ENGINEERING MATHS

TOPIC :

W ✓1. Linear Algebra.
Tu/Sun 2. Probability & Statistics.
Th 3. Calculus.
4. Differential equations.
5. Complex Variables.
6. Numerical Methods.
7. Laplace Transforms.

1–2: CS

3–5: → differentiability, → Integration

- Maths → 15M out of 100M.
  (10 to 12) is target.

Maths + A/RE = 10 + 15 = 25M.

1. LINI

LA → Wed
Calculus → Thurs.
Prob/Stats → Sunday ✓

1. LINEAR ALGEBRA (Matrices)

(i) Intro to Matrices.
(ii) Transpose of a matrix.
(iii) determinants.
(iv) Inverse of a Matrix.
(v) Rank of a matrix.
(vi) System of equations.
(vii) Eigen Values & Eigen Vectors.
(viii) Caley-Hamilton Theorem.

(i) INTRO TO MATRICES

Algebra : Branch of maths which deals with variables.

eg: $x^2 - 3x + 2 = 0$ deg(x) = 2.

Linear Algebra: deals with linear eqⁿs.

→ Linear Algebra is a branch of mathematics which deals with linear equations.
↳ degree = 1

eg.
$x + y = 2$
$x = 1$
$x + y + z = 3$
$x + z = 9$

→ Matrices is a basic tool which is used to analyze linear equations.
∵ Conventional process is too length.

i → row no.
j → col no.

* Types of Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$$

→ upper
→ principle diagonal.
→ lower.
→ Square Matrix (m=n)

diagonal ⇒ $i = j$
off-diagonal ⇒ $i \neq j$
upper-diagonal ⇒ $i < j$.
lower-diagonal ⇒ $i > j$.

1) Diagonal Matrix

Square matrix in which all the off-diagonal elements are equal to 0.

$\boxed{a_{ij} = 0 \text{ for } i \neq j}$

eg: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → all off-diag are 0.

2) Scalar Matrix

Diagonal Matrix in which all the diagonal elements are equal.

$\boxed{\begin{matrix} a_{ij} = 0 \text{ for } i \neq j \\ a_{ij} = K \text{ for } i = j \end{matrix}}$

eg: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

3) Identity Matrix / Unit Matrix

$\boxed{\begin{matrix} a_{ij} = 0 \text{ for } i \neq j \\ a_{ij} = 1 \text{ for } i = j \end{matrix}}$ → diagonal elements must be 1.

→ Identity Matrix is also called as Unit Matrix / Multiplicative identity matrix.

$\therefore AI = IA = A$.

→ 1: mult. identity.

4) Null Matrix / Zero Matrix / Additive Identity

$\boxed{a_{ij} = 0 \ \forall\ i, j}$

→ Null Matrix is also called as Additive Identity matrix.

$A + O = O + A = A$

5) Upper Triangular Matrix

A Square Matrix in which all the elements below Principal diagonal are equal to 0, then the matrix is called as upper Triangular matrix.

$\boxed{a_{ij} = 0 \text{ for } i > j}$

eg: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

6) Lower Triangular Matrix

$\boxed{a_{ij} = 0 \text{ for } i < j}$

7) Idempotent Matrix

Idempotent $\Rightarrow A^2 = A$.
Involutary $\Rightarrow A^2 = I$

* Multiplication of 2 Matrices

2 Matrices are confirmable for multiplication if the:

$\boxed{\text{no. of columns in 1st Matrix} = \text{No. of rows in 2nd Matrix}}$

* Properties of Multiplicⁿ of Matrices:

(i) Matrix multiplication may/may not satisfy commutative Property.
$\because$ if AB exists, there is no guarantee that BA will exist

$A: 2\times3 \quad B \to 3\times4 \qquad AB: 2\times4 \ ; \ BA: \times$

(ii) Matrix multiplication satisfies associative Property.

$\boxed{A(BC) = (AB)C}$

(iii) The eq^n $AB = 0$, doesnt necessarily implies that atleast one of the matrix of A & B is a zero Matrix.

ie. The product of 2 Non-zero matrices may result in a Zero Matrix.

eg: $\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(iv) If the product of 2 Non-Zero matrices is a Zero Matrix, then both matrices must be Singular Matrices.

$D(A) = 1 \cdot 1 - 1 \cdot 1 = 0$

$D(B) = -1 - (-1) = 0$

(v) Any Square Matrix can be expressed as product of its lower triangular and upper triangular matrix.

$\boxed{A = LU}$ → Upper Δ Matrix

L → Lower Δ matrix

if $A_{m \times n} * B_{n \times p} = AB_{m \times p}$

$\boxed{\begin{array}{l} \text{No. of mul} = mp(n) \\ \text{No of add} = mp(n-1) \end{array}}$

* LU decomposition of matrix.

Q1: $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$: the diagonal elements of U are considered as both ①, then $l_{22} = ?$

Soln:

$A = \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$

1. Elish method
2. Gauss method.

$A = LU$.

$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$

$l_{11} = 2$.

$l_{11} \cdot u_{12} = 2$. $\quad u_{12} = 1$

$l_{21} = 4$

$l_{21} \cdot u_{12} + l_{22} = 9$.

$4 \cdot 1 + l_{22} = 9$.

$\boxed{l_{22} = 5}$

Q2: Matrix A has 'x' rows, & 'x+5' columns.
Matrix B has 'y' rows & '11-y' columns.

If both AB & BA exists.

$x \cdot x+5 \quad , y \cdot 11-y$

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$x+5=y$
$11-y=x$

$x=11-y$
$2x+5=11$
$\boxed{x=3,\ y=8}$

A: $x \times (x+5)$ B: $y \times (11-y)$

$x+5=y$ — (i)
$x=11-y$ — (ii)

eg3: Consider X is an idempotent Matrix. Then which of the following statement is true?

a) $I+X \rightarrow$ idempotent
b) $I-X \rightarrow$ idempotent
c) Both
d) None

X is idempotent
$X^2=X$.

a) $(I+X)^2=I+X$
b) $(I-X)^2=I-X$

$(I+X)^2=I^2+X^2+2IX$
$\Rightarrow I+X+2X$
$=I+3X$ ✗

$(I-X)^2=I^2+X^2-2IX$
$=I+X-2X$
$=I-X$ ✓

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eg4: (imp) $A=[a_{ij}]_{m\times n}$ is defined by:
$a_{ij}=i+j$.
then the sum of all the elements of the matrix is?

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$\Rightarrow \begin{bmatrix} 1+1 & 1+2 & \cdots & 1+n \\ 2+1 & 2+2 & \cdots & 2+n \\ \vdots & & & \\ m+1 & m+2 & \cdots & m+n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & & & \\ m & m & & m \end{bmatrix}_{m\times n} + \begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & & n \\ \vdots & \vdots & & \vdots \\ 1 & 2 & & n \end{bmatrix}_{m\times n}$$

$$\Rightarrow \frac{m(m+1)}{2}\cdot n + \frac{n(n+1)}{2}\cdot m$$

$$\Rightarrow \frac{mn}{2}[m+1+n+1]$$

$$= \frac{mn}{2}[m+n+2]$$

* Transpose of Matrix

- The transpose of a Matrix is obtained by interchanging rows with columns / vice-versa.
- It is denoted by A^T.

⇒ If the order of matrix A is $m \times n$ then, order of A^T is $n \times m$.

⇒ if the element of A is a_{ij} then, the element of A^T is a_{ji}.

* Properties of Transpose of Matrix:

(1) $(A^T)^T = A$.

(2) $(A+B)^T = A^T + B^T$

(3) $(AB)^T = B^T \cdot A^T$.

(4) $(K \cdot A)^T = K(A^T)$ where, $K \to$ constant.

* Symmetric Matrix

⇒ $A^T = A$. when, $\boxed{a_{ji} = a_{ij}}$

(Refer Priya Note)

Complex Matrices.

* Conjugate Transpose Matrix

Transpose conjugate Matrix. $A^\theta / (\bar{A})^T$.

* Properties of Conjugate Transpose Matrix:

(1) $(A^\theta)^\theta = A$.

(2) $(A+B)^\theta = A^\theta + B^\theta$.

(3) $(AB)^\theta = B^\theta \cdot A^\theta$.

(4) $(KA)^\theta = \bar{K} A^\theta$ where, 'K' is complex no.

Real T
- → $A^T = A$ (Symm) → $a_{ji} = a_{ij}$.
- → $A^T = -A$ (Skew)
- → ortho $A^T = A^{-1}$.

Complex M
- → Hermitian ⇒ $A^\theta = A$. → $\overline{a_{ji}} = a_{ij}$
- → Skew-Hermitian ⇒ $A^\theta = -A$ → $\overline{a_{ji}} = -a_{ij}$.
- → Unitary ⇒ $A^\theta = A^{-1}$.

eg 1: If A and B are square matrices of same order and if A is symmetric then:

$B^T \cdot AB$ is ?

$A^T = A$.

$\Rightarrow (B^T \cdot AB)^T = B^T A^T B$

$\Rightarrow B^T (AB)$.

∴ symmetric

if $(B^T AB)^T = B^T AB$

$(B^T AB)^T = -B^T AB$

eg 2: every diagonal element of a Hermitian Matrix is:

a) Purely real — $2 \to 2$.
b) purely imaginary — $i \to -i$
c) complex — $2+i \to 2-i$.
d) None.

$A^\theta = A$.

$\overline{a_{ji}} = a_{ij}$.

$\overline{a_{ii}} = a_{ii}$

$\bar{c} = c$ (~~complex~~ conjugate = element).

eg 3: Every diagonal element of a Skew Hermitian matrix is:

a) Purely imaginary.

Conj.

$i \longrightarrow -i$

$A^\theta = -A$.

$\overline{a_{ji}} = -a_{ij}$

$\overline{a_{ii}} = -a_{ii}$.

$\boxed{\bar{c} = -c}$. (purely img).

$A^\theta = -A$

a_{ij}

$\overline{a_{ji}} = -a_{ij}$

$\overline{a_{ii}} = a_{ii}$

eg 4: If A is a Hermitian Matrix, then iA is ?

$A^\theta = A$.

$(iA)^\theta = iA \Rightarrow H$.

$(iA)^\theta = -iA \Rightarrow S\text{-}H$.

$(iA)^\theta = \bar{i} A^\theta =$

$K \to$ complex

$\Rightarrow -iA^\theta$

$\Rightarrow -iA$

∴ Skew-Hermitian

* Minor of a Matrix : → Cell

deleting some rows / columns from original matrix

Q How many 2x2 possible from 3X3 ?

$$ {}^3C_2 \times {}^3C_2 = \underline{\underline{9}} . $$

eg2: How many 2X3 minors are possible in the following matrix ?

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 1 \\ 1 & 4 & 3 & 6 & 2 \\ 4 & 6 & 2 & 0 & -1 \end{bmatrix}_{4\times5} \to 2\times3 .$$

$\Rightarrow {}^4C_2 \times {}^5C_3$

$\Rightarrow {}^4C_2 \times$

$\frac{\cancel{2}\cancel{4}\times\cancel{3}}{\cancel{2\times1}} \times \frac{5\times\cancel{4}\,2}{\cancel{2}\times\cancel{2}} \Rightarrow \underline{\underline{60}}$

6×10

* Minor of an element

The minor of an element a_{ij} is obtained by deleting corresponding ith row & jth column & considering the determinant of remaining matrix.

→ Denoted by M_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$$

minor of a_{11} $(M_{11}) \Rightarrow \cdot\, a_{22}a_{33} - a_{32}a_{23}$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

• $M_{23} \Rightarrow a_{11}a_{32} - a_{31}a_{12}$

* CoFactor of an element

The cofactor of an element a_{ij}:

$$a_{ij} \Rightarrow \boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

if:

$i+j$ = even ;	$C_{ij} = M_{ij}$
$i+j$ = odd ;	$C_{ij} = -M_{ij}$

Minor

(iii) DETERMINANTS

It is defined as the sum of the product of the elements of the raw/column with the corresponding co-factors.

Δ with resp. to 1st raw.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

calculate determ in 6 ways.

w.r.t. R1 $\Delta = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$

w.r.t. C2 $\Delta = a_{12} \cdot C_{12} + a_{22} \cdot C_{22} + a_{32} \cdot C_{32}$

value: $a_{11} \cdot (a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

* Properties of determinants [Vimp]

(1) If A is a Square Matrix, then:

(i) $|A| = |A^T|$

(2) If 2 Parallel lines of a determinant are interchanged, the determinant retains its numerical value but opp in sign.

eg: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = 27.$

$\begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 0 \\ 4 & 5 & 6 \end{vmatrix} = -27.$

$\begin{vmatrix} 1 & 3 & 2 \\ 7 & 0 & 8 \\ 4 & 6 & 5 \end{vmatrix} = 27 \quad -(-27)$

(3) In a determinant, if a raw/column is completely 0, then the determinant value will be 0.

$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0.$

(4) In a determinant, if 2 Parallel lines are identical to each other, then the determinant value will be 0.

$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{vmatrix} = 0.$

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(5) In a determinant, if 2 Parallel lines are proportional to each other, then the determinant value will be 0.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{vmatrix} = 0 . \qquad 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0.$$

(6) If the determinant contains consecutive ~~terms~~ numbers, then the value of the determinant = 0.
[Valid for $\geq$ 3rd order determinants]

$$\begin{vmatrix} 1991 & 1992 & 1993 \\ 1994 & 1995 & 1996 \\ 1997 & 1998 & 1999 \end{vmatrix} = 0$$

Proof:

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_2$

$$\begin{vmatrix} 1991 & 1992 & 1993 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} \Rightarrow 0$$

(7) If each element of a line is multiplied by same factor then the whole determinant is multiplied by that factor.

Imp

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 27 \ ; \quad \begin{vmatrix} K & 2K & 3K \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 27K.$$

$$\begin{vmatrix} K & 2K & 3K \\ 4K & 5K & 6K \\ 7K & 8K & 9K \end{vmatrix} = 27K^3 .$$

(8) The determinant of [upper triangular, lower triangular, diagonal, scalar matrix] is equal to the Product of the diagonal elements.

$$A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 2 \end{vmatrix} \ne 0.$$

lower Δ

lower Triangular matrix

$|A| = 1 \cdot 1 \cdot 1 \cdot 2 = 2$

(9) Determinant of Skew Symmetric matrix of even order is a Perfect Square.

$A^T = -A$.

(10) Determinant of Skew Symmetric matrix of odd order is $\underline{\underline{0}}$.

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11) Determinant of orthogonal Matrix $\Rightarrow \pm 1$
maybe 1 or -1.

12) If A & B are 2 Sq Matrices of same order.
then $|AB| = |A| \cdot |B|$.

13) If A is a non-singular matrix:
(i) $|A^n| = |A|^n$.
(ii) $|A^{-1}| = |A|^{-1} = 1/|A|$.

- If A is a nxn Square matrix
(i) $|\text{Adjoint } A| = |A|^{n-1}$.
(ii) $|\text{Adj}(\text{Adj} A)| = |A|^{(n-1)^2}$.

14) If A is a nxn Square matrix, &
K is a constant.

$$\boxed{|KA| = K^n |A|}$$

eg1. If A is 2x2 Sq. Matrix ; $|A| = 4$; $|3A| = ?$

$|A| = 4$. $\quad K=3, n=2$.

$|3A| = 3^2 |A|$
$\Rightarrow \underline{\underline{36}}$.

eg2. Let A is a 3x3 Square Matrix ; with
$|A| = 5$,
if $B = 4A^2$; then $|B| = ?$

Soln:
$|B| = |4A^2| = 4^3 |A^2|$.
$\Rightarrow 4^3 |A|^2$
$\Rightarrow 4^3 \times 5^2 \Rightarrow 64 \times 25$
$\Rightarrow \underline{\underline{1600}}$

eg3. Evaluate following determinants:

(i) $\begin{array}{c|c} \begin{vmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \\ 1 & -2 & 0 & 1 \end{vmatrix} & \begin{matrix} 2 \\ 4 \\ 1 \\ 3 \end{matrix} \end{array}_{4\times 4}$

Calculate Δ for this.

(ii) $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

$\Rightarrow 0 + 0 + 0 + 1 \cdot (-1)^{3+4} \cdot \begin{vmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix}$

$\Rightarrow -1[0 - 1(-1) + 0] = \underline{\underline{-1}}$

$\Rightarrow 1(-1)^{2+1}\begin{vmatrix}1 & 2 & 3\\ 3 & 0 & 1\\ 0 & 1 & 2\end{vmatrix} + 0 +$

$3\cdot(-1)^{2+3}\begin{vmatrix}0 & 1 & 3\\ 2 & 3 & 1\\ 3 & 0 & 2\end{vmatrix} + 0.$

$\Rightarrow -1[1\cdot(-1) - 2\cdot 6 + 3\cdot 3] +$

$-3[-1\cdot(1) + 39]$

$\Rightarrow -1(-1-12+9) + -3(-28)$

$\Rightarrow 4 + 84 = \underline{88}$

Q4: The value of the determinant:

$\begin{vmatrix}1+b & b & 1\\ b & 1+b & 1\\ 1 & 2b & 1\end{vmatrix}$ is ?

$C_1 \rightarrow C_1 + C_2 + C_3.$

$\Rightarrow \begin{vmatrix}2+2b & b & 1\\ 2+2b & 1+b & 1\\ 2+2b & 2b & 1\end{vmatrix}$

$\Rightarrow 2+2b\begin{vmatrix}1 & b & 1\\ 1 & 1+b & 1\\ 1 & 2b & 1\end{vmatrix} \Rightarrow \underline{0}$

* Inverse of Matrix

→ If A is a non-singular ($|A| \neq 0$.) Matrix, then its inverse is given by:

$$\boxed{A^{-1} = \frac{adj.(A)}{|A|}}$$

→ Inv. exits for only non-Singular matrix.

* CoFactor Matrix

The matrix obtained by replacing each element with its corresponding cofactor is called CoFactor Matrix.

$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$

$cof(A) = \begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix} \xrightarrow{T} Adj(A).$

$$\boxed{adj(A) = [cof(A)]^T}$$

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* Properties of inverse of a Matrix:

(1) $(A^{-1})^{-1} = A$

(2) $A * A^{-1} = A^{-1} * A = I$

(3) $AB = I \implies A = B^{-1}$ and $B = A^{-1}$

(4) $(A^{-1})^{T} = (A^{T})^{-1}$

(5) $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

Note: 1) if $A = n \times n$ non-singular Matrix.

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| I_n$$

where, I_n = identity matrix of $n \times n$.

2) if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

eg 1: If $A = \begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ then $A^{-1} = ?$

$13 - (-i^2)$.
$13 + i^2$.

$\rightarrow \operatorname{adj} A = \begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

$\rightarrow |A| = 13 - 1 = 12$.

$\implies \boxed{A^{-1} = \frac{1}{12} \begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}}$

eg 2: If $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$ then $a + b = ?$

$\rightarrow \operatorname{adj} A = \begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix}$

$\rightarrow |A| = 6$.

$\therefore A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$

$a = 0.1/6$.

$b = 1/3$

$\therefore a+b = \frac{1}{3} + \frac{0.1}{6} = \frac{2.1}{6}$ //

$\Rightarrow 0.345$

eg3: If A, B, C, D, $\longrightarrow$ n×n matrix each with non-zero determ.

if $ABCD = I$
then $B^{-1} = ?$

$B^{-1} = CDA$.
$C^{-1} = DAB$

if, $CEDBGAF = I$ ✓

$B^{-1} = GAFCED$
$G^{-1} = AFCEDB$
$D^{-1} = BGAFCE$.

Proof:

$* \ ABCD = I$
$A^{-1}ABCD = A^{-1}I$
$BCDD^{-1} = A^{-1}D^{-1}$
$BCC^{-1} = A^{-1}D^{-1}C^{-1}$
$B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1}$
$\boxed{B^{-1} = CDA}$

eg4: Let $A = [a_{ij}]_{5\times5}$ is defined by:

$\boxed{a_{ij} = i^2 - j^2}$

$\begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & \\ 8 & & 0 \end{bmatrix}$.

then $A^{-1} = ?$

diagonal element $= 0$ $\rightarrow$ Null
$\rightarrow$ Skew-Symmetric

can't be Null:

Since, $a_{12} = 1^2 - 4$
$\Rightarrow -3 \neq 0$.

$\therefore$ Definitely Skew Symmetric $\rightarrow$ order $= \cancel{6}\ 5$
$\therefore |D| = 0$

$\because a_{ij} = -a_{ji}$
$\Rightarrow i^2 - j^2 = -(j^2 - i^2)$.

$\therefore |A| = 0$, $\longrightarrow$ Singular Matrix

$\therefore A^{-1}$ doesn't exist. ✓

eg5: If matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$

$-1C$

$Adj\ A = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & K & 7 \end{bmatrix}$ $K = ?$

$K \rightarrow a_{32}$

$K \rightarrow a_{23}$
Co:

$K = \text{adj}A_{32} = C_{23}$

$(-1)^{2+3} \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix}$

$\Rightarrow \underline{\underline{-5}}$. Ans.

eg 6: $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}_{3\times3}$; $A^{-1} = ?$

Start from how many rows with & least column with

Simple Technique

Soln:

$IM = \begin{bmatrix} 1 & 3 & 0 & 1 \\ -1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 2 \\ 1 & 3 & 0 & 1 \end{bmatrix}$ (Intermed. M)

compute & determ.

$\text{Adj}A \Rightarrow \begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$

$|A| = 1(5) - 1(4)$
$= 1$.

$A^{-1} = \begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$.

→ By multiplying first row of A with 1st col of Adj(A), or 2nd row (A) with 2nd col of Adj(A), or 3(A) with 3c (Adj(A)).

→ we get determinant.

$C = AB$.
$C_{12} \to$ 1st row of A * 2nd col of B

$A(\text{adj}A) = |A| I_n$

$A(\text{adj}A) = \begin{bmatrix} |A|_{11} & 0 & 0 \\ 0 & |A|_{22} & 0 \\ 0 & 0 & |A|_{33} \end{bmatrix}$

3rd row of A * 3rd col of adj A.

$\Rightarrow |A| = [1\ 0\ -1] \begin{bmatrix} 5 \\ -6 \\ 4 \end{bmatrix} = \underline{\underline{1}}$.

$\Rightarrow |A| = 2 - 3 + 2 = \underline{\underline{1}}$.

eg 7: if $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$; $A^{-1} = ?$

$IM = \begin{bmatrix} 3 & 2 & 3 & 3 \\ 1 & 4 & 4 & 1 \\ 4 & 1 & 2 & 4 \\ 3 & 2 & 3 & 3 \end{bmatrix}$

$|A| = 2\ 3\ 4$ 10, -15, 5

$= 20 - 45 + 20$

$|A| = -5$

$\text{Adj}A \Rightarrow \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$

$A^{-1} \Rightarrow \boxed{\begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}}$

(V) Rank of a Matrix → Vimp

determ. approach

- Rank of a Matrix is equal to ~~the~~ order of the highest non-zero square submatrix.
- A number 'R' is said to be rank of the matrix if it satisfies following conditions:

(i) There exists atleast 1 non-zero square matrix of order R whose determinant ≠ 0.

(ii) If the matrix contains any submatrices of order 'R+1' and above, then the value of those determinants must be equal to 0.

eg.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$$

$|A| \neq 0 \longrightarrow \rho(A) = 3$ [full rank]

if $|A| = 0 \Rightarrow \rho(A) < 3$.

Find 2x2 Matrix → a of them

if atleast one is non-zero ⇒ $\rho(A) = 2$

if all is zero.

$\rho(A) = 1$ (Given non-zero matrix).

else

$\rho(A) = 0$.

Q. $R(A_{7\times7}) = 3$ then $|A| = ?$

$|A| = 0$; if it wud have been non-zero, $R \to 7$.

$R(6\times6) = 0$

$R(5\times5) = 0$

$R(4\times4) = 0$

$R(3\times3) \neq 0$.

* Properties of Rank of a Matrix:

1) $\rho(O_{n\times n}) = 0$ [Null Matrix]

2) $\rho(I_{n\times n}) = n$ (full rank) $|I_{n\times n}| = 1 \neq 0$

3) $\rho(AB) \leq \min(\rho(A), \rho(B))$.

4) If A is a $m\times n$ ⇒ ← rectangular

$\rho(A) \leq \min(m, n)$

5) $\rho(A) = \rho(A^T)$.

6) A matrix with rank r contains 'r' linearly independent vectors.

7) Rank of Matrix = no. of non-zero rows in the row-echelon form of a matrix.

or col in col-echelon form.

$\begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$

Rank by row operations approach. → upper Δ matrix.

* Linearly dependent / Independent Vectors

A set of vectors are said to be linearly dependent if we can express 1 vector as linear combination of remaining vectors.

eg:
$\vec{a} = \bar{i} + 2\bar{j} + 3\bar{K}$
$\vec{b} = 2\bar{i} + \bar{j} - \bar{K}$.
$\vec{c} = 3\bar{i} + 3\bar{j} + 2\bar{K}$.

$\vec{c} = \vec{a} + \vec{b}$
∴ linearly dependent

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{bmatrix} \Rightarrow 0 \text{ LD.} \quad \neq 0 \text{ LID.}$$

eg1: The vectors $\bar{i} + \bar{j} + \bar{K}$, $2\bar{i} + 3\bar{j} + \bar{K}$, $5\bar{i} + 6\bar{j} + 4\bar{K}$ are:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 6 & 4 \end{vmatrix} = 1(6) - 1(3) + \cancel{6} - 3$$
$= 6 - 3 - 3$
$\Rightarrow 0$

∴ linearly dependent

[determinant applicable for only Square Matrix]
Inverse → only Sq. Matrix.

eg 2: Calculate the ranks of the following matrices:

a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}_{3\times3}$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 4R_1$
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}$

$R_3 \to R_3 - R_2$
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ ✓ ✓ ×

$\boxed{\rho(A) = 2}$

Row echelon form

b) $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{3\times4}$. → Rank defined.

$R_2 \to 2R_2 - 3R_1$
$R_3 \to 2R_3 - R_1$
$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$

$R_3 \to 5R_3 + R_2$
$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\boxed{\rho(A) = 2}$. rank ≤ 3. min(3,4).

imp

eg3 $A = [a_{ij}]_{n \times n}$ with $n \geq 3$ is defined by : $\boxed{a_{ij} = ij}$,

then $\rho(A) = ?$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ $\quad R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \therefore \boxed{\rho(A) = 1}$

Note: In a matrix, if all the rows are proportional to each other, then $\boxed{\text{rank} = 1}$.

Q The rank of a 5x7 matrix in which every element $= -1$

$\rightarrow \rho(A) = 1$

Q Calculate ranks of the following matrices ?

1) $\begin{bmatrix} 1-a & a^2 & a^3 & \cdots & a^n \\ 1-a & a^2 & a^3 & \cdots & a^n \\ \vdots \\ 1-a & a^2 & a^3 & \cdots & a^n \end{bmatrix}$ $\rightarrow$ all rows proport.

$\therefore \boxed{\rho(A) = 1}$

$\therefore$ if $a = 0$ $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$

b) $\begin{bmatrix} a & a^2 & a^3 & \cdots & a^n \\ a & a^2 & a^3 & \cdots & a^n \\ a & a^2 & a^3 & \cdots & a^n \\ \\ a & a^2 & a^3 & \cdots & a^n \end{bmatrix}$ if $a = 0 \Rightarrow \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 \\ 0 & \cdots \end{bmatrix}$

$\therefore \rho(A) = 0$

if $a \neq 0 \therefore \rho(A) = 1$

$\therefore$ (d) depends on a value ✓

(VI) SYSTEM OF EQUATIONS

If the system of equations has a solution then it is called as consistent System otherwise, it is called as inconsistent System.

eg: $x + y = 2$
$x - y = 0$

$x = 1, y = 1$

$\rightarrow$ consistent

unique   infinite

$x + y = 2$
$2x + 2y = 4$

$x + y = 2$
$2x + 2y = 5$; $x + y = 2.5$

No Solution

$\rightarrow$ Inconsistent

* Homogeneous System of Eq^n s

consistent system have [always solution].

Consider the system of Eq^n s:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots\cdots a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots a_{nn}x_n = 0 \end{cases}$$

Write the system in the form of:

$AX = 0$, where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

A → Coefficient Matrix; X → Solutn.

Zero Solution satisfies anyway.

Homogeneous → consistent (Soln)
- unique / trivial sol^n: $\rho(A) = n$ (no of var) or, $|A| \neq 0$
- Infinite / non-trivial: $\rho(A) < n$ or, $|A| = 0$.

$$B \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = 0 \text{ (Homogeneous).}$$

* Non-Homogeneous System of Eq^n s

may be consist / incons.

Consider the System of Eq^n s

$$\begin{rcases} a_{11}x_1 + a_{12}x_2 + \cdots\cdots a_{1n}x_n = b1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots a_{2n}x_n = b2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots a_{nn}x_n = b_n \end{rcases}$$

$$\boxed{AX = B}$$

$$* \ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} * \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

A → Coefficient Mat. X → Sol^n Mat. B → Col Mat

Zero Solution isnt satisfying.

* Consider Augmented matrix $[A/B]$ a

$$[A/B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

** Non-Homogeneous
- Consistent $[\rho(A) = \rho(A/B)]$
  - Unique $[\rho(A) = \rho(A/B) = n]$
  - infinite $[\rho(A) = \rho(A/B) < n]$
- Inconsistent (No Soln) $[\rho(A) \neq \rho(A/B)]$

* Non-Homogeneous System of 2 Variables

Consider the system of eq^n s:

$$\left.\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}\right\} \text{RHS} \neq 0 \text{ [Non-homo]}$$

Unique	Infinite	No Solution
$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}} \neq \frac{c_1}{c_2}$
 intersecting	infinite	 parallel to each other.

Unique:	infinite	No Soln:
$x+y=2$	$x+y=2$	$x+y=2$
$x-y=0$	$2x+2y=4$	$2x+2y=5$
		$m \to -1$

Q1 The system of eq^n s:

$$4x+2y=7$$
$$2x+y=6 \quad \text{has [No solution]}$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$
$$2 = 2 \neq 7/6$$

Q2: Consider the system of eq^n s:

$$3x + 2Ky = -2$$
$$Kx + 6y = 2$$

The value of K ? such that system has infinite Solution:

$$\frac{3}{K} = \frac{K}{3} = -1 \qquad \boxed{K = -3}$$

Q3: Consider the matrix $A = \begin{bmatrix} K & 2K \\ K^2-K & K^2 \end{bmatrix}$ (x_1, x_2)

and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

No. of distinct values of K for which System $AX = 0$ will have non-trivial [infinite]

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$Kx_1 + 2Kx_2 = 0$ → Homogeneous
$(K^2-K)x_1 + K^2x_2 = 0.$

∴ Non-trivial → infinite $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}}$

$\frac{K}{K^2-K} = \frac{2K}{K^2}$

$K^2 = 2K^2 - K$

$\boxed{K = 0, 2}$

Q. The solⁿ of the system of eqⁿs :

$x + 2y + 3z = 20.$
$7x + 3y + z = 13$
$x + 6y + 2z = 0$

a) $x=-2, y=2, z=8$
b) $x=2, y=-3, z=8.$
c) $x=2, y=3, z=-8$
d) $x=8, y=2, z=-3$

∴ (b) by Substitution

Q. The value of x_3 obtained by solving the system of eqⁿs:

$x_1 + 2x_2 - 2x_3 = 4$
$2x_1 + x_2 + x_3 = -2$
$-x_1 + x_2 - x_3 = 2$

$$[A/B] = \left[\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 1 & 1 & -2 \\ -1 & 1 & -1 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + R_1$

$$\left[\begin{array}{cccc} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 3 & -3 & 6 \end{array}\right]$$

A, B

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

$\rho(A) = 3$
$\rho(A/B) = 3$
$n = 3$
∴ Unique

$x_1 + 2x_2 - 2x_3 = 4$
$-3x_2 + 5x_3 = -10.$
$2x_3 = -4$

$\boxed{x_3 = -2}$

* $\Delta \to$ eliminating last col
$\Delta_1 \to$ eliminating first column.

Unique solⁿ $\Rightarrow \Delta \neq 0.$
Infinite solⁿ $\Rightarrow \Delta = 0, \Delta_1 = 0.$
No solⁿ $\Rightarrow \Delta = 0, \Delta_1 \neq 0.$

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Q1: The System of equations

$$x+2y+z=6$$
$$2x+y+2z=6$$
$$x+y+z=5$$

has ____ soln?

$$[A/B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right]$$

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 6 \\ 1 & 2 & 6 \\ 1 & 1 & 5 \end{vmatrix}$$

$\Delta = 0$, $\Delta_1 = 2(10-6) - 1(-1) + 6(-1)$
$= 8 + 1 - 6 = 3$

[No soln]

Q2: The System of eqns:

$$x_1 - 4x_2 - x_3 = -3$$
$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 + 2x_2 + 5x_3 = 2$$

$$\Delta = \begin{vmatrix} 1 & -4 & -1 \\ 2 & -1 & 3 \\ 3 & 2 & 5 \end{vmatrix}$$

$\Rightarrow 1(-11) + 4(1) - 1(7)$
$= -11 + 4 - 7 = \underline{\underline{-14}}$

Unique soln

Q3. The system of equations:

$$x + 2y + z = 4$$
$$2x + y + 2z = 5$$
$$x - y + z = 1$$

$$\Rightarrow \Delta = 0 \Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} \Rightarrow 0$$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 2 & 5 \\ -1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow 2(-3) - 1(1+5) + 4(3)$
$-6 - 6 + 12 = \underline{0}$

$\therefore$ Infinite Solutions

Q4 The system of eqn:

$$-x + 5y = -1$$
$$x - y = 2$$
$$x + 3y = 3$$

$$[A/B] = \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right]$$

$\Delta_{\bar{n}_n}$ |

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$R_2 \to R_2 + R_1$
$R_3 \to R_3 + R_1$

$$[A|B] = \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2$
$\rho(A|B) = 2 \qquad n = 2$.

$\therefore \rho(A) = \rho(A|B) = n$.

$\therefore$ Unique Solution

Q2 For what value of 'a', the system of equations:

$2x + 3y = 4$
$x + y + z = 4$
$x + 2y - z = a$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right]$$

has a solution is:

$\Delta = 0$; $\quad \Delta_1 = \begin{vmatrix} 3 & 0 & 4 \\ 1 & 1 & 4 \\ 2 & -1 & a \end{vmatrix}$

$\Rightarrow 3(a+4) + 4(-3)$
$= 3a + \cancel{12} - \cancel{12} = 0$ $\quad \boxed{a=0}$

$\because \Delta = 0$; the system will have infinite no. of solutions;
for ∞ no. of Solutions;
$\boxed{\Delta_1 = 0}$

* EIGEN VALUES & EIGEN VECTORS

* Eigen Value

$A \to n \times n$ Square Matrix and λ is a Scalar Matrix.

$\therefore A - \lambda I \Rightarrow$ Characteristic Matrix.

and
$|A - \lambda I| = 0 \Rightarrow$ Characteristic Eq^n.

* Roots of the Char. Eq^n are called as the Eigen Values.

* Eigen Vector

If λ is an Eigen Value corresponding to matrix A, then there exists a non-zero vector X such that
$\boxed{AX = \lambda X}$

$X \to$ eigen Vector corresponding to eigen Value (λ).

* Properties of Eigen Values & Eigen Vectors

(1) Sum of the eigen values = Trace of a Matrix
[Sum of elements in PD]

(2) Product of the eigen values = Determinant of Matrix

(3) Eigen Values of A and A^T } are same.

(4) Eigen Values of upper triangular & lower triangular, diagonal, Scalar matrix = diagonal elements.

(5) The Eigen values of Symmetric matrix are always real.

(6) The Eigen Values of Skew-Symmetric matrix is either purely imaginary or 0.

(7) The Eigen values of Orthogonal Matrix are unit modulus.

(8) If $a+ib$ is an eigen value of a real matrix; then $a-ib$ must be another eigen value.

(9) If λ is an eigen value of Matrix A:

(i) $K\lambda$ is an eigen value of KA.

(ii) $\lambda^n \longrightarrow A^n$.

(iii) $1/\lambda \longrightarrow A^{-1}$.

(iv) $|A|/\lambda \longrightarrow adj\,A$.

(v) $\lambda \pm K \longrightarrow A \pm KI$.

$A^{-1} = \frac{adj\,A}{|A|}$

$adj\,A = |A| \cdot A^{-1}$

$adj\,A = \frac{|A|}{\lambda}$.

(10) The eigen vectors of A and A^n are same.
but diff eigen values.

$A \longrightarrow \lambda$, $A^n \longrightarrow \lambda^n$ } $A \rightarrow X$, $A^n \rightarrow X$. } same EV.

(11) The eigen Vectors corresponding to distinct eigen values of Symmetric Matrix are orthogonal to each other

$\boxed{\cdot \text{ product b/w EV of S.M} = 0}$

* Problems

Q. Consider the matrix $A = UV^T$, where
$U = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
then eigen values of Matrix is?

$2\times1 \quad 1\times2 \quad 2\times2$

$A = UV^T = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$

$A = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$

$\Rightarrow A - \lambda I = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1-\lambda & 1\\2 & 2-\lambda\end{bmatrix}$

$\therefore |A - \lambda I| = 0$

$\therefore (1-\lambda)(2-\lambda) - 2 = 0.$

$\lambda^2 - 3\lambda = 0$ $\qquad \lambda^2 - 3\lambda + 0 = 0$

$\boxed{\lambda = 0, 3}$

* ~~upto~~ for 2x2 Matrix

$\boxed{\lambda^2 - Tr(A)\lambda + |A| = 0}$ $\rightarrow \lambda^2 - 3\lambda + 0 = 0.$

$\because a, b$

$x^2 - (a+b)x + ab = 0$

$\therefore \lambda^2 - Tr(A)\lambda + |A| = 0.$

* for 3x3

Cubic Eqn.

$-\lambda^3 + Tr(A)\lambda^2 - [m_{11} + m_{22} + m_{33}]\lambda + |A| = 0.$

if options given : check sum & product

Q. Calculate eigen values for the following Matrices.

1) $\begin{bmatrix}-1 & 4\\4 & -1\end{bmatrix}$ $\Rightarrow \lambda^2 + 2\lambda - 15 = 0.$

$\Rightarrow \lambda^2 + 5\lambda - 3\lambda - 15 = 0.$

$\lambda(\lambda+5) - 3(\lambda+5) = 0.$

(a) 3, −5 $\qquad \lambda = 3, -5$ (a)

Product = 6
Sum = 7

2) $\begin{bmatrix}2 & -1\\-4 & 5\end{bmatrix}$ $\Rightarrow \lambda^2 - 7\lambda + 6$

$= \lambda^2 - \lambda - 6\lambda + 6$

$\lambda(\lambda-1) - 6(\lambda-1)$

(b) 1, 6 $\qquad$ (b) 1, 6

3) $\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$ $\Rightarrow$ $\qquad S \rightarrow 3.$ $D \rightarrow 0$ $\qquad$ (c)

$\Rightarrow -\lambda^3 + 3\lambda^2 - [0 + 0 + 0]\lambda + 0 = 0.$

or $3\lambda^2 - \lambda^3 = 0.$

$\lambda^2(3 - \lambda) = 0$

(c) 0, 0, 3

4) $\begin{bmatrix}1 & -1 & 5\\0 & 5 & 6\\0 & -6 & 5\end{bmatrix}$ $\qquad 1(25+36) + ~~00~~ 5($

$\boxed{\begin{matrix}S \rightarrow 11\\P \rightarrow 61\end{matrix}}$

$\Rightarrow -\lambda^3 + 11\lambda^2 - [61 + 5 + 5]\lambda + 61 ~~\phi~~.$

$-\lambda^3 + 11\lambda^2 - 71\lambda + 61 = 0$

(c) $1, 5 \pm i6.$

$S = 1 \cdot i \quad P =$
p23.

5) $\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

no need of charc eqn.

$\Rightarrow -\lambda^3 + \lambda^2 - [-3 + -3 + -8]$

× a) $3, 3+5i, 6-i$
× b) $-6+5i, 3+i, 3-i$
× c) $3+i, 3-i, 5+i$
✓ d) $3, -1+3i, -1-3i$ ✓

Q. The eigen values of a 3x3 Matrix P are:
$2+\sqrt{-1}, 3$
then $|P| = ?$

3 Eigen Values

$\Rightarrow 3, 2+i, 2-i \Rightarrow \underline{15}$ (Product).

Q. 1 of the eigen value of the matrix

$$\begin{array}{c} 10 \rightarrow \\ 10 \rightarrow \\ 10 \rightarrow \end{array} \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 2 \\ 5 & 2 & 3 \end{bmatrix} \text{ is ?}$$

Soln:

3 eigen Values. $\boxed{\therefore 1 \text{ eigen Value} = 10}$

$\boxed{\text{If Sum of all the rows = Constant, then that constant must be one of the Eigen Value.}}$

$A \rightarrow \lambda_1, \lambda_2$
$A^2 \rightarrow \lambda_1^2, \lambda_2^2$.

Q. Consider a non-singular 2x2 matrix A with Trace (A) = 4.
Trace $(A^2) = 5$
then : $|A| = ?$

2 eigen values.

Soln:

$A \rightarrow 2\times 2$
$\Downarrow$
λ_1, λ_2
$\underbrace{A^2 \rightarrow \lambda_1^2, \lambda_2^2}_{\text{Property}}$

$\lambda_1 + \lambda_2 = 4$ —— (1)
$\lambda_1^2 + \lambda_2^2 = 5$ —— (2)
$\downarrow$
$\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 = 16$.
$5 + 2|A| = 16$..

$\boxed{|A| = 5.5}$

Q. If $(1, 2, -1)$ are eigen values of A, then the trace & determinent of the matrix $B = A - A^{-1} + A^2$ is ?

$\underline{1, 2, -1} \Rightarrow A$.

$\lambda \rightarrow A$.
A

$B = A - A^{-1} + A^2$.

$\lambda \rightarrow A$

$B = \lambda - \frac{1}{\lambda} + \lambda^2$.

$B(\lambda)_1 = 1$.
$B(\lambda)_2 = 5.5$
$B(\lambda)_3 = 1$.

$2 - \frac{1}{2} + 4$

$\boxed{\begin{array}{l} \text{Trace} = 7.5 \\ |B| = 5.5 \end{array}}$

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Q. If Matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ → Upper Δ Matrix

Then the eigen values of A^{-1} are ?

$\boxed{\lambda = (1, 2, 0)} \longleftarrow A$

$\left(1, \frac{1}{2}, \infty\right) \longleftarrow A^{-1}$

EV can't be ∞

Now, $|A| = 0$.

∴ Inverse doesn't exist

[How EV calculated]

∴ (d) doesn't exist.

Q. The eigen values of a 2×2 Matrix X are −2, −3.

Then eigen values of the matrix:

$(X + I)^{-1}(X + 5I)$ are :

$\lambda \rightarrow X$

$\lambda + 5 \rightarrow X + 5I$

∴ $\frac{X + 5I}{X + I} = B$.

$\lambda = (-2, -3)$.

if $\lambda \rightarrow X$.

$\frac{\lambda + 5}{\lambda + 1} \longrightarrow B$.

$3/_{-1}$ $\quad$ $2/_{-2}$

∴ $B \rightarrow \lambda(-3, -1)$.

$\boxed{\begin{array}{c} \lambda \longrightarrow A \\ \lambda \pm K \longrightarrow A \pm KI \end{array}}$

Q. For Matrix A satisfying the eqn below:

The Eigen Values are:

$$[A] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 0 \\ 4 & 5 & 6 \end{bmatrix}.$$

(a) (1, 0, 0)

(b) (1, i, −i)

(c) (1, 1, −1)

(d) (1, 1, 0)

Soln

$|A B|$

$|A|\,|B| = |C|$. → Interchange 2 ↔ 1 row

$|A|\,K = -K$

$\boxed{|A| = -1}$

→ Substitute in options (c).

* Problems on Eigen Vectors

Q. The value of 'P' such that the vector:

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigen vector of the matrix:

$\begin{bmatrix} 4 & 1 & 2 \\ P & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}$ is ?

Scanned with CamScanner

$AX = \lambda X$ Eigen value, Vector.

$$\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$4+2+6 \quad \lambda$ —(1)
$p+4+3 = 2\lambda$ —(2)
$14-8+30 \quad 3\lambda$ —(3)

$\Rightarrow 12 = \lambda \qquad 7+p = 24.$
$7+p = 2\lambda \qquad \boxed{p=17}$
$36 = 3\lambda.$

$\boxed{\lambda = 12}$ one of the Eigen Value

Q The eigen vectors of the matrix :
$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form of
$\begin{bmatrix} 1 \\ a \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix}$; then $a+b = ?$

$\underline{\lambda = 1, 2}$ (UTM)

$AX = \lambda X.$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+2a \\ 2a \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix}$$

$1+2a = 1.$
$\boxed{a=0}$ ✓

$AX = \lambda X.$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+2b \\ 2b \end{bmatrix} = \begin{bmatrix} 2 \\ 2b \end{bmatrix}.$$ ✓

$2b = 1.$
$\boxed{b = 1/2}$ ✓

$\boxed{a+b = 1/2}$ Ans ✓

Q A matrix has eigen values $[-1, -2]$ and the corresponding eigen vectors are : $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$;
then the matrix is ?

$S = -3$
$P = 2$

Sum (a) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (b) Prod $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ ✓ $-4+$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ ✓ (d) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ ✓

Scanned with CamScanner

$AX = \lambda X$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$a - b = -1$ — (i)

$c - d = 1$ — (ii)

→ Check options.

(d) ✓

Q. The matrix $A = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}_{3\times3}$ has ($A^T = A$)

3 distinct Eigen Values and one of its Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; then which one of the following can be another eigen vector of A?

a) $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ ✓(c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

∵ Matrix → Symmetric

Eigen Values → distinct.

∴ orthogonal.

Take dot Product with all options.

(c) $1 + 0 \cdot 0 + 1 \cdot (-1) = \underline{\underline{0}}$ ✓

Q. The eigen Vectors of the matrix: $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ are:

$\lambda^2 - 5\lambda + \cancel{2}4 = 0$.

$\lambda = (\cancel{3},\cancel{2})$.

$\lambda^2 - \lambda - 4\lambda + 4 = 0$.

$\lambda(\lambda - 1) - 4(\lambda - 1) = 0$

$\lambda = 4, 1$ ✓

→ $AX = \lambda X$.

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda = 4$

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$2x_1 + 2x_2 = 4x_1 \implies x_1 = x_2$ ✓

$x_1 + 3x_2 = 4x_2$.

Scanned with CamScanner

$x_1 = K,\ x_2 = K.$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K \\ K \end{bmatrix} \Rightarrow K\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Uparrow$

$\underline{\lambda = 1}$.

$\frac{x_1}{x_2} = \frac{-2}{1}$

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$2x_1 + 2x_2 = x_1$ $\quad x_1 = -2x_2.$

$x_1 + 3x_2 = x_2$

$\boxed{x_2 = K,\ x_1 = -2K}$

$\therefore \begin{bmatrix} -2K \\ K \end{bmatrix} \Rightarrow K\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ✓ $\quad \frac{x_1}{x_2} = -2$ ✓

$\Uparrow$ ✓.

Q. One of the eigen vector of the matrix:

$\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ is ? $\quad -30+18.$

(a) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\text{Soln.}}$

$\lambda^2 - \lambda + -12 = 0.$

$\lambda^2 + 3\lambda - 4\lambda - 12 = 0.$

$\boxed{\lambda = -3, 4}.$

$\lambda(\lambda+3) - 4(\lambda+3) =$

$\lambda = -3$ corr. eign value

$\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 9 \end{bmatrix}$

$\rightarrow AX = \lambda X.$

$\Rightarrow$ [$\quad$ ~~X~~ use shortcut.

$$\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

a) $\begin{bmatrix} 7 \\ 15 \end{bmatrix} = \lambda\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ X.

b). $\begin{bmatrix} 28 \\ 72 \end{bmatrix} = \lambda\begin{bmatrix} -2 \\ 9 \end{bmatrix}$ X.

c) $\begin{bmatrix} -12 \\ -24 \end{bmatrix} = \lambda\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ X $\quad \rightarrow$ 1 +ve.

✓(d) $\begin{bmatrix} -3 \\ -3 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -3$ (Eigen value)

Q. The eigen Vector of the matrix:

$\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ ✓(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} -8 \\ 18 \end{bmatrix}$ $\quad \begin{bmatrix} -10 \\ 25 \end{bmatrix}$ $\quad \begin{bmatrix} -10 \\ 5 \end{bmatrix}$ $\quad \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$\Rightarrow -5\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

* Caley Hamilton Theorem

- Every Square Matrix satisfies its characteristic equation.
- By using Caley Hamilton theorem, we can calculate Higher Powers of A as well as A^{-1}.

Q. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$; then A satisfies

Char eqn: $\lambda^2 + 3\lambda + 2 = 0$.

$$\Downarrow$$

$$A^2 + 3A + 2 = 0.$$

$$A^2 + 3A + 2I = 0$$

cant add constant only.

Q2 The char eqn of a 3x3 Matrix P is given by:

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0.$$

then $P^{-1} = ?$

$P^3 + P^2 + P + 1 = 0$. (Caly-H. Th).

$\rightarrow P^3 + P^2 + P + I = 0$. (Multi P^{-1} boths)

$(P^3)P^{-1} + (P^2)P^{-1} + P(P)^{-1} + P^{-1} = 0$.

$\Rightarrow P^2 + P + I + P^{-1} = 0$.

$$\boxed{P^{-1} = -(P^2 + P + I)}$$

Q. A 3x3 Matrix P such that:

$P^3 = P$,

then the eigen values of P are:

(a) 1, 1, −1.
(b) 1, 0.5 + j 0.866, 0.5 − j 0.866.
(c) 1, −0.5 + j 0.866, −0.5 − j 0.866.
(d) 0, 1, −1.

$\lambda \longrightarrow P$

$\lambda^3 \longrightarrow P^3$

$\Rightarrow \lambda^3 = \lambda$.

$\lambda^3 - \lambda = 0$.

$\lambda(\lambda^2 - 1) = 0$.

$\therefore \boxed{\lambda = 0, \pm 1}$

$\lambda \rightarrow P$

$\lambda^3 \rightarrow P^3$

$\lambda^3 = \lambda$

Q. If matrix $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$; then $A^3 = [\;?\;]$

(a) $15A + 12I$. $= -30 + 12 = -18$ ✗
(b) $19A + 30I$ $= -38 + 30 = -8$ ✓
(c) $17A + 15I$ $= -34 + 15$ ✗
(d) $17A + 21I$. $= -34 + 21 = -13$ ✗

Soln:

$\lambda^2 + 5\lambda + 6 = 0$.

$A^2 + 5A + 6I = 0$.
$A^2 = -5A - 6I$.

$A^3 = -5A^2 - 6A$.
$\Rightarrow -5(-5A - 6I) - 6A$.

$\boxed{A^3 = 19A + 30I}$ ✓

Trick:

$\lambda^2 + 5\lambda - 6 = 0$.
$(-2, -3)$.

M-2

λ
$A \Rightarrow -2, -3$.
$A^3 \Rightarrow -8, -27$ ✓ } Eigen values.

Now. (-2) in all option.

$\lambda^9 = 511\lambda + 510$.

Q. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$; then $A^9 = ?$

a) $511A + 510I$
b) $309A + 104I$
(c) $154A + 155I$
(d) $154A + 159I$

$\lambda^2 + 3\lambda + 2 = 0$.
$A \longrightarrow \lambda = (-1, -2)$.
$A^9 \longrightarrow \lambda^9 = (-1, -512)$.

if $A = -1$ (a) -1. ✓
(b) ✗
(c) ✗
(d) ✗

$\therefore$ (a) $511A + 510I$ ✓

$\lambda \to A$.
$\lambda^3 \to A^3$

$\lambda \to A^3 = 19A + 30I$.

$\lambda^3 = 19\lambda + 30$.

2) CALCULUS

* Limits

$f(x)$ approaches $x=a$ → approaching to a & strictly $x \neq a$.

$\boxed{\lim_{x\to a} f(x) = f(a)}$ limit of funct^n.

Q. $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

$\Rightarrow 0\cdot\sin\left(\frac{1}{0}\right)$

$\Rightarrow 0\sin(\infty) \Rightarrow 0$ → finite value.

$\therefore 0 \times FV = \underline{\underline{0}}$

	(i/p) domain	(o/p) range
$\sin x$	$(-\infty,\infty)$	$[-1,1]$

$\sin\infty \in [-1,1]$

* Left Hand Limits & Right Hand Limits

- The LHL of $f(x)$ at $x=a$ is given by:

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$$

- The RHL of $f(x)$ at $x=a$ is given by:

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$

- A function is said to be having a limit at $x=a$; if $\boxed{\text{LHL at } x=a = \text{RHL @ } x=a}$

* If, LHL $\neq$ RHL, limit dne.

. $\lim_{x\to 2} f(x)$: x approaching to 2, but strictly speaking $x\neq 2$.

. $\lim_{x\to 2^-} f(x)$:

→ $x = 1.9999 \approx 2$

. $\lim_{x\to 2^+} f(x)$

→ $x = 2.00001 \approx 2$.

Q1. $f(x) = \begin{cases} x^2 ; & x<1 \\ 2x-1 ; & x>1 \end{cases}$ → not continuous

then $\lim_{x\to 1} f(x) = ?$

Soln

LHL:

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x^2 = 1$ ✓ ($x<1$)

RHL

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 2x-1 = 1$ ✓ ($x>1$)

$\therefore$ LHL = RHL $\therefore$ exists $= \underline{\underline{1}}$. Ans.

$\underset{x\to a^-}{Lt} f(x) = f(a-h)$.

Q. $\underset{x\to 0}{Lt} \dfrac{e^{1/x}-1}{e^{1/x}+1}$

$\underset{x\to 0^-}{Lt} f(x) = \underset{h\to 0}{Lt} f(0-h)$.

(a) 1
(b) −1
(c) 0
(d) dne.

Soln.

LHL $\rightarrow \underset{x\to 0^-}{Lt} \dfrac{e^{1/x}-1}{e^{1/x}+1}$.

$\Rightarrow \underset{h\to 0}{Lt} \dfrac{e^{-1/h}-1}{e^{-1/h}+1} \Rightarrow \dfrac{-1}{1} = -1$

$\therefore \boxed{LHL = -1}$

RHL $\rightarrow \underset{x\to 0^+}{Lt} \dfrac{e^{1/x}-1}{e^{1/x}+1}$. $\quad \underset{h\to 0}{Lt} f(0+h)$

$\Rightarrow \underset{h\to 0}{Lt} \dfrac{e^{1/h}-1}{e^{1/h}+1}$

$\Rightarrow \underset{h\to 0}{Lt} \dfrac{\cancel{e^{1/h}}[1-1/e^{1/h}]}{\cancel{e^{1/h}}[1+1/e^{1/h}]}$

$\Rightarrow 1$

$\therefore \boxed{RHL = 1}$

$\because LHL \neq RHL$.

$\therefore$ Limit doesnot exist.

* Standard Limit

1) $\underset{x\to 0}{Lt} \dfrac{\sin x}{x} = 1$
2) $\underset{x\to 0}{Lt} \dfrac{\tan x}{x} = 1$
3) $\underset{x\to 0}{Lt} \dfrac{x}{\sin x} = 1$
4) $\underset{x\to 0}{Lt} \dfrac{\sin ax}{x} = a$.
5) $\underset{x\to 0}{Lt} \dfrac{x}{\sin ax} = \dfrac{1}{a}$
6) $\underset{x\to a}{Lt} \dfrac{x^n - a^n}{x-a} = na^{n-1}$.
7) $\underset{x\to 0}{Lt} \dfrac{e^{ax}-1}{x} = a$.
8) $\underset{x\to 0}{Lt} \dfrac{a^x-1}{x} = \ln_e a$.
9) $\underset{x\to a}{Lt} \log[f(x)] = \log \boxed{\underset{x\to a}{Lt} f(x)}$

Q1. $\underset{x\to 0}{Lt} \dfrac{\sin^2 x}{x}$

2. $\underset{x\to \pi/4}{Lt} \dfrac{\tan 2(x-\pi/4)}{x-\pi/4}$ $\quad \frac{1}{3}-1$

3. $\underset{x\to 8}{Lt} \dfrac{x^{1/3}-2}{x-8}$ $\quad \dfrac{x^{1/3}-8^{1/3}}{x-8} = \dfrac{1}{3}(8)^{-2/3}$

$= \dfrac{1}{3} 2^{-2}$

1) $\underset{x\to 0}{Lt} \frac{\sin x}{x} \cdot \sin x \Rightarrow 1 \cdot 0 = \underline{0}$.

2) Lt $x - \pi/4 = h$.

$\underset{h\to 0}{Lt} \frac{\tan 2h}{h} \Rightarrow \underline{\underline{2}}$.

3) $\underset{x\to 8}{Lt} \frac{x^{1/3} - 8^{1/3}}{x - 8}$ $\qquad \frac{x^n - a^n}{x-a} = na^{n-1}$

$\Rightarrow \frac{1}{3} \cdot (8)^{-2/3}$

$\Rightarrow \frac{1}{3} \cdot 2^{\cancel{3}(-2)/\cancel{3}}$

$\Rightarrow \frac{1}{3} \cdot \frac{1}{4} = \underline{\underline{1/12}}$.

* INDETERMINANT FORMATS

The formats like:

$$\left[\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty\times 0, \infty-\infty, 0^0, \infty^0\right]$$

are called as Indeterminant formats.

eg: $\underset{x\to 0}{Lt} \frac{\sin x}{x} \Leftarrow \frac{0}{0}$.

[Apply L'Hospital's rule]

$\Rightarrow \frac{\cos x}{x} = 1$.

- For $(0/0, \infty/\infty, 0\times\infty, \infty\times 0)$. Apply L'Hospital Rule.
- For $(\infty - \infty) \rightarrow$ Take LCM.
- For $(0^0, \infty^0) \rightarrow$ directly Ans = 1.

* L'Hospital Rule

$\underset{x\to a}{Lt} \frac{f(x)}{g(x)} = 0/0$ or ∞/∞.

$\underset{x\to a}{Lt} \frac{f'(x)}{g'(x)} = 0/0$ $\qquad$ (LHR).

$\underset{x\to a}{Lt} \frac{f''(x)}{g''(x)} = K$.

(Apply L' Hospital; until you get val $\neq \frac{0}{0}$ or $\frac{\infty}{\infty}$)

Q 1) $\underset{x\to 0}{Lt} \frac{1}{10}\left[\frac{1-e^{-i5x}}{1-e^{-ix}}\right]$ $\qquad -e^{-i5x} \cdot -i5x$.

2) $\underset{x\to \pi/4}{Lt} \frac{\cos x - \sin x}{x - \pi/4}$

3) $\underset{x\to 0}{Lt} \frac{e^{2x}-1}{\sin 4x}$ $\qquad e^{-ax} = e^{-ax} \cdot -a$

4) $\underset{x\to 0}{Lt} \frac{\sin x}{x e^x}$ $\qquad \frac{e^{2x} \cdot 2}{\cos 4x \cdot 4}$

5) $\underset{x\to 0}{Lt} \dfrac{e^x - (1 + x + x^2/2)}{x^3}$

6) $\underset{\alpha\to 0}{Lt} \dfrac{x^\alpha - 1}{\alpha}$

7) $\underset{x\to 0}{Lt}\ x \log(\sin x)$

8) $\underset{x\to \infty}{Lt}\ x \tan(1/x)$

9) $\underset{x\to 0}{Lt} \dfrac{1-\cos x}{x^2}$

---

1) $\underset{x\to 0}{Lt} \cdot \dfrac{1}{10}\left[\dfrac{\cancel{i}e^{-i5x}\cdot(\cancel{-i}5)}{\cancel{i}e^{-ix}\cdot(\cancel{-i})}\right]$

$\Rightarrow \dfrac{1}{2}\left[\dfrac{e^{-i5x}}{e^{-ix}}\right]$

$\Rightarrow \dfrac{1}{2}\times 1 = \underline{\underline{1/2}}$

2) $\underset{x\to \pi/4}{Lt} \dfrac{\cos x - \sin x}{x - \pi/4} \in 0/0$

$\Rightarrow \underset{x\to \pi/4}{Lt} \dfrac{-\sin x - \cos x}{1}$

$\Rightarrow -\left[\dfrac{2}{\sqrt{2}}\right] \Rightarrow \underline{\underline{-\sqrt{2}}}$

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3) $\underset{x\to 0}{Lt} \dfrac{e^{2x}-1}{\sin 4x} \in 0/0$

Soln

$\Rightarrow \underset{x\to 0}{Lt} \dfrac{e^{2x}\cdot \cancel{2}}{\cos 4x \cdot \cancel{4}_2}$

$\Rightarrow \dfrac{1}{2}$ //

---

4) $\underset{x\to 0}{Lt} \dfrac{\sin x}{x e^x}$ $0/0$

$\boxed{\begin{array}{l}\underline{\underline{M-2}} \\ \underset{x\to 0}{Lt} \dfrac{\sin x}{x} \times \\ \underset{x\to 0}{Lt}\ 1/e^x\end{array}}$

$\Rightarrow \underset{x\to 0}{Lt} \dfrac{\cos x}{x e^x + e^x} \Rightarrow \dfrac{1}{0+1} \Rightarrow \underline{\underline{1}}$

---

5) $\underset{x\to 0}{Lt} \dfrac{e^x - (1+x+x^2/2)}{x^3}$

$\Rightarrow \underset{x\to 0}{Lt} \dfrac{e^x - (1+2x)}{3x^2}$

$\Rightarrow \underset{x\to 0}{Lt} \dfrac{e^x - (2)}{6x}$

$\Rightarrow \underset{x\to 0}{Lt} \dfrac{e^x}{6} \Rightarrow 1/6$ //

$\dfrac{a^x-1}{x}$

$\ln x$

---

6) $\underset{\alpha\to 0}{Lt} \dfrac{x^\alpha - 1}{\alpha} \in 0/0$ $\quad n^x \quad a^x$

$\Rightarrow \underset{\alpha\to 0}{Lt} \dfrac{x^\alpha \cdot \log x}{1}$ $\quad \underset{x\to 0}{Lt} \dfrac{a^x-1}{ax}$

$\Rightarrow \underline{\log x}$ $\quad \Rightarrow \log a.$

7) $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$ $\quad \infty \times 0$

$\Rightarrow \lim_{x\to\infty} \frac{\tan(1/x)}{1/x}$ $\quad \in 0/0$

$\Rightarrow 1/x = \theta$

$\therefore \lim_{h\to 0} \frac{\tan h}{h} = \underline{\underline{1}}$

8) $\lim_{x\to 0} x\log(\sin x)$

$\Rightarrow \lim_{x\to 0} \log(\sin x)^x$

$\Rightarrow \lim \log \lim_{x\to 0} (\sin x)^x$

$\Rightarrow \underline{\log 1 = 0}$

9) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\quad \in 0/0$

$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2}$

$= \lim_{x\to 0} \frac{2}{4} \frac{\sin x/2}{\frac{x}{2}} \cdot \frac{\sin x/2}{\frac{x}{2}}$

$= \underline{\underline{1/2}}$

$\frac{\sin x}{2x}$ $\quad \frac{\cos x}{2}$

$1/2$

$*$ $\lim_{x\to 0} \frac{1-\cos ax}{x^2} = \frac{a^2}{2!}$

$\lim_{x\to 0} \frac{1-\cos ax}{x^{4}} = \frac{a^2}{4!}$ even

$\boxed{\frac{1-\cos x}{x} = 0}$

if, $\lim_{x\to 0} \frac{1-\cos ax}{x^3} = 0$ odd

Q) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^4}$ $\quad \in 0/0$

$\lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^4}$

or, $\lim_{x\to 0} \frac{\tan x}{x} \left(\frac{1-\cos x}{x^3}\right)$

$\Rightarrow 1 \times 0 = \underline{\underline{0}}$

$*$ $\underline{\infty - \infty \text{ Model}}$ $\quad$ (LCM)

Q) $\lim_{x\to 0} \left[\frac{1}{x} - \frac{1}{e^x - 1}\right]$ $\quad \in \infty - \infty$

$\Rightarrow \lim_{x\to 0} \frac{e^x - 1 - x}{x \cdot (e^x - 1) \cdot x/x}$ $\quad 0/0$

$\Rightarrow \frac{e^x - 1 - x}{x^2}$

$\Rightarrow \frac{e^x - 1}{2x} \Rightarrow \frac{e^x}{2} = 1/2$

Step 1: Take LCM

Step 2: Multiply & divide by x.

Step 3: Apply L'Hospital's rule

* 0^0 Model $\rightarrow 1$.

Q1 $\lim_{x\to 0}(\sin x)^{x} = 1$

2. $\lim_{x\to 1}(\log x)^{\log x} = 1$

3. $\lim_{x\to a}(x-a)^{x-a} = 1$

$(0/0)$

* ∞^0 Model $\rightarrow 1$

1) $\lim_{x\to \pi/2}(\tan x)^{\cos x} = 1$.

2) $\lim_{n\to \infty} n^{1/n} = 1$

3) $\lim_{x\to \infty}(x+e^{-x})^{1/x} = 1$

* 1^∞ Model

$\frac{b}{x}\cdot ax$

1) $\lim_{x\to 0}(1+ax)^{b/x} = e^{ab}$.

2) $\lim_{x\to \infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$

3) $\lim_{x\to 0}(\cos x + a\sin bx)^{c/x} = e^{abc}$.

apply it for $0^0 / \infty^0 / 1^\infty / \infty^0$

$f(x)^{g(x)}$ Model

$$\boxed{\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}}$$

Q1 $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$. $\in 1^\infty$

$\Rightarrow e^{\lim_{x\to 0}\frac{1}{x^2}\left[\frac{\sin x}{x}-1\right]}$

$\Rightarrow e^{\lim_{x\to 0}\frac{\sin x - x}{x^3}}$

$\Rightarrow e^{\lim_{x\to 0}\frac{\cos x - 1}{3x^2}}$

$\Rightarrow e^{1/3 \times -1/2} \Rightarrow e^{-1/6}$.

$\frac{1}{3}\left[\frac{1-\cos x}{x^2}\right)$

$\frac{1}{3}\times\frac{1}{2}$

$\frac{1}{6}$

$\frac{-\sin x}{6x}$

$\frac{-\cos x}{6}$

$\frac{1-\cos x}{x^2} = \frac{1}{2}$

* Continuity

A function $f(x)$ is said to be continuous at $x=a$:

(1) $f(x)$ must have a defined value at $x=a$.

(2) Lim value at $x=a$ must be equal to the funct^n^ value at $x=a$

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

$$\Downarrow$$

right continuous.

- $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$.

Left continuous

$\therefore$ Left cont. & Right continuous = continuous.

Q. If $f(x) = (1+3x)^{1/x}$ for $x \neq 0$
$= e^3$ for $x=0$

Then check the continuity for $x=0$.

$x\neq 0$ ~~dotal~~ limit

$\lim_{x\to 0} (1+3x)^{1/x} \Rightarrow \in 1^\infty$

$\Rightarrow e^{\frac{1}{x}(3x)} \Rightarrow e^3$. ✓

$\therefore$ Limit = funct Value

$\therefore$ Continuous ✓

Q1 If $f(x) = (x+1)^{\cot x}$ for $x\neq 0$
$= K$ for $x=0$.

is continuous at $x=0$.

Find $K=?$

Sol^n^

$\lim_{x\to 0} (x+1)^{\cot x}$. $\in 1^\infty$.

$\Rightarrow e^{\cot x \cdot x}$

$\Rightarrow e^{\frac{\cos x \cdot x}{\sin x}}$

$\Rightarrow e^1 \Rightarrow e$

$\therefore \boxed{K=e}$ ✓

Q Which one of the following function is continuous at $x=3$?

(a) $f(x) = \begin{cases} 2, & x=3 \\ x-1, & x>3 \\ \frac{x+3}{3}, & x<3 \end{cases}$ ✓

LHL = 2
RHL = 2
$f(3) = 2$

(b) $f(x) = \begin{cases} 4, & x=3 \\ 8-x, & x\neq 3 \end{cases}$ ✗

$f(3) = 4$
lim = 5
$\therefore$ not continu.

(c) $f(x) = \begin{cases} x+3, & x\leq 3 \\ x-4, & x>3 \end{cases}$ ✗

LHL = 6
RHL = −1
$f(3) = 6$. $\therefore$ Limit dne.

(d) $f(x) = \frac{1}{x^3-27}$, for $x \neq 3$. ✗

↳ f(3) not defined.

* DIFFERENTIATION

→ A function f(x) is said to be differentiable at $x=a$, if LHD at $x=a$ = RHD @ $x=a$.

$$\boxed{LHD|_{x=a} = RHD|_{x=a}}$$

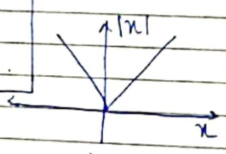
→ If a funct^n is differ. at $x=a$, then it must be continuous at $x=a$ but the converse of this statement may not be true.

L ↑ C ↓ D — may / may not; may not.

$\boxed{\text{* Differentiability guarantee continuity but continuity doesnot guarantee differentiability.}}$

(i) $\boxed{\frac{d}{dx}|x| = \frac{|x|}{x} ; x \neq 0.}$

$|x| = x ; x>0$
$-x ; x<0.$
$0 ; x=0.$



|x| diff everywhere except 0.

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. $|2x+3|$ is not differentiable at $-3/5$.

⇒ $|x|$ is continuous everywhere, differential everywhere except $|_{x=0}$

⇒ $|2x+3|$ is continuous everywhere, diff. everywhere except $|_{x=-3/2}$.

⇒ $|4x-5|$ is not differentiable $|_{x=5/4}$

Q. If $f(x) = \frac{1}{1+|x|}$ then at $x=0$; f(x) is:
(a) continuous & diff.
(b) continuous but not diff.
(c) differentiable but not continuous.
(d) neither cont. nor differentiable.

Sol^n

$$f(x) = \begin{cases} \frac{1}{1+x} ; & x>0. \\ \frac{1}{1-x} ; & x<0. \\ 1 ; & x=0 \end{cases}$$

Continuity ✗

LHL ⇒ $\frac{1}{1-x} = 1$. $f(0)=1$.

RHL ⇒ $\frac{1}{1+0} = 1/1$.

∴ continuous at $x=0$

$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

· differentiability

$x<0$ LHD $= \frac{-1}{(1-x)^2} \times -1$

$\Rightarrow \underline{\underline{1}}$

$x>0$ RHD $\frac{-1}{(1+x)^2} \Rightarrow \underline{\underline{-1}}$

∴ Not differentiable at $x=0$.

Q. If $f(x)=2x+1$

$$f(x)=\begin{cases} 2x+1; & x\le 1 \\ \alpha x^2+\beta x; & x>1 \end{cases}$$

is differentiable for every value of x. Then α, β ?

Soln.

∴ differentiable at $x=1$

· LHD $= 2$

· RHD $= 2\alpha x+\beta$

$\Rightarrow 2\alpha+\beta$

$(\text{LHD}=\text{RHD})_{x=1}$

∴ $2\alpha+\beta = 2$ — (i).

(a) $\alpha=0, \beta=2$

(b) $\alpha=3, \beta=\cancel{4}9$

(c) $\alpha=4, \beta=-1$

(d) $\alpha=-1, \beta=4$.

Check continuity

$\alpha+\beta=3$ ✓

(d)

LHL = RHL

Q. If $f(x)=x\sin x$ satisfies:

$f''(x)+f(x)+t\cos x=0$

then t ?

Soln.

$f(x)=x\sin x$.

$f'(x)=x\cos x+\sin x$.

$f''(x)=-x\sin x+\cos x+\cos x$.

$\Rightarrow \cancel{-x\sin x}+2\cos x+\cancel{x\sin x}+t\cos x=0$

$(t+2)\cos x=0$.

$\boxed{t=-2}$

Q. If $x=e^{y+e^{y+\cdots \infty \text{ terms}}}$.

$\frac{dy}{dx}=?$

Soln.

$\Rightarrow x=e^{y+x}$

$1=e^{y+x}\left[\frac{dy}{dx}+1\right]$

$\Rightarrow 1=x\left[\frac{dy}{dx}+1\right]$

$\Rightarrow \frac{1}{x}-1=\frac{dy}{dx}$

∴ $\boxed{\frac{dy}{dx}=\frac{1-x}{x}}$

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* Parametric differentiation

If $x = f(\theta)$; $y = g(\theta)$

then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

1. If $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ then $\frac{dy}{dx} = ?$

Sol^n^

$\frac{dx}{d\theta} = a(1-\cos\theta)$

$\frac{dy}{d\theta} = a(\sin\theta)$

$\frac{dy}{dx} = \frac{a\sin\theta}{1-\cos\theta} = \frac{2\,\cancel{S}C}{2\,\cancel{S}S}$ ⇒

$\boxed{\frac{dy}{dx} = \cot\theta/2}$. Ans

* Partial differentiation

$z = f(x, y)$

first order PD ⇒ $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Second order PD ⇒ $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \cdot \partial y}$

Q. if $f = y^x$ then:

$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x=2, y=1}$

⇒ $\frac{\partial f}{\partial x} = y^x \log y$

⇒ $\frac{\partial^2 f}{\partial x \partial y} = \frac{y^x}{y} + \log y \cdot x y^{x-1}$

⇒ $\frac{1^2}{1} + \log 1$

⇒ $\underline{\underline{1}}$ Ans.

Q. if $z = xy\ln(xy)$: then the relation between $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$\frac{xy}{xy} \cdot y$

$\frac{\partial z}{\partial x} = x\left[\frac{\cancel{y}}{\cancel{xy}} \cdot y\right] + \ln(xy) \cdot y$

$\frac{\partial z}{\partial x} \Rightarrow y(1 + \ln(xy))$ —— (i)

$\frac{\partial z}{\partial y} = \frac{\cancel{xy}}{\cancel{xy}} \cdot x + \ln(xy) \cdot x$

$\frac{\partial z}{\partial y} \Rightarrow x(1 + \ln(xy))$ —— (ii).

∴ $\boxed{x\frac{\partial z}{\partial x} = y \cdot \frac{\partial z}{\partial y}}$. (relation)

Q. If $u = f(x+cy) + g(x-cy)$
then:
u_{xx} / u_{yy} ?

$u_x = f'(x+cy)\cdot 1 + g'(x-cy)$.

$\rightarrow u_{xx} = f''(x+cy) + g''(x-cy)$.

$u_y = f'(x+cy)\cdot c + g'(x-cy)(-c)$.

$\rightarrow u_{yy} = f''(x+cy)\cdot c^2 + g''(x-cy)c^2$.

$\therefore \frac{u_{xx}}{u_{yy}} = \frac{f''(x+cy) + g''(x-cy)}{c^2[f''(x+cy) + g''(x-cy)]}$

$\Rightarrow 1/c^2$.

$\therefore \boxed{\frac{u_{xx}}{u_{yy}} = \frac{1}{c^2}}$.

* Total Derivative

If $z = f(x, y)$ where, $x = g(t)$, $y = h(t)$ then the TD of z wrt t is given by:

$\because$ more than 1 Var. $\therefore$ Partial:

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$x = g(t) \quad y = h(t)$

$\downarrow \quad \downarrow$

$\frac{dx}{dt} \quad \frac{dy}{dt}$.

$\therefore \boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$

Q. if $z = x^2 + y^2$. where, $x = t^2+1$, $y = t^2-1$

then $\frac{dz}{dt} = ?$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$\Rightarrow 2x * 2t + 2y\cdot 2t$.

$\Rightarrow 4xt + 4yt$.

$\therefore 4t(x+y)$.

$\Rightarrow 4t(2t^2)$.

$\Rightarrow 8t^3$.

$\therefore \boxed{\frac{dz}{dt} = 8t^3}$

* MAXIMA – MINIMA

Procedure to calculate maxima & minima of given function:

Step 1: Consider given function as $f(x)$.

Step 2: Calculate $f'(x)$

Step 3: Equate $f'(x) = 0$

Step 4: Consider the root as $x = a$

Step 5: Calculate $f''(x)$ & $f''(a)$.

if $f''(a) < 0$; funct^n = maximum at $x = a$.

& max value $= f(a)$

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Q $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b = 0$

$\boxed{x = -b/2a}$

→ if $f''(a) > 0$, then function is said to be having a minima at $x=a$ & the minimum value $= f(a)$

→ if $f''(a) = 0$, the function is said to be having a Saddle Point [neither minima nor maxima] $|_{x=a}$

Q The maximum & minimum values of the function :
$f(x) = x^3 - 6x^2 + 9x + 25$ are :

$f'(x) = 3x^2 - 12x + 9$
$\Rightarrow x^2 - 4x + 3 = 0$
$x^2 - x - 3x + 3 = 0$
$x(x-1) - 3(x-1) = 0$
$\boxed{x = 3, 1}$

$f''(x) = 6x - 12$. $x = 3$: minima.
$x = 1$: maxima.

max value $= 29$ $f(1)$
min value $= 25$ $f(3)$

Q The maximum value of the function $f(x) = x^2 e^{-x}$ occurs at $x = ?$

$f'(x) = x^2 \cdot e^{-x} \cdot (-1) + e^{-x} \cdot 2x$
$\Rightarrow e^{-x}[2x - x^2] = 0$
$\therefore \boxed{x = 2, 0}$ $\neq 0$

$x^2 = 2x$
$x = 2, 0$

$f''(x) = e^{-x}[2 - 2x] + [2x - x^2] e^{-x} \cdot -1$
$\Rightarrow e^{-x}[2 - 2x] - e^{-x}[2x - x^2]$

$\Rightarrow 1 \cdot 2 - 1$ ~~maxima~~

$\Rightarrow f''(x)|_0 = 2 \therefore$ minima

$\Rightarrow \boxed{\begin{array}{l} \text{maxima} = 2 \\ \text{minima} = 0 \end{array}}$

$\therefore f(2) = 4e^{-2} \Rightarrow 4/e^2$

$\therefore \boxed{\text{max value} = \frac{4}{e^2}}$

Q Max value of $f(t) = e^{-t} - 2e^{-2t}$ occurs at $t = ?$

$f'(t) = -e^{-t} + 4 e^{-2t}$
$\Rightarrow -e^{-t} + 4e^{-2t}$
$-e^{-t}[1 + 4e^{-t}$

$-e^{-t}[1-4e^{-t}]=0$

$\therefore 1-4e^{-t}=0$

$e^{-t}=1/4$

$\Rightarrow e^{t}=4$

$\therefore \boxed{t=\ln_e 4}$ → only maxima

Q. The minimum value of $f(x)=e^x+e^{-x}$ is ?

$f'(x)=e^x-e^{-x}=0$

$t=\frac{1}{t}$ $t^2=1$

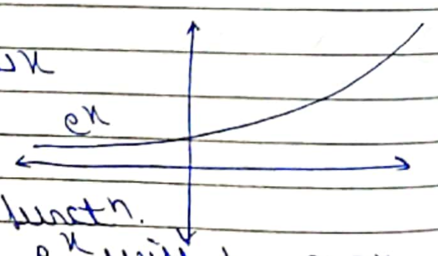
$\therefore e^x=+1$

$\underline{\underline{x=0}}$ → only 1 value

$\therefore$ Min Value $=2$

Q. The maximum value of the function $f(x)=\frac{e^{\sin x}}{e^{\cos x}}$

$f(x)=e^{\sin x-\cos x}$



$\because e^x$ is increasing functn.

if x is maximum, e^x will be max

$\therefore \sin x-\cos x$ should also be maximum

$\Rightarrow \sin x-\cos x \longrightarrow \sqrt{2}$ (max value)

$\therefore \boxed{\text{Max value of } f(x)=e^{\sqrt{2}}}$

- Max value of : → RL.
  $a\sin x+b\cos x=\sqrt{a^2+b^2}$.
- Max value of
  $\sin x-\cos x=\sqrt{1^2+(-1)^2}=\sqrt{2}$.

---

* Procedure to calculate local maxima / local minima or maxima / minima in the interval $[a,b]$.

1. Consider given function as $f(x)$.
2. Calculate $f'(x)$
3. Equate $f'(x)=0$.
4. Consider the roots as $x=c,d$

→ If $c,d\in[a,b]$.
then maximum value $=\max\{f(a), f(b), f(c), f(d)\}$

Min value $=\min\{f(a), f(b), f(c), f(d)\}$

→ If $c,d\notin[a,b]$; then:
min value $=\min(f(a), f(b))$

max value $=\max((f(a), f(b))$

Q. The min value of $f(x) \to y = x^2$ in the interval $[1,5]$ is:

$f(x) = x^2$.
$f'(x) = 2x$. $\quad 2x = 0$.
$\boxed{x=0} \to$ out of interval

min value $= \min\left(f(1), f(5)\right)$
$= \min(1, 25)$
$\Rightarrow \underline{\underline{1}}$.

Q. The max value of function:
$f(x) = x^3 - 9x^2 + 24x + 5$ in $[1,6]$

$f'(x) = 3x^2 - 18x + 24 = 0$.
$x^2 - 6x + 8 = 0$.
$x^2 - 4x - 2x + 8 = 0$
$x(x-4) - 2(x-4)$ -

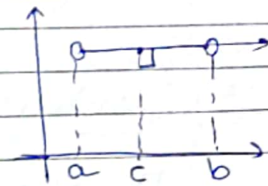
$\underline{\underline{x = 2, 4}}. \in [1,6]$.

$\max\left[21, 25, 21, 41\right] = \underline{\underline{41}}$.

$8 - 36 + 48 + 5$.
$64 - 114 +$

Q.* MEAN VALUE THEOREM

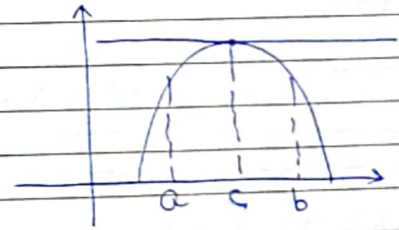
1) Rolls Mean Value Theorem:



If $f(x)$ is:
(i) continuous in $[a,b]$.
(ii) differentiable in (a,b)
(iii) $f(a) = f(b)$

Then, there exists $c \in (a,b)$.
such that : $\boxed{f'(c) = 0}$.

Here, at the point 'c', tangent is parallel to X axis.



2) Lagrange's Mean Value Theorem

If $f(x)$ (i) is continuous in $[a,b]$.
(ii) differential in (a,b).

Then there exists $c \in (a,b)$ such that:

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Here, the tangent is parallel to the line joining of the points:

$(a, f(a))$, $(b, f(b))$.

(3) Couchy's Mean Value Theorem

If $f(x), g(x)$ are two functions such that:

(i) continuous in $[a,b]$.

(ii) differentiable in (a,b)

(iii) $g'(x) \neq 0 \ \forall \ (a,b)$

then there exists, $c \in (a,b)$ such that:

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

Q. The value of c for:

$f(x) = e^x(\sin x - \cos x)$ in $\left[\pi/4, 5\pi/4\right]$ is:

Soln:

$f\left[\frac{\pi}{4}\right] = e^{\pi/4}[0] = 0$.

$f\left[\frac{5\pi}{4}\right] = e^{5\pi/4}\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 0$.

$\therefore$ Apply Rolls

$\therefore$ $e^x \rightarrow$ only at ∞ (not def) $\rightarrow$ not in int.

$\sin x - \cos x \rightarrow$ always defined

$\therefore$ continuous.

Now, $e^x(\cos x + \sin x) + (\sin x - \cos x)e^x$

$\Rightarrow 2e^x \sin x$. $\rightarrow [\pi/4, 5\pi/4]$.

Now, $\therefore$ differentiable.

$\therefore$ $f'(c) = 0$

$2e^c \sin c = 0$.

$\sin c = 0$

$c = \pm n\pi$; $\boxed{c = \pi}$ $\rightarrow$ belongs.

Q2. The value of c for:

$f(x) = 1 - x^2 + x^3$ in $[-1, 1]$ is:
a b.

Soln:

$f(1) = 1$

$f(-1) = 1 - 1 - 1 = -1$. $\neq$.

$\therefore$ Apply Lagrange's Theorem:

$\therefore$ continuous.
$\therefore$ differentiable } polynom defined everyw.

$-2x + 3x^2 \rightarrow$ (poly)

$\therefore$ $f'(c) = \frac{f(b)-f(a)}{b-a}$

$-2c + 3c^2 \Rightarrow \frac{1+1}{1+1}$ $\therefore 3c^2 - 2c = 1$

$\therefore 3c^2 - 2c - 1 = 0$.

$c = \left(1, -1/3\right)$

$\therefore \boxed{c = -1/3}$ Ans → which is present in the $[-1, 1]$.

Q. The value of c for the function:

$f(x) = \frac{1}{x}$; $g(x) = \frac{1}{x^2}$ in $[1, 2]$

↳ not defined at 0 (not in interval).

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{-1/c^2}{-2/c^3} = \frac{1/2 - 1}{1/4 - 1}$$

$$\Rightarrow \frac{c}{2} = \frac{1/2}{3/4}$$

$$\Rightarrow \boxed{\frac{c}{2} = \frac{4}{3}} \in [1, 2]$$

* SERIES EXPANSION

1) Taylor Series Expansion of :

f(x) about x = a is given by:

$$* \quad f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots$$

$$\boxed{\text{coeff of } (x-a)^n = \frac{f^n(a)}{n!}}$$

2) Mclaren Series expansion of :

f(x) about x = 0 is:

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots$$

$$\boxed{\text{Coeff of } x^n = \frac{f^n(0)}{n!}}$$

3) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$ (odd f)

4) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\ldots$ (even f)

5) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

$\boxed{e^{i\pi} = -1}$

Q. In the series expansion of: $f(x) = e^x$; about $x = 2$ the coefficient of $(x-2)^4 = ?$

Coeff : $(x-2)^4 \longrightarrow \dfrac{f^4(2)}{4!}$ $a=2, n=4$.

$\dfrac{f^4(2)}{4!}$

$f(x) = e^x$.
$f'(x) = e^x$. $\Rightarrow \dfrac{e^2}{24}$

$\therefore \boxed{\text{Coeff} = e^2/24}$ ✓

Q. In the series expansion of $f(x) = e^x + \sin x$ about $x = \pi$, coeff of

$(x-\pi)^2 \longrightarrow \dfrac{f^2(\pi)}{2!}$

$\Rightarrow \dfrac{e^\pi}{2}$.

$f'(x) = e^x + \cos x$
$f^2(x) = e^x - \sin x$.

$\therefore \boxed{\text{Coeff} = \dfrac{e^\pi}{2}}$ ✓

* INTEGRAL CALCULUS

1. Indefinite Integration. ⟶ CS
2. Definite Integration.
3. Multiple Integrals.
4. Applications of Integration.

* Indefinite Integration

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$.
2. $\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + C$.
3. $\int a^x\,dx = \dfrac{a^x}{\log a} + C$
4. $\int \dfrac{1}{x} = \log x + C$
5. $\int \sin x = -\cos x + C$.
6. $\int \cos x = \sin x + C$.
7. $\int \sec^2 x = \tan x + C$.
8. $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$.
9. $\int \sec x \tan x\,dx = \sec x + C$.
10. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$

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11. $\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$.

12. $\int \frac{1}{\sqrt{a^2+x^2}} = \sinh^{-1}(x/a)$

13. $\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a)$ Vimp

14. $\int \frac{f'(x)}{f(x)} = \log f(x) + C$

15. $\int e^x[f(x)+f'(x)]dx = e^x.f(x) + C$

16. * Integration by Parts

$\Rightarrow \int f(x)g(x)dx$.

$\Rightarrow f(x)\int g(x) - \int f'(x)\left[\int g(x)dx\right]dx$.

↳ Apply ILATE.

1) $\int \frac{(1+x)e^x}{\cos^2(xe^x)} dx$.

$xe^x = t$.

$xe^x + e^x = \frac{dt}{dx}$

$e^x(1+x)dx = dt$.

$\Rightarrow \int \frac{dt}{\cos^2 t}$

$\Rightarrow \tan t + C$.

$\therefore \tan(xe^x) + C$.

2) $\int \frac{\sec^2 x\, dx}{\sqrt{16+\tan^2 x}}$.

t^2

$\tan^2 x = t$.

$\frac{dt}{dx} =$

Let $\tan x = t$.

$\sec^2 x\, dx = dt$.

$\frac{dt}{dx} = \sec^2 x$

$\frac{1}{\sqrt{a^2+x^2}}$

$\therefore \int \frac{dt}{\sqrt{16+t^2}} = \sinh^{-1}\left(\frac{t}{4}\right)$

$\Rightarrow \sinh^{-1}\left(\frac{\tan x}{4}\right)$ //

3) $\int \frac{x^2}{x-1} dx$.

$\Rightarrow \int \frac{(x^2-1)+1}{x-1}$

$\Rightarrow \int (x+1)dx + \int \frac{1}{x-1} dx$.

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$\Rightarrow \frac{x^2}{2} + x + \ln(x-1) + C$

(2) $\int \frac{x e^x \, dx}{(x+1)^2}$

$\Rightarrow \int \frac{e^x[(x+1) - 1]}{(x+1)^2}$

$\Rightarrow \int e^x \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]$

$\Rightarrow \int e^x [f(x) + f'(x)] = e^x f(x)$

$\therefore e^x \cdot \frac{1}{x+1} + C$

$\therefore \boxed{\frac{e^x}{x+1} + C}$

(3) $\int \frac{1}{e^x - 1} dx$

$\Rightarrow \int \frac{1}{\frac{1}{e^x} - 1} dx$

$\Rightarrow \int \frac{e^{-x}}{1 - e^{-x}} dx$ — $e^{-x} \leftarrow f'(x)$, $1 - e^{-x} \leftarrow f(x)$

$\Rightarrow$ $\underline{\ln(1 - e^{-x})}$

Q $\int \frac{1 + \sin^2 x}{1 + \cos 2x} dx$ — $1 + \cos 2x \to 2\cos^2 x$

$\sec^2 x - \tan^2 x = 1$

$\Rightarrow \frac{1}{2}\left[\int \sec^2 x \, dx + \int \tan^2 x \, dx\right]$

$\Rightarrow \frac{1}{2}\left[\tan x + \int \sec^2 x - 1\right]$

$= \frac{1}{2}\left[\tan x + \tan x - x\right]$

$\Rightarrow \left[\tan x - \frac{x}{2} + C\right]$ Ans

* $\int \tan x \, dx = \int \frac{-(-\sin x)}{\cos x}$

$\Rightarrow -\ln|\cos x| + C$

$= \ln|\sec x| + C$

$\therefore \boxed{\int \tan x = \ln|\sec x| + C}$

3] PROBABILITY & STATISTICS

1) Intro to Probability.
2) Conditional Probability.
3) Bayes Theorem
4) Distribution
5) Statistics.

1) INTRO TO PROBABILITY

Random experiment: An experiment whose outcome can't be predicted with certainity is called as random experiment.

eg: tossing a coin, throwing a dice, selecting a card from deck

* Sample Space

All the possible outcomes of an experiment is called as Sample Space.

eg: tossing a coin $S = \{H, T\}$
throwing a dice $S = \{1, 2, 3, 4, 5, 6\}$

* Event:

Event is a set containing some/all the possible outcomes present in Sample Space.

eg: E = getting a prime no. after rolling dice
$S = \{1, 2, 3, 4, 5, 6\}$
$E = \{2, 3, 5\}$

$\therefore \boxed{E \subseteq S}$

* PROBABILITY.

Probability of an event E is defined as no. of ways, an event can happen to the no. of ways a Sample Space can happen.

$\boxed{P(E) = \frac{n(E)}{n(S)}}$ $\boxed{0 \leq P(E) \leq 1}$

* Prob 0 = impossible event
Prob 1 = certain event.

* Types of Events

1) Complimentary Events:

The event E^C is said to be complementary event of E, if it contains all the outcomes present in S, but not in E.

$S = \{1, 2, 3, 4, 5, 6\}$

- E = getting an even no. after rolling a dice
$E = \{2, 4, 6\}$
- E^C = getting an odd no.
$E^C = \{1, 3, 5\}$

$\therefore \boxed{E + E^C = S}$

$\Rightarrow P(E) + P(E^C) = P(S)$

$\therefore \boxed{P(E) + P(E^C) = 1}$ // Complimentary rule.

2) Equally Likely Events

2 or more events are said to be equally likely events if their probabilities are equal.

eg: In coin experiment, the prob of getting H/T is equally likely. (1/2)
. In dice experiment, the prob of getting any no is equally likely. (1/6)

3) Mutually exclusive / disjoint Events

- 2 events A & B are said to be mutually exclusive if $A \cap B = \phi$

$\boxed{P(A \cap B) = 0}$

- 2 events A & B are said to be mutually exclusive if A occurs, B cannot occur and vice versa.

eg: In coin experiment, the prob of getting H/T is mutually exclusive.
In dice experiment, the prob of getting any no is mutually exclusive.
if 1 occurs, rem 5 can't occur & vice-

4) Independent events

2 events A & B are said to be independent events if $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

. Mutually exclusive events are corresponding to same Sample Space, whereas independent events are corresponding to different Sample Space.

eg: tossing 2 coins.

* Addition Theorem of Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

- Addition Th of Prob. is applied when {or, either or, neither nor, atleast once} is present in the problem

Q. The prob that a leap year will have 53 sundays is ?

. Leap year = 366 days

$\Rightarrow \underbrace{52 \text{ weeks}} + \underbrace{2 \text{ days}}$

52 Sundays covered

S,M M,T, T,W W,Th Th Fri Fri S, Sat Sun.

$\therefore \boxed{P(E) = \frac{2}{7}}$

Q2 A Husband & wife appear for an interview. The probability of husband getting selected is $\frac{1}{5}$ & the prob. of wife is getting selected is $1/7$.
In this situation:
(i) What is the prob. that only one of them getting selected?
(ii) Atleast one of them getting selected.
(iii) Both of them getting selected.
(iv) None getting selected.

$P(H) = 1/5$, $P(W) = 1/7$
$P(H^c) = 4/5$, $P(W^c) = 6/7$

H	W
S	S
S	N
N	S
N	N

$\frac{1}{5}\times\frac{1}{7} + \frac{1}{5}\times\frac{1}{7}$

$\Rightarrow$ (i) $\underbrace{\frac{1}{5}\times\frac{6}{7}}_{\text{simultaneously}} + \frac{4}{5}\times\frac{1}{7} = 2/7$

second choice/alternative.

$\therefore$ *

$1 - \frac{4}{5}\cdot\frac{6}{7} = 1 - \frac{24}{35}$

(ii) These are independent events.
(Husband Selec. doesn't depend on wife Selection).

Addition Theorem

$P(H \cup W) = P(H) + P(W) - P(H \cap W)$
$= \frac{1}{5} + \frac{1}{7} - \frac{1}{5}\cdot\frac{1}{7} = \frac{11}{35}$.

(iii) $\frac{1}{5}\times\frac{1}{7} = \frac{1}{35}$ [1 − None]

(iv) $\frac{4}{5}\times\frac{6}{7} = \frac{24}{35}$.

* P(atleast one of them selected) = 1 − P(None of them selected)

Q3 Person A can solve 80% of Gate Qs.
Person B can solve 60% of Gate Qs.
If a Question is selected at random. What is the probability of the Qs can be solved?

A	B
S	U
U	S
S	S
U	U

$P(A) = 0.8$ $P(B) = 0.6$
$P(A^c) = 0.2$ $P(B^c) = 0.4$

* Here, the Q will be solved, if atleast one of them solves the Question.

P(atleast one of them solving) = 1 − P(none of them solving).

$\Rightarrow 1 - 0.08$
$\Rightarrow 0.92$. // independent events

Q4 The probability of husband giving truth is 0.9, & the probability of wife giving truth is 0.01. What is the probability that both will give contradict opinions (Narrating on the same scene).

$P(H) = 0.9$, $P(W) = 0.01$
$P(HT) = 0.1$, $P(WT) = 0.99$.

$P(\text{contradict}) = 0.9 \times 0.99 + 0.01 \times 0.1$
$= 0.892$

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Q5. A card is selected randomly from a pack of 52 cards. The probability that the selected card is spade or face card is?

use Addition

52 cards

13 Hearts, 13 Diam, 13 Spade, 13 Clubs

Red, Black

not independent event. ∴ Addition

2, 3, 4, …, 10, A (9)

K, Q, J = Face Cards.

∴ Total = 12 face cards

A → Spade
B → Face card

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$

$\frac{13}{52} \times \frac{5}{52} = \frac{22}{52}$

$= \frac{22}{52} = \frac{11}{26}$ //

Q6. A no is selected randomly from 1 to 100 (both are included). The probability that the selected no. is divisible by 2 or 3 or 5 is?

Apply Addition Theorem:

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$

$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$\Rightarrow P(A \cup B \cup C) = \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{6}{100}$

$- \frac{10}{100} + \frac{3}{100} = \frac{74}{100} = 0.74$

Q7. A coin is tossed until Head appears for the first time. The probability that the required no of tosses is odd is?

independent

$\Rightarrow \{1, 3, 5, 7, \ldots\}$ → head appears in 1st toss is?

Prob of Head in 1st toss

$P(\text{I}^{st} \text{ toss}) = 1/2$

$P(\text{III}^{rd} \text{ toss}) = P(TTH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^3$

$P(\text{V}^{th} \text{ toss}) = P(TTTTH) = \left(\frac{1}{2}\right)^5$

⋮

$\therefore P(\text{odd}) = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \ldots$

$\Rightarrow \frac{1/2}{1 - 1/4} = \frac{1}{2} \times \frac{4}{3} \Rightarrow 2/3$ //

$\frac{1/2}{1 - 1/4} = \frac{1}{2} \times \frac{4}{3}$

Q8. A dice is rolled 2 times. The probability that 2nd turn results in a value higher than 1st toss.

(1,2), (1,3), (1,4), (1,5), (1,6)
(2,3), (2,4), (2,5), (2,6)
(3,4), (3,5), (3,6)
(4,5), (4,6)
(5,6)

upper triangular matrix.

$P = \frac{15}{36} = 5/12$. $\frac{5\times 6}{2} = 15/36$.

Q9. A bag contains 5 Black Balls, 7 Red balls what is the probability of drawing 2 balls of same colour.

$\frac{{}^5C_2}{{}^{12}C_2} + \frac{{}^7C_2}{{}^{12}C_2}$ ✓.

alternate choice.

Q10. A bag contains 5 red balls, 5 black balls. In the first draw, 1 ball is selected at random & discarded without noticing its colour. The probability of drawing red ball in the second draw is?

5R, 5B. $P(R_1 \cap R_2) + P(B_1 \cap R_2)$.

Case 1: Red is drawn first

$\frac{{}^5C_1}{{}^{10}C_1} \times \frac{{}^4C_1}{{}^9C_1} = 2/9 = \frac{4}{18}$

Case 2: Black is drawn first.

$\frac{\cancel{5}}{\cancel{10}_2} \times \frac{5}{9} = \frac{5}{18}$

2 other choices

$1/2$. ✓

Q11. A no. is selected randomly from 1 to 100, both are included. The probability that the selected no. will not contain digit 7.

[7, 17, 27, 37, 47, 57, 67, 77, 87, 97]

from 100 to 999.

8 9 9

$\{0, 1, 2, \cancel{7}, \ldots 8, 9\}$ → 9 digits.

$\Rightarrow \frac{648}{900} \longrightarrow (9\times10\times10)$ ✓

$\Rightarrow \frac{18}{25}$. Ans

Q12. In a row, Boys & Girls wants to sit in alternative position, the probability of this arrangement if row consists:

(i) 4B, 3G.   B G B G B G B =

(ii) 3B, 3G.

B G B G B G B

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G.. G – G –

$^nP_n = n!$

(ii) B $\underline{\ \ }$ B $\underline{\ \ }$ B $\underline{\ \ }$ B, with G under each gap

$\Rightarrow \dfrac{4! \times 3!}{7!}$ Ans

(iii) 3B, 3G

$2 \times \left[\dfrac{3! \times 3!}{6!}\right] \Rightarrow \dfrac{\cancel{6} \times \cancel{6} \times \cancel{2}}{\cancel{6} \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2}} \Rightarrow \dfrac{1}{10}$

∵ row may start with boy/Girl

Q.13 What is the probability that Virat Kohli & Anushka Sharma will have ~~divorce~~ same birth month is

VK ⟶ 12 choice = 12/12.

AS ⟶ 1 choice = 1/12.

∴ $1 \times 1/12 = 1/12$.

[As we Cartesian Product Method].

2) * Conditional Probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$
$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Prob. of event B happening after event A.

* P(A/B) = Probability of event A happening after event B.

* P(B/A) = Probability of event B happening after event A.

$P(A \cap B)$ ⟶ Joint Probability of A & B.

P(A) = marginal Prob. of A.

P(B) = marginal prob. of B.

* Multiplication theorem of Probability

$P(A \cap B) = P(B) \times P(A/B)$

$\Rightarrow P(A) \times P(B/A)$.

AD

(i) $P(\text{only } A) = P(A \cap B^c)$.

$\Rightarrow P(A) - P(A \cap B)$.

(ii) $P(\text{only } B) = P(B \cap A^c)$

$= P(B) - P(A \cap B)$

(iii) $P(A \text{ and } B) = P(A \cap B)$

(iv) $P(\text{Neither } A \text{ nor } B) = P(A^c \cap B^c)$
$= P(A \cup B)^c$
$= 1 - P(A \cup B)$

Q1 An exam consists 2 papers: P1 and P2. The probability of a student failing in P1 is : ~~2~~ 0.~~6~~3 & that in P2 is 0.2. Given that a student has failed in P2. Then probability of failing in P1 is 0.6. The prob. of a student failing in both the papers is?

$P(P_1) = 0.3$ $\qquad P\left(\frac{P_1}{P_2}\right)$.

$P(P_2) = 0.2$

$P\left(\frac{P_1}{P_2}\right) = 0.6$. $\qquad P(P_1 \cap P_2)$

P_2 → happened earlier (Q). $\qquad P(P_1) \cdot P(P_2/P_1)$

$\qquad 0.3 \times$

$\Rightarrow \frac{P(P_1 \cap P_2)}{P(P_2)} = 0.6$

$\therefore \boxed{P(P_1 \cap P_2) = 0.12}$. Ans

Q2 P & Q are considering to apply for a job. The prob that P applies for the job is 1/4. The prob that P applies for the job given that Q applies for job is 1/2. And the prob. that Q applies for job given P applies for job is 1/3. Then the prob that P doesnot apply the job given that Q doesnot apply for the job is:

→ $P(P) = \frac{1}{4}$; $P\left(\frac{P}{Q}\right) = \frac{1}{2}$; $P\left(\frac{Q}{P}\right) = \frac{1}{3}$.

. $P\left(\frac{P^c}{Q^c}\right) = ?$

$\frac{P(P \cap Q)}{P(Q)} = \frac{1}{2}$ $\qquad \frac{P(P \cap Q)}{P(P)} = \frac{1}{3}$

$\Rightarrow P(Q) = \frac{1}{6}$. $\qquad P(P \cap Q) = \frac{1}{12}$.

$\frac{\bar{P}\bar{Q}}{(\overline{P+Q})}$ $\qquad \frac{\bar{P}\bar{Q}}{\overline{P+Q}}$

$\therefore P\left(\frac{P^c}{Q^c}\right) = \frac{P(P^c \cap Q^c)}{P(Q^c)}$

$\Rightarrow \frac{1 - P(P \cup Q)}{1 - P(Q)}$

$\Rightarrow \frac{2/3}{5/6} = \frac{4}{5}$. Ans

$P(P \cup Q) = P(P) + P(Q) - P(P \cap Q)$

$\Rightarrow \frac{1}{4} + \frac{1}{6} - \frac{1}{12} = 1/3$

Q3 2 dices are rolled. Let 'r' denote the sum of the nos appearing on the 2 dices, then calculate:

(i) $P(r > 6)$

(ii) $P(r/3 \text{ is integer})$

(iii) $P\left(\frac{r=8}{r/4 \text{ is integer}}\right)$

(iv) $P\left(\frac{r=9}{r/5 \text{ is integer}}\right)$

Range : 2 to 12

$2 \longrightarrow (1,1)$

$3 \longrightarrow (1,2),(2,1)$

$4 \longrightarrow (1,3),(2,2),(3,1)$

$5 \longrightarrow (1,4),(2,3),(3,2),(4,1)$

$6 \longrightarrow (1,5),(2,4),(3,3),(4,2),(5,1)$

$7 \longrightarrow (1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$

$8 \longrightarrow (2,6),(3,5),(4,4),(5,3),(6,2)$

$9 \longrightarrow (3,6),(4,5),(5,4),(6,3)$

$10 \longrightarrow (4,6),(5,5),(6,4)$

$11 \longrightarrow (5,6),(6,5)$

$12 \longrightarrow (6,6)$

RE

r	2	3	4	5	6	7	8	9	10	11	12
P(X=r)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

(i) $P(r>6) = P(r=7) + P(r=8) + \dots P(r=12)$

$\Rightarrow \frac{6}{36} + \frac{5}{36} + \dots \dots \frac{1}{36}$

$\Rightarrow \frac{6\times 7}{2} \Big/ 36 \Rightarrow \frac{21}{36}$ //

(ii) $P(r/3) = P(3) + P(6) + P(9) + P(12)$

$= \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36}$

$\Rightarrow \frac{1}{3}$ //

(iii) $P\left(\frac{r=8}{r/4 \text{ is integer}}\right) =$

$P(4) + P(8) + P(12)$

$= \frac{3}{36} + \frac{5}{36} + \frac{1}{36} = 1/4$

$\Rightarrow \frac{P[(r=8) \cap (r/4 \text{ is integer})]}{P(r/4 \text{ is integer})}$

$\Rightarrow \frac{P(r=8)}{1/4} = \frac{5/36}{1/4} = 5/9$

$r=9$ $\quad r=5,10$

(iv) $P\left(\frac{r=9}{r/5 \text{ is integer}}\right) \Rightarrow \frac{P[r=9 \cap r/5 \text{ is int}]}{P[r/5 \text{ is int}]}$

$7/36 \Rightarrow \frac{P[\emptyset]}{7/36} = 0$

$\Rightarrow \frac{4/36}{7/36} = \frac{4}{7}$ //

$\frac{4 \times 3}{{}^{12}C_2}$

not. (i)

Q4 A box contains 4 White Balls, 3 red Balls. In succession 2 balls are randomly selected & removed from the box. Given that, the first removed ball is white. What is the probability that the second removed ball is red?

4W, 3R

$\downarrow$ remove 1W.

3W, 3R.

$\Rightarrow \frac{{}^3C_1}{{}^6C_1} = \underline{\underline{1/2}}$ Ans

4 computing prob of getting white

$\frac{{}^4C_1}{{}^7C_1} \times \frac{{}^3C_1}{{}^6C_1}$

$\frac{4^2}{7} \times \frac{3}{6}$

Q5. 2 coins are tossed. If atleast one of the outcome is known to be head, then what is the prob. that both outcome are head.

if 2 coins ; SS → HH, HT, TH, $\cancel{TT}$.

Now, SS' = {HH, HT, TH}.

$\therefore P = \frac{\text{fav. case}}{\text{sample space}} = 1/3$.

$\therefore \boxed{P = 1/3}$

Let A, B, C be 3 mutually exclusive events, occur in sample space. Let E be event which is pass. in all 3 event A, B, C; then accord. to BT

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3) BAYE'S THEOREM → defective comp from A, B, C suppliers.

(case) Let E be the event.

$$P(A/E) = \frac{P(A) \cdot P(E/A)}{P(A) \cdot P(E/A) + P(B) \cdot P(E/B) + P(C) \cdot P(E/C)}$$

Q1 There are 2 Bags. B1 contains 2 red Balls, 5 Green Balls; B2 contains 2 red Balls, 6 G Balls. A person tosses a coin; if it is Head; he goes to B1 and draws a ball. If it is Tail, he goes to B2 & draws a ball. In this situation:

(i) What is the prob. of drawing red ball?

(ii) Given that the drawn ball is red, what is the probab. that it come from B1?

2R
5G
B1

2R
6G
B2

Head Prob Tail Prob.

(i) $P(\text{red Ball}) = \underbrace{\frac{1}{2} \times \frac{2}{7}}_{\text{Bag 1}} + \underbrace{\frac{1}{2} \times \frac{2}{8}}_{\text{Bag 2}}$

1/2 → R 2/7, G; 1/2 → R 2/8, G

$\Rightarrow \underline{\underline{15/56}}$

copy.

(ii) $P\left(\frac{\text{Bag 1}}{\text{red Ball}}\right) = \frac{P(\text{Bag 1} \cap \text{Red})}{P(\text{red})} = \frac{1/2 \times 2/7}{15/56}$

$\Rightarrow 8/15$. Ans

Q2 3 companies X, Y, Z supply computers to an University. The percentage of computers supplied by them & the probability of those being defective are tabulated below:
Given that a computer is defective, what is the probability that it is supplied by company Y.

Company	% of comp supplied	Probabil. of being defective
X	60%	0.01
Y	30%	0.02
Z	10%	0.03

$P(\text{defective}) = \underbrace{0.6 \times 0.01}_{X} + \underbrace{0.3 \times 0.02}_{Y} + \underbrace{0.1 \times 0.03}_{Z}$
$\Rightarrow \underline{0.015}$

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$P\left(\frac{\text{comp. Y}}{\text{defective}}\right) \Rightarrow \frac{0.3 \times 0.02}{0.05} = 2/5 = \underline{0.4}$

Q3 There are 3 coins, out of those; 2 are unbiased coin & one is a biased coin with 2 heads.
A person randomly selects a coin and tossed the coins twice. If 2 heads appeared, what is the prob. that the selected coin is biased coin?

3/3, 1/4
1
1/3

2H from biased coin.

$P(2\text{ Heads}) = \underbrace{\frac{2}{3} \times \frac{1}{4}}_{\text{Select UB coin}} + \underbrace{\frac{1}{3} \times 1 \times 1}_{\text{Select B coin}} \Rightarrow 1/2$.

$\Rightarrow P\left(\frac{B.C.}{2\text{ Heads}}\right) = \frac{1/3}{1/2} = 2/3$. Ans.

4) ## DISTRIBUTIONS.

* Random Variable:
The variable which is assigned to all the possible outcomes of an experiment is called as random variable.

RV is divided into 2 types:
(i) discrete RV.
(ii) continuous RV.

(i) Discrete RV

If the RV considers only integer value, then it is called discrete RV.
eg: no of students in class.
No. of heads appear after tossing 4 coins.

(iii) Continuous RV

If the RV considers all the possible real values, then it is called as continuous RV.

eg: weight of student in class.
Volume of a body.

* DISCRETE DISTRIBUTIONS

Basic discrete distribution. Properties:

(i) $\sum P(X) = 1$ → avg(X).

(ii) mean = $E(X) = \sum X P(X)$ ✓

(iii) Variance = $V(X) = E(X^2) - [E(X)]^2$.

$\Rightarrow \sum X^2 P(X) - \left[\sum X P(X)\right]^2$.

(iv) SD = $\sqrt{V(X)}$.

if X, Y are any 2 random Var; and a, b are constants.

(i) $E(aX) = aE(X)$.

(ii) $E(aX+b) = aE(X) + b$.

(iii) $V(aX) = a^2 V(X)$.

(iv) $V(ax+b) = a^2 V(X) + 0$ ∵ b (constant) value don't change.

Q. A r.v. X takes the values: {1 2 3} with prob shown:

X	1	2	3
P(X)	$\cancel{3(2x)}$ $\frac{2+5p}{5}$	$\frac{1+3p}{5}$	$\frac{1.5+2p}{5}$

then calculate p, E(X).
↳ mean.

$\sum P(X) = 1$.

∴ $\frac{2+5p}{5} + \frac{1+3p}{5} + \frac{1.5+2p}{5} = 1$.

$4.5 + 10p = 5$.

$10p = \frac{5}{10} \Rightarrow 0.025$.

∴ $\boxed{p = 0.05}$ ✓

mean = $E(X) = \sum X \cdot P(X)$.

$\Rightarrow 1\left[\frac{2+5p}{5}\right] + 2\left[\frac{1+3p}{5}\right] + 3\left[\frac{1.5+2p}{5}\right]$.

$\Rightarrow \frac{17p + 8.5}{5} = \frac{8.5 + 0.85}{5}$

$\Rightarrow \underline{1.87}$ ✓

Q. A RV ~~to~~ X takes the values [1, 2, 3] with the probability shown below: then calculate SD(X)

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X	1	2	3
$P(X)$	0.3	0.6	0.1

→ discrete Values.

$E(X) = \sum X \cdot P(X)$

$\Rightarrow 0.3 + 1.2 + 0.3 \Rightarrow \underline{\underline{1.8}}$

$\boxed{E(X) = 1.8}$

$\boxed{V(X) = E(X^2) - E(X)^2}$

$E(X^2) = \sum X^2 \cdot P(X)$

$\Rightarrow 0.3 + 2.4 + 0.9$

$\Rightarrow 3.6$

$\boxed{E(X^2) = 3.6}$ ✓

$Var = E(X^2) - [E(X)]^2$

$\Rightarrow 3.6 - [1.8]^2$

$\Rightarrow \underline{\underline{0.36}}$ ✓

$\boxed{SD = 0.6} \rightarrow \sqrt{Var}$.

Q3 An Exam Paper has 150 MCQs of 1M each with each Question having 4 choices. Each incorrect answer gives −0.25M. Suppose thousand students choose all their answers randomly with uniform probability, then the total of the expected marks of all the students is ? — E(Marks)

Let X denotes the marks obtained for a single Q.

X	1	−0.25
$P(X)$	1/4	3/4

$E(X) = \sum X\,P(X)$

$= 1\left(\frac{1}{4}\right) - 0.25\left(\frac{3}{4}\right)$

$\Rightarrow \frac{1}{16} \times 150 \times 1000 = 9375.$

↑ 1 Q [Expected marks Per Q]

Q4 A Person rolls a dice until 3 is observed for the first time. Let X denotes no. of times dice is thrown the expected value of X is ? E(X)

X	1	2	3	4 …	… n
$P(X)$	$1/6$	$\frac{5}{6}\cdot\frac{1}{6}$	$\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}$		$\left(\frac{5}{6}\right)^{n-1}\frac{1}{6}$

Getting 6 in 3rd dice.

$E(X) = \sum X\,P(X)$

$\Rightarrow 1\left(\frac{1}{6}\right) + 2\left(\frac{5}{6}\cdot\frac{1}{6}\right) + 3\left(\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}\right) + \dots \infty$

$\Rightarrow \frac{1}{6}\left[1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + \dots \infty\right]$

$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots \infty$.

$\Rightarrow \frac{1}{6}\left[\left(1-\frac{5}{6}\right)^{-2}\right] = \frac{1}{6} \times 6^2 = 6$

* BINOMIAL DISTRIBUTION

- Prob. of obtaining 'r' successes out of 'n' trials by using binomial distribution is given by:

$$\boxed{P(X=r) = {}^nC_r\, p^r q^{n-r}}$$

where, p = prob. of success
q = prob. of failure.

$\therefore$ $p+q=1$

- In Binomial distribution,
mean = $E(X) = np$
variance = $V(X) = \underbrace{n p}_{E(X)} q$, $q<1$

→ In BD, $\boxed{\text{mean} > \text{Variance}}$

→ If p denotes prob. of success, q denotes prob. of failure; then the BD is given by:

$(q+p)^n$

→ BD is applicable:

(i) Only when 2 outcomes are possible.
(ii) Probability of success must remain constant from trial to trial.

* In the problems related to coin/dice/sampling with replacement. Prob. of success remains constant from trial to trial, so we can apply Binomial distribution.

Q1. A coin is tossed 4 times (4 trials). What is the prob of getting heads exactly 3 times? (3 success).

$p = 1/2$
$q = 1/2$
$r = 3$
$n = 4$.

$\therefore P(X=r) = {}^nC_r\, p^r q^{n-r}$

$\Rightarrow {}^4C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^1$

$\Rightarrow \frac{4 \times 1}{8} \times \frac{1}{2} = 1/4$

Q2. A dice is rolled 3 times. What is the prob of getting face 4 atleast once.

atleast 1 $\Rightarrow X \geq 1$
atmost 1 $\Rightarrow X \leq 1$

$p = 1/6, q = 5/6, n = 3,$

$P(X \geq 1) = 1 - P(X=0)$

$1 - \left[{}^3C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3\right]$

$1 - \frac{5^3}{6^3} \Rightarrow \frac{1 - 125}{216} = \frac{91}{216}$

$\therefore \boxed{P(X \geq 1) = \frac{91}{216}}$

Q. A coin is tossed ∞ no. of times. What is the prob. of getting 4th head in the 10th toss?

diff from 4 heads in 10 tosses.

(Prob of 3 Heads in 9 toss)

(9 tosses $\Rightarrow$ 3 Heads) X (Head in 10th toss)

$p = 1/2, q = 1/2, r = 3, n = 9.$

$\Rightarrow {}^9C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^6 \cdot \times \frac{1}{2}$

$\Rightarrow \frac{{}^9C_3}{2^{10}} \Rightarrow 0.082$

* POISSON DISTRIBUTION

- The probability of obtaining a success given by PD is given by:

$$\boxed{P(X=R) = \frac{e^{-\lambda}\lambda^r}{r!}}$$

- In case of PD:

mean $= E(X) = \lambda$

Variance $= V(X) = \lambda$

$\therefore \boxed{\text{mean} = \text{Variance}}$

→ Poisson distribution is used to express rare occurrences [Prob of Success is very small & no. of approximations are large].

Prob (Succ

Q. The no. of accidents occuring in a city in a month follows Poisson distribution with mean as 5.2. The prob. of occurence of ~~<2~~ atmost 2 accidents in a randomly selected month is:

Let X denotes the no. of accidents occuring in a city.

$\lambda = 5.2$.

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$\Rightarrow \frac{e^{-5.2}(5.2)^0}{0!} + \frac{e^{-5.2}(5.2)^1}{1!} + \frac{e^{-5.2}(5.2)^2}{2!}$$

$$\Rightarrow e^{-5.2} + e^{-5.2} \cdot 5.2 + e^{-5.2}\frac{(5.2)^2}{2}$$

$$\Rightarrow e^{-5.2}\left[1 + 5.2 + \frac{(5.2)^2}{2}\right].$$

$$\Rightarrow \underline{\underline{0.1087}}.$$

Q. The probability of a resistor being defective is 0.02. There are 50 such resistors in a circuit. The probability of more than 2 defective resistors is ?

Let X denotes the no. of defective resistors

$$P(X>2) = 1 - \left[P(X=0) + P(X=1) + P(X=2)\right]$$

$$\Rightarrow 1 - \left[\frac{e^{-1}1^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!}\right]$$

$$\Rightarrow 1 - \frac{5}{2}e^{-1} \approx \underline{\underline{0.08}}$$

$100 \rightarrow \cancel{0.02}\ 2$
$50 \rightarrow 1$

$\lambda = 1$

$\frac{2}{100} \times 50$

$\frac{2}{100} \times 50$

* CONTINUOUS DISTRIBUTIONS

1. Basic continuous Distributions

If X = continuous RV, then the continuous distr of X is defined by the prob

$$\boxed{P(-\infty < x < \infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1}$$ → Prob.

* Properties:

(1) Mean $= E(x) = \int_{-\infty}^{\infty} x f(x)\,dx.$

(2) Variance $= V(x) = E(x^2) - [E(x)]^2$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2.$$

(3) $P(x>a) = \int_{a}^{\infty} f(x)\,dx$

(4) $P(a<x<b) = \int_{a}^{b} f(x)\,dx$

(5) $P(x<a) = \int_{-\infty}^{a} f(x)\,dx.$

(6) Probability distribution function $F(x)$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

→ probabil. density functn

Q. A probability density function is given by:

$f(x) = \frac{A}{x^y}$, where A, y are constants with $y > 1$ and $1 \le x \le \infty$, & $f(x) = 0$ for $-\infty < x < 1$

$$f(x) = \begin{cases} A/x^y, & 1 \le x < \infty. \\ 0, & -\infty < x < 1. \end{cases}$$

For $f(x)$ to be a valid density function, $A = ?$

Soln:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\Rightarrow \int_{-\infty}^{1} (0) + \int_{1}^{\infty} \frac{A}{x^y}\,dx = 1$$

x^{-n} $\quad x^{-(y-1)}$ $\quad 0^{-1}$ $\quad \frac{1}{x^{(y-1)}}$

$$\Rightarrow A \int_{1}^{\infty} x^{-y}\,dx = 1.$$

$$A \cdot \left[\frac{x^{-y+1}}{-y+1}\right]_1^{\infty} = 1 \qquad \left(\frac{1}{x^{(y-1)}}\right)_1^{\infty}$$

$$\Rightarrow \frac{A}{1-y}\left[x^{-\infty} - x^{0}\right] = 1$$

$$\Rightarrow \frac{A}{1-y}(0-1) = 1 \qquad \therefore \boxed{A = y-1}$$

Q. If $f(x) = K(5x - 2x^2)$ for $0 < x < 2$
$= 0$, otherwise.

then $P(X > 1)$?

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\Rightarrow \int_{-\infty}^{0} f(x) + \int_{0}^{2} f(x) + \int_{2}^{\infty} f(x)\,dx.$$

$$\int_{0}^{2} K(5x - 2x^2) = 1$$

$$K \cdot \left[\frac{5x^2}{2} - \frac{2x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow K\left[10 - \frac{16}{3}\right] = 1.$$

$$\therefore \boxed{K = \frac{3}{14}}$$

$$\therefore P(X > 1) = \int_{1}^{\infty} f(x)\,dx = \int_{1}^{\infty} \frac{3}{14}(5x - 2x^2).$$

$$\Rightarrow \frac{3}{14}\int_{1}^{2}(5x - 2x^2) + \int_{2}^{\infty}$$

$$\Rightarrow \frac{3}{14}\left[\frac{5x^2}{2} - \frac{2x^3}{3}\right]_1^2 = \frac{3}{14}\left[\frac{5\times 4}{2} - \frac{16}{3} - \frac{5}{2} - \frac{2}{3}\right.$$

$\Rightarrow \frac{3}{14}\left[10-\frac{16}{3}-\frac{5}{2}+\frac{2}{3}\right]$ $\begin{array}{r}45\\-32\\\hline 13\end{array}$

$\Rightarrow \frac{3}{14}\left[\frac{60-32-15-\cancel{8}\,4}{6}\right]$

$= \frac{3}{14} \times \frac{17}{6} \Rightarrow \underline{\underline{17/28}}$. Ans

Q3. X is a RV with density function:

$$f(x) = \begin{cases} 0.2, \text{ for } |x| \le 1 \\ 0.1, \text{ for } 1 < |x| \le 4 \\ 0, \text{ otherwise.} \end{cases}$$

then $P(0.5 < x < 5)$?

Soln.

$\int_{0.5}^{5} f(x)\,dx$.

$\Rightarrow \int_{0.5}^{1} 0.2\,dx + \int_{1}^{4} 0.1\,dx + \cancel{\int_{4}^{5} 0}$.

$\Rightarrow \cancel{0.2x + 0.1x}$

$\Rightarrow 0.2 \times 0.5 + 0.1 \times 3$

$\Rightarrow \underline{\underline{0.4}}$

2. Uniform Distribution

A continuous RV, X is said to be following uniform distribution in the [a,b] if its PDF is given by:

$$\boxed{f(x) = \frac{1}{b-a}}$$

In case of uniform distribution:

(i) Mean $\longrightarrow E(x) = \frac{b+a}{2}$.

(ii) Variance $\longrightarrow V(x) = \frac{(b-a)^2}{12}$. $\frac{8}{\sqrt{12}}$

Q1. The SD of a RV which is uniformly distributed in the interval [2,10].

Variance $= \frac{8^2}{12} \Rightarrow \frac{64}{12} = \frac{16}{3}$

$SD = \sqrt{\text{Variance}} = \frac{4}{\sqrt{3}}$. Ans.

Q2. If $X \longrightarrow [0,1]$, X: uniformly distribute RV then $E(x^3) = ?$

$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$.

$$E(x^3) = \int_{-\infty}^{\infty} x^3 f(x)\,dx.$$

$$\Rightarrow \int_0^1 x^3 \cdot dx$$

$[a,b] \Rightarrow f(x) = \frac{1}{b-a}$

$[0,1] \Rightarrow f(x) = \frac{1}{1-0} = 1$

$$\Rightarrow \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}. \underline{\underline{\text{Ans}}}.$$

3. <u>Exponential Distribution</u>

A continuous RV, X is said to be following exponential distribution with parameter λ, if its PDF is given by:

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0.$$
$$= 0 \quad , \text{ otherwise.}$$

<u>In case of exponential distribution:</u>

- Mean = $E(X) = 1/\lambda$.
- Variance = $V(X) = 1/\lambda^2$.

Teacher's Sign / Remark

Q. The duration in minutes of a telephone conversation follows exponential distribution, with:

$f(x) = \frac{1}{5} e^{-x/5}, \quad x \geq 0$ $\quad \lambda = 1/5$

The probability that the telephone conversation will exceed 5 min is?

Let X denotes the duration of telephone conversation in minutes.

Soln.

$$P(X > 5) = \int_5^\infty f(x)\,dx$$

$$\Rightarrow \frac{1}{5}\int_5^\infty e^{-x/5}\,dx$$

$e^{-x/5}$

$0 \; -1$

$$\Rightarrow \frac{\cancel{1}}{\cancel{5}}\left[\frac{e^{-x/5}}{\cancel{-1/5}}\right]_5^\infty \quad \frac{-1}{\cancel{5}}$$

$$\Rightarrow \frac{-1}{\cancel{25}}$$

$$\Rightarrow -1\left[-\frac{1}{e}\right] \Rightarrow \frac{1}{e} = 0.3678$$

4. Normal Distribution → Gaussian dist.

A continuous RV X is said to be follow normal distribution with parameters $[\mu, \sigma^2]$, if its PDF is given by:

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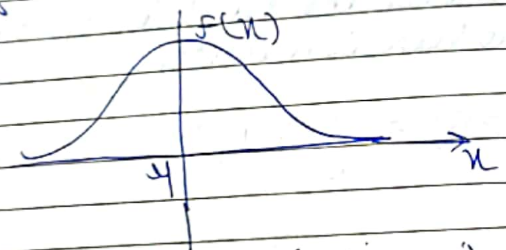
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ for } -\infty < x < \infty$$

→ In case of Normal dist.:

$$\left.\begin{aligned} \text{mean} &= E(x) = \mu \\ \text{Variance} &= V(x) = \sigma^2 \end{aligned}\right\}$$

→ Symmetrical abt mean.



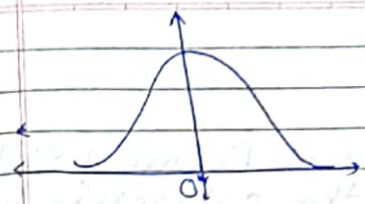
→ In Normal distribution, if we consider $\boxed{Z = \frac{x-\mu}{\sigma}}$; we will get Stand. Normal distribution, whose pdf is given by:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \text{ for } -\infty < z < \infty$$

→ In case of Standard ND;

$$\boxed{\begin{aligned} \text{mean} &= 0 \\ \text{variance} &= 1 \end{aligned}}$$

∴ mean = 0, symmet. abt 0.

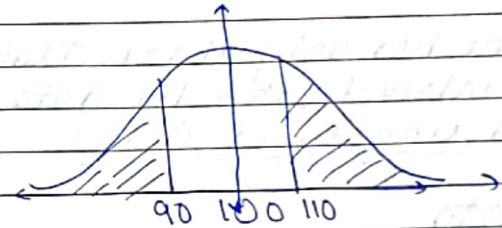


Q For a rv following Normal distribution, mean $(\mu) = 100$;

If $P(X \geq 110) = \alpha$

then: $P(90 < x < 110) = ?$

Soln:



$P(90 < x < 110) = 1 - P(x \geq 110) - P(x \leq 90)$

$\Rightarrow 1 - \alpha - \alpha$

$\Rightarrow 1 - 2\alpha$. Ans

* STATISTICS

- According to Proff. Fisher (Father of Stats), Stats is defined as the collection & analysis of data.
- Data is divided into 2 types:
  (i) Ungrouped data / raw data.
  (ii) Grouped data.

(i) Ungrouped data

- If the data has only observations with any class intervals, then the data is called as ungrouped data.

(ii) Grouped data

If the data has class Intervals along with frequency, then the data is called as Grouped data.

i) Mean

$n_1 \longrightarrow X_1$ → for n_1 observ, mean is X_1.

$n_2 \longrightarrow X_2$

$$\boxed{\overline{X} = \frac{n_1 X_1 + n_2 X_2}{n_1 + n_2}}$$ → overall mean.

* In case of grouped data:

→ mean $(\overline{X}) = \frac{\Sigma x_i f_i}{\Sigma f_i}$

· x indicates midpoint of the class

Q Calculate mean of the following data:

Flow Rate	frequency (f)	x	fx
7.5 - 7.7	1	7.6	7.6
7.7 - 7.9	5	7.8	39
7.9 - 8.1	35	8.0	280
8.1 - 8.3	17	8.2	1392
8.3 - 8.5	12	8.4	1008
8.5 - 8.7	10	8.6	86
	$\Sigma f = 80$		

$\Sigma fx = \frac{652.8}{80} = 8.16$

2) Median

- Median is the middle value of the data in the sense that, the total no. of observations < median = no. of observations greater than median.
- If n is odd, median $= \frac{n+1}{2}$ value.

eg: 1, 3, 5, 7, 9

Median $\longrightarrow \frac{5+1}{2} = \frac{6}{2} = 3^{rd}$ value $= 5$

→ If n is even, then median = $\dfrac{\frac{n}{2}\text{ value} + \left(\frac{n}{2}+1\right)\text{ value}}{2}$

eg. 1, 3, 5, 7, 9, 10

$\dfrac{5+7}{2} = 6$

median = $\dfrac{\left(\frac{6}{2}\text{ value}\right) + \left(\frac{6}{2}+1\text{ value}\right)}{2}$

Q1 The speeds observed on a road junction are 72, 36, 23, 49, 58, 61, 19, 89, 24, 14. Median is ?

14, 19, 23, 24, 36, 49, 58, 61, 72, 89

⇒ $\dfrac{85}{2} = 42.5$ $\quad \dfrac{5^{th}, 6^{th}}{2} = \dfrac{36+49}{2}$

* Mode

- The observation which repeats max no. of times in the given data is called as mode.

eg: 1, 3, 9, 8, 6, 1, 4, 3, 2, 8, 4, 1

$\boxed{\text{mode} = 1}$

Q The marks obtained by 100 students are given in the following table. Calculate mean, median & mode.

SNo	Marks obtained	No. of Stud.	
1	25	20	0–20
2	30	20	21–40
3	35	40	41–80
4	40	20	81–10

Word Ungroup

∴ CI not same

Mean = $\dfrac{20\times25 + 20\times30 + 40\times35 + 40\times20}{100}$

$\boxed{\text{Mean} \Rightarrow 33}$

- Median = (even ab) $\dfrac{\left(\frac{100}{2}\text{ value}\right) + \left(\frac{100}{2}+1\text{ value}\right)}{2}$

$= \dfrac{35+35}{2} = 35$

$\boxed{\text{Median} = 35}$

$\boxed{\text{Mode} = 35}$ Ans

* Standard deviation → deviation of data from mean.

Standard deviation:

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

$\boxed{\text{Variance} = (SD)^2}$

* SD indicates spread of the data around mean.
Low SD indicates that data is closely clustered around mean

High SD indicates that the data is dispersed over a wider range of values

Q. Calculate SD of the following data.

32, 98, 70, 25, 75.

$\bar{y} = 60$

$$\sigma = \sqrt{\frac{1}{5}[784 + 1444 + 100 + 1225 + 225]}$$

$$= \sqrt{755.6} \Rightarrow \underline{\underline{27.48}}$$

* Coefficient of Variation

$$\boxed{\text{Coefficient of Variation} = \frac{\sigma}{\bar{y}}} \rightarrow SD$$

Q. The statistics of the runs scored by 4 Batsmen are provided in the following Table; who is the consistent batsman

Batsmen	Avg	SD	
Dhawan	31.2	5.21	0.16
Rohit	46.0	6.35	0.13
Kohli	54.4	6.22	0.11
Dhoni	17.9	5.90	0.32

Kohli: consistent

* Regression Analysis

- Regression is the estimation of unknown values of 1 variable from the known value of another variable
- The line of regression of y on x is given by:

$$\boxed{y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})}$$

regression coeff of y on x [byx].

→ x on y ⇒ $\boxed{x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})}$

b_{xy}.

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r: coefficient of correlation.

* Coefficient of correlation : (Similarity)

→ It is the numerical measure of correlation. Its value lies between −1 and +1.

$r = -1 \Rightarrow$ both var. are exactly opp.
$r = +1 \Rightarrow$ both var. are exactly same
$r = 0 \Rightarrow$ No relation.

→ Coefficient of correlation (r) is the GM of 2 regression coefficients.

$$\boxed{r = \sqrt{b_{xy} \cdot b_{yx}}}$$

$$\therefore r = \sqrt{\frac{r \cdot \cancel{\sigma_y}}{\cancel{\sigma_x}} \cdot \frac{r \cancel{\sigma_x}}{\cancel{\sigma_y}}} \Rightarrow \sqrt{r^2} = r$$

* The angle between any 2 regression lines is given by:

$$\boxed{\tan\theta = \frac{1-r^2}{|r|} * \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}}$$

$r = 0 \Rightarrow \tan\theta = \infty$
$\therefore \theta = \pi/2$

* ∴ The 2 regression lines are perpendicular to each other.

$r = \pm 1$; $\tan\theta = 0$; $\theta = 0$.
So, the two regression lines are ∥ to each other.

IMP TOPICS :

1 – System of Eqⁿs.
2 – Eigen Values & Eigen Vectors
3 – Baye's Theorem
4 – Calculation of Mean & Variance.
9 – Maxima & Minima

INTEGRATION – RAVI

Process of finding functn given its derivative

$F'(x) = f(x)$
$\int f(x) = F(x) + C$ → in indefinite int

$\Rightarrow \int x = \frac{x^2}{2} + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1}x + C$

$\int \frac{1}{1+x^2} dx = -\cot^{-1}x + C$

$\int \cot x \, dx = \log|\sin x| + C$ Cell → Form

$\int \tan x \, dx = -\log|\cos x| + C$

Q1. $\int x^6 dx = \frac{x^7}{7} + C$

Q2. $\int \frac{x^3 - 1}{x^2} dx$

$\int \frac{x^3}{x^2} - \frac{1}{x^2} dx$

$\Rightarrow \int x\,dx - \int \frac{1}{x^2} dx$ → $\int x^{-2} = \frac{x^{-1}}{-1}$

$\Rightarrow \frac{x^2}{2} + C - \frac{x^{-1}}{-1}$

$\Rightarrow \frac{x^2}{2} + \frac{1}{x} + C$

Q3. $\int \cos 2x \, dx$

$\Rightarrow \frac{1}{2} \int \sin 2x \, dx + C$

$\therefore \frac{1}{2} \sin 2x + C$

Q4. $\int \operatorname{cosec} x (\operatorname{cosec} x + \cot x) dx$

$\Rightarrow \int \operatorname{cosec}^2 x + \int \operatorname{cosec} x \cot x$

$\Rightarrow -\cot x - \operatorname{cosec} x + C$

* Methods of integration

1. Integration by substitution

Q. $\int \tan x \, dx$

$\Rightarrow \int \frac{\sin x}{\cos x} dx$

$\Rightarrow \cos x = t$
$-\sin x = \frac{dt}{dx}$

$\Rightarrow \int \frac{-dt}{t}$

$\Rightarrow -\int \frac{1}{t} dt$

$\Rightarrow -\ln(\cos x)$,

$\Rightarrow \therefore \log|\sec x| + C$

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if options given, differ options.

Q1 $\int \sin^3 x \cos^2 x \, dx$.

$\Rightarrow \int \sin x \cdot \sin^2 x \cdot \cos^2 x \, dx$.

$\Rightarrow$ let $\cos x = t$.
$-\sin x \, dx = dt$.

$\Rightarrow \int \cos^2 x (1 - \cos^2 x) - dt$.

$\Rightarrow -\int t^2 (1 - t^2) \, dt$

$\Rightarrow -\int t^2 - t^4 \, dt$

$\Rightarrow -\left[\frac{t^3}{3} - \frac{t^5}{5}\right]$

$-\frac{t^3}{3} + \frac{t^5}{5} + C$ //

$\Rightarrow -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$ //

* Integration by Parts

$$\int f(x) g(x) \, dx = f(x) \int g(x) - \int \left(f'(x) \int g(x) dx\right) dx$$

$\hookrightarrow$ which to choose $f(x), g(x)$

ILATE.

Inverse trigo functn.

$\xrightarrow[f(x)]{}$ priority.

4

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$\overset{\to f(x)}{\text{I L (A) T E}}$.

$\Rightarrow \int x e^x dx \to$ (A)

4Q $\int \frac{\log x}{x^2}$ $\quad$ $\overset{\to f(x)}{\text{ILATE}}$.

$\log x \int \frac{1}{x^2} - \int \frac{1}{x} \int \frac{1}{x^2} dx$.

$-\frac{\log x}{x} - \int \frac{1}{x} \cdot \frac{-1}{x} dx$.

$-\frac{\log x}{x} + \int \frac{1}{x^2} dx$

$\Rightarrow -\frac{1}{x}(\log x + 1) + C$

x^{-2}
$\frac{x^{-1}}{-1}$
$-\frac{1}{x}$

x^{-2}
$\frac{x^{-1}}{-1}$

Q Example 6

Q $\int x \cos x \, dx$. $\quad$ I LATE.

$\hookrightarrow f(x)$

$x \int \cos x - \int 1 \cdot \int \cos x \, dx$.

$\Rightarrow x \sin x - \int \sin x \, dx$.

$\Rightarrow x \sin x + \cos x + C$.

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* example 7

* Integration by Partial fractions

ratio of 2 polynomial

Q $\int \frac{1}{x(x+1)} dx$.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = Ax + A + Bx.$$

$$A+B=0 ; \boxed{A=1, \quad B=-1}$$

$$\int \frac{1}{x} - \frac{1}{x+1} \Rightarrow \log|x| - \log|x+1| + C$$

* example 8

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{x^2}{(x^2+1)(x^2+4)} dx \rightarrow \text{Calc}$$

* ~~example~~

* Intro to Definite integrals

$$\int_a^b f(x)dx = F(b) - F(a)$$

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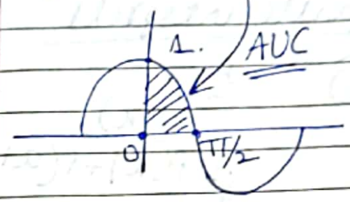
$$\int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} \Rightarrow 1 - 0 = 1$$



Properties :

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ → Calc

2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \forall c \in (a,b)$

3. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

4. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \frac{b-a}{2}$.

5. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & ; \text{ if } f(x) \text{ is even} \\ 0 & ; f(x) \text{ is odd} \end{cases}$

6. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 & , \text{ if} \end{cases}$

Properties → cell

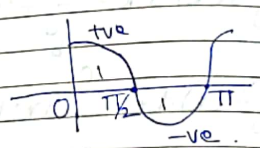
* Example 1 on Definite integrals

cell

Q $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}$ $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}$

$\Rightarrow \frac{b-a}{2}$

Q $\int_0^{\pi} |\cos x|\, dx$.



$\Rightarrow \int_0^{\pi/2} |\cos x| + \int_{\pi/2}^{\pi} |\cos x|\, dx$ $1+1=2$

$\Rightarrow \int_0^{\pi/2} \cos x + \int_{\pi/2}^{\pi} -\cos x\, dx$.   ii) $\int_0^{\pi} \cos x = 0$

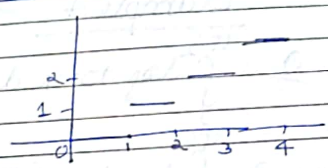
$\Rightarrow [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$

$\Rightarrow (1-0) - (0-1)$

$1+1=2$

Page 8

Q $\int_0^n [x]\, dx$. → Step function



$\int_0^1 0\cdot dx + \int_1^2 1\cdot dx + \int_2^3 2\, dx + \cdots \int_{n-1}^{n} (n-1)\, dx$

$\Rightarrow 0 + 1(2-1) + 2(1) + (n-1)$

$0 + 1 + 2 + \cdots (n-1)$

$\frac{n(n-1)}{2}$ //

* Example 3

Q $\int_0^1 x(1-x)^{99}\, dx$. $\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$

$\int_0^1 (1-x)(x)^{99}$.

$\Rightarrow \int_0^1 x^{99} - x^{100}$

$\Rightarrow \left[\frac{x^{100}}{100} - \frac{x^{101}}{101}\right]_0^1$

$\Rightarrow \frac{1}{100} - \frac{1}{101} \Rightarrow \frac{1}{10100}$ //

Example 4 → Call

Q $\int_0^{\pi/2} \log \tan x \, dx$. —— (I)

$\Rightarrow \int_0^{\pi/2} \log \tan(\pi/2 - x)\, dx$.

$\Rightarrow \int_0^{\pi/2} \log(\cot x)\, dx$ —— I

adding : $2I = \int_0^{\pi/2} \log(1) \Rightarrow \int_0^{\pi/2} 0$.

$\underline{\underline{0}}$ ✓

Example 5 → Call

Q $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

$x = \tan t$. $\quad x \to 0$ to 1 .

$dx = \sec^2 t \, dt$. $\quad t \to 0$ to $\pi/4$

$\Rightarrow \int_0^{\pi/4} \log(1 + \tan t)\, dt = I$ —(i)

$\Rightarrow \int_0^{\pi/4} \log(1 + \tan(\frac{\pi}{4} - t))\, dt =$

$\Rightarrow \int_0^{\pi/4} \log\left[1 + \frac{1 - \tan t}{1 + \tan t}\right] dt = I$

$\Rightarrow \int_0^{\pi/4} \log\left(\frac{2}{1+\tan t}\right) dt = I$ —— (2)

$2I = \int_0^{\pi/4} \log 2$

$2I = \log 2 (t)_0^{\pi/4}$

$2I = \log 2 \left(\frac{\pi}{4}\right)$

$\boxed{I = \frac{\pi \log 2}{8}}$ ✓

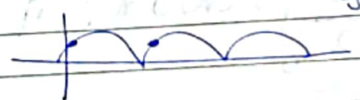
Example 6

$I_1 = \int_0^{\pi} f(\cos^2 x)\, dx$ $\quad I_2 = 3I_1$

$I_2 = \int_0^{3\pi} f(\cos^2 x)\, dx$. $\Rightarrow 3\int_0^{\pi} f(\cos^2 x)$.

$\boxed{\int_0^{na} f(x)\, dx = n \int_0^{a} f(x)\, dx \quad \text{if } f(a+x) = f(x)}$

$f(a+x) = f(x)$.



$\boxed{\cos(\pi + x) = -\cos}$

when $f(x) = f(a+x)$.

$\cos^2 x = \cos^2(\pi + x)$. ✓

$(-\cos x)^2 = \cos^2 x$

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$\sin(\pi - x)$
$\cos(\pi - x) = \cos x$

even $= f(x) = f(-x)$
add $= f(x) = -f(-x)$

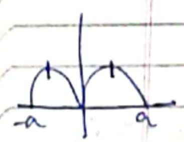
* Example 7 on DI

Q $\displaystyle\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right) dx.$

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & , f(x) \text{ is even} \\ 0 & , f(x) \text{ is add.} \end{cases}$$

$f(-x) = \cos x \log\left(\frac{1-x}{1+x}\right).$

$\Rightarrow \cos x \log\left(\left(\frac{1+x}{1-x}\right)^{-1}\right).$



$\Rightarrow -\cos x \log\left(\frac{1+x}{1-x}\right).$

$\therefore$ add $\therefore$ $\boxed{I=0}$

Q $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ Cell

$$\int_0^a x \cdot f(x)dx = \frac{a}{2}\int_0^a f(x)dx \; ; \text{ if } f(a-x) = f(x)$$

$f(\pi - x) = \dfrac{\sin(\pi - x)}{1+\cos^2(\pi - x)} = \dfrac{\sin x}{1+\cos^2 x}$

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$f(a+x) = f(x)$

$\pi \displaystyle\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$\dfrac{\pi}{2}\displaystyle\int_0^{\pi}$

$-\dfrac{\pi}{2}\displaystyle\int_{1}^{-1} \frac{dt}{1+t^2}$

$-\dfrac{\pi}{2}\left[\tan^{-1} t\right]_{1}^{-1}$

$-\dfrac{\pi}{2}\left[-\dfrac{\pi}{4} - \dfrac{\pi}{4}\right] \Rightarrow -\dfrac{\pi}{2} \times -\dfrac{\pi}{2} \Rightarrow \pi^2/4$

* Example 9 Cell

$\dfrac{n-1}{n} \; \dfrac{n-3}{n-2}$

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$$

Q1 $\displaystyle\int_0^{\pi/2} \sin^8 x\, dx$ $\quad n = 8$ (even).

$\Rightarrow \dfrac{7}{8} \cdot \dfrac{5}{6} \cdot \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2} = \dfrac{35\pi}{256}$

Q2 $\displaystyle\int_0^{\pi/2} \sin^5 x\, dx$ $\quad 5 =$ add.

$\dfrac{n-1}{n}$

$\dfrac{4}{5} \times \dfrac{2}{3} \times 1 = 8/15.$

Q3 $\displaystyle\int_0^{\pi/2} \cos^3 x\, dx$

$\Rightarrow \dfrac{2}{3} \times 1 = 2/3$

Q4 $\displaystyle\int_0^{\pi/2} \cos^6 x\, dx = \dfrac{5}{6} \times \dfrac{3}{4} \times \dfrac{1}{2} \times \dfrac{\pi}{2}$

$= 5\pi/32$

$m, n = +ve$.

* Example 10

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\cdots 2/1 \; (n-1)(n-3)\cdots 2/1}{(m+n)(m+n-2)\cdots 2/1} K$$

$K = \pi/2$, $m \& n$ = even.
$K = 1$, otherwise.

Q.1. $\int_0^{\pi/2} \sin^8 x \cos^3 x \, dx$

$$\Rightarrow \frac{(7\times5\times3\times1)(2)}{11\times9\times7\times5\times3\times1} \times 1 = 2/99$$

Q2 $\int_0^{\pi} \sin^4 x \cdot \cos^3 x \, dx = 0$.

$$\boxed{2\int_0^{\pi/2} \sin^4 x \cdot \cos^3 x \, dx \text{ if } f(x) = \text{even} \quad \text{when } f(\pi - x) = f(x)}$$

$\sin^4(\pi - x) \cdot \cos^3(\pi - x)$.
$\sin^4 x \; (-\cos x)^3 \Rightarrow -ve$
$\therefore$ odd

Q3 $\int_{-2\pi}^{2\pi} \sin^4 x \cos^8 x$ $\quad \boxed{f(-x) = f(x)}$ even

$$\Rightarrow 2\int_0^{2\pi} \sin^4 x \cos^8 x.$$

Cell

$\Rightarrow f(2\pi - x) = f(x)$

$\sin^4(2\pi - x) \cdot \cos^8(2\pi - x)$.

$\Rightarrow \sin^4 x \cdot \cos$

$$\Rightarrow 4\int_0^{\pi} f(x)\,dx.$$

$\underline{f(\pi - x) = f(x)}$. even.

$$\Rightarrow 8\int_0^{\pi/2} f(x)\,dx$$

$\sin^4(\pi - x) = \cos^8(\pi - x)$

$\Rightarrow$

$$8\left[\frac{3\cdot1\cdot7\cdot5\cdot3\cdot1}{12\cdot10\cdot8\cdot6\cdot4\cdot2} \times \pi/2\right] = \frac{7\pi}{25}$$

Q Gate 2009

$$\int_0^{\pi/4} \frac{1 - \tan x}{1 + \tan x}\, dx$$

$\tan[a-b] = \tan-$

$$\Rightarrow \int_0^{\pi/4} \tan\left[\frac{\pi}{4} - x\right]$$

$\pi/4 - x$.

$\tan\left(\frac{\pi}{4} - \frac{\pi}{4} + x\right)$.

$$\Rightarrow \int_0^{\pi/4} \tan x \; = \log|\sec x|_0^{\pi/4}.$$

$\Rightarrow \log(\sec \pi/4) - \log(\sec 0)$
$\Rightarrow \log\sqrt{2} - \log 1$.
$\Rightarrow \log(\sqrt{2})$

$\Rightarrow \frac{1}{2}\log 2$

$\frac{1-\tan x}{1+\tan x} = \tan(\pi/4 - x)$

Q. Gate 2014

ILATE

Cell

$\int_0^{\pi} x^2 \cos x \, dx$

$\Rightarrow x^2 \int_0^{\pi} \cos x - \int_0^{\pi} 2x \int_0^{\pi} \cos x \, dx$

$\Rightarrow x^2 [\sin x]_0^{\pi} - \int 2x [\cancel{\cos x}\ \sin x]_0^{\pi} dx.$

$\Rightarrow x^2[0] - \int 2x$

$\Rightarrow 0 - 2\int_0^{\pi} x \sin x \, dx.$ $\quad -x \sin(\pi - x)$

$\int_0^a f(x)dx = \frac{a}{2}\int_0^a f(x)dx$

-2π

if $f(a-x) = f(x)$.

Q. Gate 2011

$\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x} \Rightarrow \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}}$

$\Rightarrow \int_0^{\pi/2} e^{2ix}$

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$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ Euler's

$\Rightarrow \left[\frac{e^{2ix}}{2i}\right]_0^{\pi/2}$

$[x,$

$\Rightarrow \frac{1}{2i}\left[e^{2ix}\right]_0^{\pi/2}$

$= \frac{1}{2i}\left[e^{i\pi} - e^0\right]$

$= \frac{1}{2i}\left[\cos\pi + i\sin\pi - 1\right]$

$\Rightarrow \frac{1}{2i}\left[-1-1\right] = \frac{-1}{i} \Rightarrow \frac{-1}{i} \Rightarrow \frac{-i}{i^2} = \frac{-i}{i^2}$

$\Rightarrow \frac{i^2}{i} = i$

Q. Gate 2015

$\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2} dx.$

$1/x = t$ $\qquad \frac{-1}{x} = t$

$\frac{-1}{x^2} dx = dt$

$\int_{\pi}^{\pi/2} \cos t(-dt) = \int_{\pi/2}^{\pi} \cos t \, dt$

$\Rightarrow (\sin t)_{\pi/2}^{\pi} = 0 - 1 = -1$

* Introduction

⇒ Sample Space & Events

* Random Experiment outcome not certain (Set of outcomes known)

eg tossing a coin
Rolling a dice
Picking an object from box (without looking)

* For each RE, a Sample Space. Set [Set] of all possible outcomes

$S_1 = \{H, T\}$
$S_2 = \{1, 2, 3, 4, 5, 6\}$
$S_3 = \{HH, HT, TH, TT\}$
$S_4 = 36$ o/ps of (2 dice rolled)

· finding life of bus
$S_5 = \{0, 1, \ldots \infty\}$

* Event: Any subset of SS (Set)

$E_1 = \{H\}$ // getting Head
$E_2 = \{1, 3, 5\}$ // getting odd no
$E_3 = \{HH, HT, TH\}$ // atleast 1 H
$E_4 = \{12, 21\}$ // Sum = 3
$E_5 = \{[3, 6]\}$ // atleast 3 atmost

$E_1 \cup E_6$
⇒ $E_1 \cap E_2 = \phi$ (mutually exclusive Events) [Same SS]

⇒ $\overline{E_1} = S - E_1$ S_2 = dice

Set of odd   Set of even

* Probabilities to events

Sample event, {1}
Compound Event ⟶ {12, 20

eg
· $S = \{H, T\}$ Fair Coin: equally likely to get H or T
$1/2$ $1/2$

· $\{H\} = E$
$P(E) = 1/2$ $P(E_2) = 1/2$ (assign no to event)

⟶ $S = \{HH, HT, TH, TT\}$
$1/4$

* For each event E, in S we assign a no. P(E) such that,
→ $0 \le P(E) \le 1$
→ $P(S) = 1$ $\sum P(E) = 1$
→ for any seq of events $E_1 \ldots$ that are mutu exclusive

then $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$

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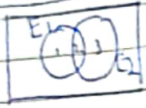
$E_1 = \{HH\}$, $P = 1/4$

$E_2 = \{TT\}$, $P = 1/4$.

$\therefore E_3 = E_1 \cup E_2 = 1/4 + 1/4 = \underline{1/2}$

(not mutually excl.



$E_1 \cup E_2 = (1,2) \cup (2,3)$ — 2.

$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

eg. $E_1 = \{HH, HT\}$ // 1st coin is head

$E_2 = \{HH, TH\}$ // 2nd coin is head

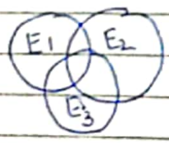
$E_3 = \{HT, TH, HH\}$ // atleast 1 head.

$E_1 \cup E_2$.

$P(E_3) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \underline{\underline{3/4}}$

* 

- $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

* $P(S) = 1$ // certain than one in SS comes as outcome.

* E, E^c are mutually exclusive.

$S = E \cup E^c$.

$\boxed{P(S) = P(E) + P(E^c) = 1}$

eg: Prob of pass = 0.6

Prob of fail = 0.4.

* **Random Variables**

eg $S = \{HH, HT, TH, TT\}$ Tossing 2 coins.

$X \rightarrow$ count # times H turns up. ↖ RV

(functn of the o/p).

X: 0 1 2.

Prob distrib of RV: $P(X)$: 1/4 1/2 1/4 → $P(X=2)$ [HH].

$P(X=0)$ $P(X=1)$

eg: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. 3 coins (eq. likely) 1/8.

$X \longrightarrow$ count # tails.

X:	0	1	2	3
$P(X)$:	1/8	3/8	3/8	1/8

$P(X=0)$ $P(X=3)$

→ $X(HHT) = 1$.

X = functn which take SS as i/p & o/p a number $= \{0\ 1\ 2\ 3\}$

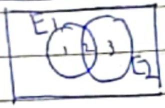
→ $\sum P(X) = 1$

$E_1 = \{HH\}$, $P = 1/4$

$E_2 = \{TT\}$, $P = 1/4$.

$E_3 = E_1 \cup E_2 = 1/4 + 1/4 = \underline{\underline{1/2}}$.

(not mutually excl.



$E_1 \cup E_2 = (1,2) \cup (2,3)$ — 2.

$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

eg. $E_1 = \{HH, HT\}$ //1st coin is head

$E_2 = \{HH, TH\}$ //2nd coin is head

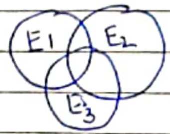
$E_3 = \{HT, TH, HH\}$ //atleast 1 head.

$E_1 \cup E_2$.

$P(E_3) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \underline{3/4}$

* 

- $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

* $P(S) = 1$ //certain than one in SS comes as outcome.

* E, E^C are mutually exclusive.

$S = E \cup E^C$.

$\boxed{P(S) = P(E) + P(E^C) = 1}$

eg: Prob of pass = 0.6
Prob of fail = 0.4.

* Random Variables

eg $S = \{HH, HT, TH, TT\}$ Tossing 2 coins.

$X \rightarrow$ count # times H turns up. ← RV

(functn of the o/p).

X:	0	1	2
Prob distrib of RV: $P(X)$:	1/4	1/2	1/4

$P(X=0)$, $P(X=1)$, $P(X=2)$ [HH].

eg: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. 3 coins (eq. likely) 1/8.

$X \longrightarrow$ count # tails.

X:	0	1	2	3
$P(X)$:	1/8	3/8	3/8	1/8

$P(X=0)$, $P(X=3)$

→ $X(HHT) = 1$.

X = functn which take SS as i/p & o/p a number = {0 1 2 3}

→ $\sum P(X) = \underline{1}$

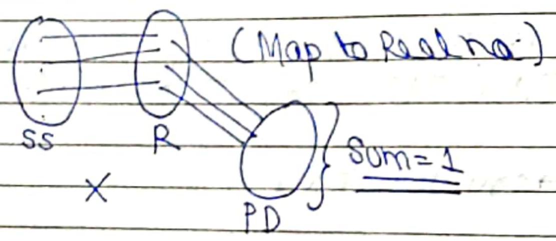
$Z(11) = 2$
$Z(66) = 12$

eg: $S = \{11, 12, \dots 66\}$, 1/36

Z: Sum of values on 2 dice.

Z:	2	3	4	5	6	7	8	9	10	11	12
prob distribn → P(Z):	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

* RV → discrete RV → RV values are pts [finite, countable set].
→ continuous RV → RV gives range.
X: lifetime of car
[20, 80].
(∞ points in range)



⇒ X = count # heads.
$X = \{0, 1, 2\}$
P(X) 1/4 2/4 1/4 $\Sigma 1$ ✓

Q flip coin until we get 1st Head
N = # times to flip a coin until getting a head.

		TH	FTH		
N:	1	2	3	4	n
P(N)	p	$(1-p)p$	$(1-p)^2p$	$(1-p)^3p$	$(1-p)^{n-1} \cdot p$

P(N=1)
∵ indep events ∴ multiply.

$P(H) = p,\ P(T) = 1-p$.

$\Sigma P(n) = p + (1-p)p + (1-p)^2 p + \dots (1-p)^{n-1} p$

$\Rightarrow p(1 + (1-p) + (1-p)^2 + \dots \infty)$.

$\Rightarrow p\left[\frac{1}{1-1+p}\right] = 1$. Ans

* **Bernoulli Random Variable**

. 2 values = X = 0, 1.

$SS = \{1, 2, 3, 4, 5, 6\}$ (1 → Success, rest → Failure)

Pick 1, & 1 comes in dice = Success.

$P(S) = 1/6,\ P(F) = 5/6$.

$P(\underbrace{X=1}_{\text{Success}}) = 1/6,\ P(\underbrace{X=0}_{\text{Failure}}) = 5/6$.

Bernoulli RV

In exp. whose outcome can be classified as either a success or a failure, let X = 1 if the outcome is success & X = 0, if it is a failure, the prob mass functn of X:

$P(X=0) = 1-p$

$P(X=1) = p$; p: prob of success

$X \to$ Bernoulli RV

$X \to 0, 1$ } discrete RV.

Q) 2 coins

{HH, HT, TH, TT}

if 2 heads = win / loss

$P(X=1) = 1/4$: ~~fortune~~ success

$P(X=0) = 3/4$: failure.

Q {H, T}

Head = success

$P(X=1) = 1/2$

$P(X=0) = 1/2$.

⇒ divide SS into success & failure

* Binomial RV

- Collection of Bernoulli exp

Q if H: Success.

flip n coins. Out of n coins, some will be success. X counts the # success.

n = 4, Getting 1 success ? $P(H)=1/2$, $P(T)=1/2$

HTTT + THTT + TTHT + TTTH

$$\Rightarrow \boxed{\frac{4!}{3!} \cdot p(1-p)^3} \to P(X=1)$$

$$P(X=2) \longrightarrow \boxed{\frac{4!}{2!2!} p^2(1-p)^2}$$

* if we conduct n exp, if every exp. there can be S/F independently,

X = if we try the count # of success.

→ Defn → coll (n, p)

Prob Mass functn of $X \to$ discrete RV.

$$\boxed{P(X=r) = {}^nC_r \, p^r q^{n-r} \; ; \; r = 1, 0 \cdots n}$$

$$\Rightarrow \sum_{r=0}^{n} {}^nC_r \, p^r q^{n-r} \approx 1$$

* Binomial Random Variable (eg)

$p = 0.1$

a) exactly one is defective, in SS of 3.

$P(X=1) = {}^nC_1 \, p^1 q^{n-1}$

$\Rightarrow {}^3C_1 (0.1)(0.9)^2$

b) at most 1 defective

$P(X \leq 1) = P(X=0) + P(X=1)$

$= {}^3C_0 (0.1)^0 (0.9)^3 + {}^3C_1 (0.1)(0.9)^2$

(c) atleast 1

$P(X \geq 1) = 1 - P(X=0)$

$\Rightarrow 1 - {}^3C_0 (0.1)^0 (0.9)^3$

* Poisson Random Variable

X taking on one of the values 0, 1, 2 .. n is said to be Poisson rv, with parameter λ,

$$P(r) = \boxed{P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}}$$

$r = 0, 1, 2 \cdots \infty$

$$\sum_{r=0}^{\infty} P(X=r) = 1$$

Note: Poisson RV is used to approximate Binomial RV when $n \uparrow, p \downarrow$

$\lambda = np \rightarrow$ Poisson instead of Binomial with λ

* example: at least 1 error. $[\lambda = 1]$

$P(r \geq 1) = 1 - P(r=0)$

$\Rightarrow 1 - \frac{e^{-\lambda} \cdot \lambda^r}{r!}$

$1 - \frac{e^{-\lambda} \cdot \cancel{\lambda^0}}{} = 1 - e^{-\lambda} = 1 - 1/e$

$\because P(X=0) + P(X=1) + P(X=2) + \cdots \infty = 1$

* $P(X=0) = e^{-1} = 1/e \leftarrow$ Prob that no errors

eg2 Poisson Random V.

• $\lambda = 3$

$\rightarrow P(r=0) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}$

$\Rightarrow \frac{e^{-3} \cdot 3^0}{0!} = 1/e^3$ Ans

$\rightarrow P(r=1) = \frac{e^{-\lambda} \cdot \lambda}{1!} \Rightarrow e^{-3} \cdot 3 = 3/e^3$

$\rightarrow P(r \geq 1) = 1 - P(r=0)$ [Atleast 1 accident]

$= 1 - 1/e^3$

* Continuous Random Variable

$\Rightarrow$ X: denote sum of 2 dice.

X = 1 2 3 4 . . . 12. [discrete Values]

- countable no. of values.
- Countable Set.

$\rightarrow$ X: years of material

X = [1850 – 1865]

$\rightarrow$ take a continuous range (uncountable set).

$\rightarrow$ X: lifetime of a car

X = [1, 20]

interested in Range

$P \in$ [range]. ✓

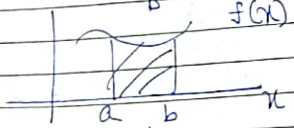
$P \in$ [point] ✗

* X take uncountable set of values & there exits a non negative funct^n $f(x)$ called Prob density funct^n having the property that for any set 'B' of real numbers

$$P(x \in B) = \int_B f(x)\,dx$$

→ Prob density functn

→ AUC



→ $P(x \in [a,b])$

Q $X : [a, b]$

- $P(a \le X \le b) = \int_a^b f(x)\,dx$ → Pdf.
- $P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$ → AUC $[-\infty, a]$

$\therefore P(X = a) = \int_a^a f(x)\,dx = 0$ ← (Prob of rv at a point)

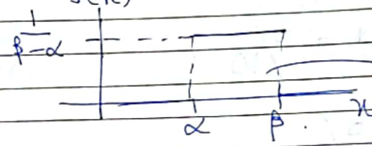
But, discrete RV, Prob(point) $\neq 0$ [may not be 0]

$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$ ⇒ Prob = 1

* Uniform Random Variable (CRV)

- X is a URV on the interval (α, β) if its PDF is given by:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \text{if } \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$



f(x) → $A = \frac{1}{\beta - \alpha} \cdot \beta - \alpha = 1$

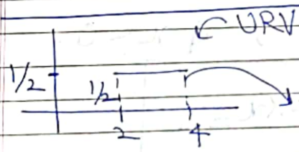
* $\int_{-\infty}^{\infty} f(x)\,dx = 0 + \int_{\alpha}^{\beta} f(x)\,dx + 0$

$\Rightarrow \therefore \boxed{AUC = 1}$ $\quad \underline{\underline{\Sigma P = 1}}$

or, $\int_{\alpha}^{\beta} \frac{1}{\beta - \alpha}\,dx = 0$

$\frac{1}{\beta - \alpha} \cdot [x]_{\alpha}^{\beta} = \underline{\underline{1}}$

*



$\frac{1}{2}$ → uniform in this range.

$$f(x) = \begin{cases} 1/2, & [2, 4] \\ 0, & \text{otherwise} \end{cases}$$

X = cont. URV.

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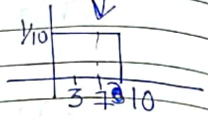
* Eg on Uniform Random Variable

Q range $(0,10) \rightarrow f(x) = \begin{cases} \frac{1}{10} & ; [0,10] \\ 0 & , \text{otherwise} \end{cases}$



a) $X<3$.

$P(x<3) = \int_{-\infty}^{3} f(x)\,dx \Rightarrow 0$.

$= \int_{0}^{3} f(x)\,dx = 3/10$

Vimp

b) $x>7. = 3/10$

c) $1<x<6$.

$\Rightarrow 5 \times \frac{1}{10} = 1/2$.

* Exponential Random Variable

A CRV whose pdf, for $\lambda>0$

$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & , \text{if } x \geq 0 \quad [0,\infty) \\ 0 & , \text{if } x<0. \end{cases}$

Q $\int_{-\infty}^{\infty} f(x)\,dx = \boxed{\int_{-\infty}^{\infty} \lambda e^{-\lambda x} = 1}$

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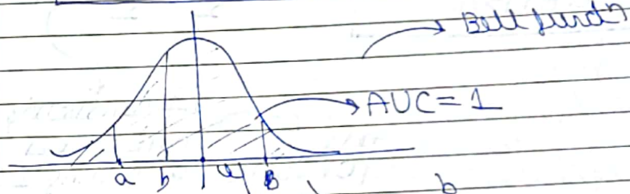
Q $P \in [a,b]$.

$\therefore P(a \leq X \leq b) = \int_{a}^{b} \lambda \cdot e^{-\lambda x}\,dx$.

* Normal RV

X is normally dist with μ, σ^2

$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}}$



$\int_{a}^{b} f(x)\,dx$.

$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

- $P(x \leq a) = \int_{-\infty}^{a} f(x)\,dx$
- $P(x \geq a) = \int_{a}^{\infty} f(x)\,dx$

* Expectation → mean
→ to guess which value would come out (most prob.)

X = getting #H.

X	0	1	2
P(X)	1/4	1/2	1/4

$\sum X \cdot P(X)$.

$\boxed{E(X) = \sum_{i: P(i)>0} i\, P(X=i)} = 0 + \frac{1}{2} + \frac{1}{2} \Rightarrow 1$.

b4 conducting exp. more chances is of getting 1 (HT/TH)
true (for some cases)

Q X: no. when roll a dice

X	1	2	3	4	5	6
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

$E(X) = \frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\cdot\frac{1}{6}$

$\Rightarrow \frac{6\times 7}{2} \times \frac{1}{6} \Rightarrow 3.5$

~~3/4 will have High Prob~~

∴ Expectation term not correct

* E(X) of Bernoulli RV (Discrete RV).

parameter: p → P(X=1) [Success].

X	0	1
P(X)	1−p	p

→ $E(X) = 0 + p = p$.
→ Exp. #success in 1 trial.

1 trial —— p succe.
1/p trial —— 1 succe.

∴ $\boxed{1 \text{ success} — 1/p \text{ trials}}$ exp #

Q Prob of success = 0.2, how many trials are required to get 1 success? [in Bernoulli].
→ 5 trials.

* E(X) of Binomial RV

n Trials, how many success is counted by X.

$P(X=r) = {}^nC_r\, p^r q^{n-r}$ $\quad q = 1-p$

$r \in [0, n]$.

→ $E(X) = \sum_{i=0}^{n} i \cdot P(X=i)$

$$\Rightarrow \sum_{i=0}^{n} i \cdot {}^nC_i \, p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} \frac{i \cdot n!}{(n-i)!\, i!} p^i (1-p)^{n-i}$$

$$\Rightarrow \frac{n!}{(n-i)!\,(i-1)!} p^i (1-p)^{n-i}$$

$$np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} p^{i-1} (1-p)^{n-i}$$

$$\underline{i-1=K}$$

$$\Rightarrow np \sum_{K=0}^{n-1} \underbrace{\frac{(n-1)!}{(n-1-K)!\,K!}}_{{}^{n-1}C_K} p^K (1-p)^{n-1-K}$$

$$\Rightarrow \underline{np}$$

if Prob of single Success is p, & n trials, Prob of then #Success = np with high Prob

Q n=10, P= 0.6
Expected #success ? $\Rightarrow \underline{\underline{6}}$

$\sum i \cdot P(X=i)$

* Poisson random Variable

λ.

$$f(x) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}$$

$$E(X) = \sum_{i=0}^{\infty} i \left(\frac{e^{-\lambda} \cdot \lambda^i}{i!} \right)$$

$$\Rightarrow \sum_{i=1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^i}{(i-1)!}$$

$$\Rightarrow \lambda \cdot e^{-\lambda} \cdot \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$$

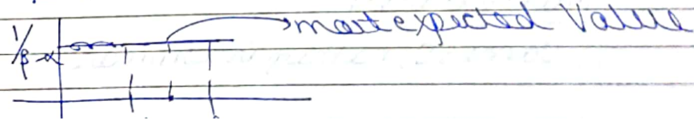
$$\Rightarrow \lambda \cdot e^{-\lambda} \cdot \sum_{K=0}^{\infty} \frac{\lambda^K}{K!} \qquad K=i-1$$

$$\boxed{E(X) = \lambda}$$

* Expectⁿ of Continuous RV

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$$

* Exp of Uniform RV



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$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha} \cdot dx$$

$$= \frac{1}{\beta-\alpha} \cdot \frac{(\beta^2-\alpha^2)}{2}$$

$$\Rightarrow \frac{\alpha+\beta}{2} \cdot \text{Ans}$$

* Expectn of Exponential RV

$$E(X) = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx = 1/\lambda$$

* E(X) of Normal RV (μ, σ^2)

$$\boxed{E(X) = \mu}$$

* Independent Events → if an event has already occured it won't change the prob of next event

Fair Coin: $P(H) = P(T) = 1/2$.

HH

$P(H_1 \cap H_2) = P(H_1) \times P(H_2/H_1)$.

H_1: Head in 1st flip

$= \frac{1}{2} \cdot \frac{1}{2} = 1/4$.

· Mut. ex. v/s indep.

sameness, nothing in common.

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→ 2nd outcome is independent of 1st outc.

* (A, B) are 2 events

$$\left.\boxed{\begin{array}{l} P(A/B) = P(A) \\ P(B/A) = P(B) \end{array}}\right\} \therefore A \,\&\, B \text{ are independents.}$$

$P(B/A)$ → Prob of B given A happend

→ $P(A \cap B) = P(A) \cdot P(B/A) = P(A) \cdot P(B)$.

$\boxed{P(A \cap B) = P(A) \cdot P(B)}$ (only when A, B are ind. events).

$\boxed{P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)}$

* If any 2 events: $P(A \cap B) = P(A) \cdot P(B)$, then we can say: these 2 events are independent & converse.

* if n events: $E_1, E_2, E_3 \cdots E_n$ for any subset it shud hold: → independent.

$$P(E_i \cap E_j \cdots E_l) = P(E_i) \cdot P(E_j) \cdots P(E_l).$$

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independent events.

Do exp with replacement

eg1

3G, 2R

Q. $P(G_1 \cap R_2)$ — Red ball in Second trial

$\underbrace{\quad}_{\text{independent events}}$

$\Rightarrow P(G_1) \times P(R_2)$

$\Rightarrow \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$

Q. $P(G_1 \cap G_2 \cap G_3)$ — G ball in 1st T & G ball in 2nd T..

$\rightarrow \left(\frac{3}{5}\right)^3$

Q.

A	B	C
3R 2G	3R 5G	1R 4G

$P(G_A R_B G_C)$ → 1 ball from each bag. → G ball from C.

$\Rightarrow P(G_A \cap R_B \cap G_C)$: Ind. events.

$\Rightarrow \frac{2}{5} \times \frac{3}{8} \times \frac{4}{5}$

Q. We draw 1 ball from each bag.
$P(2 \text{ Green}, 1 \text{ Red})$.

A	B	C	
R	G	G	$= \frac{3}{5}\cdot\frac{5}{8}\cdot\frac{4}{5} +$
G	R	G	$= \frac{2}{5}\cdot\frac{3}{8}\cdot\frac{4}{5} +$
G	G	R	$= \frac{2}{5}\cdot\frac{5}{8}\cdot\frac{1}{5}$

3 mutually exclusive events

even 1 No: you are rejected.

eg2: convince 3 people to marry a girl.

$P(\text{Dad}) = 1/2$ [All think Ind.]

$P(\text{Mom}) = 1/3$

$P(\text{Brother}) = 1/4$.

$P(\text{Marry}) = P(D \cap M \cap B)$. YYY.

$= P(D) \cdot P(M) \cdot P(B)$

$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{24}$.

eg3: A & B are independent.

$P(A \cap B) = P(A) \cdot P(B)$

* if A, B are independent, then so all:

for any # events

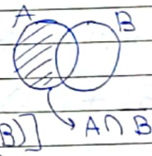
(i) $A, \bar{B}$

(ii) $B, \bar{A}$

(iii) $\bar{A}, \bar{B}$.

} all are independent.

4. $P(A \cap \bar{B})$



$\Rightarrow P(A) - P(A \cap B)$

$\Rightarrow P(A) - [P(A) \cdot P(B)]$

$\therefore P(A)[1 - P(B)]$

$\therefore P(A) \cdot P(\bar{B})$

4. $P(B \cap \bar{A}) = P(B) - P(A \cap B)$

$= P(\bar{A}) \cdot P(B)$.

4. $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$\Rightarrow 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [P(A) + P(B) - P(A) \cdot P(B)]$.

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$\Rightarrow 1-P(A)-P(B)[1-P(A)]$

$= [1-P(A)][1-P(B)]$

$= P(\bar{A})\cdot P(\bar{B})$

* if $E_1, E_2 \cdots E_n$ are indep., $\bar{E_1}, \bar{E_2}, \bar{E_3}$ are also independent.

eg 4. InterView: 2 contest.

$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{3}$ (Getting Job).

Selⁿ of A is independent with selⁿ of B.

Q What is the Prob: both get selected?

$P(A \cap B) = \frac{1}{6}$

Q exactly one get selected.

$\frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

$P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$P(A)\cdot P(\bar{B}) + P(\bar{A})\cdot P(B)$ ∵ independ.

Q Prob that none get selected.

$P(\bar{A} \cap \bar{B}) = P(\bar{A})\cdot P(\bar{B}) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$

Q atleast 1 get selected.

$1 - P(\bar{A} \cap \bar{B}) = \frac{2}{3}$

Q atmost 1 selected.

$\frac{1}{3} + \frac{1}{2}$

1 Key is right.

ever if tried

eg 5 lock : 8 Keys. (Let Key stay)

Prob that open lock in 7th key?

independent event. [Prob remains same]

$P(\bar{L_1} \cap \bar{L_2} \cap \bar{L_3} \cap \bar{L_4} \cap \bar{L_5} \cap \bar{L_6} \cap L_7)$

$\Rightarrow P(\bar{L_1})\cdot P(\bar{L_2})\cdot P(\bar{L_3}) \cdots P(L_7)$

$\Rightarrow \left(\frac{7}{8}\right)^6 \cdot \frac{1}{8}$ Ans.

* $P(A \cap B) =$

$P(A) =$ $P(B) =$

Verify: $P(A \cap B) = P(A) \times P(B)$ ⟶ independ.

eg: $S = \{HH, HT, TH, TT\}$ ← flip 2 coins.

E_1 = Getting Head on 1st Coin. $\{HH, HT\}$

E_2 = Getting Head on 2nd coin. $\{TH, HH\}$

$P(E_1) = \frac{1}{2}$

$P(E_2) = \frac{1}{2}$

$P(E_1 \cap E_2) = \frac{1}{4}$

~~Got Head~~ HH.

$P(E_1 \cap E_2) = P(E_1) * P(E_2)$

∴ E_1, E_2 are independent. ∵ Fair coin

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eg 7: Derive if events are indep.

X: Rolling 2 dice.

SS = { (1,1), (1 2) (1 3 .... (1 6)
(2 1) (2 2) 2 3 ..... 2 6
31 32 33 ..... 3 6

61 62 63 66 }

- E_1 = Getting a 4 on 1st dice = $6/36 = 1/6$

$\boxed{P(E_1) = 1/6}$

- E_2 = Sum of two dice is '6'
: {15, 51, 24, 42, 33}

$\boxed{P(E_2) = 5/36}$ $\frac{5}{36} \cdot \frac{1}{6}$.

$P(E_1 \cap E_2) = \underline{1/36}$
{42}

$\boxed{P(E_1 \cap E_2) \neq P(E_1) \cdot P(E_2)}$

∴ $E_1 \cap E_2$ are dependent.

Assm!! if dice 1 = 6, dice 2 X. [no chance to get 6].

Sum = 6, depends on 1st dice.

- E_3 = Sum on the dice is 7.
(16) (61) (25) (52) (34 43)

$\boxed{P(E_3) = 1/6}$

$P(E_1 \cap E_3) = \underline{1/36}$.
(43)

∴ E_1, E_3 are independ events

∵ whatever happens on 1st dice, the chance is there, to get 7.

eg 8: 4 balls numbered 1 2 3 4
E = {1 2} F = {1 3} G = {1 4}
Is E, F, G are independent? $^3C_2 + {}^3C_3$

$P(E \cap F) = P(E) \cdot P(F)$ ✓

$P(F \cap G) = P(F) \cdot P(G)$ ✓

$P(E \cap G) = P(E) \cdot P(G)$ ✓

$P(E \cap F \cap G) = P(E) \cdot P(F) \cdot P(G) = 1/8$ ✗

(1) = 1/4

∴ ${}^nC_2 + {}^nC_3 + {}^nC_4 \dots {}^nC_n$.

∴ $\sum_{i=2}^{n} {}^nC_i$

- $P(1) = P(2) = P(3) = P(4) = 1/4$.

$P(E) = 1/2$
$P(F) = 1/2$
$P(G) = 1/2$

→ $P(E \cap F) = P(E) \cdot P(F) = 1/4$ ✓
{1}

∴ E, F, G are not independent

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eg 9: 50 G, 50 D — 50 Good Bulbs are ? (replacement). 2 bulbs taken with rep.

$SS = \{GG, GB, BG, BB\}$ (all equally likely)

$\frac{1}{2}\cdot\frac{1}{2}$, 1/4, 1/4, 1/4.

- A = event such that 1st B is G
  $\{GG, GB\}$
  $P(A) = 1/2$
- B = 2nd B is Good
  $\{GG, BG\}$
  $P(B) = 1/2$
- $C = \{GG, BB\}$
  $P(C) = 1/2$

Q Are A, B, C independent ?

$P(A\cap B) = P(A)\cdot P(B)$ ✓ (1/4)
$P(B\cap C) = P(B)\cdot P(C)$ ✓
$P(A\cap C) = P(A)\cdot P(C)$ ✓
$P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)$ ✗
1/4 ≠ 1/8

∴ A, B, C are not independent events (But Pairwise independent).

eg 10: Flip fair coin 5 times (independent)
Prob no 2H/2T consecutively.

$\{\underbrace{HTHTH}_{1/32}, \underbrace{THTHT}_{1/32}\} = 1/16$.

Q Pass — P 0.5, M 0.6, C 0.7. (Independent).

→ $\underbrace{P(P\cap M\cap C)}_{\text{All Sub Passed compulsory}} = 0.5\times0.6\times0.7$.

→ Prob that he fails exactly 1.
$0.5\times0.6\times0.7 + 0.5\times0.4\times0.7 + 0.5\times0.6\times0.3$

→ fails in 3 courses $= \underbrace{P(\bar{P}\cap\bar{M}\cap\bar{C})}_{\text{also indep}}$.

eg 11: if A, B, C are 3 independent events
Pairwise indep +
$P(A\cap B\cap C) = P(A)P(B)P(C)$.

Prove that: A and $(B\cup C)$, B and $(A\cup C)$, C and $(A\cup B)$ are indep.

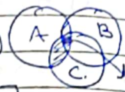
$\rightarrow P(A \cap (B \cup C)) = P(A) \cdot P(B \cup C)$

$= P((A \cap B) \cup (A \cap C))$

$= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$

$= P(A) \cdot P(B) + P(A) P(C) - P(A \cap B \cap C)$

$\Rightarrow P(A) \cdot P(B) + P(A) P(C) - P(A) P(B) P(C)$



$\rightarrow A \cap B \cap C$

$\Rightarrow P(A) [P(B) + P(C) - P(B) P(C)]$

$= \underline{\underline{P(A) \cdot P(B \cup C)}}$. Hence proved.

---

eg 12: $P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 1/4$
all these are indep.?

a) $(A \cup \bar{B}) \cap C$ $\leftarrow (A \cup \bar{B}), C$ are ind.
b) $P(\bar{A} \cup B \cup \bar{C})$
c) $P(\bar{A} \cap \bar{B} \cap \bar{C})$
d) $(\bar{A} \cup \bar{B}) \cap C$.

all are independent

a) $P(A \cup \bar{B}) \cdot P(C)$
$\Rightarrow [P(A) + P(\bar{B}) - P(A) P(\bar{B})] \cdot P(C)$
$= \left(\frac{1}{2} + \frac{2}{3} - \frac{1}{3}\right) \cdot \frac{1}{4}$

b) $P(\bar{A}) + P(\bar{B}) + P(\bar{C}) - P(\bar{A} \cap B) - P(B \cap \bar{C})$
$- P(\bar{A} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C})$

c) Same as b
d) Same as a.

---

* A & B are mutually exclusive events.

$\underbrace{P(A \cap B)}_{\phi \; = \; 0} = \underbrace{P(A) \cdot P(B)}_{\neq 0}$ $\therefore$ Not Independent.

$\boxed{A = SS}$

* $P(A) = 1$ [certain event]
A is independent of any event B.

$\underbrace{P(A \cap B)}_{SS \cap B} = \underbrace{P(A)}_{1} \cdot P(B)$. [Proved].

$P(B)$.

$\rightarrow$ If flip a coin, what is Prob of getting H/T?

* $P(A) = 0$ [imposs. event].
A is independent of any event

$\underbrace{P(A \cap B)}_{\phi \; = \; 0} = \underbrace{P(A)}_{0} \cdot P(B)$ ✓

* Total Probability

Q.

A	B	C
2G	1G	3G
3R	4R	2R

Choose a box & then a ball pick. What is Prob that the ball is Green.

Prob of ball is Green depends on the box chosen.

$P(A) = P(B) = P(C) = 1/3$

$P(G/A) = 2/5$

$P(G/B) = 1/5$

$P(G/C) = 3/5$

$P(G) = ?$ ← (Total Prob).

→ SS: (AG), (AR)
(BG), (BR)
(CG), (CR)

3 mutually exclusive events

E1 E2 E3

A is spread across E_1, E_2, E_3.

P(A) solved used total prob theorem

$A = (E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A)$ → mut. exclus

$P(A) = P((E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A))$

E_1: prob of picking A.
A: prob of getting Green Ball.

$\Rightarrow P(A) = P(E_1 \cap A) + P(E_2 \cap A) + P(E_3 \cap A).$

Total Prob Theorem (dep)

$\boxed{P(G) = P(A \cap G) + P(B \cap G) + P(C \cap G)}$

$\Rightarrow P(A) \cdot P(G/A) + P(B) \cdot P(G/B) + P(C) \cdot P(G/C)$

$\Rightarrow \frac{1}{3} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{3}{5}$

(Total Probability.

∵ A & G are dependent

* Rolling 2 dice

→ 6 m-ex events.

$$SS = \begin{bmatrix} 11, & 12 & \cdots & 16 \\ 21, & 22 & \cdots & 26 \\ \vdots & & & \\ 61, & 62 & \cdots & 66 \end{bmatrix} \; 36$$

Prob that second ~~ball~~ dice is 2

⇒ divide SS into mutually exclusive events.
(what happens on 1st die)
↓
6 Parts.

Total prob (indepen)

$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + P(E_3 \cap A) + \cdots P(E_6 \cap A)$

$\underbrace{\text{1 on 1st}}_{\text{independent events}}$

$= P(E_1) \cdot P(A) + P(E_2) \cdot P(A) + P(E_3) \cdot P(A) + \cdots$

$\Rightarrow \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \cdots \cdots \frac{1}{6} \cdot \frac{1}{6}$

$= 1/6$

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$P(G_1 \cap R_2) + P(R_1 \cap R_2)$.

eg 1: | 2G 3R | A | 3R 2G | B

draw a ball from A, add to B, & then draw from B. what is the prob that ball is red

SS | R_1R_2 (E_1) | G_1R_2 (E_2) → A
| R_1G_2 | G_1G_2 |

$P(A) = P(E_1 \cap A) + P(E_2 \cap A)$. (dependent event)

$P(A) \Rightarrow P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)$

$\Downarrow$

$\Rightarrow$ $\boxed{P(R_2) = P(G_1) \cdot P(R_2/G_1) + P(R_1) \cdot P(R_2/R_1)}$

Total Prob of getting Red ball $\Rightarrow \frac{2}{5} \times \frac{3}{6} + \frac{3}{5} \times \frac{4}{6}$

3R 3G | 4R 2G

eg2. | O O R 3Balls |

- 1 ball definitely red.
- Pick a ball

What is the prob that it is red.

→ 1) (R) R R 2) (R) N R (3) (R) N N

E_1 E_2 E_3

N = Non Red

3 mutually exclusive ...

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E_1 | E_2 | E_3

getting red ball given E_2

$TP = P(E_1) \cdot P(R/E_1) + P(E_2) \cdot P(R/E_2) + P(E_3) \cdot P(R/E_3)$

$TP \Rightarrow \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3}$

dependent events.

eg3. | 4 Fair coins | 3 UF coins |

We take a coin & flip it. P(Head)?

$P(H) = P(T) = 1/2$

$P(H) = 1/3$
$P(T) = 2/3$

P(H) depend on the E happended like t

SS | FH (E_1) | UH (E_2) |
| FT | UT |

$TP = P(F) \cdot P(H/F) + P(U) \cdot P(H/U)$

$\Rightarrow \frac{4}{7} \cdot \frac{1}{2} + \frac{3}{7} \cdot \frac{1}{3}$

Prob of getting 2 Heads?

$\Rightarrow$

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eg4: 2 coins

SS = FF UF FU UU

FF	UU	F&U
		HH

$\rightarrow \dfrac{12\ SS}{4\times 3}$

$P(HH) = \underbrace{P(FF \cap HH)}_{\text{dependent}} + P(UU \cap HH) + P(FU \cap HH)$

$\Rightarrow P(FF)\cdot P(HH/FF) + P(UU)\cdot P(HH/UU) + P(FU)\cdot P(HH/FU)$

$\Rightarrow \cancel{\frac{4}{7}\frac{3}{6}}$

Total Prob of getting 2 heads $\Rightarrow \left(\dfrac{{}^4C_2}{{}^7C_2}\right)\left(\dfrac{1}{2}\cdot\dfrac{1}{2}\right) + \left(\dfrac{{}^3C_2}{{}^7C_2}\right)\left(\dfrac{1}{3}\cdot\dfrac{1}{3}\right) + \left(\dfrac{{}^4C_1\cdot{}^3C_1}{{}^7C_2}\right)\cdot\left[\dfrac{1}{2}\cdot\dfrac{1}{3}\right]$

eg5: Answer / Guess

p: prob that she knows the answer

guess answer $\longrightarrow$ correct: prob: $1/m$.

KC	GC / GW

Prob that she answers it correct.

$P(C) = P(K\cap C) + P(G \cap C)$

$= \underset{p}{P(K)}\cdot \underset{1}{P(C/K)} + \underset{1-p}{P(G)}\cdot \underset{1/m}{P(C/G)}$

$\Rightarrow p + (1-p)\dfrac{1}{m}$

$1-p$: $1/m$, $1-1/m$

eg 6: A: reached F

B C Semi $\rightarrow$ B will win

$P(B) = 1/3$, $P(C) = 2/3$

if fight $A \wedge B \longrightarrow P(A) = 2/3$

$A\ C \longrightarrow P(A) = 1/4$

What is the prob that A wins?

	c_1	c_2
SS:	BA	CA
	BB	CC

$\rightarrow$ 1st match: C win, 2nd m: A wins.

$P(A) = \underbrace{P(B\cap A) + P(C \cap A)}_{\text{indepen}}$

$= P(B)\cdot P(A/B) + P(C)\cdot P(A/C)$

$P(A) \Rightarrow 1/3 \cdot 2/3 + 2/3 \cdot 1/4$

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* Baye's Theorem

2R 3G	1R 4G	3R 2G
A	B	C

a) ~~Q. Total Prob of getting Red ball~~ P(R)

b) ~~Q. P(R/A)=~~

a) Prob. of getting red ball given that A is chosen?

$$P(R/A) = 2/5$$

b) What is the probability of getting a red ball from bag A. (imp)

Bag A chosen then Red ball chosen. $P(A \cap R) = P(A) \cdot P(R/A)$.

c) What is the Probability of getting a red ball? $P(R)$

$$TP = P(A \cap R) + P(B \cap R) + P(C \cap R)$$

d) What is the conditional Probability that Bag A is chosen given that Red ball is drawn.

$$P(A/R)$$

Reverse Prob

[Predict the Cause]

known info
$P(A) = P(B) = P(C) = 1/3$
$P(R/A) = 2/5$
$P(R/B) = 1/5$
$P(R/C) = 3/5$

$$P(A/R) = \frac{P(A \cap R)}{\underbrace{P(A \cap R) + P(B \cap R) + P(C \cap R)}_{\text{3 sources for event R to occur.}}}$$

$$\boxed{P(A/R) = \frac{P(A) \cdot P(R/A)}{P(A) \cdot P(R/A) + P(B) \cdot P(R/B) + P(C) \cdot P(R/C)}}$$

→ Baye's formula.

den = Prob. of

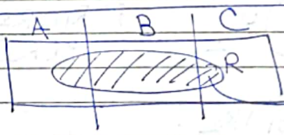
$$P(A/R) = \frac{P(A) \cdot P(R/A)}{P(R)} \longrightarrow \text{Total Prob}$$

$$P(R) \cdot P(A/R) = P(A) \cdot P(R/A)$$

$$P(R \cap A) = P(A \cap R)$$

Proved

OR



the new SS

$P(A/R)$ ∵ R has occur

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SS

$P(A/R)$ $\qquad$ $P(R)$

$\dfrac{P(A \cap R)}{\text{Total}} = \dfrac{P(A\cap R)}{P(R)}$

* BT used to find Src of event.

Q $P(B/R) = \dfrac{P(B \cap R)}{P(R)}$: (Total Prob)

eg1: A: 2R, 3G $\quad$ B: 3R, 2G

SS

R_1R_2	G_1R_2
R_1G_2	G_1G_2

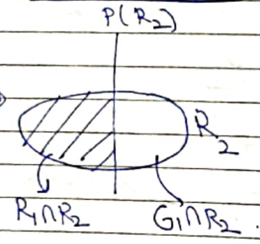
$P(R_2)$

$\cdot P(R_2) = P(R_1 \cap R_2) + P(G_1 \cap R_2)$
$= P(R_1)\cdot P(R_2/R_1) + P(G_1)\cdot P(R_2/G_1)$
$= \frac{2}{5}\cdot\frac{4}{6} + \frac{3}{5}\cdot\frac{3}{6}$

Given that we got red ball^2, what

Reverse Prob

$P(R_1/R_2)$
$= \dfrac{P(R_1 \cap R_2)}{P(R_2)}$
$= \dfrac{P(R_1)\cdot P(R_2/R_1)}{P(R_2)} = \dfrac{2/5 \cdot 4/6}{2/5\cdot 4/6 + 3/5 \cdot 3/6}$



$\dfrac{P(R_1\cap R_2)}{P(R_2)}$

Q $P(R_1/G_2) = \dfrac{P(R_1 \cap G_2)}{P(G_2)} \longrightarrow$ Total Prob.

$P(R_1 \cap G_2) + P(G_1 \cap G_2)$.

eg2: 5F $\quad$ 4U. $\quad P(H) = 1/3$, $P(T) = 2/3$

cal: $P(H)$? [Total Prob].

$P(H) = P(F \cap H) + P(U \cap H)$
$= P(F)\cdot P(H/F) + P(U)\cdot P(H/U)$
$\Rightarrow \frac{5}{9}\cdot\frac{1}{2} + \frac{4}{9}\cdot\frac{1}{3}$

Rev Prob BT.

$P(F/H) = \dfrac{P(F \cap H)}{P(H)} = \dfrac{P(F)\cdot P(H/F)}{P(H) = (TP)}$.

$P(F/T) = \dfrac{P(F\cap T)}{P(T)} = \dfrac{P(F\cap T)}{P(U \cap T) + P(F \cap T)}$

$P(U/H) = \dfrac{P(U \cap H)}{P(H)}$

$P(U/T) = \dfrac{P(U \cap T)}{P(T)}$.

eg3: n Green, m Red

1st time $\begin{cases} \text{if Green, add K Green \& to bag } (G+KG) \\ \text{if red, add } R+KR \text{ to the bag.}\end{cases}$

$SS = \left\{ \begin{array}{c|c} G_1R_1 & R_1R_2 \\ G_1R_2 & R_1G_2 \end{array} \right\}$

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Q $P(R_2) = ?$ Prob that 2nd ball is Red

$P(G_1 \cap R_2) + P(R_1 \cap R_2)$

$= P(G_1) \cdot P(R_2/G_1) + P(R_1) \cdot P(R_2/R_1)$

$\Rightarrow \left(\frac{n}{n+m}\right) \cdot \left(\frac{m}{n+m+K}\right) + \left(\frac{m}{m+n}\right) \cdot \left(\frac{m+K}{m+n+K}\right)$

Q $P(G_2/G_1) = \frac{n+K}{m+n+K}$

$P(G_2/R_1) = \frac{n}{m+n+K}$

- $P(G_2) = P(R_1 \cap G_2) + P(G_1 \cap G_2)$
  $\Rightarrow P(R_1) \cdot P(G_2/R_1) + P(G_1) \cdot P(G_2/G_1)$

BT [Find Cause] $\rightarrow$ $P\left(\frac{R_1}{R_2}\right) = \frac{P(R_1 \cap R_2)}{P(R_2)} = TP$.

$P\left(\frac{G_1}{R_2}\right) =$

$P\left(\frac{R_1}{G_2}\right) =$

$P\left(\frac{G_1}{G_2}\right) =$ ✓

eg4: Knows $\longrightarrow p$.
guess $\longrightarrow 1-p$.

$P(R/G) = 1/m$, $P(W/G) = 1 - 1/m$

$P(C) = P(K \cap C) + P(G \cap C)$

$= P(K) \cdot P(C/K) + P(G) \cdot P(C/G)$

$\Rightarrow p + (1-p) \frac{1}{m}$

Prob that she gave correct ans

$P(G/C) = \frac{P(G \cap C)}{P(C)} \longrightarrow TP$.

$\Rightarrow \frac{P(G) \cdot P(C/G)}{P(C)}$

$\Rightarrow \frac{(1-p) \cdot 1/m}{P(C)}$

$P(K/C) = \frac{P(K \cap C)}{P(C)}$

$= \frac{P(K) \cdot P(C/K)}{} = \frac{p}{P(C)}$

$$\boxed{P(K/C) = \frac{p}{p + (1-p) 1/m}}$$ ✓

eg5: Bayes T

A F.   B C.   (1/3, 2/3)

$P(A/B) = 2/3$
$P(A/C) = 1/3$

SS: A BB   CC
BA   CA.

$P(A) = P(B \cap A) + P(C \cap A)$.

$\Rightarrow P(B) \cdot P(A/B) + P(C) \cdot P(A/C)$

$= 1/3 \cdot 2/3 + 2/3 \cdot 1/3$

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