* LINEAR ALGEBRA - MIT

$\Rightarrow$ $2x - y = 0$ —(i)
$-x + 2y = 3$ —(ii)

① Matrix form: $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$; $AX = B$

② Row Picture:
[Plot 2 lines in 2D space, intersⁿ = Solⁿ]

③ Column Picture
$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ (LC of cols) imp

$\Rightarrow \begin{bmatrix} 2x - y = 0 \\ -x + 2y - z = -1 \\ -3y + 4z = 4 \end{bmatrix}$ → ∴ 3 Planes meet at a Point, & that can be a solution [max].
∴ 2 Planes at max, form a line.

---

* Solⁿ for $AX = B$; for every B, exists if the linear combination of the colⁿ of A, fill the entire nD space.
→ if the colms lie in the same plane: LC won't give 3D space, it will be in that Plane.

imp → $\boxed{AX = \text{LC of col of A}}$ ↳ ∴ B can be any other Plane.

---

* Elimination with Matrices

$\boxed{\text{Purpose of eliminⁿ: } A \longrightarrow U}$ (Upper Δ Matrix)

imp (i) Success: n Pivots, failure: < n Pivot.
(ii) if Pivot = 0, exchange with row (with non zero Pivot below)

eg → Pg 5 [Success / failure of elimination]

- After bring to U form, use Back Substitution to compute the solution.

column Vector

imp

$c_1 c_2 c_3$

* $\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix}3\\4\\5\end{bmatrix}$ = LC of columns of Matrix
  $3C_1+4C_2+5C_3$
* $[1\ 2\ 3]\begin{bmatrix} \\ \\ \end{bmatrix}$ = LC of rows of Matrix
  $1r_1+2r_2+3r_3$

RHS [B]

row Vector

* Elimination using Elimination Matrices

$$\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}\boxed{1}&2&1\\3&8&1\\0&4&1\end{bmatrix}=\begin{bmatrix}1&2&1\\0&2&-2\\0&4&1\end{bmatrix}$$

$r_2=-3r_1+r_2=r_2-3r_1$ (Sub $3r_1$ from r_2) E_{21}

$\boxed{\text{eg}}$ → Pg 7 [E_{21}, E_{32}]

∴ $\boxed{(E_{32}\cdot E_{21})A=U}$ imp

[exchange row/col]

* Permutation Matrix: $\overset{P}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}c&d\\a&b\end{bmatrix}$ ①

[Exchange rows] $r_2=1r_1+0r_2$

imp

② [Exchange cols] → $\begin{bmatrix}a&b\\c&d\end{bmatrix}\overset{P}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}=\begin{bmatrix}b&a\\d&c\end{bmatrix}$

$c_1=0\cdot c_1+1\cdot c_2$.

* Inverse = Reverse the operⁿs
  $r_2=-3r_1+r_2 \xrightarrow{\text{inv.}} r_2=r_2+3r_1$

* Multiplication of 2 Matrices

① $[\ ]_A\cdot[\ ]_B=[\ \square\ ] \to a_{34}$
$=row_3(A)\times col_4(B)$

② $\underset{m\times n}{[\ ]_A}\cdot\underset{n\times p}{\overset{c_1}{[\ ]_B}}=\overset{c_1^c}{[\ ]_C}$ → col of C are comb of columns of A
$\boxed{A\times c_1=c_1^C}$

③ $r_1\,[\ ]_A\times[\ ]_B=r_1\,[\ ]_C$ → rows of C are comb of rows of B
$\boxed{r_1\times B=r_1^C}$

imp

④ $\begin{bmatrix}2&7\\3&8\\4&9\end{bmatrix}\begin{bmatrix}1&6\\0&0\end{bmatrix}=\begin{bmatrix}2\\3\\4\end{bmatrix}[1\ 6]+\begin{bmatrix}7\\8\\9\end{bmatrix}[0\ 0]$

$\Sigma(col(A)\times row(B))$

* Inverses: $\boxed{AA^{-1}=A^{-1}A=I}$
  // for Sq. Matrix, Left Inv = Right inv.

Q How to find if Matrix A is invertible?

⇒ for Non-invertible / Singular Matrices:
$\boxed{AX=0}$, $X\neq 0$ imp. $\boxed{|A|=0}$

$\boxed{\text{eg}}$ → Pg 11 [2] Proof

* Find Inverse of Invertible Matrix

$$\underset{A}{\begin{bmatrix}1&3\\2&7\end{bmatrix}}\cdot\underset{A^{-1}}{\begin{bmatrix}a&c\\b&d\end{bmatrix}}=\underset{I}{\begin{bmatrix}1&0\\0&1\end{bmatrix}}$$

$\boxed{E*[A\ I]=[I\ A^{-1}]}$ // Gauss Jordan Idea → $\boxed{E=A^{-1}}$

imp

$\boxed{\text{eg}}$ → Pg 12 [GJ Idea to Solve Inverse Matrix]

* $\boxed{(AB)^{-1}=B^{-1}A^{-1}}$, $\boxed{(A^T)^{-1}=(A^{-1})^T}$

* A: 3×3
  $E_{32}(E_{31}(E_{21}A))=U$ [No Row exchanges].
  ∴ $A=[E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}]\cdot U$.

imp → ∴ $\boxed{A=LU}$ // Basic factorization of Matrix

If no row exchanges: $L\to(E_{21}^{-1}E_{31}^{-1}\dots)$

* How Expensive is Elimination?
  # operations on n×n Matrix A.

$\underset{100^2}{\begin{bmatrix}\square& \\ & \end{bmatrix}}+\underset{99^2}{\begin{bmatrix}\square\end{bmatrix}}\dots \Rightarrow \boxed{O(n^3)}$ imp

imp

* Permutations → identity matrix with reordered rows.

- $A = 3\times3$ matrix
- P_{12}: Permutation M, that exchange $r1$ & $r2$.
  - 6 or $3!$ Permutatⁿ Matrix exists.

→ $n\times n$ Matrix → $n!$ Permutⁿ Matrices.

→ ($P_{12}, P_{13}, P_{23} \dots$) forms a closed group, imp

$\boxed{P^{-1} = P^T}$ (Orthogonal M)

imp $\boxed{PA = LU}$ // With row exchange.

→ does row exchanges, when pivot $= 0$.

$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $A \cdot P \rightarrow$ exchange rows of A.

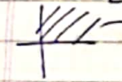
* Transpose : $\boxed{(A^T)_{ij} = A_{ji}}$

→ Symmetric Matrix : $A^T = A$

→ $R^T R$ is always symmetric

* Vector Spaces : A Space of Vectors, with any LC between them, are still in the Space. ∴ $\boxed{\text{Closed under LC}}$ → imp

eg: R^3: All Vectors with 3 comp in 3D Space. [add/mult with const], still in R^3.

eg:  → $\begin{bmatrix} 2 \\ 3 \end{bmatrix} x - 7 = \begin{bmatrix} -14 \\ -21 \end{bmatrix}$, out of Space

∴ Not Vector Space.

* SubSpace : A Vector Space inside VS.

eg: line thru origin in $R^2/R^3 \dots R^n$.

Possible SS of R^3 imp
- → all of R^3
- → Plane/ line through origin
- → only origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

* Column Space : Create SubSpaces out of A

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$; $\underbrace{c\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c'\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}}$

all LCs of columns form the column Space $C(A)$ in R^3.

→ [at max, forms plane] ∵ 2 Vectors

imp
- if, P & L = Subspace
  1) $P \cup L$:  [Not a SS], if line not in Plane. (goes out of Space)
  2) $P \cap L$: [It's a SubSpace] → Origin

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ → LC of these columns = Column Space.

[they form Plane, doesn't fill R^4 at all] → 2D imp [depend]

Note: System of Eqⁿs : $AX = B$ can only be solved when B is a vector in the column Space of A / $B = $ LC of col of A.

→ Not Independent columns ∴ they form a 2D Subspace in R^4; Even if we throw C3, we can get the same column Space.

* NullSpace of A [N(A)] $\boxed{|A| = 0 \rightarrow X \neq 0}$

- It contains all Solutions (X) to $AX = 0$.

eg: $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $X \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

∴ $\boxed{c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \text{Null Space [Line in } R^3]}$ imp

$\boxed{\text{Column Space} \rightarrow \text{2D Subspace in } R^4}$

imp

* $AX=0$, always has a Null Space.

eg → Pg 25 [NullSpace/SS has to go through origin]

* Solve $AX=0$ [for N(A)] $\boxed{r_1+r_2=r_3}$

$A \to U$

$$A=\begin{bmatrix}\boxed{1}&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}$$

$\because C_2=2C_1$
$\therefore$ Column space is a 3D subspace in R_3.

$$\begin{bmatrix}1&2&2&2\\0&0&\boxed{2}&4\\0&0&2&4\end{bmatrix} \to \begin{array}{cccc}p&f&p&f\end{array}\begin{bmatrix}\boxed{1}&2&2&2\\0&0&\boxed{2}&4\\0&0&0&0\end{bmatrix}_U$$

$\boxed{\text{Rank}(A)=\#\text{Pivots}}$

imp

$\Rightarrow x_1+2x_2+2x_3+2x_4=0$
$2x_3+4x_4=0$

$X=\begin{bmatrix}x_1\\1\\x_3\\0\end{bmatrix}$ → any fV.

$\therefore \left(c\begin{bmatrix}-2\\1\\8\end{bmatrix}+c'\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}\right)$ → NullSpace [LC of spcl solutions].

* If $A: m\times n$, rank $=r$.
#pivots $=r$, # free columns $=n-r$.

* Reduced Row Echelon form (R) [cleaner U]

$A \to R$

$$\underset{A}{\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}} \xrightarrow{\text{Elim.}} \begin{array}{cccc}p&f&p&f\end{array}\underset{R}{\begin{bmatrix}1&2&0&-2\\0&0&1&2\\0&0&0&0\end{bmatrix}}$$

Null Space dir. computed → $\begin{bmatrix}-F\\I\end{bmatrix}$

eg → Pg 27

$$N(A)=\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

Note: Solution to $AX=0/UX=0/RX=0$ are all same.

* $\boxed{R=\begin{bmatrix}I&F\\0&0\end{bmatrix}\ ;\ \underset{(N)}{X}=\begin{bmatrix}-F\\I\end{bmatrix}\ \text{(mixed)}}$ imp

eg → Pg 28 [Solve $AX=0$, (row exchange), compute Null Space (with or without RRF)].

* Solving $AX=B$ → Complete Solution.
- Perform elimination on Augmented Matrix.

Note: $AX=B$ is solvable:
· exactly, when B is in Column Space of A
imp → if comb of rows of A gives 0 row, then the same combn of entries of B must give 0.

[$AX=B$]

1) $X_{particular}$: Set all FV $=0$, then solve for PV.

eg → Pg 30 [$X_{particular}$]

2) $\boxed{X_{NullSpace}+X_{particular} \Rightarrow \text{Complete Sol}^n}$

imp

→ if Plane not thru origin, & line any dir = not SS.

[Rank tells us about no of solns]

imp

① Full Column Rank ($r=n<m$)   [# eqns > variables]   [no rel^n b/w cols]
- Pivot in every column ; free V $=0$, $\boxed{N(A)=0}$ only 0 vector.
$R=\begin{bmatrix}I\\0\end{bmatrix}$; only $X_{particular}$

$\therefore$ Unique $=0$ or 1 Solution.

② Full Row Rank ($r=m<n$)   [eqns < # Varib]   [any free V can be put to obt. LC $=0$]
- Pivot in m row/col. $\therefore$ $n-m$ free V. → $\therefore$ N(A) exists.
$R=[\,I\ F\,]$ (partly mixed)
$\therefore$ always a solution, ∞ sol^{n}s.

③ Full Rank ($r=m=n$) $\boxed{R(A/B)=n}$
- $R=I$
- unique / 1 Solution definitely.

④ $r<m$, $r<n$ $\boxed{R(A/B)<n}$
$R=\begin{bmatrix}I&F\\0&0\end{bmatrix}$
$\therefore$ 0 or ∞ Solutions → $X=\begin{bmatrix}-F\\I\end{bmatrix}$

KEY CONCEPTS

$\boxed{AX=0;\ |A| \neq 0}$

① * Independence: Vectors $x_1, x_2, x_3 \cdots x_n$ are linearly independent if no combination gives 0 vector $\boxed{N(A)=0}$ [No free V]

→ if 0 vector $= V_1, V_2$ ∴ $V_1 + 0V_2 = 0$
[V_1, V_2 are linearly dependent]

** When $V_1, V_2 \cdots V_n$ are the columns of A, they are independent if the null space $(A) = 0$.
[Rank $= n$, free $V = 0$]

** They are dependent if $AC = 0$, for some non zero C in $N(A)$. [free $V \neq 0$, Rank $< n$]
[some comb of col^ns add to 0]

imp

② * Spanning a Space: Vectors $V_1, V_2 \cdots V_n$ span a space: Space consists all combinations of those vectors. eg: col^ns span the column space.

③ Basis: for a VS is a sequence of Vectors: $V_1, V_2 \cdots V_d$ with 2 properties:
(i) they are independent
(ii) they span the Space.

imp

eg → Pg 36 [Bias for R3 VS]

Note: R^n: n Vectors give a bias.
for $n \times n$ matrix: those cols give a bias for R^n, if A is invertible ($N(A)=0$)

imp

invertible: $|A| \neq 0$: rank $= n$, free $V = 0$, $N(A) = 0$.
[n Basis] ∴ definitely independent cols.

* Given a Space [R^n], every basis for the space has the same no of Vectors [Dimension]

eg: $\begin{array}{c} \begin{array}{cccc} p & p & f & f \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \end{array}$ [$C(A) = 2D$ Subspace in R^3].

– What is the basis of $C(A)$?
(C_1, C_2), (C_1, C_3), (C_2, C_4) : Basis.

* rank(A) = # of pivot cols
= Dimension of the Column Space
= No of Vectors in each Basis of $C(A)$

[Dimension of Col Space]

imp

* Dimension of Null Space (A)
dim $N(A)$ = # free $V = n - r = 2$
Sol^ns of $AX = 0$, can form a 2D SS in R^4

eg → Pg 38

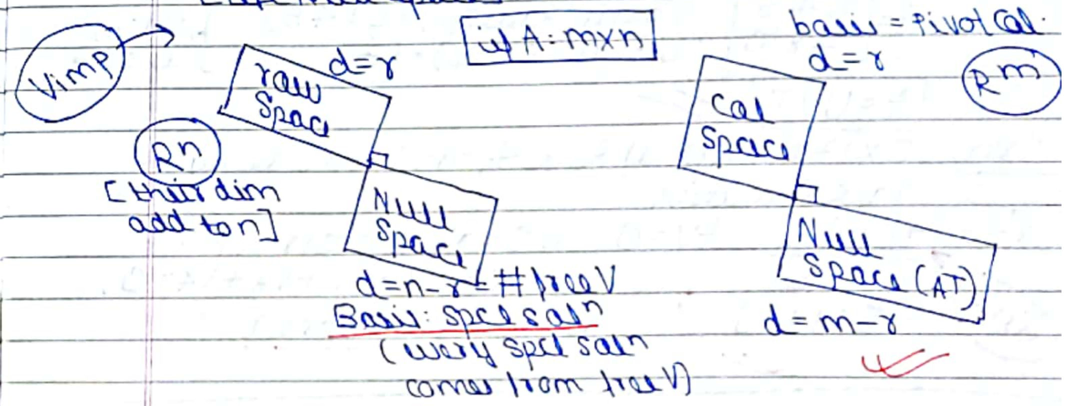
* THE 4 Fundamental SubSpaces

A: $m \times n$

1) Column Space: $C(A)$ in R^m
2) Null Space: $N(A)$ in R^n
3) Row Space / all comb. of col$(A^T) = C(A^T)$ in R^n
4) Null Space (A^T): $N(A^T)$ in R^m
[left Null Space]

imp

Vimp



eg → Pg 39 [Last eg]

Note: Row Space of A / REF will be same: Basis of row space will be the first r rows of A. (imp)

* Left Null Space / $N(A^T)$: Comb of rows to give 0
  $N(A)$: Comb of columns to give 0.

MATRIX SPACE

* A New Vector Space ($R^{n\times n}$)
  - All 3x3 Matrix form the matrix Space (M)

  $\underline{\text{Basis}(M)}$: $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \ldots\ldots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$

  (imp) dim of 3x3 MS [$R^{3\times3}$] → $\boxed{\dim(M)=9}$

  Subspaces of M:
  - (i) Symmetric 3x3 SubSpace: $\dim(S)=6$.
  - (ii) Upper Δ 3x3 SubSpace: $\dim(U)=6$
  - (iii) Diagonal 3X3 SubSpace: $\dim(d)=3$
    [$S\cap U$]

$$\boxed{\overset{6}{\dim}(S)+\overset{6}{\dim}(U)=\dim(\overset{3}{S\cap U})+\dim(\overset{9}{S+U})}$$

(imp) → any S, any U for the entire M.

* Rank 1 Matrices [Building Blocks of Matrices]

$A=\begin{bmatrix}1&4&5\\2&8&10\end{bmatrix}$ V^TBasis (Row Sp.) $=[1\ 4\ 5]$, U Basis (Col Sp) $=\begin{bmatrix}1\\2\end{bmatrix}$ } ∵ dim is same

$\boxed{A=UV^T}$

Note: 5x17 matrix of Rank 4, can be made by 4 rank 1 matrix.

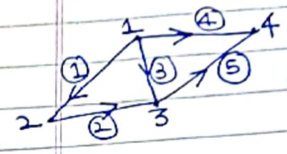
eg → Pg 45 $AV=0$, if S = all Vectors in R^4 with $V_1+V_2+V_3+V_4=0$, is it a subspace?

(imp) $A=[1\ 1\ 1\ 1]$

$x_1, x_2, \ldots$ = emf
$y_1, y_2, \ldots$ = current

[Not so imp]

* Graphs, Network, Incidence Matrix



$$\begin{array}{c|c} & \begin{matrix}1&2&3&4\end{matrix} \leftarrow \text{nodes}\\ \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{bmatrix}-1&1&0&0\\0&-1&1&0\\-1&0&1&0\\-1&0&0&1\\0&0&-1&1\end{bmatrix}\end{array}$$

Rows 1, 2, 3: loop [rows are dependent]. Rows 1–5 are the edges.

Incidence Matrix (A)

$\boxed{AX=0}$ $\{e\}$ eqn (i) $= \begin{bmatrix}x_2-x_1\\x_3-x_2\\x_3-x_1\\x_4-x_1\\x_4-x_3\end{bmatrix}$ → potential diff across edges.

1) $\boxed{N(A)}$ → $\dim = \underset{[4-3]}{1}$, Basis $= c\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ (every node constant pot)

2) $\boxed{\dim C(A)}=3$ [rank]

$\boxed{A^Ty=0}$ $= \begin{array}{c}1\\2\\3\\4\end{array}\begin{bmatrix}-y_1-y_3-y_4\\y_1-y_2\\y_2+y_3-y_5\\y_4+y_5\end{bmatrix}$ → At Node flow, incom. Curr = out Curr

(Kirchoff's Current law).

3) $\boxed{N(A^T)\text{: Basis}}$ → 2 Vectors: $\begin{bmatrix}1\\1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\\1\end{bmatrix}$
(due to 2 loops)

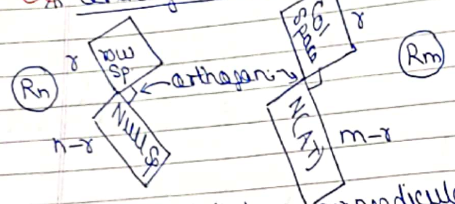
4) Row Space (A)

Note: dim (Row Space) = dim (Col Space) = no. of indep. edges (No loops: Tree) = 3

∴ $\boxed{\#\text{nodes} - \#\text{edges} + \#\text{loops} = 1}$ // Euler's Form

(Potent.) (Ohm's) (Kirchoff's)
• $e=AX$, $y=Ce$, $A^Ty=f$
∴ $\boxed{A^TCAx=f}$ → Imp eqn. **

* Orthogonal Vectors & SubSpaces



⇒ Orthogonal Vectors: perpendicular Vectors.

imp * if $x^Ty = 0$; x & y are orthogonal.

* $|x|^2 + |y|^2 = |x+y|^2$ // Pythogoras th.

∴ $\boxed{x^Ty = 0 \longrightarrow \text{orthogonal}}$ // from Pyth.

* 0 Vector & any Vector is orthogonal

* SubSpaces S & T are orthogonal, if every Vector in S, is orthogonal to every vector in T. imp

eg → Pg 53 [Prove row Sp ⊥ Null Sp.]

* Orthogonal SS in 3D

row Space = 1D (Line)
Null Space = 2D (Plane) } Normal to Plane.

Note: Row Sp. & Null Sp are orthogonal complements of each other.

* Solving AX=b "when No Soln exists" (b not in CS of A) $\boxed{A\hat{X} = p}$

$p = Pb$

$= \frac{A^TA}{A \cdot A^T} b$

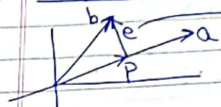
· When m>n (no. of eqns > unknowns)
convert: $AX = b \longrightarrow \boxed{A^TA\hat{X} = A^Tb}$

[A^TA → Square, Symmetric Matrix
rank(A) = rank(A^TA)
N(A) = N(A^TA)] imp

⇒ Use concept of Projection

p: Projectn of b onto C(A)
P: projection matrix.

* Projection onto SubSpaces [2D]



[error(e) = b − p]

$\boxed{p = Pb}$ [Pb has to be in the C(P)]

① $\boxed{p = Xa}$ (X: some multiple)

② $a^T(b - Xa) = 0 \longrightarrow \boxed{X = \frac{a^Tb}{a^Ta}}$ imp

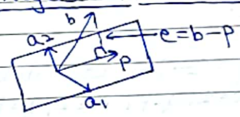
3. $\boxed{p = Pb} = a \cdot \frac{a^Tb}{a^Ta} \Rightarrow \boxed{P = \frac{aa^T}{a^Ta}}$ // P = Proj. Matrix.

→ C(P) = line through a ∴ Pb = p.

→ [P is Symmetric & $P^2 = P$ (double Projectn)] imp

* Why Project? ∵ AX=b has no soln, choose closest vector in C(A): p, ∴ $\boxed{A\hat{X} = p}$ has solution. imp — Projection of b.

* Projection into SubSpaces [3D]



· a_1, a_2: basis of C(A) (plane)

① $\boxed{A\hat{X} = p}$ // Projn of b onto plane. [We need to solve]

∴ b is ⊥ to a_1 & a_2.

∴ $\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{X}) = 0$

2. $A^T(\underbrace{b - A\hat{X}}_{e}) = 0$ ∵ e is in N(A^T), ∴ ⊥ to C(A)

$\boxed{A^TA\hat{X} = A^Tb} \longrightarrow \boxed{\hat{X} = A \cdot (A^TA)^{-1}A^Tb}$ ②

3. $p = Pb \longrightarrow \boxed{P = A(A^TA)^{-1}A^T}$ ③ // P = Proj. Matrix. imp

Note: If A = Square Invertible, $C(A) = \mathbb{R}^n$, ∴ Pb in Col. Space, ∴ $P = I$ imp

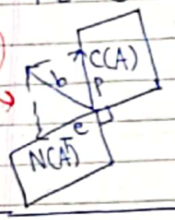
→ P is Symmetric & $P^2 = P$.

* **Application:**
1. Least Squares fitting by a line
   - Find the best line
eg → Pg 59 [Appln of Projection] ← imp

* **Projection Matrices & Least Squares**

$Pb = b$: if b already in CS
$Pb = 0$; if $b \perp CS$ (b in N(AT))

imp

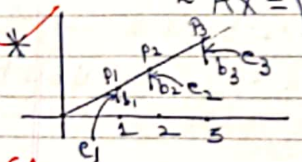


$\boxed{P = A(A^TA)^{-1}A^T}$

$\cdot\ p + e = b$
$\underbrace{Pb\ \ (I-P)b}_{\text{Both are Projection Matrices}}$

Least Best Line ⇒ To find the best fit line: $AX = b$. ($b \neq C(A)$)
$A^TA\hat{X} = A^Tb$ ← (Solve)
$\approx A\hat{X} = p$ ←

*

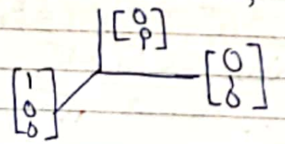


→ $y = C + Dt$. [eqn of line].
($t=1, y=P_1, t=2, y=P_2$).

eg → 61
Solving $A^TA\hat{x} = A^Tb$ gives 2 eqns : $\boxed{f = e_1^2 + e_2^2 + e_3^2}$
imp →
$3C + 6D = 5 \longrightarrow \partial f/\partial C = 0$
$6C + 14D = 11 \longrightarrow \partial f/\partial D = 0$.

* A^TA → Symmetric Matrix
if A = invertible, → A^TA = invertible.

* ⇒



→ Orthonormal Vectors. (unit modulus & orthog)

* **Orthonormal Vectors** $[q_1, q_2 \cdots q_n]$

⇒ $q_i^T q_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$, ⊥ to each other. $\therefore |q|^2 = 1$ → Orthogonal Matrix

⇒ $Q^TQ = \underbrace{\begin{bmatrix} q_1 & — \\ \vdots & \\ q_n & — \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$

$\therefore \boxed{Q^TQ = I,\ Q^T = Q^{-1}}$ ← imp

Q → Square Matrix with Orthonormal cols.
[Orthogonal Matrix]

eg: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ ← orthogonal Matrix (Q).
· product = 0 (⊥)

* $\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$ → these 3 cols are orthonormal basis for the 3D SubSpace

⇒ **Projection Matrix** [if A = orthogonal Matrix (Q)]

imp
- $P = Q(\underbrace{Q^TQ}_{I})^{-1}Q^T = QQ^T = I$ (if Q = Sq. Matrix) $Q^T = Q^{-1}$
- $P = QQ^T$ (if Q = rectangular Matrix)
  [But idempotent & Symmetric]

⇒ $A^TA\hat{x} = A^Tb$ [Imp Q - for finding Best line].
· $\underbrace{Q^TQ}_{I}\hat{x} = Q^Tb$ ⇒ $\boxed{\hat{x} = Q^Tb}$ // Much Simplify
imp

* **Graham Schmidt**
⇒ make matrix a, b orthogonal vectors
$(a, b) \longrightarrow \underbrace{(A, B)}_{\perp} \longrightarrow \underbrace{q_1 = \frac{A}{|A|},\ q_2 = \frac{B}{|B|}}_{\text{Orthonormal Vectors}}$

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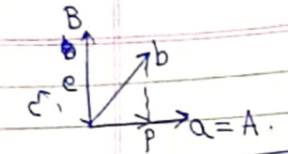
$\boxed{p = Pb}$



$$\boxed{\begin{aligned} A &= a \\ B &= b - p = b - \frac{A^Tb}{A^TA} \cdot A \end{aligned}}$$

imp

$$\boxed{C = c - \underbrace{\frac{A^Tc}{A^TA} \cdot A}_{\text{C's component in A}} - \frac{B^Tc}{B^TB} \cdot B}$$ → C's comp. in B

* GS adjusted $a, b \longrightarrow A, B$ [shifted b to 90°]
  CS still same.
* $A = LU \xrightarrow[GS]{} A = QR$

* DETERMINANTS → No. associated with every Square Matrix.

1. $|I| = 1$
2. exchange odd rows = −ve sign
   even rows = +ve sign

imp

$$\boxed{\det(\text{Perm. Matrix}) = \begin{cases} 1, & \text{if even \# exchange} \\ -1, & \text{if odd \# exchange} \end{cases}}$$

3. a) $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix} = t|A|$.
   b) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ ← imp

   // det is linear funct^n of each row.
4. if 2 equal rows, $|A| = 0$.
5. Apply any row-transformation / col trans.
   ⟹ det doesnot change.
6. if Row of zeros, $|A| = 0$.
7. Upper Δ, lower Δ, Scalar, Diagonal ⟹
   det = Product of diagonals
   ∵ By Possible row transf ⟹ $\begin{vmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_n \end{vmatrix} = d_1 d_2 d_3 \cdots d_n |I|$

imp

$\boxed{\text{det = Product of the Pivots}}$

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8. if $|A| = 0$, $A \to$ singular
   $|A| \neq 0$, $A \to$ non-singular / Invertible / n pivots / rank = n.
9. $|AB| = |A| \cdot |B|$
   $|A^{-1}| = 1/|A|$
   $|KA| = K^n |A|$ ← imp
10. $|A| = |A^T|$ (Proved: $A = LU$, $A^T = U^TL^T$. (Dig Prod.

* Determinant Formula

$\Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{0}$ (Linear Prop.)

$\Rightarrow ad - bc$ ∵ [1 for every row/col]

$\Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \cdots\cdots \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$ (3!)

i) $\boxed{(n \times n) \text{ Matrix} \longrightarrow n! \text{ ways to calc. Det}}$

imp

ii) $\boxed{\text{Cofactor of } a_{ij} = (-1)^{i+j} \det \begin{pmatrix} n-1 \text{ Matrix} \\ \text{with row } i, \\ \text{col } j \text{ erased} \end{pmatrix}}$

// Recursive formula.

$\Rightarrow \boxed{|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}}$ // along any row/col.

Cofactor Formula

* $\boxed{|A_n| = |A_{n-1}| - |A_{n-2}|}$ ← imp

eg → Pg 75 [Eq of this formula]

* Cramer's rule

Scanned with CamScanner

* Inverse

$$\boxed{A^{-1} = \frac{1}{|A|}[\text{cofactor}]^T}$$ → Adjoint

Prove: $A\cdot(\text{Adj}) = |A| I_n$ to prove above formula.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ & & & \vdots \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} d_1 & & & \\ & d & & \\ & & d & \\ & & & d_{\cdot\cdot} \end{bmatrix}$$

imp

$\because a_{11}C_{11} + a_{12}C_{12} + \cdots a_{1n}C_{1n} = |A|$.

* Cramer's Rule

$AX = b$; $X = A^{-1}b = \boxed{\frac{1}{|A|} \cdot C^T \cdot b = X}$

$\hookrightarrow C_{11}b_1 + C_{21}b_2 + \cdots$

$X_1 = \frac{|B_1|}{|A|}, X_2 = \frac{|B_2|}{|A|} \cdots$ $\boxed{B_j = A \text{ with col } j \text{ replaced by } b}$

$\boxed{X_j = \frac{|B_j|}{|A|}}$ imp

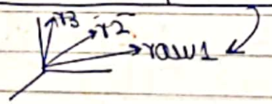
$B_1 = \begin{bmatrix} 1 & n-1 \\ b & \text{col} \\ 1 & \text{of } A \end{bmatrix}$ find all these determ to get x.

[Very time consuming]

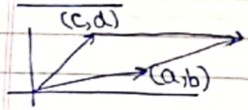
* APPLICATIONS OF DETERMINANT

1) Volume

$\boxed{|A| = \text{Volume of a Parallelopiped}}$ imp

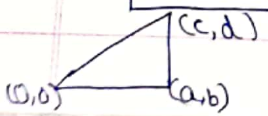


2) Area



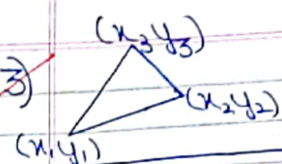
$\text{Area} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\therefore \boxed{|A| = \text{Area of } ||\text{gm (with origin)}}$



$\Rightarrow \boxed{\frac{1}{2}\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{Area of } \Delta\text{gl}}$

3)



$\text{area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ imp

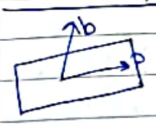
* Eigen Values & Eigen Vectors

→ E.Vector is a certain vector x, if multiplied with A, Ax comes out parallel with X.

$\therefore \boxed{AX = \lambda X}$ $\lambda \to$ E.Value, $X \to$ E.Vector.

Note: If A is singular, $|A| = 0, X \neq 0$ then $\lambda = 0$ imp

⇒ E.Values/E.Vectors of Projection Matrix



- $P\cdot X = X$ (if X is in the plane)

$\therefore \boxed{\lambda = 1}$, EV = all vectors in the C(A)

- $PX = 0$ (if $X \perp$ to Plane)

$\therefore \boxed{\lambda = 0}$

* Sum of Eigen Values = Trace.

Product of Eigen Values = Determinant.

* $AX = \lambda X \longrightarrow \underbrace{(A - \lambda I)}X = 0$. imp

↳ has to be singular, for non-zero eigen vector.

$\therefore |A - \lambda I| = 0$. // Characteristic equation.

$\boxed{\lambda^2 - (\text{Trace})\lambda + |\det| = 0}$

→ A	λ
A^2	λ^2
$A+I$	$\lambda+1$
$A+B$	$\lambda_1 + \lambda_2$ } × ($\because$ EV are different).
$A \cdot B$	$\lambda_1\lambda_2$

$\boxed{\begin{array}{l} \text{Symmetric matrix} \longrightarrow \text{real eigen Values} \\ \text{skew Symmetric} \longrightarrow 0 \text{ or imaginary EVs} \end{array}}$

* if repeated eigen Values ⇒ dependent eigen Vector.

* Eigen Values / EVectors

- For nxn Matrix A, $|A-\lambda I|$ will be a polynomial of degree n.
- EigenSpace of A, corr. to λ

$\boxed{N(A-\lambda I) : \text{Null Space}}$ ← imp

0 Vector + all EVs of $A-\lambda I$.

$\boxed{\text{eg}}$ → Pg 3

* If $v_1, v_2 \cdots v_r$ are E.Vectors that corr to distinct E.Values $\lambda_1, \cdots \lambda_r$ of an nxn matrix A, then the set $\{v_1, v_2 \cdots v_n\}$ are linearly independent.
EVector from diff EValues are LI.

* A is invertible, if 0 is not an EValue.

imp → $\boxed{\text{EValue}=0}$ → $|A|=0$, Singular,
$|A-\lambda I|=0 \therefore |A|=0$.
as EVector $= N(A)$

* Matrix Diagonalization

→ If A, B are similar, then same Char. Polynom, Same EValue, same multiplicity.
⇒ $\boxed{A = PBP^{-1}}$, B = invertible matrix

→ A is diagonalizable, if A is similar to diagonal matrix; $A = PDP^{-1}$

imp → A nxn matrix A is diagonalizable, if it has n linearly independent EVectors.

$\boxed{\text{Cols of } P = n \text{ LI EVectors of } A}$
Diagonals → EValue

$\boxed{\text{eg}}$ → Pg 6 [2Qs]

* Linear Transformation

of A: mxn matrix. $T: \underbrace{R^n}_{\text{domain}} \longrightarrow \underbrace{R^m}_{\text{codomain}}$

. for a given x in R^n, the vector $T(x)$ in R^m is called the image of x.

$\boxed{\text{eg}}$ → Pg 8 [Range ≠ Codomain]

* A transf T is linear if:
(i) $T(U+V) = T(U) + T(V)$
(ii) $T(cU) = c\cdot T(U)$.
f is invertible (one-one + onto) if pivots in every row/col, rank = n, $|D| \neq 0$.

* Application of EVectors & EValues

① Fibbonacci Sequ: → {0 1 1 2 3 5 8 13 ⋯}

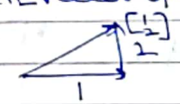
$F_n = F_{n-1} + F_{n-2}$

mostly at higher Values, doesnt rotate much.
EValue = 1.61803 (Golden Ratio).
[term × 1.61803 = next term]

* 3Blue/1Brown → Essence of LA

Vector ← CS [ordered set of no.s]
Vector ← Math [vector opern can be done]

. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ← x comp, ← Y comp



* Linear Algebra: language to describe Space.

→ $\hat{i} + 2\hat{j}$. (any other basis can be chos)

basis vectors of xy coordinate System.

* LC $\rightarrow a\vec{v}+b\vec{w}$
Span of v & w : Set of all their LC.
Span of 2 LI vectors in R^3 is a Plane.
Span of 2 LI vectors redundant.
* Linearly dep : 1 vector redundant.
Basis : Set of LI vectors that span the Space

* $\underbrace{\text{Linear}}_{\text{line remain lines, fixed origin}}$ $\underbrace{\text{Transformation}}_{\text{function}}$ → Squishing/morphing Space

i/p ⟶ o/p.

– Grid lines remain Parallel & even Spaced after transformation.

$A=\begin{bmatrix}1 & 3\\ -2 & 0\end{bmatrix}$ → Squishes & transforms the Space such that: trans $\hat{i}$, trans $\hat{j}$

trans $\hat{i}$ trans $\hat{j}$

$x_1\hat{i}+x_2\hat{j}$ = new o/p.

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = x\begin{bmatrix}a\\ c\end{bmatrix}+y\begin{bmatrix}b\\ d\end{bmatrix}$

$t\hat{i}$ $t\hat{j}$ … $\hat{i}$ … $\hat{j}$

* rotate Space by 90° $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

* Shear $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

$\boxed{\text{Matrix: transformation of Space such as Grid lines remain Parallel \& evenly Spaced.}}$

* Matrix mult as Composition

$\underbrace{\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}}_{\text{shear}}\underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{\text{rotn}}\begin{bmatrix}x\\ y\end{bmatrix}$ → First rotate, then shear.

* $t\hat{i}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}$, $t\hat{j}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1\\ 0\end{bmatrix}$

$\begin{bmatrix}t\hat{i} & t\hat{j}\end{bmatrix}$ → ans

∴ associative, but not commutative.

* 3D Linear Transformation : record transformed basis into $t\hat{i}, t\hat{j}, t\hat{k}$, &
o/p = $x_1 t\hat{i} + x_2 t\hat{j} + x_3 t\hat{k}$.
→ Usage : Computer Graphics, Robotics.

* The Determinant
How much area/volume of the grid scales after the linear transformation?
∴ |D| = 3, Area scales by 3.
if, |D| < 0 : Orientation of Space is inverted, [Flipping of Paper].

* 3D Linear Transformation : how much Vol. scales?
[Cube ⟶ Parallelopiped]
if det = 0 : line/Plane/Point. [cols/rows are LD].
∴ rank < n, free V exists, $N(A) \neq 0$.

$\boxed{|M_1M_2| = |M_1|\cdot|M_2|}$

* Linear System of Equations
→ Ax = b. $\vec{x}$ → A transf → $\vec{b}$
inverse transf.
$x = bA^{-1}$

if, |A| = 0 ; A^{-1} dne, cant unsquish a line to get a plane.
but if, Ax = b, |A| = 0 ; Soln exists if b in that line. (b in C(A) = line not Plane).

* Rank → [dim of col Space/row Space]
→ If Linear Transf squishes space into line = Rank 1
into a Plane = Rank 2.

* If a Linear Transf. squishes space onto a plane then there is a full line of vectors [through origin, ⊥ to Plane]

Plane(2) + Line(1) = Space(3)

R(A) N(A) $V(A) = R^3$.

* $A\vec{X} = \vec{0}$, $|A| = 0$; $\boxed{N(A) = X}$

Null Space = Kernel

* Eigen Vector: Vector which remains at its own span after linear transf (just scales ↑↓).

Eigen Value: Scale factor.

→ if $\lambda = -1/2$; flipped & squished by $1/2$.

* $(A - \lambda I)\vec{v} = 0$

↳ for $V \neq 0$, $(A - \lambda I)$ needs to squish space to a lower dim / $|A - \lambda I| = 0$.

$\boxed{V = N(A - \lambda I)}$

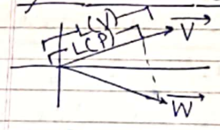
* for a rotatn matrix: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, there exists 0 eigen Vectors.

* A single Eigen Value can have many Eigen Vectors

eg: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\lambda = 2$ (every Vector in Plane is an eigen Vector).

* Use Eigen Vectors as basis of the Coordinate System.

* Dot Products & Duality



$\boxed{V \cdot W = L(V) * L(P)}$

$\boxed{V \cdot W = |V| \cdot |W| \cos\theta}$

if, Same dir^n → dot is +ve.

* Norms - LA

- ML uses tensors as basic units of rep.
- 2 reasons to use Norms → how big a Vector/tensor is ($|V| = 5$) → to estim. how close one tensor is to another.

* Norm is a function that f

(i) $f(x) = 0 \longrightarrow x = 0$

(ii) $f(x+y) \leq f(x) + f(y)$ [Triangle Inequality]

(iii) $\forall a \in R$, $f(ax) = \|a\| \cdot f(x)$

length also inc. by a.

$\boxed{f(\vec{V}) \longrightarrow +\text{ve Scalar Quantity}}$

* Vector Norms

1. Euclidean Norm [2 Norm]

$\|V\|_2 = (V_1^2 + V_2^2 + \cdots V_n^2)^{1/2}$

. 2 norm or L2 norm / regular distance.

2. 1 Norm:

$\|V\|_1 = |V_1| + |V_2| + \cdots |V_n|$

3. p Norm

$\|V\|_p = (|V_1|^p + |V_2|^p + \cdots |V_n|^p)^{1/p}$

4. ∞ - Norm / max Norm

$\|V\|_\infty = \max(|V_1|, |V_2|, \ldots \ldots |V_n|)$.

* Matrix Norm

$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$: Forbenius Norm: $\sqrt{1^2 + 2^2 + 2^2 + 0^2} = 3$

. 1 no. that rep size of matrix.

* Cauchy Schwarz Inequality

$|\vec{x} \cdot \vec{y}| \leq \|x\| \cdot \|y\|$

$|\vec{x} \cdot \vec{y}| = |x| \cdot |y|$ if, $\vec{x} = c\vec{y}$

eg → Pg 33 [Proof]

* HyperPlane

→ **2D Space**: Line seperates into 2 regions

$w_1x_1 + w_2x_2 + w_0 = 0$ → Line

→ **3D Space**: Plane sepr. into 2 regions.

$w_0 + w_1x_1 + w_2x_2 + w_3x_3 = 0$

∴ [Line in 2D = Plane in 3D = HyperPlane in nD]

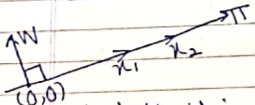
divides nD into 2 regions.

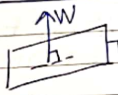
→ $w_0 + w_1x_1 + w_2x_2 + \cdots w_nx_n = 0$

$\pi : \boxed{w_0 + w^Tx = 0}$

if, Plane through origin: $w^Tx = 0$ (w: col vector)

$$\boxed{w \cdot x = w^Tx = \|w\|\|x\|\cos\theta_{w,x} = 0}$$

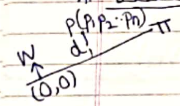




π $w^Tx=0$
[Eqn of Hyper Plane].

Geometrical intuition of w

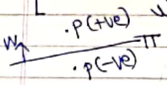
* Distance of Point from Plane



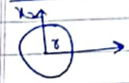
$$\boxed{d = \frac{w^Tp}{\|w\|}} = \boxed{\frac{w \cdot p}{\|w\|}}$$

* A hyperPlane divides a Space into 2 regions called as Half Spaces.

⇒ [if $\vec{p}$ in the same dirn as $\vec{w}$, $d = +ve$
if app dirn as $\vec{w}$, $d = -ve$.]



* Circle



$x_1^2 + x_2^2 = r^2$: eqn of circle

if, $x_1^2 + x_2^2 < r^2$ (inside), $> r^2$ (outside).

⇒ $x_1^2 + x_2^2 + x_3^2 = r^2$ → Eqn of Sphere.

⇒ $(x_1^2 + x_2^2 + x_3^2 + \cdots x_n^2 = r^2)$ → Eqn of HyperSphere

* Dot Product

$a = [a_1\ a_2\ a_3 \cdots a_n]$
$b = [b_1\ b_2\ b_3 \cdots b_n]$

$a \cdot b = a_1b_1 + a_2b_2 + \cdots a_nb_n$

$\Rightarrow [a_1\ a_2 \cdots a_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a^Tb$.

$$\therefore \boxed{a \cdot b = a^Tb = \sum_{i=1}^{n} a_ib_i = \|a\|\|b\| \cdot \cos\theta}$$

if, orthogonal ⇒ $a^Tb = a \cdot b = 0$.

⇒ $\boxed{a \cdot a = \|a\|^2}$