1. Show that the intersection of a family of topologies on X forms a topology for X.
Proof Let $(\mathcal{T}_\alpha)_{\alpha\in I} \subseteq P(X)$ be a family of topologies for X. We NTS: $\bigcap_{\alpha\in I}\mathcal{T}_\alpha$ is a topology for X.

- For *each* $\alpha \in I$, we have $X, \varnothing \in \mathcal{T}_\alpha$. Thus, $X, \varnothing \in \bigcap_{\alpha\in I}\mathcal{T}_\alpha$.
- Let $(U_l)_{l\in L}$ be an arbitrary collection of sets in $\bigcap_{\alpha\in I}\mathcal{T}_\alpha$. Then, $(U_l)_{l\in L} \subseteq \mathcal{T}_\alpha$ for **each** $\alpha \in I \implies \bigcup_{l\in L} U_l \in \mathcal{T}_\alpha \quad \forall\alpha \in I$
$\implies \bigcup_{l\in L} U_l \in \bigcap_{\alpha\in I}\mathcal{T}_\alpha$.
- Let $(A_k)_{k=1}^n$ be a finite collection of sets in $\bigcap_{\alpha\in I}\mathcal{T}_\alpha$; then $\bigcap_{k=1}^n A_k \in \mathcal{T}_\alpha$ for each $\alpha \in I \implies \bigcap_{k=1}^n A_k \in \bigcap_{\alpha\in I}\mathcal{T}_\alpha$.

Thus, $\bigcap_{\alpha\in I}\mathcal{T}_\alpha$ is a topology for X.

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2. Let X be a set, and define $\mathcal{T} = \{U \subseteq X : (X \setminus U) \text{ is finite }\} \cup \varnothing$. Show that $\mathcal{T}$ is a topology for X. This is called the *cofinite topology* for a set.
Proof

- First, $X \setminus (X) = \varnothing$ is finite, whereby $X \in \mathcal{T}$; by definition, $\varnothing \in \mathcal{T}$ also.
- Let $(A_l)_{l\in L}$ be an arbitrary collection of sets in $\mathcal{T}$. Then, $X \setminus A_l$ is finite for all $l \in I$. Select any two indices $l_1, l_2 \in L$; since $(X \setminus A_{l_1}) \cap (X \setminus A_{l_2})$ is finite and $\bigcap_{l\in L}(X \setminus A_l) \subseteq (X \setminus A_{l_1}) \cap (X \setminus A_{l_2})$, we see that $\bigcap_{l\in L}(X \setminus A_l) = X \setminus (\bigcup_{l\in L} A_l)$ is finite, whereby $(\bigcup_{l\in L} A_l) \in \mathcal{T}$
- Let $(A_k)_{k=1}^n$ be a finite collection of sets in $\mathcal{T}$. Then, $X \setminus A_k$ is finite for each $k \in [1:n]$, whereby $\bigcup_{k=1}^n (X \setminus A_k) = X \setminus (\bigcap_{k=1}^n A_k)$ is finite. Hence, $\bigcap_{k=1}^n A_k \in \mathcal{T}$.

Thus, the cofinite topology is indeed a topology for X.

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3. Let $\mathcal{T}$ be the cofinite topology for $\mathbb{Z}$. (i) Show that the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = n, \;\; n \in \mathbb{N}$ converges to *every* point in $\mathbb{Z}$. (ii) Describe all the convergent sequences in $\mathbb{Z}$.

Ans. **(i)** We are given $\mathcal{T} = \{U \subseteq \mathbb{Z} : (\mathbb{Z} \setminus U) \text{ is finite}\}$. Let $z \in \mathbb{Z}$ be any integer, and consider any nbhd U_z of z, where $z \in U_z \in \mathcal{T}$.
Since $U_z \in \mathcal{T}$, $(\mathbb{Z} \setminus U_z)$ is finite. Let $\mathbb{Z} \setminus U_z = \{z_1, ..., z_p\}$. Choose the largest integer in this set : $M = \max\{z_1, ..., z_p\}$. Then for any positive integer $n > M$, we have $a_n = n \notin (\mathbb{Z} \setminus U_z)$. Thus $n > M \implies a_n \in U_z$. Hence $a_n \to z$.

(ii) Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(\mathbb{Z}, \mathcal{T})$: $a_n \to z$ for some $z \in \mathbb{Z}$.
We claim that no terms of $(a_n)_{n\in\mathbb{N}}$ **other than** z can be repeated infinitely many times.
To see this, assume there exists a term $p \neq z$ of a_n which is repeated infinitely many times: $a_i = p = a_j$ for infinitely many indices (i, j) where $i \neq j$.

Let U_z be any nbhd of z, so that $z \in U_z \in \mathcal{T}$ and $(\mathbb{Z} \setminus U_z)$ is finite. Then, the set $G_z = (U_z \setminus \{p\})$ is a nbhd of z, since $\mathbb{Z} \setminus (U_z \setminus \{p\}) = (\mathbb{Z} \setminus U_z) \cup \{p\}$ is finite and contains z.
Since $a_n \to z$, there exists an N such that $n > N \implies a_n \notin (\mathbb{Z} \setminus G_z)$. However, $p \in (\mathbb{Z} \setminus G_z)$, whence $a_n \notin (\mathbb{Z} \setminus G_z) \implies a_n \neq p$ for $n > N$. But $a_n = p$ for infinitely many $n \in \mathbb{N}$, a contradiction! This proves our claim.

Also note that:
1. If $a_n \to z$, **and** z *is* repeated infinitely many times, then the limit z of (a_n) is **unique**.
2. On the other hand, **if** (a_n) **converges and no term** of (a_n) is repeated infinitely many times, then $a_n \to z'$ for *every* $z' \in \mathbb{Z}$.

Next we show the converse direction.
Let (a_n) be a sequence where **at most one** term is repeated infinitely many times. We claim that (a_n) converges in $(\mathbb{Z}, \mathcal{T})$. There are two distinct possibilities:
1. Suppose **exactly one** term of (a_n) is repeated infinitely many times: $a_n = z_0$ for infinitely many $n \in \mathbb{N}$, and all other terms of (a_n) are repeated only finitely many times. We claim that $a_n \to z_0$.
To see this, let U_{z_0} be any nbhd of z_0. Then $F = (\mathbb{Z} \setminus U_{z_0})$ is **finite**, and $z_0 \notin F$. Let $F = \{f_1, ..., f_s\}$. Since each $f_i \in F$ is repeated at most finitely many times in (a_n), we can set $N = \max\{n \in \mathbb{N} : a_n \in F\}$ to see that $n > N \implies a_n \notin F \implies a_n \in U_{z_0}$. Thus, $a_n \to z_0$.
2. Now suppose no terms of (a_n) are repeated infinitely many times. Here, $a_n \to z$ for any $z \in \mathbb{Z}$. To see this, pick any z and a nbhd U_z, where $G = (\mathbb{Z} \setminus U_z)$ is finite. Then $G = \{g_1, ..., g_s\}$. Let $N = \max\{n \in \mathbb{N} : a_n \in G\}$. For any $g_i \in G$, $a_n = g_i$ for at most finitely many $n \in \mathbb{N}$, whereby $N < \infty$. Now, $n > N \implies a_n \notin G \implies a_n \in U_z$. Thus, $a_n \to z$.

Hence we arrive at the following characterization of convergent sequences in $(\mathbb{Z}, \mathcal{T})$:

(a_n) converges in $(\mathbb{Z}, \mathcal{T}) \iff$ **At most one** term of (a_n) is repeated infinitely many times.

I) Further, a characterization of sequences with *unique* limits:

$\exists! z : a_n \to z \iff a_n = z$ for infinitely many $n \in \mathbb{N}$ and **no other term** is repeated infinitely many times.

II) Lastly, sequences converging to *every* point in $\mathbb{Z}$:

$a_n \to z$ for every $z \in \mathbb{Z} \iff$ No term of (a_n) is repeated infinitely many times.

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4. Let S be a nonempty subset of the set X. Determine $\overline{S}$ when: (i) X has the discrete topology, (ii)X has the indiscrete topology, (iii) X has the cofinite topology.
Ans. **(i)** Recall that in the discrete topology, every set is closed. Further,

S is closed $\iff \overline{S} = S$. Hence, when $\mathcal{T} = \mathcal{P}(X)$, $\overline{S} = S$.

(ii) In the indiscrete topology, $\mathcal{T} = \{\varnothing, X\}$. Let $p \in X$. Then the only nbhd of p in $(X, \mathcal{T})$ is X itself! Thus: $p \in X \implies \forall V_p, V_p = X \implies V_p \cap S = X \cap S = S \neq \varnothing$, whereby $p \in \overline{S}$. Then, $X \subseteq \overline{S}$, so $\overline{S} = X$.

(iii) Let $\mathcal{T} = \{U \subseteq X : (X \setminus U) \text{ is finite }\}$. **Assume that X is infinite**, otherwise $\mathcal{T} = \mathcal{P}(X)$ takes us to case-**(i)**. Note that for any set A in a topological space $(X, \mathcal{T})$:
1. $A \in \mathcal{T} \iff A$ is open.
2. $A \notin \mathcal{T} \iff A$ is not open. (This does not imply that A is closed !! Eg: The discrete metric space, $(0,1]$ in $\mathbb{R}$).
3. A is closed $\iff (X \setminus A)$ is open.
4. A is open $\iff (X \setminus A)$ is closed.
5. There is a sequence (x_n) in A such that $x_n \to p \implies p \in \overline{A}$.

First, let S be **finite**. Then, $(X \setminus (X \setminus S)) = S$ is finite, whereby $(X \setminus S) \in \mathcal{T}$, and its complement : $(X \setminus (X \setminus S)) = S$, is **closed**. Since S is closed, $\overline{S} = S$
Now, let S be **infinite**. Then we can construct a sequence $(s_n) : s_n \in S$ such that **no term of s_n is repeated infinitely many times**. By **Problem-3**, $s_n \to p$ for any $p \in X$, whereby $p \in \overline{S}$. Hence, $X \subseteq \overline{S}$, or: $\overline{S} = X$.

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5. Let $(X, \mathcal{T})$ be a metrizable topological space and p, q are distinct points in X. Show there exist nbhds V_p, V_q of p and q respectively such that $V_p \cap V_q = \varnothing$.

Ans. Since $(X, \mathcal{T})$ is metrizable, there exists a metric d such that:
$U \in \mathcal{T} \iff U$ is $d-$open. Since $p \neq q$, $d(p,q) > 0$. Let $r < \frac{d(p,q)}{2}$ and $r' < \frac{d(p,q)}{2}$.
We claim that $B(p;r') \cap B(q;r) = \varnothing$. To see this, let $x \in B(p;r') \cap B(q;r)$.
Then, $d(x,p) < r'$ and $d(x,q) < r$. However:
$d(p,q) \leq d(p,x) + d(x,q) < r' + r < \frac{d(p,q)}{2} + \frac{d(p,q)}{2}$, yielding the absurdity $d(p,q) < d(p,q)$. Letting $V_p = B(p;r')$ and $V_q = B(q;r)$, we get $V_p \cap V_q = \varnothing$.

Thus: $(X, \mathcal{T})$ is a **metrizable** $\implies \exists$ disjoint nbhds for any two distinct points in X.

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6. Let X be an **infinite** set and $\mathcal{T}$ be the cofinite topology on X. Prove that the property described in problem-5 does *not* hold for open sets in X. Conclude that **the cofinite topology for X is *not* metrizable**.

Ans. We are given $\mathcal{T} = \{U \subseteq X : (X \setminus U) \text{ is finite}\} \cup \varnothing$. Since X is infinite, $U \in \mathcal{T} \implies U$ is infinite when $U \neq \varnothing$. Thus, all nonempty open sets in this cofinite topology are infinite.
Now, let $p, q \in X$ where $p \neq q$, and let V_p, V_q be any two nbhds of p, q respectively. Suppose $V_p \cap V_q = \varnothing$. Then $V_p \subseteq (X \setminus V_q)$. However, the **infinite** set V_p cannot be a subset of the **finite** set $(X \setminus V_q)$! Thus, $V_p \nsubseteq (X \setminus V_q) \implies$

$V_p \cap V_q \neq \varnothing$. Hence, *any* two nbhds of any points p, q in X will intersect. Thus, $(X, \mathcal{T})$ is not metrizable.

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7. Let $(X, \mathcal{T})$ be a topological space, and $S \subseteq X$. Show that $\overline{S}$ is the intersection of all the closed sets in X that contain S.

Ans. First, let $x \in \bigcap\{F : S \subseteq F,\ F \text{ is closed in } X\}$. Let V_x be any nbhd of x in X. Suppose $V_x \cap S = \varnothing$. Then $S \subseteq (X \setminus V_x)$. Since $(X \setminus V_x)$ is a closed set containing S, by our assumption, $x \in (X \setminus V_x)$, an absurdity! Thus, $V_x \cap S \neq \varnothing$, whereby $x \in \overline{S}$. Hence, $\bigcap\{F : S \subseteq F,\ F \text{ is closed in } X\} \subseteq \overline{S}$.
Now in the converse direction, let $x \in \overline{S}$, and let F be any closed set where $S \subseteq F$. Let V_x be any nbhd of x. Then, $V_x \cap S \neq \varnothing$. Furhter, $(V_x \cap S) \subseteq (V_x \cap F)$, whereby $V_x \cap F \neq \varnothing$. Hence, $x \in \overline{F} = F$.
Thus: $\overline{S} \subseteq \bigcap\{F : S \subseteq F,\ F \text{ is closed in } X\}$.

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8. Show that $\text{int}(S)$ is the union of all the open sets contained in S.
Proof. First, let $p \in \text{int}(S)$. Then , there exists a nbhd V_p of p such that $V_p \subseteq S$. Thus, $p \in \bigcup\{U \subseteq S : U \text{ is open }\}$, whereby $\text{int}(S) \subseteq \bigcup\{U \subseteq S : U \text{ is open }\}$.
In the converse direction, if $p \in\subseteq \bigcup\{U \subseteq S : U \text{ is open }\}$, then there is an open set U containing p such that $U \subseteq S$, whereby $p \in \text{int}(S)$.

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9. Prove that : (i) $\overline{X \setminus S} = X \setminus \text{int}(S)$, (ii) $\text{int}(X \setminus S) = X \setminus \overline{S}$.
Ans. Refer to previous notes on *dual identities*.

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10. Let X be a set. Define $\mathcal{T} = \{U \subseteq X : (X \setminus U) \text{ is at most countable}\} \cup \varnothing$. Prove that $\mathcal{T}$ is a topology for X. This is called the *co-countable topology* for X.
Proof:

- First, $\varnothing \in \mathcal{T}$, and since $X \setminus X = \varnothing$ is finite, $X \in \mathcal{T}$.
- Let $(U_\alpha)_{\alpha \in I}$ be a collection of sets in $\mathcal{T}$, so that $F_\alpha = (X \setminus U_\alpha)$ is at most countable for each $\alpha \in I$. Then, $X \setminus (\bigcup_{\alpha \in I} U_\alpha) = \bigcap_{\alpha \in I}(X \setminus U_\alpha)$ $= \bigcap_{\alpha \in I} F_\alpha$. For any two indices $\alpha_1, \alpha_2 \in I$, $\bigcap_{\alpha \in I} F_\alpha$ is contained in $F_{\alpha_1} \cap F_{\alpha_2}$, which is at most countable. Thus, $\bigcap_{\alpha \in I} F_\alpha = X \setminus (\bigcup_{\alpha \in I} U_\alpha)$ is countable, hence $(\bigcup_{\alpha \in I} U_\alpha) \in \mathcal{T}$.
- Let $(U_k)_{k=1}^n$ be a finite collection of sets in $\mathcal{T}$. Then, $F_k = X \setminus U_k$ is finite for each $k \in [n]$. A **finite** union of at most countable sets is at most countable (can be shown by induction on the number of sets), whereby $\bigcup_{k \in [n]} F_k = X \setminus \bigcap_{k \in [n]} U_k$ is at most countable. Hence $\bigcap_{k \in [n]} U_k \in \mathcal{T}$.
