Linear Algebra [MIT]

* Lec 1 | MIT – Gilbert Strang

Text : Intro to LA.

→ n linear eqn's, n unknowns
- Row Picture
- Column Picture
- Matrix form (algebra way).

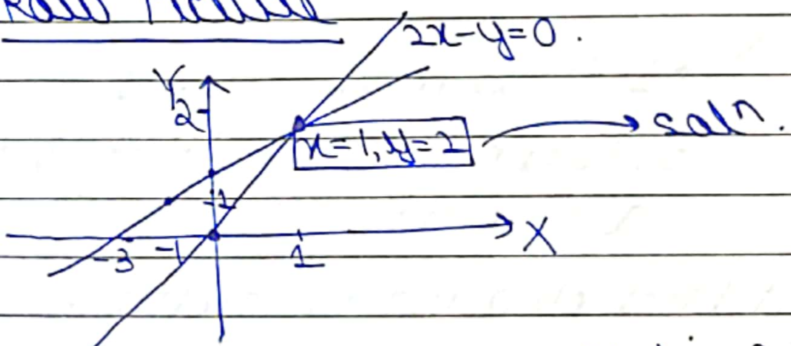
$\boxed{\text{2 eqn's, 2 unknowns}}$

Eg. $2x - y = 0$
$-x + 2y = 3$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\text{coeff matrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$AX = B$. ← Matrix form.

* Row Picture

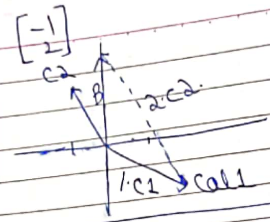


* Column Picture

→ combine 2 vectors [Linear Combination of the columns].

$$x\underset{C_1}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} + y\underset{C_2}{\begin{bmatrix} -1 \\ 2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

$1\cdot c1 + 2\cdot c2 = $ result



→ Column Picture.

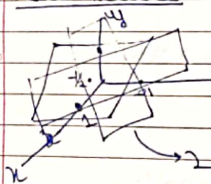
3 eqns, 3 unknowns

$2x - y = 0$
$-x + 2y - z = -1$
$-3y + 4z = 4$

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$

$AX = B$ ← Matrix Form.

- Row Picture → Plane [each row gives us a Plane in 3X3].

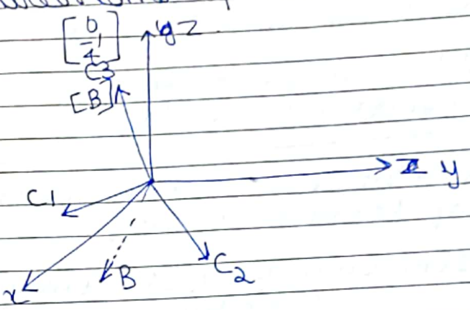


→ 2 Planes meet in a line.
∴ 3 Planes meet in a Point & that a Solution.
→ not that clear in row Picture.

- Column Picture

$$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

- Linear combn of 3 (3 dim Vectors)



$(x, y, z) \longrightarrow (0, 0, 1)$ [Soln]
[Point, where 3 Planes meet].

- RHS ⟶ $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

Now, Soln: $(x = 1,\ y = 1,\ z = 0)$.
∴ in Column Pic (C1, C2, C3 same, just the LC changes)

Q) Can I solve $AX = B$, for every B?
Is there a solution?
OR
Do the Linear combinations of the colns fill 3 dimension Space?

AX ⟶ Linear Combination of Col

for this A, ⟶ Yes.
↳ Non-Singular.

- If the 3 colⁿs lie in the same plane their combⁿs will also lie in the same plane.
  - $\therefore$ LC won't give 3D Space.
  - $\therefore A$ = Singular, not invertible.

Q. 9D Space → Vector with 9 components
[LC of 9 columns (9D Space)].

can RHS (B) always pass? [Can 9 column comb fill 9D Space?]
- if A: singular, rows are not independent,
  or $C_9 = \lambda \cdot C_8$. X
  [We can't find a solⁿ].

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* AX = B [Matrix form]
[Matrix * Vector] =

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

1) $\underbrace{1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix}}_{\text{LC of columns way}}$

2) row at a time [dot Product] row x col.

$\boxed{\text{AX} \rightarrow \text{is a combination of columns of A}}$

2) * Elimination with Matrices

- System of Eqⁿs
- Method: Elimination
  - way s/w package solves eqⁿs.

if, elimination may succeed / not [if its good matrix].

Q
$x + 2y + z = 2$
$3x + 8y + z = 12$
$4y + z = 2$
→ AX = B

Success
3 Pivots

1st pivot

$$\underbrace{\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} \rightarrow \begin{matrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{matrix} \rightarrow \begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{matrix}$$

2nd pi

2 eqⁿs in y & z.

⇒ A → U [Upper Δ Matrix]

(Pivots can't be 0)

* Purpose of elimination: A → U (Upper Δ Matrix)

- Failure of eliminⁿ (can't get 3 pivots)
  - if 1st Pivot = 0, exchange rows.
    [If we have a non zero below zero pivot].

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} \rightarrow \quad \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

[2 Pivots]
$\therefore$ failure.

* Back Substitution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \text{augmented matrix.}$$

A | B.

$$\downarrow$$

[Row echelon form].

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

U C

$(A \longrightarrow U, B \longrightarrow C)$.

$$x + 2y + z = 2 \longrightarrow (1)$$
$$2y - 2z = 6 \longrightarrow (2)$$
$$5z = -10 \longrightarrow (3)$$

$\boxed{z = -2, y = 1, x = 2}$ ← Back substitution [Solving in rev order, ∵ system is triangular]

* Matrices elimination.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

- $\begin{array}{c} c_1 c_2 c_3 \\ \begin{bmatrix} & & \end{bmatrix} \end{array} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \text{LC of columns of Mat} = 3C_1 + 4C_2 + 5C_3$.

- $\underset{1\times3}{[1\,2\,3]} \underset{3\times3}{\begin{bmatrix} & & \end{bmatrix}} = \underset{1\times3}{1r_1 + 2r_2 + 3r_3} = \text{LC of rows}$.

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① $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow$ Sub $3r_1$ from r_2.

S1:

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow 2,3 \text{ [mul row 2 x col 3]}$$

E_{21} [fix 21 pos].

$r_2 = -3r_1 + r_2 + 0 \cdot r_3$

$r_2 = r_2 - 3r_1$

S2: Subtract $2r_2$ from r_3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

E_{32} U.

∴ $r_3 = r_3 - 2r_2$.

- $\boxed{E_{32}(E_{21} \cdot A) = U}$ Associative law.

$\underbrace{(E_{32} E_{21})}_{E} A = U$

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- Matrix → changes 2 rows/ exchange rows. : Permutation Matrix (P).

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

P → r

- exchange column instead

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

NA

∴ [~~left row opern~~]

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$C_1 = 0 \cdot C_1 + 1 \cdot C_2$.

[right col opern]

- $\underbrace{(E_{32} E_{21})}_{E} A = U$.

how to get A from U.

* Inverse Matrix

$$\underset{A^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$R_2 = -3r_1 + r_2$

$r_2 = r_2 - 3r_1$

Inv

$r_2 =$ add $3r_1$ to r_2.

$r_2 = 3r_1 + r_2$.

L3 : Multiplication & Inverse Matrices

* Multiply 2 Matrices

$$\underset{A}{r_3\begin{bmatrix} a_{31} \ \cdots \end{bmatrix}} \underset{B}{\begin{bmatrix} b_{14} \\ \vdots \end{bmatrix}}^{C_4} = \underset{C = A \cdot B}{\begin{bmatrix} [a_{ij}] \end{bmatrix}} \to a_{34} \Rightarrow r_3 \times C_4.$$

regular way

$C_{34} = (\text{row}_3(A) \cdot \text{col}_4(B))$

$\Rightarrow a_{31}b_{14} + a_{32}b_{24} + a_{33}b_{34} \ldots$

$C_{34} \Rightarrow \sum_{K=1}^{n} a_{3K} b_{K4}$

$\rightarrow \underset{m \times n}{A} \ \underset{n \times p}{B} = \underset{m \times p}{C}$.

Col. way

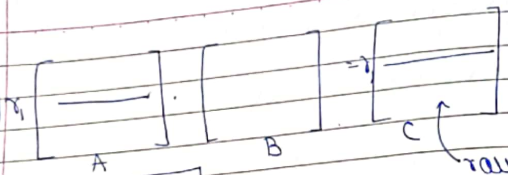
$$\underset{m \times n}{A\begin{bmatrix} & \\ & \end{bmatrix}} \ \underset{n \times p}{B\begin{bmatrix} C_1 & \cdots \end{bmatrix}} = \underset{m \times p}{C\begin{bmatrix} C_1^C & \cdots \end{bmatrix}}$$

col of C are comb of column of A.

$M \times C_1 = C_1^C$. = LC of Col of A.

∴ Mul A by p coln, one by one to get the p col of C.

raw way



$\boxed{r_1 \times B = r_1^C}$

raws of C are comb of raws of ~~A~~ B.

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col of A × raw of B = m×p Matrix.
m×1 1×p

$$\underset{A}{\begin{bmatrix}2\\3\\4\end{bmatrix}} \cdot \underset{B}{\begin{bmatrix}1 & 6\end{bmatrix}} = \underset{C}{\begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}}$$

→ raws = mul of B
col = mul of A

multiple [col^n s].

4th way

AB = sum of (cols of A × raws of B)

$$\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix}\begin{bmatrix}1 & 6\\0 & 0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix}$$

Mul by Blocks

$$\underset{A}{\left[\begin{array}{c|c}A_1 & A_2\\ \hline A_3 & A_4\end{array}\right]} \underset{B}{\left[\begin{array}{c|c}B_1 & B_2\\ \hline B_3 & B_4\end{array}\right]} = \left[\begin{array}{c|c} & \\ & \end{array}\right]$$

→ $A_1B_1 + A_2B_3$

20×20 20×20

Each Block : 10×10.

* Inverses

- Square matrices (A)

if A^{-1} exists: $A^{-1}A = I$
↳ A is invertible / non-singular

∴ $\boxed{AA^{-1} = I = A^{-1}A}$ → Commutative.

for Sq Matrix, Left inv = right inv.
Rec Matrix, No.

⇒ Singular case (No inverse)

$A = \begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}$ → why this is not invertible?

Let, $AB = I$ ↳ columns are comb of cols of A.

column pict → ① [We cant identity matrix].
∵ c_1, c_2 lie on same line.

② ∵ In can find a vector X, with $AX = 0$. // Key eq^n.

$X \neq 0$

$$\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}\overset{X}{\begin{bmatrix}3\\-1\end{bmatrix}} = \begin{bmatrix}0\\0\end{bmatrix}$$

↳ $r_p =$ $c_1 = 3c_1 - c_2$

⇒ $IX = 0$. if, X is non-zero
↳ $X = 0$ ∴ A not invertible

For non-invertible matrices / singular Matrix

$\boxed{AX = 0}$; s.t. $X \neq 0$.

* Invertible Matrix

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

$A \times C_j$ of $A^{-1} = C_j$ of I

∴ 2 get 2 RHS

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{— (i)}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{— (ii)}$$

* Gauss-Jordan Idea : Solve 2 eqn's at once.

$$\left[\begin{array}{cc|cc} \cancel{1} & \cancel{2} & \cancel{1} & \cancel{0} \\ \cancel{3} & \cancel{7} & \cancel{0} & \cancel{1} \end{array}\right]$$

$$\downarrow$$

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ \boxed{2} & 7 & 0 & 1 \end{array}\right]}_{A \quad | \quad I} \rightarrow \left[\begin{array}{cc|cc} 1 & \boxed{3} & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right].$$

$\downarrow$ (Top elim^n)

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]}_{I \quad | \quad A^{-1}}.$$

$$E \times [A \ I] = [I \ A^{-1}]$$

2Es (in this case)

elimination matrices.

∴ $\boxed{E = A^{-1}}$

$E \cdot A = I$.

$E \Rightarrow A^{-1}$.

⇒ Solving n eqn's at the same time

* Lecture 4

$A \longrightarrow U$ (Elimination)

⇒ $(AB)^{-1}$

if A, B is invertible what's inverse of AB ?

$$\boxed{(AB)^{-1} = B^{-1} A^{-1}}$$

Proof

$(AB)(B^{-1}A^{-1}) = I$

• $A \cdot A^{-1} = I$ $\qquad I^T = I$

$(A^{-1})^T \cdot A^T \Rightarrow I$ (Transpose both sides).

inverse of $A^T = (A^T)^{-1}$.

∴ $\boxed{(A^T)^{-1} = (A^{-1})^T}$

* A = LU is the most basic factorization of a Matrix.

Q. $\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}$ ∴ $\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$

if 4, anything, 2nd pivot=0, ∴ elimin fails to give ans

$r_2 \rightarrow r_2 - 4r_1$

$\boxed{A = LU}$.

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}.$$

$(E_{21})^{-1}$

$r_2 = 4r_1 + r_2$.
$= r_2 + 4r_1$

No pivot=0.

* A: 3X3.

$E_{32} \cdot (E_{31}(E_{21} \cdot A)) = U$. (no row exchanges.)

$A = \underbrace{[E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}]}_{L} U \Rightarrow LU$

Product of inverse of the elem. matrices.

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Q. $$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}.$$

$$\underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \underline{L}$$

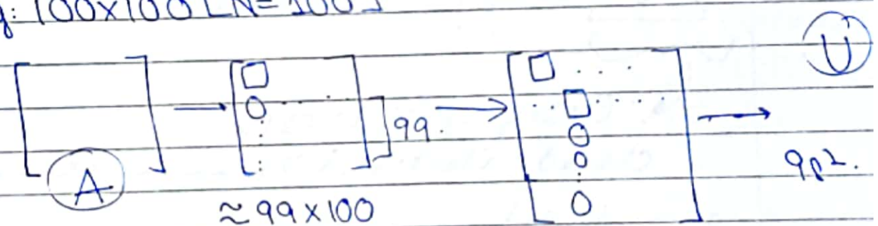
$R_3 = 1 \cdot R_3$.

- $EA = U$
  $A = LU$.

* $\boxed{A = LU}$
If no row exchanges, the multipliers go directly into L → $[E_{21}^{-1} E_{31}^{-1} \ldots]$.
[Using elimination].

* How expensive is elimination? (# operations) on $n \times n$ matrix A.
mult + sub.

eg: 100x100 [N=100]



$\approx 99 \times 100$
$\approx (100)^2$.

99^2

$$\sum_{n=1}^{n=100} n^2 = \frac{n(n+1)(2n+1)}{6} = \underline{\underline{O(n^3)}}$$

(A)

$\therefore \frac{1}{3}n^3$ → numbers are getting smaller.

$\int x^2 = x^3/3$

* what about B?

A/B. → 100+99+98+ ··· 1 = O(n²).
→ O(n³).

* If row exchanges

* Transposes & Permutations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cdot P_{12}$ → PM that exchanges row 1 & row 2.

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P_{12}, P_{13}, P_{23},

$\therefore$ 1 2 3 ⇒ 3! = 6 P exists.

→ ∴ Group of Matrices,
closed, identity, inverse all exist.

$P_{12} \xrightarrow{\text{inverse}} P_{21}$

$\therefore \boxed{P^{-1} = P^T}$

Q How many 4×4 Permutations?
4! = 24 Ps

* Lecture 5

— Vector Spaces

* Permutations (P): execute row exchanges
[if we get a zero pivot]

A = LU → doesn't count for row exchanges.

$\boxed{PA = LU}$ → with row exchanges (0 pivot if).

does row exchanges, into order, where pivot ≠ 0.

// any invertible A

* P = identity matrix with reordered rows,
if (n×n) → n! possible reordings
(all n×n permutations)
— They are all invertible.

$\therefore \boxed{P^{-1} = P^T}$ $\boxed{P \cdot P^T = I}$

Q
$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$
3×2 → 2×3

* Transpose

$(A^T)_{ij} = A_{ji}$

* Symmetric Matrices

↳ transposing doesn't change the matrix

- $A^T = A$

$$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$$

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* R^TR is always <u>symmetric</u> — R: Rectang Matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 10 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

$\therefore (R^TR)^T = R^T(R^T)^T = R^TR$. <u>Proof</u>.

$\therefore$ Symmetric

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* Vector Spaces & SubSpaces

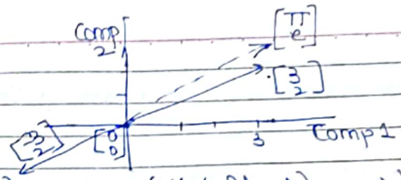
→ a Space of Vectors, allow me to do vector operations, do LCs.

eg:

R^2 : add 2 dim real vectors.

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix} \ldots\ldots$

$\begin{bmatrix} 3+\pi \\ 2+e \end{bmatrix}$



R^2 : plane (XY Plane) ⟶ Vector Space.
: all 2 dim real vectors

⟶ (if point (0,0) removed from space, not pass X, $\because \begin{bmatrix} 3 \\ 2 \end{bmatrix} \times 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

* R^3 = all vectors with 3 real components

eg: $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

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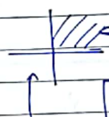
* R^n = all column vectors with n real components.
[we can take any comb: add / multiply → (with rules) we still in R^n]

⇒ <u>Not a Vector Space</u>



← 1/4 (all vectors with only +ve components).

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \times -7 = \begin{bmatrix} -14 \\ -21 \end{bmatrix}$ X (not in V Space).

not Vector Space
(not closed under Scaler Mult).

$\therefore$ VS → closed under linear comb. ✓

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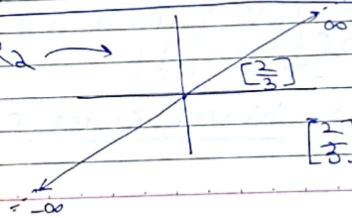
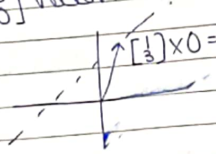
* R_2 ⟶



Vector Space inside R_2: Subspace of R_2.

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \times 3 = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ → Line.

- To be a subspace, line in R_2 must go thr $\begin{bmatrix}0\\0\end{bmatrix}$ vector.



* Possible Subspaces of R^2 → 2D Vector Space

1. all of R^2 (whole 2D Space). (P)
2. any line through $\begin{bmatrix}0\\0\end{bmatrix}$ vector (L) (not R_1)
3. any origin $\begin{bmatrix}0\\0\end{bmatrix}$ vector (Z)

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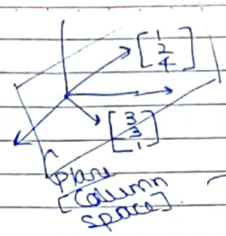
* SS of R^3

1. all of R^3
2. any plane/line through origin.
3. any origin $\begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓

Q $A = \begin{bmatrix}1 & 3\\2 & 3\\4 & 1\end{bmatrix}$ Create SubSpaces out of Matrix A.

- All colmns are vectors in R_3.
  (Take all the LCs form a SubSpace)
  $2\begin{bmatrix}1\\2\\4\end{bmatrix} + 3\begin{bmatrix}3\\3\\1\end{bmatrix} \ldots$
  Called : Column Space $C(A)$



[Take all comb of thes 2 vectors → column Space] [Plane]

→ If C_1, C_2 : same line column Space: Line
∴ depends on Vectors

→ R_{10}, ...

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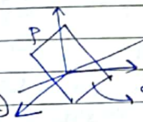
* Lecture 6: Column Space & Null Space

Center of Linear Algebra — SubSpaces

- VS : Bunch of Vectors, which can be added / mult K & result stays in the Space [all LCs]

eg: R_3 ↓ VS [Vector Space]



SubSpace [Plane thru $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a SS].

- SubSpace : Vectors inside R_3, which is also a VS.
  ∴ VS inside VS = SubSpace

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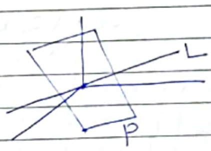
* P & L = SubSpaces

1) - $P \cup L$ = all Vectors in P or L or Both
   Not SS



∴ If we add, some from P, some from L, we will be out from SS

2) $P \cap L$: All vectors in both P & L. → Subspace

only common $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (in this case)

Note: if subspaces S & T,
$S \cap T$ is a SubSpace.

$V \& W$ in S, any LC will be in S.

Column Space of Matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}_{4 \times 3}$$

→ columns of this matrix.

• [CS of A, is a SubSpace in R4] : C(A)

3 vectors

all linear combination of these 3 columns.

Q. Is the CS filling the whole R4 ?
[SS] [VS].

→ we can't get whole 4D Space.

Q. Does AX = B, always have a solution for every B ? Which B

→ NO

→ can LC of these 3 vector fill all R4 ?

$$AX = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

• 4 eqns, 3 unknowns

Q. which RHS allow me to solve this ??
[B]

• $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ → definitely yes $AX = 0$.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$ [Do by Col S].

→ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} / \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} / \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ or any LC of columns

Note: System of Eqns $AX = B$ can only be solved, when B is a vector in the column Space of A.
B = LC of columns of A.

Q. Are these 3 columns independent?

If we take LC of these 3 columns, do we get a 3D Subspace? Or, any of col not contributing anything new?

Or,
Can I throw away any col to get the same Column Space?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$ → CS is 2D Subspace of R4.

Line Line. → C3 is C1 + C2
∴ it doesn't contribute new
[dependencies].

diff SubSpace

* NullSpace of A N(A)

It contains all Solutions (X) to AX=0.

$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

Subspace in R3.

Column Space in R4

$$\Rightarrow \underset{A}{\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}}\underset{X}{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

• $X\to\begin{bmatrix}0\\0\\0\end{bmatrix}$ [Zero Vector].

∴ Null Space contains $\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix}$

$\begin{bmatrix}2\\2\\-2\end{bmatrix}$, or $\begin{bmatrix}c\\c\\-c\end{bmatrix}$

$\underbrace{c\begin{bmatrix}1\\1\\-1\end{bmatrix}}_{Ans}$ ← ∴ [Any multip. of]

∴ Null Space is a line in R3

$\begin{bmatrix}1\\1\\-1\end{bmatrix}$ $c\begin{bmatrix}1\\1\\-1\end{bmatrix}$

* Check that the soln to AX=0, always give a SubSpace? $x+x^*$ = also belong to SS.

If $A\underset{v}{x}=0$ & $A\underset{w}{x^*}=0\to A(\underset{v}{x}+\underset{w}{x^*})=0$.

∴ If $v\in SS, w\in SS$; $v+w\in SS$.

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$A(v+w)=0$ ✓

* If $AV=0$, then $A(kV)=0$. ✓

Q. $\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ $\boxed{AX=B}$

Do the solutions form a SubSpace? "NO".

$\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}}_{\text{many soln.}}$

[Plane/Line doesnt go through origin]

∵ Zero Vector is not a solution ∴ can't be SS.

* SS have to go thru origin

* Lecture 7: Solve AX=0, Pivot Variables, Spec Soln

Q. $A=\begin{bmatrix}\boxed{1}&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}_{3\times4}$ (4 Vectors 3 comp)

· C_2 mult of C_1 (dependent).

· 3rd row = dependent.

Solve AX=0, By elimination; we aren't changing the soln / Null Space.

Zero Pivot.

$$\begin{bmatrix}1&2&2&2\\0&\boxed{0}&\boxed{2}&4\\0&0&2&4\end{bmatrix}\to\begin{array}{cccc}P&F&\text{Pivot col}&\text{free col}\\\downarrow&\downarrow&\downarrow&\downarrow\end{array}\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}$$

Below 0 ∴ can't be exchanged ⇒ dependency of column process

U [echelon form] (Non zeros in staircase form)

* Rank of A = #pivots = 2

$UX = 0$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
$2x_3 + 4x_4 = 0$

free var.

$X = \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix}$ $x_1 \to -2$, $x_3 \to 0$

$\therefore X = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (S1) is a soln to $AX = 0$. [in the Null Space]

S2: any mult. $X = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (any ∞ line is).

· Now assign any value to free V.

$\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ free variable

(S2) $\therefore X = c'\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ in the null space

Q. All Soln to $AX = 0$?

free V → can be given any values.

$c\begin{bmatrix} -2 \\ 1 \\ 8 \end{bmatrix} + c'\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ → all LCs of the Special Coln.

if, rank = 2 : # of pivot V

if $m \times n$; rank = r; $\boxed{\text{no. of free V} = n - r}$

* Reduced row Echelon form (R)

↳ Zeros above & below the pivot.

– more cleaner U.

$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

∵ R3 was comb of R1 & R2 ∴ 0s ⇒ row of 0s

$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

eqn

[reduced row echelon form].

1) Pivot cols : 1 & 3
free cols : 2 & 4.

2) Matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ sitting in pivot rows & columns.

· $x_1 + 2x_2 - 2x_4 = 0$ → $RX = 0$
$x_3 + 2x_4 = 0$

Note: Soln to $AX = 0$, $UX = 0$, $RX = 0$ are Same.
[Just elimination]

$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$ Pivot Cols (I) $\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}$ free Cols (F)

$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ Null Space $\begin{bmatrix} -F \\ I \end{bmatrix}$

2 spcl Solns
[2 free Variables]

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reduced row echelon form

$* \; R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

- r pivot rows.
- r pivot cols.
- n−r free cols.

$RN = 0$.

$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

columns are the special solutions.

$RX = 0$.

$[I \; F] \begin{bmatrix} X_{pivot} \\ X_{free} \end{bmatrix} = 0$.

$\boxed{X_{pivot} = -F \cdot X_{free}}$

Q. $A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$ → $C3 = C1 + C2$

PC PC free col

∴ 2 Pivots

∴ $\boxed{rank = 2}$

$\begin{array}{ccc} p & p & f \end{array}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (row exch).

$\begin{array}{ccc} p & p & f \end{array}$

U, free col.

$X = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ → set FV to 1 → from eqns. [next page] (Back Subst-)

$x_1 + 2x_2 + 3x_3 = 0$.

$2x_2 + 2x_3 = 0$.

∴ Null Space $(X) = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (line) → above

* Keep going to RRE.

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{cc} I & F \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ → div by 2. → $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ R

∴ $X = \begin{bmatrix} -F \\ I \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

N, above.

* Lecture 8. Solving $AX = B$ → Complete Solution.

Q. $x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$

$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$

$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$

$(r_3 = r_1 + r_2)$.

∴ $b_3 = b_1 + b_2$

$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$ → augmented Matrix $= [A|b]$

(Perform elimination)

if ... we can find ...

$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right] \quad \begin{matrix} 1 \\ 3 \\ 0 \end{matrix}$$

$R(A) = R(A/B) < n$

$\therefore \infty$ solution.

$b_3 = b_1 + b_2$.

* Condⁿ on b, that make eqⁿ solvable?
* Solvability condⁿ on RHS:

→ $AX=B$ is solvable, exactly when if B is in Column Space of A.

→ If comb of rows of A, gives zero row; then the same combination of the entries of B must give 0.

* To find complete Soln to $AX=B$.

① $X_{particular}$: Set all free variables to 0 then solve $AX=B$ for pivot variables.

Here: $[x_2 = 0, x_4 = 0]$

$x_1 + 2x_3 = 1$

$2x_3 = 3$

$x_3 = 3/2$

$x_1 = -2$

$X_{particular} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$. Ans

② $(X_{nullspace} + X_{particular})$: complete solution.

* $AX_p = B$

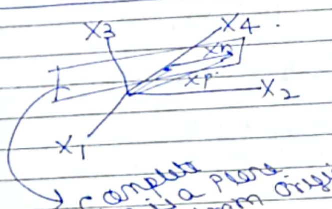
$+\ AX_n = 0$

$A(X_p + X_n) = B$. [Complete Solⁿ].

$$X_{complete} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}}_{X_p} + \underbrace{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{Special Solⁿs (Null Space)}}$$

X_n [Sub Space] [2 D SS inside R4].

* Plot all Solⁿ X : R4



$AX=B$ solⁿs form a SS?

⇒

* Complete solⁿ is a plane shifted from origin ∴ Not SubSpace.

* m×n matrix A of rank r [# of pivots].

$\boxed{r \leq m}$, $\boxed{r \leq n}$

* Full Column Rank ($r = n$) [every colⁿ: 1 Pivot]

- Pivot in every colⁿ.
- Free Variables = 0 ; N(A) = only 0 vector

Particular Solⁿ

- $AX=B \rightarrow X = X_{particular}$ ⇒ Unique Solution if it exists ∴ 0 or 1 solⁿ.

Q $A = \begin{bmatrix} \boxed{1} & 3 \\ 2 & \boxed{1} \\ 6 & 1 \\ 5 & 1 \end{bmatrix}_{4\times 2}$ rank = 2 = n. [independent cols] [Nothing in Null Space].

Row Reduced echelon form → $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ → $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

→ AX=B ? does solⁿ exist.

if $B = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

✱ Full Row Rank $\begin{bmatrix} \quad \end{bmatrix}_{m\times n}$

means $\boxed{r=m}$

I can solve AX=B, for which RHS ? every B

→ left with n−r free Variables [n−m]

Q $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$ → rank = 2 [2 Pivots].

$R = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$ — $\underbrace{\quad}_{I}$ $\underbrace{\quad}_{F}$

✱ $r = m = n$. [Full Rank]

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ → invertible matrix. $|A| \neq 0$.

$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\boxed{N(A) = 0 \text{ Vector}}$

- Conditions to Solve AX=B ?

$r=m=n$ $R=I$ 1 Solⁿ Full Rank	$r=n<m$ $R=\begin{bmatrix} I \\ 0 \end{bmatrix}$ (0 or 1 sol/n) Full Col Rank	$r=m<n$ $R=\begin{bmatrix} I & F \end{bmatrix}$ partly mixed. · [always Solⁿ] ∞ Full row Rank

Rank tells about no. of Solutions

$r<m, r<n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ → 0≠0 for some bs possible.

(0 or ∞ solutions).

* Lecture 9 [Independence, Basis, Dimension]

1. Linear Independence
2. Space Spanning
3. Basis of SS & dimension.

Q. $A: m \times n$, with $m < n$.
[more unknowns than eq^n s]

There, there are nonzero sol^n s to $AX = 0$

There will be free col^n s.

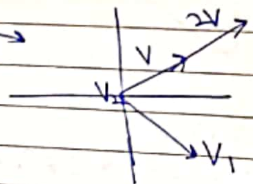
$AX = 0$. rank $= m$ (atmost)
$\therefore$ free V $= n - m$.
[any Value]. ✓

* Independence

→ Vectors $x_1, x_2, \dots x_n$ are linearly independ if no combination gives 0 vector.
(except the zero combination] (all cs $= 0$)

$c_1x_1 + c_2x_2 + \dots c_nx_n \neq 0$.

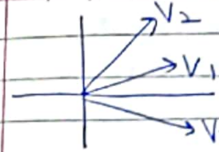
eg: R^2 →



(i) V & 2V are dependent
$2V - V = 0$.

(ii) $0V_1 + 6V_2 = 0$.
[dependent]
if, 0 vector is present, all dependent.

(iii)



(independent) V_1 & V_2

→ dependent V_1, V_2, V_3
[3 Vectors in Plane have to be dependent]

$AX = 0$ → $A = \begin{array}{ccc} V_1 & V_2 & V_3 \\ \hline \end{array}$ $\begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix}_{2\times 3} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$ → $m < n$.
[free V.] ✓

[3 Vectors, 2 comp].

$c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 2.5 \\ -1 \end{bmatrix} = 0$.

- Columns are dependent, if something in NS.
$\therefore$ NS → any value of Free Var.

Dependent

$\therefore$ free V $= 0$.

** When $V_1, V_2, \dots V_n$ are the columns of A, they are independent if the nullspace of A only the zero Vector. $\boxed{\text{Rank} = n}$

** They are dependent if $AC = 0$, for some nonzero c in null space of A.
$\boxed{\text{Rank} < n}$

free Variables yes.

* Bunch of Vectors to Span a Space

Spanning a Space: Vectors $V_1, V_2 \dots V_e$ span a space means: the space consists of all combinations of those vectors.

→ The columns of a matrix span the column Space.

Basis fo... (everything ...)

* Basis for a Vector Space is a sequence of Vectors : $V_1, V_2, V_3 \ldots V_d$ with 2 properties :
(i) they are independent.
(ii) they span the space.

eg: 3D Space (R^3)

· One basis is: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ (i) indep.

columns

[I] . What N(I) → 0 vector.
∴ independent.

· 2nd basis is: $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ (ii) indep.

→ comb = plane
not 3D ∴ span ✗

$\begin{bmatrix}3\\3\\8\end{bmatrix}$ take any vector that's not in the plane.

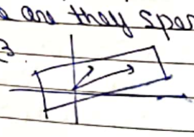
** R^n ; n vectors give basis of nxn matrix with those cols :
invertible. $|A| \neq 0$. ✓

Q $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ What Space, will they be a basis for? : Plane
[? ho are they span].

imp

plane inside R3.

2D subspace inside R3.



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* $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$

= If matrix is invertible, their columns for a basis for R3. (∵ cols = independent)
∴ many bases.
(but they all have same no of vectors
$R_3 \longrightarrow 3$.

Given a Space,
→ Every basis for the space has the same no of vectors.
[Dimension of the space].

eg: Space is C(A), find the basis?

$$\begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \\ \end{array}$$
$$\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}$$
$$\begin{array}{cccc} p & p & f & f \end{array}$$

Rank=2

do they form basis of C(A)?
→ not independent.
→ they span.

2D CS in R4.

$N(A) = \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$ ∴ ≠ 0 ∴ not ind.

Basis: C_1, C_2
C_1, C_3
C_2, C_4

* rank(A) = # of pivot columns
= Dimension of the column Space(A)
(not dim of A)
= 2
∴ 2 Vectors in Basis.

· Another basis of the C(A)

$\begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}7\\5\\7\end{bmatrix}$ → 2 & independent.

Sum of All cols.

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\# of pivot.

- $\boxed{\dim C(A) = \text{rank}}$

* Dimension of Null Space

letting me what way the colum are dep. ⇒ add to 0

dim 2. $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ = free V: 0 & 1

$\underbrace{\qquad\qquad}_{\text{2 Spcl Solution}}$

they are indep. span Null Space ∴ basis.

- $\boxed{\dim N(A) = \#\text{ free Variables} = n - r}$

** Lec 10. The 4 fundamental SubSpaces

C(A), N(A), ...

- standard basis of R3 → $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

⇒ $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$ → ∴ 2 of basis ⇒ not independent

→ Row Space: 2D

∴ 2 Pivots, rank = 2 ; not indep rows.

* 4 SubSpaces

A: m×n

1) Column Space C(A) in R^m
2) Null Space N(A) in R^n
3) Row Space → all comb of rows

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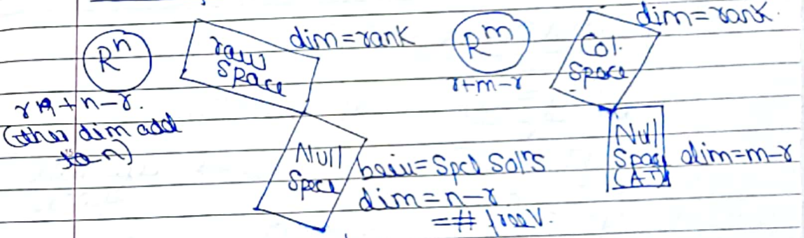
⇒ all comb of the columns of AT = Row Space → C(AT)

$\boxed{\text{Row Space} = C(A^T)}$ in R^n.

(4) Null Space (AT) ⟶ N(AT) [Left Null Space] in R^m. ✓

⇒ 4 SubSpaces [Picture]



- Basis ? dimension ? ✓ — [every spcl solⁿ come from a free Variable]

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$ → row Space = 2D ∴ rank = 2.

∴ column Space = 2D. ✓

∴ 2 Pivot Col.

* (I) $\begin{array}{cccc} p & p & f & f \end{array}$

$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

I F

0

R

$\boxed{C(R) \neq C(A)}$.

diff column Spaces.

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$\therefore$ raw opern's are same

but, raw space will be same.

$\therefore$ Basis for the raw space of A or R is the first r rows of R.

$\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$: Best Basis for Raw Space

any LCs of rows will be in raw space only.

* Left Null Space - $N(A^T)$

$\boxed{A^T y = 0}$ y is $N(A^T)$.

$y^T A = 0^T$ $\therefore$ Left Null Space

$\begin{bmatrix} y^T \end{bmatrix}\begin{bmatrix} & A & \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

* $A^T y = 0$; Basis? Gauss Jordan

rref $[A_{m\times n} \quad I_{m\times m}] \longrightarrow [R_{m\times n} \quad E_{m\times m}]$

$EA = R$

Earlier, R was I then E was A^{-1}

• $E[A\,I] = [I\,A^{-1}]$ raw reductn

(I) Pg 39

* $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow[\text{redn}]{\text{same raw}} \begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix}$

I E

Left NS = comb of rows that give 0 row.
NS = comb of colms to give 0 col.

dim = m − r = 3 − 2 = 1 : 1 Vector in Basis
$\Rightarrow [-1\ 0\ 1]$

* A new Vector Space $(R^{n\times n})$.

— All 3x3 Matrices
"Every matrix is a Vector".
$\therefore$ they obeys the rules.
} Matrix Space (M)

• A + B, cA, (not A·B for now, $\therefore$ vector mul not defined)

→ Subspaces of M

• All upper Δ $^{3\times3}$ Matrices / All symmetric $^{3\times3}$ Matrices / diagonal Matrices (D) $^{3\times3}$.

→ dim (D) = 3.

$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&7\end{bmatrix}$

→ independent Matrices
$\therefore$ They span the SS of D.
$\therefore$ Basis.
(can form any Diag Matrix with this).

* Lecture 11 [Matrix Spaces]

M = all 3x3 matrices.

. SS: Symmetric 3x3 / Upper Δ 3x3

[∵ sym + sym = sym]

∵ sym x K = sym]

* Basis for M = all 3x3 (Dim(M=9))

9 in Basis [Stand basis]

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\cdots\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

∴ dimension = 9 ⟶ How many belong to the sym 3x3 SS?

* Symmetric 3x3 SubSpace.

dim(Sym) = 6.

$$\begin{bmatrix}x&x&x\\&x&x\\&&x\end{bmatrix}$$

⟹ dim(Upper Δ 3x3 SS) = 6

$$\begin{bmatrix}x&x&x\\&x&x\\&&x\end{bmatrix}$$

↳ all the 6 basis present in the big Basis (M).

* $\boxed{S \cap U = \text{diagonal } 3\times3 \text{ SS}}$

dim(d) = 3

dim(S∩U) = 3

$$\begin{bmatrix}x&&\\&x&\\&&x\end{bmatrix}$$

↳ Vectors that are Symmetric / upper Δ.

* (S ∪ U) = Matrices in S or U.

$\underbrace{6D \quad 6D}_{\text{diff dirn}}$ ~~not SS~~

⟹ Sum

S + U = any element of S + any element U

d=6 d=6. (Symm + UΔ).

⟹ all 3x3 S. (M)

$\boxed{\dim(S+U)=9}$

RL

$$\boxed{\underset{6}{\dim(S)} + \underset{6}{\dim(U)} = \underset{3}{\dim(S\cap U)} + \underset{9}{\dim(S+U)}}$$

* $\frac{d^2y}{dx^2} + y = 0$. ⟵ Null Space of DE.

Solns: y = cos x, sin x,

. complete soln: y = $C_1 \cos x + C_2 \sin x$ (LC)

Basis: sin x, cos x , dim(Soln Space) = 2

∵ Second order eqn.

Point:

Here cos x, sin x are vectors.

[Basis pairs in LDE].

Building Matrices

* Rank 1 Matrices

$\underset{2\times 3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow$ rank 1.

Basis for the row Space $= [1\ 4\ 5]$.

- $\dim C(A) = 1$.
  Basis $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow \underset{2\times 1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{1\times 3}{[1\ 4\ 5]}$.

* Rank 1 Matrix

$\boxed{A = UV^T}$; U = Column Vector. V^T = row Vector

Note: 5X17 matrix of rank 4, can be made by 4 rank 1 matrices ✓

Q. M = all 5X17 matrices $\rightarrow$ max rank = 5.
Subset of rank 4 matrices

If we add 2 rank 4 matrices, is it rank 4? (No), max rank $\rightarrow$ 5.

~~r(A)+r(B)~~

$\boxed{r(A+B) \le r(A) + r(B)}$

* Subset of Rank 1 matrices not a SubSpace. ∵ It can add up to rank 2.

Q. In R^4 $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$ $AV = 0$

S = all Vectors in R4 with $V_1 + V_2 + V_3 + V_4 = 0$.
It is a SubSpace? $\rightarrow$ dim=3

Yes
- Scalar mult. ✓
- $V + W \leftrightarrow S$.

- Basis for S $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\rightarrow$ for V

1) S = null Space of $A = \underset{4\times 1}{[1\ 1\ 1\ 1]}$

rank = 1, $\dim(N(A)) = n - r = 4 - 1 = 3$.
(3 Vectors in Basis).

* $\boxed{A = [1\ 1\ 1\ 1]}$

1) ~~row Space = dim = 1~~

pi fre $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right.$

dim(row Space) = 1
(a line in R4)

3 Spcl Solns. (Spcl Basis).

2) Column Space = C(A) $\rightarrow$ in space: R^1.
$\rightarrow$ mult of 1 $\rightarrow R^1$.
$\boxed{\dim = 1}$

3) Null Space (A^T) $\rightarrow \{0\}$. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} X = 0$.

$\boxed{\dim = 0}$ ✓

* Graph $\longrightarrow G=(V,E)$

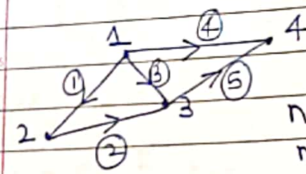
{nodes, edges}

Q How far nodes can be separated ?

* Lec 12 : Graphs, network, Incidence Matrix

Appl^n of Linear Algebra.

LA uses matrices which comes from Appl^n.



$G=(V,E)$

$n=4$; $m=5$
nodes edges.

⇒ Incidence Matrix (A)

$$\begin{array}{c|c} & \text{nodes} \rightarrow 1 \;\; 2 \;\; 3 \;\; 4 \\ A = & \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

edges

$\left.\begin{matrix}1\\2\\3\end{matrix}\right\}$ loop. [rows are dependent]. ∴ LD rows.

Null Space ? [How to comb cols to get 0].

If colms are independent, $N(A)=0$

$AX=0$. → heat flow/ current flow etc.

$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ potential at the nodes

$$AX = \begin{bmatrix} x_2-x_1 \\ x_3-x_2 \\ x_3-x_1 \\ x_4-x_1 \\ x_4-x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{e=AX}$ $\quad e$ → potential differences across the edges. [current].

$N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ → constant potential.

∴ Basis $= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$; → 1 Dimensional

$\boxed{\dim N(A) = 1}$

* Suppose we ground node 4. ≡ (potential = 0)

$\boxed{\text{rank}(A)=3}$; $\dim C(A)=3$.

⇒ Null Space of A^T $\boxed{A^T Y=0}$

$\boxed{\dim = 2}$ $(m-r)$

$A^T \rightarrow n \times m$
4×5

$$\underset{A^T \quad 4\times5}{\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

curvature

• $\boxed{A^T y = 0}$ // Kirkoff's current Law.

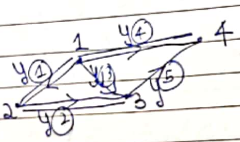
current $\propto$ P.D. (Ohms Law)

Null Space A^T. (dim=2). $\boxed{y = Ce}$

Note: $x_2 - x_1, \ldots, x_n - x_n$ (PD) $e = AX$ $\xrightarrow{\text{Ohms}}$ currents: $y_1, y_2, y_3, y_4 \ldots$

→ in eq out.

$$* \quad A^T y = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} -y_1 - y_3 - y_4 \\ y_1 - y_2 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



→ At node 4, the total flow = 0 ∵ conserv^n of charge.

* Basis of $N(A^T)$ → 2 vector

from Graph → $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ → ∵ 2 Loops.

but outer Loop, $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

[Sum of those 2] ∴ not indep.

* Row Space (A) or $C(A^T)$

$\boxed{\text{dim}=3}$ → 3 indep rows.

$$\begin{array}{ccccc} p & p & & p & \\ \end{array}$$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

A^T.

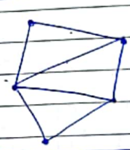
→ 3 indep. columns = 3 indep edges in G. = no Loop. [Tree]

$\dim N(A^T) = m - r$: # no. of independent loops

$\boxed{\#\text{Loops} = \#\text{edges} - \#\text{nodes} - 1}$

⇒ $\boxed{\#\text{nodes} - \#\text{edges} + \#\text{loops} = 1}$

Euler's Formula.

Q 

$5 - 7 + 3 = 1$ ✓.

PD
$e = AX$
$y = Ce$ ← Ohms Law Const
$A^T y = f$ → if batteries [current from outside]

} → 3 eq^ns

∴ $\boxed{A^T C A x = f}$

// Basic eq^n of Applied Math.

* Lec 13: Quiz Review

Q u, v, w are nonzero vectors in R^7. They span a subspace of R^7. Possible dim of subspace?

↳ 1 or 2 or 3.
can't be 0, ∵ nonzero vectors.

Q. U = 5X3 Matrix. $r = 3$ Pivots (U = R)

1) What N(U)?

$N(U) = 0$ vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$

5X3.

2) B = 10X3 Matrix

$B = \begin{bmatrix}U\\2U\end{bmatrix} \xrightarrow{\text{Row Red.}} \begin{bmatrix}U\\0\end{bmatrix}$ — rank B = 3.

$C = \begin{bmatrix}U & U\\U & 0\end{bmatrix} \xrightarrow{\text{Row}} \begin{bmatrix}U & U\\0 & -U\end{bmatrix} \rightarrow \begin{bmatrix}U & 0\\0 & U\end{bmatrix}$ — 6 Pivots [Reduced Row echelon form].

rank(B) = 3.
rank(C) = 6.

3) $\boxed{\dim[N(C^T)] = 4}$

$C = \underset{m\ \ n}{10 \times 6}$.

$m - r = 10 - 6 = \underline{4}$

Q $\underset{3\times3}{A}\ \underset{3\times1}{X} = \begin{bmatrix}2\\4\\2\end{bmatrix}$, $X = \begin{bmatrix}2\\0\\0\end{bmatrix} + c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\1\end{bmatrix}$ — $N(A) = 2$

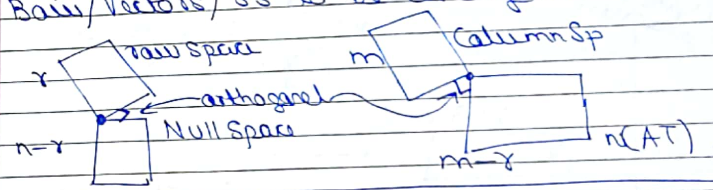
What's dim of Row Space (A)?

$\dim N(A) = 2$.
$\dim R(A) = 1$.

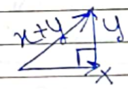
$\Rightarrow A\begin{bmatrix}2\\0\\0\end{bmatrix} = \begin{bmatrix}2\\4\\2\end{bmatrix}$ $\quad A = \begin{bmatrix}1 & -1 & 0\\2 & -2 & 0\\1 & -1 & 0\end{bmatrix}$

* Lec 14: Orthogonal Vectors & SubSpaces

Basis/Vectors/SS to be orthogonal.



* Orthogonal Vectors (perpend. Vectors).



(dot Product)
$x^T.y = x_1y_1 + x_2y_2 + \dots$
$\Rightarrow 0$.

Pythagoras

$\boxed{\text{if } (x^Ty = 0)}$ // Orthogonal.

$|x|^2 + |y|^2 = |x+y|^2$ ∴ if orthogonal.

$X = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $\quad |X|^2 = 14$

$\boxed{\|X\|^2 = \underset{1\times3}{X^T}\underset{3\times1}{x}}$ $\therefore$ 1×1

$X = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $\quad Y = \begin{bmatrix}2\\-1\\0\end{bmatrix}$ $\quad \boxed{X^T Y = 0}$ $\therefore$ orthogonal vectors.

$\|X\|^2 = 14 \quad \|Y\|^2 = 5$

$X+Y = \begin{bmatrix}3\\1\\3\end{bmatrix}$ (Proved)

$\|X+Y\|^2 = 19.$

* $X^T X + Y^T Y = (x+y)^T(x+y)$ (when orthogonal)

Pyth theorem

$\Rightarrow \cancel{X^T X} + \cancel{Y^T Y} + \underset{s}{X^T Y} + \underset{s}{Y^T X}.$

$\therefore 2X^T Y = 0.$

$\therefore \boxed{x^T y = 0}$ ← Pythag. lead me to this.

* if $x = 0$, y = anything $\therefore$ orthogonal.

$\therefore$ (0 vector orthogonal to all) ✓

* SubSpace (S) is orthogonal to SubSpace (T)

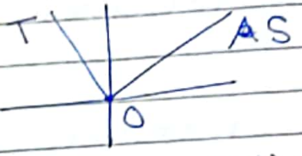
means: every vector in S is orthogonal to every vector in T.

eg: blackB, Floor [xz & xy plane].

(these 2 planes aren't orthogonal).

$\therefore$ Vectors in S aren't $\perp$ to vect in T.

- SubSpaces in R^2. S & T are orthogonal



* Row Space is orthogonal to Null Space [X in]

$AX = 0$

$\underset{A}{\begin{bmatrix} r1 \\ r2 \\ rm \end{bmatrix}} \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Row Space = all LC of Rows

$\therefore X$ orthogonal to $\underbrace{r1, r2 \cdots rm}_{\text{all rows}}$

$c_1(r_1)^T x = 0$

$c_2(r_2)^T x = 0$

as, $c_1(r_1^T)x + c_2(r_2)^T x = 0.$

$(c_1 r_1 + c_2 r_2 + c_3 r_2)^T x = 0.$

$\therefore$ X is orthogonal to row Space.

* Orthogonal SubSpaces in 3D.

[2D SS,

rS = line, nS = line ; X

$n=3, r=1$

- $A = \begin{bmatrix}1 & 2 & 5\\ 2 & 4 & 10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$ $\quad \dim N(A) = 2 \quad n-1$

→ plane $\perp$ to [1 2 5] Vector (N) Normal Vector

$\dim(R(A)) = 1$

$\dim(N(A)) = 2.$

Fundamental LA [Part 2]

* NullSpace & RowSpace are orthogonal complements in R^n.

NullSpace contains all Vectors that are $\perp$ to row Space.

* Solve $AX=b$, when no Solution

b not in CS of A.

$m > n \rightarrow$ No Soln. [lot of info, separate noise from info]
[eq^ns] [unkn.]

but Soln ?

$\Rightarrow A^TA$ → $\cdot$ ~~|BA|~~^{2}
$n\times m$ $m\times n$

$\cdot$ $n\times n$ Square M

$\cdot$ Symmetric.

$(A^TA)^T = A^TA$

$AX=B$

$\boxed{A^TAX = A^TB}$ → Central Eqn.

Hope, this one will have soln.

Q $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ (A)

3eq^ns, 2UK

X

B need to be in CS, can be solved.

[Most Vectors won't be on that plane]

$m=3$
$n=2$
$m>n$.
rank = 2

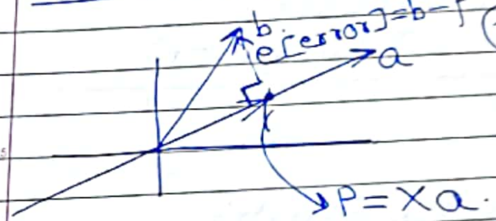
$\boxed{rank(A^TA) = rank(A)}$

$\cdot$ $A^TA = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$ → invertible.
[not always invertible].
rank = 2, $|A| \neq 0$

$N(A^TA) = N(A)$
$rank(A^TA) = rank(A)$
rank = n → A

Note: → A^TA is invertible, exactly if, A has independent columns.

* Lec 15: Projections onto SubSpaces



(2D)

a: 1D SubSpace.

x = some multiple.

$a^Te = 0$

$\boxed{a^T(b - xa) = 0}$ (a is perp to p).

$x a^Ta = a^Tb$

$\boxed{x = \frac{a^Tb}{a^Ta}}$ (1) , $\boxed{p = xa}$ (2) p → projection.

$\boxed{p = a \cdot \frac{a^Tb}{a^Ta} \rightarrow Pb}$ P: projection matrix.

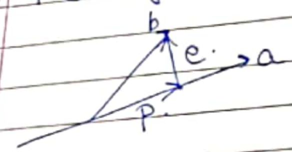
$\cdot$ if $b \rightarrow 2b$, $p \rightarrow 2p$
$a \rightarrow 2a$, p still same.

(3) $\boxed{P = \frac{aa^T}{a^Ta}}$ → matrix, → no.

Properties of Projection Matrix

$P = \frac{a a^T}{a^T a}$ $\quad$ $p = Pb$ $\quad$ p on a.

→ $C(P)$ = line through a.
→ rank$(P) = 1$
→ P is symmetric. $\quad$ * $\boxed{P^T = P. \quad P^2 = P}$



$Pb = p$.

→ ∴ if project twice, same line.
$P^2 = P$ (Idempotent)

---

* Why project?

∵ $AX = b$ may have no Solution.

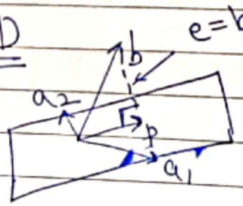
AX has to be in $C(A)$ ∴ b is not.
& we choose the closest Vector in $C(A)$

∴ Solve $A\hat{X} = p$ $\quad$ $\boxed{\hat{X} = \text{best possible}}$
p → Projection of b onto $C(A)$

---

* 3D

$e = b - p$ $\quad$ $\boxed{b \text{ not in } CS}$



a_1, a_2: basis for plane.

plane of a_1, a_2 = colSpace of
$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$

• e is ⊥ to the Plane.

$p = \hat{x}_1 a_1 + \hat{x}_2 a_2$
$\Rightarrow A\hat{X}$

$\boxed{p = A\hat{X}}$

Q $p = A\hat{x}$, Find $\hat{x}$.

Key: $b - \underbrace{A\hat{x}}_{p}$ is perpendicular to plane.

e is ⊥ to a_1 & a_2 also. (everything in Plane)

∴ $a_1^T(b - A\hat{x}) = 0$ —— (i)
$a_2^T(b - A\hat{x}) = 0$. —— (ii)

$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

↓

$A^T\underbrace{(b - A\hat{x})}_{e} = 0$. $\quad$ $a^T(b - xa) = 0$ → 2D, Line.

$A^T e = 0$.

geometry → $\boxed{\therefore e \text{ is in } N(A^T). \quad e \text{ is } \perp \text{ to the } C(A)}$ $\quad$ CS ⊥ $N(A^T)$

• $\boxed{\underbrace{A^T A}_{n \times n} \hat{x} = A^T b}$ —— eqn (Key eqn). $\quad$ $a^T a \hat{x} = a^T b$

$\rightarrow A^T A\hat{x} = A^T b$: Key eqⁿ

$\hat{x} = (A^TA)^{-1}A^Tb$ P: projection

$p = A\hat{x} = A\underbrace{(A^TA)^{-1}A^T}b \rightarrow \frac{aa^T}{a^Ta}$ in 1D (Here no)

Projection Matrix in nD: $\boxed{P = A(A^TA)^{-1}A^T}$

$\Rightarrow A \cdot A^{-1}(A^T)^{-1}A^T \longrightarrow I$ (if A is sq. invertible)

A: not Square $\begin{bmatrix} a_1 & a_2 \end{bmatrix}$ ∴ not invertible ∴ $(A^TA)^{-1} = A^{-1}$ ✗

- if A is invertible Square Matrix: $C(A) \rightarrow$ whole of R^n. if, project b onto whole Space. ∴ b already in CS. $\underline{e=0}$

∴ $\boxed{P = I}$

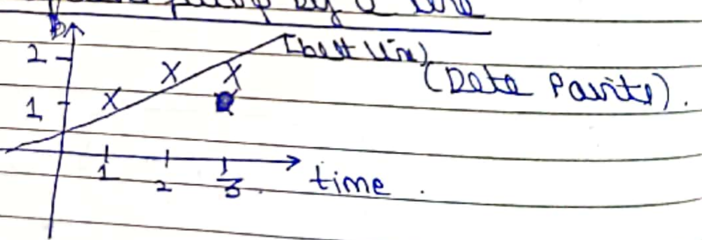
→ P is Symmetric, $P^T = P$.

→ $P^2 = P$ (Geometrically) ✓

∴ $A(A^TA)^{-1}\underbrace{A^T \cdot A(A^TA)^{-1}}_{\times}A^T$.

Application:

* Least Squares fitting by a line



- fit points: (1,1), (2,2), (3,2) (by a line) [Find best line]

- $b = C + Dt$ = line

$C + D = 1$

$C + 2D = 2$

$C + 3D = 2$ → this not in C(A)

$\rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ → 2D subspace.

$A \quad X \quad = b$. [No Solⁿ, 3eqⁿ, 2UK]

∴ Solve; $A^TA\hat{x} = A^Tb$. (Mul both by A^T)

* Lec 16: Projection Matrices & Least Squares

- $P \cdot b = b$; if b already in CS
- $Pb = 0$. if b ⊥ column space [no comp in CS]. b in $N(A^T)$.

$Pb = A(A^TA)^{-1}\underbrace{A^T \cdot b}_{0} \Rightarrow 0$ (if $b \perp C(A)$)

$Pb = A(A^TA)^{-1}A^T \cdot AX = AX = b$.

∴ b in C(A)

$\underline{Ax = b}$

p: nearest [illegible] C(A)

Imp

- $p + e = b$

$Pb \quad \underbrace{(I-P)}b$

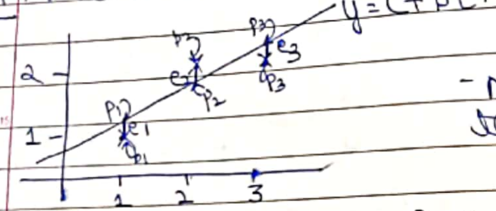
Project to C(A)

Projection onto the Perpend. Space

$(I-P)^2 =$ [illegible]

$\boxed{P = A(A^TA)^{-1}A^T}$

Aim: Find the best straight line. $y = C + Dt$.



- minimise the error to find best C & D

$$\left.\begin{aligned} C+D&=1\\ C+2D&=2\\ C+3D&=2 \end{aligned}\right\}\ 3\ eq^n s.$$

- $AX = b$

$$\underset{A}{\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}}\underset{X}{\begin{bmatrix}C\\D\end{bmatrix}} = \underset{b}{\begin{bmatrix}1\\2\\2\end{bmatrix}}$$

// No Solution. $\because$ [b ∉ C(A)]

Basis for CS = 2

- $|AX-b|^2 = |error|^2$ (e vector).

$= e_1^2 + e_2^2 + e_3^2$ (minimise this)

Linear Regression // Sum of squares → measure of error.

Camlin Page 61 Date / /

✳ If an outlier datapoint, that won't be the best line (∵ giant error huge). ← Statistical term.

the points on the line → $\underbrace{P_1, P_2, P_3}$

- components that lie on line
- if these were put in Rhs of eq^ns, eq^n could be solved.

Q. Find $\hat{X} = \begin{bmatrix}C\\D\end{bmatrix}$, P.

$\boxed{A^TA\hat{X} = A^Tb}$ → most imp eq^n to use, if got any error/Noise.

$A^TA: \begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix} = \begin{bmatrix}3&6\\6&14\end{bmatrix}$ (Symmetric/idempotent/invertible).

$A^Tb: \begin{bmatrix}5\\11\end{bmatrix}$. $\hat{X} = \begin{bmatrix}C\\D\end{bmatrix}$.

$$\therefore \left.\begin{aligned} 3C + 6D &= 5 \quad (i)\\ 6C + 14D &= 11 \quad (ii) \end{aligned}\right\}\ \underline{\underline{\text{Normal eq}^n\text{s}}}$$

$e = AX - b$.

Or min: $e_1^2 + e_2^2 + e_3^2$.

$f(C,D) = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

↓ use calculus.

$\frac{\partial f}{\partial C} = 0$ — (i), $\frac{\partial f}{\partial d} = 0$ — (ii)

Awsm

$3C + 6D = 5$
$6C + 4D = 11$
$\boxed{D = 1/2,\ C = 2/3}$ → the best fit line.

is $\frac{2}{3} + \frac{1}{2}t$

* $p_1 = 7/6$, $p_2 = 5/3$, $p_3 = 13/6$

* $e_1 = -1/6$, $e_2 = 1/3$, $e_3 = -1/6$

$p + e = b$ $\qquad$ $\underline{e = b - p}$

$$\underset{b}{\begin{bmatrix}1\\2\\2\end{bmatrix}} = \underset{p}{\begin{bmatrix}7/6\\5/3\\13/6\end{bmatrix}} + \underset{e}{\begin{bmatrix}-1/6\\2/6\\-1/6\end{bmatrix}}$$

[True Point] [Proj. Points] [error Points].

* p & e are ⊥ to each other.

$-7/36 + 20/36 + -13/36 = 0$.

∴ orthogonal

* e is ⊥ to all vectors in whole Col Sp.

$\begin{bmatrix}1\\1\\1\end{bmatrix}$, a basis

∴ $-1/6 + 2/6 - 1/6 = 0$ ✓.

∴ Just Solve $A\hat{x} = p$.

[A matrix is invertible, if its ...

* $A^TA = \begin{bmatrix}3 & 6\\6 & 14\end{bmatrix}$ invertible?

Q** if A has independent cols, then A^TA is invertible
(rank = n for invertible)

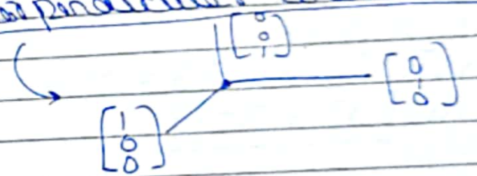
Proof:
Supp $A^TAx = 0$
To prove, x must be 0. → A^TA: cols are independent

Trick: $x^TA^TAx = 0$
$(Ax)^T \cdot Ax = 0$. $\qquad y^Ty = 0 \Rightarrow \|y\|^2 = 0$. ∴ $y = 0$

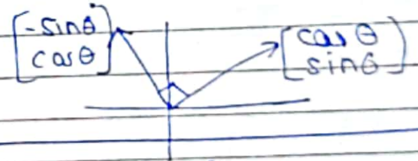
∴ $Ax = 0$ →

. if A has independent cols, $Ax = 0$, then x is 0

* Columns are definitely independent if they are perpendicular unit vectors like (orthonormal vectors)



A^TA ? → I (A with these 3 cols)


← orthonormal vectors.

* Lec 17 (Last lec on Orthogonality)

. Orthogonal basis
. Orthogonal matrix (Q).

$q_1, q_2 \cdots q_n$: orthogonal vectors.

$$\Rightarrow q_i^T q_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$ Orthonormal vectors.

New Matrix : Q
[cols are orthonormal]

* $Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$

$$Q^TQ = \underbrace{\begin{bmatrix} q_1^T - \\ \vdots \\ q_n^T - \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} q_1 & \cdots & q_n \\ | & & | \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ 0 & 0 & 1 & \cdots \\ & & & 1 \end{bmatrix}}_{I}$$

$\Rightarrow$

$\Rightarrow$ We call Q: orthogonal matrix, when its a square matrix.

Property: if Q is square, then $Q^TQ = I$
tells us $\boxed{Q^T = Q^{-1}}$

eg. $$Q = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \times \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{Q^T} = I.$$

eg. $\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{Q}$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ Q (not orthogonal Matrix).

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ ✓

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$ ← fav orthogonal Matrix. [Adhemar]

$\Rightarrow$ Why orthogonal matrix?

eg: (Rectangular)

$$\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$ → these 3 cols are orthonormal, & these are the orthonormal basis for the 3D space they span. [they are obv. independent].

$\boxed{Q \text{ has orthonormal columns}}$ (convention).

Project onto its column space, $P = ?$

$$P = Q\underbrace{(Q^TQ)}_{I}^{-1}Q^T = \underline{\underline{QQ^T}}$$ $P = QQ^T$

$P = I$ if Q is square $\because Q^T = Q^{-1}$

if, Q is rectangle $\neq$ $P = QQ^T$

- symmetric
- idempotent $P^2 = P$.

$$\underbrace{(QQ^T)(QQ^T)}_{I} = QQ^T.$$

$= QQ^T$ ✓

- $A^TA\hat{x} = A^Tb$ [Imp eqn simplified].

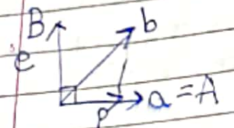
Now, A if Q [orthonormal columns]

$\Rightarrow \underbrace{Q^TQ}_{I}\hat{x} = Q^Tb$.

$\therefore \boxed{\hat{x} = Q^Tb}$ ✓. → $\hat{x}_i = q_i^Tb$.

Graham-Schmidt

- make the matrix orthogonal.



orthogonal vectors A, B → orthonormal vectors $q_1 = \frac{A}{|A|}$, $q_2 = \frac{B}{|B|}$

· independent vectors a, b.

$$\rightarrow A = a$$
$$\rightarrow B = b - \overbrace{\underbrace{\frac{A^Tb}{A^TA}}_{P} \cdot A}^{P}$$

$P = \frac{ATA}{ATA}$

· $A^TB = 0$ (to be orthogonal)

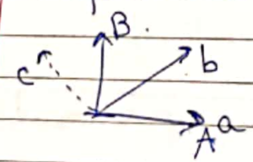
$\therefore A^T\left(b - \frac{A^Tb}{A^TA} \cdot A\right)$

$\Rightarrow A^Tb - A^Tb = 0$. Proved

A & B are orthog

---

* independent vectors a, b, c now.



· $C = c - \underbrace{\frac{A^Tc}{A \cdot A} A}_{c\text{'s comp in } A} - \underbrace{\frac{B^Tc}{B^TB} \cdot B}_{c\text{'s comp in } B}$

$C \perp A, C \perp B$.

$\frac{C}{|C|}$ = unit vector.

eg: $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ → 2 independent vector

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$.

A

$B = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{b} - \underbrace{\frac{A^Tb}{A^TA}}_{3/3 = 1} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{A} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ B $A \perp B$

$q_1 \cancel{=}$ $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}}_{\text{orthonormal cols}}$.

· $C(Q) \overset{\text{same}}{=} C(a, b)$. [just adjusted b to 90°].

· Gram-Schmidt made these into orthonormal basis.

· $\underline{A = LU}$ → $\boxed{A = QR}$ (Exp of Gram Schmidt).

$\underset{a \quad b}{\begin{bmatrix} a_1 & a_2 \end{bmatrix}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} a_1^Tq_1 & a_2^Tq_1 \\ a_1^Tq_2 & a_2^Tq_2 \end{bmatrix}$

R = upper triangular

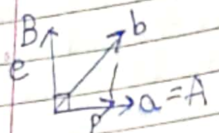
$a_1^Tq_2 = 0$. $\therefore q_2 \perp a_1$

* $A = QR$

$$\underset{3\times2}{\begin{array}{cc} a_1 & a_2 \\ \hline 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{array}} = \underset{3\times2}{\begin{array}{cc} q_1 & q_2 \\ \hline 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{array}} \cdot \underset{2\times2}{\begin{bmatrix} 3/\sqrt{3} & \\ 0 & \end{bmatrix}}$$

Graham-Schmidt

- make the matrix orthogonal.



$a, b \longrightarrow$ orthogonal vectors A, B $\longrightarrow$ orthonormal vectors

independent vectors a, b

$q_1 = \frac{A}{|A|}$ $q_2 = \frac{B}{|B|}$

$\rightarrow A = a$

$\rightarrow B = b - \underbrace{\frac{A^Tb}{A^TA} A}_{p}$

$P = \frac{A^TA}{A^TA}$

- $A^TB = 0$ (to be orthogonal)

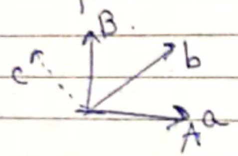
$A^T\left(b - \frac{A^Tb}{A^TA} A\right)$

$\Rightarrow A^Tb - A^Tb = 0$. Proved

A & B are orthog.

---

* independent vectors a, b, c now.



- $C = c - \frac{A^TcA}{A \cdot A} - \frac{B^Tc}{B^TB} B$

$\frac{A^Tc}{A \cdot A}A$ → c's comp in A

$\frac{B^Tc}{B^TB}B$ → c's comp in B

$C \perp A, C \perp B$.

$\frac{C}{|C|}$ = unit vector

Camlin Page 67

eg: $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ → 2 independent vector

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$.

$a \to A$

$B = \underset{b}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} - \underbrace{\frac{A^Tb}{A^TA}}_{\frac{3}{3} = 1} \underset{A}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ B $A \perp B$

$q_1 = \cancel{A}$

$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}}_{\text{orthonormal cols}}$.

- $C(Q) \overset{\text{same}}{=} C(a, b)$. [just adjusted b to 90°]

Gram-Schmidt made these into orthonormal basis.

- $\underline{\underline{A = LU}} \longrightarrow \boxed{A = QR}$ (Exp of Gram Schmidt).

$\underset{a \quad b}{\begin{bmatrix} a_1 & a_2 \end{bmatrix}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} a_1^Tq_1 & a_2^Tq_1 \\ a_1^Tq_2 & a_2^Tq_2 \end{bmatrix}$

R = upper triangular

$a_1^Tq_2 = 0$. $\because q_2 \perp a_1$

* $A = QR$

$\underset{3\times2}{\begin{array}{cc} a_1 & a_2 \\ \left[\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 0 \\ 2 \end{matrix}\right] \end{array}} = \underset{3\times2}{\begin{array}{cc} q_1 & q_2 \\ \left[\begin{matrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{matrix}\right. & \left.\begin{matrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{matrix}\right] \end{array}} \underset{2\times2}{\begin{bmatrix} 3/\sqrt{3} & \\ 0 & \end{bmatrix}}$

* Lec 18: Properties of Determinants

- Conc. on Square matrices.
- no. associated with every Sq. matrix.

$\boxed{\det A = |A|}$

* invertible, when $|A| \neq 0$.

⇒ 3 properties of det:

1. $\det(I) = 1$
2. exchange 2 rows:
   . reverse sign of determinant.

⇒ $\det(\text{Permutn Matrix}) = \begin{cases} 1, & \text{if even \# exchange} \\ -1 & \text{odd \# exchange} \end{cases}$

① $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

② $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$ → perm. matrix

⇒ $\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$

3. → multiply 1 row by t.

a) $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} \Rightarrow t \cdot |A| \Rightarrow t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

b) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Det. is linear for each row. (if all other rows are same).
. each row separately.

4. If 2 equal rows, the det is 0.

($7 \times 7 \rightarrow r_1 = r_3$).

. exchange those rows → same matrix
two

(det same, sign didn't change)
∴ 0. Ans $\boxed{|A| = 0}$

5. Subtract $l \times$ row i from row k
(& determinant doesn't change).

∴ $|A| = |U|$ (in elimination as b4).

eg: $\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$ (3b)

Proof:

$\Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} + -l \cancel{\begin{vmatrix} a & b \\ a & b \end{vmatrix}} \rightarrow 0$

⇒ det

6. Row of zeros ⟶ $\det A = 0$

$5 \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}$

$5|A| = |A|$. $|A| = 0$. ✓

1,5,3a

7. $U = \begin{vmatrix} d_1 & \times & \times & \times \\ 0 & d_2 & \times & \times \\ 0 & 0 & \ddots & \times \\ 0 & 0 & 0 & d_n \end{vmatrix}$ → upper Δ Matrix.

$\boxed{|U| = d_1 d_2 \cdots d_n}$
[Product of pivots].

$A \xrightarrow[\text{elimin}^n]{} U$
(row exchange) → then switch the sign too.

Proof: (make 0s in place of *) by possible elimin^n too.
[left with a Diagonal matrix].

(30) → $\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots d_n \end{bmatrix} = d_1 d_2 \cdots d_n \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix}$ I 1.

$\Rightarrow \prod_{i=1}^{n} d_i$

8. $\det(A) = 0$
exactly when, A is singular.
By elim, we get row of zeros [even if 1 diagonal elem = 0 after elimin^n].

$\det(A) \neq 0$
when A is invertible.
→ when elimin^n provides full set of pivots.

$\boxed{A \longrightarrow U \longrightarrow D \rightarrow d_1 d_2 \cdots d_n.}$
$\underbrace{\qquad\qquad}_{\text{elim}^n}$

eg: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \longrightarrow \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} \cdot b \end{vmatrix} \longrightarrow$ → if a is 0, row exchange

→ ad − bc

9. $\det(A \cdot B) = \det(A) \cdot \det(B)$

$\det(A^{-1})$ $\quad A^{-1} A = I$
$\det(A^{-1}) \cdot \det(A) = 1$.
$\Rightarrow \frac{1}{\det(A)}$

$\boxed{|A^{-1}| = \frac{1}{|A|}}$ → imp

eg: $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$.

→ $|A^2| = |A|^2$.
→ $|2A| = 2^n |A|$ $\quad$ if A: n×n Matrix.
∵ factoring 2 out of n rows.
area, volume.

10. $|A^T| = |A|$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

∴ if a col. is all 0s ⇒ $|A| = 0$.
∴ exchanging 2 cols ⇒ reverses the sign.

Proof: $|A^T| = |A|$
$|U^T L^T| = |LU|$
$\Rightarrow |U^T||L^T| \Rightarrow |L| \cdot |U|$
→ product of diagonals anyway

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* Lec 19: Determinant Formulas & Cofactors

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$ ← Linear Prop.

(1) $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$, (2) $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

Permut.

$$\Rightarrow \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & \cdot \\ 0 & \cdot \end{vmatrix}$$

$0 + ad + -bc + 0$

$\therefore det = ad - bc$

if 3x3.

$\underset{1^{st}\,row}{1 \longrightarrow 3P} \longrightarrow \underset{2^{nd}\,r.}{9P} \longrightarrow \underset{3rd}{27P}$. (LC)

$$\Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}$$

pick 1 for every row/col

$a_{11}a_{22}a_{33} + - \underbrace{a_{11}a_{32}a_{23}}$

$$+ \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$$ $\because$ 1 row exch

$-a_{12} \cdot a_{21} \cdot a_{33}$. $+\underbrace{a_{12}a_{23}a_{31}}$

$\because$ 2 row exch.

$$- \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$+a_{13}a_{21}a_{32}$ $\quad -a_{13}a_{22}a_{31}$

$\therefore$ n! ways to calculate determ.

$\{ + \setminus \quad - / \}$

→ for 4x4, + → / ?

* For nxn Matrix:

Big Formula: $\boxed{|A| = \sum_{n!\ terms} \pm a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}}$

$\underbrace{(\alpha, \beta, \gamma \cdots \omega)}_{cols.}$ = perm of $(1, 2 \cdots n)$.

eg: $\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$ $\xrightarrow{\text{Big For}}$

4! terms. $\quad \left.\begin{matrix}(4,3,2,1) = +1 \\ (3214) = -1\end{matrix}\right\}$ 2 terms

$\therefore D = 0 \ (1-1)$ $\quad \because$ 22 terms were 0.

→ Singular Matrix..

(rows & cols are dependent).

$\therefore r_1 - r_2 + r_3 - r_4 = 0$

* Cofactors → in pieces

determinant small part

from pg 72

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}$$

$|d|$ = cofactor [like some, recursive call].

$$\therefore \boxed{\text{Cofactor of } a_{ij} = (-1)^{i+j} \cdot \det \underbrace{\begin{pmatrix} n-1 \text{ Matrix} \\ \text{with row } i, \\ \text{col } j \text{ erased} \end{pmatrix}}_{\text{Minor}}}$$

$$\therefore \boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots a_{1n}C_{1n}}$$

along row 1 (Cofactor Formula)

$$* \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d + b(-c).$$

cofactor of a_{11}

⇒ det = Product of pivots
= Big formula
= cofactor formula

eg

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1$

$|A_2| = 0$

$|A_3| = 1 \cdot 0 - 1\,1 = -1$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} -1$$

(same det with row exch).

$$|A_4| = 1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$\Rightarrow -1 - 1(0)$ $\quad |A_4| = 1 \cdot |A_3| - 1|A_2|$

$\Rightarrow -1$

imp

$$\therefore \boxed{|A_n| = |A_{n-1}| - |A_{n-2}|}$$

$$|A_4| = |A_3| - |A_2| = -1$$

$\therefore$ $|A_5| = 0$
$|A_6| = 1$
$|A_7| = 1$
$|A_8| = 0$.

period 6.

$\therefore |A_6| = |A_1|$.

* Cramer's Rule, Inv: Lec 20

1. Inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

C_{11}, C_{21}, C_{22}, C_{12}

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot \underbrace{[\text{Cofactor}]^T}_{\text{Adjoint}}}$$

$* \; 3\times3$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det A} C^T$$

C^T → product of $n-1$ entries.

$\det A$ → product of n entries.

⇒ How to check: $A \cdot A^{-1} = I$.

$A \cdot C^T = (\det A) I$. Proof

$$\begin{bmatrix} a_{11} & \cdots & a_{n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & & C_{n1} \\ C_{12} & & C_{n2} \\ \vdots & & \vdots \\ C_{1n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

a_{ij}.

Cofactor Formula $\left\{ \begin{array}{l} \bullet\, a_{11}C_{11} + a_{12}C_{12} + \cdots \\ \quad \cdots a_n C_{1n} \\ \Rightarrow \det \end{array} \right.$

• $a_{11} \cdot C_{12} + a_{2} \cdot C_{22} + \cdots a_{1n} C_{2n}$ ⇒ 0.

from 2×2

its like $\det(A_S)$ → if $A_S = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$

⇒ $ab - ba \rightarrow |A_S|$.

Q. Non Singular Matrix add 1 to the a_{11}. What happens to A^{-1}? → use the formula.

Q. $AX = b$ $\qquad \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$

$X = A^{-1} b = \frac{1}{|A|} \cdot C^T b$ → $C_{11}b_1 + C_{21}b_2$.

* Cramer's Rule

$X_1 = \frac{\det B_1}{\det A}$ → $\therefore C \cdot b$ = determ of something (for sure).

$X_2 = \frac{\det B_2}{\det A}$

$\Rightarrow B_1 = \begin{bmatrix} b_1 & n-1 \text{ col of } A \end{bmatrix}$ ← A: $C_1 \xrightarrow{\text{rep}} b$

$\therefore C_{11} = \begin{bmatrix} a_{22} & \cdots & a_{n2} \\ \vdots & & \\ a_{2n} & \cdots & a_{nn} \end{bmatrix}$

from det, coming through the mat

$|B_1| = b_1 \cdot C_{11} + b_2 \cdot C_{21} + \cdots$

$\therefore B_j = \begin{bmatrix} \text{A with col } j \text{ replaced by } b \end{bmatrix}$

$\therefore \boxed{x_j = \frac{\det B_j}{\det A}}$

Cramer's Rule.

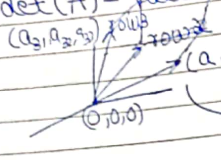
(Just find these determ to get x)

→ Very Time Consuming. (Elimin^n takes less time).

* Volume → Appln of Determinant.

3x3

→ det(A) = Volume of Box (Parallelopiped).



→ Parallelopiped.

if det (A) = −ve, V = |det|. (abs value)

- Sign of det → changing the edges (exc).
  - Left Handed & Right Handed Box

---

1) * A = I → Box = Cube. (Unit Cube)

|A| = 1 ✓

2) * Orthogonal Matrix (Q).

Unit Cube.

[Just turned in Space).

V = 1

⇒ |Q| = ?   · $Q^TQ = I$

Proof

$|Q^T| \cdot |Q| = 1$.

$|Q|^2 = 1$.

$\boxed{|Q| = 1}$ $\boxed{|Q| = \pm 1}$.

3) * Non Cubes

if I double an edge / double the first row

V   D : double

---

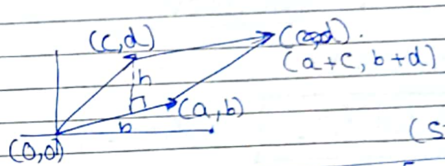
* No Longer. |A| = Volume of Box

4) ⇒ Parallelopiped

$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Shows Vol. has this Property.

*

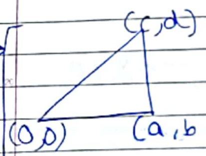


(start from 0,0)

→ Area of ||gm = b × h = det $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

⇒ ad − bc

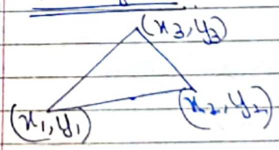
just you need to know the Coord.

→ 

⇒ $\frac{1}{2} \det \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= \frac{1}{2}(ad - bc)$

* Triangles



area = $\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

little elimn,

2 row → 1 = 0

3 row → 1 = 0.

≈ moving Δ to origin.

* Lec 21 : Eigen Values & Eigen Vectors

- Ax

i/p: x

o/p: Ax

certain vectors, where Ax comes out parallel to x (Same dir^n) → Eigen Vectors

$\Rightarrow Ax$ parallel to X.

$\therefore \boxed{Ax = \lambda x}$ λ = prop.

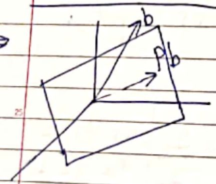
λ → E·Value. x → Eigen Vector

* Eig. Val 0 ?

$AX = 0$

E·Vectors are in the $N(A)$

Note: if A is singular $[|A| = 0]$, $\rightarrow X \neq 0$, then $\lambda = 0$: Eigen Value.

$\Rightarrow$ 

Q. E·Values of Projection Matrix? E·Vectors

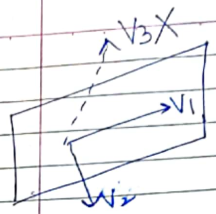
* if X in the Plane, will be an eigen Vector

$PX = X$

$\lambda = 1$

$EV = P$

– All E·Vectors in the $C(A)$



- V_1, V_2 = Plane.
- Any X that is $\perp$ to Plane

$PX = 0$.

$\therefore$ $\boxed{\begin{array}{l}\lambda = 0 \\ \text{E·Vector} = P\end{array}}$

$\therefore \lambda s \longrightarrow 0, 1$

* Permut^n M

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $X = ?$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$AX = X$

$AX = -X$

$AX = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda = 1$

$\lambda = -1$, $X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $AX = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -X$.

* Sum of E·Values = Trace. $\sum \lambda s = \text{Trace}$

(if $n \times n$ ⟶ max n Eigen Values)

* How to Solve $AX = \lambda X$, (2 UK)

$\Rightarrow \underbrace{(A - \lambda I)}_{\text{has to be singular, else } X = 0.} X = 0$. if, $x \neq 0$, $|A| = 0$.

- $|A - \lambda I| = 0$. // Characteristic Eq^n

Eigen Value Eq^n.

Eigen Vector

⇒ Find λ first from Char eq^n.

2x2

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$\Rightarrow (3-\lambda)^2 - 1 = 0$

$(3-\lambda)^2 = 1$

$\lambda^2 - 6\lambda + 8 = 0$ $\boxed{\lambda = 2, 4}$

Trace Determinant

$\boxed{\lambda^2 - \text{Trace}\,\lambda + |\det| = 0}$ → Char eq^n of 2x2.

$(A - \lambda I)X = 0$.

$\lambda = 4$ ⇒ $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} X = 0$. E·Vector → N(A).

⇒ $\boxed{X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 4}$

$\lambda = 2$ → $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X = 0$.

$\boxed{X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \lambda_2 = 2}$

→ bcw (∵ dot of EVs $\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3A.

* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\lambda_1 = 1, x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1, x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

∴ E·Vectors same, E·values = $\cancel{\lambda 3} + \cancel{3\lambda}$ $\lambda + 3$.

⇒ If $AX = \lambda X$

then $(A + 3I)X = \lambda X + 3X = (\lambda + 3)X$.

* if $AX = \lambda X$. B has E·Values $= \alpha_1$.

$BX = \alpha X$

$(A+B)X = (\lambda + \alpha)x$. X False.

∴ x can't be E·Vector of B

∴ $BY = \alpha Y$.

A+B / A·B → are not, $\lambda_1 + \lambda_2 / \lambda_1\lambda_2$.

Because, E Vectors are different.

* $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (orthog matrix). (rotates every vector by 90°)

skew symmet. matrix.

EVs & E·Vectors.

$\lambda_1 + \lambda_2 = 0$, $\lambda_1\lambda_2 = 1$.

$\lambda^2 + 1 = 0$.

$\boxed{\lambda = \pm i}$

complex conjugate of each other.

E·Vector?

(∵ rotated 90° → same dirn)

∴ Not poss.

* $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ → $\boxed{\lambda_1 = 3, \lambda_2 = 3}$ (repeated E.Value)

$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$

$\lambda = 3, 3$ — equal roots.

degenerate matrix → 1 Eigen Vector (dependent)

$(A - \lambda I)X = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $X_2 = $ [no 2nd independent eigen Vector. x_2]

* Diagonalization & Powers of A

S: E.Vector matrix. x_1, x_2 = E.Vector

Suppose n linearly independent E.Vector of A. Put them in cols of S

$AS = A \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & & \lambda_{..} & 0 \\ 0 & & & \lambda_n \end{bmatrix}}_{\text{E.Value Matrix } [\Lambda]}$

$\Rightarrow S\Lambda$

$\boxed{AS = S\Lambda}$

$AS = S\Lambda$ ← diagonalisation

$\Rightarrow \boxed{S^{-1}AS = \Lambda}$ (Provid: S is invertible)

$\boxed{A = S\Lambda S^{-1}}$

Q $AX = \lambda X$ — A^2? What are E.Values and E.Vectors?

$A^2X = \lambda \cdot AX = \lambda^2 X$.

E.Value of $A^2 \longrightarrow \lambda^2$.

E.Vector of $A^2 \longrightarrow X$.

* Intro to Eigen Values & Eigen Vectors

$T(\vec{v}) = \lambda \vec{v}$ → eigen Vector; λ → eigen Value assoc. with E Vector

$T(\vec{v_1}) = \vec{v_1}$ $\boxed{\lambda = 1}$

$T(\vec{v_2}) = -\vec{v_2}$, $\boxed{\lambda = -1}$

$AX = \lambda X$. (X → E.Vector.)

[dir/line they span doesn't get changed]

Geek's

* Diagonalization of Matrices

* Eign Values & Eign Vectors

$AX = \lambda X$. A: $n \times n$ Matrix
x in R^n.
λ is Scalar.

St. Scaling X by λ

→ if $x \neq 0$, & satisfies $AX = \lambda X$, X is said to be eign Vector of A with eign Value λ.

- $AX = \lambda X$.
$AX - \lambda X = 0$
$(A - \lambda I)X = 0$

$\because x \neq 0$, we need to find non trivial soln to it.

$|A - \lambda I| = 0$. $\because$ has free V. $N(A) \neq 0$.
[not invertible].

* $\det(A - \lambda I) = 0$.

eg: $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

$\lambda^2 + 6\lambda - 16 = 0$.
$\boxed{\lambda = 8, -2}$

* For $n \times n$ matrix A, $|A - \lambda I|$ will be a polynomial of degree n.

to find eign Vector. $\lambda = 8$

$(A - 8I)X = 0$. $AX = 0$.

$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $N(A)$?

$\begin{bmatrix} -1 & 3 \\ 0 & \cancel{-8} \end{bmatrix}$ $\cancel{-18}0$

$-x_1 + 3x_2 = 0$. $x_2 = 0$
$-18x_2 = 0$

$x_1 = 3x_2$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$\frac{x_1}{x_2} = \frac{3}{1}$.

$\therefore$ any mult. of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is eign Vector.

Q. $\lambda = -2$
$(A - (-2)I)X = 0$.

$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$.

$\begin{bmatrix} 9 & 3 \\ 0 & 0 \end{bmatrix}$ $\underbrace{\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}}$ Ans.

* Given an eigen value λ of Matrix A, we solve $(A-\lambda I)X = 0$ to find the associated eigen vectors. → homogeneous System.
$\therefore$ EV is $N(A-\lambda I)$.

⇒ The eigenSpace of A corresponding to λ is $N(A-\lambda I)$. [Null Space].

EigenSpace consists of 0 Vector & all EVs of $A-\lambda I$.

⇒ Basis of ES = Basis of $N(A-\lambda I)$.

eg. $\begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 8 & -8 & 14 \end{bmatrix}$ $\lambda = \{0, \underbrace{2, 2}_{\text{repeated eigen values}}\}$.

2: multiplicity 2

$\lambda = 0 \longrightarrow X = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (basis for ES of A ass. with $\lambda = 0$)

$\lambda = 2 \longrightarrow X = \underbrace{\begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\text{basis of ES of A with } \lambda = 2}$: 2 linearly indep. eigen Vectors.

Th: If $v_1, \ldots, v_r$ are eigen Vectors that correspond to distinct eigen Values $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, \ldots, v_r\}$ is linearly independent.

EV from diff E-Values are linearly independent.

$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}$: Linearly independent.

Q. is $\lambda = 3$, an EV of [ ] or EV of [ ]?

$|A - 3I| \stackrel{?}{=} 0$.
if Yes, 3 is an EValue.

* $|A - \lambda I|$ is called the characteristic equatn.

* Invertible Matrix Th: A is invertible if & only if 0 is not an eigen Value of A.

$|A - \lambda I| = 0$ satisfied by EV.
$|A| = 0$. $\therefore$ non-invertible.

* Matrix Diagonalization

if A & B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $A = PBP^{-1}$

Th If A, B are similar then same Char. Polynomial, same EValue (same multiplic).

* A is diagonal matrix.
$A = PDP^{-1}$

$A^2 = (PDP^{-1}) \cdot (PDP^{-1})$.
$= PD(I)DP^{-1}$
$A^2 = PD^2P^{-1}$

$A^3 = PD^3P^{-1}$

eg: $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow D^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$ $D^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}$

if A is similar to D, computing powers of A also easy

* A is said to be diagonalizable if A is sim. to diagonal Matrix,

$A = PDP^{-1}$ for some invertible matrix P & some diagonal matrix D.

* An $n \times n$ Matrix A is diagonalizable if & only if A has n linearly independent eigen Vectors.

In fact, $A = PDP^{-1}$ with D as diagonal Matrix if, cols of $P \longrightarrow$ n linearly independent EVectors of A

In this case, diagonal entries of D are eigen Values of A that correspond, respectively to the eigen Vectors in P.

eg: $A = \begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$ diagonalizable?

$\lambda = 8, -2$

2 distinct E.Values, 2 LI e. vectors $\rightarrow$ ∴ diagonalizable

$\lambda = 8$: $AX = \lambda X$. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (look closely, these are orthogonal EV). ∴ A is symm.

$\lambda = -2$: $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$

$\boxed{P = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}}$, $\boxed{D = \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix}}$

8 −2 (order).

Verify $A = PDP^{-1}$.

$A = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}$

$\lambda = 0, 2, 2$.

A doesn't have 3 distinct EV, we donot know whether A is diag. or not / 3 LI EVectors or not.

$\lambda = 2 \longrightarrow$ 2 Linearly Indep Eigen Vectors. $\begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 0 \longrightarrow$ 1 EV. $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

∴ 3 LI EV, ∴ Diganilizable

$P = \begin{bmatrix} -1 & 4/3 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

∴ $A = PDP^{-1}$ [Verified]

Q $A = \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix}$ $\lambda = 4, -2, -2$.

$\lambda = \cancel{4}\,-2 \longrightarrow$ 1 EV only, ∴ 1 free V.

∴ 3 EVs ✗, ∴ Not diagonalizable

Page 7

* Linear Transformations

$\underset{m\times n}{} \; \underset{n\times 1}{} \; \underset{m\times 1}{}$

$T(x) = AX$, where A: m×n Matrix.

$$\underset{R^n}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \longrightarrow T \longrightarrow \underset{R^m}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

$T: R^n \longrightarrow R^m$

Q $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T(x) = AX$ ← A: Permⁿ Matrix.

if $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$AX = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$T\left(\begin{bmatrix} 4 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$\boxed{T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}}$ → swapping order of element, (∵ Permⁿ Matrix)

reflect a vector across line $y = x$.

(Used in computer Graphics).

* A transformation T from R^n to R^m is a rule that assigns to each vector X in R^n a vector $T(x)$ in R^m.
R^n : domain of T, R^m : codomain of T.

→ for a given x in R^n, the vector $T(x)$ in R^m is called the image of x.

→ Set of all images $T(x)$ is called range of T.

range is subset of codomain.
[onto / into].

Domain = R^2.
Codomain = R^2.
Range of T = R^2. [∵ swapping rows)

Q $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ → abv ∵ determinant = 0, matrix squishes the space onto a line
~~C(A) = line in R^2~~

Domain of T = R^2.
Codomain = R^2.

$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$T\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 2(x_1 + 2x_2) \end{bmatrix}$

$\begin{bmatrix} kx \\ 2kx \end{bmatrix} \leftarrow \boxed{y = 2x}$

∴ Vectors are projected onto line $y = 2x$

∴ Range ⟶ all vectors of form $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 8 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

* How do we determine if given vector is in the range of T?

$T(X) = b$?

$AX = b$? [is b in C(A)].

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 2 & 4 & 10 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 0 & -2 \end{array}\right]$$

$R(A) \neq R(A/B)$

$\therefore$ No Solution (inconsistent)

* Linear Transf.

A transf T is linear if:

(i) $T(u+v) = T(u) + T(v)$, for u, v in dom(T)

(ii) $T(cu) = c\,T(u)$.

$T(X) = Ax \longrightarrow$ Linear Tran.

Q $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$; $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$T: R_2^2 \longrightarrow R^3$

$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$: colns of I_2

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 \cdot T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2\, T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \therefore \quad T(X) = AX$$

* I_n: Identity Matrix

Colms of I_n are $e_1\, e_2\, e_3 \ldots\ldots e_n$.

* Th: Let $T: R^n \longrightarrow R^m$ be a linear Transf.

Then there exists a unique matrix A such that:

$T(X) = Ax$ for all x in R^n.

in fact, A is the $m \times n$ matrix whose jth col is the vector $T(e_j)$

$A = [\,T(e_1)\ T(e_2) \cdots T(e_n)\,]$

$\hookrightarrow$ Standard Matrix for Liner Transf T.

Inverse

* f is not invertible if its not one-one / onto.

→ if values in codomain, which are not in range

→ if many to one.

$2^2 = 4$, $(-2)^2 = 4$

* A mapping $T: R^n \longrightarrow R^m$ Onto [b is in C(A)]. there is some x in R^n such that $T(x) = b$.

Since $T(X) = Ax$, this means that for every b in R^m, there is some x in R^n, such that $Ax = b$. [Pivot in each row]

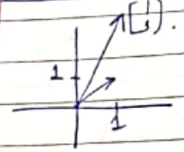
Scanned with CamScanner

One-one

- $T(X)=AX$, this means that for every b in R^m, there cannot be an infinite no. of solutions to the system $AX=b$
- The system $AX=b$ can have no free V. [Must be pivot in each col of A]

* Applications of EigenVectors & E-Values

⇒ App^n s of Matrix



$\begin{bmatrix}1\\1\end{bmatrix}$.

[scaled & rotated]

$\begin{bmatrix}3 & -2\\-1 & 4\end{bmatrix} \times \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$

in out

$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ will rotate, $\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ will scale.

Linear Transf.

$AX=b$

[Linear Algebra].

- A vector which is only scaled by that matrix, is called Eigen Vector of that matrix.

  Scale factor = E·Value.

Appl^n of Matrix:

1. Solves System of Eq^ns.

$\begin{pmatrix}3 & -2\\-1 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$ [tell the i/p Vector, which on transf. becomes $\begin{pmatrix}1\\3\end{pmatrix}$]

$T(X)=b$

$X = A^{-1}B$

* Appl^ns of EVs

- Matrix linear transformation makes a vector either scale/rotate. [diff length/angle]
- If the vector just scales after the matrix transf, that vector is just the eigen Vector

Q. Tell the eigen Vector of $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ rot^n matrix?

→ dne. except $\begin{bmatrix}0\\0\end{bmatrix}$ [imaginary]

⇒ if EVs are complex, then any input Vector will be rotated by some amount.

Ap① Fibonacci Sequence.

0 1, 1, 2, 3, 5, 8, 13, 21.

832040, next term

$F_n = F_{n-1} + F_{n-2}$

$\begin{bmatrix}F_n\\F_{n-1}\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}F_{n-1}\\F_{n-2}\end{bmatrix}$

$(8,5) \xrightarrow[\times\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}]{} (13,8)$

mostly at higher Values, doesnt rotate much
Evalue = 1.61803 (Golden Ratio)

$$\begin{bmatrix} 13 \times 1.618 \\ 8 \times 1.618 \end{bmatrix} = \begin{bmatrix} \text{next} \\ \text{Same} \end{bmatrix}$$

2)
* Basis for a Vector Space

- minimal representative set.

· Linear indep.

$\{v_1 v_2 \ldots v_p\}$ is said to be LI, if the only soln to equation is:

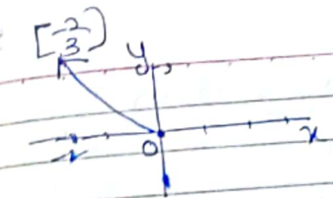
$c_1v_1 + c_2v_2 + \cdots c_pv_p = 0$ — only soln.
$c_1 = c_2 = c \ldots\ldots 0$

* LINEAR ALGEBRA . – 3Blue 1 Brown

– Vectors
· Arrows pointing in Space (len/dirn) (Phy)
· Ordered List of no. [CS] $\begin{bmatrix} 2600 \\ ₹120 \end{bmatrix}$ house → 2D List.
·
· Add/mult 2 nos : [Math]
$\begin{pmatrix} \vec{v} + \vec{w} \\ 2\vec{v} \end{pmatrix}$

· in LA, these vector rooted at origin.
$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \leftrightarrow$ 

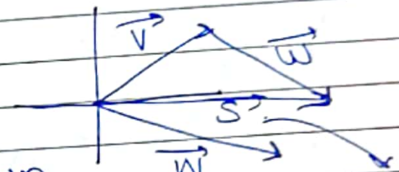
2D Space



in 3D

(every V associated with ordered triplet of nos)

*



Addn

$S = \text{sum} = \vec{v} + \vec{w}$.
[Vector w moved from origin].

$$\begin{matrix} x\text{ comp} \to \\ y\text{ comp} \to \end{matrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

4S on X
1S on Y.

Mult by no.

Scaling

(Scale the Vector)
$\vec{V} \longrightarrow 2\vec{V}$.

$-1.8 \longrightarrow$ reverse the vector then Scale

$$2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

* [Language to describe Space/data Points in Space.]

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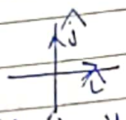
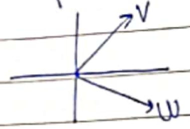
* **Linear Combination, Span & Basis Vectors**



$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3\hat{i} - 2\hat{j}$.

$\hat{i}, \hat{j}$ (basis Vectors) of XY coordinate System.

* what if we choose diff Basis Vectors.



v, w basis OK.

$\begin{bmatrix} -0.8 \\ 1.3 \end{bmatrix}$

(diff cordinate System)

* Linear comb of v, w

$a\vec{v} + b\vec{w}$

scalars.

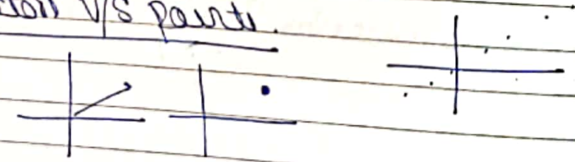
* The Span of $\vec{v}$ & $\vec{w}$ is the set of all their linear combinations.

$a\vec{v} + b\vec{w}$.

(if v w same line, span is that line) else, R^2. [2 dim Vector Space].

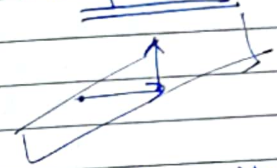
what all vectors you can reach with $\vec{v}$ & $\vec{w}$.

* Vectors v/s points.



* Think of sets of Vectors as points

* Span of 2 Vectors in a R^3 is plane.



[tip sits on the flat sheet].

[if Span of 3 Vectors]

$a\vec{v} + b\vec{w} + c\vec{u} \longrightarrow R^3$. (if lin indep.

→ if 3rd Vector on plane = plane.

if 3rd v not on plane = R^3.

[access any 3D vector]

* No vector should be redundent. Or linearly depend or 1 V = LC of other Vectors.

Basis: Set of linearly independent Vectors that Span the full Space.

$\hat{i} : \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

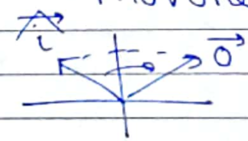
* Linear Transformation → movement

function

ip ⟶ op.

2 prop.

- Lines remain lines
- Origin fixed.



$T(\vec{v}) = \vec{r}$

[squishing/morphing Space] (AWSM!!)

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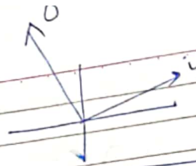
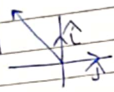


* $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ 

[Grid lines remain Parallel & evenly Spaced] (after Trans)

$\vec{V} = -1\hat{i} + 2\hat{j}$

(transf $\vec{V} = -1$ (tran $\hat{i}$) $+ 2$ (tran $\hat{j}$))

$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ ∵ coord. System changes. ∴ $\hat{i}$ & $\hat{j}$ changes / transform$}$

∴ LC can be described by $\hat{i}$ transf & $\hat{j}$ ~~transf~~.

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \\ \text{transf } \hat{i} \quad \text{transf } \hat{j} \end{array}$$ 2x2 Matrix.

10:59

$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5\underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\hat{i}\text{ transf}} + 7\underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\hat{j}\text{ transf}}$

• $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$

Trans. Versions of basis Vectors. $\begin{bmatrix} a \\ c \end{bmatrix}$ $\begin{bmatrix} b \\ d \end{bmatrix}$

(Matrix Vector multiplicn).

---

* rotate Space by 90.

$$\begin{array}{cc} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \text{tr } \hat{i} \quad \text{tr } \hat{j} \end{array}$$

* Shear

$$\begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ \hat{i} \quad \text{tr } \hat{j} \end{array}$$

Q What does transf of $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ looks like?

$\hat{i} \longrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ trans $\hat{i}$ transf $\hat{j}$: $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

if dep. colns: tr $\hat{i}$, tr $\hat{j}$ will be a line.

[Tran squishes all 2D Space into a line. [1 D Span]

* Matrix gives a lang to describe transf.

$\boxed{\text{Transformations of Space.}}$ (Such as Grid lines rem. Parallel & evenly Spaced).

* Matrix multiplicn as Composition

Composition of rotation & Shear.

$$\begin{array}{cc} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \\ \text{tr } \hat{i} \quad \text{tr } \hat{j} \end{array}$$

$$\underbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}}_{\text{Shear}}\left(\underbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}}_{\text{Rot}^n}\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&-1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

rotation then shear action

← right to left
(first rotate then shear)

$f(g(x))$

$$\begin{array}{cc}M_2 & M_1\end{array}$$
$$\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1&-2\\1&0\end{bmatrix}=\begin{bmatrix}2&0\\1&-2\end{bmatrix}$$

$tr\ \hat{i}=\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$

$tr\ \hat{j}=\begin{bmatrix}0&2\\1&0\end{bmatrix}\underbrace{\begin{bmatrix}-2\\0\end{bmatrix}}_{tr\ \hat{j}}=\begin{bmatrix}0\\-2\end{bmatrix}$

coord.

* transf M_1 & then M_2.

> Matrix mult: Apply 1 transf, & then the other one.

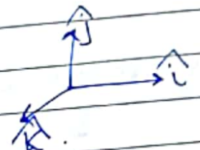
is $M_1M_2 \overset{?}{\neq} M_2M_1$.
rot, shear — shear, rotate.
(overall effect diff.)

ASSOC: $(AB)C = A(BC)$.

first transf C, then B then A
∴ Proved [Trivial]

* 3D Linear Trans

$$\begin{bmatrix}2\\6\\-1\end{bmatrix}\xrightarrow{T(\vec{V})}\begin{bmatrix}3\\2\\0\end{bmatrix}$$



record trans basis & write a 3x3 matrix.
∴ just 9 nos

Q rotate 90° along Y axis.

$\hat{i}_t \to \begin{bmatrix}0\\0\\-1\end{bmatrix}$ $\quad \hat{j}_t \to \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\quad \hat{k}_t \to \begin{bmatrix}1\\0\\0\end{bmatrix}$.

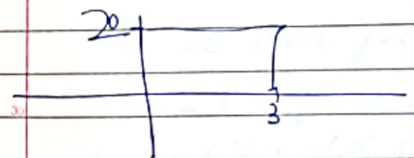
$x\hat{i}+y\hat{j}+z\hat{k}$.

$\underbrace{[\quad]}_{\text{second}}\ \underbrace{[\quad]}_{\text{first trans}}$

– Computer Graphics & Robotics.

* The Determinant.

How much area/regions inc/dec after the linear transformⁿ happens? [ratio]



$$\begin{bmatrix}3&0\\0&2\end{bmatrix}$$
$\hat{i}_t$ $\quad \hat{j}_t$

∴ Area inc by 6.

Determinant: Scale factor of Area transf [after linear Transf].

∵ if det = 3, incr. area by a factor of 3.

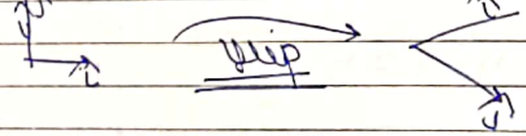
$$\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 0$$

∴ squashes every vector onto a line. $(1 \to 0)$.

Area = 0

Q: Det = −ve, how can you scale area by a negative no.?

Orientⁿ of space is inverted. flipping of paper.



* 3D transf ?

How much volume it gets scaled?

$(1 \times 1 \times 1) \xrightarrow{LT} \text{Parallelopiped}$.

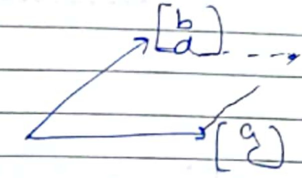
cube

det = 0 ⟶ Line/Plane/point
∴ (Volume = 0)
⟶ cols are linearly dependent.

if det < 0 ⟶ use right hand rule
if left hand, flip orientⁿ.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

⟶ Area of 11gm [how much area scaled].



* $[\;\;]_{3\times3}$

* $|M_1 M_2| = |M_1| \times |M_2|$. (Explain in 1 sentence).

* Inverse matrix, C(A), Rank, N(A)

Use of LA: Manipulate Space − Graphics
· Usefulness of Matrices [Linear Syst of eqⁿs].

xy ✗ sin x ✗ x^2 ✗

· Linear System of EQ's

$AX = b$

$$\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$\boxed{A\vec{x} = \vec{v}}$

$\vec{x} \longrightarrow$ Space of A [transf] $\longrightarrow \vec{v}$.

Inverse

Q $\underset{A}{\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{\vec{v}}{\begin{bmatrix} -4 \\ -1 \end{bmatrix}}$.

1) $|A| \neq 0$ [space doesn't get squashed into 0 area]

∴ Ans $X = VA^{-1}$

A = clock 90
A^{-1} = inv. clock 90

$\therefore \boxed{A^{-1}A = I}$ (graphically same part)

transf that does nothing $\hat{i}, \hat{j}$.

- $A\vec{x} = A\vec{V}$

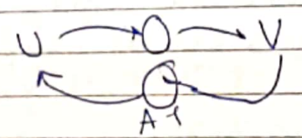
$\cancel{A^{-1}A}\vec{x} = A^{-1}\vec{V}$

$\boxed{\vec{x} = A^{-1}\vec{V}}$ $V \longrightarrow \bigcirc \longrightarrow x$

---

* [$Ax = b$, and $|A| \neq 0$: unique Sol^n]

$\therefore$ if $|A| \neq 0$, A^{-1} def exists.



u lands on V after Linear Transf.

But if $|A| = 0 \rightarrow$ Line.
A^{-1} dne $\therefore$ cant unsquish a line to get a plane

but:

$Ax = b$, if $|A| = 0$.

Sol^n exists if b in that line / $C(A)$.

---

* Rank $\rightarrow$ [No of dim of o/p Space / Col Space].

if linear transf st. squishes Space into line = rank 1.

squishes space into plane = rank 2.

like, if o/p of 3x3 Matrix fill whole R^3, rank = 3

* Column Space. $\rightarrow$ output Vector.

$\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$ LC of cols fill whole R^2 Space.
$t\hat{i}$ $t\hat{j}$.

---

* if Rank = # col = n

[Full Rank]

— If a 3D transf. squishes the space onto a plane, then there is a whole line of vectors, (thru origin), $\perp$ to Plane.

if line, then $N(A) =$ Plane.
$C(A)$.

$\therefore$ [Null Space / Kernel] $N(A)$ dim $= n - r$.

$A\vec{X} = \vec{V}$.

if $A\vec{X} = 0$ $\boxed{X = N(A)}$

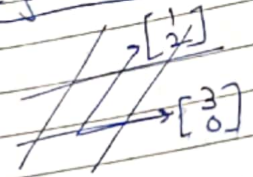
---

* Non Square matrices as transf

$\begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix}$ $\rightarrow$ 2D column Space in R3.
$\hat{i}$ $\hat{j}$

* Eigen Values & Eigen Vectors :

$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$



EVect → Vector which remains on its own span, after linear transf. (just scales)

EValue → $\lambda_1 = 3$, $\lambda_2 = 2$

(others get rotated).

* EValue $= -1/2$,
[flipped & squished by 1/2)
(but stay on line, didn't get rotated off it).

* g: Rotate a cube
(axis of rotation = EVector, $\lambda = 1$]
∵ don't stretch

$A\vec{V} = \lambda\vec{V}$
Matrix → A; Scalar mult. → λ

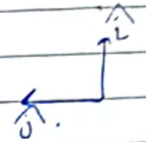
$(A - \lambda I)\vec{V} = 0$.

for non-zero EV.
$|A - \lambda I| = 0$
or, A needs to squish space to a lower dimension,
for $V \neq 0$.
[V will be $N(A - \lambda I)$]

$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \to \lambda = 2, 3$

* There could be no Eigen Vectors

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



(rotate by 90°)
∵ rotate every vector out of its own span. [imag. Eigen Values)

⇒ Shear

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

[1 1]

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → $\vec{E}$Vector.

$\lambda = 1$

* A single EValue can have more than a line full of eigen vectors.

Eg: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. $\lambda = 2$
(But every Vector in Plane is eigenvector).

* Eigen Basis

$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ → diagonal matrix.
[All basis vectors are eigen vector, with diag. entries → EValues.

$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdots \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{100\text{-times}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 3^{100} & 0 \\ 0 & 2^{100} \end{pmatrix}$

* Use Eigen Vectors as basis of the Coordi System. [15:22]

* Change of Basis.

$\vec{\uparrow}$ $\begin{bmatrix}3\\2\end{bmatrix} = 3\hat{i} + 2\hat{j}$.

[i, j : Basis of Coordinate System]

* Another Basis.

She drew her own Grid from this basis [same origin]

$b_2 \begin{bmatrix}-1\\1\end{bmatrix}$ $\begin{bmatrix}2\\1\end{bmatrix} b_1$ ← from our pers.

from her persp.
$b_1 = \begin{bmatrix}1\\0\end{bmatrix}$
$b_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$a b_1 + b b_2$

* How do you trans b/w coordinate sys

her $\begin{bmatrix}-1\\2\end{bmatrix} = -1 b_1 + 2 b_2$ $\quad b_1 = \begin{bmatrix}2\\1\end{bmatrix}, b_2 = \begin{bmatrix}-1\\1\end{bmatrix}$

$= -1\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$ our sys

if $\begin{bmatrix}3 & 1\\2 & 2\end{bmatrix}$ in my system,
what would this be in her System?

$\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1/3 & 1/3\\-1/3 & 2/3\end{bmatrix}$

* $\underbrace{\begin{bmatrix}1/3 & 1/3\\-1/3 & 2/3\end{bmatrix}}_{\text{Inv chang of basis}} \underbrace{\begin{bmatrix}3\\2\end{bmatrix}}_{\text{our syst}} = \underbrace{\begin{bmatrix}5/3\\1/3\end{bmatrix}}_{\text{her system}}$.

$\underbrace{\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}}_{\text{Chang of Basis}} \underbrace{\begin{bmatrix}-1\\2\end{bmatrix}}_{\text{her System}} = \underbrace{\begin{bmatrix}-4\\1\end{bmatrix}}_{\text{our System}}$.

* Jenn's 90° rotation of Space not $\underbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}_{\text{our matrix}}$

$\left[\text{inv chang of basis} \cdot \text{rot Mat.} \cdot \text{Chng of Basis} \cdot \vec{V}\right]$

→ transf matrix in her language.

11:21

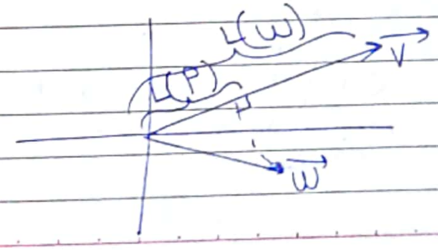
* Dot Products & Duality

Only with LT, we can underst.

$\begin{bmatrix}2\\7\\1\end{bmatrix} \cdot \begin{bmatrix}8\\2\\8\end{bmatrix} = \begin{bmatrix}16\\14\\8\end{bmatrix} = 38$

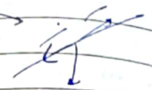
p·v = Dot

$\begin{bmatrix}4\\1\end{bmatrix} \begin{bmatrix}2\\-1\end{bmatrix}$

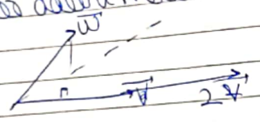


$\boxed{L(p) \cdot L(w) = \text{dot } P}$

if Vectors in same dir^n: dot > 0
else, dot < 0
⊥ , dot = 0



* Order doesn't matter



$2\vec{v} \cdot \vec{w} = 2(v \cdot w)$

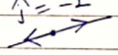
* But why Proj^n?

[Think in terms of LTs]

$$\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4 \cdot 1 + 3 \cdot -2 = \underline{\underline{-2}}$$

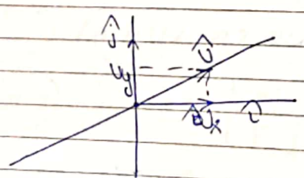
1x2

$\hat{i} = 1$
$\hat{j} = -2$



Single

$\begin{bmatrix} 2 \\ 7 \end{bmatrix} \rightarrow L(\vec{v}) \longrightarrow [\quad]$



$$\begin{bmatrix} U_x & U_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = U_x \cdot x + U_y \cdot y$$

$\hat{i}$ tran $\hat{j}$ tran

* Any time you have 2D to 1D linear transf, apply a dot.

duality

* Dot is used to understand proj, whether or not vectors point to same dir^n.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

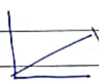
~~2D to 1D~~ tr

$$\begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

* Norms – LA

- ML uses tensors as basic units of rep.
- 2 reasons to use norms
  - how big a vector/tensor is (for scalar: no.)

∴ Norm : Mapping from Vector/Tensor ⟶ +ve Scalar.



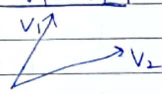
$v\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\boxed{\|V\|=5}$ (len of Vector) → norm

- to estimate how close one tensor is to another [how big diff is?]

$\boxed{\text{Cat}}$ 60 60 ⟶ [ ] 3600 px/s

$\boxed{\Delta V = V_1 - V_2}$



$\|\Delta V\| = \text{Ans}$

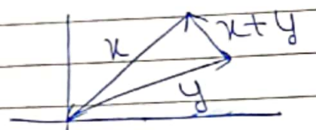
* Norm is the generalizn of the notion of len of vectors.

A norm is a function f that: [Properties]

(i) $f(x) = 0 \longrightarrow x = 0$ [0 vector].

$\|x\| = 0$

(ii) $f(x+y) \leq f(x) + f(y)$ [Triangle Ineq]



$\|x+y\| \leq |x| + |y|$. (School M)

∵ [SD is abv. Straight line].

f: Normal

(iii) $\forall a \in R$, $f(ax) = \underbrace{|a| f(x)}_{\text{length also incr by } \alpha}$.

$\boxed{\begin{array}{l} f(\vec{v}) \longrightarrow +\text{ve Scalar} \\ \text{Norm} \end{array}}$.

eg: Vector Norms

1) Euclidian Norm: [2 norm] → Natural

$\|V\|_{2} = (V_1^2 + V_2^2 + V_3^2 + \cdots V_n^2)^{1/2}$

- 2 norm or L^2 norm
- Corr to our notion of dist.

2) 1-norm

$\|V\|_1 = |V_1| + |V_2| + \cdots |V_n|$

3) p-norm ($p >= 1$)

$\|V\|_p = (|V_1|^p + |V_2|^p + \cdots |V_n|^p)^{1/p}$.

4) ∞-Norm / max Norm

- find the abs value of max comp.

$\|V\|_\infty = \max(|V_1|, |V_2| \cdots |V_n|)$

(limit of p norm)

* can be extended for matrix.

* Frobenius norm. $\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

$FN(M) = \sqrt{1^2 + 2^2 + 2^2 + 0^2} = 3$

$\boxed{\|A\|_F \neq \|A\|_2}$

[Matrix L_2 Norm]

1 no. that rep size of the matrix.

* Cauchy Shwarz Inequality

$\underbrace{\vec{x}, \vec{y}}_{\text{non-zero}} \in R^n$.

$$\left[\begin{array}{l} |\vec{x}\cdot\vec{y}| \le \|x\|\cdot\|y\| \\ |\vec{x}\cdot\vec{y}| = |x|\cdot|y| \quad \text{if } \vec{x} = c\vec{y} \end{array}\right]$$

$\boxed{\text{Cauchy Swartz Inequality}}$

Proof: $p(t) = |t\vec{y} - \vec{x}|^2 \ge 0$.

$$\left\{ |\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2} \right. \rightarrow \ge 0 \text{ [Norm]}$$

$$\left\{ |v|^2 = \vec{v}\cdot\vec{v} \right.$$

$\Rightarrow \underbrace{(t\vec{y} - \vec{x})\cdot(t\vec{y} - \vec{x})}_{\text{dot Pro.}}$

$\Rightarrow t\vec{y}\cdot t\vec{y} - \vec{x}\cdot t\vec{y} - t\vec{y}\cdot\vec{x} + |x|^2$.

$\Rightarrow \underbrace{(\vec{y}\cdot\vec{y})}_{a} t^2 - \underbrace{2(\vec{x}\cdot\vec{y})}_{b} t + \underbrace{\vec{x}\cdot\vec{x}}_{c} \ge 0$.

$at^2 - bt + c \ge 0$.

$p\left(\frac{b}{2a}\right) = \cancel{a}\left(\frac{b^2}{4a\cancel{^2}}\right) - b\cdot\left(\frac{b}{2a}\right) + c \ge 0$.

$\ne 0$
> 0.

$\frac{b^2}{4a} - \frac{b^2}{2a} + c \ge 0$.

$\frac{-b^2}{4a} \qquad c \ge \frac{b^2}{4a}$

$b^2 \le 4ac$

$\cancel{4}(|y|^2 \cdot |x|^2) \ge \cancel{4}(\vec{x}\cdot\vec{y})^2$

$\boxed{|x|\cdot|y| \ge |\vec{x}\cdot\vec{y}|} \leftarrow$ Cauchy Shwartz Inequality.

if $\vec{x} = c\vec{y}$.

$|\vec{x}\cdot\vec{y}| = |c\vec{y}\cdot\vec{y}| = |c|\cdot|\vec{y}\cdot\vec{y}| = |c||y|^2$

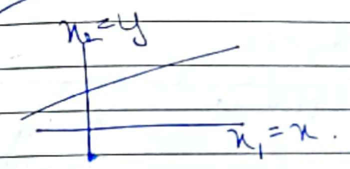
$\Rightarrow \underbrace{|cy|}_{x}\cdot|y|$

$\therefore |\vec{x}\cdot\vec{y}| = |x|\cdot|y|$.

* HyperPlane

→ Line → separate into 2 regns

2D



$\cdot\ y = mx + c$

$\cdot\ \boxed{ax + by + c = 0}$

$ax_1 + bx_2 + c = 0$ ⟶ Gen Eqn of Line.

$\therefore w_1x_1 + w_2x_2 + w_0 = 0 \longrightarrow 2D$

⟶

in 3D : eqn of plane → $ax + by + cz + d = 0$.

$w_1x_1 + w_2x_2 + w_3x_3 + w_0 = 0$.

$\boxed{\text{line in 2D = Plane in 3D} \cdots = \text{hyperPlane in nD}}$

* nD hyperplane

$$W_0 + W_1x_1 + W_2x_2 + \cdots W_nx_n = 0$$

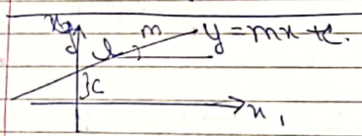
Summ^n → $W_0 + \sum_{i=1}^{n} W_i x_i = 0$

Vector → $W_0 + \underbrace{[W_1 W_2 \cdots W_n]}_{W_{1\times n}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X_{n\times 1}} = 0$

$$W_0 + W_1x_1 + W_2x_2 + \cdots W_nX_n = 0$$

$\Rightarrow \boxed{W_0 + W^Tx = 0}$ $\quad W_n = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix}$

π → Plane in any D space.

* 

c = y intercept
m = slope.

$$W_0 + W_1x_1 + W_2x_2 = 0$$

$$x_2 = \underbrace{-\frac{W_0}{W_2}}_{c} - \underbrace{\frac{W_1}{W_2}}_{m} x_1$$

if line is passing thru origin ⇒ c = 0

∴ $\boxed{W_0 = 0}$

∴ eq^n of line through origin in 2D → 2D: $W_1x_1 + W_2x_2 = 0$

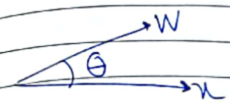
3D: $W_1x_1 + W_2x_2 + W_3x_3$

$\boxed{nD: W_1x_1 + W_2x_2 + \cdots W_nx_n = 0}$

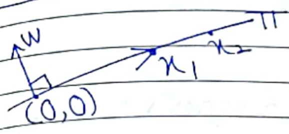
∴ $\boxed{W^Tx = 0}$ → Eq^n of Hyperplane passing through origin.

* $\pi_n : W^Tx = 0$ $\quad W = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$W \cdot x = W^Tx = |W| \cdot |x| \cdot \cos\theta_{W,x} = 0.$$



$\boxed{\theta = 90°}$

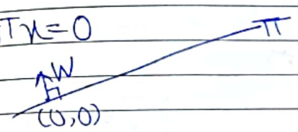
* 

$W \cdot x_1 = 0$.
$W \cdot x_2 = 0$

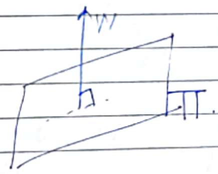
if $W \perp \pi$
then $W \cdot x_i = 0 \ \forall \ x_i \in \pi$.

Geometrical In of W

∴ $W^Tx = 0$



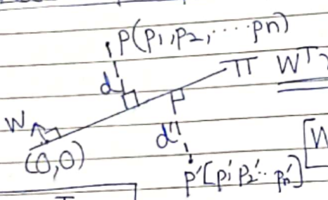
∴ $\hat{W} = \frac{W}{|W|}$



* HalfSpaces

→ Distance of a point from a plane

Reqⁿ 1



$W^T x = 0$

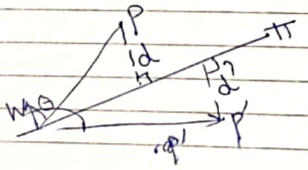
$\begin{bmatrix} W_{n\times 1} & P_{n\times 1} \end{bmatrix}$

Reqⁿ 2 $\boxed{d = \frac{W^T p}{\|W\|}}$

$d = \frac{W \cdot p}{\|W\|}$ if $|W| = 1$;

* A hyperplane divides a space into 2 regions called halfspaces

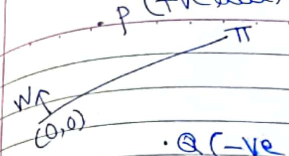
Line : 2D → 2 HS
Plane : 3D → 2 HS.



$\boxed{W \cdot p = |W|\,|p| \cos\theta}$

$d = \frac{W \cdot p}{|W|} = +ve$ $d' = \frac{W \cdot p'}{|W|} = -ve$

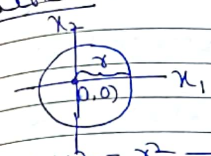
(sin. tells which halfspace the point is in?)



· Q (−ve dist)

[Like line Namely into I.I.L.]
: dot → WH of LA

* Circle



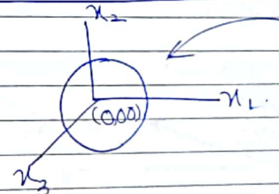
$x^2 + y^2 = r^2$ ⟶ Eqⁿ of circle.
$(x-h)^2 + (y-K)^2 = r^2$ $O(h,K)$

· $P(x_1, x_2)$ ⟶ inside / outside ?

$x_1^2 + x_2^2 - r^2 \leq 0 \Rightarrow$ P inside the circle.
$x_1^2 + x_2^2 - r^2 > 0 \Rightarrow$ P outside the circle
$x_1^2 + x^2 - r^2 = 0 \Rightarrow$ P lies on circle.

* 3D

x_1, x_2, x_3



$x_1^2 + x_2^2 + x_3^2 = r^2$ ⟵ Eqⁿ of Ball.

* nD

$x_1, x_2, x_3 \ldots \ldots x_n$.

$\sum_{i=1}^{n} x_i^2 = r^2$

(nD Sphere → $x_1^2 + x_2^2 + \cdots x_n^2 = r^2$
Hypersphere

* Dot Product

$a = [a_1, a_2 \cdots a_n]$
$b = [b_1, b_2 \cdots b_n]$.

add $\rightarrow a+b = [a_1+b_1, a_2+b_2 \cdots a_n+b_n]$

Mult $\rightarrow$ dot
Mult $\rightarrow$ cross [not much used in ML]

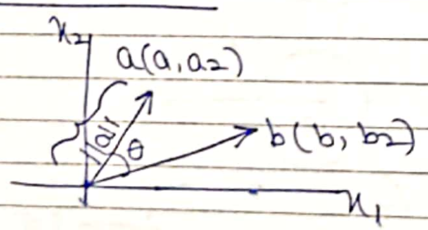
$$\left(a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 + \cdots a_nb_n\right)$$

$$\Rightarrow \underset{1\times n}{[a_1 a_2 a_3 \cdots a_n]} \underset{n\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}} = \underset{1\times 1}{[\ ]}$$

$\therefore$ $\underline{a^Tb}$ ($\because$ a = col vector).

$$\therefore \boxed{a \cdot b = a^Tb = \sum_{i=1}^{n} a_i b_i}$$

* Geometrical



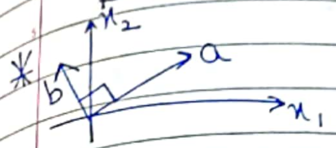
$a \cdot b = \underbrace{|a|}_{\text{norm}} \cdot |b| \cdot \cos\theta$

norm of a = dist of a from origin.

$a \cdot b = a_1b_1 + a_2b_2 = |a||b| \cos\theta$

$$\boxed{\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2}{|a| \cdot |b|}\right)}$$

*



$a \cdot b = |a||b| \cos 90$

$\therefore \boxed{a \cdot b = 0}$

as, $\underline{\underline{a^Tb = 0}}$

$$a \cdot b = \sum_{i=1}^{n} a_i b_i$$

$$\therefore \boxed{\theta_{a,b} = \cos^{-1}\left(\frac{\sum_{i=1}^{n} a_i b_i}{|a||b|}\right)}$$

- $a \cdot a = a_1a_1 + a_2a_2 + \cdots a_na_n$
  $= a_1^2 + a_2^2 + \cdots a_n^2$
  $\Rightarrow \underline{\underline{|a|^2}}$.

Scanned with CamScanner