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A Complete and Robust Approach to Axisymmetric Method of Characteristics for Nozzle Design

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The method of characteristics was developed for supersonic flow solutions in channels beginning in the 1930's and continuing at a vigorous pace at least into the 1980's. Nevertheless, the open literature does not contain an unabridged derivation of this fundamental nozzle design tool and there are no open sources which describe robust implementation strategies. This paper fills both of these gaps in the literature and provides the technical details required for the accurate use of, for example continuity, in nozzle design exercises over a wide range of expansion ratios.

Nomenclature

A	=	area
F	=	thrust
M	=	Mach number
$\dot{m}$	=	mass flow rate
p	=	pressure
p_t	=	total pressure
R	=	specific gas constant
u	=	axial velocity
v	=	radial velocity
γ	=	ratio of specific heats
ρ	=	density
ε	=	switch for two dimensional or axisymmetric flow
θ	=	flow angle
μ	=	Mach angle

I. Introduction

An excellent review of various nozzle types suitable for rocket propulsion applications is presented by Östlund and Muhammad-Klingmann.[1] For shock-free nozzles, the primary loss in the high-speed, diverging section of the nozzle is related to the fact that for all practical nozzles there is a non-axial component to the flow velocity at the exit plane of the nozzle. For well-designed nozzles, the loss associated with this geometric effect can be minimized. The classical foundation for this type of analysis is the method of characteristics.
in 1929, Ludwig Prandtl and Adolf Busemann devised a graphical approach for solving supersonic irrotational flows which can be used for flow turning in nozzles without the generation of shocks. This shock free nozzle solution is achieved by canceling expansion waves rather than by implementing turns which would otherwise be compressive. This method for supersonic flow analysis has become known as the method of characteristics and has been used since the 1930's for nozzle design and analysis and for many other applications. The method of characteristics is essentially a numerical approach; however, unlike traditional finite differencing schemes, this approach allows for the determination of the wall angles necessary for shock free nozzle flow directly from the flow solution. This feature makes the method of characteristics particularly useful for nozzle design.

Mathematically, the method of characteristics as applied to the inviscid flow equations is a branch of absolute differential calculus in which a certain class of partial differential equations which apply to field variables in a given

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domain can be reduced to ordinary differential equations or even algebraic equations if their application is restricted to specific lines within the domain. The partial differential equations governing irrotational supersonic flow in two dimensions and in axisymmetric situations represent one such case in which this mathematical approach can be applied with great success.

The method was first proposed and developed as a general solution approach for partial differential equations by mathematicians Jacques Saloman Hadamard in 1903 and Tullio Levi-Civita in 1932.[5] Prandtl and Buseman first implemented this approach graphically to solve the two dimensional supersonic nozzle problems in 1929. In 1947, Shapiro and Edelman outlined a comprehensive graphical and analytical approach to the two-dimensional method of characteristics.[6] In 1949 a method was proposed for developing an analytical solution to the problem of developing an all axial supersonic stream using the axisymmetric method of characteristics.[7] By 1954, a theoretical adaptation of the mathematics to both two dimensional and axisymmetric flows had been established.[8] Guentert and Netmann at NASA Lewis[9] implemented the analytical approach in practice for the design of axisymmetric method of characteristic nozzles and showed that a practical solution can be developed by starting from an estimated pressure distribution along the nozzle centerline. This approach allows the designer to specify the exit Mach number (and exit pressure). Guderley and Hantsch[10,11] formulated an approach for developing a nozzle for maximum thrust with a given length. Using this formulation, Rao[12,13,14] developed a straightforward, albeit iterative, solution method for finding the shape of the compressive portion of the nozzle contour for a maximum thrust, fixed length nozzle for a given ambient pressure by using Lagrange multipliers. Such nozzles are often referred to as thrust optimized contour nozzles. This approach was widely used in the 1960's and 1970"s; however, these nozzle designs are not shock free internally and the pressure spikes during the starting transient can create substantial mechanical loads on the nozzle.

In this paper, a complete method of characteristics approach is presented with examples.

II. Derivation

Presented in this section is a partial derivation of the axisymmetric method of characteristics. For the conference paper, the complete derivation will be included. For the purposes of the abstract, only a sketchy outline of the results is included. The starting point for axisymmetric solution is Sauer's solution[20] for nozzle throat conditions. This is a solution is analytic and can be implemented to produce reliable results for the starting line for the method of characteristics as long as the radius of the nozzle wall contour upstream of the throat is at least twice the nozzle radius. A detailed treatment of Sauer's solution is included at the end of the paper.

If there are no gradients in total enthalpy or entropy, the right side of both Eq. 3.36 and Eq. 3.44 is zero and $\partial v/\partial x = \partial u/\partial y$ which guarantees that the velocity can be represented as the gradient of a potential or

$$\vec{V} = \vec{\nabla}\Phi \qquad 1$$

and the governing flow equation for nozzle flow is:

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} - \left(\frac{2uv}{a^2}\right)\frac{\partial u}{\partial y} + \left(1-\frac{v^2}{a^2}\right)\frac{\partial v}{\partial y} = -\varepsilon\frac{v}{y} \qquad 2$$

Where ε = 1 for axisymmetric flow and 0 for two dimensional flow. Solutions to Eq. 2 can be developed by considering the variation of u and v as follows:

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du \qquad 3$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \frac{\partial u}{\partial y}dx + \frac{\partial v}{\partial y}dy = dv \qquad 4$$

The system of equations, 2 – 4 is a linear system which can be solved using a variety of methods. The physics based approach for describing the domain along which solutions exists described by Anderson fills the interpretive gaps left by other presentations. To understand the solution from a physics perspective, Eq. 4 is solved for the derivative of the u velocity in the x direction:

$$\frac{\partial u}{\partial x}=\frac{\frac{2uv}{a^2}\frac{\partial u}{\partial y}-\left(1-\frac{v^2}{a^2}\right)\frac{\partial v}{\partial y}+\varepsilon\frac{v}{y}}{\left(1-\frac{u^2}{a^2}\right)} \qquad 5$$

For a general numerical approach to solving the *partial differential equation* given in Eq. 2, a grid based solution could be developed using Eq. 4 by recognizing the there are no derivatives in x on the right side. Therefore, the u component of velocity could be calculated at downstream locations strictly from a know set of conditions along a vertical line (which could be used to calculate $\partial v/\partial y$ *and* $\partial u/\partial y$) using a Taylor Series expansion. The value for $\partial v/\partial x$ is obtained from the condition for irrotationality, $\partial v/\partial x=\partial u/\partial y$ and values for v could then also be developed using a Taylor Series expansion.

This method breaks down, however, for cases in which u = a because the denominator in Eq. 3.49 becomes zero and the derivative becomes at least indeterminate and could be discontinuous. Keep in mind that the direction x is entirely a construct of the problem formulation and can be chosen to be in any direction x' as shown in Figure 1 at the outset without compromising the validity of Eqs 2 – 5 if they are written in the x' coordinate system for two dimensional flow. (We must be careful to recognize that this is not generally true for $\varepsilon \neq 0$ on a macroscopic scale but in the limit as the dimensions become small in the vicinity of point p, axissymetric flow is approximately two dimensional and we will show momentarily that the angle at which a singularity exists can be calculated for axisymmetric flow in the reference frame where the equations are valid for this flow on a macroscopic scale.) If the coordinate system happens to be chosen such that for a particular given location in the flow, the flow velocity perpendicular to the direction along which the grid data is available for advancing the solution is equal to the sound speed, then the derivative of the velocity in that initially arbitrarily chosen direction as described by equation 3.49 is discontinuous and possibly indeterminate. For a given supersonic flow which satisfies the assumptions laid out thus far, the numerical method briefly alluded to above would suffice with programming to avoid the singularity. However, the beauty of this method is that if a direction is chosen to *deliberately* produce a singularity in equation 5, an alternative set of constraints can be constructed which reduce the partial differential equation in 2 into an ordinary differential equation or even an algebraic equation if applied along the line of the singularity. The auxiliary constraint equation arises from the fact that the velocity is physically finite everywhere and for this condition to be satisfied, the numerator Eq. 5 must also be zero for example. A complete solution to the system of equations given in 2 -4 can be developed using this approach by considering the flow condition depicted in Figure 1.

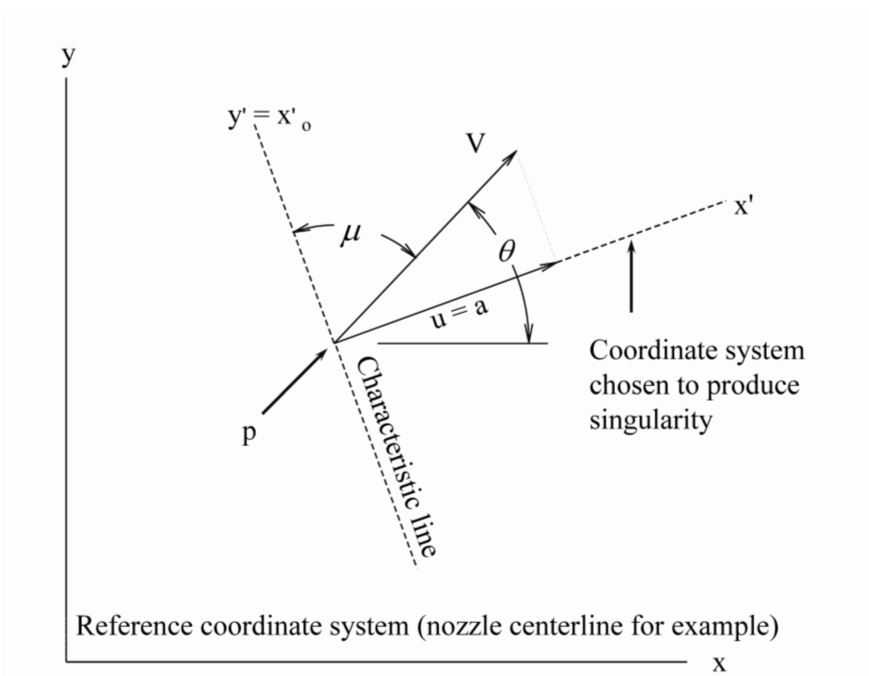

Figure 1: Coordinate system for the application of the method of characteristics

It is more convenient to develop solutions based on the orientation of the coordinate system to the velocity vector V which is fixed at a constant flow angle θ in the reference coordinate system for steady flow regardless of the orientation chosen for the x' system. The Mach angle μ is defined such that sin μ = a/V = 1/M and for this flow sin μ = u'/V. As shown in Figure 1, the Mach angle provides the orientation between the velocity vector and the x' coordinate system. If the flow is isentropic, it is also shock free and the velocity field is continuous in the vicinity of the flow field point of interest denoted by p. For this reason, the line $y' = x'_o$ represents a line along which, in the vicinity of p, the singularity will exist. Such lines are referred to as *characteristic* lines along which solutions can be developed using the ordinary differential equations or algebraic equations which result from the existence of the singularity condition. Notice that the angle between the characteristic line and the x axis of the fixed reference coordinate system is θ + μ. This is true regardless of how the initial fixed coordinate system is chosen; however, for axisymmetric flow the x axis must be chosen such that the x axis is the axis of symmetry in order for Eq. 2 to hold macroscopically. Along these lines the derivatives of all of the flow variables, are indeterminate but finite in the x' system. To continue to develop a solution throughout the flow field for the x' system using this methodology would require that the coordinate system constantly be rotated to accommodate the local flow conditions. Alternatively, a solution is sought in the fixed reference frame.

To efficiently develop a solution to the system of equations Eqs. 2, 3 and 4 in the reference coordinate system, it is convenient to recognize that this system can be treated as simultaneous linear algebraic equations in the variables $\partial u/\partial x$, $\partial u/\partial y$, and $\partial v/\partial y$. The solution to this linear set of equations can be found succinctly using Cramer's rule. For example:

$$\frac{\partial v}{\partial y}=\frac{\begin{vmatrix}\left(1-\frac{u^2}{a^2}\right) & -\frac{2uv}{a^2} & -\varepsilon\frac{v}{y}\\ dx & dy & du\\ 0 & dx & dv\end{vmatrix}}{\begin{vmatrix}\left(1-\frac{u^2}{a^2}\right) & -\frac{2uv}{a^2} & \left(1-\frac{v^2}{a^2}\right)\\ dx & dy & 0\\ 0 & dx & dy\end{vmatrix}}=\frac{N}{D} \qquad 6$$

For a flow solution in the vicinity of the point p of interest, solutions for the derivatives can be found along any line of slope characterized by arbitrarily chosen values for dx and dy. If the denominator in Eq. 6 is zero the singularity condition resurfaces. In this case, the u component of velocity is not being varied by rotating the coordinate system but rather by adjusting the values of dx and dy to find a slope within the reference coordinate system to create the singularity. Once again, the numerator must also be zero for the velocity to remain finite in this domain along the line which produces the singularity. To determine the combinations of dy and dx which lead to a singularity, the denominator is expanded and set to zero.

$$\left(1-\frac{u^2}{a^2}\right)(dy)^2+\frac{2uv}{a^2}dydx+\left(1-\frac{v^2}{a^2}\right)(dx)^2=0 \qquad 7$$

Dividing by dx^2 gives

$$\left(1-\frac{u^2}{a^2}\right)\left(\frac{dy}{dx}\right)_{char}^2+\frac{2uv}{a^2}\left(\frac{dy}{dx}\right)_{char}+\left(1-\frac{v^2}{a^2}\right)=0 \qquad 8$$

Equation 8 is a quadratic expression in the slope $(dy/dx)_{char}$ for the characteristic lines. Solving for the slope gives:

$$\left(\frac{dy}{dx}\right)_{char}=\frac{\frac{-2uv}{a^2}\pm\sqrt{\left(\frac{2uv}{a^2}\right)^2-4\left[\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{a^2}\right)\right]}}{2\left(1-\frac{u^2}{a^2}\right)}$$

Simplifying and recognizing that $(u^2 + v^2)/a^2 = M^2$ gives

$$\left(\frac{dy}{dx}\right)_{char}=\frac{\frac{-uv}{a^2}\pm\sqrt{\left[\frac{u^2+v^2}{a^2}\right]-1}}{1-\frac{u^2}{a^2}}=\frac{\frac{-uv}{a^2}\pm\sqrt{M^2-1}}{1-\frac{u^2}{a^2}} \qquad 9$$

Notice that real solutions for the slope of the characteristics exist only for supersonic flows. For a Mach number of unity, only one characteristic will exist but for Mach numbers greater than unity two characteristic slopes exists, allowing for a practical approach to develop a flow field solution along the crossing characteristic lines.

A more succinct expression for the slopes of the characteristic lines can be developed using the characteristic flow angles, θ and μ. The first step in developing this simplification is to write the u and v velocity components in terms of the flow angle of the velocity vector and the magnitude of the velocity vector as $u = V\cos\theta$ and $v = V\sin\theta$. Substituting these expressions into Eq. 9 gives:

$$\left(\frac{dy}{dx}\right)_{char}=\frac{\frac{-V^2\cos\theta\sin\theta}{a^2}\pm\sqrt{\frac{V^2}{a^2}\left[\cos^2\theta+\sin^2\theta\right]-1}}{1-\frac{V^2}{a^2}\cos^2\theta} \qquad 10$$

Recognizing from the definition of the Mach angle that $V^2/a^2 = M^2 = 1/\sin^2\mu$, Eq. 10 becomes

$$\left(\frac{dy}{dx}\right)_{char}=\frac{\frac{-\cos\theta\sin\theta}{\sin^2\mu}\pm\sqrt{\frac{\left[\cos^2\theta+\sin^2\theta\right]}{\sin^2\mu}-1}}{1-\frac{\cos^2\theta}{\sin^2\mu}} \qquad 11$$

From trigonometry,

$$\sqrt{\frac{\left[\cos^2\theta+\sin^2\theta\right]}{\sin^2\mu}-1}=\sqrt{\frac{1}{\sin^2\mu}-1}=\sqrt{\frac{1-\sin^2\mu}{\sin^2\mu}}=\sqrt{\frac{\cos^2\mu}{\sin^2\mu}}$$

Using this result and multiplying numerator and denominator by $\sin^2\mu$, Eq. 11 becomes

$$\left(\frac{dy}{dx}\right)_{char} = \frac{-\cos\theta\sin\theta \pm \cos\mu\sin\mu}{\sin^2\mu - \cos^2\theta} = \frac{-2\cos\theta\sin\theta \pm 2\cos\mu\sin\mu}{2\sin^2\mu - 2\cos^2\theta} \qquad 12$$

As a reminder, the salient half angle relationships are:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\mu = 2\sin\mu\cos\mu$$

$$2\sin^2\mu = 1 - \cos 2\mu$$

$$2\cos^2\theta = 1 + \cos 2\theta$$

Substituting into Eq. 12 gives

$$\left(\frac{dy}{dx}\right)_{char} = \frac{-\sin 2\theta \pm \sin 2\mu}{1 - \cos 2\mu - 1 - \cos 2\theta} = \frac{\sin 2\theta \mp \sin 2\mu}{\cos 2\mu + \cos 2\theta} \qquad 13$$

The trigonometric identities for summing the sines and cosines of two angles are:

$$\sin 2\theta \mp \sin 2\mu = 2\sin\left(\frac{2\theta \mp 2\mu}{2}\right)\cos\left(\frac{2\theta \pm 2\mu}{2}\right) = 2\sin(\theta \mp \mu)\cos(\theta \pm \mu)$$

$$\cos 2\mu + \cos 2\theta = 2\cos(\theta - \mu)\cos(\theta + \mu)$$

Substituting these identities into Eq. 13 gives

$$\left(\frac{dy}{dx}\right)_{char} = \frac{2\sin(\theta \mp \mu)\cos(\theta \pm \mu)}{2\cos(\theta - \mu)\cos(\theta + \mu)} = \tan(\theta \mp \mu) \qquad 14$$

This equation indicates that the characteristic lines along which solutions can be developed are at angles of θ - μ and $\theta + \mu$ to the fixed reference axes, in agreement with the graphical result obtained above for the characteristic line at angle $\theta + \mu$. This derivation establishes the existence of the second characteristic line at angle $\theta - \mu$. The characteristic line at angle $\theta + \mu$ is known as the C_+ or "left running" characteristic because of the "+" sign on the angles and because it run to the left of a particle following the streamline. The characteristic line at angle $\theta - \mu$ is known as the C_- or "right running" characteristic. These two characteristics are shown schematically in Figure 2. Notice that Eq. 13 holds for both two dimensional flows and axisymmetric flows because the axisymmetric term –v/y does not appear in the denominator of Eq. 3.50. Also, notice that the slope is dependent on the local flow angle and the local Mach number and as those local flow conditions change from point to point in the flow, the slope of the characteristics change and thus the characteristics are curved for non uniform supersonic flow, as indicated in Figure 2.

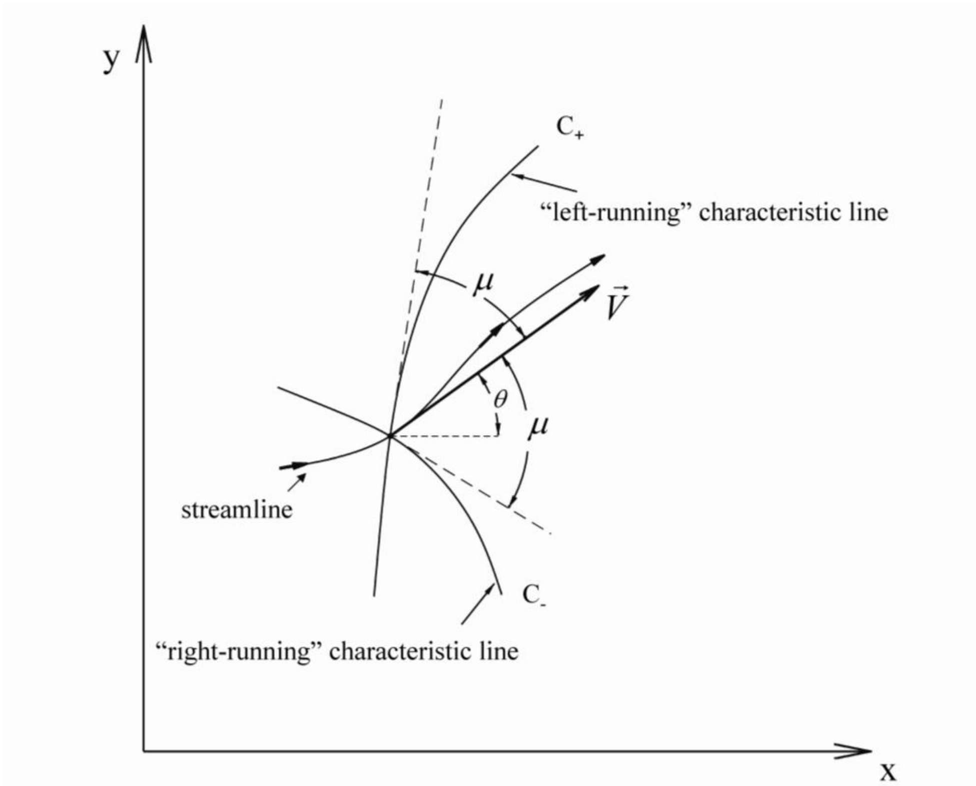

Figure 2: Left and right running characteristic lines

Characteristics for a near uniform Mach 2 Nozzle

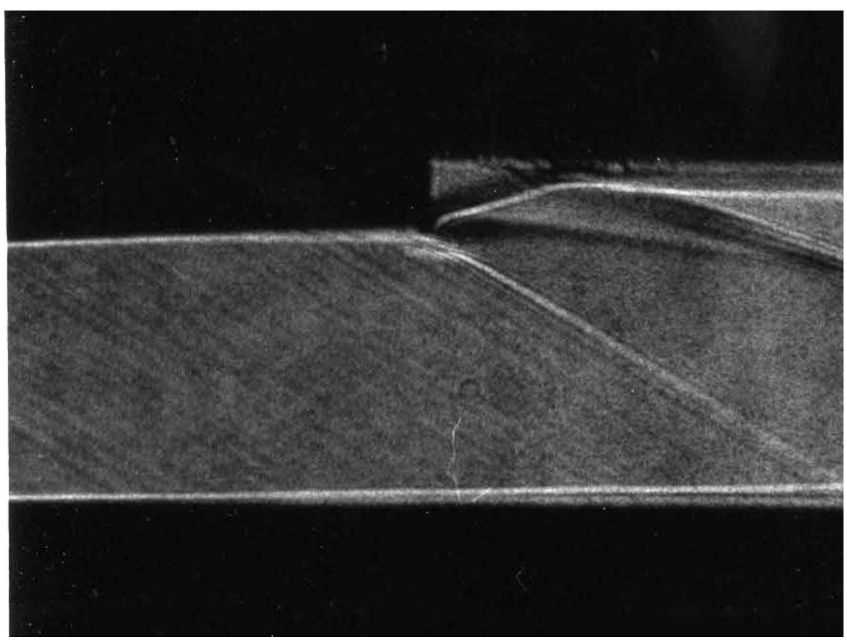

Figure 3: Mach lines visible along the characteristics in a Mach 2 nozzle exit

Equation 14 can only be used to determine the directions along which solutions exist. To develop the solution for the flow variables, the numerator in Eq. 6 is set to zero to satisfy the equation for the singularity condition. Expanding the numerator in Eq. 6 and setting the result to zero gives:

$$\left(1-\frac{u^2}{a^2}\right)(dvdy-dudx)+\frac{2uv}{a^2}dvdx+\varepsilon\frac{v}{y}dx^2=0 \qquad 15$$

Dividing by dvdx and rearranging gives

$$\left(1-\frac{u^2}{a^2}\right)\left(\frac{dy}{dx}-\frac{du}{dv}\right)+\frac{2uv}{a^2}+\varepsilon\frac{v}{y}\frac{dx}{dv}=0$$

Solving for du/dv gives

$$\frac{du}{dv}=\frac{dy}{dx}+\frac{\frac{2uv}{a^2}}{1-\frac{u^2}{a^2}}+\frac{\varepsilon\frac{v}{y}}{\left(1-\frac{u^2}{a^2}\right)}\frac{dx}{dv} \qquad 16$$

The flow relationships developed using Eq. 14 only apply along the characteristic lines because setting the numerator to zero would be arbitrary but for the singularity condition. This allows for:

$$\frac{dy}{dx}=\left(\frac{dy}{dx}\right)_{char}$$

Substituting from Eq. 9 for the slope of the characteristics gives:

$$\frac{du}{dv}=\left[\frac{\frac{-uv}{a^2}\pm\sqrt{M^2-1}}{1-\frac{u^2}{a^2}}\right]+\frac{\frac{2uv}{a^2}}{1-\frac{u^2}{a^2}}+\frac{\varepsilon\frac{v}{y}}{\left(1-\frac{u^2}{a^2}\right)}\frac{dx}{dv}$$

which can be simplified to

$$\frac{du}{dv}=\left[\frac{-\frac{uv}{a^2}\mp\sqrt{M^2-1}}{1-\frac{u^2}{a^2}}\right]-\frac{\varepsilon\frac{v}{y}}{\left(1-\frac{u^2}{a^2}\right)}\frac{dx}{dv} \qquad 17$$

Eliminating the slope of the characteristic line between Eqs. 9 and 14 gives

$$\left[\frac{-\frac{uv}{a^2}\mp\sqrt{M^2-1}}{1-\frac{u^2}{a^2}}\right]=\tan(\theta\pm\mu)$$

and Eq. 17 becomes

$$du=\tan(\theta\pm\mu)dv-\frac{\varepsilon\frac{v}{y}}{\left(1-\frac{u^2}{a^2}\right)}dx \qquad 18$$

In summary, the two principle results from the method of characteristics are the well published characteristics slopes given by

$$\left(\frac{dy}{dx}\right)_{char} = \tan(\theta \mp \mu) \tag{14}$$

and the compatibility equations given by

$$du = \tan(\theta \pm \mu)dv - \frac{\varepsilon \frac{v}{y}}{\left(1 - \frac{u^2}{a^2}\right)} dx \tag{18}$$

The ordinary differential compatibility equations are typically further reduced to algebraic equations for two dimensional treatments with ε = 0, but must be solved using numerical approaches for axisymmetric flow.

The closest implementation of this set of equations is by Zucrow and Huffman but some significant gaps exists between the Zucrow implementation and a robust, practical solution.

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III. Solver Related Discussion

A key issue associate with the solution to the compatibility equations is the determination of the nozzle wall boundary using either flow tangency or continuity. The Rao approach hinges on continuity but very few details about how Rao actually performed continuity calculations are published in the literature. Flow tangency is highly dependent on initial conditions and higher order solutions for flow angle at the wall.

A second key issue for robustness is the development of the characteristic net in the vicinity of the wall. Academic treatments tend to produce solutions for anecdotal single case examples which cannot generally be expanded to arbitrary expansion ratios. A robust approach for characteristic net construction will be presented in the paper.

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IV. Results

Some initial results from the implementation efforts developed by the authors for this paper are shown in Figures 4 and 5 for a low expansion ratio and a high expansion ratio respectively.

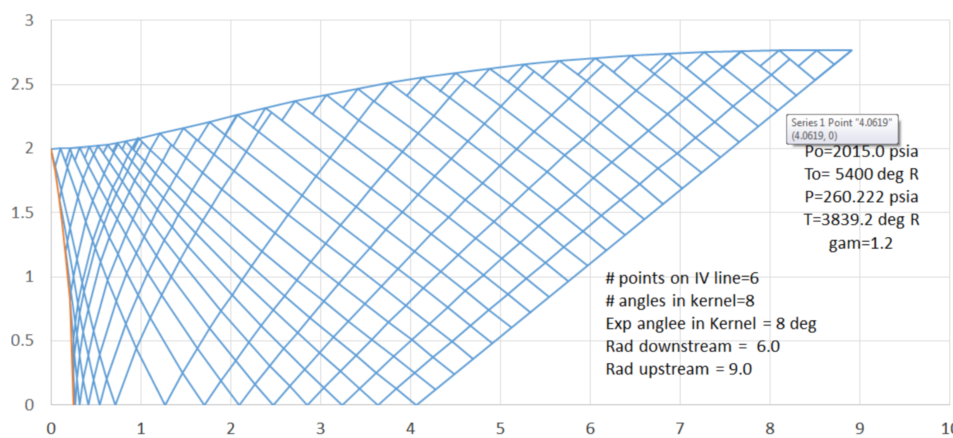

Figure 4: Low Expansion Ratio Implementation of the Method of Characteristics for Nozzle Design

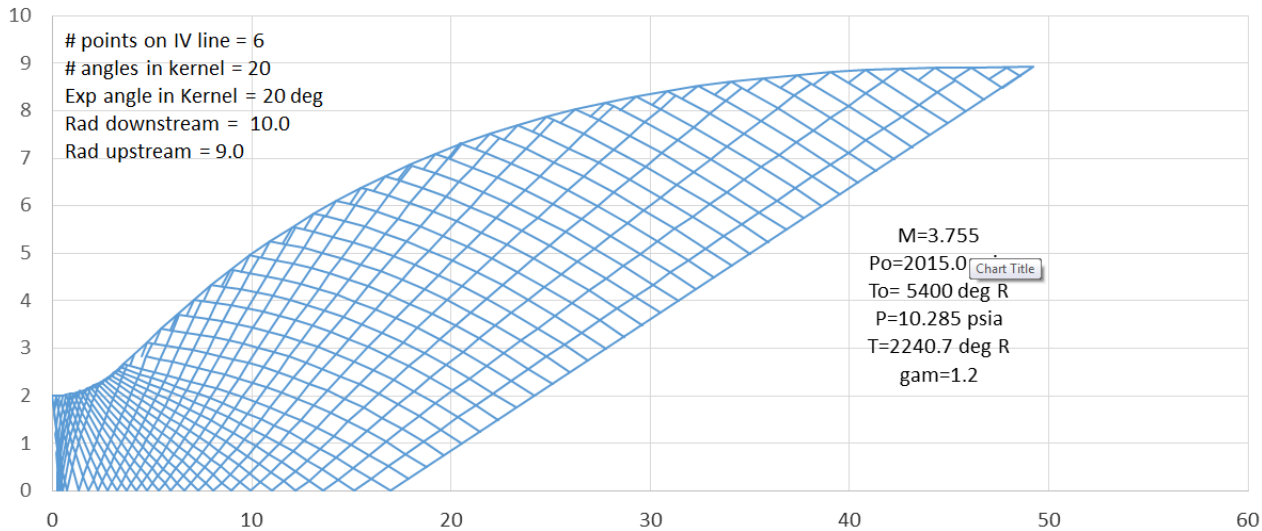

Figure 5: High Expansion Ratio Implementation of the Method of Characteristics for Nozzle Design

V. Conclusion

This paper presents both a more comprehensive treatment of the theory of the method of characteristics for axisymmetric flows and a practical guide to robust implementation of the axisymmetric method of characteristics.

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