# 高等数学



## 数列

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

## 三角函数

$$1 + \tan^2 \alpha = \sec^2 \alpha$$

## 二项式定理

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k$$

## 常用不等式

$$|a+b| \le |a| + |b|$$

$$|a-b| \ge \big||a| - |b|\big|$$

$$\sqrt{ab} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$$

$$e^x \ge x + 1$$

$$\ln x \le x - 1$$

$$\frac{1}{1+x} \le \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x} \quad (x > 0)$$

## 常用导数

$$(\tan x)' = \sec^2 x \quad ; \quad (\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x \quad ; \quad (\csc x)' = -\csc x \cot x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad ; \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2} \quad ; \quad (\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

$$\left[\ln\left(x + \sqrt{x^2+1}\right)\right]' = \frac{1}{\sqrt{x^2+1}}$$

$$\left[\ln\left(x + \sqrt{x^2-1}\right)\right]' = \frac{1}{\sqrt{x^2-1}}$$

## 反函数的导数

$$y_x' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{x_y'}$$

$$y_{xx}'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{1}{x_y'}\right)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{1}{x_y'}\right)}{\mathrm{d}y} \cdot \frac{1}{x_y'} = \frac{-x_{yy}''}{\left(x_y'\right)^3}$$

## 变限积分求导

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(t)\mathrm{d}t\right] \\ &= f\left[\varphi_2(x)\right]\varphi_2'(x) - f\left[\varphi_1(x)\right]\varphi_1'(x) \end{aligned}$$

## 曲率

$$k = \frac{|y''|}{\left[1 + (y')^2\right]^{\frac{3}{2}}}$$

$$R = \frac{1}{k} = \frac{\left[1 + (y')^2\right]^{\frac{3}{2}}}{|y''|}$$

$$\begin{cases} x_0 = x - \frac{y'\left(1+(y')^2\right)}{y''} \\ y_0 = y + \frac{1+(y')^2}{y''} \end{cases}$$

## N 阶导数公式

$$\begin{aligned} \left(\mathrm{e}^x\right)^{(n)} &= \mathrm{e}^x \\ \left(a^x\right)^{(n)} &= a^x(\ln a)^n(a > 0, a \neq 1) \\ \left(\frac{1}{1-x}\right)^{(n)} &= \frac{n!}{(1-x)^{n+1}} \\ \left(\frac{1}{1+x}\right)^{(n)} &= \frac{(-1)^n n!}{(1-x)^{n+1}} \\ (\sin kx)^{(n)} &= k^n \sin\left(kx + n \cdot \frac{\pi}{2}\right) \\ (\cos kx)^{(n)} &= k^n \cos\left(kx + n \cdot \frac{\pi}{2}\right) \\ (\ln x)^{(n)} &= (-1)^{n-1}\frac{(n-1)!}{x^n}(x > 0) \\ \left[\ln(1+x)\right]^{(n)} &= (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}(x > -1) \\ \left[(x+x_0)^m\right]^{(n)} &= m(m-1)(m-2)\cdots(m-n+1)(x+x_0)^{m-n} \\ \left(\frac{1}{x+a}\right)^{(n)} &= \frac{(-1)^n n!}{(x+a)^{n+1}} \end{aligned}$$

## Taylor 公式（简单无穷小展开）

$$
\begin{aligned}
\mathrm{e}^x &= 1 + x + \frac{x^2}{2!} + o(x^2) \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4) \\
\tan x &= x + \frac{x^3}{3} + o(x^3) \\
\arcsin x &= x + \frac{x^3}{6} + o(x^3) \\
\arctan x &= x - \frac{x^3}{3} + o(x^3) \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \\
a^x &= 1 + (x \ln a) + \frac{(x \ln a)^2}{2!} + o(x^2) \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + o(x^3) \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + o(x^3) \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + o(x^2)
\end{aligned}
$$

## 无穷小比阶

$$
\begin{aligned}
\arcsin ax \sim \sin ax &\sim ax \\
\arctan ax \sim \tan ax &\sim ax \\
\ln(1+x) &\sim x \\
\sqrt[b]{1+ax} - 1 &\sim \frac{a}{b}x \\
(1+ax)^b - 1 &\sim abx \\
\sqrt{1+x} - \sqrt{1-x} &\sim x \\
1 - \cos x &\sim \frac{x^2}{2} \\
x - \ln(1+x) &\sim \frac{x^2}{2} \\
\tan x - \sin x &\sim \frac{x^3}{2} \\
\tan x - x &\sim \frac{x^3}{3} \\
x - \arctan x &\sim \frac{x^3}{3} \\
x - \sin x &\sim \frac{x^3}{6} \\
\arcsin x - x &\sim \frac{x^3}{6}
\end{aligned}
$$

## Taylor 公式（完全展开）

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, -\infty < x < +\infty\,. \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots, -\infty < x < +\infty\,. \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots, -\infty < x < +\infty\,. \\
\tan x &= x + \frac{x^3}{3} + o(x^3) \\
\arcsin x &= x + \frac{x^3}{6} + o(x^3) \\
\arctan x &= x - \frac{x^3}{3} + o(x^3) \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots, -1 < x \leqslant 1\,. \\
a^x &= \sum_{n=0}^{\infty} \frac{(x \ln a)^n}{n!} \\
&= 1 + (x \ln a) + \frac{(x \ln a)^2}{2!} + \cdots + \frac{(x \ln a)^n}{n!} + \cdots, -\infty < x < +\infty\,. \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
&= 1 + x + x^2 + \cdots + x^n + \cdots, -1 < x < 1\,. \\
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n \\
&= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots, -1 < x < 1\,. \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + \cdots, \qquad
\begin{cases}
x \in (-1,1) & \text{当}\, \alpha \leqslant -1\,, \\
x \in (-1,1] & \text{当}\, -1 < \alpha < 0\,, \\
x \in [-1,1] & \text{当}\, \alpha > 0\,.
\end{cases}
\end{aligned}
$$