Three Questions on Separability

1. Question: Are the irrationals separable?

Question. Is the metric space $\big(\mathbb{R}\setminus\mathbb{Q},\, d\big)$ separable, where $d(x,y)=|x-y|$?

Answer. Yes, it is separable.

Proof (explicit countable dense subset). Define

$$D := \mathbb{Q}+\sqrt{2} = \{q+\sqrt{2} : q\in\mathbb{Q}\}.$$

Then D is countable (it is the image of the countable set $\mathbb{Q}$ under $q\mapsto q+\sqrt{2}$), and in fact $D\subseteq\mathbb{R}\setminus\mathbb{Q}$ since if $q+\sqrt{2}\in\mathbb{Q}$ then $\sqrt{2}=(q+\sqrt{2})-q\in\mathbb{Q}$, a contradiction.

Now fix an irrational $x\in\mathbb{R}\setminus\mathbb{Q}$ and $\varepsilon>0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q\in\mathbb{Q}$ such that

$$\big|q-(x-\sqrt{2})\big|<\varepsilon.$$

Then $q+\sqrt{2}\in D$ and

$$\big|(q+\sqrt{2})-x\big| = \big|q-(x-\sqrt{2})\big|<\varepsilon.$$

So every ε-ball around x contains a point of D, hence D is dense in $\mathbb{R}\setminus\mathbb{Q}$. Therefore $\mathbb{R}\setminus\mathbb{Q}$ is separable. $\square$

2. Question: What's the general idea behind the $\mathbb{Q}+\sqrt{2}$ trick?

Question. We basically used a workaround: start with $\mathbb{Q}$ (countable dense in $\mathbb{R}$), then force irrationality by adding $\sqrt{2}$. What's the general principle?

Answer. The principle is: *translate a known dense countable set by an irrational number.* Translations preserve density, and an irrational translate of $\mathbb{Q}$ lies entirely inside the irrationals.

Lemma 1 (Translation preserves density)**.** *Let $(\mathbb{R},|\cdot|)$ have its usual metric and fix $a\in\mathbb{R}$. If $A\subseteq\mathbb{R}$ is dense in $\mathbb{R}$, then $A+a:=\{x+a : x\in A\}$ is dense in $\mathbb{R}$.*

Proof. Let $U\subseteq\mathbb{R}$ be nonempty open. Then $U-a:=\{u-a : u\in U\}$ is also nonempty open. Since A is dense, $(U-a)\cap A\neq\emptyset$. Pick $x\in(U-a)\cap A$. Then $x+a\in U\cap(A+a)$, so $U\cap(A+a)\neq\emptyset$. Thus $A+a$ is dense. $\square$

Forcing irrationality. If $a\notin\mathbb{Q}$, then $\mathbb{Q}+a\subseteq\mathbb{R}\setminus\mathbb{Q}$, because if $q+a\in\mathbb{Q}$ for some $q\in\mathbb{Q}$ then $a=(q+a)-q\in\mathbb{Q}$, a contradiction.

Conclusion. For *any* irrational a, the set $\mathbb{Q}+a$ is a countable dense subset of $\mathbb{R}\setminus\mathbb{Q}$.

3. Question: Can a separable metric space have a nonseparable dense subspace?

Question. Give a simple example of a separable metric space which has a nonseparable dense subspace.

Answer. *No such example exists in metric spaces.* In fact, every subspace of a separable metric space is separable (so separable metric spaces are *hereditarily separable*).

Theorem 1 (Separable metric spaces are hereditarily separable)**.** *If (X, d) is a separable metric space, then every subspace $Y \subseteq X$ is separable (in particular, every dense subspace is separable).*

Proof. Since X is separable, fix a countable dense subset $D = \{x_1, x_2, \dots\} \subseteq X$. Consider the collection of balls with centers in D and rational radii:

$$\mathcal{B} := \{B(x_n, r) : n \in \mathbb{N},\ r \in \mathbb{Q}_{>0}\}.$$

This set $\mathcal{B}$ is countable. We claim it is a base for the topology of X.

Let $U \subseteq X$ be open and let $y \in U$. Then there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq U$. Because D is dense, choose some $x_n \in D \cap B(y, \varepsilon/2)$. Now pick a rational $r \in \mathbb{Q}_{>0}$ with $0 < r < \varepsilon/2$. Then

$$y \in B(x_n, r) \subseteq B(y, \varepsilon) \subseteq U,$$

so indeed $\mathcal{B}$ is a countable base for X; i.e. X is second-countable.

Now let $Y \subseteq X$ be any subspace. Then

$$\mathcal{B}_Y := \{B \cap Y : B \in \mathcal{B}\}$$

is a countable base for Y in the subspace topology, so Y is second-countable. Finally, any second-countable space is separable: enumerate the (nonempty) basic open sets $\mathcal{B}_Y = \{U_1, U_2, \dots\}$ and choose $y_k \in U_k$ for each k with $U_k \neq \emptyset$. The set $\{y_k\}$ is countable and dense in Y. Hence Y is separable. □

Remark. The phenomenon *can* occur in general topological spaces (non-metrizable ones), but not in metric spaces.