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Application of De-Laval Nozzle Transonic Flow Field Computation Approaches

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***Abstract*—**A supersonic expansion cannot be achieved within a convergent-divergent nozzle if the flow velocity does not reach that of the sound at the throat. The computation of the flow field characteristics at the throat is thus essential to the nozzle developed thrust value and therefore to the aircraft or rocket it propels. Several approaches were developed in order to describe the transonic expansion, which takes place through the throat of a De-Laval convergent-divergent nozzle. They all allow reaching good results but showing a major shortcoming represented by their inability to describe the transonic flow field for nozzles having a small throat radius. The approach initially developed by Kliegel & Levine uses the velocity series development in terms of the normalized throat radius added to unity instead of solely the normalized throat radius or the traditional small disturbances theory approach. The present investigation carries out the application of these three approaches for different throat radiuses of curvature. The method using the normalized throat radius added to unity shows better results when applied to geometries integrating small throat radiuses.

***Keywords*—**De-Laval nozzles, transonic calculations, transonic flow, supersonic nozzle.

I. Introduction

THE thrust of a propulsion engine mainly lies on the moment imparted to the combustion gases which are accelerated in a continuous manner from subsonic to highly supersonic speeds. The aerodynamic design of supersonic De-Laval exhaust nozzles is usually considered in three parts: the subsonic section or convergent, the throat, and the supersonic divergent section.

The design of the contraction although important in itself does not have any effect on that of the supersonic duct. The throat region however is of considerable importance because of the necessary flow conditions required to carry out the supersonic computations [1].

The throat flow region of a De-Laval nozz1e has been widely studied [2]-[6]. Various expansion techniques have been applied to describe the transonic flow field. All of these methods are essentially the same, being perturbations about the one-dimensional flow through the so-called normalized throat wall radius of curvature ($R=\rho_{tu} / y_t$).

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II. Transonic Flow Solution Approaches

A. Sauer's Approach

Sauer's approach [7] is based on the theory of small disturbances. It solves the equation of the small disturbances for a compressible flow for a two-dimensional as well as an axially symmetric flow. The solution is proposed in terms of inverse of R to solely the first order, thus neglecting all the terms starting from the second order. The method is interesting and has been applied by designers during several years. Its main advantage lies in the fact that it can be applied to flows around an arbitrary profile. It however does not give any information on the direction of the flow or the distribution of the isobars, its main disadvantage being the fact that it diverges completely for small values of the parameter *R*.

TABLE I
Nomenclature and Units

Symbol	Unit	Quantity
C_D	[-]	Discharge coefficient
P_a	[N/m^2]	Ambient pressure
P_t	[N/m^2]	Total pressure
r, z	[-]	Transformed normalized coordinates
R	[-]	Normalized throat radius of curvature
R_G	[J/kgK]	Ideal-gas constant
T_t	[K]	Total temperature
u, v	[m/s]	Axial and radial velocity components
$\underline{u}, \underline{v}$	[-]	Normalized (with respect to sonic velocity) axial and radial velocity components
u_0, u_w	[m/s]	Throat and wall axial velocities
x, y	[m]	Axial and radial Cartesian system coordinates
y_t	[m]	Throat nozzle radius
y_e	[m]	Exit nozzle radius
α	[-]	Constant
γ	[-]	Specific heat ratio
δ	[-]	=0,1 for a two-dimensional or an axially-symmetric flow respectively
ρ_{tu}	[m]	Upstream throat radius of curvature
η, ξ	[-]	Toroïdal coordinates

Sauer's two components of the velocity are given by:

$$\begin{cases} \underline{u}(x,y) = \alpha x + \dfrac{(\gamma+1)\alpha^2 y^2}{2(1+\delta)} + ... \\ \underline{v}(x,y) = \dfrac{(\gamma+1)\alpha^2 xy}{(1+\delta)} + \dfrac{(\gamma+1)^2 \alpha^3 y^3}{2(1+\delta)(3+\delta)} + ... \end{cases} \tag{1}$$

with:

$$\alpha = \sqrt{\frac{1+\delta}{(\gamma+1)\rho_{tu} y_t}}$$

B. Hall's Approach

Hall [8] developed a transonic solution based on the small perturbation theory applying it to an irrotational, perfect gas. The velocity components were expressed in cylindrical coordinates in terms of inverse powers of the normalized throat wall radius of curvature. Their normalized (with respect to the sonic velocity) expressions are found to be of the form:

$$\begin{cases} u = 1 + \dfrac{u_1(r,z)}{R} + \dfrac{u_2(r,z)}{R^2} + \dfrac{u_3(r,z)}{R^3} + \ldots.. \\ v = \sqrt{\dfrac{\gamma+1}{2R}} \left[\dfrac{u_1(r,z)}{R} + \dfrac{u_2(r,z)}{R^2} + \dfrac{u_3(r,z)}{R^3} + \ldots. \right] \end{cases} \tag{2}$$

where:

$$z = \frac{x}{y_t} \sqrt{\frac{2R}{(\gamma+1)}} \tag{3}$$

$$r = \frac{y}{y_t} \tag{4}$$

and:

$$u_1 = \frac{1}{2} r^2 - \frac{1}{4} + z \tag{5-a}$$

$$v_1 = \frac{1}{4} r^3 - \frac{1}{4} r + rz \tag{5-b}$$

$$u_2 = \frac{2\gamma+9}{24} r^4 - \frac{4\gamma+15}{24} r^2 + \frac{10\gamma+57}{288} + z \left(r^2 - \frac{5}{8} \right) - \frac{2\gamma-3}{6} z^2 \tag{5-c}$$

$$\begin{aligned} v_2 = {} & \frac{\gamma+3}{9} r^5 - \frac{20\gamma+63}{96} r^3 \\ & + \frac{28\gamma+93}{288} r + z \left(\frac{2\gamma+9}{6} r^3 - \frac{4\gamma+15}{12} r \right) + rz^2 \end{aligned} \tag{5-d}$$

$$\begin{aligned} u_3 = {} & \frac{556\gamma^2+1737\gamma+3069}{10368} r^6 - \frac{388\gamma^2+1161\gamma+1881}{2304} r^4 \\ & + \frac{304\gamma^2+831\gamma+1242}{1728} r^2 - \frac{2708\gamma^2+7839\gamma+14211}{82944} \\ & + \left(\frac{52\gamma^2+51\gamma+327}{384} r^4 - \frac{52\gamma^2+75\gamma+279}{192} r^2 \right. \\ & \left. + \frac{92\gamma^2+180\gamma+639}{1152} \right) + z^2 \left(-\frac{7\gamma-3}{8} r^2 + \frac{13\gamma-27}{48} \right) \\ & + \frac{4\gamma^2-57\gamma+27}{144} z^3 \end{aligned} \tag{5-e}$$

$$\begin{aligned} v_3 = {} & \frac{6836\gamma^2+23031\gamma+30627}{82944} r^7 - \frac{3380\gamma^2+11391\gamma+15291}{13824} r^5 \\ & + \frac{3424\gamma^2+11271\gamma+15228}{13824} r^3 - \frac{7100\gamma^2+22311\gamma+30249}{82944} r \\ & + z \left(\frac{556\gamma^2+1737\gamma+3069}{1728} r^5 - \frac{388\gamma^2+1161\gamma+1181}{576} r^3 \right. \\ & \left. + \frac{304\gamma^2+831\gamma+1242}{864} r \right) + z^2 \left(\frac{52\gamma^2+51\gamma+327}{192} r^3 \right. \\ & \left. - \frac{52\gamma^2+75\gamma+279}{192} r \right) - z^3 \left(\frac{7\gamma-3}{12} r \right) \end{aligned} \tag{5-f}$$

C. Kliegel and Levine's Approach

By noticing that the expansion parameter $(1/R)$ in Hall's transonic solution is introduced through the wall boundary condition that requires the flow angle to be equal to the local wall slope, it was concluded that this latter condition couldn't be exactly satisfied in cylindrical coordinates. Moreover, the radial wall velocity (v) being proportional to the boundary slope can become large in the vicinity of throat for nozzles having small values of R. On that basis, Kliegel and Levine [9] developed a solution in a system of toroïdal coordinates (η, ξ) in which both axis and wall are coordinate lines. In this system, the axis and nozzle wall are represented by the curves $\eta = 0$ and $\eta = \eta_w = \text{Constant}$ respectively, and the toroïdal coordinates are defined as:

$$\begin{cases} \dfrac{y}{y_t} = \sqrt{1+\dfrac{2}{R}} \dfrac{\sinh\eta}{(\cosh\eta + \cos\xi)} \\ \dfrac{x}{y_t} = \sqrt{1+\dfrac{2}{R}} \dfrac{\sin\xi}{(\cosh\eta + \cos\xi)} \end{cases} \tag{6}$$

where: $-\pi \le \xi \le +\pi$ and $-\infty \le \eta \le +\infty$

Using this toroïdal coordinate system and developing the solution in terms of $(1/R+1)$ instead of $(1/R)$, led to:

$$\begin{aligned} u = 1 + \frac{u_1(r,z)}{R+1} + \frac{1}{(R+1)^2} [u_1(r,z) + u_2(r,z)] \\ + \frac{1}{(R+1)^3} [u_1(r,z) + 2u_2(r,z) + u_3(r,z)] + \ldots \end{aligned} \tag{7-a}$$

$$\begin{aligned} v = \sqrt{\frac{\gamma+1}{2(R+1)}} \left\{ \frac{v_1(r,z)}{R+1} + \frac{1}{(R+1)^2} \left[\frac{3}{2} v_1(r,z) + v_2(r,z) \right] \right. \\ \left. + \frac{1}{(R+1)^3} \left[\frac{15}{8} v_1(r,z) + \frac{5}{2} v_2(r,z) + v_3(r,z) + \ldots \right] \right\} \end{aligned} \tag{7-b}$$

with:

$$\begin{cases} z = \sqrt{\dfrac{2R}{\gamma+1}}\dfrac{x}{y_t} = \sqrt{\dfrac{2R+4}{\gamma+1}}\dfrac{\sin\xi}{(\cosh\eta+\cos\xi)} \\ r = \dfrac{y}{y_t} = \sqrt{1+\dfrac{2}{R}}\dfrac{\sinh\eta}{(\cosh\eta+\cos\xi)} \end{cases} \quad (8)$$

In this case, the expansion parameter $[1/(R+1)]$ remains less than 1 for any value of the normalized wall radius of curvature R.

The relations describing the throat axis and wall velocities, and the nozzle discharge coefficient for the two above-mentioned methods are:

Hall:

$$\begin{cases} u_0 = 1 - \dfrac{1}{4R} + \dfrac{10\gamma+57}{288R^2} - \dfrac{2708\gamma^2+7839\gamma+14211}{82944R^3} \\ u_w = 1 + \dfrac{1}{4R} + \dfrac{14\gamma+15}{288R^2} - \dfrac{2364\gamma^2+4149\gamma+2241}{82944R^3} \\ C_D = 1 + \dfrac{\gamma+1}{R^2}\left[\dfrac{1}{96} - \dfrac{8\gamma+21}{2304R} + \dfrac{754\gamma^2+2123\gamma+2553}{276480R^2}\right] \end{cases}$$

Kliegel and Levine:

$$\begin{cases} u_0 = 1 - \dfrac{1}{4(R+1)} + \dfrac{10\gamma-115}{288(R+1)^2} - \dfrac{2708\gamma^2+2079\gamma+415}{82944(R+1)^3} \\ u_w = 1 + \dfrac{1}{4(R+1)} - \dfrac{14\gamma-57}{288(R+1)^2} - \dfrac{2364\gamma^2-3915\gamma+14377}{82944(R+1)^3} \\ C_D = 1 + \dfrac{\gamma+1}{(R+1)^2}\left[\dfrac{1}{96} - \dfrac{8\gamma+27}{2304(R+1)} + \dfrac{754\gamma^2-757\gamma+3633}{276480(R+1)^2}\right] \end{cases}$$

III. Numerical Implementation and Applications

The three approaches were included in different subroutines and integrated in the master logic program initially provided by Zucrow and Hoffman [10] using the method of characteristics thus enabling the solution of an appropriate set of difference equations. The modified Euler Predictor-corrector algorithm with iteration used to integrate the coupled set of ordinary differential equations describing the supersonic flow with constant γ is stable and yields a solution of second order accuracy.

The application of the transonic methods described above is carried out for two study cases represented by the propulsion nozzles equipping the first stages of research rocket engines.

The main input data used to perform the computations are summarized in Table II.

TABLE II
Input Data for the De-Laval Nozzles Computations

	P_t	T_t	P_a	Y_t	y_e	R_G	γ
Nozzle1	69	2800	1.013	0.069	0.223	320	1.20
Nozzle 2	54	2500	1.013	0.088	0.197	320	1.20

IV. Results and Discussions

The sonic lines are of great importance to the supersonic computations from which they start. The sonic lines obtained by application of the three methods are plotted in Figs. 1 and 2 for nozzles 1 and 2 respectively.

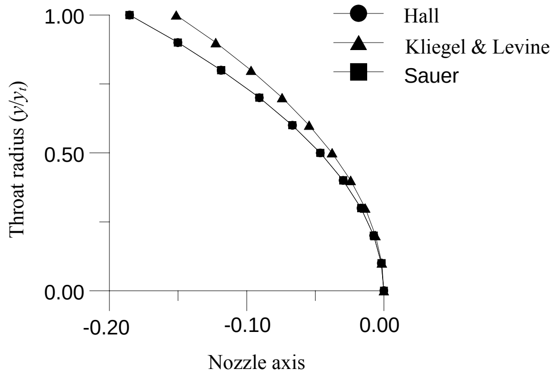

Fig. 1 Sonic lines for Nozzle 1

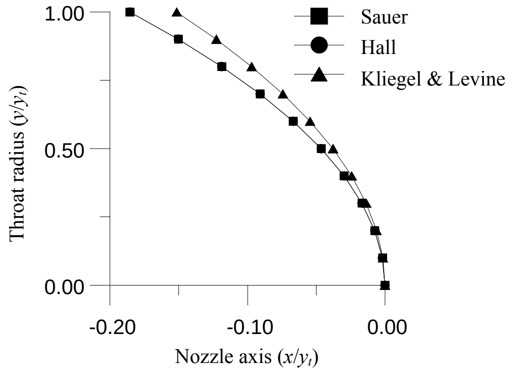

Fig. 2 Sonic lines for Nozzle 2

Unlike the one-dimensional theory where they are simulated as straight vertical lines, they are found here to be parabolic showing the two-dimensional aspect of the solution. They start exactly at the throat (x/y_t=0) and move backwards to intersect the nozzle wall. Sauer and Hall's sonic lines are almost the same because using the same system of coordinate. Due to the use of the inverse powers of the normalized throat radius of curvature added to unity i.e. $1/(R+1)$, Kliegel and Levine's curve is shifted towards the divergent section.

The parabolic aspect of the solution may also be shown by plotting the isobars within the throat region (Figs. 3 and 4). Hall's and Kliegel and Levine's solutions are compared and found to be very close. Sauer's solution does not provide a direct mean of determining the pressure distribution. These results are found to be compatible with those obtained by Back and Cuffel [11] who chose a value of R equal to 0.625.

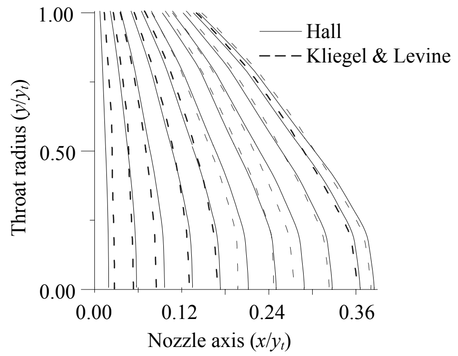

Fig. 3 Nozzle 1 throat region isobars

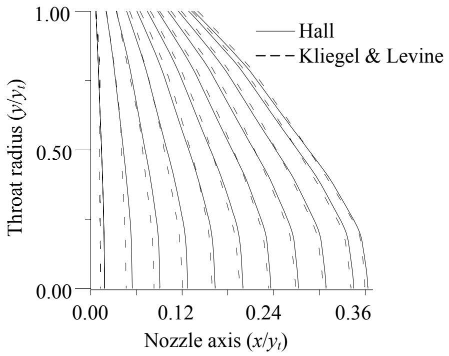

Fig. 4 Nozzle 2 throat region isobars

The influence of R is shown through the representation of the sonic lines for different values (the sonic lines corresponding to the values of R = 0.5, 0.8, 1.0, 1.5 and 2.0 are represented) of this parameter. Figs. 5 and 6 show the results obtained through the application of Sauer's approach for nozzles 1 and 2 respectively.

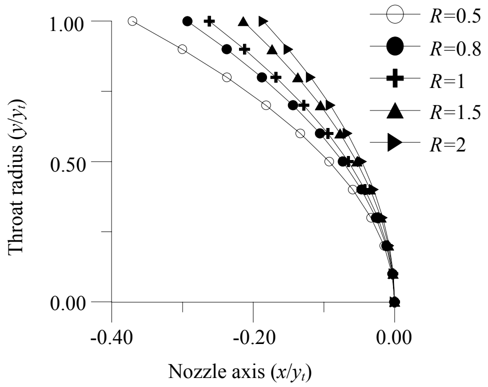

Fig. 5 Sauer's solution for Nozzle 1

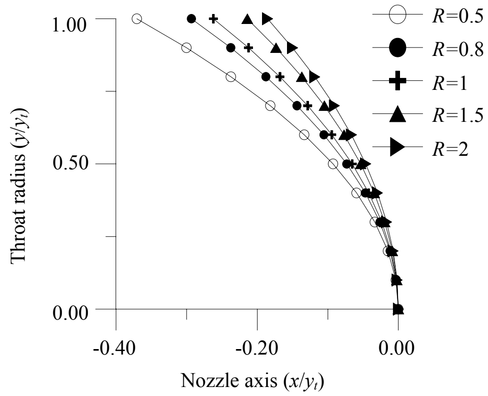

Fig. 6 Sauer's solution for Nozzle 2

It can be seen that the parabolic aspect of the sonic lines is preserved thought they are shown to be moving towards the convergent section with the values of R diminishing. Moreover, for $R \leq 0.5$, the parabolic aspect of the solution is lost leading to the divergence of the method. The same discussion may be carried out for the results brought from the application of the approach of Hall (Figs. 7 and 8).

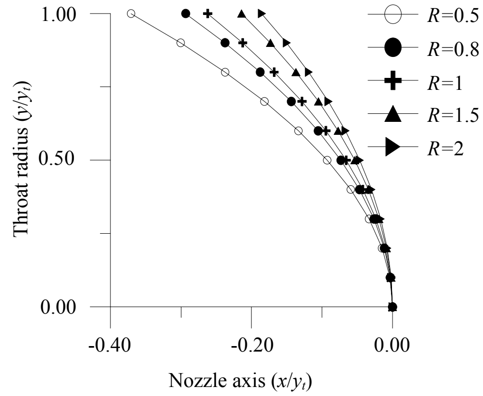

Fig. 7 Hall's solution for Nozzle 1

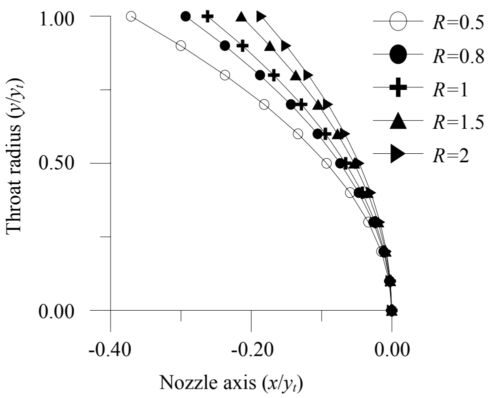

Fig. 8 Hall's solution for Nozzle 2

One of the main advantages of Kliegel and Levine's method lies on its ability to produce nice parabolic sonic lines representing the two-dimensional aspect of the transonic solution regardless of the value of R. This is mainly due to the fact that the normalized throat radius of curvature does not have as much influence on of Kliegel and Levine's approach as it has on Sauer's and Hall's methods.

Figs. 9 and 10 show the solution produced by Kliegel and Levine's method for values of R approaching zero.

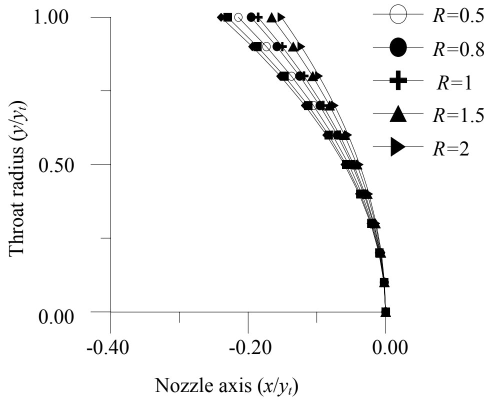

Fig. 9 Kliegel & Levine's solution for Nozzle 1

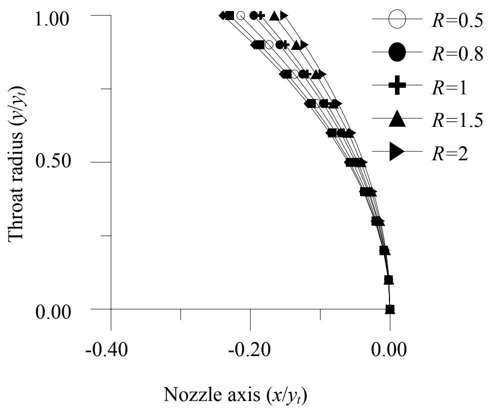

Fig. 10 Kliegel & Levine's solution for Nozzle 2

This important feature can be proven further by producing the distribution of the discharge coefficient (Table III). It may be seen that the difference in terms of the discharge coefficient (C_D) starts to be perceivable for values of the normalized wall radius of curvature less than unity ($R < 1$) between Sauer-Hall and Kliegel-Levine approaches. The results represented in Table III are plotted in Fig. 11. It can be seen that for $R \geq 1$, the curves representing the distribution of the discharge coefficient versus the normalized wall radius of curvature R are quite close to each other, whereas for $R \prec 1$ they start diverging.

TABLE III
DISTRIBUTION OF THE DISCHARGE COEFFICIENT AS FUNCTION OF R

R	C_D-Sauer	C_D-Hall	C_D-K & L
0.070	******	******	0.9021
0.072	******	******	0.9200
0.092	******	******	0.9520
0.138	******	******	0.9640
0.230	******	******	0.9730
0.345	******	******	0.9799
0.460	0.8300	0.8500	0.9844
0.541	0.9186	0.9101	0.9856
0.632	0.9554	0.9215	0.9886
0.690	0.9554	0.9413	0.9897
1.000	0.9781	0.9750	0.9933
1.500	0.9901	0.9908	0.9960
2.000	0.9944	0.9946	0.9973
2.500	0.9964	0.9963	0.9980

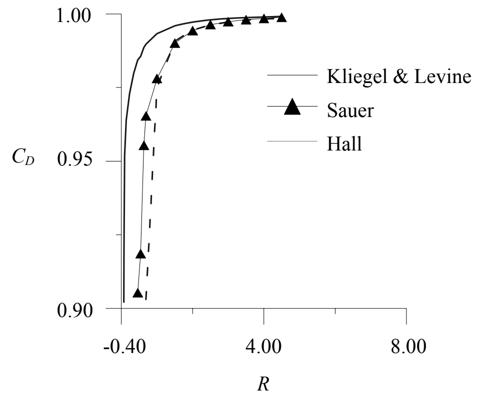

Fig. 11 Discharge coefficient C_D versus normalized wall radius of curvature R

V. CONCLUSION

Several methods may be used to describe the behavior of the flow field in the transonic region of a nozzle. The present study interests itself to the most used approaches developed by Sauer, Hall, and Kliegel and Levine. Developed and integrated into a master logic program, they are applied to two study cases allowing their testing for diverse radii of curvature.

The two-dimensional aspect of the solution is shown through the plotting of the sonic lines, which are found to be parabolic, thus moving away from the one-dimensional solution, which used to simulate them as straight vertical lines. The method of Kliegel-Levine shows a clear advantage over those of Sauer and Hall by the fact that it does converge regardless of the value of the normalized throat radius of curvature. It may thus be applied successfully to nozzles having small radii of curvature. The normalized throat radius of curvature parameter seems to be having a major influence over the flow field, which may appear in the shape and the position of the sonic line and therefore the variation of the discharge coefficient. Sonic and pressure (or Mach) lines may be used for starting the computations of the supersonic flow field using the method of characteristics.

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REFERENCES

[1] H. Kbab, "Développement et comparaison des méthodes de calcul des écoulements transsoniques", MSc dissertation, Department of aeronautics, Saad Dahlab University, Blida, 2001.

[2] Th. Von Karman, "The similarity law of transonic flow", *J. Math. And Phys*., 26, pp. 182-190, 1947.

[3] H. Yuan, and Y. He, "Transonic potential flows in a convergent divergent approximate nozzle", *J. Math. Anal. Appl.,* (2) 353, pp. 614-626, 2009.

[4] A.G. Kuz'min, "*Boundary-Value Problems for Transonic Flow*", John Wiley & sons, West Sussex, 2002.

[5] V.S. Oliveira Leith, P.A. Sovicro, and D. Bastos-Netto, "Numerical evaluation of the flow inside the transonic nozzle of a direct-connect supersonic combustion research facility", *18th ICME*, Nov. 6-11, 2005.

[6] C. Wenxiang, "Applying numerical solution to analyze the performance of nozzles", *ICEET'2009*, 16-18 Oct. 2009.

[7] A., Sauer, "General Characteristics of the Flow through Nozzles at Near Critical Speeds", *NACA TM-1147*, 1947.

[8] I. M. Hall, "Transonic Flow in Two-Dimensional and Axially–Symmetric Nozzles", *Quarterly. Journal of Mechanics and Applied Mechanics*, Vol. XV, Pt. 4, pp 487-508, 1962.

[9] J. R. Kliegel, and J. N. Levine, "Transonic Flow in Small Throat Radius of Curvature Nozzles", *AAIA journal*, Vol. 7, N° 7, pp. 1375-1378, 1969.

[10] M. J. Zucrow, and J. D. Hoffman, "*Gas Dynamics*", John Wiley & sons, 1976, vol. 2.

[11] L. H. Back, and R.F. Cuffel, "Flow Coefficients For Supersonic Nozzles With Comparatively Small Radius of Curvature Throats', *AAIA. Journal*, Vol. 8, N° 2, pp.196-198, 1970.

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