Let $\mathscr{C}$ be a set of every possible out comes of a random experiment and $\mathscr{C}$ be the sample space.

Let $P(C): \mathscr{C} \longrightarrow [0,1]$
$C \subseteq \mathscr{C}$

st

$P(C) \geq 0$

$P(C_1 \cup C_2 \ldots) = P(C_1) + P(C_2) \ldots\ldots$ where C_i are disjoint

$P(\mathscr{C}) = 1$

$P(C)$ is called the probability set function.

Thm 1 : For each $C \in \mathscr{C}$ $P(C) = 1 - P(C^*)$

C & C^* are disjoint

Thm 2 :

$P(\phi) = 0$

$P(\phi \cup \mathscr{C})$

Thm 3 : C_1 & C_2 are subset of $\mathscr{C}$ st $C_1 \subset C_2$

$P(C_2) = P(C_1 \cup (C_1^* \cap C_2))$

Thm 4

For each $C \subset \mathscr{C}$ $0 \leq P(C) \leq 1$

$\phi \subseteq C \subseteq \mathscr{C}$

Thm 5

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

$$P(C_1 \cup C_2) = P(C_1) + P(C_1^* \cap C_2)$$

$$\cancel{P(C_1 \cup C_2) = P(C_2) + P(C_2^* \cap C_1)}$$

$$P(C_2) = P(C_1 \cap C_2) + P(C_1^* \cap C_2)$$

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

Random variables

Defination :- Given a random experiment with a sample space $\mathcal{C}$

A function X which assigns to each element $c \in \mathcal{C}$ one and only one real numbes

$X(c) = x$ is called random variable.

The space of X is the set of real numbers $\mathbb{A} = \{x;\ x = X(c),\ c \in \mathcal{C}\}$

if $\mathcal{C}$ themselves have real numbers $X(c) = c$

so $\mathbb{A} = \mathcal{C}$

Let A be a subset of $\mathbb{A}$

Define $P_r(X \in A)$

$P(C)$ where $C = \{c;\ c \in \mathcal{C}$ and $X(c) \in A\}$

Notice $P_X(A)$ or $P_r(X \in A)$ will satisfy all the theorems.

Defination :- Given a random experiment with sample $\mathcal{C}$

Consider two random variables X_1 & X_2 st

$X_1(c) = x_1 \quad X_2(c) = x_2 \quad c \in \mathcal{C}$

$\mathcal{A} = \{(x_1, x_2) \mid x_1 \in X_1(c) \text{ \& } x_2 \in X_2(c), c \in \mathcal{C}\}$

$P_r[(X_1, X_2) \in A)$ A ~~be~~ a subset of $\mathcal{A}$

$P(\{c : X_1(c) = x_1 \text{ \& } X_2(c) = x_2\})$

Example

Let $\mathcal{C} = (0,1)$ Let $P(C) = \int_C dx$

is $C = (\frac{1}{4}, \frac{1}{2})$

$P(C) = \int_{1/4}^{1/2} dx$

Define $X(c) = 3c + 2$

$\mathcal{A} = \{x ; 2 < x < 5\}$

Let $A = (2, 3)$

$\therefore C = (0, \frac{1}{3})$

$\Rightarrow P(C) = \int_0^{1/3} dx$

Discrete random variable.

Let x denote a random variable with one dimensional space $\mathbb{A}$.

suppose $\mathbb{A}$ has finitely many points.

Let $f(x)$ be a function

$$f : \mathbb{A} \rightarrow (0,1)$$

st

$$\sum_{A} f(x) = 1$$

when even $A \in \mathbb{A}$ it can be expressed in terms such a

$f(x)$ by

$$P(A) = \sum_{x \in A} f(x)$$

Continous type of random variable

Let $\mathbb{A}$ be one dimentional and $f : \mathbb{A} \rightarrow (0,1)$ st

$$\int_{\mathbb{A}} f(x)\, dx = 1$$

x is said to be of continous type

if $A \subset \mathbb{A}$ then

$$P(A) = \int_{A} f(x)\, dx$$

The notion of pdf of one variable X can be extended to notion of pdf of two or more random variables

X & Y are two discrete type or of the continous type we have a distribution.

$$P(A) = Pr((x,y) \in A) = \sum_{A}\sum f(x,y)$$

$$P(A) = \iint_{A} f(x,y)\,dx\,dy$$

This notion can be extended further to n random variables

If $f(x)$ is the pdf of a continous type of random variable X and is A is a set $\{x;\ a<x<b\}$ the $P(A) =$ $Pr(X \in A)$ or $P(a<X<b)$ is formulated as

$$P(X \in A) = \int_{a}^{b} f(x)\,dx$$

if A is singleton $\{x=a\}$

$$P(X \in A) = \int_{a}^{a} f(x)\,dx = 0$$

This fact enable us to write

$$Pr(a<X<b) = Pr(a \leq X \leq b)$$

This helps us to change the value of the pdf of a continous type of random variable at a single point without altering the distribution of X.

$$f(x) = e^{-x} \quad x > 0$$
$$0 \quad \text{otherwise}$$

can be written s

$$f(x) = e^{-x} \quad x \geq 0$$
$$0 \quad \text{otherw.}$$

More generally if two probality density functions of a random variables of the continous type differ only on a set having probality zero, the two probality set function are exactly same.

The Distribution Function

Let variable X have the probability set functions P(A).

Take x to be a real numbe and consider the set $(-\infty, x]$ for such P(A) The probability depends upon on point x.

$$F(x) = Pr(X \leq x)$$

$$= \sum_{\omega \leq x} f(\omega)$$

$$= \int_{-\infty}^{x} f(z)\, dz$$

we shall use $F(\infty)$ & $F(-\infty)$ to denote $\lim_{x \to \infty} F(x)$ & $\lim_{x \to -\infty} F(x)$ respectively

Some properties

i) $0 \leq F(x) \leq 1$ because $0 \leq P(A) \leq 1$

ii) F(x) is non decreasing function of x

ie if $x \geq y$ Then $F(x) \geq y$

$$\Rightarrow \{\alpha \leq x\} = \{\alpha : \alpha \leq y\} \cup \{\alpha : y < \alpha \leq x\}$$

$$P(\{\alpha : \alpha \leq x\}) = P(\{\alpha : \alpha \leq y\}) + P(\{\alpha : y < \alpha \leq x\})$$

$$F(x) = F(y) + Pr(X \in (y, x])$$

$$\therefore \quad F(x) \geq F(y)$$

iii)

$$Pr(y < X \leq x) = F(x) - F(y)$$

Remark

$$Pr(X = b) = \lim_{h \to 0} Pr(b-h < X \leq b)$$

$$= \lim_{h \to 0} F(b) - F(b-h)$$

$$= F(b) - F(b^-)$$

iv) F is right continous

$\Rightarrow$ Let $h > 0$

$$\lim_{h \to 0} Pr(a < X \leq a+h) = \lim_{h \to 0} F(a+h) - F(a)$$

$$\lim_{h \to 0} \{x; a < x \leq a+h\} = \phi$$

suppose not ϕ

say $k \in \lim_{h \to 0} (a, a+h]$

$\therefore$ take h ~~smaller than~~ $= \frac{k-a}{2}$

hence $k \notin (a, \frac{a+k}{2}]$

$0 = F(a+) - F(a)$

$\therefore$ right continous