Chapter 1 Problems

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Problem 1.1

For the distribution of ages in the example in Section 1.3.1

$$\begin{aligned}
N(14) &= 1 \\
N(15) &= 1 \\
N(16) &= 3 \\
N(22) &= 2 \\
N(24) &= 2 \\
N(25) &= 5
\end{aligned}$$

(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$ Total number of kids is $1+1+3+2+2+5 = 14$.

$$\begin{aligned}
\langle j^2 \rangle &= \frac{1}{14}(14^2 + 15^2) + \frac{2}{14}(22^2 + 24^2) + \frac{3}{14}16^2 + + \frac{5}{14}25^2 \\
&= \frac{3217}{7} \approx 459.57 \\
\langle j \rangle &= \frac{1}{14}(14 + 15) + \frac{2}{14}(22 + 24) + \frac{3}{14}16 + + \frac{5}{14}25 \\
&= 21 \\
\langle j \rangle^2 &= 441
\end{aligned}$$

(b) Determine Δj for each j and use Equation 1.11 to compute the standard deviation

$$\begin{aligned}
\sigma^2 = \langle (\Delta j)^2 \rangle &= \frac{1}{14}((-7)^2 + (-6)^2) + \frac{2}{14}(1^2 + 3^2) + \frac{3}{14}(-5)^2 + + \frac{5}{14}4^2 \\
&= \frac{130}{7}
\end{aligned}$$

(c) Check using part a and part b

$$\begin{aligned}
\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2 &= \frac{130}{7} \\
&= \langle (\Delta j)^2 \rangle
\end{aligned}$$

1 Problem 1.3

Consider the Gaussian Distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

where A, a and λ are positive real constants (necessary integrals are in the back of the book)

(a) Use equation 1.16 to determine A We have that $\int_{-\infty}^{\infty} \rho(x) = 1$ so we have

$$\begin{aligned}
\int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2} dx &= 2A \int_0^{\infty} e^{-\lambda x^2} dx \\
&= 2A\sqrt{\pi}\frac{1}{2\sqrt{\lambda}} \\
&= A\sqrt{\frac{\pi}{\lambda}} = 1
\end{aligned}$$

Thus solving for A we get

$$A = \sqrt{\frac{\lambda}{\pi}}$$

(b) Find $\langle x \rangle, \langle x^2 \rangle$, and σ

Let $u = x - a$. Then $\frac{du}{dx} = 1, du = dx$, and

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} Axe^{-\lambda(x-a)^2} dx \\
&= \int_{-\infty}^{\infty} A(u+a)e^{-\lambda u^2} du \\
&= \int_{-\infty}^{\infty} Aue^{-\lambda u^2} du + \int_{-\infty}^{\infty} Aae^{-\lambda u^2} du
\end{aligned}$$

Since the first integral in the sum is integrating an odd function ($ue^{\lambda u^2} = -((-u)e^{\lambda(-u)^2})$), that integral is zero. This leaves

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} Aae^{-\lambda u^2} du \\
&= a\int_{-\infty}^{\infty} Ae^{-\lambda u^2} du \\
&= a
\end{aligned}$$

Using the formula in the back of the book, we have that

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 A e^{-\lambda(x-a)^2} dx \\
&= \int_{-\infty}^{\infty} (u+a)^2 A e^{-\lambda u^2} du \\
&= \int_{-\infty}^{\infty} u^2 A e^{-\lambda u^2} du + \int_{-\infty}^{\infty} 2auAe^{-\lambda u^2} du + \int_{-\infty}^{\infty} a^2 A e^{-\lambda u^2} du \\
&= 4A\sqrt{\pi}(\frac{1}{2\sqrt{\lambda}})^3 + a^2 \\
&= \frac{1}{2}\frac{\sqrt{\lambda}}{\sqrt{\pi}}\sqrt{\pi}\frac{1}{\sqrt{\lambda}^3} + a^2 \\
&= \frac{1}{2\lambda} + a^2
\end{aligned}$$

And lastly,

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda} + a^2 - a^2 = \frac{1}{2\lambda}$$

2 Problem 1.5

Consider the Gaussian wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

Where A, ω, and λ are real coefficients.

(a) Normalize Ψ

Firstly, $|\Psi| = |Ae^{-\lambda|x|}|$ since $|e^{i\omega t}| = |cos(\omega t) + isin(\omega t)| = 1$. Thus $|\Psi|^2 = A^2 e^{-2\lambda|x|}$ so

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi|^2 dx &= \int_{-\infty}^{\infty} A^2 e^{-2\lambda|x|} dx \\
&= 2A^2 \int_{0}^{\infty} e^{-2\lambda x} dx \\
&= 2A^2 \frac{1}{-2\lambda} \int_{0}^{\infty} e^{-2\lambda x}(-2\lambda) dx \\
&= 2A^2 \frac{1}{-2\lambda} \int_{0}^{-\infty} e^u du \\
&= \frac{2A^2}{-2\lambda}[e^u]_0^{-\infty} \\
&= \frac{A^2}{-\lambda}(0-1) = \frac{A^2}{\lambda} = 1
\end{aligned}$$

Thus, we have that $A = \sqrt{\lambda}$ and Ψ is normalized.

(b) Determine $\langle x \rangle$ and $\langle x^2 \rangle$

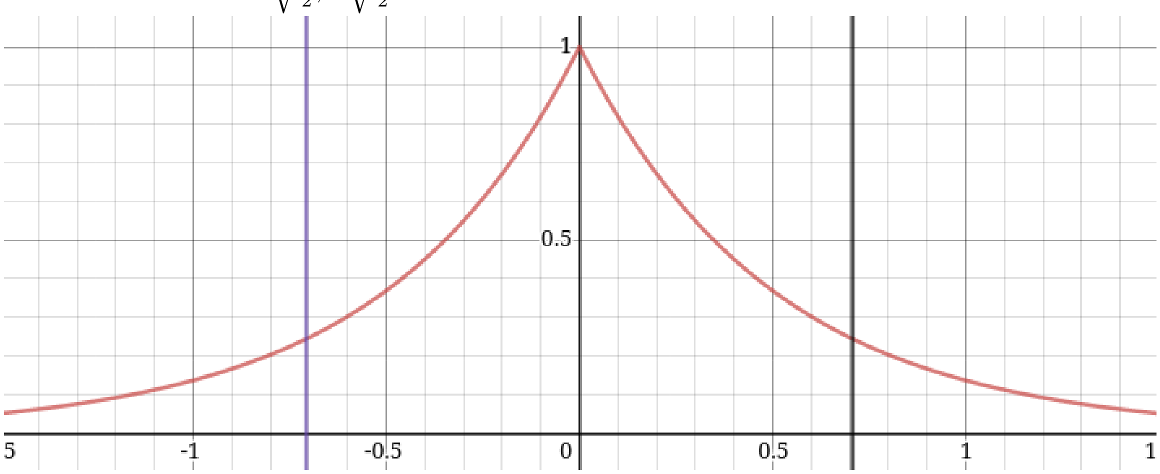
$$\begin{aligned}\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx &= \int_{-\infty}^{\infty} \frac{x}{\lambda} e^{-2\lambda|x|} dx \\ &= \frac{1}{\lambda} \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx \\ &= 0\end{aligned}$$

The last step follows from integrating an odd function symmetrically around 0. Now to find $\langle x \rangle^2$:

$$\begin{aligned}\langle x \rangle^2 = \int_{-\infty}^{\infty} x^2|\Psi|^2 dx &= \frac{1}{\lambda} \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx \\ &= \frac{2}{\lambda} \int_{0}^{\infty} x^2 e^{-2\lambda x} dx \\ &= \frac{2}{\lambda} 2(\frac{1}{2\lambda})^3 \\ &= \frac{1}{2\lambda^4}\end{aligned}$$

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x \rangle + \sigma)$ and $(\langle x \rangle - \sigma)$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

Firstly, $\sigma^2 = \frac{1}{2\lambda^4}$ so $\sigma = \frac{1}{\sqrt{2}\lambda}$. Looking at a graph of $|\Psi|^2$ for $\lambda = 1$ marked at $x = \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}$:



The probability x is greater than $\sqrt{\frac{1}{2}}$ is

$$\int_{0.5}^{\infty} e^{-2x} dx = \left[-\frac{1}{2} e^{-2x}\right]_{(1/\sqrt{2})}^{\infty} = \frac{1}{2e^{\sqrt{2}}} \approx 0.12156$$

And the same value is taken for the probability than x is less than $\sqrt{\frac{1}{2}}$ by symmetry. Thus the probability that $x \notin [\langle x\rangle - \sigma, \langle x\rangle + \sigma]$ is $\frac{1}{e}^{\sqrt{2}} \approx 0.24312$.

3 Problem 1.7

Calculate $\frac{d\langle p\rangle}{dt}$. This should equal $\left\langle -\frac{\partial V}{\partial x}\right\rangle$.

$$\frac{d\langle p\rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \left(\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial^2 \Psi}{\partial t \partial x} \right) dx$$

First simplifying the second term in the integral we find

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial t \partial x} &= \left[\Psi^* \frac{\partial \Psi}{\partial t} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial t} dx \\
&= \int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial t} dx
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d\langle p\rangle}{dt} &= \int_{-\infty}^{\infty} \left[\left(i\hbar \frac{\partial \Psi}{\partial t} \right)^* \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial x}\left(i\hbar \frac{\partial \Psi}{\partial t} \right) \right] dx \\
&= \int_{-\infty}^{\infty} \left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \right)^* \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial x}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \right) \right] dx \\
&= \int_{-\infty}^{\infty} \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial x}\frac{\partial^2 \Psi}{\partial x^2} \right) + V\Psi^* \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial x} V\Psi \right] dx \\
&= \int_{-\infty}^{\infty} \left[-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x} \right) + V\frac{\partial |\Psi|^2}{\partial x} \right] dx \quad \text{(using the product rule)} \\
&= \frac{-\hbar^2}{2m}\left[\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} V\frac{\partial |\Psi|^2}{\partial x} dx \\
&= \int_{-\infty}^{\infty} V\frac{\partial |\Psi|^2}{\partial x} dx \quad \text{(since } \Psi \text{ and its derivatives vanish at infinity)} \\
&= \left[V|\Psi|^2 \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial V}{\partial x}|\Psi|^2 dx \quad \text{(using integration by parts)} \\
&= \left\langle -\frac{\partial V}{\partial x} \right\rangle
\end{aligned}$$

4 Problem 1.9

A particle of mass m has the wave function

$$\Psi(x,t) = Ae^{-a\left[(mx^2/\hbar)+it\right]}$$

(a) Find A.

$$\begin{aligned}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 &= \int_{-\infty}^{\infty} |Ae^{-a\left[(mx^2/\hbar)+it\right]}|^2 dx \\ &= A^2 \int_{-\infty}^{\infty} |e^{-amx^2/\hbar}|^2 |e^{-ait}|^2 dx \\ &= 2A^2 \int_{0}^{\infty} e^{-2amx^2/\hbar} dx \\ &= 2A^2 \int_{0}^{\infty} e^{-x^2/k^2} dx \text{ for } k = \sqrt{\frac{\hbar}{2am}} \\ &= 2A^2 \sqrt{\pi} \frac{k}{2} \\ &= A^2 \sqrt{\frac{\pi\hbar}{2am}} = 1\end{aligned}$$

Thus we have $A = \left(\frac{\pi\hbar}{2am}\right)^{\frac{1}{4}}$.

(b) For what potential energy function, $V(x)$, is this a solution to the Schroedinger Equation?

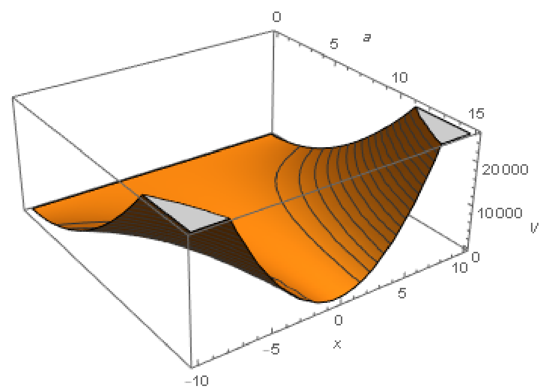
First we compute $\frac{\partial\Psi}{\partial t}, \frac{\partial^2\Psi}{\partial x^2}$:

$$\begin{aligned}\frac{\partial\Psi}{\partial t} &= Ae^{-amx^2/\hbar}\left(-aie^{-ait}\right) = (-ai)\Psi \\ \frac{\partial^2\Psi}{\partial x^2} &= Ae^{-ait}\frac{\partial^2}{\partial t^2}\left(e^{-amx^2/\hbar}\right) \\ &= Ae^{-ait}\left[\left(\frac{-2amx}{\hbar}\right)^2 - \frac{2am}{\hbar}\right] e^{-amx^2/\hbar} \\ &= \left[\left(\frac{2amx}{\hbar}\right)^2 - \frac{2am}{\hbar}\right]\Psi\end{aligned}$$

Plugging into the Schroedinger Equation we get

$$\begin{aligned}i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \\ i\hbar(-ai)\Psi &= -\frac{\hbar^2}{2m}\left[\left(\frac{2amx}{\hbar}\right)^2 - \frac{2am}{\hbar}\right]\Psi + V\Psi \\ a\hbar &= -2ma^2x^2 + a\hbar + V \\ V &= 2ma^2x^2\end{aligned}$$

Graphing this we see that the potential is greatest the further away from the likely regions $x = 0$ and increases quadtratically, proportional to a.



(c) Calculate the expectation values of x, x^2, p, and p^2.

$$|\Psi|^2 = A^2 e^{-2amx^2/\hbar}$$

This is a gaussian distribution with $\langle x \rangle = 0$, $\sigma^2 = \hbar/(4am)$ so

$$\langle x^2 \rangle = \sigma^2 + \langle x \rangle^2 = \hbar/(4am)$$

For $\langle p \rangle$,

$$\begin{aligned}
\langle p \rangle &= -i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x} dx \\
&= -i\hbar \frac{-2am}{\hbar} \int_{-\infty}^{\infty} x \Psi^* \Psi dx \\
&= 0 \text{ since } \langle x \rangle = 0
\end{aligned}$$

For $\langle p^2 \rangle$,

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[\left(-i\hbar \frac{\partial}{\partial x} \right)^2 \right] \Psi dx \\
&= \int_{-\infty}^{\infty} \Psi^* (-\hbar^2) \frac{\partial^2}{\partial x^2} \Psi dx \\
\frac{\partial^2 \Psi}{\partial x^2} &= \frac{-2am}{\hbar} \Psi + \left(\frac{2amx}{\hbar} \right)^2 \Psi \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} 2am\hbar \Psi^* \Psi dx + \int_{-\infty}^{\infty} -4a^2 m^2 x^2 \Psi^* \Psi dx \\
&= 2am\hbar - 4a^2 m^2 \langle x^2 \rangle = 2am\hbar - am\hbar \\
&= am\hbar
\end{aligned}$$

(d) Find σ_p and σ_x. Is their product consistent with the uncertainty principle?

$$\begin{aligned}
\sigma_p^2 &= am\hbar \\
\sigma_x^2 &= \hbar/(4am) \\
\sigma_x\sigma_p &= \sqrt{am\hbar\frac{\hbar}{4am}} = \hbar/2
\end{aligned}$$

Which is the lower bound of the uncertainty inequality! It appears then that Gaussian-like wave functions are those which minimize the uncertainty principle.