Frequency Stability Degradation of an Oscillator Slaved to a Periodically Interrogated Atomic Resonator

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***Abstract*—Atomic frequency standards using trapped ions or cold atoms work intrinsically in a pulsed mode. Theoretically and experimentally, this mode of operation has been shown to lead to a degradation of the frequency stability due to the frequency noise of the interrogation oscillator.**

In this paper a physical analysis of this effect has been made by evaluating the response of a two-level atom to the interrogation oscillator phase noise in Ramsey and multi-Rabi interrogation schemes using a standard quantum mechanical approach. This response is then used to calculate the degradation of the frequency stability of a pulsed atomic frequency standard such as an atomic fountain or an ion trap standard. Comparison is made to an experimental evaluation of this effect in the LPTF Cs fountain frequency standard, showing excellent agreement.

I. Introduction

The development of new passive frequency standards using trapped ions or cold atoms has produced devices with a potential fractional frequency stability of the order of $10^{-13}\tau^{-1/2}$ or better. In these new types of standards, the internal interrogation process is discontinuous and periodic, and the control of the interrogation oscillator also is periodic. The frequency of this oscillator is compared to that of the atomic resonance during a part of duration T_i of the operating cycle only, and its frequency is controlled at the end of each cycle.

In the late 1980s, Dick [1], at the Jet Propulsion Laboratory, derived the atomic response to the oscillator frequency fluctuations using a geometrical approach. Furthermore, it was shown that the oscillator frequency noise at Fourier frequencies, which are close to multiples of $1/T_c$, is down-converted, leading to a degradation of the frequency stability.

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In this paper, we evaluate the atomic response to a variation of the frequency of the interrogation oscillator for different interrogation schemes. This response is used in two companion papers to derive the equation for the frequency stability limitation of the standard [2], [3]. The resulting model is experimentally verified using an atomic fountain standard [4].

II. Time Dependence of the Sensitivity to Oscillator Frequency Fluctuations

Let δP be the change of the probability that a transition occurred at the outcome of the atomic interaction with the noisy microwave field of the interrogation oscillator. This change is related to a fluctuation, $\delta\omega(t)$, of the frequency of the interrogating oscillator during the interaction in the following way:

$$\delta P = \frac{1}{2}\int_{\text{int.}} g(t)\delta\omega(t)dt. \tag{1}$$

This equation defines $g(t)$, the sensitivity function to frequency fluctuations of the interrogating field $\delta\omega(t)$. This equation assumes that all the atoms are subjected to the same phase perturbation. The interrogation occurs during an interaction time T_i. The physical meaning of this function can be obtained by calculating the effect of an infinitesimally small phase step $\Delta\phi$ at time t in the oscillator signal, which can be expressed as a frequency variation $\delta\omega(t) = \Delta\phi\delta(t-t')$. This produces a change $\delta P(t, \Delta\phi)$ in the probability that a transition occurred, and $g(t)$ is given by:

$$g(t) = 2\lim_{\Delta\phi\to 0}\delta P(t, \Delta\phi)/\Delta\phi. \tag{2}$$

In control system terminology, $g(t)$ is the response of the atomic system to a phase step of the interrogation oscillator, or the impulse response with respect to a frequency change occurring at time t.

A. A Simple Example of $g(t)$

We consider a Ramsey interrogation scheme where the atoms have a resonance frequency ω_0 and experience successively two microwave fields of frequency ω and Rabi frequency b, each for a duration τ_p, separated by a time T.

Assuming $T \gg \tau_p$ and $\Omega_0 = (\omega - \omega_0) \ll b$, the probability that the atomic transition occurred can be written as:

$$P \cong \frac{1}{2} \sin^2 b\tau_p [1 + \cos(\Omega_0 T + \Delta\phi)], \tag{3}$$

where $\Delta\phi$ is a phase step that can take place at any time between the two microwave interactions.

It is worth noting that the interrogating field is stepwise frequency modulated with frequency $1/2T_c$, where T_c is the time for one cycle of operation, to generate the servo error signal required to lock the interrogation oscillator on the atomic transition. Its (angular) frequency is $\omega_0 + \Delta\omega + \omega_m$ or $\omega_0 + \Delta\omega - \omega_m$ according to the half period of modulation considered, where $\Delta\omega \ll b$ is the difference between the oscillator frequency and the resonance frequency ω_0 and ω_m is the modulation depth. If we consider a small phase step $\Delta\phi \ll 1$, applying (3) to the generic k-th cycle, we obtain:

$$P_k \cong \frac{1}{2} \sin^2 b\tau_p [1 + (\cos((\Delta\omega + (-1)^k \omega_m)T) - \sin((\Delta\omega - (-1)^k \omega_m)T)\Delta\phi)]. \tag{4}$$

When the interrogation oscillator is locked to the resonance, $\Delta\omega \approx 0$. Then the variation of the signal due to $\Delta\phi$ becomes:

$$\delta P_k \cong \frac{(-1)^k}{2} \sin^2 b\tau_p \sin(\omega_m T)\Delta\phi \tag{5}$$

and using the relation (2) we obtain

$$g(t) = \begin{cases} (-1)^k \sin^2 b\tau_p \sin\omega_m T & 0 \le t \le T, \\ 0 & T \le t \le T_c. \end{cases} \tag{6}$$

This function is periodic, with period $2T_c$. As explained later and in two companion papers [2], [3], a meaningful quantity in this process is $g(t)/g_0$ where g_0 is the mean value of $g(t)$ over the cycle time T_c:

$$g_0 = \frac{1}{T_c} \int_0^{T_c} g(t)dt \approx (-1)^k \frac{T}{T_c} \sin^2 b\tau_p \sin\omega_m T. \tag{7}$$

We thus have:

$$g(t)/g_0 = \begin{cases} \frac{T_c}{T} & 0 \le t \le T, \\ 0 & T \le t \le T_c. \end{cases} \tag{8}$$

The ratio $g(t)/g_0$ is thus periodic with period T_c; and, under the validity of the previous assumption, its value is independent of experimental parameters such as the Rabi frequency and the modulation depth. It depends only on the ratio between the cycle time T_c and the interrogation time T. In a modern atomic frequency standard, such as an ion trap or an atomic fountain, the cycle duration is the sum of an unavoidable dead time plus the interrogation time; therefore the function $g(t)$ it not a constant during each cycle. As explained in the following and in companion papers [2], [3], this causes degradation of the frequency stability of the locked oscillator. Clearly, this very simple model neglects the response of the atoms to the field during the microwave interactions of duration τ_p.

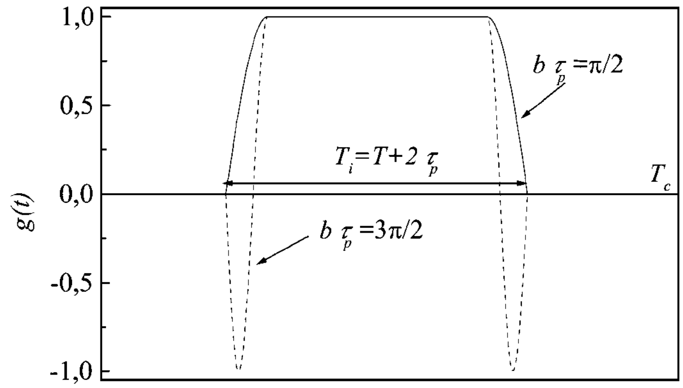

Fig. 1. The function $g(t)$, assumed centered in the cycle period, for the case of a Ramsey interrogation scheme with $b\tau_p = \pi/2$ (solid line) and $b\tau_p = 3\pi/2$ (dashed line), with $T = 0.5$ s, $T_c = 1$ s, $\tau_p = 15$ms, and $\Omega_0 = -\omega_m$.

B. The Sensitivity Function for the Ramsey Interrogation

A more general approach to the calculation of $g(t)$ can be performed using the density matrix formalism for a two-level atom (see the Appendix). The function $g(t)$ can be calculated analytically for the Rabi or Ramsey interrogation scheme, if the Rabi frequency b is constant during the microwave pulses. We limit ourselves to the Ramsey case, which is commonly used. Under the conditions $T \gg \tau_p$ and $\Omega_0 \ll b$ we have:

$$g(t) = \begin{cases} a \sin bt & 0 \le t \le \tau_p \\ a \sin b\tau_p & \tau_p \le t \le T + \tau_p \\ a \sin b(T + 2\tau_p - t) & T + \tau_p \le t \le T + 2\tau_p \\ 0 & T + 2\tau_p \le t \le T_c \end{cases} \tag{9}$$

where $a = -\sin\Omega_0 T \sin b\tau$ and $\Omega_0 = \pm\omega_m$ according to the half period of modulation considered. Fig. 1 shows the variation of $g(t)$ for $b\tau_p = \pi/2$ and $b\tau_p = 3\pi/2$. Unlike the simple model of (6) the shape of $g(t)$ is strongly dependent on the microwave power applied during the microwave pulses.

Another case that is relevant is the atom-field interaction in a multi-λ cylindrical cavity, resonating in the TE_{01n} mode [5], which is used in the PHARAO prototype [6]. In this device, balls of cold cesium atoms will be launched along the axis of a cavity exited in such a mode. During their interrogation, the atoms experience a microwave field whose amplitude is proportional to $\sin(n\pi t/T_i)$, where T_i is the total interaction time. In this case, $g(t)$ must be calculated numerically (see the Appendix). Fig. 2 shows the variation of $g(t)$ for $n = 3$. Here, the operating parameters are chosen to provide the maximum slope of the resonance curve. This is achieved for $b_c T_i/n = 3.66$ and $\omega_m T_i = 2.31$, where b_c is the Rabi frequency at an anti-node of the microwave field.

It is clear that the shape of $g(t)$ depends on the type of interrogation scheme and on the details of the interaction such as field power and frequency detuning.

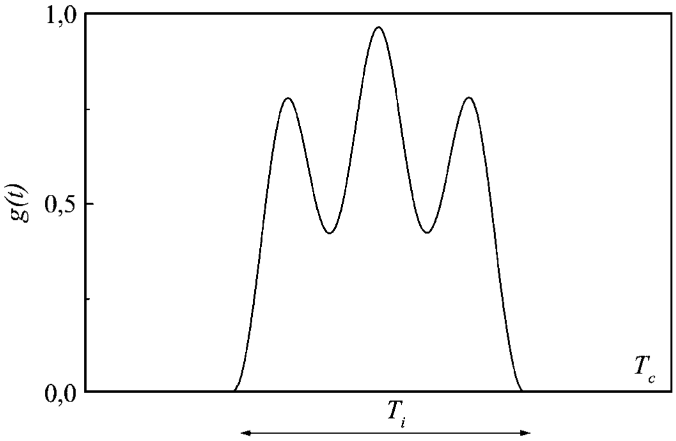

Fig. 2. The function $g(t)$, assumed centered in the cycle period, for the case of a multi-λ interrogation scheme in a TE_{013} cavity, with $T_i = 0.53$ s, $T_c = 1$ s, $b_c T_i/3 = 3.66$, $\omega_m T_i = 2.31$, and $\Omega_0 = -\omega_m$.

III. Limitation of the Frequency Stability Due To Sampling

The control loop being closed, frequency corrections are applied to the interrogation oscillator at discrete times t_k, at the end of each cycle. It is possible to show that the spectral components of the interrogation oscillator phase noise around frequencies m/T_c are translated to frequencies below $1/T_c$ [2], [3]. This spectrum folding is at the origin of the frequency stability degradation of the atomic frequency standard. For very low Fourier frequencies, the down-converted noise spectrum can be assumed white.

The Allan variance of the locked interrogation oscillator is related to the frequency noise spectral density of the free running oscillator and to the harmonic content of the function $g(t)$ [2], [3] in the following way:

$$\sigma^2_{y\,\mathrm{lim}}(\tau) = \frac{1}{\tau}\sum_{m=1}^{\infty}\left(\frac{g_m^{c2}}{g_0^2} + \frac{g_m^{s2}}{g_0^2}\right) S_y^f(m/T_c) \tag{10}$$

where $\sigma^2_{y\,\mathrm{lim}}(\tau)$ is a lower limit to the achievable stability. Here $S_y^f(m/T_c)$ is the one-sided power spectral density of the relative frequency fluctuations of the free running interrogation oscillator at Fourier frequencies m/T_c, and the parameters g_0, g_m^s, and g_m^c are defined by:

$$\begin{aligned}\begin{pmatrix} g_m^s \\ g_m^c \end{pmatrix} &= \frac{1}{T_c}\int_0^{T_c} g(\xi)\begin{pmatrix}\sin 2\pi m\xi/T_c \\ \cos 2\pi m\xi/T_c\end{pmatrix} d\xi, \\ g_0 &= \frac{1}{T_c}\int_0^{T_c} g(\xi) d\xi.\end{aligned} \tag{11}$$

It is possible without loss of generality to simplify (10) by applying a time translation to the function $g(t)$ in order to obtain a cosine series. Fig. 3 shows the coefficients $(g_m/g_0)^2 = ((g_m^{s2}+g_m^{c2})/g_0^2)$ versus the rank m for the function $g(t)$ in the Ramsey interrogation scheme for the two cases $b\tau_p = \pi/2$, which provides the optimal interrogation condition, and $b\tau_p = 3\pi/2$, which is used in the fountain frequency standard to evaluate the power dependent shifts. It is assumed that $\tau_p = 0.015$s, $T = 0.5$s, and $T_c = 1$s. Fig. 4 shows the coefficients $(g_m/g_0)^2$ versus the rank m for the function $g(t)$ assuming interrogation in a multi-λ TE_{013} cavity, with $b_c T_i/3 = 3.66$ and $\omega_m T_i = 2.31$. It is worth noting that, for m larger than about 10, $(g_m/g_0)^2$ decreases as m^{-6} in this case. This property provides very good immunity against the white phase noise of the oscillator.

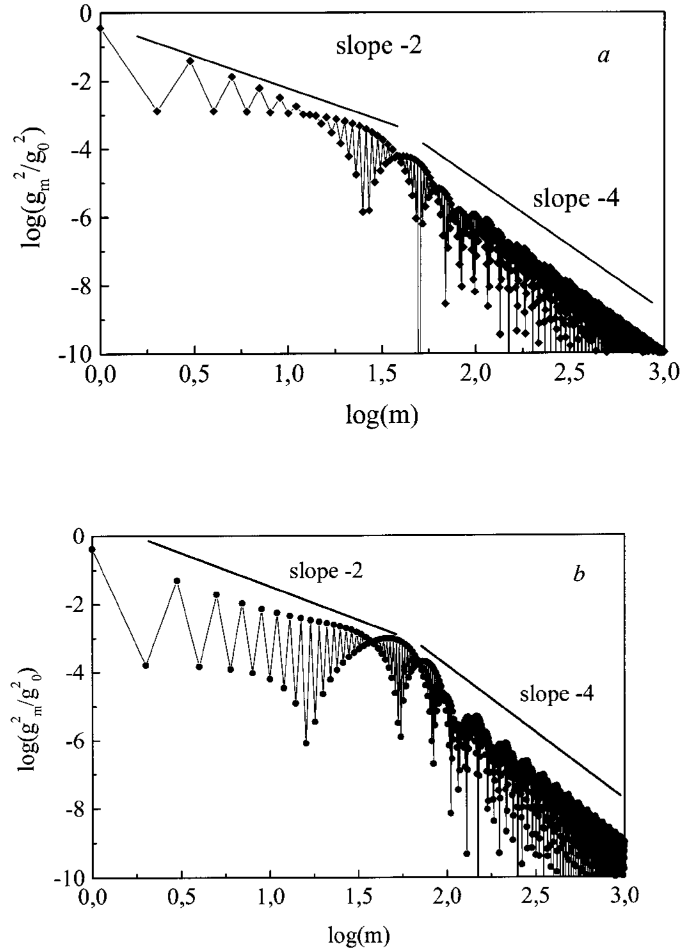

Fig. 3. Calculated spectrum of the function $g(t)/g_0$ for the case of Ramsey interrogation: (a) $b\tau_p = \pi/2$ (b) $b\tau_p = 3\pi/2$, with $T = 0.5$ s, $T_c = 1$ s, and $\tau_p = 15$ ms.

IV. Experimental Evaluation of the Frequency Stability Degradation

In order to verify the model and provide evidence of the down-conversion effect, we have made various measurements with the interrogation oscillator used with the LPTFs Cs atomic fountain. We have purposely degraded its spectrum with different types of frequency noise. Fig. 5 is a schematic of the experimental set-up. We have used three different sources of noise: a white noise signal in the range of 0.1 Hz to 1600 Hz (f^0) with which various low-

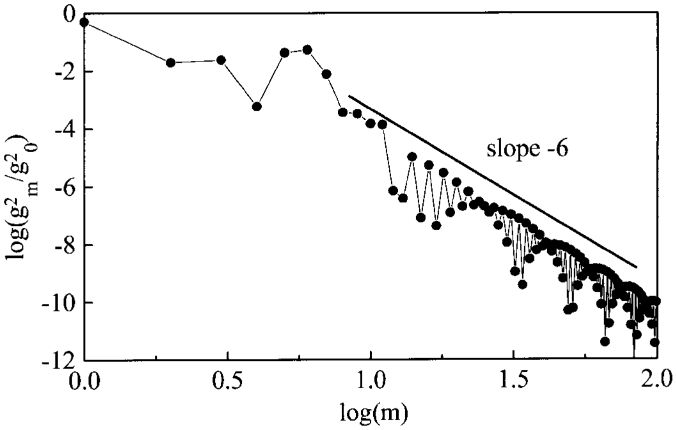

Fig. 4. Calculated spectrum of the function $g(t)/g_0$ for the case of multi-λ interrogation in a TE_{013} cavity, with $T_i = 0.53$ s, $T_c = 1$ s, $b_cT_i/3 = 3.66$, and $\omega_mT_i = 2.31$.

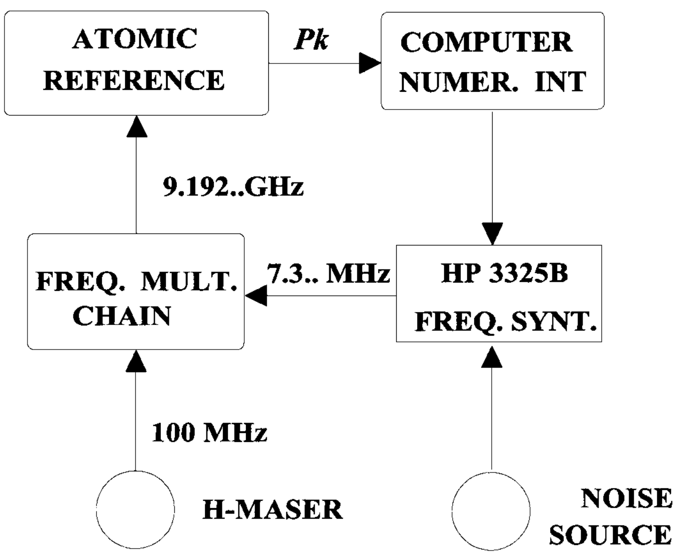

Fig. 5. Schematic of the atomic fountain frequency lock loop.

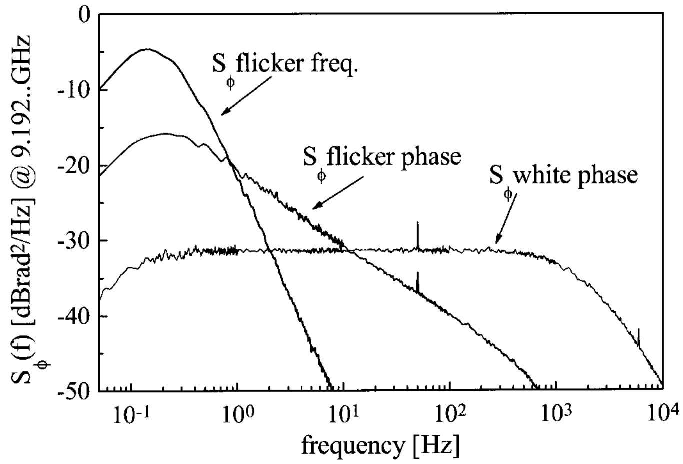

Fig. 6. Measured phase noise spectral density of the degraded interrogation oscillator for the three types of noise used in the experiment.

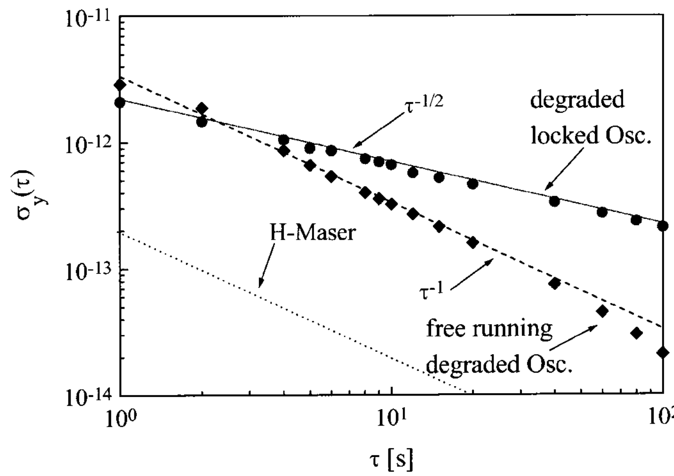

Fig. 7. Fractional frequency stability of the free running and of the locked interrogation oscillator.

pass filters can be used; a flicker noise generator (f^{-1}) in the range of 0.5 to 100 Hz; and a generator with spectral density proportional to f^{-3}, for Fourier frequencies from 0.5 to 100 Hz. Fig. 6 shows the corresponding phase noise power spectral densities. We use the noise generators to drive the phase modulation input of an offset synthesizer. The phase noise added to the synthesizer is transferred to the interrogation oscillator spectrum at 9.192 GHz.

It is worth noting that, in our measurement set-up, the interrogating oscillator is obtained by phase-locking a frequency multiplication chain to the reference H-maser, as shown in Fig. 5. The frequency-locking of the interrogation oscillator to the atomic resonance is obtained by controlling the central frequency of the offset synthesizer used to generate the difference between the 92nd harmonic of the 100 MHz H-maser signal and the hyperfine frequency of the cesium atom.

The frequency stability of the atomic fountain measured against the hydrogen maser is then obtained by calculating the Allan standard deviation on the frequency corrections applied to the offset synthesizer. As a consequence, the interrogating oscillator has to be degraded only for frequencies larger than or equal to the cycle frequency. For low frequencies (i.e., for times long compared to the cycle period), the noise sources are high pass filtered. This avoids the long-term degradation and is particularly effective for the flicker frequency noise which gives a flat Allan variance. As shown in Fig. 7, the fractional frequency stability of the free running oscillator behaves as τ^{-1}, whereas the stability of the locked oscillator is proportional to $\tau^{-1/2}$. This clearly shows that the frequency stability of the locked oscillator is dominated by the aliasing noise for integration times longer than 10 to 20s. We measured the stability for two conditions: $b\tau_p = \pi/2$ and $b\tau_p = 3\pi/2$. Tables I and II report the calculated values using (10) and the measured data for the flicker phase and flicker frequency noises. For these colored noises, (10) shows that the frequency stability is mainly limited by the first term of the series, which depends only on the ratio between the interrogation time T_i and the cycle time T_c. In order to verify precisely the model, we need a measurement that is more sensitive to

TABLE I
$b\tau_p = \pi/2$

Type of noise	σ_y meas.(1s)	σ_y calc.(1s)
Frequency flicker	$2.4\ 10^{-12}$	$2.3\ 10^{-12}$
Phase flicker	$3.0\ 10^{-12}$	$2.9\ 10^{-12}$

TABLE II
$b\tau_p = 3\pi/2$

Type of noise	σ_y meas.(1s)	σ_y calc.(1s)
Frequency flicker	$2.8\ 10^{-12}$	$2.4\ 10^{-12}$
Phase flicker	$4.8\ 10^{-12}$	$4.6\ 10^{-12}$

the shape of $g(t)$. In the case of white phase noise, the frequency noise spectrum behaves as f^2 and consequently the down-conversion is strongly dependent on the harmonic content of $g(t)/g_0$. Changing the white phase noise spectrum by low-pass filtering the noise source with different cut-off frequencies improves the sensitivity of the measurements. Fig. 8 reports the calculated and measured results, which agree within the limits of the measurement errors, estimated to be about 20%. Measurements performed with the filtered white phase noise confirm the validity of the model, even for different interrogation oscillator levels, i.e., for $b\tau_p = \pi/2$ and $3\pi/2$. It is interesting to note that, for large filter cut-off frequency, the degradation ratio is equal to 3. For the colored noises the degradation does not depend on the microwave power.

V. Discussion

As shown before, the degradation depends on the harmonic content of $g(t)$. A constant value of the sensitivity function over the cycle would eliminate this effect. Unfortunately, the operation of the fountain frequency standard requires unavoidable dead times.

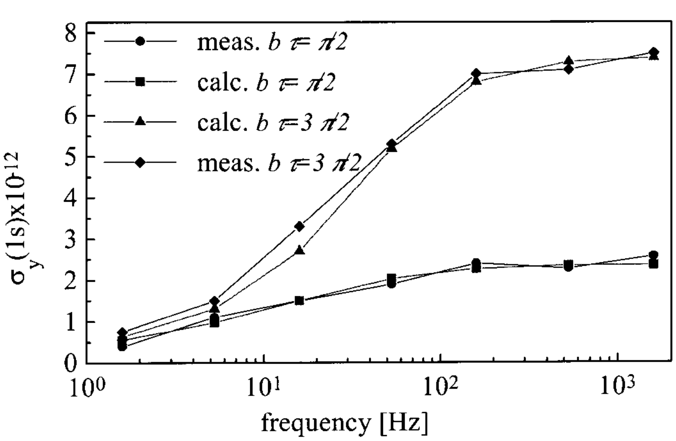

Fig. 8. Measured and calculated frequency stability versus the cut-off frequency of the white phase noise for $b\tau_p = \pi/2$ (circles, squares) and $b\tau_p = 3\pi/2$ (diamonds, triangles).

The level of the effect would obviously be reduced with an oscillator, showing a much improved spectral purity, such as a cryogenic sapphire oscillator [7], [8]. A reduction of this "intermodulation" noise has already been obtained in a Rb cell standard [9] and in a thermal Cs beams clock [10] by filtering at $1/T_c$ the frequency noise of the interrogation oscillator. In a cesium fountain, in the PHARAO set-up or in a trapped ion frequency standard, it will not be feasible to use notch filters to reject the oscillator noise at such low Fourier frequencies ($\leq$1 Hz).

If we consider the state of the art of quartz oscillators, the phase noise below a few Hertz is mainly limited by flicker frequency noise. In this case it is easy to obtain an approximate relationship between the flicker floor of the interrogation oscillator σ_y^{LO} and the frequency stability of the standard $\sigma_{y\,\mathrm{lim}}(\tau)$ versus the duty cycle, defined as $d = T/T_c$. In the case of the Ramsey interrogation with $T \gg \tau_p$ and $0.4 < d < 0.7$ we have:

$$\begin{aligned}\sigma_{y\,\mathrm{lim}}(\tau) &\cong \frac{\sigma_y^{LO}}{\sqrt{2\ln(2)}}\left|\frac{g_1}{g_0}\right|\sqrt{\frac{T_c}{\tau}} \\ &= \frac{\sigma_y^{LO}}{\sqrt{2\ln(2)}}\left|\frac{\sin(\pi d)}{\pi d}\right|\sqrt{\frac{T_c}{\tau}}.\end{aligned} \tag{12}$$

Apparently the only way to reduce this detrimental effect is to increase the duty cycle. For a given duty cycle, the degradation is proportional to $\sqrt{T_c}$. This result is most significant for trapped ions standards and for the PHARAO clock, where $T_c \sim$ 3–10s.

To illustrate the beneficial effect of the increase of the interrogation duty cycle d, or of the release of several balls during each cycle in the case of the PHARAO clock in space, we have made some numerical calculations, in which the quartz oscillator is assumed to show a frequency noise spectral density given by:

$$S_y^f(f) = 3.2\ 10^{-29} f^2 + 1.0\ 10^{-27} f + 3.2\ 10^{-26}/f. \tag{13}$$

In the case of the Ramsey method of interrogation, the total interrogation time is $T_i = T + 2\tau_p$. Fig. 9 shows the variation of $\sigma_{y\,\mathrm{lim}}$ versus T_i/T_c for three sets of parameter values, and Fig. 10 shows the variation of $\sigma_{y\,\mathrm{lim}}$ versus the number of balls. It is assumed that the interrogation occurs in a TE_{013} cavity and that the time interval between two successive ball releases is $T_i/6$. Again, three sets of parameter values are considered.

Thus it is possible, in an atomic resonator based on the interrogation of atoms launched sequentially, to reduce the limiting effect we have considered. This may be accomplished by a proper design of the resonator leading to as large as possible duty cycle and/or by launching several clouds of atoms during one cycle. For the case of ion traps, the use two parallel traps has been proposed [11].

One may also note that, in the two examples given, $\sigma_{y\,\mathrm{lim}}$ is smallest for $1/T_c = 1$ Hz. This value is the close to the Fourier frequency, f_Q, for which $S_y^f(f)$ of the VCXO shows a minimum. With the data of (11), we have

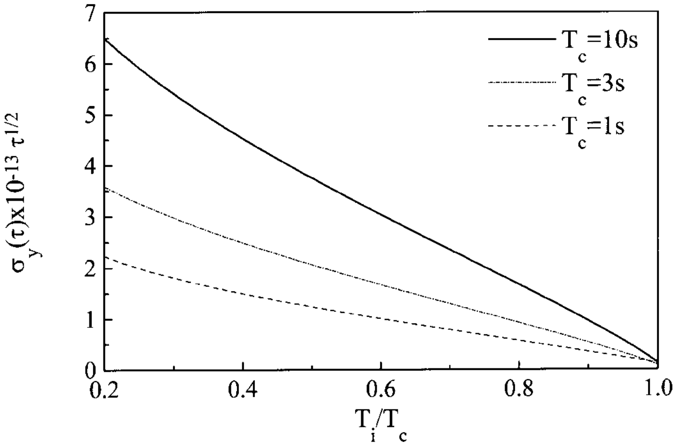

Fig. 9. Frequency stability versus the interrogation duty cycle for the Ramsey interrogation for different cycle lengths. With $T_i = 0.5T_c$ and $\tau_p = 0.015T_c$.

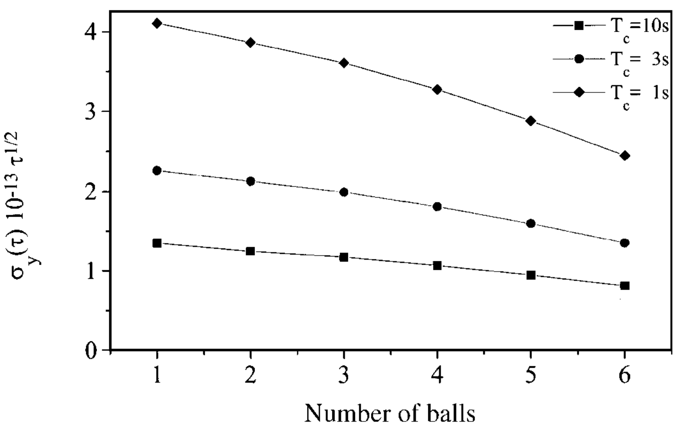

Fig. 10. Calculated frequency stability versus the number of launched atomic balls for the multi-λ case. The atomic balls are separated by $T_i/6$ and $T_i = 0.53T_c$, $b_cT_i/3 = 3.66$, and $\omega_m T_i = 2.31$.

$f_Q = 5$ Hz. This suggests that, whenever possible, the characteristics of the atomic resonator and of the VCXO should be matched. This is achieved when the condition $f_Q \approx 1/T_c$ is fulfilled.

VI. Conclusions

In this paper we have developed a quantum mechanical calculation of the atomic response to the phase noise of the interrogation oscillator in a two level atom. This model has been used in the calculation of the frequency stability degradation in a pulsed atomic frequency standard due to the down-conversion of the frequency noise of the interrogating oscillator. We also have compared results of calculations based on this model with experimental values obtained by using the LPTF Cs atomic fountain with a purposely degraded oscillator. The theory and the experiments agree within the limits of measurement errors. For a state-of-the-art 5 or 10 MHz BVA quartz oscillator, the excess noise due to the sampling process limits the frequency stability of an atomic fountain to about $10^{-13}\tau^{-1/2}$ for a cycle time of 1 s. Better results could be achieved using cryogenic sapphire oscillators. The model shows that the limitation comes primarily from the flicker frequency noise of the oscillator and that the characteristic white phase floor does not affect the results.

There do not seem to be any obvious signal processing techniques that could be applied to reduce the consequences of this detrimental effect.

Appendix

As shown in [12], the change of the quantum state of the atoms interacting with a quasi-resonant field $b\cos(\omega t+\Delta\phi)$ can be represented in a matrix form. We have, in general,

$$\begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{pmatrix} = \mathcal{R}[b, \Delta\phi, \Omega_0, t] \begin{pmatrix} a_1(0) \\ a_2(0) \\ a_3(0) \end{pmatrix}, \tag{A-1}$$

where $a_1(t)$ and $a_2(t)$ denote the atomic coherence and $a_3(t)$ the relative population difference of the two levels involved in the transition. The column matrices at the right and at the left represent the atom properties at the beginning and at the end of an interaction of duration t, respectively. $\mathcal{R}[b, \Delta\phi, \Omega_0, t]$ is a 3x3 matrix whose elements depend on t, the Rabi frequency b, the phase ϕ, and the detuning from the atomic resonance. We have [see (A-2) top of next page]:

Therefore, the population difference of atoms submitted to various amplitude and phase conditions during their interaction with the magnetic microwave field can be calculated from matrix products. The variation of the probability that a transition took place is related to the matrix elements in the following way:

$$P(t, \Delta\phi) = \frac{1}{2}\left(1 - \frac{a_3(t, \Delta\phi)}{a_3(0)}\right). \tag{A-3}$$

According to (1) the sensitivity function $g(t)$ is:

$$\begin{aligned} g(t) &= 2 \lim_{\Delta\phi \to 0} \delta P(t, \Delta\phi)/\Delta\phi \\ &= \left.\frac{\partial a_3(t, \Delta\phi)}{\partial \Delta\phi}\right|_{\Delta\phi=0} \frac{1}{a_3(0)}. \end{aligned} \tag{A-4}$$

The effect of the small phase step $\Delta\phi$ occurring at a given time during the interaction can be expressed easily. In the case of a Ramsey interrogation where the Rabi frequency b is constant, we can write:

$$\begin{aligned} a(t, T, \tau_p, \Delta\phi, b, \Omega_0)_1 = {} & \mathcal{R}[b, \Delta\phi, \Omega_0, \tau_p]\mathcal{R}[0, \Delta\phi, \Omega_0, T] \\ & \times \mathcal{R}[b, \Delta\phi, \Omega_0, \tau_p - t]\mathcal{R}[b, 0, \Omega_0, t]a(0) \end{aligned}$$

if the phase step takes place during the first microwave interaction;

$$\begin{aligned} a(t, T, \tau_p, \Delta\phi, b, \Omega_0)_2 = {} & \mathcal{R}[b, \Delta\phi, \Omega_0, \tau_p]\mathcal{R}[0, 0, \Omega_0, T] \\ & \times \mathcal{R}[b, 0, \Omega_0, \tau_p]a(0) \end{aligned}$$

$$\mathcal{R}[b,\Delta\phi,\Omega_0,t] = \begin{pmatrix} \begin{pmatrix} \cos\Omega t + \\ \frac{b^2\cos^2\Delta\phi}{\Omega^2}(1-\cos\Omega t) \end{pmatrix} & \begin{pmatrix} -\frac{\Omega_0}{\Omega}\sin\Omega t + \\ -\frac{b^2\cos\Delta\phi\sin\Delta\phi}{\Omega^2}(1-\cos\Omega t) \end{pmatrix} & \begin{pmatrix} -\frac{b\cos\Delta\phi\Omega_0}{\Omega^2}(1-\cos\Omega t) \\ +\frac{b\sin\Delta\phi}{\Omega}\sin\Omega t \end{pmatrix} \\ \begin{pmatrix} \frac{\Omega_0}{\Omega}\sin\Omega t + \\ -\frac{b^2\cos\Delta\phi\sin\Delta\phi}{\Omega^2}(1-\cos\Omega t) \end{pmatrix} & \begin{pmatrix} \cos\Omega t + \\ \frac{b^2\sin^2\Delta\phi}{\Omega^2}(1-\cos\Omega t) \end{pmatrix} & \begin{pmatrix} \frac{b\cos\Delta\phi}{\Omega}\sin\Omega t + \\ \frac{b\Omega_0\sin\Delta\phi}{\Omega^2}(1-\cos\Omega t) \end{pmatrix} \\ \begin{pmatrix} -\frac{b\cos\Delta\phi\Omega_0}{\Omega^2}(1-\cos\Omega t) \\ -\frac{b\sin\Delta\phi}{\Omega}\sin\Omega t \end{pmatrix} & \begin{pmatrix} -\frac{b\cos\Delta\phi}{\Omega}\sin\Omega t \\ +\frac{b\sin\Delta\phi\Omega_0}{\Omega^2}(1-\cos\Omega t) \end{pmatrix} & \begin{pmatrix} 1-\frac{b^2}{\Omega^2}(1-\cos\Omega t) \end{pmatrix} \end{pmatrix},$$

$$\Omega = \sqrt{b^2+\Omega_0^2}. \quad \text{(A-2)}$$

if the phase step takes place during the period T; and

$$a(t,T,\tau_p,\Delta\phi,b,\Omega_0)_3 = \mathcal{R}[b,\Delta\phi,\Omega_0,\tau_p - t]\mathcal{R}[b,0,\Omega_0,t] \times \mathcal{R}[0,0,\Omega_0,T]\mathcal{R}[b,0,\Omega_0,\tau_p]a(0)$$

if the phase step takes place during the last microwave interaction. We suppose that the atomic population is prepared in a single quantum state at the beginning of the interaction, i.e., $a_1(0) = a_2(0) = 0$, $a_3(0) = -1$. At the end of the interaction we measure the relative population difference $a_3(t)$. The function $g(t)$ can be easily obtained by expanding $a_3(t)$ to first order to respect to $\Delta\phi$. We thus have:

$$g(t) = \begin{cases} \left.\frac{\partial}{\partial\phi}a_3(t,T,\tau_p,\Delta\phi,b,\Omega_0)_1\right|_{\Delta\phi=0} & 0\le t\le\tau_p \\ \left.\frac{\partial}{\partial\phi}a_3(t,T,\tau_p,\Delta\phi,b,\Omega_0)_2\right|_{\Delta\phi=0} & \tau_p\le t\le T+\tau_p \\ \left.\frac{\partial}{\partial\phi}a_3(t,T,\tau_p,\Delta\phi,b,\Omega_0)_3\right|_{\Delta\phi=0} & T+\tau_p\le t\le T+2\tau_p \\ 0 & T+2\tau_p\le t\le T_c. \end{cases} \quad \text{(A-5)}$$

It is worth noting that, after the end of the interaction, and up to the end of the interrogation cycle T_c the value of the function is null. With the conditions $T \gg \tau_p$ and $\Omega_0 \ll b$ we obtain:

$$g(t) = \begin{cases} a\sin bt & 0 \le t \le \tau_p \\ a\sin b\tau_p & \tau_p \le t \le T+\tau_p \\ a\sin b(T+2\tau_p - t) & T+\tau_p \le t \le T+2\tau_p \\ 0 & T+2\tau_p \le t \le T_c, \end{cases} \quad \text{(A-6)}$$

where the constant a has the same meaning as in (9). When the microwave amplitude is not a constant during the interaction, two different methods can be implemented. One may divide the interaction time into elementary intervals during which the amplitude is assumed a constant, or the differential equations describing the evolution of $a_1(t')$, $a_2(t')$, and $a_3(t')$ can be integrated numerically. In the case of atoms traveling along the axis of a TE_{01n} microwave cavity, and, assuming that the phase step $\Delta\phi$ is very small [12], these equations take the form:

$$\begin{aligned} \frac{\partial a_1(\xi)}{\partial\xi} + \frac{\Omega_0 T_i}{n}a_2(\xi) - b_c\Delta\phi\frac{T_i}{n}a_3(\xi)\sin(\pi\xi) = 0 \\ \frac{\partial a_2(\xi)}{\partial\xi} - \frac{\Omega_0 T_i}{n}a_1(\xi) - b_c\Delta\phi\frac{T_i}{n}a_3(\xi)\sin(\pi\xi) = 0 \\ \frac{\partial a_3(\xi)}{\partial\xi} + \frac{b_c T_i}{n}(a_1(\xi)\Delta\phi + a_2(\xi))\sin(\pi\xi) = 0, \end{aligned} \quad \text{(A-7)}$$

with:

$$\xi = \frac{nt'}{T_i}. \quad \text{(A-8)}$$

In these equations, T_i is the time of flight across the cavity and b_c is the Rabi frequency at an antinode point of the microwave field. The value of the frequency sensitivity function $g(t)$, at the time t is computed by integrating the differential system with $\Delta\phi = 0$ for $0 < \xi < nt/T_i$ and $\Delta\phi \neq 0$ for $nt/T_i < \xi < n$. The change of the population difference is then calculated at the outcome of the atom-field interaction and $g(t)$ is obtained from (A-4).

In the case of multi-ball operation, the total sensitivity function $g_{mb}(t)$ is obtained in the following way:

$$g_{mb}(t) = \sum_{j=1}^{n_b} g(t - j\Delta t), \quad \text{(A-9)}$$

where n_b is the number of released atomic balls and Δt is the time interval between two successive balls.

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