Pyralis Rocket Engine

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Spike Contour Algorithm

Purpose

This report describes and algorithm for designing the contour of the spike of an aerospike nozzle. The algorithm also predicts the thermodynamic properties of the working fluid along the spike, and the performance of the nozzle.

Model Summary

The algorithm analyzes the flow through the supersonic section of the aerospike nozzle. It examines a series of characteristics in the expansion fan which accelerates the flow. It finds the spike contour required to maintain an equal mass flow though each characteristic. The underlying thermodynamic model assumes that the ratio of specific heats γ and the molar mass of the working fluid are constant through the nozzle, and that the expansion is adiabatic and isentropic. These assumptions greatly simplify the calculations but introduce some error. The approach of the model is based on a paper written by C.C. Lee for NASA's Marshal Space Flight Center [1].

Relevant Requirements

List all system requirements which drive your design decisions here.

Algorithm Details

Definitions

The aerospike nozzle described here is cylindrically symmetric. The shroud is the outer solid boundary of the nozzle. The spike is the inner solid boundary of the nozzle. The throat is formed by the corner of the shroud (called the lip) and the spike. The exit plane of the nozzle contains the tip of the spike and is orthogonal to the centerline of the nozzle [Figure 2].

Supersonic Expansion Process in an Aerospike Nozzle

In an aerospike nozzle, the working fluid is expanded in the supersonic section of the nozzle by a Prandtl-Meyer expansion fan. The expansion fan emanates from the corner of the nozzle shroud and accelerated the flow from $M = 1$ at the throat to $M = M_e > 1$ at the exit plane [Figure 1].

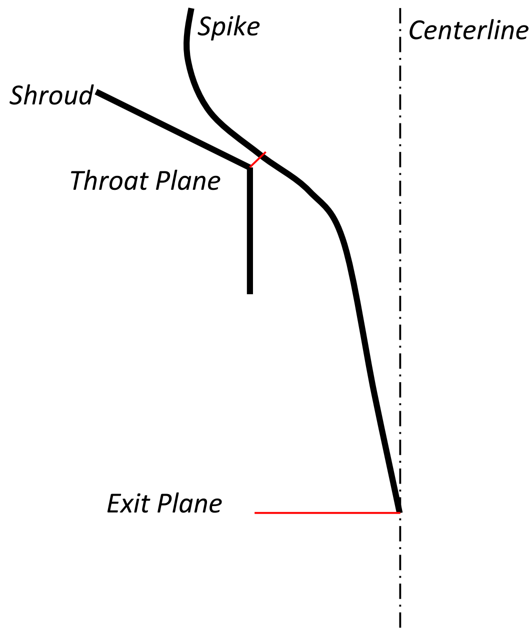

Figure 2: Important features in the aerospike nozzle

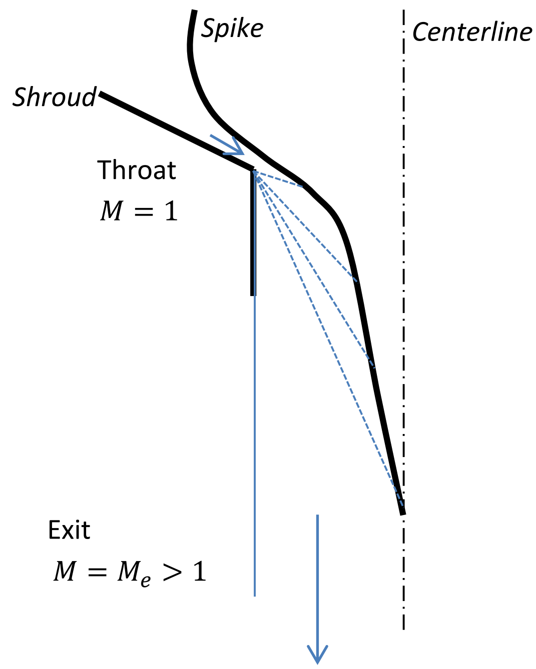

Figure 1: The expansion fan in the aerospike nozzle

Algorithm Inputs

The following parameters are inputs to the algorithm:

- p_c the combustion chamber pressure.
- T_c the combustion chamber temperature.
- p_a the ambient air pressure.
- $\frac{A_e}{A_t} = \epsilon$ the expansion ratio of the nozzle.
- N the number of points to examine when determining the contour of the spike.
- γ the ratio of specific heats in the working fluid.
- m_{molar} the molar mass of the working fluid.

Exit Mach Number, Exit Pressure, and Expansion Ratio

Given the area expansion ratio ϵ, the exit Mach number can be found via an inversion of Stodola's Aera-Mach Equation as describe in [2]. Once the exit Mach number is known, the exit pressure is found by the equation for isentropic expansion:

$$p_e = p_c \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{-\gamma}{\gamma-1}}$$

Alternatively, if the desired exit pressure is given as an input to the algorithm, the required area expansion ratio can be found by Equation 3-25 in [3]:

$$\epsilon = \frac{A_e}{A_t} = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} \left(\frac{p_e}{p_c}\right)^{\frac{1}{\gamma}} \sqrt{\left(\frac{\gamma+1}{\gamma-1}\right)\left(1-\left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right)}$$

Then the exit Mach number is the found by:

$$M_e = \sqrt{\frac{2}{\gamma-1}\left(\left(\frac{p_c}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right)}$$

Shroud Geometry

The shroud makes and angle δ with the horizontal. At the throat ($M = 1$), the velocity of the flow ($\vec{v}_t$) is along the shroud surface and also make an angle δ with the horizontal [Figure 3: Shroud].

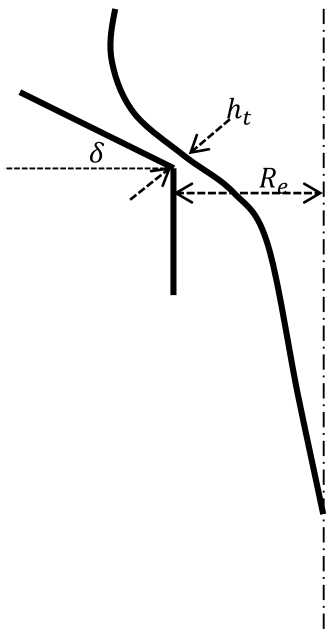

Figure 3: Shroud geometry

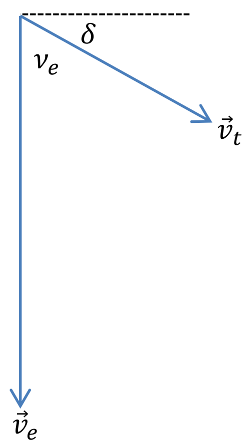

Figure 4: Turning angle through the expansion fan

After the expansion fan has accelerated the flow to $M = M_e$, we want the velocity of the flow ($\vec{v}_e$) to be parallel to the centerline of the nozzle. As the initial and final Mach numbers of the expansion fan are known $(1, M_e)$, the angle through which the flow will be turned by the expansion fan ν_e can be found by the Prandtl-Meyer relation:

$$\nu_e = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M_e^2-1)}\right) - \tan^{-1}\left(\sqrt{M_e^2-1}\right)$$

The sum of the expansion fan turning angle and the shroud slope angle must be a right angle in order for the flow to be parallel to the centerline after the expansion fan. Thus:

$$\nu_e + \delta = \frac{\pi}{2}$$

$$\delta = \nu_e - \frac{\pi}{2}$$

Now the ratio of the throat gap h_t to the shroud radius R_e can be found, using equation 12 from [1]:

$$\frac{h_t}{R_E} = \frac{\epsilon - \sqrt{\epsilon(\epsilon - \sin(\delta))}}{\epsilon \sin(\delta)}$$

Throat Conditions

As the throat Mach number is known to be 1, the fluid temperature, pressure, density and velocity can be found at the throat:

$$T_t = T_c \left(1 + \frac{\gamma - 1}{2}\right)^{-1}$$

$$p_t = p_c \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{-\gamma}{\gamma - 1}}$$

$$\rho_t = \frac{p_t}{R\, T_t}$$

$$v_t = \sqrt{\gamma\, R\, T_t}$$

Where $R = \frac{R_{universal}}{m_{molar}}$ is the specific gas constant of the fluid.

Spike Contour

A series of N Mach numbers in the set $[1, M_e)$ are examined. An expansion fan expands the flow through an infinite number of Mach lines, which meet at the corner of the shroud. Each Mach number M_x defines a particular Mach line which makes an angle $\phi_x = \nu_e - \nu_x + \mu_x$ with the centerline [Figure 5]. ν_x is the Prandtl-Meyer angle at the current Mach number:

$$\nu_x = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1}\left(\sqrt{\frac{\gamma - 1}{\gamma + 1}(M_x^2 - 1)}\right) - \tan^{-1}\left(\sqrt{M_x^2 - 1}\right)$$

ν_e is the Prandtl-Meyer angle at the exit Mach number:

$$\nu_e = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1}\left(\sqrt{\frac{\gamma - 1}{\gamma + 1}(M_e^2 - 1)}\right) - \tan^{-1}\left(\sqrt{M_e^2 - 1}\right)$$

And μ_x is the Mach angle at the current Mach number M_x. It is the angle between the flow velocity direction and the local Mach line when the flow has expanded to M_x.

$$\mu_x = \sin^{-1}\left(\frac{1}{M_x}\right)$$

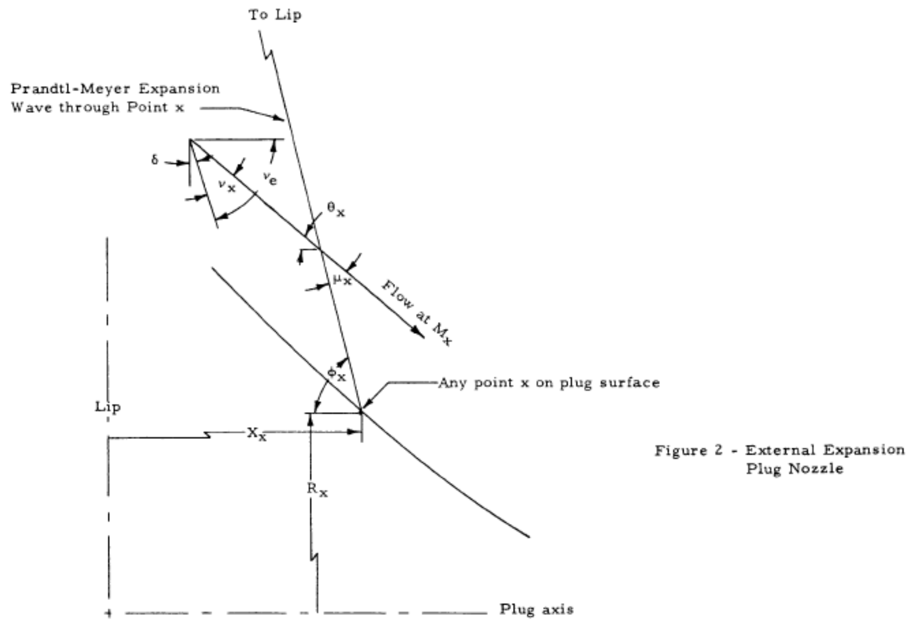

Figure 5: Using a characteristic to define a point on the surface of the spike contour. Figure taken from [1]. Note that the orientation of this figure is rotated 90° form the orientation of the other figures in this report.

The expansion fan is cylindrically symmetric about the centerline of the nozzle, so each Mach line in the expansion fan can be revolved about the centerline to form a frustum. The flow is traveling at Mach number M_x when it passes through this frustum, and the side of the frustum makes an angle ϕ_x with the centerline. The top of the frustum is set at the lip of the shroud and has a radius R_e. The base of the frustum is where the Mach line meets the contour of the spike, at a radius R_x. The height of the frustum is the axial distance from the shroud lip to the intersection of the Mach line and the spike contour, and is X_x. The side area of the frustum is [Equation 20 from [1]]:

$$A_x = \frac{\pi\,(R_e^2 - R_x^2)}{\sin(\phi_x)}$$

If we can find this A_x, we can solve for R_X, X_x and fix a point on the spike contour. Note that mass is conserved through the expansion fan. Therefore, during steady-state operation of the nozzle, the mass flow through the throat $\dot{m}_t$ must equal the mass flow though each Mach-line frustum, $\dot{m}_x$.

$$\dot{m}_t = \dot{m}_x$$

$$\rho_t v_t A_t = \rho_x v_x A_x$$

$$A_x = \frac{\dfrac{\rho_t}{\rho_c} A_t}{\dfrac{\rho_x}{\rho_c}\dfrac{v_x}{v_t}\sin(\mu_x)}$$

By equating the two equations for A_x, R_x can be found as a function of M_x [Equation 26 from [1], derivation in Appendix 1: Derivation of $\boldsymbol{R_x}$ $\boldsymbol{M_x}$ Relation]:

$$\frac{R_x}{R_e} = \sqrt{1 - \frac{\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_x^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\sin(\nu_e - \nu_x + \mu_x)}{\epsilon}}$$

Now that R_x is known, X_x can be found from [Equation 19 from [1]]:

$$X_x = \frac{R_e - R_x}{\tan(\phi_x)}$$

Using these equations, each Mach number defines a unique point on the contour of the spike. This point is where the Mach line associated with that Mach number must intersect the spike in order to have steady mass flow though the surface defined by that Mach line.

By examining a range of Mach numbers in the set $[1, M_e)$, a set of points on the contour of the spike is found. A spline is then fit through these points to define the spike contour.

It is computationally feasible to calculate enough points that the spike contour is defined to a precision greater than that of available fabrication tools. For example, with a small engine ($R_e = 15.0\text{mm}$) and a lathe with $0.0005\text{in} = 0.013\text{mm}$ precision, approximately 1200 points are required for the radial precision of the model to exceed the tolerance to which the physical part can be manufactured.

Thermodynamic Conditions along the Spike

This model can also be used to predict the thermodynamic conditions that will exist near the surface of the spike at an axial distance X_x. The fluid flow will be travelling at Mach number M_x. The fluid's static temperature and pressure can be found by:

$$p_x = p_c\left(1+\frac{\gamma-1}{2}M_x^2\right)^{\frac{-\gamma}{\gamma-1}}$$

$$T_x = T_c\left(1+\frac{\gamma-1}{2}M_x^2\right)^{-1}$$

Specific Impulse

This model can also be used to predict the specific impulse of the engine, given the thrust force produced by integrating the spike pressure over the spike area down to some axial distance X_x. This can give a crude estimate of the I_{sp} loss due to truncating the spike at to a length X_x. The thrust force F is found by:

$$F_x = \dot{m}_t v_t \sin(\delta) + (p_t - p_a)A_t \sin(\delta) + \int_0^x (p_{x'} - p_a)\mathrm{d}A$$

The first term is the momentum thrust at the throat, the second term is the pressure thrust at the throat, and the third term is the pressure thrust along the spike from the throat to point x. Now divide all terms by the mass flow to find the specific impulse:

$$I_{sp_x} g = v_t \sin(\delta) + \frac{(p_t - p_a) A_t \sin(\delta)}{\rho_t A_t v_t} + \int_0^x \frac{(p_{x'} - p_a)}{\rho_t A_t v_t} \mathrm{d}A$$

Using thermodynamic relations, this can be rewritten as [Equation 30 from [1], note that p_e should be p_c]:

$$I_{sp_x} g = v_t \sin(\delta) + \frac{v_t \sin(\delta)}{\gamma}\left[1 - \left(\frac{p_a}{p_c}\right)\left(\frac{p_c}{p_t}\right)\right] + \frac{v_t}{\gamma}\left(\frac{p_c}{p_t}\right)\int_0^x \frac{(p_{x'} - p_a)}{p_c} \mathrm{d}A$$

Rewriting the integral in finite difference form [Equation 31 from [1], note that p_e should be p_c]:

$$I_{sp_x} g = v_t \sin(\delta)\left[1 + \frac{1}{\gamma}\left(1 - \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{p_a}{p_c}\right)\right)\right] + \frac{v_t}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{\epsilon}{2}\right)\sum_{x=1}^{N}\left[\left(\frac{p_{(x-1)} - p_a}{p_c}\right) + \left(\frac{p_{(x)} - p_a}{p_c}\right)\right]\left[\left(\frac{R_{(x-1)}}{R_e}\right)^2 - \left(\frac{R_{(x)}}{R_e}\right)^2\right]$$

For several sample cases, the specific impulse predictions by this method were found to be within 1% of the predictions made by the *Rocket Propulsion Analysis* software tool. *RPA* was set to predict the specific impulse at matched ambient pressure with the chemical composition of the fluid freezing at the throat.

Appendix 1: Derivation of R_x M_x Relation

Equate the two equations for A_x:

$$\frac{\pi\,(R_e^2 - R_x^2)}{\sin(\phi_x)} = \frac{\frac{\rho_t}{\rho_c} A_t}{\frac{\rho_x}{\rho_c}\frac{v_x}{v_t}\sin(\mu_x)}$$

Divide both sides by A_e. Note that $A_e = \pi R_e^2$ and $\frac{A_t}{A_e} = \frac{1}{\epsilon}$

$$\frac{1}{\sin(\phi_x)}\left(1 - \frac{R_x^2}{R_e^2}\right) = \frac{\frac{\rho_t}{\rho_c}}{\frac{\rho_x}{\rho_c}\frac{v_x}{v_t}\sin(\mu_x)}\left(\frac{1}{\epsilon}\right)$$

$$\frac{R_x^2}{R_e^2} = 1 - \left(\frac{\frac{\rho_t}{\rho_c}}{\frac{\rho_x}{\rho_c}\frac{v_x}{v_t}}\right)\left(\frac{\sin(\phi_x)}{\sin(\mu_x)}\right)\left(\frac{1}{\epsilon}\right)$$

Rewrite the density and velocity term:

$$\frac{\rho_t}{\rho_c} = \left(1 + \frac{\gamma-1}{2}1^2\right)^{\frac{-1}{\gamma-1}} = \left(\frac{\gamma+1}{2}\right)^{\frac{-1}{\gamma-1}}$$

$$\frac{\rho_x}{\rho_c} = \left(1+\frac{\gamma-1}{2}M_x^2\right)^{\frac{-1}{\gamma-1}}$$

Note that velocity is Mach number times the speed of sound:

$$v = M\sqrt{\gamma RT} = M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{1}{2}}\sqrt{\gamma RT_c}$$

$$\frac{v_x}{v_t} = \frac{M_x\left(1+\frac{\gamma-1}{2}M_x^2\right)^{-\frac{1}{2}}}{1\left(1+\frac{\gamma-1}{2}1^2\right)^{-\frac{1}{2}}} = M_x\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_x^2\right)\right]^{-\frac{1}{2}}$$

Combine the density and velocity ratio expressions:

$$\left(\frac{\frac{\rho_t}{\rho_c}}{\frac{\rho_x}{\rho_c}\frac{v_x}{v_t}}\right) = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{-1}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}M_x^2\right)^{\frac{-1}{\gamma-1}}M_x\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_x^2\right)\right]^{-\frac{1}{2}}}$$

$$\left(\frac{\frac{\rho_t}{\rho_c}}{\frac{\rho_x}{\rho_c}\frac{v_x}{v_t}}\right) = \frac{1}{M_x}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_x^2\right)\right]^{\frac{1}{\gamma-1}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_x^2\right)\right]^{\frac{1}{2}}$$

Note that $\frac{1}{2}+\frac{1}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}$:

$$\left(\frac{\frac{\rho_t}{\rho_c}}{\frac{\rho_x}{\rho_c}\frac{v_x}{v_t}}\right) = \frac{1}{M_x}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_x^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Now rewrite the sine term:

Note that $\sin(\mu_x) = \frac{1}{M_x}$ by the definition of Mach angle. Also, $\phi_x = \nu_e - \nu_x + \mu_x$ by definition.

$$\left(\frac{\sin(\phi_x)}{\sin(\mu_x)}\right) = \frac{\sin(\nu_e-\nu_x+\mu_x)}{\frac{1}{M_x}}$$

Inserting the rewritten density, velocity and sine terms:

$$\frac{R_x^2}{R_e^2} = 1 - \frac{\frac{1}{\cancel{M_x}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_x^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\sin(\nu_e-\nu_x+\mu_x)}{\epsilon\frac{1}{\cancel{M_x}}}$$

$$\frac{R_x}{R_e} = \sqrt{1 - \frac{\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_x^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\sin(\nu_e-\nu_x+\mu_x)}{\epsilon}}$$

References

[1] C.C. Lee. "FORTRAN Programs for Plug Nozzle Design." Prepared for the Advanced Propulsion Section, Propulsion and Mechanics Branch, P&VE Division of the George C Marshall Space Flight Center by Brown Engineering Company, Inc. March 1963.

[2] J. Majdalani and B. A. Maickie. "Explicit Inversion of Stodola's Area-Mach Number Equation." *Transactions of the ASME*, Vol. 133, July 2011, pp. 071702-1 to 071702-7.

[3] G. Stutton. *Rocket Propulsion Elements Seventh Edition*, New York, NY: John Wiley & Sons, Inc., 2001.

Subsection 2

A table:

Table: Comparison of Action Actors

Attribute	Chuck Norris	Jet Li
Mass [kg]	100	80
Punching Force [N]	10^6	10^4
Cowboy Hats [count]	1	0
Coefficient of Drag [-]	1.2	0.8

Subsection 3

A figure: