* Introduction

⇒ Sample Space & Events

* Random Experiment: outcome not certain. (set of outcomes known).

eg: tossing a coin.
Rolling a dice.
Picking an object from colln (without looking)

* For each RE, a Sample Space: [Set] Set of all possible out-comes.

$S_1 = \{H, T\}$
$S_2 = \{1, 2, 3, 4, 5, 6\}$
$S_3 = \{HH, HT, TH, TT\}$
$S_4 = 36$ o/ps if (2 dice rolled).

· finding life of car

$S_5 = \{0, 1, \dots \infty\}$.

* Event: Any subset of SS. (Set)

$E_1 = \{H\}$ // getting Head.
$E_2 = \{1, 3, 5\}$ // getting odd no.
$E_3 = \{HH, HT, TH\}$ // atleast 1 H.
$E_4 = \{12, 21\}$ // Sum = 3
$E_5 = \{[3, 6]\}$ // atleast 3, atmost 6

· $E_1 \cup E_6$.

⇒ $E_1 \cap E_2 = \phi$ (mutually exclusive Events) [Same SS]

⇒ $\bar{E_1} = S - E_1$ $S_2 = $ dice.

Set of odd · Set of even.

* Probabilities to events

Simple event, $\{1\}$.
Compound Event. → $\{12, 21\}$

eg: → $S = \{H, T\}$. Fair Coin: equally likely to get H or T.

1/2 1/2.

· $\{H\} = E$
$P(E) = 1/2$ $P(E_2) = 1/2$. (assign no to event).

→ $S = \{HH, HT, TH, TT\}$

1/4

* For each event E, in S, we assign a no. P(E) such that:

→ $0 \leq P(E) \leq 1$
→ $P(S) = 1$ $\sum P(E) = 1$.
→ for any seq of events $E_1 \cdots$ that are mutu[ally] exclusive:

union $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$

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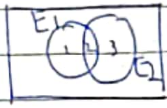
$E_1 = \{HH\}$, $P = 1/4$

$E_2 = \{TT\}$, $P = 1/4$.

$\therefore E_3 = E_1 \cup E_2 = 1/4 + 1/4 = \underline{1/2}$.



(not mutually excl.

$E_1 \cup E_2 = (1,2) \cup (2,3)$ — 2.

$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

eg: $E_1 = \{HH, HT\}$ // 1st coin is head

$E_2 = \{HH, TH\}$ // 2nd coin is head

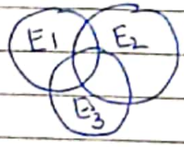
$E_3 = \{HT, TH, HH\}$ // atleast 1 head.

$E_1 \cup E_2$.

$P(E_3) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \underline{3/4}$

* 

- $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

* $P(S) = 1$ // certain than one in SS comes as outcome.

* $E, E^C$ are mutually exclusive.

$S = E \cup E^C$.

$\boxed{P(S) = P(E) + P(E^C) = 1}$



eg: Prob of pass = 0.6

Prob of fail = 0.4.

## * Random Variables

eg $S = \{HH, HT, TH, TT\}$ Tossing 2 coins.

$X \rightarrow$ count # times H turns up. ← RV

(functn of the o/p).

| | | | |
|---|---|---|---|
| X: | 0 | 1 | 2. |
| P(X): | 1/4 | 1/2 | 1/4 → P(X=2) [HH] |

Prob distrib of RV

$P(X=0)$ $P(X=1)$

eg: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. 3 coins (eq. likely) 1/8.

$X \longrightarrow$ count # tails.

| | | | | |
|---|---|---|---|---|
| X: | 0 | 1 | 2 | 3 |
| P(X): | 1/8 | 3/8. | 3/8. | 1/8 |

$P(X=0)$ $P(X=3)$

$\rightarrow X(HHT) = 1$.

$X$ = functn which take SS as i/p & o/p a number $= \{0\ 1\ 2\ 3\}$

$\rightarrow \sum P(X) = 1$

$\ldots)=12$.

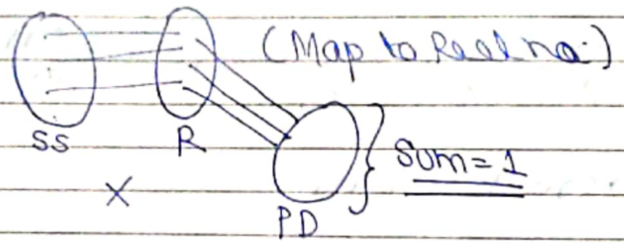
E ... 12 .... 66}, 1/36

Sum of values on 2 dice.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

* RV → discrete RV → RV values are pts [finite, countable set].
RV → continuous RV → RV gives range.

X : lifetime of car
[20, 80].
($\infty$ points in range)




⇒ X = count # heads.

$X = \{0, 1, 2\}$

P(X) 1/4 2/4 1/4 $\Sigma 1$ ✓

Q flip coin until we get 1st Head

N = # times to flip a coin until getting a head.



| | | | | | |
|---|---|---|---|---|---|
| | TH | FTH | | | |
| N: | 1 | 2 | 3 | 4 ........ | n |
| P(N) | p | $(1-p)p$ | $(1-p)^2p$ | $(1-p)^3p$ | $(1-p)^{n-1}p$ |

P(N=1)

∵ indep events ∴ multiply.

$P(H) = p$, $P(T) = 1-p$.

$\Sigma P(n) = p + (1-p)p + (1-p)^2 p + \cdots (1-p)^{n-1}p$

$\Rightarrow p(1 + (1-p) + (1-p)^2 + \cdots \infty)$.

$\Rightarrow p\left[\frac{1}{1-1+p}\right] = 1$. Ans

## * Bernoulli Random Variable

. 2 values = X = 0, 1.

SS = {1, 2, 3, 4, 5, 6}  (1 → Success; 2–6 → Failure)

Pick 1, & 1 comes in dice = Success.

$P(S) = 1/6$, $P(F) = 5/6$.

$\underbrace{P(X=1)}_{\text{Success}} = 1/6$, $\underbrace{P(X=0)}_{\text{Failure}} = 5/6$.

Bernoulli RV

In exp. whose Outcome can be classified as either a success or a failure, let X = 1 if the outcome is success & X = 0, if it is a failure, the prob mass functn of X

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$P(X=0) = 1-p$    X → Bernoulli
$P(X=1) = p$ ; p: prob of success

$X \to 0, 1$ } : discrete RV.

Q) 2 coins
{HH, HT, TH, TT}

if 2 heads = win / loss.
$P(X=1) = 1/4$ : success
$P(X=0) = 3/4$ : failure.

Q {H, T}

Head = success
$P(X=1) = 1/2$
$P(X=0) = 1/2$.

⇒ divide SS into success & failure

## * Binomial RV

- Collection of Bernoulli exp

Q if H: success.
flip n coins out of n coins, some will be success. X counts the # success.

n = 4, Getting 1 success? $P(H)=1/2$, $P(T)=1/2$
HTTT + THTT + TTHT + TTTH
⇒ $\boxed{\frac{4!}{3!} p(1-p)^3}$ → $P(X=1)$

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$P(X=2) \to \boxed{\frac{4!}{2!2!} p^2(1-p)^2}$

* if we conduct n exp, if every exp there can be S/F independently, X = if we try the count # success.

⇒ Defⁿ → (n, p)

Prob Mass functn of X → discrete RV.

$$\boxed{P(X=r) = {}^nC_r p^r q^{n-r} \; ; \; r = 1, 0 \dots n}$$

$$\Rightarrow \sum_{r=0}^{n} {}^nC_r p^r q^{n-r} \approx 1$$

## * Binomial Random Variable (eg)

$p = 0.1$

a) exactly one is defective, in SS of 3.

$P(X=1) = {}^nC_1 p^1 q^{n-1}$
$\Rightarrow {}^3C_1 (0.1)(0.9)^2$

b) at most 1 defective.

$P(X \le 1) = P(X=0) + P(X=1)$
$= {}^3C_0 (0.1)^0 (0.9)^3 + {}^3C_1 (0.1)(0.9)^2$

(c) atleast 1

$P(X \geq 1) = 1 - P(X=0)$

$\Rightarrow 1 - {}^3C_0 (0.1)^0 (0.9)^3.$

## * Poisson Random Variable

X, taking on one of the values 0, 1, 2 .. n is said to be Poisson r.v. with parameter $\lambda$.

$$P(x) = \boxed{P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}}$$

$x = 0, 1, 2 \cdots \infty.$

$$\sum_{x=0}^{\infty} P(X=x) = 1.$$

**Note:** Poisson RV is used to approximate Binomial RV

when $n\uparrow$, $p\downarrow$

$\lambda = np \longrightarrow$ Poisson instead of Binomial with $\lambda$.

## * example:

at least 1 error. $[\lambda = 1]$

$P(x \geq 1) = 1 - P(x=0)$

$\Rightarrow 1 - \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$1 - \frac{e^{-\lambda} \cdot \cancel{\lambda}}{} = \underline{1 - e^{-\lambda}} = 1 - 1/e$



$\therefore P(X=0) + P(X=1) + P(X=2) + \cdots\cdots \infty = 1$

* $P(X=0) = e^{-1} = \underline{1/e}$. $\leftarrow$ Prob that no errors

### eg2 Poisson Random V.

- $\lambda = 3$

$\rightarrow P(x=0) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$\Rightarrow \frac{e^{-3} \cdot 3^0}{0!} = 1/e^3$ Ans

$\rightarrow P(x=1) = \frac{e^{-\lambda} \cdot \lambda}{1!} \Rightarrow e^{-3} \cdot 3 = 3/e^3.$

$\rightarrow P(x \geq 1) = 1 - P(x=0)$ [Atleast 1 accident]

$= 1 - 1/e^3$

## * Continuous Random Variable

$\Rightarrow$ X: denote sum of 2 dice.

X = 1 2 3 4 . . . 12. [discrete values]

- countable no. of values.
- (countable set).

$\rightarrow$ X: years of material.

X = [1850 - 1865].

$\rightarrow$ take a continuous range (uncountable set).

$\rightarrow$ X: waiting time of a car

X = [1, 20]

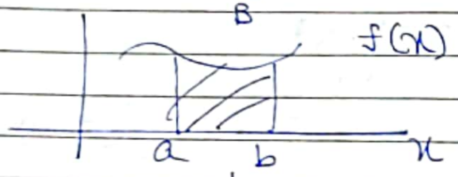
interested in range.

P ∈ [range]. ✓

P ∈ [point] ✗

* X take uncountable set of values, & there exists a non negative funct$^n$ f(x) called Prob density funct$^n$ having the property that for any set 'B' of real numbers

$P(x \in B) = \int_B f(x)\,dx$ → Prob density funct$^n$ → AUC




Q X : [a, b]

- $P(a \le X \le b) = \int_a^b f(x)\,dx$ → pdf.
- $P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$ → AUC $[-\infty, a]$

$\therefore P(X = a) = \int_a^a f(x)\,dx = 0$ (Prob of rv at a point)

But, discrete RV, Prob(point) $\ne 0$ [may not be 0]

$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$ ⇒ Prob = 1

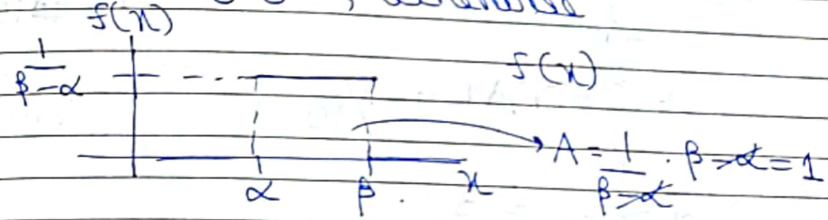




* Uniform Random Variable (CRV)

- X is a URV on the interval $(\alpha, \beta)$ if its PDF is given by:

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$



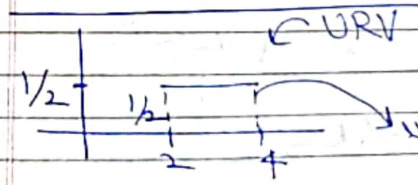

* $\int_{-\infty}^{\infty} f(x)\,dx = 0 + \int_{\alpha}^{\beta} f(x)\,dx + 0$

$\Rightarrow \therefore \boxed{AUC = 1}$ $\quad \underline{\underline{\Sigma P = 1}}$.

as $\int_{\alpha}^{\beta} \frac{1}{\beta-\alpha}\,dx = 0$.

$\frac{1}{\beta-\alpha} \cdot [x]_{\alpha}^{\beta} = \underline{\underline{1}}$

*




$\frac{1}{2}$ uniform in this range.

$$f(x) = \begin{cases} 1/2, & [2, 4] \\ 0, & \text{otherwise.} \end{cases}$$

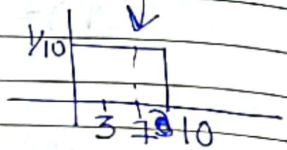
X = cont. URV.

## * Eg on Uniform Random Variable

Q range $(0,10) \rightarrow f(x) = \begin{cases} \frac{1}{10} & ; [0,10] \\ 0 & , \text{otherwise} \end{cases}$

a) $X < 3$.

$P(x<3) = \int_{-\infty}^{3} f(x)\,dx \Rightarrow 0$.




$= \int_{0}^{3} f(x)\,dx = 3/10$.

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b) $x > 7$. $= 3/10$

c) $1 < x < 6$.

$\Rightarrow 5 \times \frac{1}{10} = 1/2$.

## * Exponential Random Variable

A CRV whose pdf, for $\lambda > 0$.

$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & , \text{ if } x \geq 0 \quad [0,\infty) \\ 0 & , \text{ if } x < 0 . \end{cases}$

Q $\int_{-\infty}^{\infty} f(x)\,dx = \boxed{\int_{-\infty}^{\infty} \lambda e^{-\lambda x} = 1}$



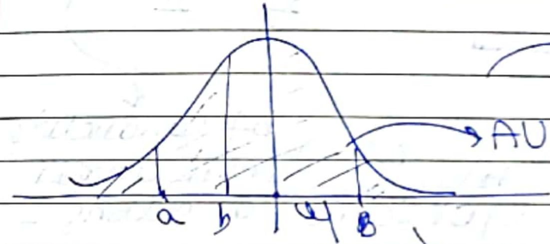
Q $P \in [a,b]$.

$P(a \leq X \leq b) = \int_{a}^{b} \lambda \cdot e^{-\lambda x} dx$.

## * Normal RV

X is normally dist with $\mu, \sigma^2$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}}$$




Bell func

AUC = 1

$\int_{a}^{b} f(x)\,dx$.

$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

- $P(x \leq a) = \int_{-\infty}^{a} f(x)\,dx$
- $P(x \geq a) = \int_{a}^{\infty} f(x)\,dx$

## * Expectation → mean

→ to guess which value would come out (most prob.)

X = getting #H.

| X | 0 | 1 | 2 |
|---|---|---|---|
| P(X) | 1/4 | 1/2 | 1/4 |

$\sum X \cdot P(X)$

$$\boxed{E(X) = \sum_{i: P(i)>0} i\, P(X=i)} = 0 + \frac{1}{2} + \frac{1}{2} \Rightarrow 1$$

b4 conducting exp. more chances is of getting 1 (HT/TH)

~~true~~ (for some cases)

Q X: no. when roll a dice.

| X | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| P(X) | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |

$$E(X) = \frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6}$$

$$\Rightarrow \frac{6\times 7}{2} \times \frac{1}{6} \Rightarrow \underline{\underline{3.5}}$$

3/4 will have High Prob

∴ Expectation term not correct

## * E(X) of Bernoulli RV (Discrete RV)

parameter: p → P(X=1) [Success]

| X | 0 | 1 |
|---|---|---|
| P(X) | 1−p | p |

→ $E(X) = 0 + p = p$

→ Exp. #success in 1 trial.

1 trial — p success

1/p trial — 1 success

∴ $\boxed{\text{1 success — 1/p trials}}$

Q Prob of success = 0.2, how many trials are required to get 1 success? exp # [in Bernoulli].

→ 5 trials

## * E(X) of Binomial RV

n Trials, how many success is counted by X?

$$P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r} \qquad q = 1-p$$

$r \in [0, n]$.

→ $E(X) = \sum_{i=0}^{n} i \cdot P(X=i)$

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$$\Rightarrow \sum_{i=0}^{n} i \cdot {}^{n}C_i \, p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{i} \frac{i \cdot n!}{(n-i)!\, i!} p^i (1-p)^{n-i}.$$

$$\Rightarrow \frac{n!}{(n-i)!\,(i-1)!} p^i (1-p)^{n-i}.$$

$$np \sum_{i=1}^{n} \frac{(n-i)!}{(n-i)!\,(i-1)!} p^{i-1} (1-p)^{n-i}$$

$\underline{i-1=K}$

$$\Rightarrow np \sum_{K=0}^{n-1} \underbrace{\frac{(n-1)!}{(n-1-K)!\,K!}}_{{}^{n-1}C_K} p^K (1-p)^{n-1-K}$$

$\Rightarrow \underline{np}.$

if Prob of single Success is p, & n trials, Prob of then #Success = np. with high Prob

Q $n=10$, $P=0.6$
Expected #success ? $\Rightarrow \underline{\underline{6}}$

$$\sum i \cdot P(X=i)$$

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## * Poisson random Variable

$\lambda$.

$$f(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$E(X) = \sum_{i=0}^{\infty} i \left( \frac{e^{-\lambda} \cdot \lambda^i}{i!} \right)$$

$$\Rightarrow \sum_{i=1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^i}{(i-1)!}$$

$$\Rightarrow \lambda \cdot e^{-\lambda} \cdot \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$$

$$\Rightarrow \lambda \cdot e^{-\lambda} \cdot \sum_{K=0}^{\infty} \frac{\lambda^K}{K!} \quad K = i-1$$

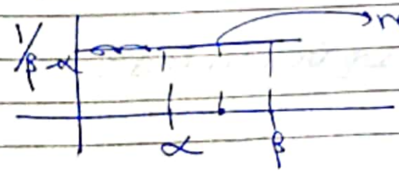
$$\boxed{E(X) = \lambda}$$

## * Expectn of Continuous RV

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$$

## * Exp of Uniform RV

most expected Value.



$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha} \cdot dx$$

$$= \frac{1}{\beta-\alpha} \cdot \frac{(\beta^2-\alpha^2)}{2}$$

$$\Rightarrow \frac{\alpha+\beta}{2} \cdot \underline{\underline{\text{Ans}}}$$

* Expectn of Exponential RV

$$E(X) = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\, dx = \underline{\underline{1/\lambda}}$$

* E(X) of Normal RV $(\mu, \sigma^2)$

$$\boxed{E(X) = \mu}$$

* Independent Events → if an event has already occurred it won't change the prob of next event

Fair coin: $P(H) = P(T) = 1/2$.

HH

$P(H_1 \cap H_2) = P(H_1) \times P(H_2/H_1)$.

Head in 1st flip $= \frac{1}{2} \cdot \frac{1}{2} = 1/4$.

· Mut. ex. indep.

sameSS, nothing in common.



→ 2nd outcome is independent of 1st outc.

* (A, B) are 2 events.

$$\boxed{\begin{array}{l} P(A/B) = P(A) \\ P(B/A) = P(B) \end{array}}$$ ∴ A & B are independents

$P(B/A)$ → Prob of B given A happend.

→ $P(A \cap B) = P(A) \cdot P(B/A) = P(A) \cdot P(B)$.

$\boxed{P(A \cap B) = P(A) \cdot P(B)}$ (only when A, B are ind. events).

$\boxed{P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)}$

* If any 2 events: $P(A \cap B) = P(A) \cdot P(B)$, then we can say: these 2 events are independent & converse.

* if n events: $E_1, E_2, E_3 \cdots E_n$ for any independent subset it shud hold:

$P(E_i \cap E_j \cdots E_l) = P(E_i) \cdot P(E_j) \cdots P(E_l)$.

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independent events.

**eg 1** $\boxed{3G, 2R}$ Ra exp with replacement.

Q $P(G_1 \cap R_2)$ — Red ball in Second trial

$\underbrace{\quad}$ independent events.

$\Rightarrow P(G_1) \times P(R_2)$

$\Rightarrow \frac{3}{5} \times \frac{2}{5} = \underline{6/25}$

Q $P(G_1 \cap G_2 \cap G_3)$ — G ball in 1st T & G ball in 2nd T...

$\Rightarrow \left(\frac{3}{5}\right)^3$

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Q

| A | B | C |
|---|---|---|
| 3R, 2G | 3R, 5G | 1R, 4G |

$P(G_A R_B G_C)$ — 1 ball from each bag; G ball from C.

$\Rightarrow P(G_A \cap R_B \cap G_C)$ : <u>Ind. events</u>.

$\Rightarrow \frac{2}{5} \times \frac{3}{8} \times \frac{4}{5}$

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Q We draw 1 ball from each bag.

$P(2\text{ Green}, 1\text{ Red})$.

| A | B | C | |
|---|---|---|---|
| R | G | G | $= \frac{3}{5}\cdot\frac{5}{8}\cdot\frac{4}{5} +$ |
| G | R | G | $= \frac{2}{5}\cdot\frac{3}{8}\cdot\frac{4}{5} +$ |
| G | G | R | $= \frac{2}{5}\cdot\frac{5}{8}\cdot\frac{1}{5}$ |

3 mutually exclusive events.

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when 1 Na: you are rejected.

**eg 2**: convince 3 people to marry a girl.

$P(\text{Dad}) = 1/2$ [All think Ind.]

$P(\text{Mom}) = 1/3$

$P(\text{Brother}) = 1/4$.

$P(\text{Marry}) = P(D \cap M \cap B)$. YYY.

$= P(D) \cdot P(M) \cdot P(B)$

$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \underline{\frac{1}{24}}$.

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**eg 3**: A & B are independent.

$P(A \cap B) = P(A) \cdot P(B)$

* if A, B are independent, then so are:

for any # events

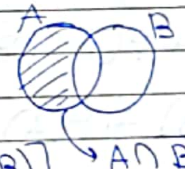
(i) $A, \bar{B}$

(ii) $\bar{B}, A$

(iii) $\bar{A}, \bar{B}$.

} all are independent.

$P(A \cap \bar{B})$



$\Rightarrow P(A) - P(A \cap B)$ — $A \cap B$.

$\Rightarrow P(A) - [P(A) \cdot P(B)]$

$\therefore P(A)[1 - P(B)]$

$\therefore \underline{P(A) \cdot P(\bar{B})}$

$P(\bar{B} \cap \bar{A}) = P(B) - P(A \cap B)$

$= P(\bar{A}) \cdot P(B)$.

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = \cancel{P(A}\ 1 - P(A \cup B)$

$\Rightarrow 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [P(A) + P(B) - P(A) \cdot P(B)]$.

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$\Rightarrow 1 - P(A) - P(B)[1 - P(A)]$

$= [1 - P(A)][1 - P(B)]$

$= P(\bar{A}) \cdot P(\bar{B})$

* if $E_1, E_2 \cdots E_n$ are indep., $\bar{E_1}, \bar{E_2}, \bar{E_3}$ are also independent.

**eg 4**: Interview: 2 contest.

$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{3}$ (Getting Job).

Seln of A is independent with seln of B.

Q What is the Prob: both get selected?

$P(A \cap B) = \frac{1}{6}$.

Q exactly one get selected.

$\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

$P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$P(A) \cdot P(\bar{B}) + P(\bar{A}) \cdot P(B)$ $\because$ independ.

Q Prob that none get selected.

$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

Q atleast 1 get selected.

$1 - P(\bar{A} \cap \bar{B}) = \frac{2}{3}$

Q atmost 1 selected.

$\frac{1}{3} + \frac{1}{2}$

**eg5**: Lock : 8 Keys. (1 Key is right.) (Let Key stay even if tried)

Prob that open lock in $7^{th}$ key?

$P(\bar{L_1} \cap \bar{L_2} \cap \bar{L_3} \cap \bar{L_4} \cap \bar{L_5} \cap \bar{L_6} \cap L_7)$ independent event.

$\Rightarrow P(\bar{L_1}) \cdot P(\bar{L_2}) \cdot P(\bar{L_3}) \cdots P(L_7)$. $\because$ [Prob remains same]

$\Rightarrow \left(\frac{7}{8}\right)^6 \cdot \frac{1}{8}$ Ans.

* $P(A \cap B) =$

$P(A) =$, $P(B) =$

Verify: $P(A \cap B) = P(A) \times P(B)$. $\rightarrow$ independ.

eg: $S = \{HH, HT, TH, TT\}$ ← flip 2 coins.

$E_1$ = Getting Head on $1^{st}$ Coin. $\{HH, HT\}$

$E_2$ = Getting Head on 2nd coin. $\{TH, HH\}$

$P(E_1) = \frac{1}{2}$

$P(E_2) = \frac{1}{2}$

$P(E_1 \cap E_2) = \frac{1}{4}$.

Got ~~Head~~ HH.

$P(E_1 \cap E_2) = P(E_1) * P(E_2)$.

$\therefore E_1, E_2$ are independent $\because$ Fair coin

**eg 7.** Desire if events are indep.

X: Rolling 2 dice.

SS = { (1,1), (12) (13 . . . . (16)
(21) (22) 23 . . . 26
31 32 33 . . . 36

61 62 63 66 }.

- $E_1$ = Getting a 4 on 1st dice = $6/36 = 1/6$
  $\boxed{P(E_1) = 1/6}$
- $E_2$ = Sum of two dice is '6'
  : {15, 51, 24, 42, 33}
  $\boxed{P(E_2) = 5/36}$

$\frac{5}{36}\ \frac{1}{6}$.

$P(E_1 \cap E_2) = 1/36$
{42}

$\boxed{P(E_1 \cap E_2) \neq P(E_1) \cdot P(E_2)}$

∴ $E_1 \wedge E_2$ are dependent.

Awsm!! ∵ if dice 1 = 6, dice 2 X. [no chance to get 6).

Sum = 6, depends on 1st dice.

- $E_3$ = Sum on the dice is 7.
  (16) (61) (25) (52) (34 43)
  $\boxed{P(E_3) = 1/6}$

$P(E_1 \cap E_3) = 1/36$
(43)

∴ $E_1, E_3$ are independ events

∵ whatever happens on 1st dice, the chance is there, to get 7.

**eg 8:** 4 balls numbered 1 2 3 4
$E = \{1\,2\}$ $F = \{1\,3\}$ $G = \{1\,4\}$
Is E, F, G are independent?

$^3C_2 + ^3C_3$

$P(E \cap F) = P(E) \cdot P(F)$ ✓
$P(F \cap G) = P(F) \cdot P(G)$ ✓
$P(E \cap G) = P(E) \cdot P(G)$ ✓
$P(E \cap F \cap G) = P(E) \cdot P(F) \cdot P(G)$. $= 1/8$ ✗
$\underbrace{}_{(1) = 1/4}$

∴ $^nC_2 + ^nC_3 + ^nC_4 \cdots ^nC_n$

∴ $\sum_{i=2}^{n} {}^nC_i$

- $P(1) = P(2) = P(3) = P(4) = 1/4$.

$P(E) = 1/2$
$P(F) = 1/2$
$P(G) = 1/2$

→ $P(E \cap F) = P(E) \cdot P(F) = 1/4$ ✓
{1}

∴ E, F, G are not independent

**eg 9** :

| 50G |
|---|
| 50D |

50 Good Bulbs ~~are~~ ?
(replacement)
2 bulbs ~~taken~~ with rep.

$SS = \{GG, GB, BG, BB\}$ (all equally likely)

$\frac{1}{2}\cdot\frac{1}{2}$, $1/4$, $1/4$, $1/4$.

- $A$ = event such that 1st B is G
  $\{GG, GB\}$
  $P(A) = 1/2$
- $B$ = 2nd B is Good
  $\{GG, BG\}$
  $P(B) = 1/2$
- $C = \{GG, BB\}$
  $P(C) = 1/2$

Q Are A, B, C independent ?

$\overset{1/4}{P(A\cap B)} = P(A)\cdot P(B)$ ✓
$P(B\cap C) = P(B)\cdot P(C)$ ✓
$P(A\cap C) = P(A)\cdot P(C)$ ✓
$P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)$. ✗
$1/4$. $\quad 1/8$

A, B, C are not independent events.
(But Pairwise independent).



**eg 10** : Flip fair coin 5 times → independent
Prob no 2H/2T consecutively

$\{\underbrace{HTHTH}_{1/32}, \underbrace{THTHT}_{1/32}\} = 1/16$

Q
| | P | M | C |
|---|---|---|---|
| Pass | 0.5 | 0.6 | 0.7 |

(Independent)

→ $\underbrace{P(P\cap M\cap C)}_{\text{All Sub Passed compulsory}} = 0.5\times0.6\times0.7$.

→ Prob that he fails exactly 1.
$0.5\times0.6\times0.7 + 0.5\times0.4\times0.7$
$+ 0.5\times0.6\times0.3$

→ fails in 3 courses $= P(\underbrace{\bar{P}\cap\bar{M}\cap\bar{C}}_{\text{also indep}})$

---

**eg 11** : if A, B, C are 3 independent events
Pairwise indep +
$P(A\cap B\cap C) = P(A)P(B)P(C)$.

Prove that: A and $(B\cup C)$
B and $(A\cup C)$
C and $(A\cup B)$ } are indep.

a+ac+ab+bc



$\rightarrow \underbrace{P(A \cap (B \cup C))} = P(A) \cdot P(B \cup C)$

$- P((A \cap B) \cup (A \cap C))$

$- P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$

$= P(A) \cdot P(B) + P(A) P(C) - P(A \cap B \cap C)$

$\Rightarrow P(A) \cdot P(B) + P(A) P(C) - P(A) P(B) P(C)$ 



$\Rightarrow P(A) [P(B) + P(C) - P(B) P(C)]$

$= P(A) \cdot P(B \cup C)$. Hence proved.

eg 12: $P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 1/4$
all these are indep.?

a) $(A \cup \bar{B}) \cap C$ ← $(A \cup \bar{B})$, C are ind.

b) $P(\overline{A \cup B \cup C})$

c) $P(\bar{A} \cap \bar{B} \cap \bar{C})$

d) $(\bar{A} \cup \bar{B}) \cap C$

all are independent

a) $P(A \cup \bar{B}) \cdot P(C)$

$\Rightarrow [P(A) + P(\bar{B}) - P(A) P(\bar{B})] \cdot P(C)$

$= \left(\frac{1}{2} + \frac{2}{3} - \frac{1}{3}\right) \cdot \frac{1}{4}$

b) $P(\bar{A}) + P(\bar{B}) + P(\bar{C}) - P(\bar{A} \cap \bar{B}) - P(\bar{B} \cap \bar{C})$
$- P(\bar{A} \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap \bar{C})$

c) Same as b

d) Same as a.

* A & B are mutually exclusive events

$\underbrace{\underbrace{P(A \cap B)}_{\phi}}_{0} = \underbrace{P(A) \cdot P(B)}_{\neq 0}$ Not Independent.

* $P(A) = 1$ [certain event] $\boxed{A = SS}$
A is independent of any event B

$\underbrace{P(A \cap B)}_{SS \cap B} = \underbrace{P(A)}_{1} \cdot P(B)$ [Proved).
$P(B)$

→ If flip a coin, what is Prob of getting H/T?

* $P(A) = 0$ [impass. event]
A is independent of any event

$\underbrace{\underbrace{P(A \cap B)}_{\phi}}_{0} = \underbrace{P(A)}_{0} \cdot P(B)$ ✓

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# * Total Probability

Q.

| A | B | C |
|---|---|---|
| 2G, 3R | 1G, 4R | 3G, 2R |

Choose a bag & then a ball pick. What is Prob that the ball is Green.

Prob of ball is Green depends on the bag chosen.

$P(A) = P(B) = P(C) = 1/3$.

$P(G/A) = 2/5$.

$P(G/B) = 1/5$

$P(G/C) = 3/5$

$P(G) = ?$ ← (Total Prob).

→ SS: (AG), (AR) ; (BG), (BR) ; (CG), (CR) : 3 mutually exclusive events

$E_1 \quad E_2 \quad E_3$ → A is spread across $E_1, E_2, E_3$.

P(A) solved using total prob theorem

$A = (E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A)$ → mut. exclu

$P(A) = P((E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A))$

$E_1$: prob of picking A.
A: prob of getting Green Ball.

$\Rightarrow P(A) = P(E_1 \cap A) + P(E_2 \cap A) + P(E_3 \cap A)$.

Total Prob theorem (dep).

$$\boxed{P(G) = P(A \cap G) + P(B \cap G) + P(C \cap G)}$$

$\Rightarrow P(A) \cdot P(G/A) + P(B) \cdot P(G/B) + P(C) \cdot P(G/C)$

$\Rightarrow \frac{1}{3} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{3}{5}$

(Total Probability)

∵ A & G are dependent

# * Rolling 2 dice

$SS = \begin{bmatrix} 11, & 12 & \cdots & 16 \\ 21, & 22 & \cdots & 26 \\ \vdots \\ 61, & 62 & \cdots & 66 \end{bmatrix}$ 36 → 6 m.ex events.

Prob that second ~~ball~~ dice is 2

$\Rightarrow$ divide SS into mutually exclusive events. (what happens on 1st die) → 6 Parts

Total Prob (indep)

$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + P(E_3 \cap A) + \cdots P(E_6 \cap A)$

$E_1$: 1 on 1st — independent events.

$= P(E_1) \cdot P(A) + P(E_2) \cdot P(A) + P(E_3) \cdot P(A) + \cdots$

$\Rightarrow \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \cdots \frac{1}{6} \cdot \frac{1}{6}$

$= 1/6$.

eg 1: 

| A | B |
|---|---|
| 2G, 3R | 3R, 2G |

$P(G_1 \cap R_2) + P(R_1 \cap R_2)$.

draw a ball from A, add to B, & then draw from B. What is the prob that ball is red

SS

| $E_1$ | $E_2$ |
|---|---|
| $R_1R_2$ | $G_1R_2$ |
| $R_1G_2$ | $G_1G_2$ |

→ A

$P(A) = P(E_1 \cap A) + P(E_2 \cap A)$. (dependent event).

$P(A) \Rightarrow P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)$

$\Downarrow$

$$\boxed{P(R_2) = P(G_1) \cdot P(R_2/G_1) + P(R_1) \cdot P(R_2/R_1)}$$

Total Prob of getting Red ball. $\Rightarrow \frac{2}{5} \times \cancel{P(3)}\ 3/6 + \frac{3}{5} \times \frac{4}{6}$

3R 3G    4R 2G

eg 2. 3 Balls (O, O, R)

- 1 ball definitely red.
- Pick a ball.

What is the prob that it is red.

→ 1) (R) R R    2) (R) R N    (3) (R) N N

$E_1$    $E_2$    $E_3$

N = Non Red

3 mutually exclusive cases.



| $E_1$ | $E_2$ | $E_3$ |
|---|---|---|
| | | |

getting Red ball given $E_2$

$TP = P(E_1) \cdot P(R/E_1) + P(E_2) \cdot P(R/E_2) + P(E_3) \cdot P(R/E_3)$

$TP \Rightarrow \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3}$

dependent events.

eg 3. We take a coin & flip it. P(Head)?

4 Fair coins: $P(H) = P(T) = 1/2$

3 UF coins: $P(H) = 1/3$, $P(T) = 2/3$

P(H) depend on the E happended first

SS

| $E_1$ | $E_2$ |
|---|---|
| FH | UH |
| FT | UT |

$TP = P(F) \cdot P(H/F) + P(U) \cdot P(H/U)$

$\Rightarrow \frac{4}{7} \cdot \frac{1}{2} + \frac{3}{7} \cdot \frac{1}{3}$

Prob of getting 2 Heads?

$\Rightarrow$

eg4: 2 coins

SS = FF UF FU UU.

| FF | UU | F&U |
|---|---|---|
| | | HH |

$\rightarrow \dfrac{12\ SS}{4 \times 3}$

$P(HH) = P(\underbrace{FF \cap HH}_{\text{dependent}}) + P(UU \cap HH) + P(FU \cap HH)$

$\Rightarrow$

$\Rightarrow P(FF) \cdot P(HH/FF) + P(UU) \cdot P(HH/UU)$

$+ P(FU) \cdot P(HH/FU)$

$\Rightarrow \frac{4}{7} \cdot \frac{3}{6}$

Total Prob of getting 2 Heads $\Rightarrow \left(\frac{4C_2}{7C_2}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right) + \left(\frac{3C_2}{7C_2}\right)\cdot\left(\frac{1}{3}\cdot\frac{1}{3}\right)$

$+ \left(\frac{4C_1 \cdot 3C_1}{7C_2}\right) \cdot \left[\frac{1}{2}\cdot\frac{1}{3}\right]$.

eg5: Answer / Guess

p: prob that she knows the answer

guess answer $\rightarrow$ correct : prob: $1/m$.

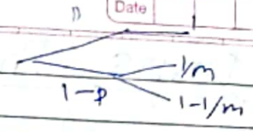
| KC | GC GW |
|---|---|

Prob that the answer it correct)

$P(C) = P(K \cap C) + P(G \cap C)$

$= \underbrace{P(K)}_{p} \cdot \underbrace{P(C/K)}_{1} + \underbrace{P(G)}_{1-p} \cdot \underbrace{P(C/G)}_{1/m}$

$\Rightarrow p + (1-p)\frac{1}{m}$






eg6:

A: reached F

B C Semi

$P(B) = 1/3$, $P(C) = 2/3$ $\rightarrow$ B will win

if fight A^B : $\longrightarrow P(A) = 2/3$

A C $\longrightarrow P(A) = 1/4$.

What is the prob that A wins?

| | $c_1$ | $c_2$ |
|---|---|---|
| SS: | BA | CA |
| | BB | CC |

$\rightarrow$ 1st match: C win
2nd m: A wins.

$P(A) = \underbrace{P(B \cap A)}_{\text{indepen}} + P(C \cap A)$

$= P(B) \cdot P(A/B) + P(C) \cdot P(A/C)$

$P(A) \Rightarrow 1/3 \cdot 2/3 + 2/3 \cdot 1/4$

# * Baye's Theorem

| 2R 3G | 1R 4G | 3R 2G |
|---|---|---|
| A | B | C |

a) ~~Total Prob of getting Red ball~~ $P(R)$

b) ~~$P(R/A)=$~~

a) Prob. of getting red ball given that A is chosen?

$$P(R/A) = 2/5$$

b) What is the probability of getting a red ball from bag A. imp

Bag A chosen then Red ball drawn. $P(A \cap R) = P(A) \cdot P(R/A)$.

c) What is the Probability of getting a red ball? $P(R)$

$$TP = P(A \cap R) + P(B \cap R) + P(C \cap R)$$

d) What is the conditional Probability that Bag A is chosen given that Red ball is drawn.

$$P(A/R)$$

Reverse Prob

[Predict the Source].

known info
$$P(A) = P(B) = P(C) = 1/3$$
$$P(R/A) = 2/5$$
$$P(R/B) = 1/5$$
$$P(R/C) = 3/5$$

$$P(A/R) = \frac{P(A \cap R)}{\underbrace{P(A \cap R) + P(B \cap R) + P(C \cap R)}_{\text{3 sources for event R to occur}}}$$

$$\therefore \boxed{P(A/R) = \frac{P(A) \cdot P(R/A)}{P(A) \cdot P(R/A) + P(B) \cdot P(R/B) + P(C) \cdot P(R/C)}}$$

Baye's formula.

den = Prob. of

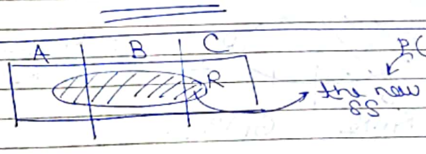
$$P(A/R) = \frac{P(A) \cdot P(R/A)}{P(R)} \longrightarrow \text{Total Prob}$$

$$P(R) \cdot P(A/R) = P(A) \cdot P(R/A)$$

$$P(R \cap A) = P(A \cap R)$$

Proved

OR




$P(A/R)$ $\because$ R has occur.

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SS 

$\rightarrow P(A/R)$ $\rightarrow P(R)$

$\frac{P(A\cap R)}{\text{Total}} = \frac{P(A\cap R)}{P(R)}$

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* BT used to find Src of event.

Q $P(B/R) = \frac{P(B\cap R)}{P(R)}$ : (Total Prob)

---

eg 1:

| A | B |
|---|---|
| 2R, 3G | 3R, 2G. |

SS

| | |
|---|---|
| $R_1R_2$ | $G_1R_2$ |
| $R_1G_2$ | $G_1G_2$ |

$P(R_2)$

Given that we got red ball, what

$\cdot P(R_2) = P(R_1\cap R_2) + P(G_1\cap R_2)$

$= P(R_1)\cdot P(R_2/R_1) + P(G_1)\cdot P(R_2/G_1)$

$= \frac{2}{5}\cdot\frac{4}{6} + \frac{3}{5}\cdot\frac{3}{6}$

Reverse Prob

$P(R_1/R_2)$

$= \frac{P(R_1\cap R_2)}{P(R_2)}$

$= \frac{P(R_1)\cdot P(R_2/R_1)}{P(R_2)} = \frac{2/5\cdot 4/6}{2/5\cdot 4/6 + 3/5\cdot 3/6}$

$R_2$

$R_1\cap R_2$ $G_1\cap R_2$.

$\frac{P(R_1\cap R_2)}{P(R_2)}$

---

Q $P(R_1/G_2) = \frac{P(R_1\cap G_2)}{P(G_2)} \longrightarrow$ Total Prob.

$P(R_1\cap G_2) + P(G_1\cap G_2)$.

---

eg 2:

5F 4U.

$P(H) = 1/3$

$P(T) = 2/3$.

eal : $P(H)$? [Total Prob].

$P(H) = P(F\cap H) + P(U\cap H)$

$= P(F)\cdot P(H/F) + P(U)\cdot P(H/U)$

$\Rightarrow \frac{5}{9}\cdot\frac{1}{2} + \frac{4}{9}\cdot\frac{1}{3}$

Rev Prob:

BT. $P(F/H) = \frac{P(F\cap H)}{P(H)} = \frac{P(F)\cdot P(H/F)}{P(H) = (TP)}$.

$P(F/T) = \frac{P(F\cap T)}{P(T)} = \frac{P(F\cap T)}{P(U\cap T) + P(F\cap T)}$

$P(U/H) = \frac{P(U\cap H)}{P(H)}$

$P(U/T) = \frac{P(U\cap T)}{P(T)}$.

---

eg 3:

n Green
m Red

1st time $\begin{cases} \text{if Green, add K Green B to bag} \quad G+KG \\ \text{if red, add } R+KR \text{ the bag.} \end{cases}$

$SS = \left\{ \begin{array}{c|c} G_1R_1 & R_1R_2 \\ G_1R_2 & R_1G_2 \end{array} \right\}$

Q $P(R_2) = ?$ Prob that 2nd ball is Red

$P(G_1 \cap R_2) + P(R_1 \cap R_2)$

$= P(G_1) \cdot P(R_2/G_1) + P(R_1) \cdot P(R_2/R_1)$

$\Rightarrow \left(\frac{n}{n+m}\right) \cdot \left(\frac{m}{n+m+K}\right) + \left(\frac{m}{m+n}\right) \cdot \left(\frac{m+K}{m+n+K}\right)$

Q $P(G_2/G_1) = \frac{n+K}{m+n+K}$

$P(G_2/R_1) = \frac{n}{m+n+K}$

- $P(G_2) = P(R_1 \cap G_2) + P(G_1 \cap G_2)$

$\Rightarrow P(R_1) \cdot P(G_2/R_1) + P(G_1) \cdot P(G_2/G_1)$

BT (Find cause) $\rightarrow$ $P\left(\frac{R_1}{R_2}\right) = \frac{P(R_1 \cap R_2)}{P(R_2)} = TP$.

$P\left(\frac{G_1}{R_2}\right) =$

$P\left(\frac{R_1}{G_2}\right) =$

$P\left(\frac{G_1}{G_2}\right) =$

eg 4: Knows $\longrightarrow p$.

guess $\longrightarrow 1-p$.

$P(R/G) = 1/m$, $P(W/G) = 1 - 1/m$

Prob that she gave correct ans $\rightarrow$ $P(C) = P(K \cap C) + P(G \cap C)$

$= P(K) \cdot P(C/K) + P(G) \cdot P(C/G)$

$\Rightarrow p + (1-p) \cdot \frac{1}{m}$

$P(G/C) = \frac{P(G \cap C)}{P(C)} \longrightarrow TP$.

$\Rightarrow \frac{P(G) \cdot P(C/G)}{P(C)}$

$\Rightarrow \frac{(1-p) \cdot 1/m}{P(C)}$

$P(K/C) = \frac{P(K \cap C)}{P(C)}$

$= \frac{P(K) \cdot P(C/K)}{} = \frac{p}{P(C)}$

$\boxed{P(K/C) = \frac{p}{p + (1-p) 1/m}}$

eg 5: Bayes T

A F.

$B \xrightarrow{1/3}$ $C \xrightarrow{2/3}$.

$P(A/B) = 2/3$

$P(A/C) = 1/3$

SS: BB CC

BA CA.

$P(A) = P(B \cap A) + P(C \cap A)$.

$\Rightarrow P(B) \cdot P(A/B) + P(C) \cdot P(A/C)$

$= 1/3 \cdot 2/3 + 2/3 \cdot 1/3$