Recall that in a metric space (X, d), if $Y \subseteq X$, we define a metric *subspace* (Y, d') of X by restricting the metric to Y: $d|_Y$. In this case, $U \subseteq Y \implies U = (V \cap Y)$ for some $V \subseteq X$, or: the subsets in Y are just sets in X intersected with Y. We also showed:

(1) U is **open** in $X \implies (U \cap Y)$ is open in Y,

(2) U is **open** in $Y \iff$ There is an open set $V \subseteq X$ such that $U = (V \cap Y)$.

Note that the *converse* of **(1)** is false. For instance: consider the space $(\mathbb{R}^2, ||.||)$ of which $Y = (\mathbb{R}, |.|)$ is a subspace. The set: $\{(x, 0) : 0 < x < 1\} = (0, 1)$ is open in $\mathbb{R}$ (indeed, it is an open ball centered at $\frac{1}{2}$ of radius $\frac{1}{2}$); but $(0, 1)$ is **not** open in $\mathbb{R}^2$. Thus, relatively open sets in a subspace Y may not be open in the parent space X.
These notions and results are generalized to the *subspace of a topological space*, as seen below.

0.1 Subpaces of Topological Spaces

Let $(X, \mathcal{T})$ be a topological space, and let $Y \subseteq X$. Note that any set in Y is some set in X intersected with Y. the collection:

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$$

defines a topology for the set Y. The space $(Y, \mathcal{T}_Y)$ is called a ***subspace*** of $(X, \mathcal{T})$. Often we will refer to Y as a subspace of X when the topologies are understood in context.
$\mathcal{T}_S$ is called the ***relative topology*** for Y inherited from $(X, \mathcal{T})$.
Each set $V \in \mathcal{T}_S$ is called ***relatively open***, and the sets $(Y \setminus V)$ are called ***relatively closed*** (in Y).

We prove that $\mathcal{T}_Y$ is a topology for Y. First, $\varnothing \cap Y = \varnothing \in \mathcal{T}_Y$. If Y is an open subset of X— that is, $Y \in \mathcal{T}$, then $Y \cap Y = Y \in \mathcal{T}_Y$ immediately. Otherwise, suppose $Y \notin \mathcal{T}_Y$. Then, $Y \not\subseteq \text{int}(Y)$, so $S = (Y \setminus \text{int}(Y))$ is non-empty. For each point $y \in (Y \setminus \text{int}(Y))$, select a nbhd U_y of this point : $y \in U_y \in \mathcal{T}$. Then, the set $B = \bigcup_{y \in S} U_y$ belongs to $\mathcal{T}$, and since $\text{int}(Y) \in \mathcal{T}$, we have $(B \cup \text{int}(Y)) \in \mathcal{T}$. Clearly, $Y \subseteq (B \cup \text{int}(Y))$, whereby $Y \cap (B \cup \text{int}(Y)) = Y$. Hence, $Y \in \mathcal{T}_Y$.
If $\{U_l\}_{l \in I}$ is a collection of sets in $\mathcal{T}_Y$, then $U_l = A_l \cap Y$ for each l in I, where $A_l \in \mathcal{T}$. Then , $\bigcup_{l \in I} U_l = \bigcup_{l \in I}(A_l \cap Y) = \{\bigcup_{l \in I} A_l\} \cap Y$, where $\{\bigcup_{l \in I} A_l\} \in \mathcal{T}$. Thus, $\{\bigcup_{l \in I} A_l\} \cap Y = \bigcup_{l \in I} U_l$ belongs to $\mathcal{T}_Y$.
Similarly we can show that whenever a finite collection of sets $\{U_k\}_{k=1}^n$ belongs to $\mathcal{T}_Y$, their intersection also belongs to $\mathcal{T}_Y$.

Suppose $(X, \mathcal{T})$ is a metrizable topology, where $\mathcal{T}$ can be determine by the metric d, and $Y \subseteq X$. Then the relative topology for Y: $\{U \cap Y : U \in \mathcal{T}\}$ inherited from X coincides with the topology for Y determined by the restriction of the metric: $d|_Y$. This is consistent with our study of metric subspaces.

As with metric topologies, subsets of X that are open in a subspace Y need not be open in X. The simplest case is when $S \subset X$ but $S \notin \mathcal{T}$. Then certainly $S \in \mathcal{T}_S$, (S is ***relatively open in*** S) — however, S is not open in X.
Similarly, since $S \setminus S = \varnothing \in \mathcal{T}_S$, S is also ***relatively closed in*** S. However, S may not be closed in X.
We saw that the open sets in a subspace S of X are precisely open sets of X intersected with S. A similar characterization holds for closed sets of a subspace.

Theorem-1 Let $(S, \mathcal{T}_S)$ be a subspace of the topological space $(X, \mathcal{T})$, and let $E \subseteq S$. Then:

(i) E is relatively closed in $S \iff$ **(ii)** $E = S \cap V$, where V is closed in X.

Proof (i) $\implies$ **(ii)** : Let E be relatively closed in S. We want to show that $E = V \cap S$ where V is closed in X, that is: $(X \setminus V) \in \mathcal{T}$.
Since E is closed in S, we have $(S \setminus E) \in \mathcal{T}_S$, that is: $(S \setminus E) = U \cap S$ for some $U \in \mathcal{T}$. Now:

$$E = S \setminus (S \setminus E) = S \setminus (U \cap S) = (S \setminus U) \cup (S \setminus S) = S \setminus U$$

so $E = S \setminus U$ where $U \in \mathcal{T}$. Since $S \subseteq X$, we have: $S \cap (X \setminus U) = S \setminus U$. Thus:

$$E = S \cap (X \setminus U)$$

where $U \in \mathcal{T} \implies U = X \setminus (X \setminus U) \in \mathcal{T}$, so that $(X \setminus U)$ is closed in X.

Now we show the converse : **(ii)** $\implies$ **(i)**. Suppose $E \subseteq S$ where $E = S \cap V$, and V is closed in X. We want to show that E is relatively closed in S, that $(S \setminus E) \in \mathcal{T}_S$.
We have: $(S \setminus E) = S \setminus (V \cap S) = (S \setminus V) \cup (S \setminus S) = (S \setminus V)$. Since $S \subseteq X$ and $V \subseteq X$:

$$(S \setminus E) = (S \setminus V) = S \cap (X \setminus V)$$

And $(X \setminus V) \in \mathcal{T}$ because V is closed in X. Thus, $(S \setminus E) \in \mathcal{T}_S$.

Relative Closures | Let $(S, \mathcal{T}_S)$ be a subspace of the topological space $(X, \mathcal{T})$, and let $E \subseteq S \subseteq X$. Recall that the closure of E in $(X, \mathcal{T})$ is the set:

$$\overline{E} = \{p \in X : \forall U_p, U_p \cap E \neq \varnothing\}$$

Here, $p \in U_p \in \mathcal{T}$, that is: U_p is a nbhd of p in $(X, \mathcal{T})$. In contrast, the ***relative closure*** of E in the subspace $(S, \mathcal{T})$ is the set:

$$\overline{E}_S = \{p \in S : \forall U_p^S, \quad U_p^S \cap E \neq \varnothing\ \}$$

Here, $p \in U_p^S \in \mathcal{T}_S$, that is: U_p^S is a nbhd of p in the ***subspace*** $(S, \mathcal{T}_S)$.

Theorem 2 Let $(S, \mathcal{T}_S)$ be a subspace of $(X, \mathcal{T})$, and $E \subseteq S$. Then:

$$\overline{E}_S = \overline{E} \cap S$$

Proof First we show that $\overline{E}_S \subseteq \overline{E} \cap S$. Let $p \in \overline{E}_S$, and let $U_p \in \mathcal{T}$ be any nbhd of p. Since $(U_p \cap S) \in \mathcal{T}_S$ is a nbhd of p in the subspace S, $(U_p \cap S) \cap E \neq \varnothing$. Since $(U_p \cap S) \cap E \subseteq (U_p \cap E)$, we have $U_p \cap E \neq \varnothing$, whereby $p \in \overline{E}$ and $p \in S$. Thus, $p \in \overline{E} \cap S$.

Now let $p \in \overline{E} \cap S$, and let U_p^S be any nbhd of p in the subspace S. Then $U_p^S = U_p \cap S$ for some nbhd of p: $U_p \in \mathcal{T}$. Then, $(U_p \cap E) \cap S \neq \varnothing$, or: $(U_p \cap S) \cap E = U_p^S \cap E \neq \varnothing$, whereby $p \in \overline{E}_S$. Thus, $\overline{E} \cap S \subseteq \overline{E}_S$.
