

Chapter 2 Problems

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Problem 2.1

Prove the following theorems:

- (a) For normalizable solutions, the separation constant E must be *real*

Assume E is complex, meaning $E = a + ib$ for some $a, b \in \mathbb{R}$. For a separable solution $\Psi(x, t) = \psi(x)e^{-iEt/\hbar}$, we have that

$$\begin{aligned} |\Psi(x, t)|^2 &= |\psi|^2 \left| e^{-i(a+ib)t/\hbar} \right|^2 \\ &= |\psi|^2 \left| e^{-iat/\hbar} e^{bt/\hbar} \right|^2 \\ &= |\psi|^2 e^{2bt/\hbar} \end{aligned}$$

For $t = 0$, we know that $\int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx = 1$ so

$$\int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx = \int_{-\infty}^{\infty} |\psi|^2 e^0 dx = \int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

Now for any $t > 0$, we have

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = \int_{-\infty}^{\infty} |\psi|^2 e^{2bt/\hbar} dx = e^{2bt/\hbar} \int_{-\infty}^{\infty} |\psi|^2 dx = e^{2bt/\hbar} = 1$$

Taking the natural log of both sides we see

$$2bt/\hbar = 0$$

and hence $b = 0$ and E is composed of only the real component.

- (b) The time independent wave function ψ can always be taken to be *real* (unlike Ψ which is necessarily complex). That doesn't mean that every solution to the time-independent Schroedinger equation is real; what it says is that if you've got one that is not, it can always be expressed as a linear combination of solutions (with the same energy) that are real. So you might as well stick to ψ s that are real.

Assume $\psi = a + ib$ is a solution to the time-independent Schroedinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \psi V = \psi E$$

Since both ψ and ψ^* solve this, we can plug both in to this equation:

$$\begin{aligned} -\frac{\hbar^2}{2m} \left(\frac{\partial^2 a}{\partial x^2} + i \frac{\partial^2 b}{\partial x^2} \right) + (a + ib)V &= (a + ib)E \\ -\frac{\hbar^2}{2m} \left(\frac{\partial^2 a}{\partial x^2} - i \frac{\partial^2 b}{\partial x^2} \right) + (a - ib)V &= (a - ib)E \end{aligned}$$

Adding these equations together we get

$$\begin{aligned} -\frac{\hbar^2}{2m} \left(2 \frac{\partial^2 a}{\partial x^2} + i \frac{\partial^2 b}{\partial x^2} - i \frac{\partial^2 b}{\partial x^2} \right) + (2a + ib - ib)V &= (a + ib - ib)E \\ -\frac{\hbar^2}{2m} \left(2 \frac{\partial^2 a}{\partial x^2} \right) + 2aV &= 2aE \\ -\frac{\hbar^2}{2m} \left(\frac{\partial^2 a}{\partial x^2} \right) + aV &= aE \end{aligned}$$

And hence $a = \frac{1}{2}(\psi + \psi^*)$ is also a solution, and has no imaginary component. It is worth noting that we should probably use $\frac{1}{\sqrt{2}}(\psi + \psi^*)$ instead, so that we still have a normalized solution.

- (c) If $V(x)$ is an even function, then ψ can always be taken to be either even or odd.

Assume we have some solution $\psi(x)$. Let $\gamma(x) = \psi(-x)$. We then have

$$\begin{aligned} \frac{\partial \gamma}{\partial x}(x) &= -\frac{\partial \psi}{\partial x}(-x) \\ \frac{\partial^2 \gamma}{\partial x^2}(x) &= \frac{\partial^2 \psi}{\partial x^2}(-x) \end{aligned}$$

If we look at γ in the time-independent equation we get

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{\partial^2 \gamma}{\partial x^2}(x) + \gamma(x)V(x) &= \gamma(x)E \\ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}(-x) + \psi(-x)V(x) &= \psi(-x)E \\ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}(-x) + \psi(-x)V(-x) &= \psi(-x)E \end{aligned}$$

Where the last equation uses the evenness of V . This equation also holds since the time independent equation holds for all $x \in \mathbb{R}$. Thus, $\psi(-x)$ is also a solution if V is even.

Now, since any linear combination of solutions is also a solution, we can use the even combination $\psi(x) + \psi(-x)$ or the odd combination $\psi(x) - \psi(-x)$ in order to get an even or odd solution to the equation.

Problem 2.2

Prove that E must exceed the minimum value of $V(x)$ at some point. Assume $E < V(x)$ for all x . Then we can solve for $\frac{\partial^2 \psi}{\partial x^2}$ to see that

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2m}{\hbar^2} \psi (V(x) - E)$$

This says that ψ and $\frac{\partial^2 \psi}{\partial x^2}$ always have the same sign, taking ψ as a real solution. Assuming ψ is C^2 , if it ever takes a positive value, then it must continue to curve upwards in the positive direction and hence would grow unbounded. Symmetrically if ψ takes on some negative value, it will grow unboundedly negative in the negative direction. Hence, since $\psi \neq 0$ we have a contradiction and $E > V(x)$ for some x .

Problem 2.3

Calculate $\langle x \rangle, \langle x^2 \rangle, \langle p \rangle, \langle p^2 \rangle, \sigma_x$, and σ_p for the n -th stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closest to the uncertainty limit?

Recall that

$$\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

Thus we compute

$$\begin{aligned} \langle x^2 \rangle &= \int_0^a x^2 |\psi_n|^2 dx \\ &= \left[\frac{1}{12} \left(\frac{3(a^2 - 2\pi^2 n^2 x^2) \sin\left(\frac{2\pi n x}{a}\right) - 6\pi a n x \cos\left(\frac{2\pi n x}{a}\right) + 4x^3}{\pi^3 n^3} + \frac{4x^3}{a} \right) \right]_0^a \\ &= \frac{a^2 (4\pi^3 n^3 + 3 \sin(2\pi n) - 6\pi n (\pi n \sin(2\pi n) + \cos(2\pi n)))}{12\pi^3 n^3} \\ &= \frac{a^2 (4\pi^3 n^3 - 6\pi n)}{12\pi^3 n^3} \\ &= \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2} \\ \langle x \rangle &= \int_0^a x |\psi_n|^2 dx \\ &= \left[\frac{a (-2\pi^2 n^2 + 2\pi n \sin(2\pi n) + \cos(2\pi n) - 1)}{4\pi^2 n^2} \right] \\ &= \frac{a}{2} \text{ (I used mathematica)} \end{aligned}$$

Thus we have

$$\begin{aligned}\sigma_x^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\ &= \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2} - \frac{a^2}{4} \\ &= a^2 \left(\frac{1}{12} - \frac{1}{2n^2\pi^2} \right)\end{aligned}$$

It makes sense that the standard deviation would scale linearly with the area.

$$\begin{aligned}\langle p \rangle &= \int_0^a \psi^* \left[-i\hbar \frac{\partial \psi}{\partial x} \right] dx \\ &= \int_0^a \frac{-2\hbar i n \pi \cos(\frac{n\pi x}{a}) \sin(\frac{n\pi x}{a})}{a^2} dx \\ &= \left[\frac{\hbar i \cos(\frac{2n\pi x}{a})}{2a} \right]_0^a \\ &= 0 \\ \langle p^2 \rangle &= \int_0^a \psi^* \left[-\hbar^2 \frac{\partial^2 \psi}{\partial x^2} \right] dx \\ &= \int_0^a \frac{1}{a^3} 2\hbar^2 n^2 \pi^2 \sin^2\left(\frac{n\pi x}{a}\right) dx \\ &= \frac{\hbar^2 n^2 \pi^2}{a^2}\end{aligned}$$

Thus $\sigma_p^2 = \langle p^2 \rangle = \frac{\hbar^2 n^2 \pi^2}{a^2}$

We can use this to check the uncertainty principle:

$$\begin{aligned}\sigma_x \sigma_p &= \left(\frac{\hbar n \pi}{a} \right) \left(a \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}} \right) \\ &> \hbar n \pi \frac{1}{2\sqrt{3}} \\ &> \hbar/2\end{aligned}$$

This quantity is minimized for $n = 1$

Problem 2.5

A particle in the infinite square well has its initial wave function an even mixture of the first two stationary states:

$$\Psi(x, 0) = A[\psi_1(x) + \psi_2(x)]$$

(a) Normalize $\Psi(x, 0)$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx = \int_{-\infty}^{\infty} A^2 (\psi_1^* \psi_1 + \psi_1^* \psi_2 + \psi_2^* \psi_1 + \psi_2^* \psi_2) dx \\ &= A^2 \left(\int_{-\infty}^{\infty} |\psi_1|^2 dx + \int_{-\infty}^{\infty} |\psi_2|^2 dx \right) \\ &= 2A^2 \end{aligned}$$

Thus $A = 1/\sqrt{2}$.

(b) Find $\Psi(x, t)$, $|\Psi(x, t)|^2$. Express the latter as a sinusoidal function of time, as in example 2.1. Let $\omega = \pi^2 \hbar / 2ma^2$.

We have that $c_1, c_2 = 1/\sqrt{2}$. Thus we have

$$\Psi(x, t) = \frac{1}{\sqrt{a}} \left[\sin\left(\frac{\pi x}{a}\right) e^{-i\omega t} + \sin\left(\frac{2\pi x}{a}\right) e^{-4i\omega t} \right]$$

See the attached mathematica notebook in which we compute that

$$|\Psi(x, t)|^2 = \frac{\sin^2\left(\frac{\pi x}{a}\right) (4 \cos\left(\frac{\pi x}{a}\right) \cos(3t\omega) + 2 \cos\left(\frac{2\pi x}{a}\right) + 3)}{a}$$

(c) Compute $\langle x \rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation?

Using mathematica we see that

$$\langle x \rangle = \frac{a}{2} - \frac{16a \cos(3t\omega)}{9\pi^2}$$

The amplitude of this oscillation is $16a/9\pi^2$ and it's frequency is 3ω .

(d) Compute $\langle p \rangle$.

Here we can use that $\langle p \rangle = m \frac{d}{dt} \langle x \rangle$:

$$\begin{aligned} \langle p \rangle &= m \frac{16a\omega}{3\pi^2} \sin(3t\omega) \\ &= \frac{8\hbar}{3a} \sin(3t\omega) \end{aligned}$$

(e) If you measured the energy of this particle, what possible values could it's energy be and with what probabilities? Find the expectation value of H . How does it compare with E_1, E_2 .

The two possible values we could get are $E_1 = \omega, E_2 = 4\omega$. Since $c_1, c_2 = 1/\sqrt{2}$, we get either E_1 or E_2 with probability $1/2$.

$$\begin{aligned}
\langle H \rangle &= \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi dx \\
&= \int_{-\infty}^{\infty} \Psi^* \frac{1}{\sqrt{2}} (E_1 \psi_1 + E_2 \psi_2) dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} [E_1 \psi_1^* \psi_1 + E_1 \psi_2^* \psi_1 + E_2 \psi_1^* \psi_2 + E_2 \psi_2^* \psi_2] dx \\
&= \frac{1}{2} \left(E_1 \int_{-\infty}^{\infty} \psi_1^* \psi_1 dx + E_2 \int_{-\infty}^{\infty} \psi_2^* \psi_2 dx \right) \\
&= \frac{E_1 + E_2}{2}
\end{aligned}$$

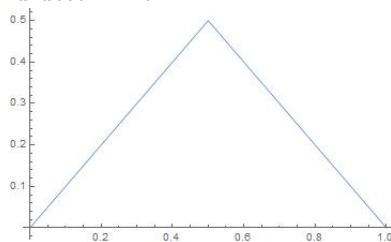
Which, as we would expect is $\langle E \rangle$.

Problem 2.7

A particle in the infinite square well has the initial wave function

$$\Psi(x, 0) = \begin{cases} Ax & 0 \leq x \leq a/2 \\ A(a-x) & a/2 \leq x \leq a \end{cases}$$

(a) Sketch $\Psi(x, 0)$ and determine A .



$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx &= A^2 \int_0^{a/2} x^2 dx + A^2 \int_{a/2}^a (a-x)^2 dx \\
&= A^2 \left([x^3/3]_0^{a/2} + [-(a-x)^3/3]_{a/2}^a \right) A \\
&= A^2 \left(\frac{a^3}{24} + \frac{a^3}{24} \right) \\
&= \frac{A^2 a^3}{12} = 1
\end{aligned}$$

Thus $A = \sqrt{12/a^3}$

(b) Find $\Psi(x, t)$.

This amounts to determining the c_n . Using mathematica we see that

$$c_n = \frac{4\sqrt{6} \sin\left(\frac{\pi n}{2}\right)}{\pi^2 n^2}$$

Thus we have

$$\Psi(x, t) = \sum_{n=0}^{\infty} \frac{4\sqrt{6} \sin\left(\frac{\pi n}{2}\right)}{\pi^2 n^2} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-itn^2\pi^2\hbar/2ma^2}$$

(c) What is the probability we measure the energy E_1 ?

This is

$$|c_1|^2 = \frac{96}{\pi^4}$$

(d) Find the expectation value of the energy:

$$\begin{aligned} \langle H \rangle &= \sum_{n=1}^{\infty} |c_n|^2 E_n = \sum_{n=1}^{\infty} \frac{96 \sin^2\left(\frac{\pi n}{2}\right)}{\pi^4 n^4} \frac{n^2 \pi^2 \hbar^2}{2ma^2} \\ &= \frac{48\hbar^2}{\pi^2 ma^2} \sum_{n=1}^{\infty} \frac{\sin^2(\pi n/2)}{n^2} \\ &= \frac{48\hbar^2}{\pi^2 ma^2} \left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots\right) \\ &= \frac{48\hbar^2}{\pi^2 ma^2} \frac{\pi^2}{8} \\ &= \frac{6\hbar^2}{ma^2} \end{aligned}$$

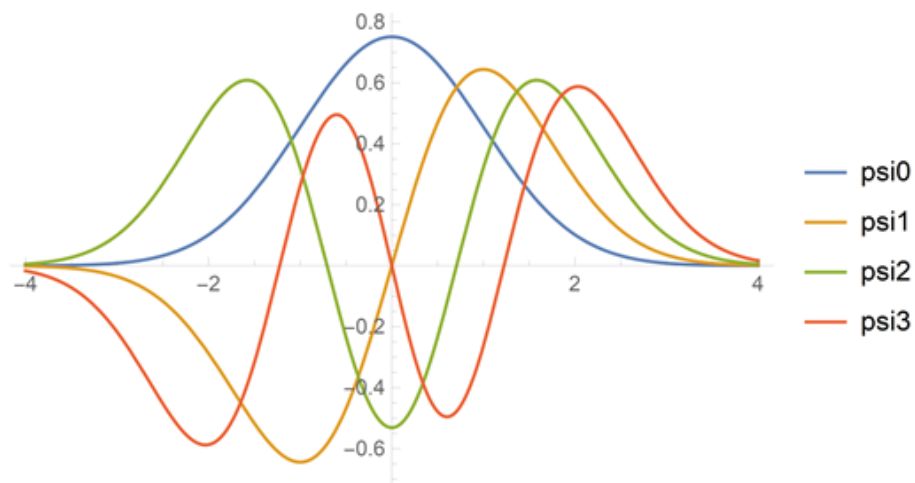
Problem 2.10

(a) Construct $\psi_2(x)$

Using mathematica we get that

$$\psi_2(x) = \frac{\sqrt[4]{m\omega} e^{-\frac{m\omega^2 x}{2\hbar}} (2mx^2\omega - \hbar)}{\sqrt{2}\sqrt[4]{\pi\hbar^5/4}}$$

(b) Sketch ψ_0, ψ_1, ψ_2 :



(c) Check orthogonality (see mathematica)

Problem 2.11

(a) Compute $\langle x \rangle, \langle p \rangle, \langle x^2 \rangle, \langle p^2 \rangle$ for ψ_0, ψ_1