Cauchy Sequences
Let (X, d) be a metric space. The sequence $\{x_n\}$ is Cauchy iff:

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0$$

That is:

$$\forall\varepsilon > 0, \exists N : (m, n) > N \Rightarrow d(x_m, x_n) < \varepsilon$$

It is easily seen that every convergent sequence is Cauchy. (X, d) is said to be complete iff every Cauchy sequence converges **to a point in** X. Thus, in a **complete** metric space: (x_n) is Cauchy $\Leftrightarrow$ (x_n) is convergent.

Theorem 2.2 If $\{x_n\}$ is a Cauchy sequence, and there exists a convergent subsequence $\{x_{n_k}\}$ of this sequence: $x_{n_k} \to x$ as $k \to \infty$, then the whole sequence converges to x.
Proof

- Let $\varepsilon > 0$. Since $x_{n_k} \to x$,

$$\exists N_1 : k > N \Rightarrow d(x_{n_k}, x) < \frac{\varepsilon}{2}$$

- Note that $n_k \geq k$ always. Since $\{x_n\}$ is Cauchy,

$$\exists N_2 : n_k \geq k > N_2 \Rightarrow d(x_{n_k}, x_k) < \frac{\varepsilon}{2}$$

- Then, for $n > \max\{N_1, N_2\}$, we have:

$$d(x_k, x) \leq d(x_k, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$$

  whereby $x_k \to x$ as $k \to \infty$.

Theorem 2.3 A closed subspace of a complete metric space is complete.
Proof
Let (Y, d') is a subspace of the metric space (X, d). Let $\{x_n\} \subseteq Y \subseteq X$ be a Cauchy sequence in this subspace. Since X is complete, x_n converges to a point in X.
Since Y is closed, every convergent sequence in Y finds its limit in Y *(Sequential Characterization of the Closure)*. Thus, every Cauchy sequence converges to a point in Y, whereby Y is complete.

Theorem 2.4 A complete subspace Y of a metric space X is closed.
Proof
Let $\{x_n\}$ be a convergent sequence in Y. Since convergent sequences are Cauchy, $\{x_n\}$ is Cauchy, whereby it converges to a point in Y (since Y is complete). Thus, every convergent sequence in Y finds its limit in Y.