Stability Wars: A-Stability in ODEs

Lado Turmanidze

January 18, 2025

In this article, I will explain how to calculate the A-Stability of a numerical method for ordinary differential equations (ODEs) and systems of ODEs. I will begin with a straightforward initial value problem (IVP) and then progress to the more complex case of systems of ODEs. I hope this step-by-step approach will help clarify the concepts involved.

1 A-Stability for $y' = -ry, y(a) = y_a$

General Derivation Algorithm for A-Stability

1. **Rewrite the Differential Equation**: The IVP is $y' = -ry, y(a) = y_a$. The analytical solution is $y(t) = y_a e^{-r(t-a)}$ with $r > 0$ ensuring that the solution decays over time.
2. **Discretize the Equation**: Substitute a numerical scheme into the differential equation. Let h be the time step and y_n be the numerical approximation at $t_n = a + n_h$.
3. **Analyze the Stability**: Define the amplification factor G, such that $y_{n+1} = Gy_n$. The method is A-stable if $|G| \leq 1$, $\forall r > 0, h > 0$.

1.1 Forward Euler Method

Discretize the equation using the Forward Euler method:

$$y_{n+1} = y_n + hf(t_n, y_n) \text{ with } f(t, y) = -ry$$

Substituting $f(t, y) \to y_{n+1} = y_n + h \cdot (-ry_n) = y_n - hry_n$

The amplification factor is:

$$G = 1 - hr$$

. To analyze stability, we require $|G| \leq 1$:

$$|1 - hr| \leq 1 \implies -1 \leq 1 - hr \leq 1 \implies 0 \leq hr \leq 2$$

Thus, the Forward Euler method is **conditionally A-stable**, as stability depends on the condition $h \leq \frac{2}{r}$

1.2 Backward Euler Method

Discretize the equation using the Backward Euler method:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \text{ with } f(t, y) = -ry$$

Substituting $f(t, y) \to y_{n+1} = y_n - hry_{n+1}$

Rearrange to solve for y_{n+1}:

$$y_{n+1}(1 + hr) = y_n \Leftrightarrow y_{n+1} = \frac{y_n}{1 + hr}$$

The amplification factor is:

$$G = \frac{1}{1 + hr}$$

Recall, for A-Stability, we require $|G| \leq 1$:

$$\left| \frac{1}{1 + hr} \right| \leq 1$$

Since $1 + hr > 0$, $\forall h > 0, r > 0$, the inequality always holds.

Thus, the Backward Euler method is **A-stable**, as stability is guaranteed for all $h > 0$.

2 Ball Motion ODEs

The motion of a ball can be described by the following ordinary differential equations:

$$\frac{dx}{dt} = v_x, \quad \frac{dy}{dt} = v_y, \tag{1}$$

$$\frac{dv_x}{dt} = -\frac{k}{m} v_x \sqrt{v_x^2 + v_y^2}, \tag{2}$$

$$\frac{dv_y}{dt} = -g - \frac{k}{m} v_y \sqrt{v_x^2 + v_y^2}. \tag{3}$$

The initial value conditions are given by:

$$\begin{aligned} x(0) = x_0, \quad y(0) = y_0, \\ v_x(0) = v_{x_0}, \quad v_y(0) = v_{y_0}. \end{aligned}$$

To calculate the A-stability of Heun's RK2 and RK4 methods for the given system of ODEs, I will rely entirely on [1].

Approach

Linearize System:

For A-stability analysis, linearization turns out to be a common practice. The nonlinear drag terms make this complex, but for simplicity, let's approximate the drag force as proportional to velocity for linear analysis:

$$\frac{dx}{dt} \approx -\frac{k}{m}v_x, \quad \frac{dx}{dt} \approx -g - \frac{k}{m}v_x$$

which can be written in matrix form representation as follows:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k}{m} & 0 \\ 0 & 0 & 0 & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g \end{bmatrix}$$

Since we have an upper triangular matrix, its eigenvalues are on the diagonal, so we get $\lambda_1 = 0, \lambda_2 = -\frac{k}{m}$.

Heun's RK2:

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right)$$

RK4:

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

with

$$k_1 = f(t_n, y_n)$$
$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$
$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$
$$k_4 = f(t_n + h, y_n + hk_3)$$

For both methods, I have to derive G and check $|G| \leq 1$, $\forall h > 0$ and eigenvalues with negative real parts.

According to [1], with $z = \lambda h$ and λ being the eigenvalue of the system,

$$G_{RK2}(\lambda) = 1 + z + \frac{z^2}{2} \text{ and } G_{RK4}(\lambda) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$$

Recall, for A-Stability, we need $|G| \leq 1$ for all $Re(z) \leq 0$:

☞ For $\lambda = 0 \implies G_{RK2} = G_{RK4} = 1 \implies A$-Stability

✌ For $\lambda = -\frac{k}{m}$, recall that a numerical method is A-stable if its stability function $G(z)$ satisfies:

$$|G(z)| \leq 1 \quad \forall Re(z) \leq 0 \text{ with } z = \lambda h$$

Neither RK2 nor RK4 are fully A-stable because their stability regions don't cover the entire left-half complex plane. However, for small enough z, they can be stable.

$$|G(z)| \leq 1 \underbrace{\Longrightarrow}_{z=\lambda h=-\frac{k}{m}h} \left|G\left(-\frac{k}{m}h\right)\right| \leq 1$$

The magnitude of z must lie within the method's stability region, meaning: $\frac{k}{m}h < z_0$ where z_0 is the leftmost point in the stability region(negative real axis). For G_{RK2} and G_{RK4} we have conditional A-Stability if the time step h satisfies:

$$h < \frac{m}{k} \cdot z_0$$

To find z_0, we set $|G(z) = 1|$ and solve:

RK2 : $G_{RK2}(z) = 1 + z + \frac{z^2}{2}$

$$G(z) = 1 \Leftrightarrow 1 + z + \frac{z^2}{2} = 1 \Leftrightarrow z\left(1 + \frac{z}{2}\right) = 0 \Leftrightarrow z \in \{-2, 0\}$$

$$G(z) = -1 \Leftrightarrow 1 + z + \frac{z^2}{2} = -1 \Leftrightarrow z^2 + 2z + 4 = 0 \Leftrightarrow z = -1 \pm i\sqrt{3}$$

RK4 : $G_{RK4}(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$. Root finding of the equations below is done with Wolfram Alpha:

$$G(z) = 1 \Leftrightarrow 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} = 1 \Leftrightarrow z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} = 0$$

$z = 0$ and $z = \frac{1}{3}\left(-4 - 10\sqrt[3]{\frac{2}{-43+9\sqrt{29}}} + \sqrt[3]{-172 + 36\sqrt{29}}\right) \approx -2.7852935$

$$G(z) = -1 \Leftrightarrow 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} = -1 \Leftrightarrow z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} = -2$$

$z \approx -2.2194 \pm 1.6873i$ or $z \approx 0.2194 \pm 2.4753i$

3 Conclusion

Heun's RK2 stability analysis shows that the amplification factor $G(z) = 1$ at $z = 0$ and $z = -2$, indicating neutral stability at these points, meaning solutions neither grow nor decay. The stability region along the real axis is $z \in [-2, 0]$. Solving $G(z) = -1$ gives complex solutions at $z = -1 \pm i\sqrt{3}$, which mark the boundary for potential instability. If the step size is too large or eigenvalues lead to z outside $[-2, 0]$, the method can become unstable. Thus, for the ball motion ODE system, RK2 is conditionally A-stable, requiring the time step h to satisfy $h \leq \frac{2m}{k}$ to maintain stability. Exceeding this threshold results in instability.

In the case of RK4, the amplification factor satisfies $G(z) = 1$ at $z = 0$ and approximately $z = -2.785$. This identifies the real-axis stability interval as $z \in [-2.785, 0]$, which is wider than that of RK2. This broader interval allows RK4 to handle larger negative eigenvalues or larger step sizes before becoming unstable. However, solving $G(z) = -1$ reveals complex roots $z \approx -2.2194 \pm 1.6873i$ and $z \approx 0.2194 \pm 2.4753i$. These complex conjugate pairs mark the boundary beyond which the method becomes unstable. Thus, while RK4 can tolerate larger time steps compared to RK2, it is not fully A-stable since its stability region does not encompass the entire left-half complex plane. Therefore, for the ball motion ODE system, RK4 is conditionally A-stable, provided the time step satisfies $h \leq \frac{2.785m}{k}$.

References

[1] Jason Frank, *WI305106 - Numerieke Wiskunde 2/Numerical Analysis 2, Chapter 10*, Utrecht University, Chair of Numerical Analysis, `https://webspace.science.uu.nl/~frank011/Classes/numwisk/ch10.pdf`, 2008.