**(I) Open covers**
Let $(X, d)$ be a metric space, and let

$$\{U_\alpha\}_{\alpha\in\tau} \ \subseteq X$$

be an arbitrary collection of sets in $X$. The collection

$$\{U_\alpha\}_{\alpha\in\tau} \subseteq X$$

is said to **cover** $X$ iff:

$$X \subseteq \bigcup_{\alpha\in\tau} U_\alpha$$

Additionally, if each $U_\alpha$ is open, then the collection $\{U_\alpha\}_{\alpha\in\tau}$ is called an **open cover** of $X$.

- Let $x \in X$. Then, the collection of open balls: $\{B(x; r) \mid r > 0\}$ is an open cover of $X$. This is an **infinite** collection because there are infinitely many positive real numbers $(r > 0)$. Further, for any $y \in X$, we can choose any $r > d(x, y)$ so that $y \in B(x; r) \Rightarrow y \in \bigcup_{r>0} B(x; r)$, so the collection covers $X$.

- Consider $\{B(x; r) \mid x \in X\}$. We can choose to fix the same $r$ for all the $x'$s in $X$, or vary its value; regardless, the collection is always an open cover of $X$. If $X$ is finite, then this cover is also finite; else, both $X$ and the open cover are infinite.

**(II) Compactness**
A set $Y$ in a metric space $(X, d)$ is said to be compact iff every open cover of $Y$ admits a finite subcover - that is:
If $\{U_\alpha\}_{\alpha\in\tau} \ \subseteq X$ is an open cover of $Y$, there exist **finitely many** (say, $n$) indices $\alpha \in \tau : \alpha_1, ..., \alpha_n$ such that $\{U_{\alpha_k} : k \in [1 : n]\}$ covers $Y$.

A metric space $(X, d)$ is **totally bounded** iff:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, x_1, ..., x_N \in X,$$

such that:

$$X \subseteq \bigcup_{i=1}^{N} B(x_i, \varepsilon)$$

That is, **for every positive radius** $r > 0$ there exists a **finite** collection of open balls of radius $r$ inside $X$ that **covers** $X$.

Next, we will see how compactness (via open covers) relates to the existence of convergent subsequences. First, we cover an important lemma and review subsequences.
Let $\{y_n\}$ be any sequence in a metric space $(X, d)$. A subsequence $\{y_{n_k}\}$ of $\{y_n\}$ is an ordered selection of its terms, indexed in $k$. The terms can be selected

via a **strictly increasing** index selector function $\sigma : \mathbb{N} \to \mathbb{N}$ which satisfies: $\sigma(k+1) > \sigma(k)$. We denote the $k-$th term chosen by $\sigma$ as: $\sigma(k) = n_k$. Clearly, $n_k \geq k \quad \forall k \in \mathbb{N}$. For example:
Given a sequence (indexed in $n$) : $y_1, \ \ y_2, \ \ y_3, \ \ y_4, \ \ y_5, \ \ y_6, ...$, we could select a subsequence by choosing only even-indexed terms of $\{y_n\}$. Here, $\sigma(k) = 2k = n_k$, so our subsequence is: $y_2, y_4, y_6 ...$, where $n_1 = 2, n_2 = 4, ..., n_k = 2k$ chosen from $n$.
We say the subsequence converges to a point $y$ iff: $\lim y_{n_k} = y$ as $k \to \infty$.
Certainly, if a sequence converges to a point, then any subsequence chosen from it will also converge to the same point (seen from $n_k \geq k$). If *all* the subsequences of a sequence converge to a given point, then the sequence also converges to that point.
There may be divergent sequences which have several convergent subsequences, but not all the subsequential limits are the same ( Eg: $x_n = (-1)^n$).

**Lemma:**
(i) Every nbhd of $y$ contains **infinitely many** points of the sequence $\{y_n\} \Leftrightarrow$
(ii) There exists a subsequence of $\{y_n\}$ converging to $y$

**Proof:** First, translating (i) into quantifiers tells us:

$$\forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N : d(y_n, y) < \varepsilon$$

This definition easily reveals (ii) $\implies$ (i). To show (i) $\implies$ (ii), we'll build a subsequence $\{y_{n_k}\}$ converging to $y$ using (i).

(Base Case) When $k = 1$, let $\varepsilon_1 = 1$ and $N_1 = 1$. Then $\exists n_1 > 1$ such that $d(y_{n_1}, y) < 1$.
(Recursive Step) At each $k-$th step where $k > 1$, set $(\varepsilon_k = \frac{1}{k}, \ \ N_k = n_{k-1})$. By (i), there exists $n_k > n_{k-1}$ such that $d(y_{n_k} y) < \frac{1}{k}$.

Thus we find indices: $n_k > n_{k-1} > ... > n_1 > 1$ such that $d(y_{n_k} y) < \frac{1}{k}$ , whereby

$$\lim_{k \to \infty} y_{n_k} = y$$

Note the critical distinction between "*all but finitely many* terms..." and "*infinitely* many terms..." of a sequence in any nbhd of $y$. The former guarantees convergence of the sequence, while the latter only ensures the existence of a convergent subsequence.

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Compactness has important consequences for boundedness and subsequential convergence.

**Theorem** TFAE:

- (1) $X$ is compact
- (2) Every sequence in $X$ has a convergent subsequence

- (3) $X$ is complete and totally bounded

We will prove the forward directions (1) $\implies$ (2) $\implies$ (3) for now.

**Proof (1) $\implies$ (2)**: Let $X$ be compact; let $\{y_j\}_{j=1}^{\infty}$ be a sequence in $X$. Suppose for contradiction that $\{y_j\}_{j=1}^{\infty}$ has no subsequence converging to a point in $X$. Then:
$\forall x \in X, \exists \varepsilon(x) > 0$ such that: **only finitely many** terms $y_j$ fall into $B(x;\varepsilon(x))$.

Clearly, $X \subseteq \bigcup_{x\in X} B(x;\varepsilon(x))$, so the collection $\{B(x,\varepsilon(x)) : x \in X\}$ is an open cover of $X$. By compactness, there exist **finitely** many points $(x_i)_{i=1}^{n}$ such that:

$$\{y_j\}_{j=1}^{\infty} \subset X \subseteq \bigcup_{k=1}^{n} B(x_k;\varepsilon(x_k))$$

But each ball $B(x;\varepsilon(x))$ contains only finitely many terms/indices of $\{y_j\}$, and there are only finitely many such balls. This implies that there are only finitely many natural numbers (used to index an infinite sequence $\{y_j\}$, a contradiction.

Thus, our first assumption was wrong **- there must exist some** $x \in X$ such that:

$$\forall \varepsilon > 0, \text{ infinitely many terms of } y_j \text{ fall in } B(x;\varepsilon(x)).$$

By our lemma, this implies that $\{y_j\}$ has a subsequence converging to $x$.

**(2) $\implies$ (3):** First, we show that $X$ is complete. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $X$. By (2), it has a convergent subsequence, whereby the entire sequence converges to the limit of this subsequence. Thus, $X$ is complete.

Let $\varepsilon > 0$ be arbitrary. We want to show that $X$ is **totally bounded-** that it can be covered by **finitely many** open balls centered at points $\{x_1, ..., x_n\}$ in $X$. We find these points iteratively as follows:

- **Step-1 :** Select any $x_1 \in X$.
  **If** $X \nsubseteq B(x_1;\varepsilon)$, continue to Step-2.
  Else, STOP. We then have finitely many ($N = 1$) balls that cover $X$.

- **Step-2 :** Select any $x_2 \in X \setminus B(x_1;\varepsilon)$.
  Note that $d(x_2, x_1) \geq \varepsilon$. If $X \nsubseteq B(x_2;\varepsilon)\bigcup B(x_1;\varepsilon)$, continue to Step-3.
  Else, stop and we have a finite cover.

  $\vdots$

- **Step-n :** At this stage, we have $X \nsubseteq \bigcup_{k=1}^{n-1} B(x_k,\varepsilon)$, so we find:
  $x_n \in X \setminus \bigcup_{k=1}^{n-1} B(x_k,\varepsilon)$; By De Morgan's Law,

  $x_n \in \bigcap_{k=1}^{n-1}(X \setminus B(x_k;\varepsilon)) \Leftrightarrow d(x_n, x_k) \geq \varepsilon$ for $1 \leq k < n$.

If $X = \bigcup_{k=1}^{n} B(x_k;\varepsilon)$, stop and we have a finite cover. Else, continue to step $(n+1)$.

If this process does not end, we will have formed an infinite sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ satisfying:

$$\exists\varepsilon > 0 \text{ such that } \forall (n,k) : n > k \geq 1,\, d(x_n, x_k) \geq \varepsilon$$

We claim this implies that $(x_n)$ cannot have a convergent subsequence. Suppose there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{n}$; then $\{x_{n_k}\}_{k=1}^{n}$ is Cauchy, whereby:

$$\forall \varepsilon* > 0, \exists N : \forall (k,l) > N, d(x_{n_k}, x_{n_l}) < \varepsilon*$$

But for $\varepsilon* = \varepsilon$, and for any $N$, by construction we have:

$$1 \leq n_l < n_k \Rightarrow d(x_{n_k}, x_{n_l}) \geq \varepsilon.$$

This yields a contradiction. Thus, the process must terminate, which provides a finite cover.