(1) Topological spaces
Let X be a set. A collection $\mathcal{T}$ of sets in X is a **topology for** X iff it satisfies the properties:

- (1) $X, \varnothing \in \mathcal{T}$
- (2) An arbitrary union of sets in $\mathcal{T}$ belongs to $\mathcal{T}$; that is:
  if $U_\alpha \in \mathcal{T}$ for each $\alpha \in I$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$, and the indexing set I can be infinite.
- (3) A **finite** intersection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

For any set X, we can always define the following two topologies:

- The Indiscrete Topology for X: $\{X, \varnothing\}$
- The Discrete Topology for X: $P(X) = \{U : U \subseteq X\}$

We verify that the discrete topology is indeed a topology. First, $X \subseteq X, \varnothing \subset X$, so (1) is satisfied. If $U_\alpha \subseteq X$ for each $\alpha \in I$, then $\bigcup_{\alpha \in I} U_\alpha \subseteq X$. Thus, if each set in the collection $(U_\alpha)_{\alpha \in I}$ belongs to $\mathcal{T}$, then their union also belongs to $\mathcal{T}$, whereby (2) is satisfied. Similarly, the intersection of a finite number of subsets of X is also a subset of X, whereby (3) is satisfied.
If $\mathcal{T}$ is a topology for X, then the pair $(X, \mathcal{T})$ is called a **topological space**, and the sets in $\mathcal{T}$ are *called* **open**.

(2) Metric Topology: If (X, d) is a metric space, then $\mathcal{T} = \{U \subseteq X : X \text{ is } d\text{-open in } X\}$ is **a metric topology** for X. Recall that U is said to be $d-$open in X iff it satisfies: $\forall u \in U, \exists r > 0 : B(u; r) \subset U$ (there's "wiggle room" around any point in U).
$\varnothing$ and X are always $d-$open; an arbitrary union of open sets is open, and a finite intersection of open sets is open. So the collection of all $d-$open subsets of the metric space (X, d) satisfies all the conditions (1)-(3) and is indeed a topology for X.

- Different metrics may determine different open sets, whereby they produce different metric topologies for X. For eg., consider two different metrics on $\mathbb{R}$:

$$d_1(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

$$d_2(x, y) = |x - y|$$

  We know that any $d-$open ball is open in a $(\mathbb{R}, d)$. Let $a \in \mathbb{R}$, and consider the d_1-open ball: $B^{d_1}(a; 1) = \{x \in \mathbb{R} : d_1(x, a) < 1\}$. By the definition of d_1, $d_1(x, a) < 1 \implies d_1(x, a) = 0 \implies x = a$, so $B^{d_1}(a; 1) = \{a\}$. Thus $\{a\}$ is open in $\mathbb{R}$ wrt. to d_1.
  But clearly, $\{a\}$ is not open with respect to d_2, since for *any* $r > 0$,

$B^{d_2}(a;r) = (a-r, a+r)$ contains points other than $\{a\}$. Indeed, any singleton set in $\mathbb{R}$ is closed wrt. d_2, while any subset of $\mathbb{R}$ is both closed *and* open wrt. d_1. Thus, d_1 and d_2 determine different metric topologies on X.

- On the other hand, equivalent metrics determine the same open sets on X, whereby they produce the same metric topology for X.
- Some topologies $\mathcal{T}$ for X cannot be determined by any metric, by which we mean: **There is no metric d such that:**

$$[A \text{ is a } d\text{−open subset of } X] \Longleftrightarrow [A \in \mathcal{T}]$$

  Such a topology is **non-metrizable.**

- For eg., the indiscrete topology : $\{X, \varnothing\}$ is never a metric topology if X has more than one element. Why? Well, the complement of a singleton set : $X \setminus \{x\}$ is always open with respect to any metric d (*). Thus, the metric topology will for X will always contain elements other than X and $\varnothing$ if $|X| > 1$.
  However, if X has only one element : $X = \{x\}$, then *any* metric will produce the indiscrete topology for X : for any $r > 0$, the open ball with respect to an arbitrary metric d is the set $B(a;r) = \{x \in X : d(a,x) < r\}$; but $y \in B(a;r) \implies y = x$, whereby the only nonempty open ball is X, and $\varnothing$ is always d−open.
- The discrete topology: $P(X)$ is always **metrizable**, under the discrete metric d_1. As discussed, *every* subset of X is open wrt. the discrete metric. Thus: $U \subseteq X$ is d_1−open $\Longleftrightarrow U \in P(X)$.

(*) We quickly prove that for any metric space (X,d), a complement of any singleton set in X is open if $|X| > 1$.
Let $x \in X$; we show that $X \setminus \{x\}$ is open. Pick any $y \in X \setminus \{x\}$. We want an $r > 0$ so that $B(y;r) \subseteq X \setminus \{x\}$. Take $r = d(x,y)$. Then, $z \in B(y;r) \implies d(z,y) < r$. But now,
$r = d(x,y) \leq d(x,z) + d(z,y) \implies 0 < r - d(z,y) \leq d(x,z) \implies d(z,x) > 0$.
Thus, $z \neq x \implies z \in X \setminus \{x\}$, whereby $B(y;r) \subseteq X \setminus \{x\}$.

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(3) Closed Sets
Recall that in a metric space (X,d) $A \subseteq X$, A is **closed** if it contains all of its adherent points : $\overline{A} \subseteq A$. Equivalently:

$$A \text{ is closed} \Longleftrightarrow X \setminus A \text{ is open.} \qquad (1)$$

The characterization (1) is used as the *definition* for **a closed set in a topological space** $(X, \mathcal{T})$: $A \in \mathcal{T}$ is closed $\Longleftrightarrow (X \setminus A)$ is open. We then have the following properties for closed sets in $(X, \mathcal{T})$:

- (1)' X and $\varnothing$ are closed sets.

- (2)' An arbitrary intersection of closed sets is closed.
- (3)' A finite union of closed sets is closed.

Nbhds, Interiors, and Open sets

Let $(X, \mathcal{T})$ be a topological space. Recall that **the sets in $\mathcal{T}$ are called open**, and $\mathcal{T}$ satisfies rules (1) – (3) on page-1.

A set $S \subseteq X$ is a **nbhd** of $x \in X$ iff: There exists $U \subset S : U$ is open ($U \in \mathcal{T}$), and $x \in U$. The open set $U \subset S$ containing x is called an **open container** for x in S.

Thus, S is a nbhd of x if it contains an open container for x.

$x \in X$ is called an **interior point** of S iff S is a nbhd of x. The **interior** of a set S, denoted int(S), is the set of all interior points of S. Clearly, int$(S) \subset S$ always.

For the following theorems, let $(X, \mathcal{T})$ be a topological space.

Theorem(2.1) (i) A subset $S \subseteq X$ is open $\Leftrightarrow$ (ii) $S = \text{int}(S)$.
That is, S is open iff S is a nbhd of each of its points.

Proof (i $\implies$ ii) Let S be open ($S \in \mathcal{T}$), and let $x \in S$. Then, $x \in S \subseteq S-$ that is, S is a nbhd of x. Thus, $x \in \text{int}(S)$. Hence $S \subseteq \text{int}(S)$, and we always have $\text{int}(S) \subset S$, so $\text{int}(S) = S$.
(ii $\implies$ i) Let $S = \text{int}(S)$. We WTS: $S \in \mathcal{T}$. For each $x \in \text{int}(S)$, there is an **open** container $U_x \subset S$ ($U_x \in \mathcal{T}$ for each $x \in \text{int}(S) = S$). By (2), $\bigcup_{x \in S} U_x \in \mathcal{T}$, whereby $\bigcup_{x \in S} U_x = S$ is open.

Thus: U is a nbhd of x if it is an open set containing x.

Theorem(2.2) int(S) is **open**: $(\text{int}(\text{int}(S)) = \text{int}(S)$

Proof

Let $x \in \text{int}(S)$. Then, x has an open container $U_x \subset S$. Our goal is to show that $U_x \subset \text{int}(S)$. Pick any $y \in U_x$. Since $U_x \subset S$ is an **open** set containing y, $U_x = V_y$, by which we mean that y has an open container $V_y \subset S$, whereby S **is a nbhd of** y. So, $y \in U_x \implies y \in \text{int}(S)$, whereby $U_x \subset \text{int}(S)$. Thus, any point in int$(S)$ has an open container insider int(S); hence, int(S) is open.

Adherence and Closure

Let $(X, \mathcal{T})$ be a topological space. $x \in X$ is *adherent* to S iff :

$$\forall U_x \subseteq X, U_x \cap S \neq \varnothing.$$

The **closure** of S is denoted: $\overline{S} = \{x \in X : x \text{ is adherent to } S\}$. If $x \in S$, then for any nbhd U_x, $x \in (U_x \cap S) \implies x \in \overline{S}$. Thus, $S \subseteq \overline{S}$ always. Recall that a subset S of X is said to be **closed** iff its complement is open ($(X \setminus S) \in \mathcal{T}$). As in the metric topology, closures are related to closed sets.

Theorem (2.3) (i) $S \subseteq X$ is closed $\iff$ (ii) $S = \overline{S}$.

Proof (i) $\implies$ (ii) : Let $S \subseteq X$ be closed, then $(X \setminus S)$ is open. Let $y \in (X \setminus S)$; then $(X \setminus S)$ is a nbhd of y, where $(X \setminus S) \cap S = \varnothing$. Since there is a nbhd of y that doesn't intersect S, $y \notin \overline{S}$. Thus, $y \notin S \implies y \notin \overline{S}$, equivalent to $y \in \overline{S} \implies y \in S$. Then, $\overline{S} \subseteq S$, whereby $S = \overline{S}$.

(ii) $\implies$ (i) : Suppose $S = \overline{S}$; we WTS that $(X \setminus S)$ is open. Let $y \notin \overline{S} = S$. Then there is a nbhd of y: $U_y \cap S = \varnothing \implies U_y \subseteq (X \setminus S)$.
Then, $\bigcup_{y \notin S}(U_y : U_y \cap S = \varnothing) = (X \setminus S)$ is open, since each U_y is open.

Theorem (2.4) Closures are closed ($\overline{\overline{S}} = \overline{S}$).
Proof Let $x \in (X \setminus \overline{S})$. Then, $\exists U_x : U_x \cap S = \varnothing$, so $U_x \subseteq (X \setminus S)$. Consider $\bigcup_{x \notin \overline{S}}\{U_x : U_x \cap S = \varnothing\} = (X \setminus \overline{S})$; since each nbhd U_x is open, $(X \setminus \overline{S})$ is open, whereby $\overline{S}$ is closed. Hence by **(2.3)**, $\overline{\overline{S}} = \overline{S}$.

Exercise-1
So far, many properties of metric spaces are transferred to a general topological space. Think about which properties of interiors, open sets and closures are *not* retained in an arbitrary topological space, and provide counterexamples or proofs for your claims.

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(4) Convergence in $(X, \mathcal{T})$: A sequence $\{x_j\}$ in X is said to converge to a point $x \in X$ iff:

$$\text{For any nbhd } U_x \text{ of } x,\ \exists N : j \geq N \implies x_j \in U_x$$

That is, all but finitely many terms of the sequence must fall into any open set containing x $-$ which again is consistent with a metric topology. In any **metrizable** topology, the limit of a convergent sequence is **unique**. This may **not** be true in a **non-metrizable** topology.

Thm (2.5) Let $S \subseteq X$. If there exists a sequence in S converging to the point $x \in X$, then $x \in \overline{S}$.
Proof Let $(x_j)_{j=1}^{\infty} \subseteq S$ where $x_j \to x$. Let U_x be any nbhd of x. Then, $\exists N : j \geq N \implies x_j \in (U_x \cap S)$; thus $U_x \cap S \neq \varnothing$. Hence $x \in \overline{S}$.

Recall the *Sequential Closure Criterion* in metric spaces, which includes the converse of **(2.5)**: if $x \in S, \exists (s_n)_{n=1}^{\infty} : s_n \to s$. If $(X, \mathcal{T})$ is **metrizable**, then the Sequential Closure Criterion holds in $(X, \mathcal{T})$. But this converse may **not** be true in an arbitrary **non-metrizable** topological space $-$ there might be a point in $\overline{S}$ such that no sequence in S converges to this point!

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(5) The Boundary
Let S be a set in X. A point $x \in X$ is called a *boundary point* of S iff $x \in \overline{S} \cap \overline{(X \setminus S)}$.

The **boundary** : $\partial S = \overline{(X \setminus S)} \cap \overline{S}$, is the set of all boundary points, adherent to both S and its complement. Clearly, the boundary is always a closed set, since it is an intersection of two closures, which are always closed sets.

Thm (2.6): $\overline{S}$ is the disjoint union of $\text{int}(S)$ and ∂S.
Proof Let $x \in \overline{S}$. Then $\forall U_x, U_x \cap S \neq \varnothing$. For any nbhd U_x, there are two disjoint cases:

- Either $U_x \subseteq S$, in which case $x \in \text{int}(S)$;
- Or $U_x \nsubseteq S \implies \exists y \in U_x : y \notin S \implies y \in U_x \cap (X \setminus S)$. Then, $x \in \overline{X \setminus S}$ and $x \in \overline{S} \implies x \in \partial S$.

Hence, $x \in \overline{S} \implies x \in \partial S \cup \text{int}(S)$.
In the other direction: $x \in \partial S \cup \text{int}(S) \implies x \in \partial S$ or $x \in \text{int}(S)$; in either case, $x \in \overline{S}$. Thus, $\overline{S} = \partial S \cup \text{int}(S)$.
