

# Chapter 1 Problems

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## Problem 1.1

For the distribution of ages in the example in Section 1.3.1

$$N(14) = 1$$

$$N(15) = 1$$

$$N(16) = 3$$

$$N(22) = 2$$

$$N(24) = 2$$

$$N(25) = 5$$

- (a) Compute  $\langle j^2 \rangle$  and  $\langle j \rangle^2$  Total number of kids is  $1 + 1 + 3 + 2 + 2 + 5 = 14$ .

$$\begin{aligned}\langle j^2 \rangle &= \frac{1}{14}(14^2 + 15^2) + \frac{2}{14}(22^2 + 24^2) + \frac{3}{14}16^2 + \frac{5}{14}25^2 \\ &= \frac{3217}{7} \approx 459.57\end{aligned}$$

$$\begin{aligned}\langle j \rangle &= \frac{1}{14}(14 + 15) + \frac{2}{14}(22 + 24) + \frac{3}{14}16 + \frac{5}{14}25 \\ &= 21\end{aligned}$$

$$\langle j \rangle^2 = 441$$

- (b) Determine  $\Delta j$  for each  $j$  and use Equation 1.11 to compute the standard deviation

$$\begin{aligned}\sigma^2 = \langle (\Delta j)^2 \rangle &= \frac{1}{14}((-7)^2 + (-6)^2) + \frac{2}{14}(1^2 + 3^2) + \frac{3}{14}(-5)^2 + \frac{5}{14}4^2 \\ &= \frac{130}{7}\end{aligned}$$

- (c) Check using part a and part b

$$\begin{aligned}\sigma^2 &= \langle j^2 \rangle - \langle j \rangle^2 = \frac{130}{7} \\ &= \langle (\Delta j)^2 \rangle\end{aligned}$$

# 1 Problem 1.3

Consider the Gaussian Distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

where  $A, a$  and  $\lambda$  are positive real constants (necessary integrals are in the back of the book)

- (a) Use equation 1.16 to determine  $A$  We have that  $\int_{-\infty}^{\infty} \rho(x) = 1$  so we have

$$\begin{aligned} \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2} dx &= 2A \int_0^{\infty} e^{-\lambda x^2} dx \\ &= 2A\sqrt{\pi} \frac{1}{2\sqrt{\lambda}} \\ &= A\sqrt{\frac{\pi}{\lambda}} = 1 \end{aligned}$$

Thus solving for  $A$  we get

$$A = \sqrt{\frac{\lambda}{\pi}}$$

- (b) Find  $\langle x \rangle$ ,  $\langle x^2 \rangle$ , and  $\sigma$

Let  $u = x - a$ . Then  $\frac{du}{dx} = 1$ ,  $du = dx$ , and

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} Axe^{-\lambda(x-a)^2} dx \\ &= \int_{-\infty}^{\infty} A(u+a)e^{-\lambda u^2} du \\ &= \int_{-\infty}^{\infty} Aue^{-\lambda u^2} du + \int_{-\infty}^{\infty} Aae^{-\lambda u^2} du \end{aligned}$$

Since the first integral in the sum is integrating an odd function ( $ue^{\lambda u^2} = -((-u)e^{\lambda(-u)^2})$ ), that integral is zero. This leaves

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} Aae^{-\lambda u^2} du \\ &= a \int_{-\infty}^{\infty} Ae^{-\lambda u^2} du \\ &= a \end{aligned}$$

Using the formula in the back of the book, we have that

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 A e^{-\lambda(x-a)^2} dx \\
&= \int_{-\infty}^{\infty} (u+a)^2 A e^{-\lambda u^2} du \\
&= \int_{-\infty}^{\infty} u^2 A e^{-\lambda u^2} du + \int_{-\infty}^{\infty} 2au A e^{-\lambda u^2} du + \int_{-\infty}^{\infty} a^2 A e^{-\lambda u^2} du \\
&= 4A\sqrt{\pi} \left( \frac{1}{2\sqrt{\lambda}} \right)^3 + a^2 \\
&= \frac{1}{2} \frac{\sqrt{\lambda}}{\sqrt{\pi}} \sqrt{\pi} \frac{1}{\sqrt{\lambda}^3} + a^2 \\
&= \frac{1}{2\lambda} + a^2
\end{aligned}$$

And lastly,

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda} + a^2 - a^2 = \frac{1}{2\lambda}$$

## 2 Problem 1.5

Consider the Gaussian wave function

$$\Psi(x, t) = A e^{-\lambda|x|} e^{-i\omega t}$$

Where  $A$ ,  $\omega$ , and  $\lambda$  are real coefficients.

(a) Normalize  $\Psi$

Firstly,  $|\Psi| = |A e^{-\lambda|x|}|$  since  $|e^{i\omega t}| = |\cos(\omega t) + i\sin(\omega t)| = 1$ . Thus  $|\Psi|^2 = A^2 e^{-2\lambda|x|}$  so

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi|^2 dx &= \int_{-\infty}^{\infty} A^2 e^{-2\lambda|x|} dx \\
&= 2A^2 \int_0^{\infty} e^{-2\lambda x} dx \\
&= 2A^2 \frac{1}{-2\lambda} \int_0^{\infty} e^{-2\lambda x} (-2\lambda) dx \\
&= 2A^2 \frac{1}{-2\lambda} \int_0^{\infty} e^u du \\
&= \frac{2A^2}{-2\lambda} [e^u]_0^{\infty} \\
&= \frac{A^2}{-\lambda} (0 - 1) = \frac{A^2}{\lambda} = 1
\end{aligned}$$

Thus, we have that  $A = \sqrt{\lambda}$  and  $\Psi$  is normalized.

(b) Determine  $\langle x \rangle$  and  $\langle x^2 \rangle$

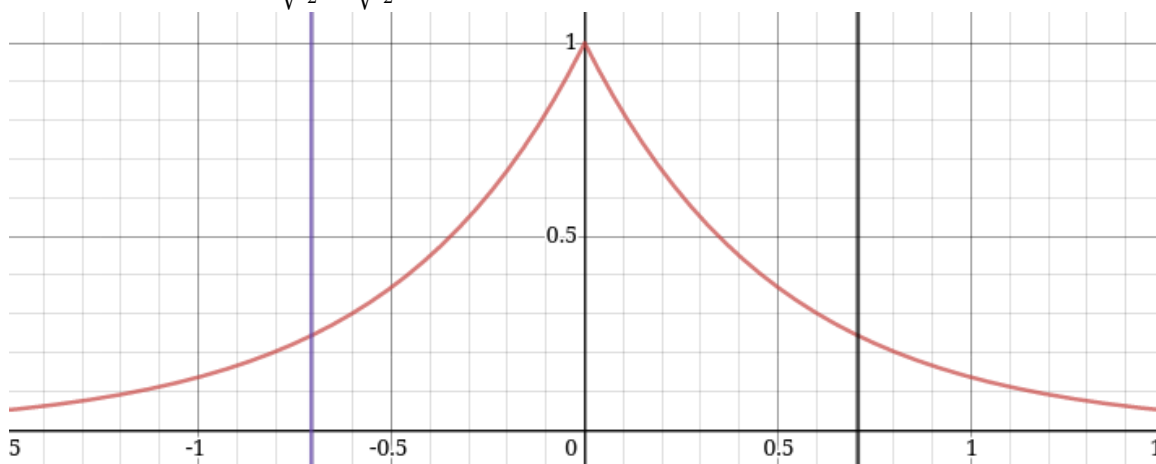
$$\begin{aligned}\langle x \rangle &= \int_{-\infty}^{\infty} x |\Psi|^2 dx = \int_{-\infty}^{\infty} \frac{x}{\lambda} e^{-2\lambda|x|} dx \\ &= \frac{1}{\lambda} \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx \\ &= 0\end{aligned}$$

The last step follows from integrating an odd function symmetrically around 0. Now to find  $\langle x \rangle^2$ :

$$\begin{aligned}\langle x \rangle^2 &= \int_{-\infty}^{\infty} x^2 |\Psi|^2 dx = \frac{1}{\lambda} \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx \\ &= \frac{2}{\lambda} \int_0^{\infty} x^2 e^{-2\lambda x} dx \\ &= \frac{2}{\lambda} 2 \left( \frac{1}{2\lambda} \right)^3 \\ &= \frac{1}{2\lambda^4}\end{aligned}$$

(c) Find the standard deviation of  $x$ . Sketch the graph of  $|\Psi|^2$ , as a function of  $x$ , and mark the points  $(\langle x \rangle + \sigma)$  and  $(\langle x \rangle - \sigma)$ , to illustrate the sense in which  $\sigma$  represents the "spread" in  $x$ . What is the probability that the particle would be found outside this range?

Firstly,  $\sigma^2 = \frac{1}{2\lambda^4}$  so  $\sigma = \frac{1}{\sqrt{2}\lambda}$ . Looking at a graph of  $|\Psi|^2$  for  $\lambda = 1$  marked at  $x = \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}$ :



The probability  $x$  is greater than  $\sqrt{\frac{1}{2}}$  is

$$\int_{0.5}^{\infty} e^{-2x} dx = \left[ -\frac{1}{2} e^{-2x} \right]_{(1/\sqrt{2})}^{\infty} = \frac{1}{2e^{\sqrt{2}}} \approx 0.12156$$

And the same value is taken for the probability than  $x$  is less than  $\sqrt{\frac{1}{2}}$  by symmetry. Thus the probability that  $x \notin [\langle x \rangle - \sigma, \langle x \rangle + \sigma]$  is  $\frac{1}{e} \approx 0.24312$ .

### 3 Problem 1.7

Calculate  $\frac{d\langle p \rangle}{dt}$ . This should equal  $\langle -\frac{\partial V}{\partial x} \rangle$ .

$$\frac{d\langle p \rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \left( \frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial^2 \Psi}{\partial t \partial x} \right) dx$$

First simplifying the second term in the integral we find

$$\begin{aligned} \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial t \partial x} &= \left[ \Psi^* \frac{\partial \Psi}{\partial t} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial t} dx \\ &= \int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial t} dx \end{aligned}$$

Thus

$$\begin{aligned} \frac{d\langle p \rangle}{dt} &= \int_{-\infty}^{\infty} \left[ \left( i\hbar \frac{\partial \Psi}{\partial t} \right)^* \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial x} \left( i\hbar \frac{\partial \Psi}{\partial t} \right) \right] dx \\ &= \int_{-\infty}^{\infty} \left[ \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi \right)^* \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial x} \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi \right) \right] dx \\ &= \int_{-\infty}^{\infty} \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2 \Psi^*}{\partial x^2} \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial x} \frac{\partial^2 \Psi}{\partial x^2} \right) + V\Psi^* \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial x} V\Psi \right] dx \\ &= \int_{-\infty}^{\infty} \left[ -\frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left( \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial x} \right) + V \frac{\partial |\Psi|^2}{\partial x} \right] dx \quad (\text{using the product rule}) \\ &= \frac{-\hbar^2}{2m} \left[ \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial x} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} V \frac{\partial |\Psi|^2}{\partial x} dx \\ &= \int_{-\infty}^{\infty} V \frac{\partial |\Psi|^2}{\partial x} dx \quad (\text{since } \Psi \text{ and its derivatives vanish at infinity}) \\ &= [V|\Psi|^2]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} |\Psi|^2 dx \quad (\text{using integration by parts}) \\ &= \left\langle -\frac{\partial V}{\partial x} \right\rangle \end{aligned}$$

### 4 Problem 1.9

A particle of mass  $m$  has the wave function

$$\Psi(x, t) = Ae^{-a[(mx^2/\hbar) + it]}$$

(a) Find A.

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx &= \int_{-\infty}^{\infty} |A e^{-a[(mx^2/\hbar)^2 + it]}| dx \\
&= A^2 \int_{-\infty}^{\infty} |e^{-amx^2/\hbar}|^2 |e^{-ait}|^2 dx \\
&= 2A^2 \int_0^{\infty} e^{-2amx^2/\hbar} dx \\
&= 2A^2 \int_0^{\infty} e^{-x^2/k^2} dx \text{ for } k = \sqrt{\frac{\hbar}{2am}} \\
&= 2A^2 \sqrt{\pi} \frac{k}{2} \\
&= A^2 \sqrt{\frac{\pi \hbar}{2am}} = 1
\end{aligned}$$

Thus we have  $A = \left(\frac{\pi \hbar}{2am}\right)^{\frac{1}{4}}$ .