Eigenvalues and Eigenvectors

Dr. Diana Matrix

April 5, 2025

1 Definition

Let A be an $n \times n$ matrix. A non-zero vector $\mathbf{v}$ is an **eigenvector** of A if:

$$A\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

for some scalar λ, called the **eigenvalue**.

2 Characteristic Polynomial

The eigenvalues of A are the roots of the characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = 0 \tag{2}$$

3 Example: 2×2 Matrix

Consider the matrix:

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \tag{3}$$

The characteristic polynomial is:

$$\begin{align} \det(A - \lambda I) &= \det \begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} \tag{4} \\ &= (3-\lambda)(2-\lambda) - 0 \cdot 1 \tag{5} \\ &= \lambda^2 - 5\lambda + 6 \tag{6} \\ &= (\lambda - 2)(\lambda - 3) \tag{7} \end{align}$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

4 Eigenvectors

For $\lambda_1 = 2$:

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0} \tag{8}$$

This gives us $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

For $\lambda_2 = 3$:

$$(A - 3I)\mathbf{v} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0} \tag{9}$$

This gives us $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

5 Diagonalization

If A has n linearly independent eigenvectors, it can be diagonalized as:

$$A = PDP^{-1} \tag{10}$$

where P contains the eigenvectors and D is a diagonal matrix of eigenvalues.

6 Applications

Eigenvalues and eigenvectors are fundamental in:

- Principal Component Analysis (PCA)
- Quantum mechanics
- Stability analysis of dynamical systems
- Google's PageRank algorithm
- Image compression

7 Conclusion

Eigenvalues and eigenvectors provide deep insights into the structure and behavior of linear transformations.