

GATE CSE NOTES

by
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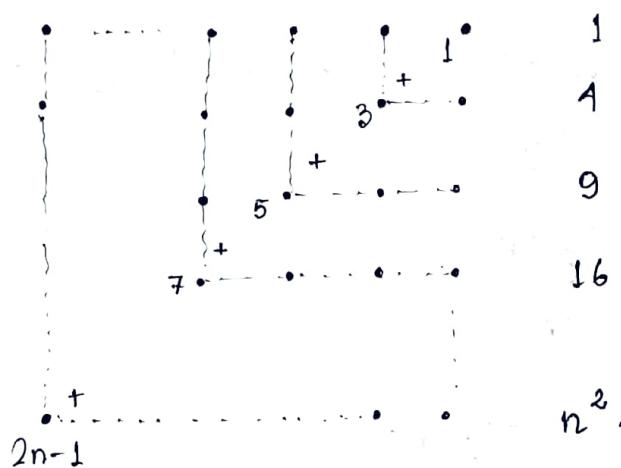
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With best wishes from Joyoshish Saha

Counting

Eg. Sum of odd numbers.

$$1 + 3 + 5 + \dots + (2n-1) = n^2.$$



- * The Product Rule
- * The Sum Rule.
- * Inclusion - Exclusion Principle / Sieve Principle / Subtraction Principle.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Generally,

$$\checkmark \quad \left| \bigcup_{i=1}^n A_i \right| = \sum_{k=1}^n (-1)^{k+1} \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} |A_{i_1} \cap \dots \cap A_{i_k}| \right)$$

- * The Pigeonhole Principle / Dirichlet drawer Principle

• Theorem 1 : If k is a positive integer & $k+1$ or more objects are placed onto k boxes, then there is at least one box containing two or more of the objects.

→ Corollary : A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

• Theorem 2. : The Generalised Pigeonhole Principle.

If N objects are placed into K boxes, then there is at least one box containing at least $\lceil \frac{N}{K} \rceil$ objects. ✓

$\rightarrow N = K(r-1)$ is the smallest integer satisfying $\lceil \frac{N}{K} \rceil \geq r$.

✓ • Theorem 3. : Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

* Binomial Theorem.

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j$$

$$\rightarrow \sum_{k=0}^n \binom{n}{k} = 2^n \quad \left| \sum \right.$$

$$\rightarrow \sum_{k=0}^n (-1)^k \binom{n}{k} = 0. \quad \left| \text{Alternating sum } \sum \right.$$

$$\rightarrow \sum_{k=0}^n 2^k \binom{n}{k} = 3^n$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \quad \left| \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \right.$$

• Vandermonde's Identity : $m, n, r \rightarrow$ nonnegative integers

$$\checkmark \quad \binom{m+n}{r} = \sum_{k=0}^r \binom{m}{r-k} \binom{n}{k} \quad \left| \begin{array}{l} r \leq m, n \\ m \text{ girls} \\ n \text{ boys} \\ \text{group of } r \end{array} \right.$$

Cor.

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2 \quad \left| \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} \right.$$

- Theorem: n, m nonnegative integers

$$r \leq n$$

$$\text{eg } \binom{5}{2} = \binom{1}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}$$

Hockey-stick identity

$$\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}.$$

- * Generating Permutations

→ Lexicographic ordering

- * Generating Combinations

- * Pigeonhole Principle (Strong form).

If $n(r-1) + 1$ objects are put into n boxes; at least one of the boxes contains r or more of the objects.

$$\rightarrow N = k(r-1) + 1$$

$N \rightarrow$ Pigeons

$k \rightarrow$ pigeonholes

$r \rightarrow$ no. of pigeons or more than it in at least one box.

pigeonhole.

Techniques.* Recurrence Relation.

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence for all integers n with $n \geq n_0$, where n_0 is a non-negative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

* Linear Homogeneous Recurrence Relation.

A linear homogeneous recurrence relation of degree k with constant coefficient is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

$$\left\{ \begin{array}{l} c_1, c_2, \dots, c_n \rightarrow \text{real numbers} \\ c_k \neq 0. \end{array} \right.$$

e.g. $f_n = f_{n-1} + f_{n-2}$ of degree two.

* Solving Linear Homogeneous Recurrence

Relations with constant coefficients.

Solution of the form $a_n = r^n$.

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0.$$

The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if & only if r is a solution of the equation (Characteristic equation).

→ Theorem. : Let c_1 & c_2 be real numbers.

$$\text{Suppose that } r^2 - c_1 r - c_2 = 0$$

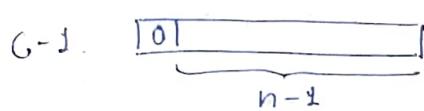
has two distinct roots r_1 & r_2 . Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \dots$ where α_1 & α_2 are constants.

→ Theorem. : Let c_1 & c_2 be real numbers with $c_2 \neq 0$. Suppose that

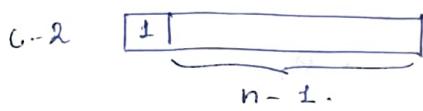
$r^2 - c_1 r - c_2 = 0$ has only one root r_0 . A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \dots$ where α_1 & α_2 are constants.

1. $T(n)$ = no. of binary strings of length n .

Recurrence Reln.



$$T(n) = T(n-1) + T(n-2)$$



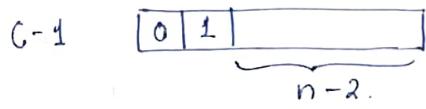
$$= 2 T(n-1)$$

$$T(1) = 2 \begin{cases} 0 \\ 1 \end{cases}$$

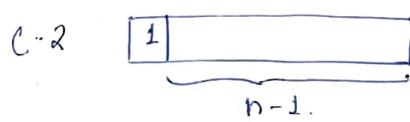
$$T(2) = 2^2 = 4$$

$$T(3) = 2^3 = 8$$

2. $T(n)$ = # binary strings without consecutive zeros.



$$T(n) = T(n-1) + T(n-2)$$



$$T(1) = 2 \begin{cases} 0 \\ 1 \end{cases} \quad T(2) = T(1) + T(1)$$

$$T(2) = 3 \begin{cases} 01 \\ 10 \\ 11 \end{cases} = 5$$

3. moods
good $i \rightarrow i+2$
bad $i \rightarrow i+1$

how many ways a person can reach n^{th} step?

$n=1 \rightsquigarrow 1$ way

$n=2 \rightsquigarrow 2$ ways

$n=3 \rightsquigarrow 3$ ways

$n=4 \rightsquigarrow 5$ ways

C-1 If a person has good mood,
find answer for $(n-2)$ places.

C-2 If has bad mood, find for $(n-1)$ places.

$$T(n) = T(n-1) + T(n-2) \quad T(1) = 1 \quad T(2) = 2$$

$T(1)$	$T(2)$	$T(3)$	$T(4)$	$T(5)$	$T(6)$	$T(7)$	$T(8)$
1	2	3	5	8	13	21	34

4. Count no. of ways of the person can climb up to n stairs for a given value m ?

$$T(n, m) = T(n-1, m) + T(n-2, m) + \dots + T(n-m, m)$$

5. G'15. Let a_n be the # of bit strings of length n containing 2 consecutive 1's. What is the recurrence reln for a_n ?

$$\rightarrow T(1) = 0, \quad a_1 = 0.$$

$$a_2 = 1. \quad - 11.$$

$$a_3 = 3 \quad - \left\{ \begin{array}{l} 011 \\ 110 \\ 111 \end{array} \right.$$

~~a)~~ $a_{n-2} + a_{n-1} + 2^{n-2}$

b) $a_{n-2} + 2a_{n-1} + 2^{n-2}$

c) $2a_{n-2} + a_{n-1} + 2^{n-2}$

d) $2a_{n-2} + 2a_{n-1} + 2^{n-2}$

$$a_3 = a_1 + a_2 + 2^1 - a. \quad = 3. \quad \checkmark$$

$$a_3 = 2a_1 + a_2 + 2^1 - c. \quad = 3. \quad \checkmark$$

$$a_4 = 8$$

0011
0110
0111
1011
1100
1101
1110
1111

Check with options.

$$a_4 = a_2 + a_3 + 2^2 - a$$

$$= 1 + 3 + 4 = 8 \quad \checkmark$$

$$a_4 = 2a_2 + a_3 + 2^2$$

$$= 9 \quad \times$$

6. a_n be the # of bit strings of length n don't contain 2 consecutive 1's.

$$\rightarrow a_1 = 2 \quad \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right.$$

$$a_2 = 3 \quad \left\{ \begin{array}{l} 00 \\ 01 \\ 10 \end{array} \right.$$

$$a_3 = 5 \quad \left\{ \begin{array}{l} 000 \\ 001 \\ 010 \\ 100 \\ 101 \end{array} \right.$$

a) $a_n = a_{n-1} + 2a_{n-2}$

~~b)~~ $a_{n-1} + a_{n-2}$

c) $2a_{n-1} + a_{n-2}$

d) $2a_{n-1} + 2a_{n-2}$

Check with options.

$$a_3 = a_1 + a_2 - b$$

$$= 5 \quad \checkmark$$

$$\left| \begin{array}{l} (-1) 0 \overbrace{\quad \quad \quad}^{n-1} \\ (-2) 10 \overbrace{\quad \quad \quad}^{n-2} \end{array} \right.$$

7. G'08. Let x_n denote # of binary strings of length n that contain no consecutive 0's.

- Which recurrence relⁿ for x_n ?

- a) $2x_{n-1}$
- b) $x_{n/2} + 1$
- c) $x_{n/2} + n$
- d) $\sqrt{x_{n-1} + x_{n-2}}$

- Value of x_5 -

- a) 5
- b) 7
- c) 8
- d) 13.

$$x_3 = x_2 + x_1 = \cancel{3+0} = 3 + 2 = 5$$

$$x_4 = \cancel{3+3} = 3 + 5 = 8$$

$$x_5 = \cancel{8+5} = 8 + 5 = 13.$$

8. G'02. Solⁿ to the recurrence relⁿ

$$T(2^k) = 3T(2^{k-1}) + 1. \quad T(1) = 1. \quad \text{is}$$

- a) 2^k
- b) $\frac{3^{k+1}-1}{2}$
- c) $3^{\log_2 k}$
- d) $2^{\log_3 k}$

$$\rightarrow \underline{k=1} \quad T(2) = 3T(2^{1-1}) + 1 = 4. \quad 2^1 = 2 \times$$

$$3^{\log_2 1} = 1 \times$$

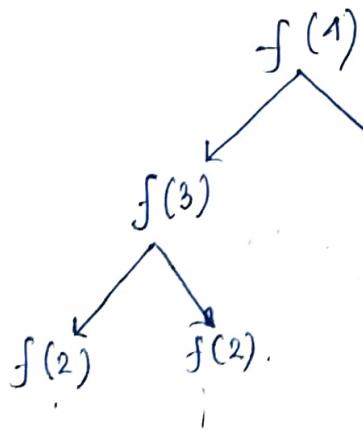
$$b) \quad \frac{3^2-1}{2} = 4 \quad \checkmark \quad 2^{\log_3 1} = 1 \times$$

• Void f(int n)

if ($n \leq 1$) return;

f(n-1);
f(n-1);

No. of recursive calls
Returning value
Time complexity
No. of comparisons
Growth of stack

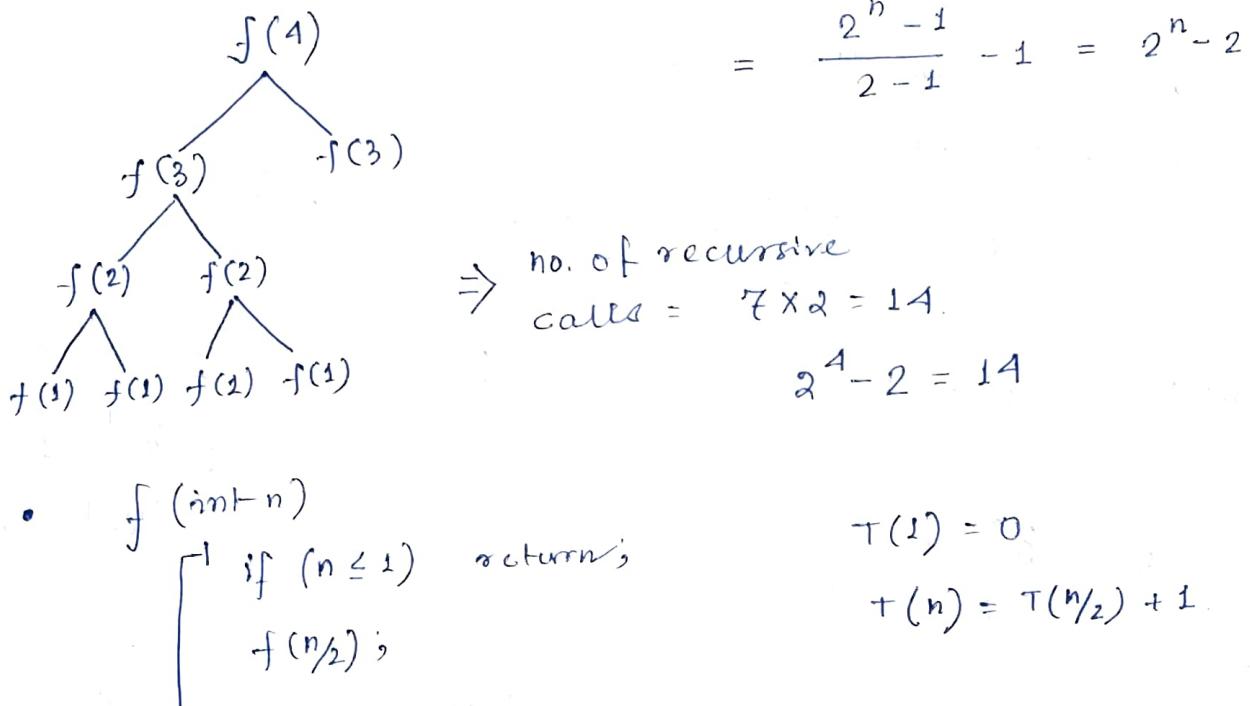


No. of recursive calls.

$$T(1) = 0.$$

$$T(n) = T(n-1) + T(n-1) + 2 \\ = 2T(n-1) + 2$$

$$\begin{aligned}
 T(n) &= 2T(n-1) + 2 \\
 T(n-1) &= 2T(n-2) + 2 && \times 2 \\
 T(n-2) &= 2T(n-3) + 2 && \times 2^2 \\
 &\vdots && \times 2^3 \\
 T(3) &= 2T(2) + 2 && \times 2^{n-3} \\
 T(2) &= 2T(1) + 2 && \times 2^{n-2} \\
 T(1) &= 0 && \times 2^{n-1} \\
 \hline
 T(n) &= 2^1 + 2^2 + 2^3 + \dots + 2^{n-2} + 2^{n-1} = \frac{2^{n+1} - 1}{2 - 1}
 \end{aligned}$$



- $f(\text{int } n)$
 - if ($n \leq 1$) return;
 - $f(n/2);$

$$\begin{aligned}
 T(n) &= T(n/2) + 1 \\
 T(n/2) &= T(n/2^2) + 1 \\
 T(n/2^2) &= T(n/2^3) + 1 \\
 &\vdots \\
 T(1) &= 0
 \end{aligned}
 \quad \left| \begin{array}{l} T(2^m) = T(2^{m-1}) + 1 \\ T(2^{m-1}) = T(2^{m-2}) + 1 \\ \vdots \\ T(2^2) = T(2^1) + 1 \\ T(2^1) = T(2^0) + 1 \\ T(2^0) = 0. \end{array} \right. \quad \begin{array}{l} n = 2^k \\ k = \log_2 n \end{array}$$

$$T(2^m) = m \Rightarrow T(n) = \log n$$

→ Theorem : Let c_1, c_2, \dots, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \dots - c_k = 0$$

has k distinct roots r_1, r_2, \dots, r_k . Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

iff

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \dots$ where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

→ Theorem : Let c_1, c_2, \dots, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \dots - c_k = 0$$

has t distinct roots r_1, r_2, \dots, r_t with multiplicities m_1, m_2, \dots, m_t , respectively, so that $m_i \geq 1$ for $i = 1, 2, \dots, t$ & $m_1 + m_2 + \dots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$$

if & only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n + \\ (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n + \dots + \\ (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n$$

for $n = 0, 1, 2, \dots$. where $\alpha_{i,j}$ are constants

for $1 \leq i \leq t$ & $0 \leq j \leq m_i - 1$.

* Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n).$$

Associated homogeneous recurrence relation.

→ Theorem. : If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$$

then every solution is of the form

$\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$$

→ Theorem. : Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + \dots + c_k a_{n-k} + f(n)$$

and

$$f(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n,$$

where b_0, b_1, \dots, b_t & s are real numbers.

When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation & its multiplicity is m , there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n.$$

* Generating Functions.

The generating function for the sequence $a_0, a_1, a_2, \dots, a_k, \dots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \dots + a_k x^k + \dots = \sum_{k=0}^{\infty} a_k x^k$$

→ Theorem. : Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and
 $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

✓ $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

✓ $f(x) g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k a_j b_{k-j} \right) x^k$.

→ Extended Binomial Coefficient.

Let u be a real number & k be a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\dots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0. \end{cases}$$

→ Theorem :

Extended Binomial Theorem. —

Let x be a real number with $|x| < 1$ & let u be a real number.

Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

$$\rightarrow \frac{1}{1-x} = 1 + x + x^2 + \dots \text{ for } |x| < 1$$

$$\rightarrow \frac{1}{1-ax} = 1 + ax + a^2x^2 \dots \text{ for } |ax| < 1.$$

$$\rightarrow \binom{-n}{\infty} = (-1)^n C(n+r-1, r). \checkmark$$

Useful generating functions.

$$\frac{G(x)}{(1+x)^n} = \sum_{k=0}^n C(n, k) x^k \quad \underbrace{C(n, k)}_{\alpha_k}.$$

$$(1+ax)^n = \sum_{k=0}^n C(n, k) a^k x^k \quad C(n, k) a^k.$$

$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^n x^k = 1 + x + x^2 + \dots + x^n \quad \begin{cases} 1 & \text{if } k \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots \quad 1.$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k \quad k+1.$$

$$= 1 + 2x + 3x^2 + \dots$$

$$\frac{1}{(1-x)^n} = \sum_{k=0}^n C(n+k-1, k) x^k \quad \begin{aligned} C(n+k-1, k) &= \\ C(n+k-1, n-1) &= \end{aligned}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \frac{1}{k!}$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k \quad (-1)^{k+1}/k.$$

$$= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots$$

Recurrence relation.

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely a_0, a_1, \dots, a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is non-negative integer.

$$\text{Q. } T(n) = 2T(n-1) - T(n-2)$$

Solution ~

\checkmark a) $T(n) = 3n$ b) 2^n c) 5

$$\rightarrow \begin{array}{ll} \text{a)} T(0) = 0 & T(2) = 2T(1) - T(0) \\ & = 2 \times 3 - 0 \\ & = 6. \checkmark \\ \text{b)} T(1) = 3 & \\ \text{c)} T(2) = 6 & T(3) = 2T(2) - T(1) \\ & = 2 \times 6 - 3 \\ & = 9. \checkmark \end{array} \quad \begin{array}{l} T(n) = 2 \times 3(n-1) - 3(n-2) \\ = 6n - 6 - 3n + 6 \\ = 3n \checkmark \end{array}$$

$$\begin{array}{ll} \text{b)} T(0) = 1 & T(2) = 2(T(1)) - T(0) \\ & = 2 \times 2 - 1 \\ & = 3 \times \\ \text{c)} T(1) = 2 & \\ \text{d)} T(2) = 4 & \\ \text{e)} T(3) = 8 & \end{array} \quad \begin{array}{l} T(n) = 2 \cdot 2^{n-1} - 2^{n-2} \\ = 2^{n-2}(2^2 - 1) \\ = 3 \times 2^{n-2} \times \end{array}$$

$$\begin{array}{ll} \text{c)} T(0) = T(1) = \dots = T(n) = 5. & T(n) = 2 \times 5 - 5 = 5 \checkmark \\ T(2) = 2 \times 5 - 5 & \\ & = 5. \checkmark \\ T(3) = 2 \times 5 - 5 & \\ & = 5 \checkmark \end{array}$$

2 solutions - $T(n) = 3n$
 $T(n) = 5.$

$$\text{Q. G'04} \quad T(1) = 1. \quad T(n) = 2T(n-1) + n \quad n \geq 2. \quad \text{evaluates to}$$

\checkmark a) $2^{n+1} - n - 2$

c) $2^{n+1} - 2n - 2$

- b) $2^n - n$

d) $2^n + n$

$$\rightarrow \begin{array}{ll} \text{a)} 2^{1+1} - 1 - 2 = 1. \checkmark & \text{b)} 2^1 - 1 = 1. \checkmark \\ \text{c)} 2^2 - 2 - 2 = 0 \times & \text{d)} 2^1 + 1 = 3 \times \end{array}$$

$T(2) = 2 + 2 = 4.$

\checkmark a) $2^3 - 2 - 2 = 4. \checkmark$ b) $2^2 - 2 = 2. \times$

$$\text{Q. G'02. } T(2^k) = 3T(2^{k-1}) + 1.$$

$$T(1) = 1.$$

a) 2^k

\checkmark b) $\frac{3^{k+1} - 1}{2}$

$\rightarrow 2^k = n$

$$T(n) = 3T(n/2) + 1.$$

By master's theorem,

$$n^{\log_3 3}$$

$$n^{\log_2 3}$$

$$O(n^{\log_2 3}).$$

$$\Rightarrow O(2^{k \log_2 3})$$

$$\Rightarrow O(3^k).$$

a) $T(1) = 1$

$$T(2^k) = 2^k + T(1) = 1$$

b) $T(2^k) = \frac{3^{k+1} - 1}{2} \quad \left| \begin{array}{l} 2^k = 1 \\ \Rightarrow k = 0 \end{array} \right.$

$$= \frac{3 - 1}{2} = 1.$$

c) $T(2^k) = x$

d) $T(2^k) = x$

$$T(2) = 3T(2^0) + 1$$

$$= 3T(1) + 1 = 1.$$

a) $T(2) = 2x$

b) $T(2) = \frac{3^2 - 1}{2} = 4$

Q. G'08 $n = 2^{2k}$ for some $k \geq 0$, the recurrence relation $T(n) = \sqrt{2}T(n/2) + \sqrt{n}$, $T(1) = 1$. evaluates to

\checkmark a) $\sqrt{n}(\log n + 1)$

c) $\sqrt{n} \log \sqrt{n}$

b) $\sqrt{n} \log n$

d) $n \log \sqrt{n}$.

$\rightarrow T(1) = 1$

a) $T(1) = 1$ c) 0

$T(2) = 2\sqrt{2}$.

b) $T(1) = 0$ d) 0.

a) $\sqrt{2}(\log_2 2 + 1) = 2\sqrt{2}$

Q. G'12. Recurrence relation capturing the optimal execution time of the Towers of Hanoi problem

with n discs is —

a) $T(n) = 2T(n-2) + 2$ c) $T(n) = 2T(n/2) + 1$

b) $T(n) = 2T(n-1) + n$. \checkmark d) $T(n) = 2T(n-1) + 1$.

$\rightarrow T(n) = 2T(n-1) + 1$.

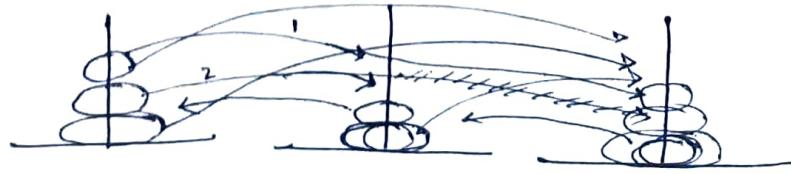
No. of movements required.

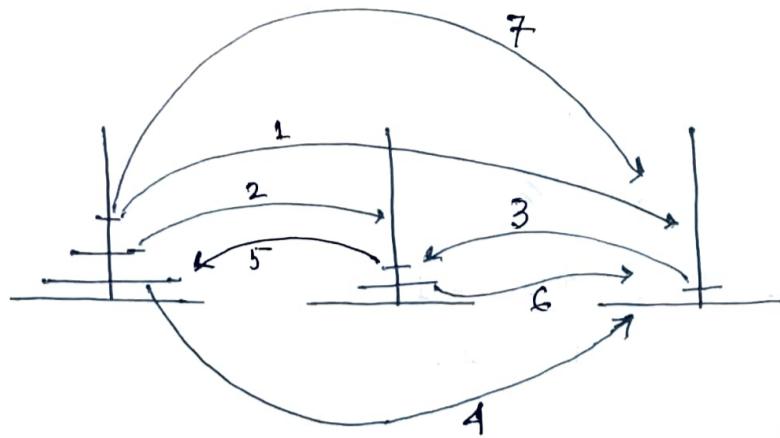
1 disc $T(1) = 1$

2 discs $T(2) = 3$

3 discs $T(3) = 7$

d) $2T(2) + 1$
 $= 7$





Q. 6'08 $x_n \rightarrow \# \text{ of binary strings of length } n$
that contain no consecutive 0's.

- a) $x_n = 2x_{n-1}$ c) $x_n = x_{n/2} + n$
b) $x_n = x_{n/2} + 1$ d) $x_n = x_{n-1} + x_{n-2}$

$$\rightarrow x_1 = 2 \quad a) x_2 = 2 \times 2 = 4 \quad \times$$

$$x_2 = 3 \quad b) x_2 = 2 + 1 = 3 \quad \Rightarrow x_3 = \times$$

$$x_3 = 5 \quad c) x_3 = 2 + 2 = 4 \quad \times$$

$$d) x_3 = x_2 + x_1 = 5 \quad \times$$

Generating functions.

Sonendra Gupta

A function whose domain is the set $\{0, 1, 2, 3, \dots\}$ of non-negative integers & whose range is a subset of \mathbb{R} , is called a discrete numeric function. (denoted as a_r).

e.g. $\{1, 9, 28, \dots\}$

$\Rightarrow \{1, 2^3+1, 3^3+1, \dots\}$

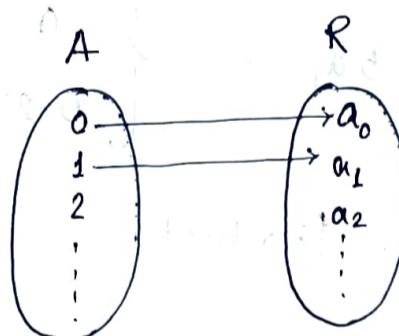
$$a_r = r^3 + 1$$

e.g. $\{0, 3, 6, 7, 15, 31, \dots\}$

$$a_r = \begin{cases} 3r & ; 0 \leq r \leq 2 \\ 2^{r-1} & ; r \geq 3 \end{cases}$$

e.g. $\{0, 3, 6, 9, 12, 15, 7, 8, 9, \dots\}$

$$a_r = \begin{cases} 3r & ; 0 \leq r \leq 5 \\ r+1 & ; r \geq 6 \end{cases}$$



$$a_r : A \rightarrow R$$

→ Operations on numeric functions:

$$\left\{ \begin{array}{l} a_r = \begin{cases} 0 & ; 0 \leq r \leq 2 \\ 2^{-r} + 5 & ; r \geq 3 \end{cases} \\ b_r = \begin{cases} 3 - 2^r & ; 0 \leq r \leq 1 \\ r+2 & ; r \geq 2 \end{cases} \end{array} \right.$$

1. Summation:

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ a_r & 0 & 0 & 0 & 2^{-3} + 5 & 2^{-4} + 5 & 2^{-5} + 5 \\ b_r & 3 - 2^0 & 3 - 2^1 & 2+2 & 3+2 & 4+2 & 5+2 \end{array}$$

$$c_r = a_r + b_r \quad 3 - 2^0 \quad 3 - 2^1 \quad 2+2 \quad 2^{-3} + 3+7 \quad 2^{-4} + 4+7 \quad 2^{-5} + 5+7$$

$$c_r = \begin{cases} 3 - 2^r & ; 0 \leq r \leq 1 \\ 4 & ; r = 2 \\ 2^{-r} + r + 7 & ; r \geq 3 \end{cases}$$

2. Multiplication by a real number:

$$a_r = \begin{cases} 0 & ; 0 \leq r \leq 2 \\ 2^{-r} + 5 & ; r \geq 3 \end{cases}$$

$$5a_r = \begin{cases} 0 & ; 0 \leq r \leq 2 \\ 5 \cdot 2^{-r} + 25 & ; r \geq 3 \end{cases}$$

3. Product:

$$\begin{array}{ccccccc} a_r & 0 & 0 & 0 & 2^{-3} + 5 & 2^{-4} + 5 & 2^{-5} + 5 \\ b_r & 3 - 2^0 & 3 - 2^1 & 2+2 & 3+2 & 4+2 & 5+2 \end{array}$$

$$c_r = a_r \cdot b_r \quad 0 \quad 0 \quad 0 \quad (2^{-3} + 5) \quad (2^{-4} + 5) \quad (2^{-5} + 5)$$

$$(3+2) \quad (4+2) \quad (5+2)$$

$$c_r = \begin{cases} 0 & ; 0 \leq r \leq 2 \\ (2^{-r} + 5)(r+2) & ; r \geq 3 \end{cases}$$

2

4. Value of a Numeric function.

$|a_r|$ denotes a value at r .

$$\text{e.g. } a_r = (-1)^r \frac{7}{r^3}, r \geq 0$$

$$|a_r| = b_0 = \frac{7}{r^3}, r \geq 0$$

5. Forward & Backward difference.



$$\Delta a_r = a_{r+1} - a_r$$



$$\nabla a_r = a_r - a_{r-1}; r \geq 1$$

$$\nabla a_0 = a_0$$

$$\text{e.g. } a_r = \begin{cases} 0 & ; 0 \leq r \leq 2 \\ 2^{-r+7} & ; r \geq 3. \end{cases}$$

$$\text{Calculating. } b_r = \Delta a_r = \cancel{\frac{1}{2^r+8}} - \frac{1}{2^{r+1}}$$

$$c_r = \nabla a_r = -\frac{1}{2^r}$$

6. Convolution:

$$c_r = a_r * b_r = \sum_{k=0}^r a_k \cdot b_{r-k} \quad \text{such that } c_0 = a_0 b_0$$

$$\text{e.g. } a_r = \begin{cases} 1 & ; 0 \leq r \leq 2 \\ 0 & ; r \geq 3 \end{cases} \quad b_r = \begin{cases} 1 & ; 0 \leq r \leq 2 \\ 0 & ; r \geq 3 \end{cases}$$

$$c_0 = a_0 b_0 = 1$$

$$c_1 = \sum_{k=0}^1 a_k b_{1-k} = a_0 b_1 + a_1 b_0 = 2$$

$$c_2 = \sum_{k=0}^2 a_k b_{2-k} = a_0 b_2 + a_1 b_1 + a_2 b_0 = 3$$

$$c_3 = 2 \quad c_4 = 1 \quad c_5 = 0$$

$$c_r = \begin{cases} 1 & ; r = 0, 1 \\ 2 & ; r = 1, 2 \\ 3 & ; r = 2 \\ 0 & ; r \geq 3 \end{cases}$$

- If $a_r = (a_0, a_1, a_2, \dots, a_r, \dots)$ is a discrete numeric function then an infinite series in terms of a parameter z written as

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_r z^r + \dots$$

is said to be a generating function of the numeric function a_r & also written in the form

$$A(z) = \sum_{r=0}^{\infty} a_r z^r$$

e.g. $a_r = (1, 3, 9, 27, \dots)$

$$a_0 = 1 \quad a_1 = 3 \quad a_2 = 9 \quad \dots$$

$$(1+\alpha)^{-1} = \frac{1-\alpha+\alpha^2}{1-\alpha^3+\dots}$$

$$A(z) = 1 + 3z + 9z^2 + 27z^3 + \dots$$

$$= 1 + 3z + 3^2 z^2 + 3^3 z^3 + \dots$$

$$= 1 + 3z + (3z)^2 + (3z)^3 + \dots$$

$$= (1-3z)^{-1}$$

- Operations on generating functions.

1. Sum. $c_r = a_r + b_r$

$$C(z) = A(z) + B(z) = \sum_{r=0}^{\infty} (a_r + b_r) z^r$$

2. Scalar multiplication

$$b_r = K a_r \Rightarrow B_z = K A_z = K \sum_{r=0}^{\infty} a_r z^r$$

3. Product

$$c_r = a_r \cdot b_r \Rightarrow C(z) = A(z) \cdot B(z) = c_r z^r$$

$$c_r = \sum_{k=0}^r a_k b_{r-k}$$

4. Multiplication of α^r with a_r

$$b_r = \alpha^r a_r \Rightarrow B(z) = A(\alpha z) = \sum_{r=0}^{\infty} a_r (\alpha z)^r$$

Numeric functions.

$A(z)$.

$$a_r = 1$$

$$1 + z + z^2 + \dots + z^r + \dots = \frac{1}{1-z}$$

$$a_r = K$$

$$K + Kz + \dots + Kz^r + \dots = \frac{K}{1-z}$$

$$a_r = \alpha^r$$

$$1 + \alpha z + \alpha^2 z^2 + \dots + \alpha^r z^r + \dots = \frac{1}{1-\alpha z}$$

$$a_r = r$$

$$z + 2z^2 + 3z^3 + \dots + rz^r + \dots = \frac{z}{(1-z)^2}$$

$$a_r = r^2$$

$$z + 2^2 z^2 + 3^2 z^3 + \dots + r^2 z^r + \dots = \frac{z(1+z)}{(1-z)^3}$$

$$a_r = r(r+1)$$

$$1 \cdot 2z + 2 \cdot 3z^2 + 3 \cdot 4z^3 + \dots + r(r+1)z^r + \dots = \frac{2z}{(1-z)^3}$$

eg. Find generating function of $5 \cdot 2^{r+2}$.

$$a_r = 5 \cdot 2^{r+2}$$

$$A(z) = 5 \cdot 2^2 + 5 \cdot 2^3 z + 5 \cdot 2^4 z^2 + 5 \cdot 2^5 z^3 + \dots$$

$$A(z) = 5 \cdot 2^2 (1 + 2z + 2^2 z^2 + 2^3 z^3 + \dots)$$

$$= \frac{20}{1-2z}$$

eg. $a_r = 2^r + 3^r$.

$$b_r = 2^r \quad c_r = 3^r$$

$$\downarrow$$

$$B(z) = \frac{1}{1-2z}$$

$$\downarrow$$

$$C(z) = \frac{1}{1-3z}$$

✓

$$A(z) = B(z) + C(z) = \frac{1}{1-2z} + \frac{1}{1-3z}$$

eg. $a_r = \begin{cases} 2^r & ; \text{ if } r \text{ is even} \\ -2^r & ; \text{ if } r \text{ is odd.} \end{cases}$

$$A(z) = 1 - 2z + 2^2 z^2 - 2^3 z^3 + 2^4 z^4 + \dots$$

$$= \frac{1}{1+2z}$$

$$CR = (-2z)$$

e.g. $(2, 3, 5, 9, 17, 33, \dots)$

$$\begin{aligned} A(z) &= 2 + 3z + 5z^2 + 9z^3 + 17z^4 + 33z^5 + \dots \\ &= (1 + z + z^2 + z^3 + z^4 + z^5 + \dots) + \\ &\quad (1 + 2z + 4z^2 + 8z^3 + 16z^4 + 32z^5 + \dots) \\ &= \frac{1}{1-z} + \frac{1}{1-2z} = \frac{2-3z}{(1-z)(1-2z)} \end{aligned}$$

e.g. $r(r+1) = a_r$

$$\begin{aligned} A(z) &= 0 + 2z + 6z^2 + 12z^3 + 20z^4 + \dots \\ &= 2z(1 + 3z + 6z^2 + 10z^3 + \dots) \\ &= 2z(1-z)^{-3} \quad | \quad (1-\alpha)^{-3} = 1 + 3\alpha + 6\alpha^2 + 10\alpha^3 + \dots \end{aligned}$$

e.g. $a_r = r \cdot 5^r$

$$\begin{aligned} A(z) &= 0 + 5z + 2 \cdot 5^2 z^2 + 3 \cdot 5^3 z^3 + 4 \cdot 5^4 z^4 + \dots \\ &= 5z[1 + 2(5z) + 3(5z)^2 + 4(5z)^3 + \dots] \\ &= 5z(1-5z)^{-2} \quad | \quad (1-\alpha)^{-2} = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \dots \end{aligned}$$

Recurrence Relations Solution

DM₂ - 34 1

Types of recurrence rel's

1. Divide & Conquer $T(n) = aT(n/b) + f(n)$.

(Can be solved by Master Method)

2. Linear recurrence $T(n) = aT(n-b) + f(n)$

(Solvable by substitution method)

Linear homogeneous recurrence ~

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

e.g. Solve $a_1 = 3, x_n = 3x_{n-1}$

$$x_n = 3^n$$

Solving Linear Recurrence

\Rightarrow Recurrence $a_n = c_1 x_{n-1} + c_2 x_{n-2} + \dots + c_k x_{n-k}$

Let $x_n = ar^n$

$$ar^n = c_1 ar^{n-1} + c_2 ar^{n-2} + \dots + c_k ar^{n-k}$$

Since, $a, r \neq 0$, dividing by ar^{n-k}

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0. \quad (\text{Characteristic polynomial})$$

$a_n = ar^n$ satisfies if r is a root of this polynomial equⁿ.

✓ Note: If $a_1r_1^n$ & $a_2r_2^n$ satisfy the recurrence their sum will also satisfy it.

So, solving linear recurrence involves:

1. Find characteristic polynomial

2. Find roots r_1, r_2, \dots, r_k of the char. poly.

3. Assuming no multiple roots, the closed form:

$$a_n = a_1r_1^n + a_2r_2^n + \dots + a_kr_k^n$$

4. Use initial values to find values of a_i 's.

Ex $a_n = 5a_{n-1} + 6a_{n-2}$ $a_0 = 1$
 $n \geq 2$ $a_1 = 3$

Char. poly. is found by letting the lowest indexed term be a_K^* , & replacing all a_i 's by a^{i-K} .

$$a_n = 5a_{n-1} + 6a_{n-2}$$

$$\left| \begin{array}{l} n-K = n-2 \\ K = 2 \end{array} \right.$$

$$\Rightarrow ar^n = 5 \cdot ar^{n-1} + 6ar^{n-2}$$

$$\Rightarrow r^2 = 5r + 6$$

$$r_1 = 6 \quad r_2 = -1$$

$$\Rightarrow a_n = a_1(-1)^n + a_2(6)^n$$

$$n=0 \Rightarrow$$

$$x_0 = a_1 + a_2 = 1 \Rightarrow a_1 = \frac{3}{7}$$

$$n=1 \Rightarrow$$

$$x_1 = -a_1 + 6a_2 = 3 \Rightarrow a_2 = \frac{4}{7}$$

Closed form \Rightarrow

$$x_n = \frac{3}{7}(-1)^n + \frac{4}{7}(6)^n.$$

Repeated roots in the Characteristic polynomial

e.g. One repeated root.

$$x_n = 4x_{n-1} + 4x_{n-2} \quad n \geq 2 \quad | \quad x_0 = 0 \quad x_1 = 1.$$

$$ar^n - 4ar^{n-1} + 4ar^{n-2} \\ \Rightarrow r^2 - 4r - 4 = 0 \Rightarrow (r-2)^2 = 0 \quad r = 2, 2.$$

(2^n) is one of the solⁿs.

$$\text{Other sol}^n \quad \beta_n = n \cdot 2^n.$$

$$x_n = a_1 2^n + a_2 \cdot n \cdot 2^n = 2^n(a_1 + n a_2)$$

$$\text{Solving, } x_n = n \cdot 2^{n-1}.$$

※ In the process of solving recurrences with repeated roots, if r is a root of the char. poly. with multiplicity $k \geq 1$, then we need to

W consider the sequences

$r^n, nr^n, n^2r^n, \dots, n^{k-1}r^n$ as part of the closed form of the solⁿ of the recurrence.

※ General Method

1. Characteristic polynomial.

2. Find roots r_1, r_2, \dots, r_m where $m \leq k$ with multiplicities $\ell_1, \ell_2, \dots, \ell_m$.

3. Closed form can be expressed as a linear combination of the sequences of form

O $n^j r_i^n$, $0 \leq j < \ell_i - 1$ & $1 \leq i \leq m$.

4. Use init. values to find constants.

eg 3rd degree ~~recu~~ polynomial:

$$x_n = 6x_{n-1} + 8x_{n-3} - 12x_{n-2}$$

$$\left| \begin{array}{l} x_0 = -1 \\ x_1 = 0 \\ x_2 = 1 \end{array} \right.$$

$$r^3 = 6r^2 + 8 - 12r$$

$$\Rightarrow r^3 - 6r^2 + 12r - 8 = 0$$

$$(r-2)^3 = 0$$

$r = 2$ with multiplicity 3.

$$x_n = a_1 2^n + a_2 \cdot n^{\cancel{2}^n} + a_3 n^2 2^n$$

$$x_n = -\frac{2^n}{8} (3n^2 - 11n + 8).$$

e.g. 3rd deg. polynomial 2 roots, one has mul. 2.

$$x_n = 4x_{n-1} + 3x_{n-2} - 18x_{n-3}$$

$$\Rightarrow r^3 - 4r^2 - 3r + 18 = 0.$$

$$r^3 + 2r^2 - 6r^2 - 12r + 6r + 18$$

$$\Rightarrow r^2(r+2) - 6r(r+2) + 6(r+2) = 0$$

$$-8 \neq 4 \cdot 4 + 6 + 18 \\ = 0.$$

$$\Rightarrow (r-3)^2(r+2) = 0$$

$$x_n = a_1 3^n + a_2 n 3^n + a_3 (-2)^n$$

$$\Rightarrow x_n = \frac{1}{15} [(12-5n)3^n + 3(-2)^n]$$

e.g. Recurrence with complex roots.

$$x_n = x_{n-1} - 4x_{n-2}$$

$$x_0 = 0$$

$$r = \sqrt{\frac{1}{4} + \frac{15}{4}} = 2$$

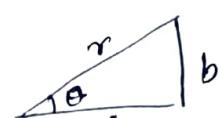
$$r^2 - r + 4 = 0$$

$$r = \frac{1 \pm i\sqrt{15}}{2}$$

$$a \pm ib = r \cos \theta \pm i r \sin \theta$$

$$= 2 e^{\pm i\theta}$$

$$\theta = \arctan \sqrt{15}$$



$$x_n = a_1 2^n \frac{e^{in\theta} + e^{-in\theta}}{2} + a_2 2^n \frac{e^{in\theta} - e^{-in\theta}}{2i} \quad | \quad r = 2 e^{\pm i\theta}$$

$$\Rightarrow a_1 2^n \cos n\theta + a_2 2^n \sin n\theta$$

$$x_n = \frac{2^{n+1}}{\sqrt{5}} \sin(n\theta).$$

$$(r \cdot e^{i\theta})^n$$

$$= r^n e^{in\theta}$$

$$| a_1 2^n e^{in\theta} + a_2 2^n e^{-in\theta} |$$

Generally, $r^2 - pr - q = 0$.

When $p^2 + 4q < 0$,

2 complex conjugate solns $(a+bi), (a-bi)$

For real-valued solns,

$$c_n = \frac{(a+bi)^n + (a-bi)^n}{2}$$

$$d_n = \frac{(a+bi)^n - (a-bi)^n}{2i}$$

if $\rho = \sqrt{a^2+b^2}$, θ is such that

$$\sin \theta = \frac{b}{\sqrt{a^2+b^2}}, \quad \cos \theta = \frac{a}{\sqrt{a^2+b^2}}.$$

then

$$c_n = \cancel{\rho^n} \rho^n \cos n\theta$$

$$d_n = \rho^n \sin n\theta$$

~~$$x_n = c_1 \rho^n \cos n\theta + c_2 \rho^n \sin n\theta$$~~

* Finding recurrence relⁿ for a sequence given
in Closed form

Closed form n^{irr^n}

Find the roots, multiplicities



Characteristic polynomial

Recurrence relⁿ

eg $x_n = (-3)^n \left(3 + \frac{4}{2^n}\right) + n^2 2^n$

roots are -3, 2 with multiplicities

2 and 3.

$$\begin{matrix} \downarrow & \downarrow \\ (1+1) & (2+1) \end{matrix}$$

Chara. poly. $(r+3)^2 (r-2)^3 \quad | r=5$

$$= r^5 - 15r^3 + 10r^2 + 60r - 72$$

$x_n = 15x_{n-2} - 10x_{n-3} - 60x_{n-1} + 72x_{n-5}$

$\overbrace{(5-5)}^{(5-3)} \qquad \overbrace{(5-2)}^{(5-1)} \qquad \overbrace{(5-1)}^{(5-0)}$

$$\text{Eq } x_n = a \sin n\theta + b \cos n\theta$$

$$x_1 = 15, x_2 = 3, x_3 = -12.$$

Find x_7

$$*\sin n\theta + \sin(n-2)\theta = 2 \sin \frac{2(n-1)\theta}{2} \cos\theta$$

$$\cos n\theta + \cos(n-2)\theta = 2 \cos \frac{2(n-1)\theta}{2} \cos\theta$$

$$x_n = 2 \cos\theta x_{n-1} - x_{n-2} \quad \begin{bmatrix} x_n + x_{n-2} = \dots \\ 2 \cos\theta \cdot x_{n-1} \end{bmatrix}$$

$$x_3 = -12 = 2 \cos\theta \cdot 3 - 15 \quad \begin{bmatrix} x_n + x_{n-2} = 2a \sin(n-1)\theta \cos\theta \\ + 2b \cos(n-1)\theta \cos\theta \end{bmatrix}$$

$$\Rightarrow \cos\theta = \frac{1}{2}.$$

$$x_n = x_{n-1} - x_{n-2} \quad x_7 = 15.$$

$$\text{or } x_1 + x_3 = a \sin\theta + b \cos\theta + a \sin 3\theta + b \cos 3\theta$$

$$3 = a \cdot 2 \sin 2\theta \cos\theta + b \cdot 2 \cos 2\theta \cos\theta$$

$$*\Rightarrow 2 \cos\theta \left[\underbrace{a \sin 2\theta + b \cos 2\theta}_{x_2} \right] = 3$$

$$\Rightarrow x_2 = 1 \Rightarrow \cos\theta = \frac{1}{2}.$$

$$\theta = 2n\pi \pm \frac{\pi}{3}$$

$$7\theta = 14n\pi \pm \frac{7\pi}{3}$$

$$7\theta = 2\pi(7n \pm 1) \pm \frac{7\pi}{3}$$

$$\checkmark \quad 7\theta = \theta \Rightarrow x_7 = x_1 = 15.$$

Eg Solve recurrence $a_n = a_{n-1} + n$, $a_0 = 4$.

$\rightarrow 4, 5, 7, 10, 14, 19, \dots$ (Telescoping)

$$a_1 - a_0 = 1$$

$$a_2 - a_1 = 2$$

$$a_3 - a_2 = 3$$

:

$$a_n - a_{n-1} = n$$

$$\begin{aligned} & (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) \\ &= \frac{n(n+1)}{2} \\ \Rightarrow a_0 + a_n &= \frac{n(n+1)}{2} \\ \Rightarrow a_n &= \frac{n(n+1)}{2} + 4 \end{aligned}$$

(by iteration)

$$a_1 = a_0 + 1$$

$$a_2 = (a_0 + 1) + 2$$

$$a_3 = ((a_0 + 1) + 2) + 3$$

:

$$a_n = a_0 + \frac{n(n+1)}{2} = 4 + \frac{n(n+1)}{2}$$

$$\boxed{\begin{aligned} a_n &= a_{n-1} + n \\ a_{n-1} &= a_{n-2} + n-1 \\ \vdots &= a_0 + 1 \\ \hline a_n &= (1+2+\cdots+n) + a_0 \end{aligned}}$$

Eg (By iteration) $a_n = 3a_{n-1} + 2$; $a_0 = 1$

$$a_1 = 3a_0 + 2$$

$$a_2 = 3(3a_0 + 2) + 2 = 3^2 a_0 + 3 \cdot 2 + 2$$

$$a_3 = 3 \{ 3(3a_0 + 2) + 2 \} + 2 = 3^3 a_0 + 3^2 \cdot 2 + 3 \cdot 2 + 2$$

$$a_n = 3 \left(3^{n-1} a_0 + 2 \cdot 3^{n-2} + \cdots + 2 \cdot 3 + 2 \right) + 2$$

$$= 3^n a_0 + (2 \cdot 3^{n-1} + \cdots + 2 \cdot 3 + 2)$$

$$= 3^n + \frac{2(1 - 3^n)}{1-3} = 3^n + \frac{2 - 2 \cdot 3^n}{-2} = 2 \cdot 3^n - 1.$$

$$a_n = \cancel{3a_{n-1}} + 2, \quad \times 1 = a_0 = 1,$$

$$a_{n-1} = \cancel{3a_{n-2}} + 2 \quad \times 3$$

$$a_{n-2} = \cancel{3a_{n-3}} + 2 \quad \times 3^2$$

$$\begin{array}{|c|c|} \hline & n-n+1 \\ \hline & n-(n-1) \\ \hline \end{array}$$

$$a_1 = \cancel{3a_0} + 2 \quad \times 3^{n-2}$$

$$a_0 = 3a_0 + 2 \quad \times 3^{n-1}$$

$$a_n = 3^n a_0 + 2(3^0 + 3^1 + 3^2 + \dots + 3^{n-1})$$

$$= 3^n + 2 \cdot \frac{-1 + 3^n}{-1 + 3}$$

$$= 2 \cdot 3^n - 1.$$

Using general solⁿ method for linear recurrence:

$$a_n = 3a_{n-1} + 2$$

$$a_n = pr^n$$

$$\Rightarrow pr^n = 3 \cdot pr^{n-1} + 2$$

~~(eg)~~ $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 2, a_1 = 3$

$$pr^n = 7pr^{n-1} - 10pr^{n-2}$$

$$\Rightarrow r^2 - 7[r] + 10 = 0 \quad r = 5, 2.$$

$$a_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot 2^n.$$

$$a_0 = \alpha_1 + \alpha_2 = 2 \quad \Rightarrow \quad a_n = \frac{7}{3} 2^n - \frac{1}{3} 3^n$$

$$a_1 = 5\alpha_1 + 2\alpha_2 = 3$$

~~(eg)~~ $F_n = F_{n-1} + F_{n-2}$ $F_0 = 0$ $F_1 = 1$

$$\Rightarrow pr^n = pr^{n-1} + pr^{n-2}$$

$$\Rightarrow r^2 - r - 1 = 0. \quad r = \frac{1 \pm \sqrt{5}}{2}$$

$$F_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

$$F_0 = 0 = \alpha_1 + \alpha_2$$

$$\Rightarrow F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n -$$

$$F_1 = 1 = \frac{1+\sqrt{5}}{2} \alpha_1 + \frac{1-\sqrt{5}}{2} \alpha_2$$

$$\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$$\text{e.g. } a_n = 6a_{n-1} - 9a_{n-2} \quad | \quad a_0 = 1, a_1 = 4$$

$$r^2 - 6r + 9 = 0$$

$$\Rightarrow (r-3)^2 = 0. \quad r = 3, 3$$

$$a_n = an 3^n + b 3^n$$

$$\Rightarrow a_n = 3^n + \frac{1}{3} n 3^n$$

* Recurrence of the form

$$Aa_n = Ba_{n-1} + c.$$

Method of Summation

Factors.

(Brilliant.org)

Summation factor s_n

$$As_n a_n = Bs_n a_{n-1} + cs_n$$

s_n chosen such that

$$s_n = \frac{A_{n-1} A_{n-2} \cdots A_2 A_1}{B_n B_{n-1} \cdots B_2 B_1}$$

$$b_n = As_n a_n \quad b_{n-1} = Bs_n a_{n-1}$$

$$b_n = b_{n-1} + cs_n$$

$$b_n = b_0 + \sum_{k=1}^n cs_k$$

$$a_n = \frac{1}{As_n} \left(A_{n=0} s_0 a_0 + \sum_{k=1}^n cs_k \right)$$

Eq

$$T_n = 2T_{n-1} + 1 \quad T_0 = 0$$

$$A_n = 1, \quad B_n = 2$$

$$S_n = \frac{1 \times 1 \times \dots \times 1}{2 \times 2 \times \dots \times 2} = \frac{1}{2^{n-1}}$$

$$\frac{1}{2^{n-1}} T_n = \frac{1}{2^{n-2}} T_{n-1} + \frac{1}{2^{n-1}}$$

$$\text{Let } b_n = \frac{T_n}{2^{n-1}}, \quad b_{n-1} = \frac{T_{n-1}}{2^{n-2}}, \quad b_0 = \frac{T_0}{2^{0-1}} = 0$$

$$b_n = b_{n-1} + \frac{1}{2^{n-1}}$$

$$b_n = 0 + \sum_{k=1}^n \frac{1}{2^{k-1}} = \frac{1 \left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1}$$

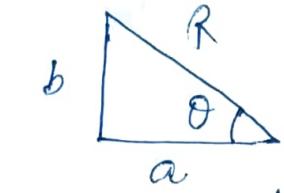
$$b_n = \frac{T_n}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$$

$$\Rightarrow T_n = \underline{\underline{2^n - 1}}$$

Complex roots $a \pm ib$, corresponding general

solⁿ $\underline{R^n [\phi_1 \cos n\theta + \phi_2 \sin n\theta]}$, $R = \sqrt{a^2+b^2}$, $\theta = \tan^{-1} \frac{b}{a}$

$$a_n = c_1 (a+ib)^n + c_2 (a-ib)^n.$$



$$= c_1 (R \cos \theta + i R \sin \theta)^n +$$

$$\sin \theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$c_2 (R \cos \theta - i R \sin \theta)^n$$

$$\cos \theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$= c_1 R^n (\cos \theta + i \sin \theta)^n + c_2 R^n (\cos \theta - i \sin \theta)^n$$

$$= c_1 R^n (\cos n\theta + i \sin n\theta) + c_2 R^n (\cos n\theta - i \sin n\theta)$$

* $\Rightarrow R^n \left[\underbrace{(c_1 + c_2)}_{\phi_1} \cos n\theta + \underbrace{(c_1 - c_2)i \sin n\theta}_{\phi_2} \right]$

$\| a_{m+2} + a_n = 0$

$$a_n = \phi_1 \cos n\theta + \phi_2 \sin n\theta$$

$$r^{n+2} + r^n = 0$$

$$\Rightarrow \phi_1 \cos \frac{n\pi}{2} + \phi_2 \sin \frac{n\pi}{2}$$

$$r^2 + 1 = 0$$

$$r = \pm \sqrt{-1}$$

$r = 0 \pm i \cdot 1 \rightarrow$ check a, b , to determine coordinate quadrant

$$R = \sqrt{1} = 1$$

$$\theta = \tan^{-1} (b/a)$$

$$= \frac{\pi}{2}$$

$$\text{Eq} \quad a_n - 2a_{n-1} + 2a_{n-2} - a_{n-3} = 0$$

$$\Rightarrow r^3 - 2r^2 + 2r - 1 = 0$$

$$r^3 - r^2 - r^2 + r + r - 1 = 0$$

$$\Rightarrow r^2(r-1) - r(r-1) + 1(r-1) = 0$$

$$\Rightarrow (r-1)(r^2 - r + 1) = 0$$

$$r = 1, \frac{1 \pm \sqrt{-3}}{2}$$

$$\frac{1}{2} \pm i\left(\frac{\sqrt{3}}{2}\right)$$

$$R = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\theta = \tan^{-1}(\sqrt{3}) \\ = \frac{\pi}{3}$$

$$a_n = \alpha (1)^n + R^n [a \cos n\theta + b \sin n\theta]$$

$$= \alpha + a \cos n\frac{\pi}{3} + b \sin n\frac{\pi}{3}$$

Non-homogeneous Linear RR

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$$

degree/order = k

- $f(n) = (b_0 n^0 + b_1 n^1 + b_2 n^2 + \dots + b_t n^t) \kappa^n$ where b_0, b_1, \dots, b_t & κ are real numbers. Particular solⁿ of the form

$$(p_0 n^0 + p_1 n^1 + p_2 n^2 + \dots + p_t n^t) \kappa^n \text{ if } \kappa \text{ is not a root of the characteristic equ}^n \text{ of associated linear homogeneous recurrence rel}^n.$$

root of the characteristic equⁿ of associated linear homogeneous recurrence relⁿ.

- solⁿ = Particular solⁿ + Homogeneous solⁿ

$$\text{eg } a_{n+2} - a_{n+1} - a_n = 4$$

$$\checkmark f(n) = 4 \cdot (4n^0)^{\frac{1}{2^n}}$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$$\text{Particular soln} = (\beta_0 n^0) \cdot 1^n = \beta_0$$

\downarrow
Homogeneous
soln.

Substituting,

$$\beta_0 - \beta_0 - \beta_0 = 4 \Rightarrow \beta_0 = -4.$$

$$\therefore a_n = \text{HS} + \text{PS}$$

$$= c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n - 4.$$

$$\text{eg } a_{n+2} - 4a_n = n$$

$$\text{HS } a_{n+2} - 4a_n = 0$$

$$r^2 - 4 = 0$$

$$r = \pm 2$$

$$\text{HS} = c_1 2^n + c_2 (-2)^n$$

$$\xrightarrow{\text{PS}} f(n) = n = (1 \cdot n^1)^{\frac{1}{2^n}}$$

$$\text{PS} = (\beta_0 n^0 + \beta_1 n)^{\frac{1}{2^n}}$$

$$= \beta_0 + \beta_1 n.$$

Substituting,

$$\beta_0 + \beta_1(n+2) - 4 \{ \beta_0 + \beta_1 n \} = n$$

$$\Rightarrow n(\beta_1 - 4\beta_1) + (\beta_0 + 2\beta_1 - 4\beta_0) = n$$

Soln. $a_n =$

$$(-\frac{1}{3} - \frac{2}{9}n) +$$

$$(c_1 2^n + c_2 (-2)^n).$$

$$\beta_1 - 4\beta_1 = 1$$

$$\Rightarrow -3\beta_1 = 1$$

$$\beta_1 = -\frac{1}{3}$$

$$-3\beta_0 - \frac{2}{3} = 0$$

$$\Rightarrow \beta_0 = -\frac{2}{9}$$

$$\text{PS} = -\frac{1}{3} - \frac{2}{9}n$$

$$\text{Lg } a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$$

HS

$$r^2 + 5r + 6 = 0$$

$$\Rightarrow r = -3, -2$$

$$HS = c_1 (-3)^n + c_2 (-2)^n$$

PS

$$f(n) \sim 3n^2$$

$$= \text{Koeff}(3 \cdot n^2) 1^n.$$

$$PS = (p_0 n^0 + p_1 n^1 + p_2 n^2) 1^n$$

$$= p_0 + p_1 n + p_2 n^2$$

Substituting,

$$p_0 + p_1 n + p_2 n^2 + 5p_0 + 5p_1 (n-1) + 5p_2 (n-1)^2 \\ + 6p_0 + 6p_1 (n-2) + 6p_2 (n-2)^2 = 3n^2$$

-24p₂
↑

$$+ \rightarrow n^2(p_2 + 5p_2 + 6p_2) + n(p_1 + 5p_1 - 10p_2 + 6p_1 - 12p_2) \\ + (6p_0 + 5p_1 + 5p_2 + 6p_0 - 12p_1 + 24p_2) = 3n^2$$

$$\begin{array}{l|l|l} \Rightarrow 12p_2 = 3 & 12p_1 - 22p_2 = 0 & 12p_0 - 7p_1 + 29p_2 = 0 \\ p_2 = \frac{1}{4} & \Rightarrow 6p_1 = 11 \cdot \frac{1}{4} & \Rightarrow 3 - \frac{77}{24} + 29p_2 = 0 \\ & \Rightarrow p_1 = \frac{11}{24} & \Rightarrow p_2 = \frac{5}{24 \cdot 29} \end{array}$$

$$PS = -\frac{97}{12 \cdot 24} + \frac{11}{24}n + \frac{1}{4}n^2$$

$$\Rightarrow 12p_0 - \frac{77}{24} + \frac{29}{4} = 0$$

$$\Rightarrow 12p_0 + \frac{-77 + 174}{24} = 0$$

$$\Rightarrow p_0 = -\frac{97}{12 \cdot 24}$$

$$\text{Sofn: } a_n \rightarrow HS + PS$$

$$\stackrel{\text{eq}}{=} a_n + 5a_{n-1} + 6a_{n-2} = 42 (4^n).$$

$$\begin{array}{l}
 \text{HS} \\
 r^2 + 5r + 6 = 0 \\
 \Rightarrow r = -2, -3 \\
 \text{HS} = \alpha(-2)^n + \beta(-3)^n
 \end{array}
 \quad \left| \begin{array}{l}
 \text{PS} \\
 f(n) = 42(4^n) = (1 \cdot 42 \cdot n^0) \cdot 4^n \\
 \text{PS} = (\beta_0) 4^n \\
 \beta_0 \cdot 4^n + 5 \cdot \beta_0 \cdot 4^{n-1} + 6 \cdot \beta_0 \cdot 4^{n-2} = 42(4^n) \\
 \Rightarrow \beta_0 \left(1 + 5 \cdot \frac{1}{4} + 6 \cdot \frac{1}{16} \right) = 42 \\
 \Rightarrow \beta_0 \cdot \frac{16 + 20 + 6}{16} = 42 \Rightarrow \beta_0 = 16
 \end{array} \right.$$

$\therefore a_n = \alpha(-2)^n + \beta(-3)^n + 16(4^n)$.

$$\stackrel{\text{eq}}{=} a_n - 6a_{n-1} + 9a_{n-2} = 3^n.$$

$$\begin{array}{l}
 \text{HS} \\
 r^2 - 6r + 9 = 0 \\
 \Rightarrow r = 3, 3 \\
 \text{HS} = \alpha \cdot 3^n + \beta \cdot n \cdot 3^n
 \end{array}
 \quad \left| \begin{array}{l}
 \text{PS} \\
 f(n) = 3^n = (1 \cdot 3^0)^{3^n} \quad \text{root same} \\
 \text{PS} = (\beta_0) 3^n \cdot n^2
 \end{array} \right.$$

Substituting,

$$\begin{aligned}
 & \beta_0 \cdot 3^n \cdot n^2 - 6 \cdot \beta_0 \cdot 3^{n-1} (n-1)^2 \\
 & + 9 \beta_0 (3^{n-2}) (n-2)^2 - 3^n \\
 \Rightarrow & \beta_0 n^2 - 6 \beta_0 \frac{1}{3} (n-1)^2 \\
 & + 9 \beta_0 \frac{1}{3^2} (n-2)^2 = 1 \\
 \Rightarrow & \beta_0 n^2 - 2 \beta_0 (n-1)^2 + \beta_0 (n-2)^2 = 1 \\
 & - 2\beta_0 + 4\beta_0 = 1 \\
 & \beta_0 = \frac{1}{2} \quad \left| \text{PS} = \frac{1}{2} 3^n \cdot n^2 \right.
 \end{aligned}$$

✓ Root of char. eqⁿ
✓ $(b_0 n^0 + b_1 n^1 + \dots + b_t n^t) \kappa^n$
✗ $(b_0 n^0 + b_1 n^1 + \dots + b_t n^t) \kappa^n \cdot n^m$
✗ multiplicity m of root κ

* Non-linear Recurrence Rel^n

$$\text{eg } \sqrt{a_n} - \sqrt{a_{n-1}} - 2\sqrt{a_{n-2}} = 0 \quad ; \quad a_0 = a_1 = 1$$

$\checkmark \quad a_n = x_n^2$

$$r^2 - r - 2 = 0$$

$$\Rightarrow x_n - x_{n-1} - 2x_{n-2} = 0 \quad (r-2)(r+1) = 0$$

$$x_n = c_1 (2)^n + c_2 (-1)^n, \quad x_0 = 1, x_1 = 1$$

$$\sqrt{a_n} = -\frac{1}{2}(2^n) + \frac{3}{2}(-1)^n$$

$$a_n = \left(-2^{n-1} + \frac{3}{2}(-1)^n \right)^2$$

$$\text{eg } a_n = 3a_{n/2} + n, \quad n = 2^k \text{ for } k \geq 1, \quad a_1 = 1$$

$$a_{2^k} = 3a_{2^{k-1}} + 2^k$$

$$a_k = 3a_{k-1} + 2^k$$

$$\frac{\text{HS}}{\cdot r} = 3$$

$$\frac{\text{PS}}{\cdot r} = \alpha 2^k \quad | \quad (-1 + \alpha)(2^k) = 3\alpha 2^{k-1}$$

$$\alpha 2^k = 3 \cdot \alpha \cdot 2^{k-1} + 2^k \quad | \quad \Rightarrow \frac{\alpha - 1}{3\alpha} = \frac{1}{2}$$

$$\Rightarrow \alpha = -2$$

$$x_k = c \cdot 3^k - 2^{k+1}, \quad x_0 = 1$$

$$a_n = 3 \cdot 3^k - 2 \cdot 2^k$$

$$= 3 \cdot 3^{\lg_2 n} - 2 \cdot n \Rightarrow 3n^{\lg_2 3} - 2n$$