Efficient Frontier

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QF600: Asset Pricing

1 Introduction

The main task of this assignment is to plot the efficient frontier and find the tangency portfolio using a dataset containing monthly nominal returns (in percentages) for ten industry portfolios over a ten-year period (Jan 2004 to Dec 2013).

2 Mean Returns and Covariance Matrix

The mean returns and standard deviation of the returns for the ten industry portfolios are calculated using simple functions readily available in Python. The table below shows a summary of mean return and standard deviation:

	Mean Return (%)	Standard Deviation (%)
NoDur	0.903	3.346
Durbl	0.733	8.362
Manuf	1.013	5.310
Enrgy	1.231	6.082
HiTec	0.766	5.381
Telcm	0.881	4.448
Shops	0.916	4.094
Hlth	0.784	3.787
Utils	0.907	3.702
Other	0.489	5.582

Figure 1: Mean Returns and Standard Deviation by Industry Portfolio

3 Minimum Variance Frontier

The minimum-variance frontier (without riskless asset) was subsequently plotted in two ways: using the analytical expression for the minimum variance frontier standard deviation, and using Python's *scipy* package to obtain the numerical frontier standard deviation. Plots cover the range from 0% to 2% in increments of 0.1% on the vertical axis.

3.1 Analytic Expression

The minimum variance frontier is given by:

$$A = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$$

$$B = \mathbf{1}^T \Sigma^{-1} \mu$$

$$C = \mu^T \Sigma^{-1} \mu$$

The variance of a portfolio with expected return $E(R_p)$ is:

$$\sigma_p^2 = \frac{1}{A} + \frac{(AC - B^2)}{A^2} \left(E(R_p) - \frac{B}{A} \right)^2$$

Thus, the standard deviation (risk) of a portfolio with expected return $E(R_p)$ is:

$$\sigma_p = \sqrt{\frac{1}{A} + \frac{(AC - B^2)}{A^2} \left(E(R_p) - \frac{B}{A} \right)^2}$$

Plotting the expected monthly return against standard deviation:

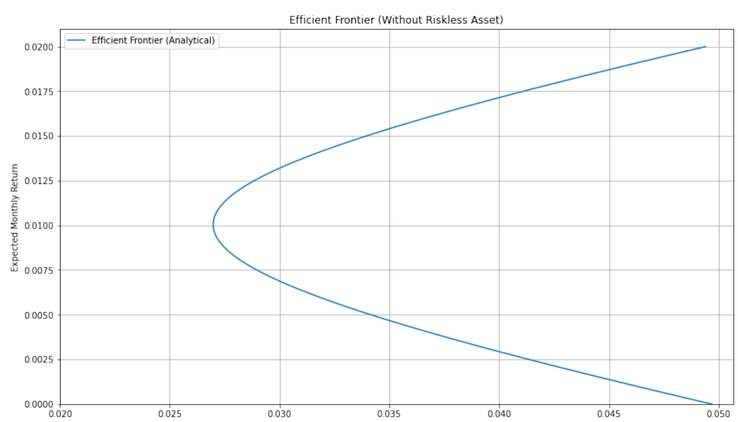

Figure 2: Minimum Variance Frontier using analytic expression

which takes the signature parabolic shape.

3.2 Numerical Optimisation

Using only the expression for portfolio returns and standard deviation, we can define an optimisation function for a fixed range of target returns (from 0 to 0.02 in this context). The function finds the portfolio weights and standard deviations which minimises variance under the constraint that weights must sum to one, and that no short-selling is allowed. The plot below shows the result:

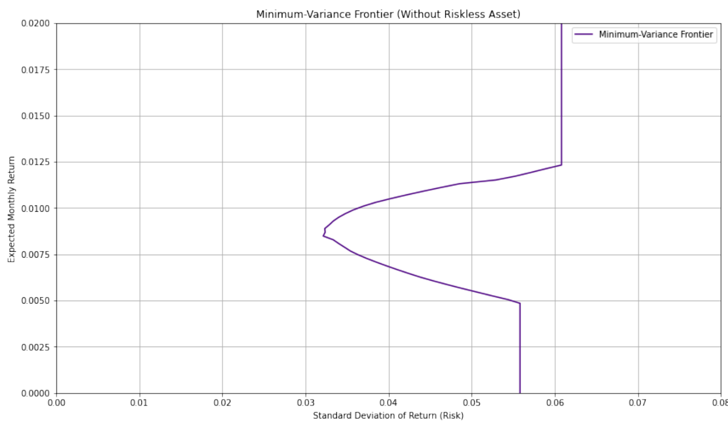

Figure 3: Minimum Variance Frontier using numerical optimisation

The shape of the plot resembles that of the analytical expression, but does not exhibit the same symmetry and has a different peak, possibly due to the limitations of least-square method that the package uses (SLSQP).

3.3 Economic Significance and Relevance

The minimum variance (efficient) frontier shows the set of portfolios that have the lowest possible risk for a given expected return. It helps investors identify the best trade-off between risk and return, particularly if they are risk averse.

4 Efficient Frontier

With a risk-free asset of interest rate 0.13% per month, we can produce a capital market line that meets the minimum variance frontier at a point (tangency portfolio).

The Capital Market Line (CML) is given by the equation:

$$E(R_p) = R_f + \frac{E(R_M) - R_f}{\sigma_M} \sigma_p$$

where:

- $E(R_p)$ is the expected return of the portfolio,
- R_f is the risk-free rate,
- $E(R_M)$ is the expected return of the market (tangency portfolio),
- σ_M is the standard deviation (risk) of the market (tangency) portfolio,
- σ_p is the standard deviation (risk) of the portfolio.

The plot below shows the efficient frontier (green), the minimum-variance frontier (blue) and the tangency portfolio (red).

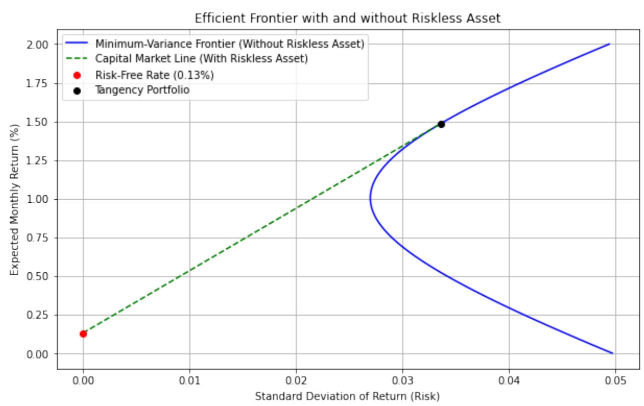

Figure 4: Capital Market Line and Minimum Variance Frontier

4.1 Economic Significance and Relevance

The efficient frontier represents the set of optimal portfolios that offer the highest expected return for a given level of risk. It encourages efficient allocation of capital and demonstrates how diversification can reduce risk.

5 Tangency Portfolio

The efficient frontier (Capital Market Line) meets the minimum variance frontier at the tangency portfolio.

The Sharpe ratio of the tangency portfolio is given by:

$$\text{Sharpe Ratio} = \frac{E(R_M) - R_f}{\sigma_M}$$

Where:

- $E(R_M)$ is the expected return of the market (tangency) portfolio,
- R_f is the risk-free rate,
- σ_M is the standard deviation (risk) of the tangency portfolio.

The Sharpe ratio for the tangency portfolio was calculated to be 0.404.

The tangency portfolio weights are shown in the table below:

Industry Portfolio	Tangency Portfolio Weight
NoDur	0.568
Durbl	-0.214
Manuf	0.714
Enrgy	0.104
HiTec	-0.363
Telcm	-0.095
Shops	0.992
Hlth	0.076
Utils	0.133
Other	-0.913

Figure 5: Table of portfolio weights for each industry

5.1 Economic Significance and Relevance

The tangency portfolio is the portfolio with the highest Sharpe ratio, meaning it provides the highest expected return for each unit of risk taken. When a risk-free asset is available, all investors, regardless of their risk tolerance, will combine the risk-free asset with the tangency portfolio to achieve their desired level of risk and return. By combining the risk-free asset with the tangency portfolio, investors can achieve better risk-adjusted returns compared to any other portfolio of risky assets.