Lab 4

STAT 517 - Winter 2023

1. Let $\{X(t) : t \geq 0\}$ be a standard Brownian motion and $M(t) = \max_{0 \leq s \leq t} X(s)$. Show that
$$P(M(t) > a \mid M(t) = X(t)) = e^{-a^2/2t}$$

2. Let $\{X(t) : t \geq 0\}$ be a standard Brownian motion and define
$$V(t) = e^{-\alpha t^2/2} X\left(\alpha e^{-\alpha t}\right)$$
   (a) Show that $\{V(t) : t \geq 0\}$ is a stationary Gaussian process. This is called the Ornstein-Uhlenbeck process.
   (b) Show that $\{V(t) : t \geq 0\}$ is a Markov process.
   (c) Show that the joint distribution of $(V(t_1), V(t_2), \ldots, V(t_k))$ for a an equally spaced grid with $t_i = t_{i-1} + \Delta$ corresponds to that of a first order autoregressive process in discrete time. Why is this useful when modeling real data?

3. Consider the dataset `mcycle` in the `R` package `MASS`, which corresponds to a series of measurements of head acceleration in a simulated motorcycle accident, used to test crash helmets

```r
library(MASS)
data(mcycle)
plot(mcycle)
```

   (a) Fit Gaussian process models with constant mean function $f(x) = \mu$ and the following covariance matrices to this data:
      i. $\gamma(x, x') = \sigma^2 e^{-\lambda|x-x'|}$.
      ii. $\gamma(x, x') = \sigma^2 \left(\nu e^{-\lambda|x-x'|} + \delta_{(x=x')}\right)$ with $\nu = 1/3$.
      iii. $\gamma(x, x') = \sigma^2 \left(\nu e^{-\lambda|x-x'|} + \delta_{(x=x')}\right)$ with $\nu = 10$.
      iv. $\gamma(x, x') = \sigma^2 e^{-\lambda(x-x')^2}$.
      v. $\gamma(x, x') = \sigma^2 \left(\nu e^{-\lambda(x-x')^2} + \delta_{(x=x')}\right)$ with $\nu = 1/3$.

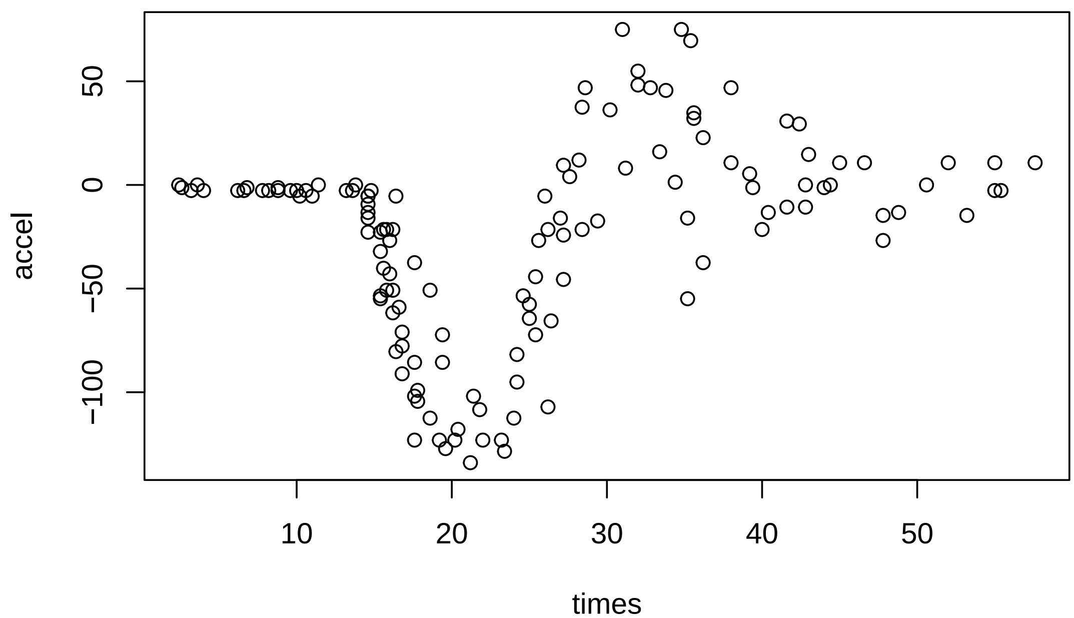

Figure 1: mcycle data.

where

$$\delta_{(x=x')} = \begin{cases} 1 & x = x' \\ 0 & \text{otherwise} \end{cases}$$

In each case, generate a plot of the estimated function in addition to providing point estimators for each of the parameters involved.

(b) Comment on the different fits and discuss what the effect of the different model parameters is on the resulting estimate