11) Two metrics are equivalent on X if they determine the same open sets in X. Show that two metrics are equivalent iff they determine the same convergent sequences.

<u>Proof)</u>

First, let's understand what the "open set" definition of equivalent metrics means.
Let X be a set, and p: $(X \times X) \rightarrow R$, d:$(X \times X) \rightarrow R$ are two metrics on X. Then:

1) <u>p and d are equivalent</u> := { [The set $U \subseteq X$ is open wrt. p] ⇔ [$U \subseteq X$ is open wrt. d] }

And by open with respect to a metric (p or d), we mean:

A set $S \subseteq X$ is **open with respect to** d if:

$$\forall x \in S,\ \exists \varepsilon > 0 \text{ such that } B^d(x;\varepsilon) \subseteq S$$

And similarly, S is **open with respect to** p if:

$$\forall x \in S,\ \exists \delta > 0 \text{ such that } B^p(x;\delta) \subseteq S$$

2) Revisit the definition of convergence: We write $x_n \rightarrow x$ wrt. (d) iff:
Every open neighbourhood (open wrt. d) of x : U_x contains all but finitely many points of x_n.
That is : For any d–open U_x, all but finitely many x_n lie in U_x ⇔ $x_n \rightarrow x$ wrt. (d)

Now, let's restate the question:
Show that:
(i) p and d determine the same open sets ⇔ (ii) p and d determine the same convergent sequences

<u>Part-1) (i) ⇒ (ii)</u> : Suppose (p and d) are equivalent, in that they determine the same open sets.
Let the sequence $\{x_n\}$ in X converge to a point x wrt. the metric p.

- We want to show: $\{x_n\}$ converges to x wrt. the metric d also.
- <u>So, we WTS:</u> For any d-open nbhd of x : U_x, all but finitely many x_n are contained in U_x.

Proof:

- Let U_x be ***any*** d–open nbhd of x (" d–open" := *open with respect to the metric d*).
- Then, by the equivalence of p and d, U_x is also open wrt. the metric p.
- Since $x_n \rightarrow x$ wrt p, and U_x is p-open, all but finitely many x_n are contained in U_x.
- Thus: all but finitely many x_n are contained in U_x, for any d-open nbhd U_x
- Thus: $x_n \rightarrow x$ wrt. d.
- To really finish the proof, we'd also need to show the direction: [x_n converges wrt. d ⇒ x_n converges wrt. p]; this is accomplished in exactly the same manner as this proof.

<u>Part–2) (ii) ⇒ (i)</u>

Suppose p and d determine the same convergent sequences. Show that they determine the same open sets. First we note that:

(S is closed wrt p) ⇔ (S^c is open wrt. p) ⇔ (S^c is open wrt. d) ⇔ (S is closed wrt. d)
That is, (the metrics p and d determine the same open sets) ⇔ (p and d determine the same closed sets)

So we will show:

(p and d determine the same closed sets) ⇒ (p and d determine the same convergent sequences)

Also recall the *__Sequential Characterization of the Closure__:

(i) [S is closed wrt. p] ⇔ (ii) [Any convergent sequence in S finds its limit in S]

Proof:

- Let S be closed wrt. p
- Take **any** convergent (wrt. p) sequence inside S : $\{x_n\}$, where $x_n \to$ x
- By [* (i) ⇒ (ii)] , x ∈ S
- Now, by assumption, the sequence $\{x_n\}$ is also convergent wrt. d: $x_n \to$ x wrt d.
- Thus any convergent (wrt.d) sequence in S finds its limit in S. Hence [By *(ii) ⇒ (i)] , S is closed wrt. d

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