热应力

弹性力学

热传导方程及其定解条件

变形和温度不耦合的热传导方程（变温影响变形，变形不产生变温）

$$\frac{\partial T}{\partial t}=\alpha\nabla^2 T+\frac{W}{c_p\rho}$$

其中，$T(x,y,z,t)$ 表示温度，$W(x,y,z,t)$ 为单位时间内每单位体积的热源的发热量，即热源强度；$\alpha=\frac{\lambda}{c_p\rho}$ 为热扩散率，c_p 为质量定压热容，ρ 为密度，λ 为导热系数，对于均匀材料，上述物理量皆为常数。

为了求解热传导方程，还需要初始条件和边界条件来构成定解问题。

- 初始条件：

$$T_{t=0}=f(x,y,z)$$

- 边界条件：

1. 已知边界处温度

$$(T)_s=\varphi(t)$$

2. 已知边界处法向热流密度：

$$(q_v)_s=\psi(t)$$

   q_v 为法向热流密度。由于热流密度在任一方向的分量，等于导热系数乘以温度在该方向的递减率，故上式又可以表示为

$$-\lambda\left(\frac{\partial T}{\partial v}\right)_s=\psi(t)$$

3. 对流换热边界条件

$$(q_v)_s=h(T_s-T_e)$$

T_s 表示弹性体边界处温度，T_e 表示周围介质的温度，根据上式也可以写成

$$-\lambda\left(\frac{\partial T}{\partial v}\right)_s=h(T_s-T_e)$$

4. 第四种边界条件（徐芝纶）

$$T_s = T_e$$

当物与之接触的另一物体以热传导方式进行热交换的情况。假定接触时完全的，即物体表面温度和接触体表面温度相同。

热应力以及弹性力学基本方程

在热应力问题，我们研究的是温度场 T 相对某一参考状态温度 T_0 的变化，即变温

$$\Delta T = T - T_0$$

我们参考温度场是均匀的，所以变温 ΔT 仍满足上述热传导方程。在后面的研究中，我们仍用 T 表述温度变化。

当弹性体发生温度变化时，弹性体体内各点的微小长度在不受约束的情况下发生正应变 αT，其中 α 为线膨胀系数，量纲为 Θ^{-1}，视为常数。

应变分量:

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha T, \quad \gamma_{yz} = \gamma_{xz} = \gamma_{xy} = 0$$

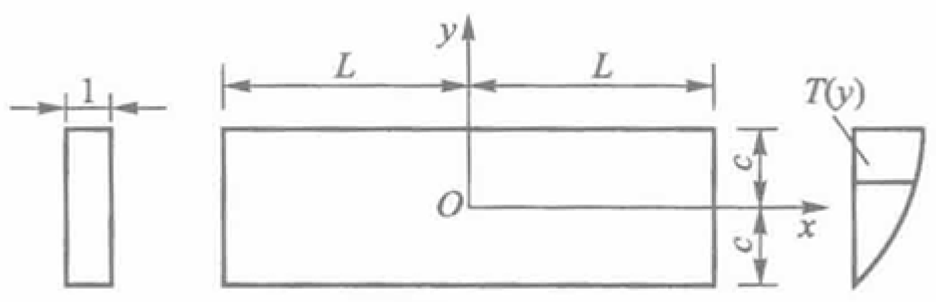

图 11-1

热弹性力学的基本方程中的平衡微分方程和几何方程同等温情况一样。

平衡微分方程

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0,$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0,$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0.$$

几何方程

$$
\begin{aligned}
&\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},\\
&\gamma_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z},\\
&\gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\\
&\gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}.
\end{aligned}
$$

本构方程:

$$
\begin{aligned}
\varepsilon_x&=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\ \ +\sigma_z)]+\alpha T,\\
\varepsilon_y&=\frac{1}{E}[\sigma_y\ -\nu(\sigma_x\ +\sigma_z)]+\alpha T,\\
\varepsilon_z&=\frac{1}{E}[\sigma_z\ -\nu(\sigma_x\ +\sigma_y)]+\alpha T,\\
\gamma_{yz}&=\frac{2(1+\nu)}{E}\tau_{yz},\\
\gamma_{xz}&=\frac{2(1+\nu)}{E}\tau_{xz},\\
\gamma_{xy}&=\frac{2(1+\nu)}{E}\tau_{xy}.
\end{aligned}
$$

用应变分量表示应力分量

$$
\begin{aligned}
\sigma_x&=\lambda\theta+2G\varepsilon_x-\frac{\alpha ET}{1-2\nu},\\
\sigma_y&=\lambda\theta+2G\varepsilon_y-\frac{\alpha ET}{1-2\nu},\\
\sigma_z&=\lambda\theta+2G\varepsilon_z-\frac{\alpha ET}{1-2\nu},\\
\tau_{yz}&=G\gamma_{yz},\\
\tau_{xz}&=G\gamma_{xz},\\
\tau_{xy}&=G\gamma_{xy},
\end{aligned}
$$

(其中 $\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$)

如果用位移分量表示

$$
\begin{aligned}
\sigma_x &= \lambda\theta + 2G\frac{\partial u}{\partial x} - \frac{\alpha E T}{1-2\nu}, \\
\sigma_y &= \lambda\theta + 2G\frac{\partial v}{\partial y} - \frac{\alpha E T}{1-2\nu}, \\
\sigma_z &= \lambda\theta + 2G\frac{\partial w}{\partial z} - \frac{\alpha E T}{1-2\nu}, \\
\tau_{yz} &= G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\
\tau_{xz} &= G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\
\tau_{xy} &= G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).
\end{aligned}
$$

如果我们考虑物体的几何形状及温度变化是轴对称的，则

平衡微分方程

$$
\begin{aligned}
\frac{\partial \sigma_\rho}{\partial \rho} + \frac{\partial \tau_{\rho z}}{\partial z} + \frac{\sigma_\rho - \sigma_\varphi}{\rho} = 0, \\
\frac{\partial \tau_{\rho z}}{\partial \rho} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{\rho z}}{\rho} = 0.
\end{aligned}
$$

几何方程

$$
\begin{aligned}
\varepsilon_\rho &= \frac{\partial u_\rho}{\partial \rho}, \\
\varepsilon_\varphi &= \frac{u_\rho}{\rho}, \\
\varepsilon_z &= \frac{\partial w}{\partial z}, \\
\gamma_{\rho z} &= \frac{\partial u_\rho}{\partial z} + \frac{\partial w}{\partial \rho}.
\end{aligned}
$$

本构方程

$$
\begin{aligned}
\sigma_\rho &= \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\theta + \varepsilon_\rho\right) - \frac{\alpha E T}{1-2\nu}, \\
\sigma_\varphi &= \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\theta + \varepsilon_\varphi\right) - \frac{\alpha E T}{1-2\nu}, \\
\sigma_z &= \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\theta + \varepsilon_z\right) - \frac{\alpha E T}{1-2\nu}, \\
\tau_{\rho z} &= \frac{E}{2(1+\nu)}\gamma_{\rho z}.
\end{aligned}
$$

如果我们考虑物体的几何形状及温度变化是球对称的，则

平衡微分方程

$$\frac{d\sigma_r}{dr}+\frac{2(\sigma_r-\sigma_t)}{r}=0$$

几何方程

$$\begin{aligned}\varepsilon_r&=\frac{du_r}{dr},\\ \varepsilon_t&=\frac{u_r}{r}.\end{aligned}$$

本构方程

$$\begin{aligned}\sigma_r&=\frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\theta+\varepsilon_r\right)-\frac{\alpha ET}{1-2\nu},\\ \sigma_t&=\frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\theta+\varepsilon_t\right)-\frac{\alpha ET}{1-2\nu}.\end{aligned}$$

例题

考虑一个等厚度的矩形薄板，将其视作平面问题，其变温为 y 的函数，而与 x 与 z 无关。由于变温仅依赖于坐标 y ，所以板内只存在 σ_x 。先考虑

$$\sigma_x'=-\alpha TE$$

也就是阻止变形，为了达到上述效果，我们在板两侧施加面力 $-\alpha TE$。但由于板的两端是自由的，所以还需要施加拉应力 αTE。因此，板内应力分布是由上述阻碍变形的面力和边界上施加的拉应力叠加而成，引入圣维南原理，边界条件可放松为主矢量和主矩的叠加：

主矢量：

$$\int_{-c}^{c}\alpha TE\,dy$$

在板内所引起应力

$$\sigma_x''=\frac{1}{2c}\int_{-c}^{c}\alpha TE\,dy$$

主矩：

$$M=\int_{-c}^{c}\alpha TEy\,dy$$

所产生的弯曲应力

$$\sigma_x''' = \frac{3y}{2c^3}\int_{-c}^{c} \alpha ETy\,dy$$

所以板内热应力为:

$$\sigma_x = -\alpha TE + \frac{1}{2c}\int_{-c}^{c} \alpha ET\,dy + \frac{3y}{2c^3}\int_{-c}^{c} \alpha ETy\,dy$$

其中 $T(y)$ 是关于 y 的函数，显然，倘若 T 为奇函数，第二项取零；倘若 T 为偶函数，第三项取零。

考察一个大的球体，设位于中心部位半径为 a 的小球内变温 T 为常数，其径向应变应为

$$\alpha T - \frac{p(1-2\nu)}{E}$$

第一项是由变温引起的膨胀，第二项是小球受外部约束，其表面受到压力 p 的作用（p 未知）

半径变化:

$$\Delta R = \alpha Ta - \frac{pa(1-2\nu)}{E}$$

考虑 p 对大球的影响，设大球半径为 b ，则应力为

$$\sigma_r = \frac{pa^3(b^3-r^3)}{r^3(a^3-b^3)}$$
$$\sigma_t = -\frac{pa^3(2r^3+b^3)}{2r^3(a^3-b^3)}$$

当 $b \gg a$ 时

$$\sigma_r = -\frac{pa^3}{r^3}$$
$$\sigma_t = \frac{pa^3}{2r^3}$$

在 $r=a$ 处考虑到 $\varepsilon_\theta = \varepsilon_\varphi = \varepsilon_t = \frac{u_r}{r}$，则

$$\Delta R = \frac{a}{E}[\sigma_t - \nu(\sigma_r+\sigma_t)]_{r=a} = \frac{pa}{2E}(1+\nu)$$

所以

$$p = \frac{2}{3}\frac{\alpha ET}{1-\nu}$$

位移解法和应力解法

以位移表示的平衡微分方程:

$$
\begin{aligned}
(\lambda+G)\frac{\partial\theta}{\partial x}+G\nabla^2 u-\frac{\alpha E}{1-2\nu}\frac{\partial T}{\partial x}=0,\\
(\lambda+G)\frac{\partial\theta}{\partial y}+G\nabla^2 v-\frac{\alpha E}{1-2\nu}\frac{\partial T}{\partial y}=0,\\
(\lambda+G)\frac{\partial\theta}{\partial z}+G\nabla^2 w-\frac{\alpha E}{1-2\nu}\frac{\partial T}{\partial z}=0.
\end{aligned}
$$

边界条件

$$
\begin{aligned}
\lambda\theta l+G\left(\frac{\partial u}{\partial x}l+\frac{\partial u}{\partial y}m+\frac{\partial u}{\partial z}n\right)+G\left(\frac{\partial u}{\partial x}l+\frac{\partial v}{\partial x}m+\frac{\partial w}{\partial x}n\right)=\frac{\alpha ET}{1-2\nu}l\\
\lambda\theta m+G\left(\frac{\partial v}{\partial x}l+\frac{\partial v}{\partial y}m+\frac{\partial v}{\partial z}n\right)+G\left(\frac{\partial u}{\partial y}l+\frac{\partial v}{\partial y}m+\frac{\partial w}{\partial y}n\right)=\frac{\alpha ET}{1-2\nu}m\\
\lambda\theta n+G\left(\frac{\partial w}{\partial x}l+\frac{\partial w}{\partial y}m+\frac{\partial w}{\partial z}n\right)+G\left(\frac{\partial u}{\partial z}l+\frac{\partial v}{\partial z}m+\frac{\partial w}{\partial z}n\right)=\frac{\alpha ET}{1-2\nu}n
\end{aligned}
$$

应力解法

$$
\begin{aligned}
\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=-\alpha E\left(\frac{1}{1-\nu}\nabla^2 T+\frac{1}{1+\nu}\frac{\partial^2 T}{\partial x^2}\right),\\
\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=-\alpha E\left(\frac{1}{1-\nu}\nabla^2 T+\frac{1}{1+\nu}\frac{\partial^2 T}{\partial y^2}\right),\\
\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=-\alpha E\left(\frac{1}{1-\nu}\nabla^2 T+\frac{1}{1+\nu}\frac{\partial^2 T}{\partial z^2}\right),\\
\nabla^2\tau_{yz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z}=-\frac{\alpha E}{1+\nu}\frac{\partial^2 T}{\partial y\partial z},\\
\nabla^2\tau_{xz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial z}=-\frac{\alpha E}{1+\nu}\frac{\partial^2 T}{\partial x\partial z},\\
\nabla^2\tau_{xy}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y}=-\frac{\alpha E}{1+\nu}\frac{\partial^2 T}{\partial x\partial y}.
\end{aligned}
$$

球体热应力问题

设圆球体内的变温 T 对球心是对称分布的，则此问题是球对称问题。以位移表示的平衡微分方程为

$$
\frac{d^2u_r}{dr^2}+\frac{2}{r}\frac{du_r}{dr}-\frac{2u_r}{r^2}=\frac{1+\nu}{1-\nu}\alpha\frac{dT}{dr}
$$

其解为

$$u_r = \frac{1+\nu}{1-\nu}\alpha\frac{1}{r^2}\int_a^r Tr^2\,dr + C_1 r + \frac{C_2}{r^2}$$

a 为内半径

对应的热应力分量:

$$\begin{aligned}
\sigma_r &= -\frac{2\alpha E}{1-\nu}\frac{1}{r^3}\int_a^r Tr^2\,dr + \frac{E}{1-2\nu}C_1 - \frac{2E}{1+\nu}\frac{C_2}{r^3} \\
\sigma_t &= \frac{\alpha E}{1-\nu}\frac{1}{r^3}\int_a^r Tr^2\,dr + \frac{E}{1-2\nu}C_1 + \frac{E}{1+\nu}\frac{C_2}{r^3} - \frac{\alpha ET}{1-\nu}
\end{aligned}$$

对于半径为 b 的实心球体，取积分下限为零，考虑正则条件 $(u_r)_{r=0}=0$，则可得 $C_2=0$
考虑边界条件 $(\sigma_r)_{r=b}=0$ 可得

$$C_1 = \frac{2(1-2\nu)\alpha}{1-\nu}\frac{1}{b^3}\int_0^b Tr^2\,dr$$

考虑空心圆球体，即边界条件满足

$$(\sigma_r)_{r=a}=0 \quad (\sigma_r)_{r=b}=0$$

利用边界条件可以解得

$$\begin{aligned}
\sigma_r &= \frac{2\alpha E}{1-\nu}\left(\frac{r^3-a^3}{(b^3-a^3)r^3}\int_a^b Tr^2\,dr - \frac{1}{r^3}\int_a^b Tr^2\,dr\right) \\
\sigma_t &= \frac{\alpha E}{1-\nu}\left(\frac{2r^3+a^3}{2(b^3-a^3)r^3}\int_a^b Tr^2\,dr + \frac{1}{2r^3}\int_a^b Tr^2\,dr - \frac{T}{2}\right)
\end{aligned}$$

热弹性应变势

我们知道这是一组非齐次偏微分方程组，他的解是由齐次通解和特解组成。引入函数 $\Phi(x,y,z)$

$$u' = \frac{\partial\Phi}{\partial x} \quad v' = \frac{\partial\Phi}{\partial y} \quad w' = \frac{\partial\Phi}{\partial z}$$

考虑

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad G = \frac{E}{2(1+\nu)}$$

有

$$
\begin{aligned}
\frac{\partial}{\partial x}\nabla^2\Phi &= \frac{1+\nu}{1-\nu}\alpha\frac{\partial T}{\partial x},\\
\frac{\partial}{\partial y}\nabla^2\Phi &= \frac{1+\nu}{1-\nu}\alpha\frac{\partial T}{\partial y},\\
\frac{\partial}{\partial z}\nabla^2\Phi &= \frac{1+\nu}{1-\nu}\alpha\frac{\partial T}{\partial z}.
\end{aligned}
$$

所以，Φ 满足微分方程

$$
\nabla^2\Phi = \frac{1+\nu}{1-\nu}\alpha T \tag{1}
$$

所以，特解所满足的应力分量可表示为

$$
\begin{aligned}
\sigma_x' &= -2G\left(\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right),\\
\sigma_y' &= -2G\left(\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2\Phi}{\partial x^2}\right),\\
\sigma_z' &= -2G\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right),\\
\tau_{yz}' &= 2G\frac{\partial^2\Phi}{\partial y\partial z},\\
\tau_{xz}' &= 2G\frac{\partial^2\Phi}{\partial x\partial z},\\
\tau_{xy}' &= 2G\frac{\partial^2\Phi}{\partial x\partial y}.
\end{aligned}
$$

而无变温对应的齐次通解为

$$
\begin{aligned}
\sigma_x'' &= 2G\left[\frac{\partial u''}{\partial x}+\frac{\nu}{1-2\nu}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\right],\\
\sigma_y'' &= 2G\left[\frac{\partial v''}{\partial y}+\frac{\nu}{1-2\nu}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\right],\\
\sigma_z'' &= 2G\left[\frac{\partial w''}{\partial z}+\frac{\nu}{1-2\nu}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\right],\\
\tau_{yz}'' &= G\left(\frac{\partial w''}{\partial y}+\frac{\partial v''}{\partial z}\right),\\
\tau_{xz}'' &= G\left(\frac{\partial u''}{\partial z}+\frac{\partial w''}{\partial x}\right),\\
\tau_{xy}'' &= G\left(\frac{\partial v''}{\partial x}+\frac{\partial u''}{\partial y}\right).
\end{aligned}
$$

将上述解叠加便可以解决这个问题

圆筒的轴对称热应力

设一个内半径为 a、外半径为 b 的长圆筒，其内的变温 T 是轴对称分布的，这是一个轴对称的平面应变问题

本构方程：

$$\begin{aligned}
&\sigma_\rho = \frac{E}{1+v}\left(\frac{v}{1-2v}\theta + \varepsilon_\rho\right) - \frac{\alpha E T}{1-2v} \\
&\sigma_\phi = \frac{E}{1+v}\left(\frac{v}{1-2v}\theta + \varepsilon_\phi\right) - \frac{\alpha E T}{1-2v} \\
&\sigma_z = \frac{Ev}{(1+v)(1-2v)}\theta - \frac{\alpha E T}{1-2v} \\
&\theta = \varepsilon_\rho + \varepsilon_\phi
\end{aligned}$$

位移表示的控制方程

$$\frac{d^2 u_\rho}{d\rho^2} + \frac{1}{\rho}\frac{du_\rho}{d\rho} - \frac{u_\rho}{\rho^2} = \frac{1+v}{1-v}\alpha\frac{dT}{d\rho}$$

对左式做整理可得 $\frac{d^2 u_\rho}{d\rho^2} + \frac{d}{d\rho}\left(\frac{u_\rho}{\rho}\right)$，积分可得 $\frac{du_\rho}{d\rho} + \frac{u_\rho}{\rho} = \frac{1+\nu}{1-\nu}\alpha T$(暂时不考虑常数项)

故取

$$u_\rho = \frac{d\Phi}{d\rho}$$

满足

$$\frac{d^2\Phi}{d\rho^2} + \frac{1}{\rho}\frac{d\Phi}{d\rho} = \frac{1+\nu}{1-\nu}\alpha T$$

这就证明了我们所选择的函数项满足轴对称的微分方程 (1)

考虑变温

$$T = \frac{T_a \ln\frac{b}{\rho}}{\ln\frac{b}{a}}$$

带入方程满足

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\Phi}{d\rho}\right) = \frac{1+\nu}{1-\nu}\frac{\alpha T_a}{\ln\frac{b}{a}}\ln\frac{b}{\rho}$$

方程的特解为

$$\Phi=\frac{K}{\ln\frac{b}{a}}\rho^2\left(\ln\frac{b}{\rho}+1\right),\quad K=\frac{1+\nu}{4(1-\nu)}\alpha T_a$$

其对应的应力分量满足

$$\begin{aligned}
\sigma_\rho'&=-\frac{2GK}{\ln\frac{b}{a}}\left(2\ln\frac{b}{\rho}+1\right)\\
\sigma_\varphi'&=-\frac{2GK}{\ln\frac{b}{a}}\left(2\ln\frac{b}{\rho}-1\right)
\end{aligned}$$

但是这个解并不满足圆筒内外壁面力为零的边界条件，他们在内外壁处给出面压力

$$\begin{aligned}
(\sigma_\rho')_{\rho=a}&=-2GK\left(2+\frac{1}{\ln\frac{b}{a}}\right)\equiv -q_1\\
(\sigma_\rho')_{\rho=b}&=-\frac{2GK}{\ln\frac{b}{a}}\equiv -q_2
\end{aligned}$$

其对应通解为圆筒内外壁受均匀拉力下的拉梅解

$$\begin{aligned}
\sigma_\rho''&=\frac{a^2b^2}{b^2-a^2}\frac{q_2-q_1}{\rho^2}+\frac{a^2q_1-b^2q_2}{b^2-a^2}\\
\sigma_\varphi''&=-\frac{a^2b^2}{b^2-a^2}\frac{q_2-q_1}{\rho^2}+\frac{a^2q_1-b^2q_2}{b^2-a^2}
\end{aligned}$$

将上述解叠加得所求热应力

平面热应力函数

平面应力状态下本构方程：

$$\begin{aligned}
\varepsilon_x&=\frac{1-\nu^2}{E}\left(\sigma_x-\frac{\nu}{1-\nu}\sigma_y\right)+(1+\nu)\alpha T,\\
\varepsilon_y&=\frac{1-\nu^2}{E}\left(\sigma_y-\frac{\nu}{1-\nu}\sigma_x\right)+(1+\nu)\alpha T,\\
\gamma_{xy}&=\frac{2(1+\nu)}{E}\tau_{xy}.
\end{aligned}$$

对于平面应变问题，则取

$$E_1=\frac{E}{1-\nu^2},\nu_1=\frac{\nu}{1-\nu},\alpha_1=(1+\nu)\alpha$$

同样引入艾里热应力函数

$$\nabla^2\nabla^2U=-\alpha E\nabla^2T$$

对于平面应变问题 $\alpha_1E_1=\frac{\alpha E}{1-\nu}$

对于极坐标形式，热应力函数与应力分量满足如下关系

$$
\begin{aligned}
\sigma_\rho &= \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} \\
\sigma_\varphi &= \frac{\partial^2 U}{\partial \rho^2} \\
\tau_{\rho\varphi} &= -\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial U}{\partial \varphi}\right)
\end{aligned}
$$

Laplace 算子满足

$$
\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}
$$

对于轴对称问题，双调和方程可简化成（此情况为平面应变）

$$
\frac{1}{\rho}\frac{d}{d\rho}\left\{\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dU}{d\rho}\right)\right]\right\} = -\frac{\alpha E}{1-\nu}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dT}{d\rho}\right)
$$

上式做四次积分可得

$$
\begin{aligned}
&\frac{1}{\rho}\frac{d}{d\rho}\left\{\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dU}{d\rho}\right)\right]\right\} = -\frac{\alpha E}{1-\nu}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dT}{d\rho}\right) \\
\overset{\text{积分}}{\Longrightarrow}\ &\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dU}{d\rho}\right)\right] = -\frac{\alpha E}{1-\nu}\frac{dT}{d\rho} + \frac{A}{\rho} \\
\overset{\text{积分}}{\Longrightarrow}\ &\frac{d}{d\rho}\left(\rho\frac{dU}{d\rho}\right) = -\frac{\alpha E}{1-\nu}\rho T + A\rho\ln\rho + B\rho \\
\overset{\text{积分}}{\Longrightarrow}\ &\frac{dU}{d\rho} = -\frac{\alpha E}{1-\nu}\frac{1}{\rho}\int T\rho d\rho + A\left[\frac{\rho}{2}\ln\rho - \frac{\rho}{4}\right] + \frac{B}{2}\rho + \frac{C}{\rho} \\
\underset{\text{舍弃常数项}}{\overset{\text{积分}}{\Longrightarrow}} &U = -\frac{\alpha E}{1-\nu}\int\frac{1}{\rho}\int T\rho d\rho + A\left[\frac{\rho^2}{4}\ln\rho - \frac{\rho^2}{4}\right] + B\frac{\rho^2}{4} + C\ln\rho
\end{aligned}
$$

可以证明此时 $\rho^2\ln\rho$ 项不满足位移的单值性条件，故舍弃。所以结果化简成

$$
U = -\frac{\alpha E}{1-\nu}\int\frac{1}{\rho}\left(\int T\rho\,\mathrm{d}\rho\right)\mathrm{d}\rho + A\ln\rho + B\rho^2
$$