L =(25+1).

Aim: To analyse and characterise wave propagation in a laminar fluid flow in presence of an obstruction

Introduction: It has been observed that once a fluid which is flowing down from a tap in the streamlined region is obstructed, stationary waves are formed.

These stationary waves can be attributed to small perturbation induced even to smoothest flows due to friction between nozzle and water (and some other imperfect nature of faucet).

These perturbations are studied under Plateau-Rayleigh instability.

Experiment:

Part A: Preliminary test

We observed that a laminar flow breaks into smaller packets down the stream

On introducing our finger as obstruction to the laminar fluid flow wave like pattern emerge on the surface of the falling stream.

Part B: Study of wavelength characteristics

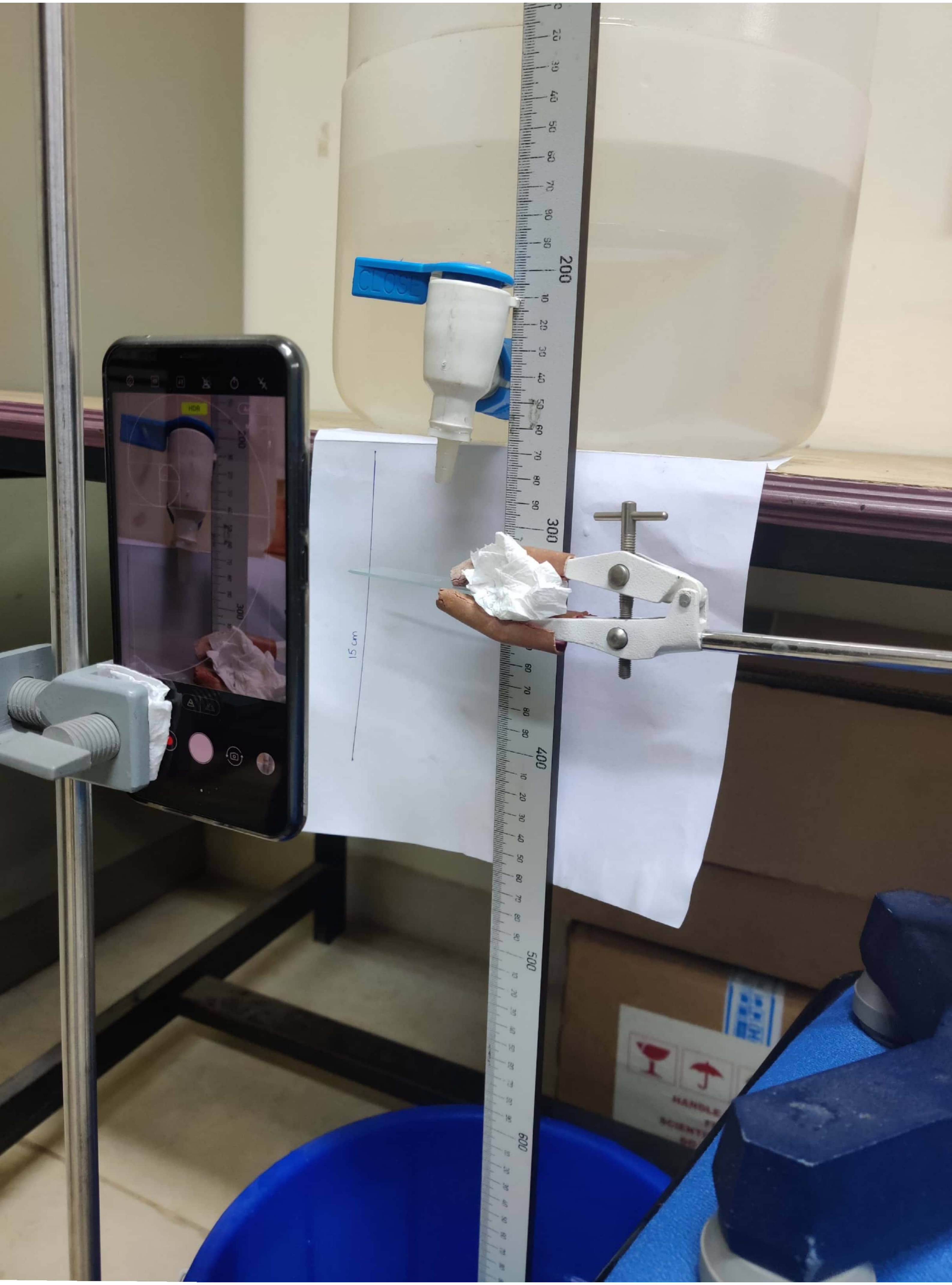
fluid under study : water

diameter of nozzle : 7 mm ± 1 mm

effective diameter of cylinderical stream : 5 mm ± 1 mm

~~Height of nozzle from obstacle : 11.9 cm ± 0.1 cm~~

Height of water column : 11.9 cm + 0.1 cm


CLOSE
15 cm

Wave part (as indicated in figure)	$h_1 = 16$ mm ± 1 mm $\lambda/2$ (mm) (node-node distance)	$h_2 = 19$ mm ± 1 mm $\lambda/2$ (mm)	$h_3 = (25 \pm 1)$ mm $\lambda/2$ (mm)	$h_4 = (30 \pm 1)$ mm $\lambda/2$ (mm)
1	(4 ± 1) mm	(3 ± 1) mm	(3 ± 1) mm	(2 ± 1) mm
2	(5 ± 1) mm	(4 ± 1) mm	(3 ± 1) mm	(2 ± 1) mm
3	–	(5 ± 1) mm	(4 ± 1) mm	(2 ± 1) mm
4	–	–	(4 ± 1) mm	(2 ± 1) mm

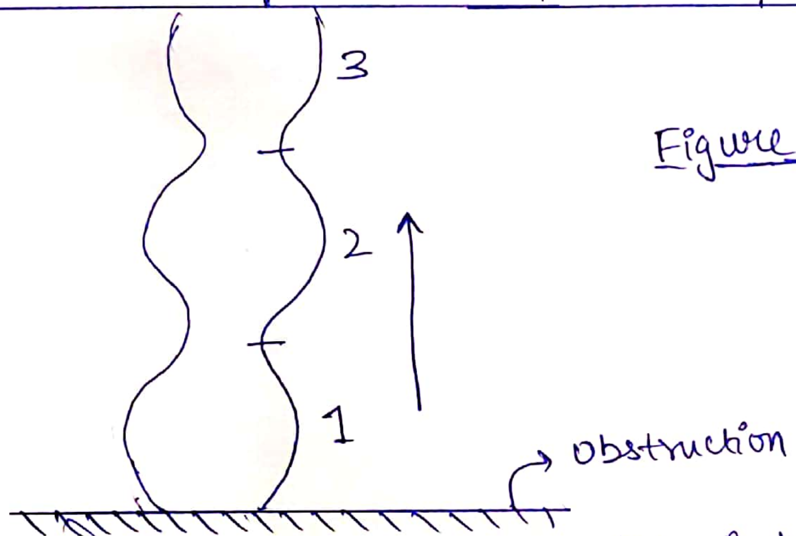


Figure 1

This figure shows the numbering of wave part

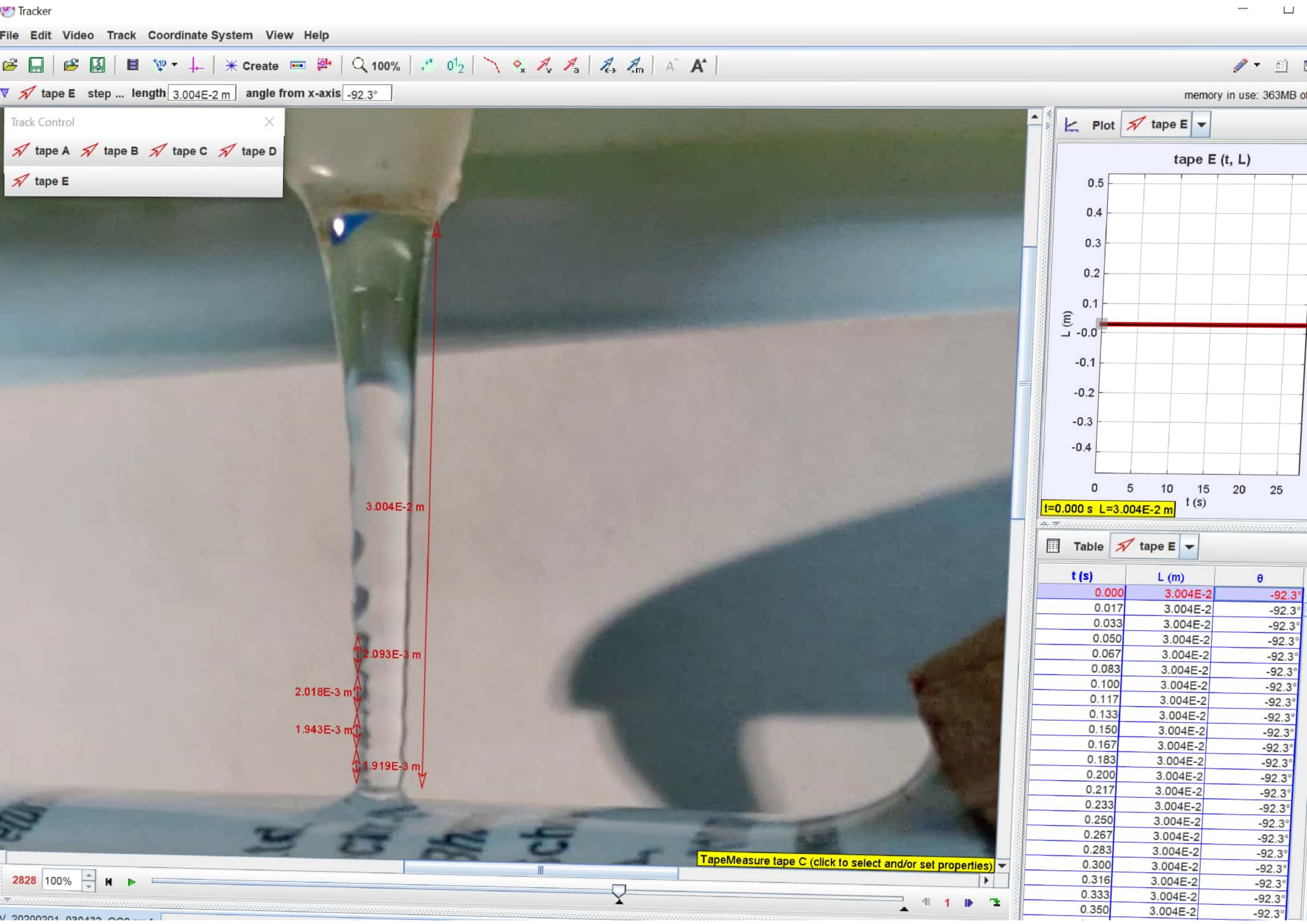
Observations

1. For a given distance between nozzle and obstruction the wavelength increase (↑) as we go up the stream i.e from wave part 1 to wave part 4.

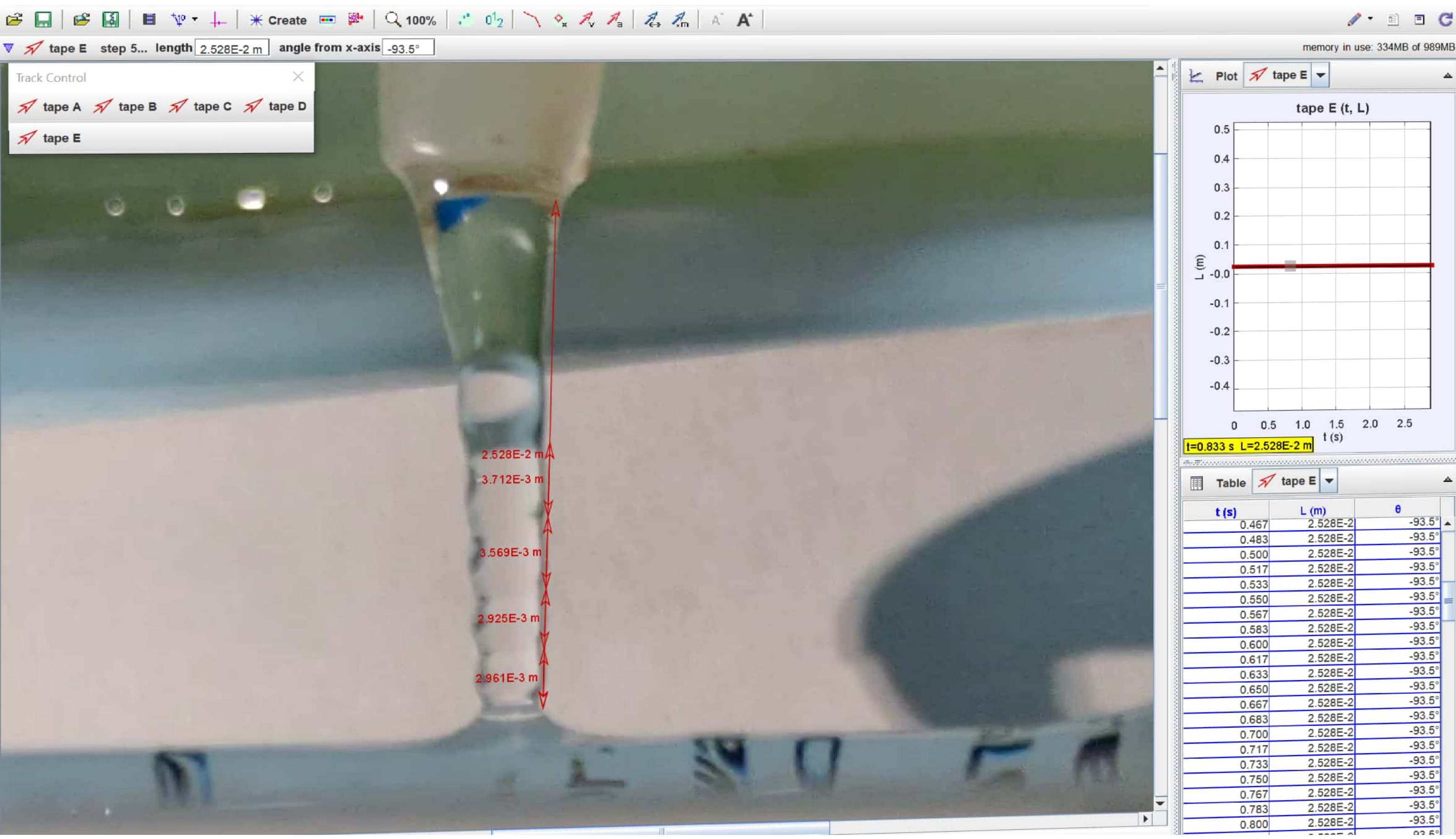
2. As the wave obstruction goes farther from the tap the wavelength decrease(↓) (or more appropriately average wavelength decreases (↓))

3. Without any obstruction the stream tends to break up into small droplets

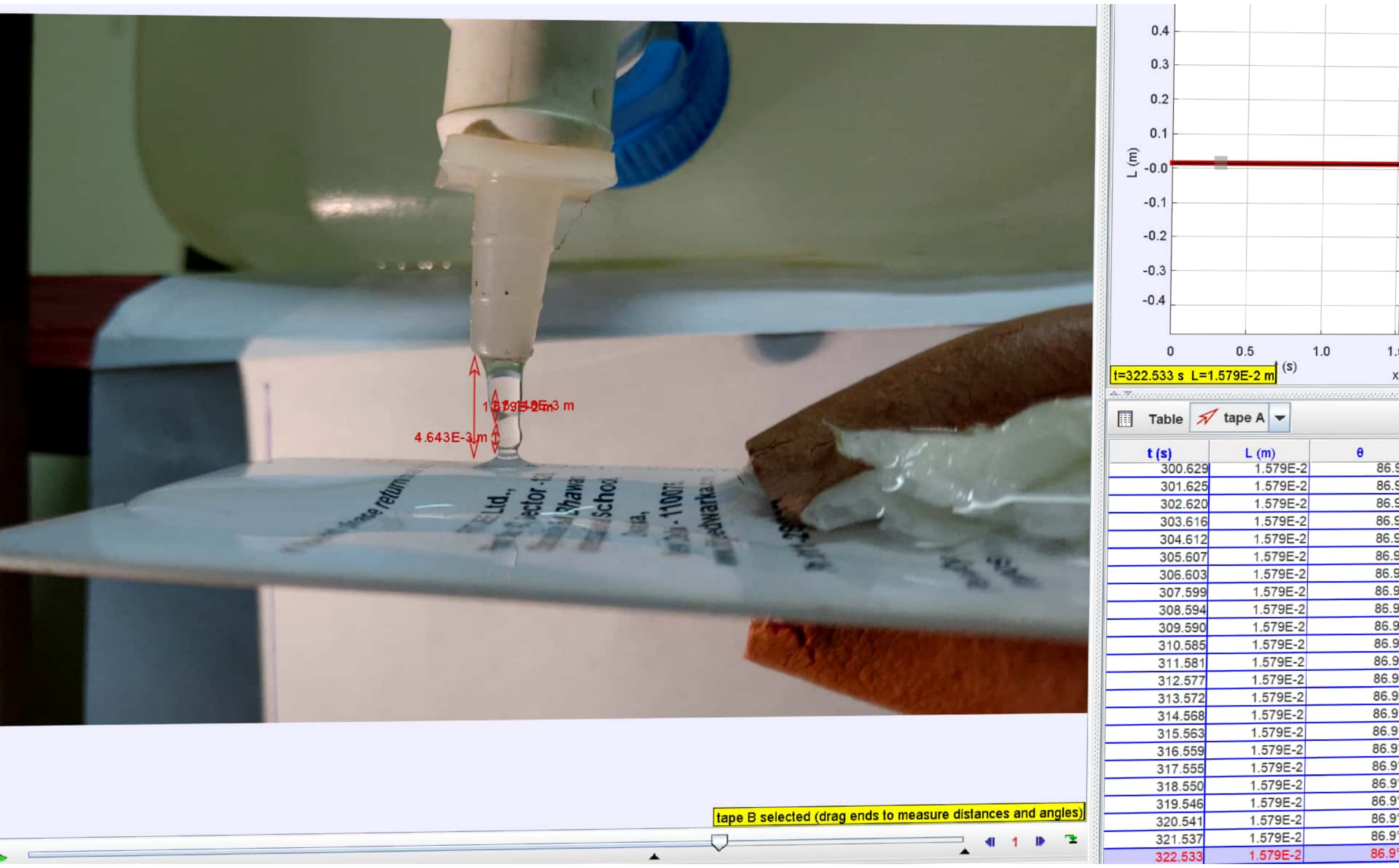
4. As we increase the radius of nozzle the effect becomes less and less observable.

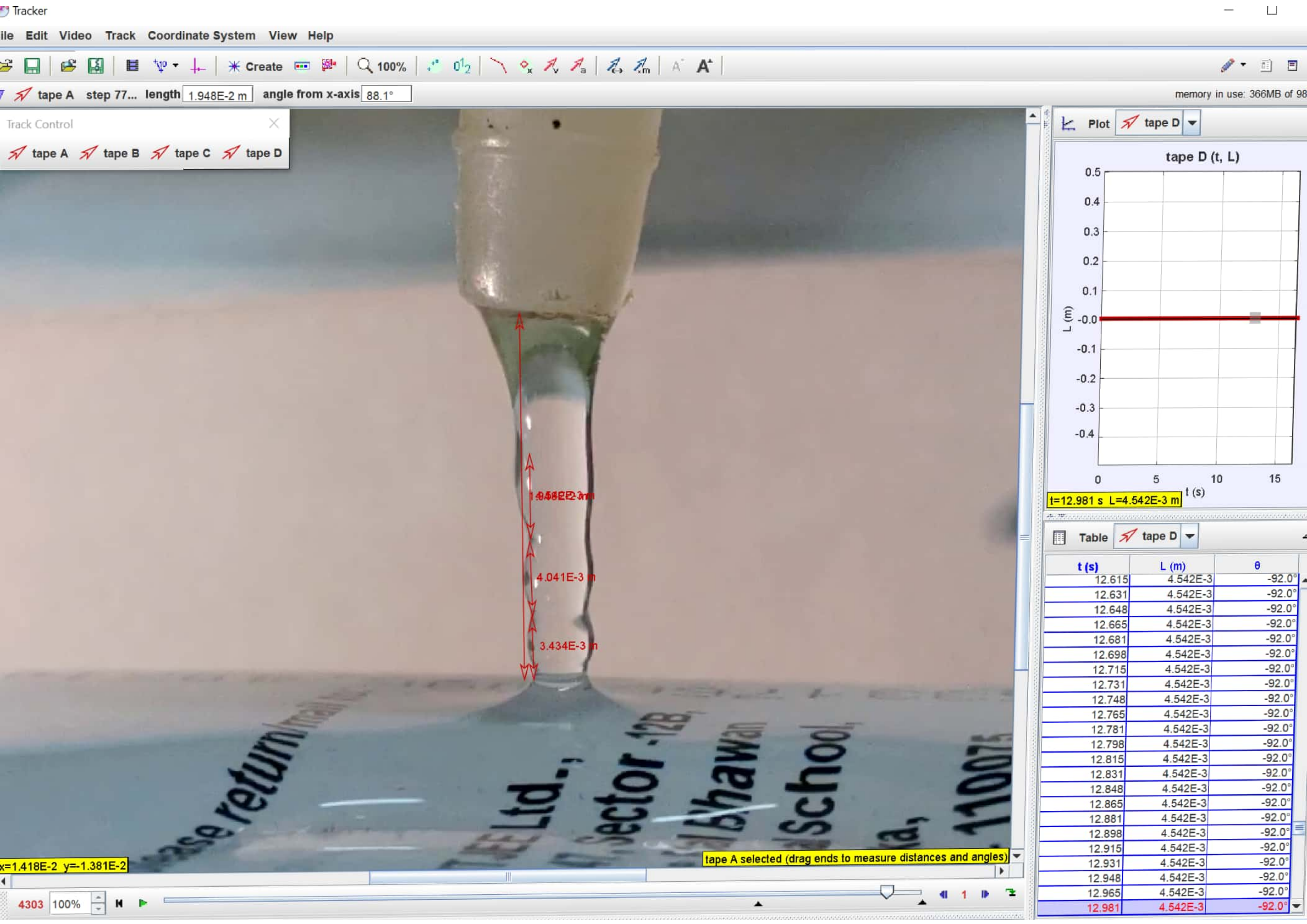


Tracker
File Edit Video Track Coordinate System View Help
Create
100%
tape E step ... length 3.004E-2 m angle from x-axis -92.3°
memory in use: 363MB
Track Control
tape A tape B tape C tape D
tape E
3.004E-2 m
2.093E-3 m
2.018E-3 m
1.943E-3 m
1.919E-3 m
TapeMeasure tape C (click to select and/or set properties)
2828
100%
1
Plot tape E
tape E (t, L)
0.5
0.4
0.3
0.2
0.1
-0.0
-0.1
-0.2
-0.3
-0.4
L (m)
0
5
10
15
20
25
t (s)
t=0.000 s L=3.004E-2 m
Table tape E
t (s)
L (m)
θ
0.000 3.004E-2 -92.3°
0.017 3.004E-2 -92.3°
0.033 3.004E-2 -92.3°
0.050 3.004E-2 -92.3°
0.067 3.004E-2 -92.3°
0.083 3.004E-2 -92.3°
0.100 3.004E-2 -92.3°
0.117 3.004E-2 -92.3°
0.133 3.004E-2 -92.3°
0.150 3.004E-2 -92.3°
0.167 3.004E-2 -92.3°
0.183 3.004E-2 -92.3°
0.200 3.004E-2 -92.3°
0.217 3.004E-2 -92.3°
0.233 3.004E-2 -92.3°
0.250 3.004E-2 -92.3°
0.267 3.004E-2 -92.3°
0.283 3.004E-2 -92.3°
0.300 3.004E-2 -92.3°
0.316 3.004E-2 -92.3°
0.333 3.004E-2 -92.3°
0.350 3.004E-2 -92.3°



Create
100%
tape E
step 5...
length 2.528E-2 m
angle from x-axis -93.5°
memory in use: 334MB of 989MB
Track Control
tape A
tape B
tape C
tape D
tape E
2.528E-2 m
3.712E-3 m
3.569E-3 m
2.925E-3 m
2.961E-3 m
Plot
tape E (t, L)
L (m)
t (s)
t=0.833 s L=2.528E-2 m
Table
t (s)
L (m)
θ
0.467 2.528E-2 -93.5°
0.483 2.528E-2 -93.5°
0.500 2.528E-2 -93.5°
0.517 2.528E-2 -93.5°
0.533 2.528E-2 -93.5°
0.550 2.528E-2 -93.5°
0.567 2.528E-2 -93.5°
0.583 2.528E-2 -93.5°
0.600 2.528E-2 -93.5°
0.617 2.528E-2 -93.5°
0.633 2.528E-2 -93.5°
0.650 2.528E-2 -93.5°
0.667 2.528E-2 -93.5°
0.683 2.528E-2 -93.5°
0.700 2.528E-2 -93.5°
0.717 2.528E-2 -93.5°
0.733 2.528E-2 -93.5°
0.750 2.528E-2 -93.5°
0.767 2.528E-2 -93.5°
0.783 2.528E-2 -93.5°
0.800 2.528E-2 -93.5°


4.643E-3 m
tape B selected (drag ends to measure distances and angles)
L (m)
t=322.533 s L=1.579E-2 m
Table
tape A
t (s)
L (m)
θ
300.629 1.579E-2
301.625 1.579E-2
302.620 1.579E-2
303.616 1.579E-2
304.612 1.579E-2
305.607 1.579E-2
306.603 1.579E-2
307.599 1.579E-2
308.594 1.579E-2
309.590 1.579E-2
310.585 1.579E-2
311.581 1.579E-2
312.577 1.579E-2
313.572 1.579E-2
314.568 1.579E-2
315.563 1.579E-2
316.559 1.579E-2
317.555 1.579E-2
318.550 1.579E-2
319.546 1.579E-2
320.541 1.579E-2
321.537 1.579E-2
322.533 1.579E-2


Tracker
ile Edit Video Track Coordinate System View Help
Create
100%
tape A step 77... length 1.948E-2 m angle from x-axis 88.1°
memory in use: 366MB of 98
Track Control
tape A tape B tape C tape D
4.041E-3 m
3.434E-3 m
x=1.418E-2 y=-1.381E-2
tape A selected (drag ends to measure distances and angles)
4303 100%
Plot tape D
tape D (t, L)
0.5
0.4
0.3
0.2
0.1
-0.0
-0.1
-0.2
-0.3
-0.4
L (m)
0
5
10
15
t (s)
t=12.981 s L=4.542E-3 m
Table tape D
t (s) L (m) θ
12.615 4.542E-3 -92.0°
12.631 4.542E-3 -92.0°
12.648 4.542E-3 -92.0°
12.665 4.542E-3 -92.0°
12.681 4.542E-3 -92.0°
12.698 4.542E-3 -92.0°
12.715 4.542E-3 -92.0°
12.731 4.542E-3 -92.0°
12.748 4.542E-3 -92.0°
12.765 4.542E-3 -92.0°
12.781 4.542E-3 -92.0°
12.798 4.542E-3 -92.0°
12.815 4.542E-3 -92.0°
12.831 4.542E-3 -92.0°
12.848 4.542E-3 -92.0°
12.865 4.542E-3 -92.0°
12.881 4.542E-3 -92.0°
12.898 4.542E-3 -92.0°
12.915 4.542E-3 -92.0°
12.931 4.542E-3 -92.0°
12.948 4.542E-3 -92.0°
12.965 4.542E-3 -92.0°
12.981 4.542E-3 -92.0°

Theoretical Analysis:-

Building a model from observations-

Increasing distance between nozzle and obstruction the average wavelength decreases. This suggests presence of multiple waves on the surface. Also, it means that the waves with lesser wavelength are more dominant down the stream, as observed from the experiment.

The waves appear stationary over long period of time, suggesting the formation of standing waves.

The theory models the perturbation due to imperfect fluid flow out of the nozzle.

The stream breaks into droplets (no obstruction) implying increase in amplitude of perturbations.

Assumptions:-

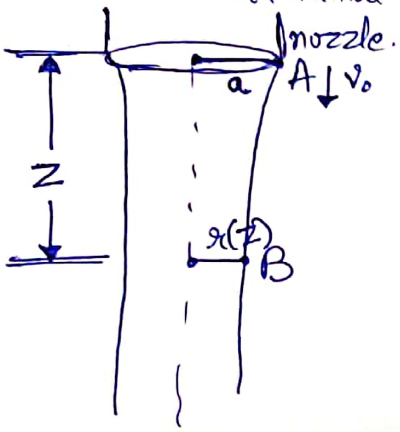
- Influence of viscosity is negligible, assuming a high Reynolds number fluid.
- The perturbations are sinosidal wave forms.
- Neglect any change in velocity, radially due to viscous forces.
- Define equilibrium state as: an infinetly long quiescent cylinder of fluid with negligible viscosity, radius of fluid column = a, density of fluid = ρ, surface tension of fluid = σ.

For the equilibrium state -

velocity $v = v(z)$: $z \equiv$ height from nozzle

$r = r(z)$: radius of circular cross-section at height z.

v_0: initial velocity of fluid.



using bernoulli's eqn -

$$\frac{1}{2}\rho v_0^2 + \rho g z + P_A = \frac{1}{2}\rho v^2 + P_B \quad (1)$$

$$\Delta P = \sigma(\nabla \cdot \hat{n})$$

$$= \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{\sigma}{r} \quad [R_2 \to \infty \text{ and } R_1 = r].$$

Hence; $P_A = P_0 + \dfrac{\sigma}{a}$ (2), $\quad P_B = P_0 + \dfrac{\sigma}{r}$ (3)

Hence; put (2) and (3) in (1);

we get;

$$\frac{v}{v_0} = \left[1 + \frac{2gz}{V_0^2} + \frac{2\sigma}{\rho v_0^2 a}\left(1 - \frac{a}{r}\right)\right]^{1/2}. \quad (4)$$

$$\text{Volume flow rate} = Q = 2\pi\int_0^r v(z)\, r(z)\, dr = \pi a^2 v_0 = \pi r^2 v(z)$$

$$\text{or} \quad \frac{r(z)}{a} = \left(\frac{v_0}{v(z)}\right)^{1/2} = \left[1 + \frac{2gz}{v_0^2} + \frac{2\sigma}{\rho v_0^2 a}\left(1 - \frac{a}{r}\right)\right]^{-1/4} \quad (5).$$

Extra: F_r = Froude Number $= \dfrac{V_0^2}{ag}$

W_e = Weber Number $= \dfrac{\rho V_0^2 a}{\sigma}$

when; $W_e \to \infty$; then: $r = a\left(1 + \dfrac{2gz}{V_0^2}\right)^{-1/4}$

and $v = v_0\left(1 + \dfrac{2gz}{V_0^2}\right)^{1/2}$

Studing perturbations

Assumptions -

\# Neglect gravity and viscosity. # radius at height z varies linearly with ε.

\# Zero external pressure

\# Pressure, velocity and radius follow a similar wave form.

Hence; $P_0 = \frac{\sigma}{a}$ — (5)

Now; $R(z) = a + \varepsilon \rho_1 + \varepsilon^2 \rho_2 \cdots$

where ε denotes perturbation amplitude.

As ε is infinitesimal; $\varepsilon \ll a$; we can neglect higher order terms and assume linearity.

we have, $\tilde{R}(z) = a + \varepsilon e^{\omega t + ikz}$ (assume sinosidal wave with exponentially varying amplitude).

ω: growth rate of instability.

→ if $\omega > 0$, then perturbation is unstable and grows

→ if $\omega < 0$, then perturbation is stable and decays.

Using Navier-Stokes eqns.-

- $\frac{\partial}{\partial t}\int_{\Omega} \rho \bar{v}\, d\Omega + \int_S \rho \bar{v}(\bar{v}\cdot\hat{n})\, dS = \Sigma f$;

$$\Rightarrow \frac{\partial \bar{v}}{\partial t} = -\frac{1}{\rho}\nabla P$$

Assume cylinderical coordinates;

$\tilde{u}_r$ = radial velocity. $\tilde{u}_z$ = axial velocity.

$\tilde{P}$ = perturbation pressure; $\tilde{u}_\theta = 0$.

Then; $\frac{\partial \tilde{u}_r}{\partial t} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial r}$ — (6) and $\frac{\partial \tilde{u}_z}{\partial t} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial z}$ — (7)

// assume only linear elements.
neglect higher order terms of ε.

• $\nabla \cdot \bar{v} = 0$;

// linearizing the equation again-
neglecting higher powers.

$$\frac{\partial \tilde{u}_r}{\partial r} + \frac{\tilde{u}_r}{r} + \tilde{u}_z = 0 \quad (8)$$

Assume, velocity and pressure share similar waveform -

$$\tilde{u}_r = R(r) e^{\omega t + ikz} \qquad \tilde{u}_z = Z(r) e^{\omega t + ikz}$$

$$\tilde{p} = P(r) e^{\omega t + ikz}$$

put and solving (6) (7) (8);

$$\omega R = -\frac{1}{\rho} \frac{dP}{dr} \quad (9) \qquad \omega Z = -\frac{ikP}{\rho} \quad (10)$$

and $$\frac{dR}{dr} + \frac{R}{r} + ikz = 0 \quad (11)$$

using (9) and (10) and eliminating P and Z in (11) to get;

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - (1 + (kr)^2) R = 0.$$

using modified Bessel's eqns. solution: order 1.

$$R(r) = C_1 I_1(kr) + d K_1(kr)$$ // $K_1(kr) \to \infty$ as $r \to 0$
hence, $d = 0$.

$$\underline{R(r) = C\, I_1(kr).} \quad (12)$$

using (9) we have $$P(r) = -\frac{\omega \rho C}{k} I_0(kr) \quad (13)$$

$$\frac{\partial \tilde{R}}{\partial t} \simeq \tilde{u}_r \Rightarrow \quad c = \frac{\varepsilon \omega}{I_1(kr)} \quad \text{—(14)}$$

At any point in wave;

$$P_0 + \tilde{P} = \sigma(\nabla \cdot \hat{n}) = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$R_1 = R_0 + \varepsilon e^{\omega t + ikx} \quad \text{—(15)}$$

$$R_2 = \frac{(1 + |r'(z)|^2)^{3/2}}{|r''(z)|} \quad ; \quad \frac{\partial r(z)}{\partial z} = i\varepsilon k e^{\omega t + ikz}$$

$|r''(z)|$ → modulas; absolute value.

and $\frac{\partial^2 (r(z))}{\partial z^2} = -\varepsilon k^2 e^{\omega t + ikz}$.

neglecting $O(\varepsilon^2)$ terms from the equations—

we have $1 + |r'(z)|^2 \simeq 1$ as $(r'(z))^2 \equiv O(\varepsilon^2)$.

hence; $R_2 = (\varepsilon k^2 e^{\omega t + ikz})^{-1}$ —(16).

$$P + \tilde{P} = \sigma\left(\frac{1}{R_0 + \varepsilon e^{\omega t + ikz}} + \varepsilon k^2 e^{\omega t + ikz}\right).$$

$$\simeq \sigma\left(\frac{1}{R_0} - \frac{\varepsilon e^{\omega t + ikz}}{R_0^2} + \varepsilon k^2 e^{\omega t + ikz}\right)$$

$$\Downarrow$$

$$\tilde{P} = -\frac{\varepsilon \sigma}{R_0^2}(1 - k^2 R_0^2) e^{\omega t + ikz} \quad \text{—(17)} \quad // \text{using (5) } P_0 = \frac{\sigma}{R_0}$$

$(R_0 = a)$.

Using (13) (14) and (17)

we find;

$$\omega^2 = \frac{\sigma}{\rho R_0^3}(kR_0)\,\frac{I_1(kR_0)}{I_0(kR_0)}\left(1-(kR_0)^2\right) \quad \text{--- (18)}$$

we can graph the function $\omega = \omega(k)$ and observe it is a concave function with maxima ~~at~~ when $\frac{d\omega}{dk} = 0$.

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solving numerically we get;

for $\omega_{max} \Rightarrow kR_0 = 0.697$

or $\lambda_{max} = \frac{2\pi}{k} = 9.02 R_0$.

For standing waves formed due to the obstruction. wave velocity must be equal to the local fluid velocity for it to appear stationary.

$v = -\omega/k$;

Hence;

$$v^2 = \frac{\omega^2}{k^2} = \frac{\sigma}{\rho k R_0^2}\,\frac{I_1(kR_0)}{I_0(kR_0)}\left(1-k^2R_0^2\right) \quad \text{--- (19)}$$

local jet speed.

Provided; v (local speed) one can predict the wavelength of the waves that travel at v and appear stationary. local speed can be ~~der~~ used by eq. (4).

Hence,

$$v^2 = v_0^2\left[1 + \frac{2gz}{v_0^2} + \frac{2\sigma}{\rho v_0^2 a}\left(1 - \frac{a}{r}\right)\right] = \frac{\sigma}{\rho k a^2}\frac{I_1(ka)}{I_0(ka)}\left(1 - k^2a^2\right).$$

Bibliography

1. MIT Lectures on fluid dynamics (Lecture 5)
2. New trends in the physics and mechanics of Biological systems
3. Wikipedia
4. Landau & Lifschitz Vol-6 fluid mechanics.