Compactness (1.5): Part-III : If (X, d) is TB and complete, every sequence within it has a convergent subsequence.

We begin by proving (3) $\implies$ (2) for our main theorem. We show that every sequence in a TB metric space has a *Cauchy* subsequence; if our metric space is **complete**, this Cauchy subsequence must converge.

Theorem 5.6 : Any sequence in a totally bounded (TB) metric space (X, d) has a Cauchy subsequence.

Proof Let $(x_j)_{j=1}^{\infty}$ be a sequence in X. We will iteratively extract subsequences from $(x_j)_{j=1}^{\infty}$ and arrive at a Cauchy subsequence; we use the symbol $\subset^*$ to denote a subsequence inclusion. Let k index the $k-$th subsequence extracted: $(x_{kj})_{j=1}^{\infty} \subset^* (x_j)_{j=1}^{\infty}$, and also denote the $k-$th term of our sequence of interest $(y_n)_{n=1}^{\infty}$, which we constructed as explained below.

- (Base Case) For $k = 1$, let the first "extracted" sequence: $(x_{1j})_{j=1}^{\infty}$ just be our original sequence $(x_j)_{j=1}^{\infty}$;
  Set $y_1 = x_{11} = x_1$.
- (Recursion) For $k \geq 2$, extract $(x_{kj})_{j=1}^{\infty}$ to satisfy:
  - (1) $(x_{kj})_{j=1}^{\infty}$ is a subsequence of $(x_{k-1j})_{j=1}^{\infty}$
  - (2) $(x_{kj})_{j=1}^{\infty}$ falls into a ball of radius $(\frac{1}{k})$ inside X.
  - (3) Set $y_k = x_{kk}$

First,we need to show that this construction can be *satisfied* for each $k \geq 2$. Certainly, condition (1) can always be satisfied. For (2), we proceed by induction.
Suppose for some $k \geq 2$, the subsequence $(x_{k-1j})_{j=1}^{\infty}$ has already been formed to satisfy (1) and (2). Since (X, d) is TB, for $\varepsilon = \frac{1}{k}$, X can be covered by ***finitely* many** (say, p) balls of radius $\frac{1}{k}$: $B_1, ..., B_p$. Thus: $(x_{k-1j})_{j=1}^{\infty} \subset X \subseteq \bigcup_{s=1}^{p} B_s$. Since the sequence $(x_{k-1j})_{j=1}^{\infty}$ has *infinitely* many terms, atleast one of these balls must contain infinitely many terms of $(x_{k-1j})_{j=1}^{\infty}$.
Let this ball be $B_l, l \in [p]$. Now, we can simply select *all* the terms in B_l *in order*, to form our $k-th$ subsequence $(x_{kj})_{j=1}^{\infty}$; let x_{k1} be the first[1] term in B_l, x_{k2} the second, and so on. Thus, $(x_{kj})_{j=1}^{\infty} \subseteq (x_{k-1j})_{j=1}^{\infty}$ falls into a ball of radius $1/k$. Setting $y_k = x_{kk}$, conditions $(1)-(3)$ are met.

Now we claim that $(y_n)_{n=1}^{\infty}$ is a Cauchy subsequence of our original sequence $(x_j)_{j=1}^{\infty}$.
For **any** $k \geq 2$, $(y_n)_{n=k}^{\infty} \subset^* (x_{kj})_{j=1}^{\infty}$, since: At the starting index $n = k$, $y_k = x_{kk} \in (x_{kj})_{j=1}^{\infty}$, and for $n > k$, $y_n = x_{nn} \in (x_{nj})_{j=1}^{\infty} \subset^* (x_{kj})_{j=1}^{\infty}$, seen by $(x_{k+1j})_{j=1}^{\infty} \subset^* (x_{kj})_{j=1}^{\infty}$ for any $k > 1$ (from$-(1)$).

[1] By "first", we mean: $k1$ is the smallest index among the indices $k-1j$ to fall in B_l, and so on (Well-Ordering of $\mathbb{N}$)

Thus, for **each** $k > 1$, $(y_n)_{n=k}^{\infty} \subset^* (x_{kj})_{j=1}^{\infty} \subseteq B(z; \frac{1}{k})$, where $z \in X$ $(-(2))$.
So whenever $n \geq k$, $\;y_n \in B(z; \frac{1}{k}) \implies d(y_n, z) < \frac{1}{k}$.
Then, $\forall k \geq 2$, $\;m, n > k \implies d(y_m, y_n) < d(y_m, z) + d(z, y_n) < \frac{2}{k}$. Thus,

For any $\varepsilon > 0$, with $N = \frac{2}{\varepsilon}$. Then $k \geq N \implies \forall m, n > N$, $d(y_m, y_n) < \varepsilon$

Whereby $(y_n)_{n=1}^{\infty}$ is a Cauchy subsequence of our original sequence $(x_j)_{j=1}^{\infty}$.