(I) Open Balls in (X, d)
Let (X, d) be a metric space. An **open ball** centered at $x_0 \in X$ of radius $r > 0$ is the set: $B(x_0; r) = \{y \in X : d(y, x_0) < r\}$
A **closed ball** with same center and radius is the set:
$B[x_0; r] = \{y \in X : d(y, x_0) \leq r\}$

The closure of an open ball in an arbitrary metric space is not necessarily the closed ball of the same center and radius (eg: in the discrete metric space). However, in $\mathbb{R}^n$, it is true that $\overline{B(x_0; r)} = B[x_0; r]$.

Further: $\bigcup\limits_{r>0} B(x_0, r) = X$ and $\bigcap\limits_{r>0} B(x_0; r) = \{x_0\}$.

(II) Interiors and Open sets
Let S be a subset of X.

- $\text{int}(S) = \{x \in S : \exists r > 0 \text{ where } B(s; r) \subseteq S\}$
- $\text{int}(S) \subseteq S$ always.
- **Open sets** If $\text{int}(S) = S$, that is: $S \subseteq \text{int}(S)$, then S is said to be open. Equivalently:

$$S \text{ is open} \Leftrightarrow \forall s \in S, \exists r > 0 : B(s; r) \subseteq S$$

- Intuitively, a point in the interior of a set will always have some "wiggle room" around it.
- $\text{int}(S)$ is always an open set: $\text{int}(\text{int}(S)) = \text{int}(S)$.
- X and $\varnothing$ are both open sets in any metric space (X, d)
- There are sets that are neither open nor closed (eg: [0,1) in $\mathbb{R}$); and there are sets that are both open and closed. For example, under the discrete metric, every set is both open and closed.
- An arbitrary union of open sets is open, while the intersection of **finitely** many open sets is open. However, the intersection of an *infinite* number of open sets may be closed: $\bigcup\limits_{n=1}^{\infty} B(x_0, \frac{1}{n}) = \{x_0\}$
- S is open $\Leftrightarrow$ S can be written as an arbitrary union of open balls in (X, d)
- If (Y, d') is a subspace of the metric space (X, d), and a set $S \subseteq Y$ is open in Y, S may not necessarily be open in the entire space X.
Eg: $(0, 1)$ is open in $\mathbb{R}$ (standard Euclidean metric), but $(0, 1)$ is closed in $\mathbb{R}^2$, since for every $x \in (0, 1), \forall \epsilon > 0, B^{\mathbb{R}^2}(x; \epsilon) \cap \mathbb{R}^2 \neq \varnothing$. That is, for any point in the interval, the ball of any positive radius centered at this point will intersect the $x - y$ plane.

(III) Adherence, Closures, Closed Sets
Let Y be a set in the metric space (X, d). A point $p \in X$ is said to be **adherent** to the set Y iff: for every $r > 0, B(p; r) \cap Y \neq \varnothing$.

- Closure(Y) , denoted $\overline{Y}$ ={adherent points of Y}
- $p \in \overline{Y} \Leftrightarrow \forall r > 0, B(p; r) \cap Y \neq \varnothing$
- $Y \subseteq \overline{Y}$ always
- If $Y = \overline{Y}$, Y is said to be closed (i.e: $\overline{Y} \subseteq Y$).
- Y is closed $\Leftrightarrow X \backslash Y$ is open. That is, a set is closed iff its complement is open.
- An arbitrary intersection of closed sets is closed; the union of **finitely** many closed sets is closed.

(IV) Sequences and Closures

- A sequence (x_n) in a metric space (X, d) is said to converge to a point x_0 in X, denoted $x_n \to x_0$, iff:

$$\lim_{n \to \infty} d(x_n, x_0) = 0$$

- Equivalently: Every $\epsilon-$ball centered at x_0 contains all but finitely many $(n > N)$ elements of the sequence x_n.
- Generalizing open balls centered at x_0 to open neighborhoods (any open set containing x_0) of x_0 (denoted U_{x_0}), $x_n \to x_0$ means: Every U_{x_0} contains all but finitely many elements of the sequence x_n.
- **Sequential Closure Criterion**:

$$x \in \overline{Y} \Leftrightarrow \text{There exists a sequence: } (x_n)_{n=1}^{\infty} \text{ in } Y \text{ such that } x_n \to x$$

- **Sequential Characterization of Closed Sets**
A is closed $\Leftrightarrow$ If (x_n) is a convergent sequence in A, then $\lim_{n \to \infty} x_n \in A$. That is, any convergent sequence contained in a closed set will find its limit inside this closed set.
- Equivalent metrics on a set X are defined to determine the same open sets in X, whereby they determine the same closed sets in X also. Thus, equivalent metrics will also determine the same convergent sequences.
- A point $p \in X$ is said to be a limit point of $S \subseteq X$ iff every $B(p; \varepsilon)$ contains infinitely many points of S. Another formulation is: $\forall \varepsilon > 0, B(p; \varepsilon)/\{p\} \cap S \neq \varnothing$. p is a limit point of S Iff there exists a sequence in S converging to p: $\{s_n\} \subseteq S : s_n \to p$ and $s_n \neq p$ for all n.

- $p \in S$ is said to be an isolated point in the set S, if $\exists r > 0 : B(p;r) \cap S = \varnothing$. For example: If $I = [0,1] \cup \{2\} \cup [3,4]$, then 2 is an isolated point (wrt. the Euclidean metric) in I.
- It can be shown that the closure of S is the disjoint union of its isolated points and its limits points.

(V) Boundaries