(I) Open covers
Let (X, d) be a metric space, and let

$$\{U_\alpha\}_{\alpha \in \tau} \subseteq X$$

be an arbitrary collection of sets in X. The collection

$$\{U_\alpha\}_{\alpha \in \tau} \subseteq X$$

is said to **cover** X iff:

$$X \subseteq \bigcup_{\alpha \in \tau} U_\alpha$$

Additionally, if each U_α is open, then the collection $\{U_\alpha\}_{\alpha \in \tau}$ is called an **open cover** of X.

- Let $x \in X$. Then, the collection of open balls: $\{B(x; r) \mid r > 0\}$ is an open cover of X. This is an **infinite** collection because there are infinitely many positive real numbers $(r > 0)$. Further, for any $y \in X$, we can choose any $r > d(x, y)$ so that $y \in B(x; r) \Rightarrow y \in \bigcup_{r>0} B(x; r)$, so the collection covers X.

- Consider $\{B(x; r) \mid x \in X\}$. We can choose to fix the same r for all the x's in X, or vary its value; regardless, the collection is always an open cover of X. If X is finite, then this cover is also finite; else, both X and the open cover are infinite.

(II) Compactness
A set Y in a metric space (X, d) is said to be compact iff every open cover of Y admits a finite subcover.

A metric space (X, d) is **totally bounded** iff:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, x_1, ..., x_N \in X,$$

such that:

$$X \subseteq \bigcup_{i=1}^{N} B(x_i, \varepsilon)$$

That is, **for every positive radius** $r > 0$ there exists a **finite** collection of open balls of radius r inside X that **covers** X.

Next we will see how compactness (via open covers) relates to the existence of convergent subsequences. First, we cover an important lemma and refresher on subsequences.
Let $\{y_n\}$ be any sequence in a metric space (X, d). A subsequence $\{y_{n_k}\}$ of $\{y_n\}$ is an ordered selection of its terms, indexed in k. The terms can be selected via a **strictly increasing** index selector function $\sigma : \mathbb{N} \to \mathbb{N}$ which satisfies: $\sigma(k+1) > \sigma(k)$. We denote the $k-$th term chosen by σ as: $\sigma(k) = n_k$. Clearly,

$n_k \geq k \quad \forall k \in \mathbb{N}$. For example:
Given a sequence (indexed in n) : $y_1, \ y_2, \ y_3, \ y_4, \ y_5, \ y_6, ...$, we could select a subsequence by choosing only even-indexed terms of $\{y_n\}$. Here, $\sigma(k) = 2k = n_k$, so our subsequence is: $y_2, y_4, y_6 ...$, where $n_1 = 2, n_2 = 4, ..., n_k = 2k$ chosen from n.
We say the subsequence converges to a point y iff: $\lim y_{n_k} = y$ as $k \to \infty$.
Certainly, if a sequence converges to a point, then any subsequence chosen from it will also converge to the same point (seen from $n_k \geq k$). If *all* the subsequences of a sequence converge to a given point, then the sequence also converges to that point.
There may be divergent sequences which have several convergent subsequences, but not all the subsequential limits are the same (Eg: $x_n = (-1)^n$).

Lemma:
(i) Every nbhd of y contains **infinitely many** points of the sequence $\{y_n\} \Leftrightarrow$
(ii) There exists a subsequence of $\{y_n\}$ converging to y

Proof: First, translating (i) into quantifiers tells us:

$$\forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N : d(y_n, y) < \varepsilon$$

This definition easily reveals (ii) $\implies$ (i). To show (i) $\implies$ (ii), we'll build a subsequence $\{y_{n_k}\}$ converging to y using (i).

(Base Case) When $k = 1$, let $\varepsilon_1 = 1$ and $N_1 = 1$. Then $\exists n_1 > 1$ such that $d(y_{n_1}, y) < 1$.
(Recursive Step) At each $k-$th step where $k > 1$, set $(\varepsilon_k = \frac{1}{k}, \ N_k = n_{k-1})$. By (i), there exists $n_k > n_{k-1}$ such that $d(y_{n_k} y) < \frac{1}{k}$.

Thus we find indices: $n_k > n_{k-1} > ... > n_1 > 1$ such that $d(y_{n_k} y) < \frac{1}{k}$, whereby

$$\lim_{k \to \infty} y_{n_k} = y$$

Note the critical distinction between "*all but finitely many* terms..." and "*infinitely* many terms..." of a sequence in any nbhd of y. The former guarantees convergence of the sequence, while the latter only ensures the existence of a convergent subsequence.

Compactness has important consequences for boundedness and subsequential convergence.

Theorem TFAE:

- (1) X is compact
- (2) Every sequence in X has a convergent subsequence
- (3) X is complete and totally bounded

We will prove the forward directions (1) $\implies$ (2) $\implies$ (3) for now.

Proof (1) $\implies$ (2): Let X be compact; let $\{y_j\}_{j=1}^{\infty}$ be a sequence in X. Suppose for contradiction that $\{y_j\}_{j=1}^{\infty}$ has no subsequence converging to a point in X. Then:
$\forall x \in X, \exists \varepsilon(x) > 0$ such that: **only finitely many** terms y_j fall in $B(x; \varepsilon(x))$.

Clearly, $X \subseteq \bigcup_{x \in X} B(x; \varepsilon(x))$, so the collection $\{B(x, \varepsilon(x)) : x \in X\}$ is an open cover of X. By compactness, there exist **finitely** many points $(x_i)_{i=1}^{n}$ such that:

$$\{y_j\}_{j=1}^{\infty} \subset X \subseteq \bigcup_{k=1}^{n} B(x_k; \varepsilon(x_k))$$

But each ball $B(x; \varepsilon(x))$ contains only finitely many terms/indices of $\{y_j\}$, and there are only finitely many such balls. This implies there are only finitely many natural numbers (used to index an infinite sequence $\{y_j\}$, a contradiction.

Thus, our first assumption was wrong - **there must exist some** $x \in X$ such that:

$$\forall \varepsilon > 0, \text{ infinitely many terms of } y_j \text{ fall in } B(x; \varepsilon(x)).$$

By our Lemma, this implies that $\{y_j\}$ has a subsequence converging to x.

(2) $\implies$ (3): First, we show that X is complete. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. By (2), it has a convergent subsequence - whereby the entire sequence converges to the limit of this subsequence. Thus, X is complete.

Let $\varepsilon > 0$ be arbitrary. We want to show that X is **totally bounded**- that it can be covered by **finitely many** open balls centered at points $\{x_1, ..., x_n\}$ in X. We find these points iteratively as follows:

- **Step-1 :** Select any $x_1 \in X$.
  If $X \nsubseteq B(x_1; \varepsilon)$, continue to Step-2.
  Else, STOP. We then have finitely many $(N = 1)$ balls covering X.
- **Step-2 :** Select any $x_2 \in X \setminus B(x_1; \varepsilon)$.
  Note that $d(x_2, x_1) \geq \varepsilon$. If $X \nsubseteq B(x_2; \varepsilon) \bigcup B(x_1; \varepsilon)$, continue to Step-3.
  Else, stop and we have a finite cover.

  $\vdots$
- **Step-n :** At this stage, we have $X \nsubseteq \bigcup_{k=1}^{n-1} B(x_k, \varepsilon)$, so we find:
  $x_n \in X \setminus \bigcup_{k=1}^{n-1} B(x_k, \varepsilon)$; By Demorgan's Law,

  $x_n \in \bigcap_{k=1}^{n-1}(X \setminus B(x_k; \varepsilon)) \Leftrightarrow d(x_n, x_k) \geq \varepsilon$ for $1 \leq k < n$.

  If $X = \bigcup_{k=1}^{n} B(x_k; \varepsilon)$, stop and we have a fintite cover. Else, continue to step $(n + 1)$.

If this process does not end, we will have formed an infinite sequence $\{x_n\}_{n=1}^{\infty}$ in X such that:

$$\exists \varepsilon > 0 \text{ such that } \forall (n,k) : n > k \geq 1,\ d(x_n, x_k) \geq \varepsilon$$

We claim this implies that (x_n) cannot have a convergent subsequence. Suppose for contradiction that there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{n}$; then $\{x_{n_k}\}_{k=1}^{n}$ is Cauchy, whereby:

$$\forall \varepsilon * > 0, \exists N : \forall (k,l) > N, d(x_{n_k}, x_{n_l}) < \varepsilon *$$

But for $\varepsilon * = \varepsilon$, and for any N, by construction we have:

$$1 \leq n_l < n_k \Rightarrow d(x_{n_k}, x_{n_l}) \geq \varepsilon.$$

Thus, the process must terminate, which provides a finite cover.