\rangle|1\rangle^{\otimes k} + \sum_{i=1}^{C}|\gamma_i\rangle|l_{\gamma_i}\rangle + \sum_{i=1}^{D}|\delta_i\rangle|l_{\delta_i}\rangle\right)\otimes|0\rangle$$

The $|\zeta\rangle$ state is only flipped for the states that have the $|\theta\rangle$ ancilla in state $|1\rangle^{\otimes k}$.

$$|\rho_2\rangle = \frac{1}{2^m}\left(\sum_{i=1}^{A}|\alpha_i\rangle|1\rangle^{\otimes k}|1\rangle + \sum_{i=1}^{B}|\beta_i\rangle|1\rangle^{\otimes k}|1\rangle + \sum_{i=1}^{C}|\gamma_i\rangle|l_{\gamma_i}\rangle|0\rangle + \sum_{i=1}^{D}|\delta_i\rangle|l_{\delta_i}\rangle|0\rangle\right)$$

The second missing edge block returns the ancilla to its initial state.

$$|\rho_3\rangle = \frac{1}{2^m}\left(\sum_{i=1}^{A}|\alpha_i\rangle|1\rangle^{\otimes k}|1\rangle + \sum_{i=1}^{B}|\beta_i\rangle|1\rangle^{\otimes k}|1\rangle + \sum_{i=1}^{C}|\gamma_i\rangle|1\rangle^{\otimes k}|0\rangle + \sum_{i=1}^{D}|\delta_i\rangle|1\rangle^{\otimes k}|0\rangle\right)$$

The positional exclusivity block changes the ancilla of states $|\beta\rangle$ and $|\delta\rangle$.

$$|\rho_4\rangle = \frac{1}{2^m}\left(\sum_{i=1}^{A}|\alpha_i\rangle|1\rangle^{\otimes k}|1\rangle + \sum_{i=1}^{B}|\beta_i\rangle|l_{\beta_i}\rangle|1\rangle + \sum_{i=1}^{C}|\gamma_i\rangle|1\rangle^{\otimes k}|0\rangle + \sum_{i=1}^{D}|\delta_i\rangle|l_{\delta 2_i}\rangle|0\rangle\right)$$

In state $|\rho_5\rangle$, the $|\alpha\rangle$ states are flipped.

$$|\rho_4\rangle = \frac{1}{2^m}\left(-\sum_{i=1}^{A}|\alpha_i\rangle|1\rangle^{\otimes k}|1\rangle + \sum_{i=1}^{B}|\beta_i\rangle|l_{\beta_i}\rangle|1\rangle + \sum_{i=1}^{C}|\gamma_i\rangle|1\rangle^{\otimes k}|0\rangle + \sum_{i=1}^{D}|\delta_i\rangle|l_{\delta 2_i}\rangle|0\rangle\right)$$

The next blocks return the ancilla registers $|\theta\rangle$, $|\zeta\rangle$ and $|\psi\rangle$ are returned to their initial states.

$$|\rho_8\rangle = \frac{1}{2^m}\left(-\sum_{i=1}^{A}|\alpha_i\rangle + \sum_{i=1}^{B}|\beta_i\rangle + \sum_{i=1}^{C}|\gamma_i\rangle + \sum_{i=1}^{D}|\delta_i\rangle\right)\otimes|1\rangle^{\otimes k}\otimes|0\rangle$$

3.4 Ancilla reutilization

CON ESTOS BLOQUES HAY VARIAS FORMAS DE HACER EL ALGORITMO. PLANTEAR QUE LAS DISTINTAS FORMAS TIENEN SUS TRADEBACKS EN NUMERO DE QUBITS Y DEPTH DEL CIRCUITO. EVALUAR DONDE PONER ESTA SECCION. (VER HCP-GROVER-V1 PARA LO QUE ESTABA ESCRITO ANTES).

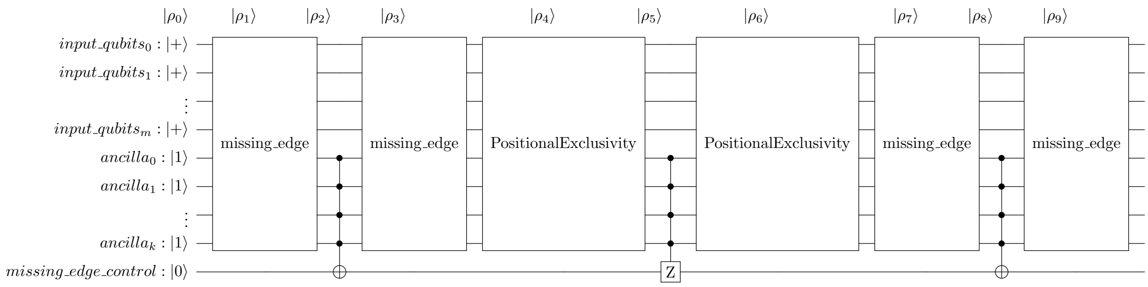


Figure 6: Oracle block diagram

4 Generalization

PARA LA GENERALIZACION DE ESTE ALGORITMO A GRAFO DE UN NUMERO ARBITRARIO DE VERTICES, SE PUEDE HACER UN MAPEO A GRAFOS DE 2^n VERTICES. TAMBIEN SE PUEDE HACER UN FORBIDDEN POSITION Y CAMBIAR EL CIRCUITO PLUS ONE PARA QUE SEA EN UN MODULO P ARBITRARIO. ESTAS DOS ALTERNATIVAS TIENEN SUS PROBLEMAS Y VIRTUDES, HACER UN ANALISIS ENTRE ELLAS. (VER HCP-GROVER-V1 PARA LO QUE ESTABA ESCRITO ANTES).

5 Results and Discussion

HACER LOS EXPERIMENTOS PERTINENTES PARA MOSTRAR EL FUNCIONAMIENTO. DISCUTIR EL ORDEN DEL ALGORITMO Y COMO SE COMPARA CON ALGORITMOS CLASICOS. (VER HCP-GROVER-V1 PARA UN EJEMPLO CON LA CORRIDA DE 3 GRAFOS)

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