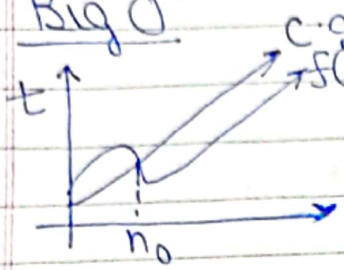
ALGORITHMS

I. Time & Space Complexity:

1) Big O → Worst Case

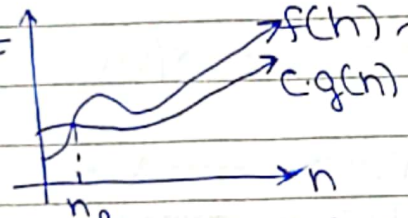


$f(n) = O(g(n))$ (imp)

$f(n) \leq c \cdot g(n)$ for $n \geq n_0$
Where, $n_0 \geq 1, c > 0$

ii) $f(n) = n$, $g(n) \rightarrow n, n^2, n^3, n^4 \ldots$
But, O → Tightest Upper Bound ∴ $O(n)$

2) Big Omega (Ω) → Best Case

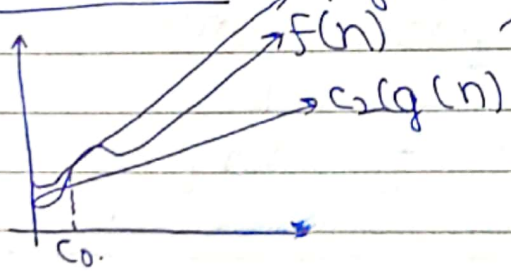


$f(n) = \Omega(g(n))$

$f(n) \geq c \cdot g(n)$, for $n \geq n_0$
Where, $n_0 \geq 1, c > 0$

ii) $f(n) = n$, $g(n) \rightarrow n, \log n, \log(\log n), 1 \ldots$
But, Ω → Tightest lower Bound ∴ $\Omega(n)$

3) Theta (θ) → Avg Case



$f(n) = \theta(g(n))$

$c_1 g(n) \leq f(n) \leq c_2 g(n), n > n_0$
where, $n_0 \geq 1, c > 0$

eg: $f(n) = 3n^2 + 2n \rightarrow \theta(n^2)$ $[O(n^2), \Omega(n^2)]$

→ When best & Worst cases are equal, Theta (θ) exists.

* Algo → Iterative : count no. of times loop runs
  → Recursive : analyse recursive function

To analyse Time

* Time complexity of Iterative Progs

1) for(i=1; i<=n; i++) → O(n)

eg → Pg4 [Gate 1991] (increment by i — inner loop)
calculate loop runs in terms of n.

eg → Pg5 [Q2] [Unrolling loop] (imp)

2) for(i=1; i<=n; i=i*2) → O(log n)

3) A()
{ while(n>1) → n=n/2;
}

n=2 — 1 t
n=4 — 2 t.
n — log n times

∴ O(log n)

eg → Pg7 [Q2] O(n log log n) (imp)

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots\frac{1}{n}=\log n$$

* Time Complexity of Recursive Progs

→ A(n)
{ if(n>1)
  return A(n-1)
}

$$T(n)=\begin{cases} T(n-1)+1 & ; n>0 \\ 1 & ; n=1 \end{cases}$$

· 2 Methods → Back Substitution
→ Tree Method

eg → Pg8 [Back Substitution Method] (imp)
$T(n)=T(n-1)+1$

· Tree Method → $T(n)=2T(n/2)+n$

eg → Pg9 [$2T(n/2)+1$] → Tree Method

$$1+2+4+8+\cdots 2^K=\frac{1(2^{K+1}-1)}{2-1}$$ (Sum of GP.) (imp)

* Compare Various functns to Analyse Time

1) Scratch out common terms
2) Log both Sides
3) Substitute Very large Values of n & compare.

Q $2^n \geq n^2$.
$n \geq 2\log n$ Sub $n=2^{100}$
$2^{100} > 200$.

$n \geq (\log n)^{100}$
$\log n \geq 100 \log\log n$
Subst. $2^{2^{300}}$
$2^{300} > 30000$

Q $n^{\sqrt{n}} \geq n^{\log n}$.
⇒ $\sqrt{n}\log n \geq \log n \log n$
⇒ $\frac{1}{2}\log n \geq \log(\log n)$. $n=2^{2^{10}}$
⇒ $512 > 10$

eg → Pg12 [last Q]
[See ∞ range of n] ; $g(n) > f(n)$.

eg → Pg 23 [Q2] (imp)
[Comparing 4 functions, & arrange them]

* MASTERS THEOREM

$T(n) = aT(n/b) + \theta(n^K \log^p n)$
$a \geq 1, b > 1, K \geq 0, p$ is real

1) if $a > b^K$, then $T(n) = \theta(n^{\log_b a})$

2) if $a = b^K$
   a) if $p > -1$, then $T(n) = \theta(n^{\log_b a} \log^{p+1} n)$
   b) if $p = -1$, then $T(n) = \theta(n^{\log_b a} \log\log n)$
   c) if $p < -1$, then $T(n) = \theta(n^{\log_b a})$

3) if $a < b^K$
   a) if $p \geq 0$, then $T(n) = \theta(n^K \log^p n)$
   b) if $p < 0$, then $T(n) = O(n^K)$

Note: $\log^2 n = \log(\log n)$
$(\log n)^2 = \log n \cdot \log n$

Q $T(n) = 2^n T(n/2) + 3^n$ ✗ a needs to be const

eg → Q5, Q6, Q14.

Q $T(n) = 64T(n/8) - n^2 \log n$ ✗
(int)

* Analysis Space Complexity

- how much max extra space required by program in terms of i/p

1) Iterative → if 1D Array (size = n), created inside prog → Space = O(n)
2) Recursive

eg → Pg 19 [Q2] (Use Tree Method) (imp)

Time → depends on no: of function calls
Space → depth of Stack / depth of Tree

* Amortized Analysis (imp) [Build Heap]

- Instead of computing cost of 1 operation, per iteration series of operations, & compute avg cost

Techniques → Aggregate
→ Accounting
→ Potential

* Aggregate Analysis

[Array list]

Prob: Insert into Array, if full: double size, copy from elements & insert

→ Worst Case = $O(n^2)$
(to insert n elements, n time for each).
→ Amortized Cost = $O(1)$

eg → Pg 21 [Example & Time Analysis]
(imp)

II] SORTING TECHNIQUES

1) * Insertion Sort

- Pick card, compare & arrange in left hand.
- Total n-1 Pass
- 1 + 2 + 3 + ... n-1] Comp.

Time: $O(n^2)$ → Worst Case (Rev Sorted)
$O(n)$ → Best Case (Sorted)

Space: $O(1)$

(imp)

(ii) If Binary Search used (Time 2)
No. of Comp = $O(\log n)$
Movement / Shift = $O(n)$.
∴ $T = O(n^2)$
Decrease comparison

(iii) Doubly Linkedlist
No of comp = $O(n)$
Movement = $O(1)$
∴ $T = O(n^2)$
Decrease movement

2) * Merge Sort Algorithm [Divide-Conquer]

- Heart of MS: Merge Algo.
- To merge 2 Sorted lists: n & m
  $T = O(n+m)$, $S = O(n+m)$
  (n+m) comp. & (n+m) copy.
- ∞ is added at the end of Sublists, to copy the other list, if 1 list is over.

eg → Pg 24 [Why ∞ & Merge] (imp)

* Merge Sort [Out of Place / Stable]

Divide & Conquer: Divide Prob into many SP, Solve the SP first, & then combine.

eg → Pg 25 [Working of MS] (imp)

⇒ Space complexity

- Merge Procedure: $O(n)$ — Extra Array.
- Recursive calls:
  Size of Stack = depth of Tree = $O(\log n)$
  ∴ Space = $O(n + \log n) = O(n)$

⇒ Time Complexity: $T(n) = 2T(n/2) + n$
Partition always at middle.

$T = \Theta(n \log n)$
(Best = Avg = Worst Case)

eg → Pg 26 [$n/\log n$ bits — Q1]

Note: To compare two n-length string, #comp = 0
eg → Pg 27

3) Quick Sort Algorithm [Divide-Conquer]

- Heart of QS: Partitioning Procedure.

Note: if i/p is small, QS → better MS

* Partitioning Procedure → $T = \Theta(n)$
1. Choose last element as pivot.
2. i = -1, j = 0
3. if A[j] > pivot, → inc j.

3. if A[j] <= pivot, inc i; & swap A[i], A[j]
4. At last, exchange A[i+1] with pivot

∴ $T = \Theta(n)$ → In any Array input.
[Partition Algo]

* Quick Sort

eg → Pg 29 [Code]

- Depending upon the Partitioning index, the SubProblems/ time complexity vary.

imp

⇒ Space Complexity
Partitioning Algo: O(1)
Recursion:
No. of Stack Records = ht of Tree
- O(logn) : Best Case
- O(n) : Worst Case

⇒ Time Complexity
$T(n) = n + T(n-K) + T(K)$
if K = middle → $T(n) = 2T(n/2) + n$: O(nlogn)
K = last/first → $T(n) = T(n-1) + n$: $O(n^2)$

Partition at middle: O(nlogn) : Best Case
Partition at last/first: $O(n^2)$: Worst Case

(ii) Ascending / Descending / Array with Eq. elem.
[$T = O(n^2)$, ht of Tree = O(n), Skewed]

Tree → n; 0, n-1; 0, n-2

Work/Level = n
#Levels = n
∴ $T = O(n^2)$

imp

(iii) Split input into ratio (eg: 1:9, 1:99, 1:999)
$T = O(n\log n)$

eg → Pg 32 [Split i/p into 1:9 ratio] or Select 1/10th Smallest elem as Pivot.

(iii) Alternate Best & Worst Case

[Worst]

n; 0, n-1; $\frac{n-2}{2}$, $\frac{n-2}{2}$

[Best]

imp

∴ $T(n) = cn + cn + 2T\left(\frac{n-2}{2}\right)$
$T(n) \Rightarrow O(n\log n)$

eg → Pg 33 [Median Q]

* HEAPS

- The DS, as an i/p to an algorithm, affects the time complexity of Algo.

⇒

	Insert	deleteMin	findMin
MinHeap	O(logn)	O(logn)	O(1)

* Heap: Almost complete binary Tree [no Gaps in Array]

- Implemented as an Array. Visualized as Tree

100 (i); 10, 20; 1, 4, 3
[Max Heap]

1	2	3	4	5	6
100	10	20	1	4	3

parent(i) = [i/2] [Parent of i
left(i) = 2i
right(i) = 2i+1

imp

eg → Pg 35 [Q1] (Is it a maxHeap, & HeapSize)

Array in dec Order: max Heap
Array in asc Order: min Heap (imp)

* Properties of a complete Binary tree

(i) ht of a node: longest edge to leaf
(ii) ht of tree = ht of root
(iii) Given ht; max nodes → $[h \Rightarrow 2^{h+1} - 1]$

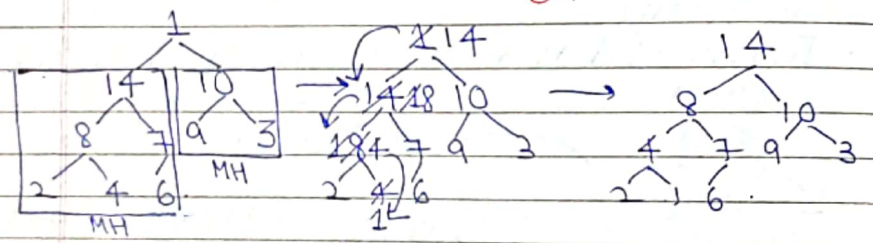
(iv) Ternary Tree: h → $(3^{h+1} - 1)/2$

(v) if n nodes
$n \xrightarrow{\text{min ht}} [\log n]$

(vi) In a CBT, leaves start from index value $\{\lfloor \frac{n}{2} \rfloor + 1 \text{ to } n\}$ (imp)
#leaves = n/2

(vii) every leaf is a heap.

* MaxHeapify: If left & right SubTree is max Heap, what to do with root so as to make the tree → maxHeap.



Max dist root travel = O(h) = O(log n).

∴ MaxHeapify: T = O(log n) = O(h)
S = O(log n)

imp → depends on level of node, where called.

* Build Max Heap

- Non leaf ∈ $\lfloor n/2 \rfloor$ to 1
- Apply MaxHeapify one by one, on the non leaf Nodes

eg → Pg 40 [eg1] (imp)

NOTE: Max no: of nodes, at height h: $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ in Heap (CBT)

∴ Build Heap = MaxHeapify on non leaf Nodes

$$\sum_{h=0}^{\log n} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot O(h) = O(n)$$

∴ Build Heap → Time Complexity = O(n)
Space Complexity = O(log n) (imp)
Max Space taken by any heapify call.

1) Extract Max / Delete Max — findMax = O(1)

(ii) Delete the root of maxHeap, & return it.
(iii) Replace root with last element, & dec Heap size.
(iii) ∵ L & R are maxHeaps, apply maxHeapify.

T = O(log n)
S = O(log n)

2) Increase Key (MaxHeap)

. IncreaseKey (A, 5, 300), [index 5 → 300]
. Worst Case: if leaf node value travels to root
∴ O(h) = O(log n) (imp)

T = O(log n)

3) Decrease Key (MaxHeap)
- if root value is decreased.
- L,R are maxHeap, so MaxHeapify
- $T = O(h) = O(\log n)$

4) Insert element (MaxHeap)
- Insert $-\infty$ at the last position.
- Increase Key $(-\infty, 300)$; & 300 travels up.
- $T = O(\log n)$

- FindMin/deleteAny/Search: $O(n)$ (imp)
  [MaxHeap]

5) Heap Sort [InPlace Sorting]

(i) Build MaxHeap.
(ii) Exchange root with last element, & dec Heap Size.
(iii) Apply maxHeapify & repeat(ii). (imp)
→ reverse sorted from behind.
$T = O(n \log n)$
∵ atleast n/2 times, the ht will be $\log n$.

6) Bubble Sort

→ Heaviest element comes to the right after 1 pass. [Stable/Adaptive/InPlace]
→ n-1 Passes.
$n-1 + n-2 + \cdots + 1 = O(n^2)$
eg → Pg 43 [Code]

- Time complexity { $O(n^2)$: Reverse Sorted
  $O(Kn)$: (if K heaviest elements are unordered, $K << n$)
  $\Rightarrow O(n)$

6) Bucket Sort
- Sort a large Set of floating no.'s in range 0 to 1.
- No. of buckets = Base = 10
- Space Complexity:
- $S = O(n+K)$; n nos, K buckets. (imp)
- Time complexity: { $T = O(n)$ [Uniform dist]
  { $T = O(n^2)$ [All map into same bucket]
- DS used → HashTable + Chaining
- eg → Pg 46 [Q]

7) Counting Sort
→ Range must be known, & keys should be in that range (K)
$S = O(K)$ (imp)
$T = O(n+K)$
→ for every no. in range, maintain a cell in array.
→ traverse the array, & count occur of num, & fill in cell.
eg → Pg 47 (imp)
* Disadvantg:
- Range must be known
- [1, 1000, 2, 3]
  (ArraySize = 1000)

8) Radix Sort

1) Each Key in Array should be/made a d-digit number.

2. Digits are numbered from 1 to b (right to left)
   eg: 153
3. for i=1 to d:
   // Use Stable Sort Algo to sort A on digit i
   [Counting Sort]

eg → Pg 48 [Example] ← imp
(When sorted on Unit digit, the 1 digit no. will be sorted)

* Time Complexity : $T = \log_b l \left(O(n+b)\right)$, $S = O(b)$ (imp)
  l: largest no.
  $\log_b l$: no. of digits
  O(n+b): Time of Counting Sort.
  O(b): Space used by counting Sort.
  b → Base
  n → No of elem.

9) Selection Sort
- find the Smallest element in each Pass & place in beginning
  $T = O(n^2)$, $S = O(1)$ → $\underbrace{n-1+n-2+n-3+\cdots 1}_{n-1} = O(n^2)$
- No of Swaps = O(n) - Max

III GREEDY TECHNIQUE

- Used for Optimization Problem (min/max)
  Optimization Problem → Greedy (want solve all, some)
  → DP (solve all, but some in exponential time)
  → O(nK)
  → O(2n)

① Fractional KnapSack Problem

→ sort n objects in P/W ratio [decreasing order] → (n log n) (simp)
→ check one by one, if it could fit in KS [full/fraction] → (n)
  T = O(n log n)
  S = O(b) → [P/W array]

eg → Pg 52, Algo → Coll

* Greedy KS using Max Heap :
  - Build max Heap (P/W) : O(n)
  - delete max n times : O(log n) x h (atmost n/2 times)

∴ T = O(n log n) [with Array / with Heap]

② Huffman Codes → [min. no of bits, to save a file]

→ Used to compress a file.
→ if 4 char : 2 bits needed] (Fixed Size encoding)
  ∴ Total 8 bits
  but, Huffman Code : (Variable Size encoding)
→ If occurences/frequencies of chars are not given, HC is used. Highest freq char gets the least no. of bits. (imp)
→ HC = Prefix Codes [code can't be prefix of any other code] eg: 0, 10, 110, 111

External Path Length = No. of bits to save file
Σ(freq x Path length)

eg → Pg 56 (imp)

ALGO

1) Make a minHeap with c[] [freq of char] : O(n)
2) for i to n:
   allocate new node z (Tree Node)
   z.left = extractmin(Q) } O(2 log n)
   z.right = extractmin(Q)
   z.freq = x.freq + y.freq } O(log n)
   Insert(Q, z)

(1) Heap

T = O(n log n)
S = O(n) → Space req to store tree.

(imp)

(2) [Sort Algo]

Sort → n log n
extract min → O(n) [Shift] } O(n²) ∵ (n-1) times
insert → O(n)

T = O(n²)

Note: If problem: pulling out min, & put back: Heap (imp)
only pull out min: Sorting Algo.

eg → Pg 58 [Prob in Huffman coding]

(3) Job Sequencing with Deadlines

1. Arrange the jobs in dec. order of Profits : O(n log n)
2. Take each job & Place it as close to its deadline (in Gantt Chart) : O(n²)

(imp)

T = O(n²)
S = { O(n) : if n << maxDeadline(d)
    { O(d) : if n > d

eg → Pg 61 [Q2]

(4) Optimal Merge Patterns

→ merge files with sorted records, so as to minimise the no. of record movements. (imp)

External Path Length = No. of record movements

Algo (i) Create min Heap : O(n)
(ii) Take 2 min, merge it, put back : O(n log n)
∴ T = O(n log n)

→ Largest record files will be moved less, nearer to root. (imp)

eg → Pg 63 [Q]

* Intro to Spanning Trees

- Simple Graph : no loops / ||l edges.
- Null Graph : no edges

E = O(V²) → Complete Graph (worst case)
V = O(log E)

⇒ Spanning Tree : Min no of edges (tree) required to connect all Vertices (n) → (n-1) edges.

⇒ No of Spanning trees → n^{n-2} STs (imp)
for Kn

* Kirchoff's theorem

→ To find no. of STs for a Not complete Graph

(i) Construct Adjacency Matrix
(ii) Diagonal Os → degree of node
(iii) Non-diagonal 1s → "-1"
(iii) Non-diagonal Os → "0".

No. of STs = Cofactor of any element
= $(-1)^{i+j} M_{ij}$.

eg → Pg 65 [Q] (imp)

* MST: ST with min cost out of the n^{n-2} STs. [Brute Force]

(5) Prim's MST Algorithm

→ Start a vertex (u), select its nbr (v).
→ From (u,v), select the nbr from the unvisited nodes & so on. (imp)
→ Tree grows one edge a time, & we never get a cycle here (∵ node selected from UV nbrs)
→ if edge wts not distinct: more than 1 MST.
eg → Pg 69 [Gate-10] (Adj. Matrix)
eg → Pg 70 [0 should be root]

(i) Prims implementation (without MinHeap)

- DS: boolean visited[V], weights[V], parent[V]
→ Run loop V times
  i. Pick vertex with minWeight from Unvisited (min), & Mark it as visited
  - explore all its unvisited nbrs
    → decide whether to update Parent & weight
      if weights[i] > edges[min][i]
      // update

(imp)

eg → Pg 71 [Algo + Example] : T = O(V²), S = O(V)

(ii) Prims implementation (with MinHeap)

eg → Algo [Call] ← (imp)
G: Graph, r: root; & eg: Pg 71

- Build Heap: O(V)
- extract min: O(V log V)
- decrease Key: (DK carried out E times)
  ∴ O(E log V)

∴ T = O(V log V + E log V + V)
T = O(E log V) [With Min Heap] (imp)

T = O(V log V + E) [With Fibanacci Heap]
O(E log V) ⟷ O(V²)

(1) Dense Graph: E = O(V²)
- O(V² log V) > O(V²)
- choose without Heap

(2) Sparse Graph: E = O(V) (imp)
- O(V log V) < O(V²)
- choose with Heap implementation

(6) Kruskals MST Algorithm

→ We may get a forest, we need to check if the edges dont form a cycle (while adding)

* Disjoint Sets

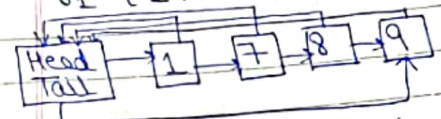
Sets which are disjoint to each other.
⇒ Operations on Disjoint Sets:
(i) Create_set(x): Create set with item(x) [Choose any item of a set as representative]
(ii) Union: S1 ∪ S2
(iii) find_set(x): find set of x

- DS for Disjoint Sets → Linked list imp
                       → Forest/Tree imp

1) LinKed list implementation

imp

$S_1 = \{1, 7, 8, 9\}$



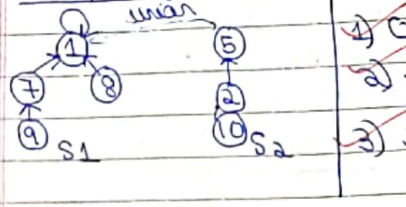
1) Create Set: O(1)
2) Find Set(7): O(1)
∵ Pointer to 7
3) Union: O(n)

∵ Point all elem of next list to head.

∴ By Amortized Analysis, with LL implementⁿ
$T = \theta(n)$ [Merge n disjoint sets, union]

2) Tree / Disjoint Forest Implementation

imp



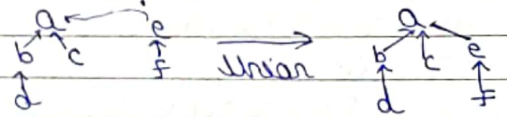
1) Create Set: O(1)
2) Find Set(x): O(d)
[d = depth]
3) Union(x,y): O(1)

* To improve findSet apⁿ, = restrict depth
- → Union By Rank
- → Path compression

(ii) Union By ~~Height~~ Rank

- Make root of tree with less nodes point to the root of tree with more nodes

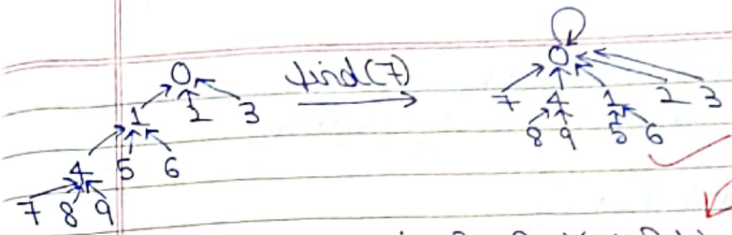
eg:



imp → Max depth possible with n nodes = log n
∴ Find Set ≈ O(log n) [Union By Rank]

(iii) Path Compression

- whenever we perform find(x), make every node visited in the path point to the root. Next, find(x): O(1)



∴ If we use Union By Rank + Path compression,

imp

Create Set: O(1)	Disjoint Set DS (tree imp)
Find Set(x): O(1)	
Union(x,y): O(1)	

5) Connected components using Disjoint Set DS

- For dynamic Graphs: Disjoint Set > BFS/DFS
- better

eg → Pg 84 [Find no of Connect Comp]

imp

$T = O(E + V)$
$S = O(V)$

6) Kruskal's MST Algorithm (Using Disjoint Sets)

Algo → Pg 84

imp

$T = O(E \log E) / O(E \log V)$
$S = O(V)$

7) Dijkstra Algorithm

- Single Source Shortest Path Algorithm
- It doesn't have the capacity to detect (-ve) edge cycles, so it is not allowed to work with -ve edge Graph (no trust on answer)

eg → Pg 87 [Algo] (Dijkstra with Heap)

imp

extract min (V times)	O(V log V)
decrease Key (E times) / Relaxation	O(E log V)

$T = O(E \log V)$
$S = O(V)$

(i) If Graph is disconnected / Graph has -ve edge cycle ⇒ Shortest Path not possible.

(8) Bellman Ford Algorithm

- Single Source Shortest Path Algo
- can detect -ve edge cycle

Graph → Bellman Ford → No (if -ve cycle) / Yes (SP)

- But, Slower than Dijkstra (imp)

⇒ Intuition: If G has n vertices SP won't have more than n-1 edges.

(i) List the edges (in some order)

(ii) Relax all the edges (n-1) times. After 1st relaxation, SP contains at most 1 edge (imp)

(iii) Relax 1 more time, if value of any node changes ⇒ -ve cycle exists.

∴ T = O(E) × O(1) → relaxation time [∵ array]

$T = O(VE)$

if, Sparse Graph: $T = O(n^2)$
Dense Graph: $T = O(n^3)$

IV DYNAMIC PROGRAMMING

dynamically decide → storing in table

* Fibonacci:

$T(n) = T(n-1) + T(n-1)$

$T = O(2^n)$ (without DP)

ht of recursion tree = n, # leaf calls = 2^n

$T = O(n), S = O(n)$ (with DP)

(1) Matrix Chain Multiplication Problem

$A_{p \times q}, B_{r \times q}$

⇒ A·B → Total mul = $p \times q \times r$
Size = $p \times r$ (C)

Problem: Parantheses mult. of n Matrices in a way, so as to min # multiplications (imp)

* For n Matrices → no. of parantheses ways = $K_{n-1} = \frac{^{2n}C_n}{n+1}$ (n=4, subst 3 in formula)

⇒ Requirements for DP:

(i) Optimal Substructure: Main Problem must be divided into Small Problem, & conquered it

(ii) Recursive formula.

(iii) Overlapping / dependent SubProblems.

* $A_1 A_2 A_3 \cdots A_n$ ← Matrix
$p_0 \times p_1, p_1 \times p_2$ ... $p_{n-1} \times p_n$ ← dimension [array]

* $A_i A_{i+1} A_{i+2} \cdots A_j \rightarrow (A_i A_{i+1} \cdots A_k)(A_{k+1} \cdots A_j)$ + $p_{i-1} p_k p_j$
($p_{i-1} \times p_k$, $p_k \times p_j$)

⇒ Recursive eqn: (imp)

$$m[i,j] = \begin{cases} 0, & i = j \\ \min_{i \le k \le j-1} \{ m[i,k] + m[k+1,j] + p_{i-1} p_k p_j \} \end{cases}$$

[min # mul to mult $A_i \cdots A_j$]

eg → Recursion Tree [Pg 94]

$T = O(2^n)$ // using Recursion without DP

Lots of overlapping SPs,

→ No. of Unique SubProblems = $O(n^2)$ (imp)

eg → Pg 95 [Q]

No of unique function calls = n^2

Time to compute 1 funct call = $O(n)$

$T = O(n^2 \times n) = O(n^3)$ ← imp

$S = O(n^2)$

- Bottom up DP (Iterative - Tabular)
- Top down DP (Memoization - Recursive)

[Actual space, bit more due to stack recursion]

② Longest Common SubSequence

RAVINDRA
AJAY

LCS = {AA} = 2

m
n

2^m SS

(ii) Brute force: $O(2^m \times n)$

* X $x_1 x_2 x_3 \ldots x_n$

Y $y_1 y_2 y_3 \ldots y_m$

Recursive Eqn: (imp)

$$C[i,j] = \begin{cases} 0 & ; \ i=0 \text{ || } j=0 \\ 1 + C[i-1, j-1] & ; \ x_i = y_i \\ \max(C[i-1,j], C[i,j-1]) & ; \ x_i \neq y_i \end{cases}$$

* Recursion tree (in worst case)

eg → Pg 99 [Recursion tree]

Depth of tree = $n+m$ (imp)

(iii) $T = O(2^{n+m})$ // recursion without DP.

Lots of overlapping SPs

→ No of Unique SPs: $O(n \times m)$

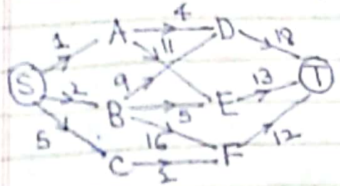
Time to compute 1 SP: $O(1)$

(iv) $T = O(mn)$, $S = O(mn)$ // with DP.

eg → Pg 101 [Bottom up DP - tabular] (imp)

(Path tracing from bottom most cell)

③ Multi Stage Graph → within a stage, no edge, edge only from S_i to S_{i+1}



Problem: Find Shortest Path from S to D.

- Dijkstra takes $O(E \log V)$ ∴ compute SP from Source to all Vertices & D ⇒ Too Slow

$$M(S,1) = \min \begin{cases} S\text{-}A + m(A,2) \\ S\text{-}B + m(B,2) \\ S\text{-}C + m(C,2) \end{cases}$$

$M(S,1)$: SP from node S [stage 1] to T

(Recursive eqn)

Vertices in next stage

eg → Pg 103 [Recursion tree]

depth = $O(n)$ (imp)

(iii) $T = O(k^n)$ // if K vertices in each stage.

// Without DP.

∴ lot of overlapping SPs ⇒ DP.

eg → Pg 103

∴ No of unique SPs = $O(n)$ (imp)

Time req to compute 1 funct call = $O(n)$

∴ $T[i] = \min_{j=i+1 \text{ to } n} \{ cost[i,j] + T[j] \}$

(ii) $T = O(n^2)$, $S = O(n)$ [1D Array]

$E = n^2$

$T = O(E)$, $S = O(V)$

// With DP.

④ 0/1 Knapsack.

KS → Fractional KS [Rice] → Greedy
KS → 0 1 KS [Parts] → DP.

Q. Why Greedy fails in 0 1 KS
eg → Pg 105 [capacity = w] (imp)

(i) Brute Force: → $T = O(2^n)$
[for every obj include/declude]

* Recursive Eqn: (imp)

$$KS(i,w) = \begin{cases} 0; & i=0 \,||\, w=0 \\ KS(i-1,w); & w_i > w \\ \max\begin{pmatrix} P_i + KS(i-1, w-w_i) \\ KS(i-1,w) \end{pmatrix} & \begin{matrix} // \text{include} \\ // \text{declude} \end{matrix} \end{cases}$$

(i th object, dat. capacity left)

eg → Recursion tree [Pg 106].

(ii) Recursion without DP.
depth = n.
$\therefore T = O(2^n), S = O(n)$
· lot of overlapping SPs, ∴ DP.

eg → Pg 106 [Bottom up DP]
· No of Unique SPs = $O(n \times W)$
· Time req to compute 1 SP: $O(1)$
· Table Size = $O(n \cdot w)$
(iii) $\therefore T = O(nw), S = O(nw)$ // with DP. (imp)

but in case, $w \approx 2^n \rightarrow T$ = exponent.
∴ depends on w : Not Polynomial Algo.

nP complete Problem.
Classes time dne, no proof that lower time cant ex

⑤ SubSet Sum Problem (imp)

Problem: Is there any SubSequence in a given Set, whose sum is S.

(i) Brute Force → $T = O(2^n)$ # SubSequences

* Recursive eqn: (imp)

$$SS(i,S) = \begin{cases} True, & S=0 \\ False, & i=0, S \neq 0 \\ SS(i-1,S); & S < a_i \quad [\text{declude}] \\ SS(i-1, S-a_i) \vee SS(i-1,S) & \\ [\text{include}] \quad [\text{declude}] & \end{cases}$$

(i th value, Sum)

eg → Pg 109 [Recursion Tree]
depth = n.

(ii) Recursion without DP → $T = O(2^n), S = O(n)$

* Lots of overlapping SPs.
* No. of Unique SPs = $O(n \times S)$
· Time required to compute 1 SP: $O(1)$

eg → Pg 109 [Bottom up DP] (imp)

$T = O(n \cdot S), S = O(nS)$ // with DP.

if $S \approx 2^n$, why to use DP?, use Brute force
∴ nP complete Problem (like 0/1 KS)

⑥ Travelling Salesman Problem

Problem: The Salesman has to visit every Vertex & return back to source (in min cost) or Find the min cost Hamiltonian Cycle. (imp)

(ii) Brute Force → $T = O(n!) = \boxed{T = O(n^n)}$
∵ $(n-1!/2)$ Hamiltonian cycles.

eg → Pg 111 [TSP example]

imp

* Recursive eqn:

$$T(i,S) = \begin{cases} \min_{j \in S} \left[C[i,j] + T(j, S-\{j\}) \right] & ; S \neq \phi \\ C[i,1] & ; S = \phi \end{cases}$$

i ↑ Set of Vertices unvisited
j ↑ (∀ v in set)
1 ↑ Source

* Lots of overlapping SPs in Recursion tree.
* No. of Unique SPs = $O(n \cdot 2^n)$
Time req to compute 1 SP = $O(n)$

$$\therefore \boxed{\begin{aligned} T &= O(n^2 2^n) \\ S &= O(n \cdot 2^n) \end{aligned}} \quad \text{// with DP}$$

imp

∴ nP Complete Problem. → $(n-1)2^n$.

$$(n-1) + (n-1)^{n-2}C_{n-3} + (n-1)^{n-2}C_{n-4} + \cdots (n-1)^{n}C_{2}$$

(7) Floyd Warshall

- All Pairs, Shortest Path Problem.
(i) Run Dijkstra for every node : $O(VE \log V)$
$\approx O(V^3 \log V)$
(ii) Run BF for every node : $O(VE)V = O(V^4)$
∴ Too Slow → Floyd Warshall

imp

* Recursive eqn:

$$d_{ij}^{K} = \begin{cases} \min\left(d_{ij}^{K-1}, d_{iK}^{K-1} + d_{jK}^{K-1}\right) & ; K \neq 0 \\ W_{ij} & ; K = 0 \end{cases}$$

- D_0: path allowed (with no item vertex)
- D_1: SP using 1 as intermediate V.
- D_2: SP using 1,2 as intermediate V.
↳ compute D_2 using D_1.

eg → Pg 115 [Bottom up DP]

O(n) # Tables = # Vertices (∵ SP with all V item)
- To compute 1 table → $O(n^2)$ Funct calls.
- each funct call: $O(1)$

imp

$$\therefore \boxed{\begin{aligned} T &= O(n^2 \times n) = O(n^3) \\ S &= O(n^2) \end{aligned}}$$

∵ Max Space only 2 Matrix at a time.

* NP Completeness

Problem → Tractable: if there exists at least one Poly algo: $O(n^K)$ / fast algo / real time S/w solutions
↓
Intractable: not tractable / slow, can solve in Poly time, but can give approx results in real time

⇒ Many probs look tractable at surface but are not

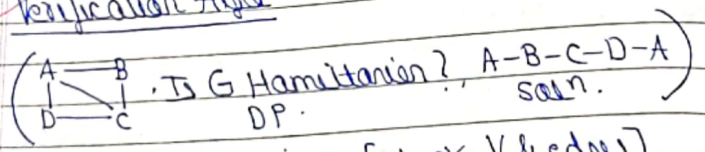
imp

Tractable	Intractable
- Shortest Path	- Longest Path
- Euler Tour	- Min Cost Hamiltonian Cycle
- 2CNF	- 3CNF $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_4 \vee \overline{x_2})$

* Optimizn → Decision Problem.
[Work on Decision Problem (Y/N) b4 Opt Prob.
(i) TSP → Is any SP covering all Vertices of length atmost K?
(ii) 01KS → Is there any sol? whose Profit is atleast K?
(If you cant solve DP, OP no chance)
⇒ We use only DP for NP-Completeness (imp)

- if OP is easy → DP is easy
- if DP is hard → OP is hard

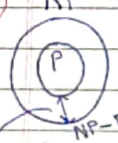
* Verification Algo



· Is G Hamiltonian? DP. A-B-C-D-A soln.

VA takes O(n) time [check V & edges]

* P: Set of Decision Problems for which there is Polynomial time algo to solve them.
* NP: Set of DPs whose solution can be verified in Poly time / O(nK) VA.
→ P: Greedy, DnC
NP (imp) (Has NP Hard some)



- if all NP-P Problems can be solved in O(n^K), P=NP
- if atleast 1 Prob of NP-P cant ever be solved in Poly time, PCNP

Rstn b/w P & NP not Known yet.

→ till today, noone could solve it Poly time / proved noone can't solve in polllll

* Poly time reduction Algo

A —Poly→ B (imp)
α β

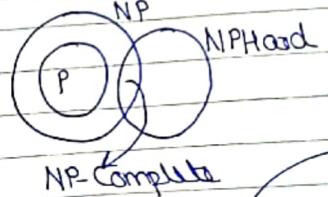
(i) if β is easy, α is also easy.
(ii) if α is hard, β is also hard.

easy = O(nK) solved

eg → Pg 8 [Q-A,B]

* NP-Hard: If every Problem in NP can be poly time reducible to a Problem 'A', than 'A' is NP-Hard. (imp)
→ if 'A' can be solved in P, every Prob in NP is P.
∴ P=NP
* NP-Complete: If it is NP & NP Hard.

* Our Belief/not Proof (imp)



* if NP-Complete, can be solved in Poly time, than ∀ NP problem in P. ∴ P=NP
* if NP-Complete, proved to never been solved.
PCNP

Task as a researcher.

* Status of NP = unknown (nothing abt Solvable time, only nK VA).

* if A=NP-Hard, & A —Poly→ B, than B=NP Hard.
* if B is already in NP, ⇒ B=NP Complete.

* Well Known NP Complete.

Circuit-SAT → SAT → 3NF SAT → Clique → Vertex Cover → HC → TSP
3NF SAT → Subset Sum.