Given a set of rings with radius R at varying z values, the 🅱-field at z' on the z-axis due to a ring at z^* is

$$\int_{z^*}^{z'} \frac{\mu_0 I dl}{4\pi} \frac{R}{(z^2+R^2)^{\frac{3}{2}}} = \frac{\mu_0}{4\pi} \frac{2\pi R^2 I}{(z^2+R^2)^{\frac{3}{2}}}\Bigg|_{z^*}^{z'}$$

By the 🅱iot-Savart law.

Thus for a solenoid with N loops, approximated as a series of current carrying loops where each loop is at some value z_i where $0 < i < N$, The net 🅱-field at a point z will be

$$\sum_{i=0}^{i=N} \frac{\mu_0}{4\pi} \frac{2\pi R^2 I}{((z)^2+R^2)^{3/2}} - \frac{\mu_0}{4\pi} \frac{2\pi R^2 I}{((z^*)^2+R^2)^{3/2}} = \text{🅱}_z$$

A derivation of the Amperian model for forces between magnetic dipoles in non-uniform magnetic fields can be found here, which states:

$$F_z = \mu \frac{\delta \text{🅱}_z}{\delta z}$$

For point-like dipole. For a very thin, bar magnet with length L, the Force would be:

$$F_z = \mu \frac{\Delta \text{🅱}_z}{\Delta z} = \mu \frac{\Delta \text{🅱}_z}{L}$$

We can determine the overall change in kinetic energy, K, by numerically approximating the following. Note: z is the location of the end of the magnet.

$$\Delta K = -\Delta U = \int_{z_0}^{z} F_z dz = \int_{z_0}^{z} \mu \frac{\text{🅱}(z) - \text{🅱}(z-L)}{L}$$

And thus we can determine the speed at which we expect a dipole in our coil gun to travel from its starting position, z_0 to the point z.

To determine the moment, $\boldsymbol{\mu}$, of the magnet, we approximate the magnet as a series of looped coils with moving charge. Given a constant charge density, $J = dI/dx$, we can use the 🅱iot-Savart law to derive the magnetic field strength at the end of a magnet, 🅱_{end}.

$$dI = Jdx$$

$$\text{🅱}_{end} = \int_{x=0}^{x=L} \frac{\mu_0 R^2 J dx}{2(R^2+x^2)^{3/2}} = \frac{\mu_0 J}{2} \frac{L}{\sqrt{L^2+R^2}}$$

Solving for the charge density and the total current moving in the magnet gives

$$J = \text{🅱}_{end} \frac{\sqrt{L^2+R^2}}{L} \frac{2}{\mu_0}$$

$$I = JL$$

$$I = 🅱_{end}\frac{2\sqrt{L^2 + R^2}}{\mu_0}$$

Thus as $\boldsymbol{\mu} = AI_{total}$,

$$\boldsymbol{\mu} = \pi R^2 🅱_{end}\frac{2\sqrt{L^2 + R^2}}{\mu_0}$$

For a given magnet with a field strength $🅱_{end}$.