D1| A metric space (X,d) is **separable** iff there exists a **countable dense subset** of X : $U \subseteq X$ where $\overline{U} = X$ and $|U| = |\mathbb{N}|$.
Equivalently, X is separable iff there is **a dense sequence :** $\{x_j\}_{j=1}^{\infty}$ **in** X (repition of x_j's are allowed - in this case, the set $U = \{x_j : j \in \mathbb{N}\}$ is finite, so it is certainly countable; we don't *need* infinitely many points).

Example: The set $\mathbb{Q}$ of rationals in $\mathbb{R}$ is countable and dense in $\mathbb{R}$ under the standard metric, whereby $\mathbb{R}$ is separable. Under the standard metric, $\overline{(\mathbb{Q} \times \mathbb{Q})} = \overline{\mathbb{Q}} \times \overline{\mathbb{Q}} = \mathbb{R} \times \mathbb{R}$ *(refer to products of metric spaces: (1.4))*, and a finite product of countable sets is countable, whereby $\mathbb{R}^2$ is separable. Similarly $\mathbb{R}^n$ is separable for any fixed n, by the dense countable set $\mathbb{Q}^n$.

Recall that $x \in \overline{A} \iff \forall n \in \mathbb{N}, B(x; \frac{1}{n}) \cap A \neq \varnothing$;
Equivalently: $\exists a_n \in A : d(a_n, x) < \frac{1}{n}$.
Another useful characterization of density: **the set** $A \subset X$ **is dense in** X **if**:
Every **nonempty open subset** of X **intersects** A $(\forall x \in X, \varepsilon > 0, \exists a \in A : a \in B(x; \varepsilon))$.

Theorem-1 A subspace of a separable space is separable.
Proof

- Let (X,d) be separable and $Y \subseteq X$ where $(Y, d|_Y)$ is a subspace of X. Let $(x_j)_{j=1}^{\infty}$ in X such that $\overline{(x_j)} = X$. We need to form a sequence in Y that is dense in Y.

- Since $Y \subseteq \overline{(x_j)}$, any nonempty open subset of Y intersects (x_j):

$$\forall y \in Y, \forall \varepsilon > 0 : B(y; \varepsilon) \cap (x_j) \neq \varnothing$$

$$\implies \forall y \in Y, \forall \varepsilon > 0, \exists j : x_j \in B(y; \varepsilon)$$

$$\text{For } \varepsilon_n = \frac{1}{n}, \ \ \exists j, \ \ \exists y \in Y : y \in B(x_j; \frac{1}{n})$$

- Thus we can find a j to each $n \in \mathbb{N}$ such that $Y \cap B(x_j; \frac{1}{n}) \neq \varnothing$. Note that repetition of js is allowed!

- Collect all indices: $I = \{(n,j) : n \in \mathbb{N}, j_n : B(x_j; \frac{1}{n}) \cap Y \neq \varnothing\}$. For each (n,j), let

$$y_{jn} \in B(x_j; \frac{1}{n}) \cap Y$$

- (*) Then, $y_{jn} \in Y$ and $d(y_{jn}, x_{jn}) < \frac{1}{n}$ for each $(n,j) \in I$

- We claim that $Y \subseteq \overline{y_{nj}}$. Let $y \in Y$ and $\varepsilon > 0$. To prove our claim, we must find a y_{nj} in the ball $B(y; \varepsilon)$.

- First, $y \in \overline{(x_j)}$, whence there is a j such that $d(y, x_j) < \frac{1}{n}$ for any n.

- By (*), y_{nj} satisfies $d(x_j, y_{nj}) < \frac{1}{n}$

- Thus, $d(y, y_{nj}) \leq d(y, x_j) + d(x_j, y_{nj}) < \frac{2}{n}$. Choose $n > \frac{2}{\varepsilon}$, to see:

$$d(y, y_{nj}) < \varepsilon \implies y_{nj} \in B(y; \varepsilon)$$

- Thus, $\overline{(y_{nj})} = Y$, whereby Y is separable.

As an example: any finite subspace of $\mathbb{R}$ is separable, being a countable dense subset of itself. The subspace of irrationals in $\mathbb{R}$ is also separable, despite being uncountably infinite!

Theorem-2 Any TB space is separable.

Proof: Let (X, d) be a TB space. Then for each $n \in \mathbb{N}$, there exist N_n points: $(x_{n1}, ..., x_{nj}, ..., x_{nN_n})$ in X so that:

$$X \subseteq \bigcup_{j=1}^{N_n} B(x_{nj}; \tfrac{1}{n}); \text{ equivalently:}$$

For each $n \in \mathbb{N}$, there exists a j such that $x \in X \implies d(x_{nj}; x) < \frac{1}{n}$.*

Let $A = \{x_{nj} : n \in \mathbb{N} | N_n, j \in [N_n]\}$. We claim that $X \subseteq \overline{A}$.
Let $x \in X$. By *, $\exists x_{nj} \in A$ such that $d(x_{nj}, x) < \frac{1}{n}$. Thus, A is dense in X. Further, there are finitely (N_n) many j's to each n, so the elements of A can be enumerated by the naturals ($|A| = |\mathbb{N}|$). Thus, X is separable.

0.1 A Base of Open Sets

Let (X, d) be a metric space. A base $\mathcal{B}$ for X is a **collection** of open sets such that any **open set** in X can be expressed as a **union of the sets in** $\mathcal{B}$. That is:

$$\mathcal{O} \subset X \text{ is open} \iff \mathcal{O} = \bigcup \{V_l : V_l \in \mathcal{B}, l \in I\}$$

- The collection of open balls in (X, d): $\mathcal{B}^1 = \{B(x; r) : r > 0\}$ is a base for (X, d); to see this, let $\mathcal{O} \subset X$ be open. Then, to each $o \in \mathcal{O}, \exists r_o > 0 : B(o, r_o) \subset \mathcal{O}$. Thus, $\mathcal{O} = \bigcup_{o \in O} B(o; r_o)$, and each $B(o; r_o) \in \mathcal{B}^1$.

- Distinguish between open covers and bases! Every base for X is an open cover, but not every open cover is a base. For instance: Fix any $r* > 0$ and consider the collection:

  $$V = \{B(x; r*) : x \in X\}$$

  Then, $X \subseteq \{B(x; r) : x \in X\}$; however, V is **not** a base for X. Let $x_0 \in X$ $r < r*$ and consider the open ball: $B(x_0; r) = \{x \in X : d(x, x_0) < r\}$. Then, any open ball in V will contain points not in $B(x_0; r)$. For a concrete example: $U_n = (-n, n)$ forms an open cover for $\mathbb{R}$; however, the interval $(0, 1)$ cannot be written as a union of some U_n-s.

- The set of open balls with rational radii: $\{B(x;r) : r \in \mathbb{Q}\}$ forms a base for $\mathbb{R}$ (prove it!)
- Consider the discrete metric space (X,d) where $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0 \Leftrightarrow x = y$. *Every* set in X is $d-$open. The collection of singletons:
$$\{\{x\} : x \in X\}$$
is **both** - a base of open sets for X, and an open cover for X. In addition: *Any* base of open sets for X must contain all singletons, since singletons are the smallest nonempty open sets in X! Thus, our singleton collection is the *minimal base* for X (the minimal base is a subset for every base of X).
- As a bonus - if X is infinite, then our singleton open cover has no finite subcover, whereby an infinite discrete metric space is never compact.

Theorem(1.5.1) : Characterization of Bases

$$\text{(1) } \mathcal{B} \text{ is a base of open sets for } X$$
$$\Longleftrightarrow$$
$$\text{(2) } \forall x \in X, \forall U_x, \exists V \in \mathcal{B} : x \in V \subseteq U_x$$

Proof
(I) (1) $\implies$ (2): Let $\mathcal{B}$ be a base for X. Let $x \in U_x$, where U_x is an open nbhd of x. Then, $U_x = \bigcup\{V : V \in \mathcal{B}\}$.* Note that it is not possible for any V to contain any points not in U_x, because * mandates *equality*, not just subset inclusion. Thus there is a $V' \in \mathcal{B}$ such that $x \in V' \subseteq U_x$.

(II) (2) $\implies$ (1): Let M be **any** open set and $m \in M$; then M is a nbhd of m. Thus there is a set $V_m \in \mathcal{B}$ such that $m \in V_m$. Now we can write $M = \bigcup\{V_m : m \in M\}$, where each $V_m \in \mathcal{B}$ is constructed by (2). Thus, $\mathcal{B}$ is a base for X.

0.2 Second-Countable Metric Space

A metric space is said to be *second-countable* if there is a base of open sets for X that is *at most countable*(finite or countably infinite). Equivalently, X is second-countable if we can form a sequence of open sets: $\{U_n : n \in \mathbb{N}\}$, such that any open set $\mathcal{O}$ can be expressed as a union of some of these U_ns.

Theorem | Second Countable iff Separable : (i) A metric space (X,d) is separable $\Longleftrightarrow$ (ii) (X,d) is second-countable.

Proof : (1) $(i) \implies (ii)$ Suppose (X,d) is separable. Then, $\exists (x_j)_{j=1}^{\infty}$ in X so that $X \subseteq \overline{(x_j)}$. We claim that the collection of open balls:

$$\mathcal{B} = \{B(x_j; \frac{1}{n}) : j \in \mathbb{N}, n \in \mathbb{N}\}$$

Is a (countable) base for X. Let $x \in X$ and U_x be a nbhd of x. Then, since U_x is open, there is an $r > 0$ so that $B(x;r) \subseteq U_x$. Since $x \in \overline{(x_j)}$, for any $\varepsilon_n = \frac{1}{n}$, there exists a j such that $x_j \in B(x; \frac{1}{n})$, equivalently: $x \in B(x_j; \frac{1}{n})$. Then for n sufficiently large,

$$x \in B(x_j; \frac{1}{n}) \subseteq B(x;r) \subseteq U_x$$

Thus by **(1.5.1)**, $\mathcal{B}$ is a base of open sets for X, and is countable.

(2) $(ii) \implies (i)$ Suppose X is second-countable. Then there exists a sequence of open sets : $\{U_n\}$ that form a base for X. For each $n \in \mathbb{N}$, select a $x_n \in U_n$. We claim that the sequence: $\{x_n\}$ is dense in X. Let A be any open set in X. Then $A = \bigcup_{k \in I} U_k$ where $I \subseteq \mathbb{N}$. Thus, there exists a $k' \in I$ so that $x_{k'} \in U_{k'} \subseteq A$. Hence, $A \cap (x_n)$ contains $x_{k'}$. Since every open set in X intersects the sequence (x_n), (x_n) is dense in X.

Lindelöf's Theorem : If (X,d) is second-countable, every open cover has a **countable** subcover.

Proof: Let $\mathcal{B}$ be a countable base of open sets for X, and $\{O_l\}_{l \in I}$ is an open cover for X. Define the subset $\gamma \subseteq \mathcal{B}$ as:

$$V \in \gamma \iff \exists l \in I : V \subseteq O_l, \text{ where } V \in \mathcal{B}$$

Since γ is a subset of a countable set $\mathcal{B}$, it is countable. We claim that γ covers X. Let $x \in X \subset \bigcup_{l \in I} O_l$. Then $\exists l' \in I : x \in O_{l'}$.
Since $O_{l'}$ is an open nbhd of x, by **(1.5.1)**, there is a set $V \in \mathcal{B}$ such that $x \in V \in O_{l'}$.
But then: $\exists l' \in I : V \subseteq O_{l'}, V \in \mathcal{B} \implies V \in \gamma$. Thus, γ **is a countable cover for** X.
Now, for each $V \in \gamma$, select **one** index $l(V) \in I$ such that $V \subseteq O_{l(V)}$. Then, the set $\{O_{l(V)} : V \in \gamma\}$ is a countable **subcover** for X.

Now we have all the ingredients to prove our main TFAE Theorem. To summarize some key results:

- (5.1) In a metric space, separable $\Leftrightarrow$ second-countable
- (5.2) TB $\implies$ separable $\implies$ second-countable
- (5.3) A subspace of any separable space is separable.

Our TFAE Theorem:

- (1.1) X is topologically compact
- (1.2) Every sequence in X has a convergent subsequence.
- (1.3) X is totally bounded and complete.

Recall: We already showed (1.1) $\implies$ (1.2) $\implies$ (1.3).
We also showed that any sequence in a TB metric space (X, d) has a cauchy subsequence; if the space is complete, then this Cauchy subsequence will converge to a point in X. This shows (1.3) $\implies$ (1.2), establishing (1.2) $\Leftrightarrow$ (1.3).

Now we must show: (1.2) $\implies$ (1.1).
Proof: Let (X, d) be a metric space where every sequence has a convergent subsequence (1.2). We want to show that X is compact. We already showed (1.2) $\Leftrightarrow$ (1.3); thus X is totally bounded.
Now: X is totally bounded $\implies$ X is second-countable (by (5.2)).
By **Lindelöf's Theorem**, every open cover of X has a countable subcover. Thus it suffices to show that any countable open cover of X has a finite subcover.
Let $\{U_k\}_{k\in\mathbb{N}}$ be a **countable open cover** for X. Suppose for contradiction that this cover has **no finite subcover**. That is:

$$\forall m \in \mathbb{N}, \exists x_m \in X \text{ where } x_m \notin \bigcup_{k=1}^{m} U_k$$

Thus, construct the sequence $\{x_m\}_{m\in\mathbb{N}}$ in X to satisfy:

$$\text{For each } m \in \mathbb{N}, x_m \in X \setminus \bigcup_{k=1}^{m} U_k, \text{ that is: } x_m \in \bigcap_{k=1}^{m} (X \setminus U_k)$$

Notice that whenever $l \geq m \geq 1$, $x_l \in X \setminus \bigcup_{k=1}^{m} U_k$. (*)
By **(1.2)**, (x_m) has a **convergent subsequence**: (x_{m_j}) converging to some $x \in X$.
By (*), For each index $m_j \geq j \geq 1$, $x_{m_j} \in X \setminus \bigcup_{k=1}^{j} U_k$ **for any** $j \in \mathbb{N}$. Further, for each $j \in \mathbb{N}$ the set $X \setminus \bigcup_{k=1}^{m} U_k$ is **closed**. By the sequential closure lemma, the limit x of our subsequence (x_{m_j}) satisfies:

$$\forall j \in \mathbb{N}, x \in X \setminus \bigcup_{k=1}^{j} U_k$$

But then, the collection $\{U_j : j \in \mathbb{N}\}$ does not cover X— a contradiction! Thus, a finite subcover must exist, whereby X is compact. We have shown that sequential compactness implies open-cover compactness.
Additionally:

Any **compact** metric space is **separable/second-countable**.
