(1.5) - II : Compactness in $\mathbb{R}^n$: The Heine-Borel Theorem

In the previous notes (Section **(1.5) - I**), we established that TFAE for any metric space (X, d):

- (1) X is compact
- (2) Every sequence in X has a convergent subsequence
- (3) X is totally bounded and complete

Now we turn our attention to $\mathbb{R}^n$ equipped with the standard Euclidean metric, which offers a convenient and simpler equivalent formulation of (3) :

$$\mathbb{R}^n \text{ is totally bounded and complete} \Leftrightarrow \mathbb{R}^n \text{ is bounded and closed}$$

Whereby **compact subsets in $\mathbb{R}^n$ are characterized by being closed and bounded**.
We already established that **a subset of $\mathbb{R}^n$ is complete iff it is closed** (Section (1.2).

We will now show that total boundedness is equivalent to boundedness in $\mathbb{R}^n$.
Note: We sometimes abbreviate "totally bounded" as "TB".

Theorem: Let $E \subset \mathbb{R}^n$. Then , (i) E is bounded iff (ii) E is totally bounded.

Proof
1.) (ii) $\implies$ (i) , i.e. (TB $\implies$ B): This direction holds for *all* metric spaces. We'll break the proof into 2 pieces.

P1) Any totally bounded metric space is bounded.
Proof: Let (X, d) be totally bounded. For $\varepsilon = 1$, find n points: $x_1, ..., x_n \in X$ so that $X \subseteq \bigcup_{k=1}^{n} B(x_k; 1)$. Pick any x, y in $X-$ then $x \in B(x_j; 1), y \in B(x_i, 1)$ for some $i, j \in [n]$. Set $M = \max\{d(x_l, x_p) : l, p \in [n]\}$. By the triangle inequality, $d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < 1 + M + 1 = M + 2$. Let $b = 2 + M$ to see that $d(x, y) < b \ \ \forall x, y \in X$.

P2) Any **subspace** of a totally bounded metric space is totally bounded.
Proof:

- Let Y be a subspace of a TB space X, and let $\varepsilon > 0$. Since X is TB, we can find $x_1, ..., x_n \in X$ so that $Y \subseteq X \subseteq \bigcup_{k=1}^{n} B(x_k; \varepsilon/2)$.
- Now, throw away all the balls in this cover of X that don't intersect Y: re-order the indices $k \in [n]$ and find an $m \in [n]$ so that $Y \subseteq \bigcup_{k=1}^{m} B(x_k; \varepsilon/2)$ and $Y \cap B(x_k; \varepsilon/2) = \varnothing$.
- From the m balls that cover Y, choose any $j-$th ball (where $j \in [m]$). Pick any $y_j \in B(x_j; \varepsilon/2)$. We claim: $B(x_j; \varepsilon/2) \subset B(y_j, \varepsilon)$, seen by: $z \in B(x_j; \varepsilon/2) \implies d(z, x_j) < \varepsilon/2$ and $d(y_j, x_j) < \varepsilon/2$, so $d(z, y_j) < d(z, x_j) + d(x_j, y_j) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

- Thus: $Y \subseteq \bigcup_{k=1}^{m} B(x_k; \varepsilon/2) \subset \bigcup_{k=1}^{m} B(y_k; \varepsilon/2)$, where the balls of radius ε with centers $y_1, ..., y_m \in Y$ cover Y.

- Restrict each of these m balls to the subspace Y (setting $B(y_j, \varepsilon)^Y = B(y_j; \varepsilon) \cap Y$), to obtain the finite $\varepsilon-$cover for Y. Thus, the subspace Y is TB in itself.

Now we can finish our proof of (TB) $\implies$ (B). Let $E \subseteq \mathbb{R}^n$, where E is totally bounded in $\mathbb{R}^n$. Invoking the proof of **(P1)**, E must be bounded.

2.) (i) $\implies$ (ii), or B $\implies$ TB in $\mathbb{R}^n$. This direction is special to $\mathbb{R}^n$. We first establish two simple lemmas required for this proof.

L1: Any open ball in $\mathbb{R}^n$ can be contained inside a closed cube.
Proof:

- Let $B(a; R)$ be an open ball in $\mathbb{R}^n$, and $x \in B(a; R)$; then $||x - a|| < R$.

- For each $i-$th component we have: $|x_i - a_i| \leq ||x - a|| < R$; By the reverse triangle inequality, $|x_i| - |a_i| \leq |x_i - a_i| < R \implies |x_i| < R + |a_i|$ for each $i \in [1:n]$.

- Let $M = R+ \max\{|a_i| : 1 \leq i \leq n\}$. Then, $|x_i| \leq M$ for each $i \in [1:n]$.

- Thus, for any $x = (x_1, ..., x_n)$ in $B(a; R)$,

$$|x_i| \leq M \;\; \forall i \in [1:n] \implies x \in [-M, M]^n$$

- Therefore, $B(a; R) \subset [-M, M]^n$, a closed cube in $\mathbb{R}^n$.

L2: $E \subset \mathbb{R}^n$ is bounded $\Leftrightarrow \exists R > 0, \exists a \in \mathbb{R}^n : E \subseteq B(a; R)$
Proof:
First, suppose that E is bounded. This means:

$$\exists M > 0 : \forall x, \forall y \in E, \;\; ||x - y|| \leq M$$

Suppose E *cannot* be contained in any open ball in $\mathbb{R}^n$. Then:

$$\forall a \in \mathbb{R}^n, \forall R > 0, \exists z \in E : ||a - z|| \geq R$$

Choose any $x \in E \subset \mathbb{R}^n$. Then for $R = M > 0$, $\exists y \in E : ||x - y|| \geq M$, which directly contradicts E being bounded. Thus, any bounded set can be contained in an open/closed ball.

Next, suppose $E \subset B(a; R)$ for some $a \in \mathbb{R}^n, R > 0$. Then, $\forall x, y \in E$:

$$||x - y|| = ||x - a + a - y|| \leq ||x - a|| + ||y - a|| < R + R = 2R$$

Thus, E is bounded as per the definition above with $M = 2R$.

These two results tell us that if $E \subset \mathbb{R}^n$ is bounded, then E can be contained in a ball of some suitably large radius R **(L2)**, and this ball can be contained in a closed cube **(L1)**. We now proceed to show that E is bounded $\implies$ E is TB.

- Let $E \subset \mathbb{R}^n$ be contained in the closed cube $[-b, b]^n$ for some large $b > 0$.
- Inside this cube $[-b, b]^n$, there are **finitely many integer lattice points**. (p is an integer lattice point in $[-b, b]^n$ if $p = (z_1, ..., z_n)$ where $-b \leq z_i \leq b$ for each $i - th$ component $i \in [n]$. Since there are finitely many (say, t) integers in the interval $[-b, b]$, the number of integer lattice points in our cube is just t^n.)
- Let $\varepsilon > 0$. Consider the collection of open balls with radius ε centered at each of the **lattice points** in the set $(\frac{\varepsilon}{2}\mathbb{Z})^n \cap [-b, b]^n$. Since there are finitely many *integer* lattice points in $(\mathbb{Z}^n \cap [-b, b]^n)$, and our $\frac{\varepsilon}{2}-$**scaled lattice points** are in bijective correspondence with these integer lattice points, the set $(\frac{\varepsilon}{2}\mathbb{Z})^n \cap [-b, b]^n$ is finite. This provides a finite $\varepsilon-$cover for E, whereby E is TB.

To summarize: So far, we have shown:

- (1) X is compact $\implies$ (2) Every sequence in X has a convergent subsequence $\implies$ (3) X is totally bounded and complete.
- The converse directions (3) $\implies$ (2) and (2) $\implies$ (1) required for our main theorem have NOT been shown yet!!
- In this note we showed that in $\mathbb{R}^n$, TB $\Leftrightarrow$ B and complete $\Leftrightarrow$ closed.

In the next note, we will show that (3) $\implies$ (2) (if X is totally bounded and complete, every sequence in X has a subsequence converging in X) through a classic diagonalization trick, and (2) $\implies$ (1).
