Chapter 1 Problems

William Arnold

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Problem 1.1

For the distribution of ages in the example in Section 1.3.1

$$\begin{aligned}
N(14) &= 1 \\
N(15) &= 1 \\
N(16) &= 3 \\
N(22) &= 2 \\
N(24) &= 2 \\
N(25) &= 5
\end{aligned}$$

(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$ Total number of kids is $1+1+3+2+2+5 = 14$.

$$\begin{aligned}
\langle j^2 \rangle &= \frac{1}{14}(14^2 + 15^2) + \frac{2}{14}(22^2 + 24^2) + \frac{3}{14}16^2 + + \frac{5}{14}25^2 \\
&= \frac{3217}{7} \approx 459.57 \\
\langle j \rangle &= \frac{1}{14}(14 + 15) + \frac{2}{14}(22 + 24) + \frac{3}{14}16 + + \frac{5}{14}25 \\
&= 21 \\
\langle j \rangle^2 &= 441
\end{aligned}$$

(b) Determine Δj for each j and use Equation 1.11 to compute the standard deviation

$$\begin{aligned}
\sigma^2 = \langle (\Delta j)^2 \rangle &= \frac{1}{14}((-7)^2 + (-6)^2) + \frac{2}{14}(1^2 + 3^2) + \frac{3}{14}(-5)^2 + + \frac{5}{14}4^2 \\
&= \frac{130}{7}
\end{aligned}$$

(c) Check using part a and part b

$$\begin{aligned}
\sigma^2 &= \langle j^2 \rangle - \langle j \rangle^2 = \frac{130}{7} \\
&= \langle (\Delta j)^2 \rangle
\end{aligned}$$

1 Problem 1.3

Consider the Gaussian Distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

where A, a and λ are positive real constants (necessary integrals are in the back of the book)

(a) Use equation 1.16 to determine A We have that $\int_{-\infty}^{\infty} \rho(x) = 1$ so we have

$$\begin{aligned}
\int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2} dx &= 2A \int_{0}^{\infty} e^{-\lambda x^2} dx \\
&= 2A\sqrt{\pi}\frac{1}{2\sqrt{\lambda}} \\
&= A\sqrt{\frac{\pi}{\lambda}} = 1
\end{aligned}$$

Thus solving for A we get

$$A = \sqrt{\frac{\lambda}{\pi}}$$

(b) Find $\langle x \rangle, \langle x^2 \rangle$, and σ

Let $u = x - a$. Then $\frac{du}{dx} = 1, du = dx$, and

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} Axe^{-\lambda(x-a)^2} dx \\
&= \int_{-\infty}^{\infty} A(u + a)e^{-\lambda u^2} du \\
&= \int_{-\infty}^{\infty} Aue^{-\lambda u^2} du + \int_{-\infty}^{\infty} Aae^{-\lambda u^2} du
\end{aligned}$$

Since the first integral in the sum is integrating an odd function ($ue^{\lambda u^2} = -((-u)e^{\lambda(-u)^2})$), that integral is zero. This leaves

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} Aae^{-\lambda u^2} du \\
&= a\int_{-\infty}^{\infty} Ae^{-\lambda u^2} du \\
&= a
\end{aligned}$$

Using the formula in the back of the book, we have that

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 A e^{-\lambda(x-a)^2} dx \\
&= \int_{-\infty}^{\infty} (u+a)^2 A e^{-\lambda u^2} du \\
&= \int_{-\infty}^{\infty} u^2 A e^{-\lambda u^2} du + \int_{-\infty}^{\infty} 2auAe^{-\lambda u^2} du + \int_{-\infty}^{\infty} a^2 A e^{-\lambda u^2} du \\
&= 4A\sqrt{\pi}(\frac{1}{2\sqrt{\lambda}})^3 + a^2 \\
&= \frac{1}{2}\frac{\sqrt{\lambda}}{\sqrt{\pi}}\sqrt{\pi}\frac{1}{\sqrt{\lambda}^3} + a^2 \\
&= \frac{1}{2\lambda} + a^2
\end{aligned}$$

And lastly,

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda} + a^2 - a^2 = \frac{1}{2\lambda}$$