GATE CSE NOTES

by

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With best wishes from Joyoshish Saha

(*) # n-length words with no consecutive 0's.

	#	
→ $n = 1$	2	0, 1
$n = 2$	3	01, 10, 11
$n = 3$	5	010, 011, 101, 110, 111
$n = 4$	8	0101, 0110, 0111, 1010, 1011, 1101, 1110, 1111.

$\underbrace{* * * *\ 10}_{n-2}$

$\underbrace{* * * * *\ 1}_{n-1}$

00
01
10
11

$$F_n = F_{n-1} + F_{n-2} \qquad F_0 = 1, \quad F_1 = 2$$

(*) Among any 6 people, there must be at least 3 mutual friends or 3 mutual strangers.

→ Proof Suppose complete graph K_6.

$$\# \text{ edges} = \frac{6(6-1)}{2} = 15$$

6 vertices ≈ 6 people

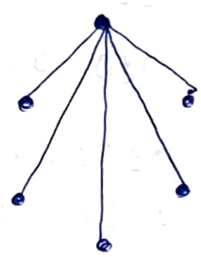
Edges – red (mutual strangers)
blue (mutual friends)

The theorem converts to ~

No matter how you color the 15 edges of a K_6 with red & blue, you can't avoid having either a red triangle - that is, a triangle all of whose 3 sides are red, representing 3 pairs of mutual strangers - or a blue triangle (3 pairs of mutual friends).

(Always, at least one monochromatic triangle).

From one vertex, we can have 0, 1, 2, 3, 4, 5 blue lines (accompanied by 5, 4, 3, 2, 1, 0 red lines).

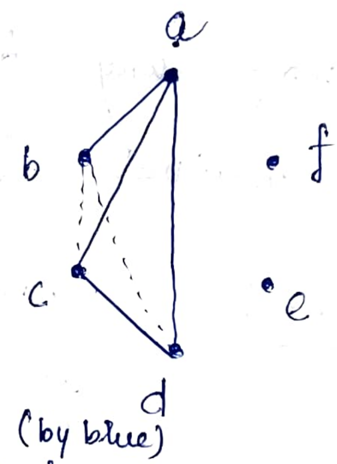


5 blue lines
0 red lines

5 pigeons (edges)
2 holes (red / blue)

By Pigeonhole principle, we either have 3+ blue lines or 3+ red lines.

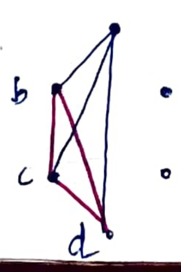
* Consider, for 3+ blue lines,



if either of, b,c ; c,d ; b,d are connected (by blue) then there are 3 mutual friends => i.e. a blue triangle. If b,c connected (by blue) then a,b,c are mutual friends. (b,c,d)

Also, when b,c,d all connected.

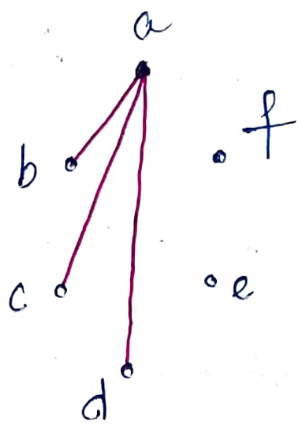
Now, if none are friends with each other, then 3 mutual strangers



b,c,d are mutual strangers.

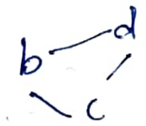
(Note: If 2 are not friends, they are strangers & vice versa. So, it's always a complete graph with red, blue edges. If there's not a blue edge there must be a red one; if not a red, one blue must be present.)

* Now, for 3+ red lines,



if any of b,c ; c,d ; b,d are strangers then a red triangle ⇒ hence 3 mutual strangers. If c,d. are strangers then a,c,d are mutual strangers.

If none are strangers (b,c,d are mutual friends), hence a blue triangle



So, 3 mutual friends.

So, we always end with a group of 3 friends or a group of 3 strangers.

[Example of Ramsey's Theorem]

* Combinatorics.

* Order matters?

Repetition?	yes	no
yes	Sequence n^k	Multisubsets
no	Arrangement ^nP_k ↳ Permutation	Subset / Combination ^nC_k

* Rule of Sum, Rule of Product

- Permutation : Ordered list, each item exactly once

- $n!$: different ways to arrange n different objects

- ^nC_k : different ways to choose k objects from n objects, where order does not matter & repetition not allowed. (# subsets of size k of $\{1, 2, \dots, n\}$)

$\frac{n!}{k!(n-k)!}$

↳ Ways to arrange n different objects $= n!$

Also, let's choose k from n → ^nC_k

now these k objects arrange in $k!$ ways

other $(n-k)$ objects arrange in $(n-k)!$ ways.

So, $n! = {}^nC_k \times (n-k)! \times k!$

$\Rightarrow {}^nC_k = \frac{n!}{k!\,(n-k)!}$

$^nC_k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$

Putting x candies in b bags.

Twelvefold way table.

distinct or identical		#candies / bag		
candies	bags	any	≤ 1	≥ 1
D	D	1 b^x	2 $\frac{b!}{(b-x)!}$ (${}^bP_x$	3 $b! \cdot S(x,b)$
I	D	4 $\binom{x+b-1}{x}$ multiset |★|	5 $\binom{b}{x}$ (${}^bC_x$	6 $\binom{x-1}{b-1}$ |★|
set partition D	I	7 $\sum_{k=1}^{b} S(x,k)$	8 1 if $x \leq b$, 0 if $x > b$	9 $S(x,b)$ partition set of ~~x~~ elems into b subsets (nonempty)
integer partition I	I	10 $\sum_{k=1}^{b} p_k(x)$	11 1 if $x \leq b$, 0 if $x > b$	12 $p_b(x)$

Answers –

1. All distinct bags (say a, b, c, ...) & all distinct candies (say 1, 2, 3, ...).

Ways to put x candies in b bags:

each candy has b choices.

$$\Rightarrow b \times b \times \dots \ x \text{ times}$$

$$= b^x$$

2. x distinct candies, b distinct bags, every bag has at most 1 candy.

each bag has a candy or it hasn't.

$$b(b-1)(b-2) \dots (b-(x-1))$$

$$= \frac{b!}{(b-x)!}$$

We select x bags out of b bags to put the candies. (order matters – ${}^bP_x$)

[or think like – $b!$ ways & $(b-x)$ remain empty, ~~so~~ they are identical objects now; so divide by $(b-x)!$]

⑤

⑤ order does not matter ${}^bC_x$ | bag A – 5 candies, bag B – 5 candies – same

5. Candies are identical, bags are distinct, every bag has at most 1 candy.

Choosing x bags to put candies among the b bags

As all the candies are identical, we choose any x bags among the b bags to put the candies. $\Rightarrow {}^{b}C_{x}$

[Also, think like - putting the candies in b! ways, then some remain vacant & some containing 1 candy. The empty & contained bags are identical themselves.

contained bags $= x$, # empty bags $= b-x$

So, divide by $x!\,(b-x)!$ $\Rightarrow \dfrac{b!}{x!\,(b-x)!}$]

8. Distinct ~~bags~~ candies, identical bags, at most 1 candy per bag

$: 0$ if $x > b$

$: 1$ if $x \le b$

11. Identical bags & candies, at most 1 candy / bag.

$: 0$ if $x > b$

$: 1$ if $x \le b$

* 4. Identical candies, distinct bags

eg. 10 identical candies, 5 distinct bags.

Sample ways 00|00000|0|0|0 (2,5,1,1,1)

or |000|0|1|000000 (0,3,1,0,6)

We need to find # ways to allocate the 10 candles & 4 bars (4 bars divide into 5 bags)

o o o | o | o o | o o o | o

1 2 3 4 5

So, we have $10 + (5-1) = 14$ places to place these objects. Choosing 4 places for the bars among 14 places. $\Rightarrow \binom{14}{4}$

— Same way for x candies, b bags

$$\binom{x+b-1}{b-1} = \binom{x+b-1}{x}$$

$$[\ {}^{n}C_{r} = {}^{n}C_{n-r}\ ]$$

6. Identical candies, distinct bags, every bag has at least 1 candy.

After putting 1 candy to each of b bags we have $x-b$ candies left.

So, now put $x-b$ identical candies in b bags (problem 4).

$$\Rightarrow \binom{(x-b)+b-1}{x-b} = \binom{x-1}{x-b} = \binom{x-1}{b-1}$$

$a \qquad\qquad\qquad\qquad b$

9. Distinct candies, identical bags, at
* least 1 candy.

$S(a,b)$: Stirling number of the second kind

(Stirling # of 1st kind count permutations of a objects with exactly b cycles).

$$S(a,b) = b\,S(a-1,b) + S(a-1,b-1).$$

basic cases :

$n \sim a \qquad r > n \Rightarrow S(n,r) = 0$

$r \sim b \qquad r = n \Rightarrow S(n,n) = 1$

$$r = n-1 \Rightarrow S(n,n-1) = \binom{n}{2}$$

$$r = n-2 \Rightarrow S(n,n-2) = \binom{n}{3} + 3\binom{n}{4}$$

$$r = 1 \Rightarrow S(n,1) = 1$$

$$r = 2 \Rightarrow S(n,2) = 2^{n-1} - 1.$$

$$S(x,y) = \frac{1}{y!}\sum_{j=0}^{y} (-1)^j \binom{y}{j} (4-j)^x$$

3. Distinct candies, distinct bags, at least 1
* candy per bag.

First, suppose they are identical bags. So, like (3), we have $S(a,b)$. Now, for distinctness of bags we have

$$b! \times S(a,b).$$

* 7. $\sum_{k=1}^{b} S(x, k)$.

12. Identical bags, candies ; at least 1 candy / bag.
*

partition number

$p_b(x)$: # partitions of the number x into exactly b parts

(Integer partition)

(No simple formula as such)

$$p_3(6) = 3 \quad \begin{cases} 4+1+1 \\ 3+2+1 \\ 2+2+2 \end{cases}$$

$$p_5(10) = 7$$

$$\begin{cases} 6+1+1+1+1 \\ 5+2+1+1+1 \\ 4+3+1+1+1 \\ 4+2+2+1+1 \\ 3+3+2+1+1 \\ 3+2+2+2+1 \\ 2+2+2+2+2. \end{cases}$$

* 10. Identical bags, candies; any # candy in each bag.

e.g. 6 id. candies, 3 id. bags.

$p_3(6)$ to use 3 bags = 3 ways

$p_2(6)$ " " 2 bags = 3 ways.

$p_1(6)$ " " 1 bag = 1 way

7 ways.

$$\sum_{k=1}^{3} p_k(6).$$

Generally, $\sum_{k=1}^{b} p_k(x)$.

* Partition number $p_k(n)$

Partition n into k parts.

for $1 < k < n$,

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$$

* Pascal's Triangle

row 0 $\quad 1$

row 1 $\quad 1 \quad 1$

$1 \quad 2 \quad 1$

$1 \quad 3 \quad 3 \quad 1$

$1 \quad 4 \quad 6 \quad 4 \quad 1$

$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$

$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$

N

Triangular number

Squares

Tetrahedral numbers

$3 \to 4$, $6 \to 9$, $10 \to 16$, $15 \to 25$

for $n \geq 0$, the n^{th} row of pascal's triangle

$$\binom{n}{0} \binom{n}{1} \binom{n}{2} \cdots \binom{n}{n}$$

for $0 < k < n$, $\boxed{{}^nC_k = {}^nC_{n-k}}$

$\boxed{{}^nC_k = {}^{n-1}C_k + {}^{n-1}C_{k-1}}$. Pascal's identity

↳ Combinatorial proof:

Choosing k from n $\approx$

how many ways to choose k

objects from n objects such that one particulare object is never chosen.

$\Rightarrow {}^{n-1}C_K$ [$n-1$ as one particular object is never chosen]

Now, # ways to choose K objects from n objects such that; that particulare object is always chosen.

$\Rightarrow {}^{n-1}C_{K-1}$ [as that object is always chosen]

So, in total ${}^{n-1}C_K + {}^{n-1}C_{K-1}$

The particulare object can be any of n objects, so, ${}^{n}C_K = {}^{n-1}C_K + {}^{n-1}C_{K-1}$.

* Row sum of Pascal's triangle 2^n.

$$\Rightarrow \boxed{{}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \dots + {}^{n}C_n = 2^n}$$

Combinatorial proof. $\Rightarrow$

ways to choose any # of objects from n objects.

total $= {}^{n}C_0 + {}^{n}C_1 + \dots + {}^{n}C_n$

each obj. can be taken or not. $\Rightarrow 2^n$

* Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} {}^nC_k \, x^k y^{n-k}$$

$(x+y)^0 = 1$

$(x+y)^1 = 1.x + 1.y$

$(x+y)^2 = 1.x^2 + 2xy + 1.y^2$

$(x+y)^3 = 1.x^3 + 3x^2y + 3xy^2 + 1.y^3$

$\vdots$

→ Combinatorial proof ⇒

x girls, y boys, n distinct candies.

ways to allocate the candies.

Each candy can go to any of the x girls or y boys. ⇒ $(x+y)^n$

[$x+y$ options for each candy of total n candies]

⇒ Now, when the girls get k candies : $\binom{n}{k} x^k y^{n-k}$

(choosing k from n for the girls)

Sum over all possible k | x choices for each of the k candies

$\Rightarrow \sum_{k=0}^{n} {}^nC_k \, x^k y^{n-k}$ | y choices for each of $n-k$ candies (that go to boys)

#) When $x = 1, y = 1$ in Binomial theorem,

$$(1+1)^n = \sum_{K=0}^{n} \binom{n}{K} 1^K 1^{n-K}$$

$$\Rightarrow \sum_{K=0}^{n} \binom{n}{K} = 2^n.$$ [Another proof]

※ In Pascal's triangle, in a row, sum of odd positions $= 2^{n-1}$ (same for even positions).

eg. 6^{th} row 1 6 15 20 15 6 1.

odd 32. even 32

So, alternating $(+,-)$ sum of ${}^nC_K$ (for $K=0$ to n) is 0.

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots \pm \binom{n}{n} = 0$$

$$\boxed{\sum_{K=0}^{n} (-1)^K \binom{n}{K} = 0}$$

Proof by Binomial theorem,

$x = -1$, $y = +1$.

$$\sum_{K=0}^{n} \binom{n}{K} (-1)^K 1^{n-K} = 0.$$

-o-

* Hockey Stick Identity.

Pascal's triangle (Flat)

1	1						
2	1	1					
3	1	2	1				
4	1	3	3	1			
5	1	4	6	4	1		
6	1	5 +	10	10	5	1	
7	1	6	15	20	15	6	1

Sum through any col.
⇒ We get the number of next row's next col.

$$ {}^{1}C_1 + {}^{2}C_1 + {}^{3}C_1 + {}^{4}C_1 + {}^{5}C_1 = {}^{6}C_2 $$

Generally,

$$ \boxed{\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}} $$

* Number of odd numbers in a row of Pascal's triangle =

$$ 2^{\wedge}(\#\,1\text{'s in the binary rep}^n \text{ of row no.}) $$

eg. 6^{th} row. $6 = (110)_2$

$\#$ odd numbers $= 2^2 = 4.$

⇒ # even numbers in a row = $(n+1) - \#\text{odd}$

* Binomial Probability

$$ {}^{n}C_r \, p^r (1-p)^{n-r} $$

* Multichoosing $\left(\!\binom{n}{k}\!\right)$ Multisubset

#ways to choose k objects from a set of n objects where order is not important, but repetition is allowed.

eg. From 3 flavors, make a double cup (containing 2 scoops, not necessarily different flavours) $\left(\!\binom{3}{2}\!\right) = 6$ 11, 12, 13, 22, 23, 33

eg. 31 flavors, triple cup $\left(\!\binom{31}{3}\!\right)$

Containing 3 flavors (all different) $= \binom{31}{3} = 4495$

$\binom{31+3-1}{31-1}$ " 2 flavors $= \binom{31}{2} \cdot 2 = 930$

$\binom{31+3-1}{3}$ " 1 flavor $= 31$

$+ \quad 5456 = \left(\!\binom{31}{3}\!\right)$

eg $\left(\!\binom{2}{3}\!\right) = 4$ | 2 flavors, 3 scoops. 111, 112, 122, 222

※ $n = 3, K = 10$ (3 flavors, 10 scoops) (Method of candies & bars) |★|

arrange 10 candies, 2 bars

o o o o | o o | o o o o

flavor 1 flavor 2 flavor 3

Arrangements possible of these candies & bars $= \binom{12}{2} = \binom{12}{10}$

Generally, n flavors, k scoops

$\Rightarrow$ k candies, $n-1$ bars

$$\boxed{\left(\binom{n}{k}\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}}$$

— Multichoose formula for multisubset
(No order, repetition allowed)

$$\left(\binom{n}{k}\right) = \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}$$

$n = n+k-1-k+1$

eg. Distribute 10 identical candies to 3 ninjas.

$\cancel{n=10}$ $\cancel{k=3}$ $n = 3$ $k = 10$

o o | o o o o | o o o o

$$\Rightarrow \binom{10+2}{2} = \binom{12}{2}$$

eg k identical candies, n ninjas, each getting at least one.

$\rightarrow$ Each ninja gets one candy first.

Remaining candies $= k-n$

$$\binom{k-n+n-1}{n-1} \quad \left(\binom{n}{k-n}\right) = \binom{k-1}{k-n} = \binom{k-1}{n-1}$$

Another explanation -

say, 10 candies, 3 ninjas, each gets ≥ 1.

o _ o _ o | o _ o | o _ o _ o _ o _ o

Now, the 2 bars go to any 2 of the 9 places

$\Rightarrow {}^{9}C_{2}$ $\quad \left[{}^{n-1}C_{K-1} \right]$

- $$\boxed{\sum_{K=0}^{m} \left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n+1}{m}\!\!\right)}$$

n flavors
at most m scoops

* Multinomial Theorem

$(x+y)^n = \sum \binom{n}{a,b} x^a y^b$ → Multinomial coeff

$a+b = n$.

$\frac{n!}{a!\,(n-a)!} = \frac{n!}{a!\,b!}$

a and b are any non-ve numbers that sum to n

$(x+y+z) = \sum \binom{n}{a,b,c} x^a y^b z^c$; a, b, c sum to n

General :

$$\boxed{(x+y+\dots+z)^n = \sum \binom{n}{a,b,\dots,c} x^a y^b \dots z^c}$$

a, b, ..., c sum to n

$\Rightarrow \binom{n}{a,b} = \frac{n!}{a!\,b!}$ $\qquad \binom{n}{a,b,\dots,c} = \frac{n!}{a!\,b!\dots c!}$

union — At least one of the varieties chosen

* Principle of Inclusion - Exclusion. (PIE)

$|A \cup B \cup C| = |A| + |B| + |C| + |ABC| - |AB| - |BC| - |CA|$

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |AB| - |BC| - |CD|$
$- |DA| - |CA| - |BD| + |ABC| + |ABD|$
$+ |ACD| + |BCD| - |ABCD|$

Proof Suppose one object is exactly in m sets among the n sets. $(1 \le m \le n)$

How many times this obj. gets counted?

$$\underset{\text{singleton}}{\binom{m}{1}} - \underset{\text{doubleton}}{\binom{m}{2}} + \binom{m}{3} - \binom{m}{4} + \dots \pm \binom{m}{m} = \binom{m}{0} = 1$$

So, the obj. gets counted once.

In $\sum_{i=1}^{n} |A_i|$, the object gets counted m times (once for each of m sets)

In $\sum_{i<j} |A_i A_j|$, $\binom{m}{2}$ times.

In $\sum_{i<j<k} |A_i A_j A_k|$, $\binom{m}{3}$ times.

$\vdots$

In $\sum_{i_1 < i_2 < \dots < i_m} |A_{i_1} A_{i_2} \dots A_{i_m}|$, $\binom{m}{m}$ times.

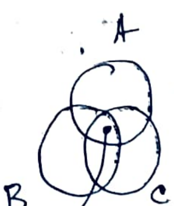
$^nC_0 - {^nC_1} + {^nC_2} - \dots {^nC_n} = 0$

$^nC_1 - {^nC_2} + {^nC_3} - \dots = {^nC_0}$

Summing up we get $\binom{m}{0} := 1$

$|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |BC| - |CA| + |ABC|.$

take a pt. (in the fig) | Counted 3 times ~~once~~ in singleton $\binom{3}{1}$ | counted ~~twice~~ 3 times in doubleton $\binom{3}{2}$ | counted once $\binom{3}{3}$



point

$^3C_1 - {^3C_2} + {^3C_3} = {^3C_0}$

Generally,

- in set theory, $\left| \bigcup_{i=1}^{n} A_i \right|$

$$\boxed{|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l| + \cdots + (-1)^{n+1} |A_1 A_2 \cdots A_n|}$$

- in probability,

PIE

$$\boxed{P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) + \cdots + (-1)^{n+1} P(A_1 A_2 \cdots A_n).}$$

eg. 6 people of different heights are getting in line to buy donuts. Compute the # of ways they can arrange themselves in line such that no 3 consecutive people are in increasing order of height, from front to back.

→ Let A be the event that the 1st, 2nd, 3rd are in ordered height,

B ~ 2nd, 3rd, 4th

C ~ 3rd, 4th, 5th

D ~ 4th, 5th, 6th

Total ways w/o any constraints $= 6! = 720$

We need to find $720 - |A \cup B \cup C \cup D|$.

By PIE,

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |AB| - |AC| - |AD|$

$- |BC| - |BD| - |CD| + |ABC| + |ABD| +$

$|ACD| + |BCD| - |ABCD|$.

$|A| =$ Choose 3 people, put them in order (one way to put 3 people in order), other 3 people can be ordered in 3! ways

$\overbrace{(- - -)}^{^6C_3}\overbrace{- - -}^{3!}$

$\binom{6}{3} \cdot 1 \cdot 3! = 120$

B $\underbrace{-(\overset{^6C_3}{- - -}) - -}_{3!}$

Same, $|B| = |C| = |D| = 120$ ...

$|AB| =$ Putting 1st, 2nd, 3rd, 4th in order.

$\overbrace{(- - - -)}^{^6C_4}\overbrace{- -}^{2!} = \binom{^6C_4} \cdot 1 \cdot 2! = 30$

$|AC| =$ Putting 1st, 2nd, 3rd, 4th, 5th in order.

$\overbrace{(- - - - -)}^{^6C_5}\overbrace{-}^{1} = (^6C_5) \cdot 1 \cdot 1 = 6$

$|AD| =$ Choosing 3 guys out of 6, then choosing 3 guys out of 3.

$\binom{6}{3}\binom{3}{3} = 20$.

$\underbrace{\overbrace{(- - -)}^{^6C_3}}_{\text{1 way order}}\underbrace{\overbrace{(- - -)}^{^3C_3}}_{\text{1 way order}}$

$|BC| = |AB| = 30$

$|BD| = |AC| = 6$

$|CD| = |AB| = 30$.

// $|ABC| = |AC| = 6$

$|ABD| =$ Everyone in order $= 1$

$|ACD| =$ " $= 1$

$|BCD| = |ABC| = 6.$

// $|ABCD| =$ Everyone in order $= 1.$

Now, $720 - |A \cup B \cup C \cup D| = \underline{349}.$

* eg 11 distinct candies to 4 children, so that each gets at least 1. (# onto f^n_s $\underset{11}{A} \to \underset{4}{B}$)

* → If no restriction # ways $= 4^{11}$
(each candy can go to one among 4)

A, B, C, D

A - child 1 gets no candy

B - child 2 gets no candy

Find - $4^{11} - |A \cup B \cup C \cup D|$

~~Child~~ 1 child gets no candy, # ways $= 3^{11} \times 4.$
(A, B, C, D) (select the child in 4 ways, give 11 candies among rest 3)

2 children don't get any candy
(AB, AC, AD, BC, BD, CD) # ways $= 2^{11} \times \binom{4}{2}.$

3 children don't get any
(ABC, ABD, BCD, ACD) # ways $= 1 \cdot \binom{4}{3}$

4 dont get any - not plausible

So, answer $= 4^{11} - 3^{11} \times 4 + 2^{11}\binom{4}{2} - \binom{4}{3}$

$4!\, S(11, 4)$
Stirling no. of 2nd kind.

eg How many integers from 1 to 100 are multiples of 2 or 3 ?

→ A be the set for multiples of 2 $|A| = 50$

B " " " " " of 3 $|B| = 33$

AB is the set that are multiples of both, hence of 6. $|AB| = 16$

By PIE, answer $= 50 + 33 - 16 = 67$

eg How many numbers b/w 1 & 1000 are not divisible by 2, 3 or 5 ?
 A B C

→ $1000 - |A| - |B| - |C| + |AB| + |AC| + |BC| - |ABC| = 266$

divisible by 2
$|A| = 500$
$|B| = 333$
$|C| = 200$

$|AB|_6 = 166$ $\quad |AC|_{10} = 100$

$|BC|_{15} = 66$ $\quad |ABC|_{30} = 33$

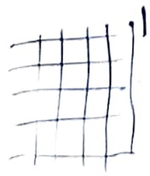
$(a,b) \to (c,d) \Rightarrow \binom{c+d-a-b}{c-a}$

* Lattice paths

possible walks from $(0,0)$ to (a,b) is

$\binom{a+b}{a}$ or $\binom{a+b}{b}$

10, 10

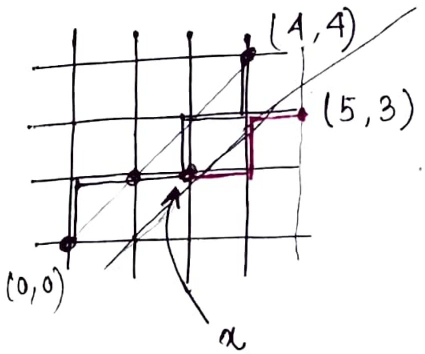


0,0

a right movements, b up movements each path sequence of length a+b. Find a places for the right among a+b places (up movements get automatically fixed)

* Lattice paths not counting the ones that cross diagonal. | C_n

$\rightarrow \binom{2n}{n} - (\#\text{violating paths})$



- reflect along a line parallel to $y=x$ passing through first violating point (α).
- observe, for (n,n) final point we always end up at $(n+1, n-1)$.
- New path : $(0,0) - (n+1, n-1)$

- Claim: # violating paths = # paths from $(0,0)$ to $(n+1, n-1)$.

(The procedure being reversible)

$\hookrightarrow \binom{2n}{n+1}$

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}$$

eg Walks from (0,0) to (10,10) that avoid (4,2).

→ how many go through (4,2)

$(0,0) \xrightarrow[4,2]{} (4,2)$ $\quad {}^{6}C_{4}$ ways

$(4,2) \xrightarrow[6,8]{} (10,10)$ $\quad {}^{14}C_{6}$ ways

Answer $= \binom{20}{10} - \binom{6}{4}\binom{14}{6}$.

Avoid (4,2) or (8,7).

$$\binom{20}{10} - \underbrace{\binom{6}{4}\binom{14}{6}}_{\text{through } (4,2)} - \underbrace{\binom{15}{8}\binom{5}{2}}_{\text{through } (8,7)}$$

0,0 → 8,7
8,7 → 10,10

$$+ \underbrace{\binom{6}{4}}_{0,0-4,2}\underbrace{\binom{9}{4}}_{4,2-8,7}\underbrace{\binom{5}{2}}_{8,7-10,10}$$

through (4,2) & (8,7)

eg How many ways can n homeworks be returned to n students such that no student gets his own homework back?

(Derangements)

Unrestricted $= n!$

One student gets own homework $= \binom{n}{1}(n-1)!$

Two students get own homework $= \binom{n}{2}(n-2)!$

$\vdots$

one or more gets their own homeworks

We need to find $n! - (A_1 \cup A_2 \cup \cdots \cup A_n)$

where A_i is the event where i students get their own homework.

$$\text{Answer} = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots$$

$$= \sum_{k=0}^{n} \binom{n}{k}(n-k)!\,(-1)^k$$

$$\boxed{D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}}$$

Derangements

Prob. that nobody gets their own homework

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \pm \frac{1}{n!}$$

As n grows, $\frac{D_n}{n!} \to 0.367879$

$= \frac{1}{e}$

(Independent of n)

* **Derangements** | Arrangement such that no object goes to its specified position.

eg. If 4 objects are there,

when unrestricted $\rightarrow 4! = 24$

$$\left.\begin{array}{llll}
1 \text{ at its correct place} & \rightarrow \binom{4}{1}(3!) & = 24 & - \\
2 \text{ " " " "} & \rightarrow \binom{4}{2}(2!) & = 12 & + \\
3 \text{ at " " "} & \rightarrow \binom{4}{3}(1!) & = 4 & - \\
4 \text{ at " " "} & \rightarrow \binom{4}{4}(0!) & = 1 &
\end{array}\right\} C_1 \cup C_2 \cup C_3 \cup C_4$$

$$N(\bar{C}_1 \bar{C}_2 \bar{C}_3 \bar{C}_4) = 4! - 24 + 12 - 4 + 1 = 9 \text{ ways.}$$

as

unrestricted arrangement	=	some at their correct places	+	no one at their correct places
$4!$		$C_1 \cup C_2 \cup C_3 \cup C_4$.		

When n objects,

$$D_n = !N = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots$$

$$= \sum_{k=0}^{n} \binom{n}{k}(n-k)!\,(-1)^k \Rightarrow \boxed{D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}}$$

Prob. that no object at its own place,

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}$$

as $n \rightarrow \infty$, $\frac{D_n}{n!} \rightarrow 1/e$. (Independent of n)

eg $n = p^2 q$ where p & q are distinct prime numbers.
How many numbers satisfy $1 \le m \le n$ & $\gcd(m,n) = 1$?
(PIE, Derangement)

→ How many relatively prime?

$n = p \cdot p \cdot q$ | number m must not be divisible by ~~m~~. p or q.

$\#$ not divisible by p $= n - n/p = p^2q - pq$
$= pq(p-1)$

$\#$ not divisible by q $= n - n/q = p^2q - p^2$
$= p^2(q-1)$

$\#$ not divisible by p & q $= n - n/pq = p^2q - p$
$= p(pq-1)$

Not divisible by p or q $= (n - n/p) + (n - n/q)$
$+ (n - n/pq)$

putting $n = p^2q$, $= p(p-1)(q-1)$.

- $\boxed{D_n = (n-1)(D_{n-1} + D_{n-2})}$ with $D_0 = 1$, $D_1 = 0$.

1, 0, 1, 2, 9, 44, ...

proof $a, b, c, \ldots n$ objects.

case 1 b goes to a's place $D(n-2)$

case 2 b goes to any other place $D(n-1)$

$D_{n-1} + D_{n-2}$

$(n-1)$ choices for b: a, c, d, ..., n

$\Rightarrow (n-1)(D_{n-1} + D_{n-2})$

eg A teacher has 6 name tags to hand out her 6 students. Prob. that at least one student gets their name tag?

→ # nobody gets their name tag =

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$6! \left(\frac{1}{0!} - \frac{1}{1!} + \cdots + \frac{1}{6!} \right) = 265.$$

$$\text{Ans} = 1 - \frac{265}{6!} = \frac{91}{144}.$$

eg 10 objects, a, b, c, d, e, f, g, h, i, j. a goes to b's place. Also, no object at its correct place.

→ # ways $= \frac{!10}{9}$ [as a could go to any of the other 9 places other than it's place, & we fixed that].