D1| A metric space (X, d) is **separable** iff there exists a **countable dense subset** of X : $U \subseteq X$ where $\overline{U} = X$ and $|U| = |\mathbb{N}|$.
Equivalently, X is separable iff there is **a dense sequence :** $\{x_j\}_{j=1}^{\infty}$ **in** X (repition of x_j's are allowed - in this case, the set $U = \{x_j : j \in \mathbb{N}\}$ is finite, so it is certainly countable; we don't *need* infinitely many points).

Example: The set $\mathbb{Q}$ of rationals in $\mathbb{R}$ is countable and dense in $\mathbb{R}$ under the standard metric, whereby $\mathbb{R}$ is separable. Under the standard metric, $\overline{(\mathbb{Q} \times \mathbb{Q})} = \overline{\mathbb{Q}} \times \overline{\mathbb{Q}} = \mathbb{R} \times \mathbb{R}$ *(refer to products of metric spaces: (1.4))*, and a finite product of countable sets is countable, whereby $\mathbb{R}^2$ is separable. Similarly $\mathbb{R}^n$ is separable for any fixed n, by the dense countable set $\mathbb{Q}^n$.

Theorem 1 A subspace of a separable metric space is separable.

Proof: Let Y be a subspace of the separable metric space X, so $Y \subseteq X = \overline{\{x_j\}}$ for some dense sequence $\{x_j\}$ in X. We want a sequence in Y that's dense in Y.

- (1) Since $Y \subseteq \overline{\{x_j\}}$ we have: $\forall y \in Y, \forall n \in \mathbb{N} : B(y; \frac{1}{n}) \cap \{x_j\} \neq \varnothing$.
- (2) Then to each $n \in \mathbb{N}, y \in Y$, **there is an index** j so that $x_j \in B(y; \frac{1}{n}) \implies d(y, x_j) < \frac{1}{n}$.
- (3) Now we collect all the indices j corresponding to n (the pairs of indices (n, j)) so that $B(x_j; \frac{1}{n}) \cap Y \neq \varnothing$ and let $y_{nj} \in Y$ fall into this intersection. Thus, form the sequence $\{y_{nj}\} \subseteq Y$ (countably many j to each n).
- (4) Pick any $y' \in Y$. By (2): for each $n \in \mathbb{N}$, there is a j so that $d(x_j, y') < \frac{1}{n}$ and by (3), y_{nj} is chosen to satisfy $d(y_{nj}, x_j) < \frac{1}{n}$.
- (5) Then: for each (n, j) pair, $d(y', y_{nj}) < d(y', x_j) + d(x_j, y_{nj}) < \frac{2}{n}$. Letting n grow large ($n \to \infty$), for a selection of $j-$s, we find a subsequence of $\{y_{nj}\}$ converging to y'.
- Thus we found a (sub)sequence $\{y_{nj}\} \subseteq Y$ converging to y', whereby $y' \in \overline{\{y_{nj}\}}$.
- y' was arbitrary, so $Y \subseteq \overline{\{y_{nj}\}}$. Thus, Y is separable.

Theorem 2 A totally bounded metric space is separable.
Proof Let (X, d) be totally bounded. Then for each $n \in \mathbb{N}$, there are finitely (m_n) many balls of radius $\frac{1}{n}$ covering X, centered at the points: $\{x_{n1}, ..., x_{nm_n}\} = \{x_{nj}\}_{j=1}^{m_n}$ (*).
Then, the set of points: $\{x_{nj}\} = \{x_{nj} : n \in \mathbb{N}, j \in [m_n]\}$ is a countable subset of X (finitely many j to each n). Denote the set of all $j-$ indices (across all n) by J.
Let $x \in X$. For each $n \in \mathbb{N}$, there are j so that $d(x_{nj}, x) < \frac{1}{n}$ (by *). Thus there is a subset $J' \subseteq J$ so that the points $\{x_{nj} : j \in J', n \in \mathbb{N}\}$ converge to x, whereby $x \in \overline{\{x_{nj}\}}$. Then $X \subseteq \overline{\{x_{nj}\}}$; thus X is separable.