

# Eigenvalues and Eigenvectors

Dr. Diana Matrix

April 5, 2025

## 1 Definition

Let  $A$  be an  $n \times n$  matrix. A non-zero vector  $\mathbf{v}$  is an **eigenvector** of  $A$  if:

$$A\mathbf{v} = \lambda\mathbf{v} \quad (1)$$

for some scalar  $\lambda$ , called the **eigenvalue**.

## 2 Characteristic Polynomial

The eigenvalues of  $A$  are the roots of the characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = 0 \quad (2)$$

## 3 Example: $2 \times 2$ Matrix

Consider the matrix:

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \quad (3)$$

The characteristic polynomial is:

$$\det(A - \lambda I) = \det \begin{pmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{pmatrix} \quad (4)$$

$$= (3 - \lambda)(2 - \lambda) - 0 \cdot 1 \quad (5)$$

$$= \lambda^2 - 5\lambda + 6 \quad (6)$$

$$= (\lambda - 2)(\lambda - 3) \quad (7)$$

So the eigenvalues are  $\lambda_1 = 2$  and  $\lambda_2 = 3$ .

## 4 Eigenvectors

For  $\lambda_1 = 2$ :

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0} \quad (8)$$

This gives us  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

For  $\lambda_2 = 3$ :

$$(A - 3I)\mathbf{v} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0} \quad (9)$$

This gives us  $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

## 5 Diagonalization

If  $A$  has  $n$  linearly independent eigenvectors, it can be diagonalized as:

$$A = PDP^{-1} \tag{10}$$

where  $P$  contains the eigenvectors and  $D$  is a diagonal matrix of eigenvalues.

## 6 Applications

Eigenvalues and eigenvectors are fundamental in:

- Principal Component Analysis (PCA)
- Quantum mechanics
- Stability analysis of dynamical systems
- Google's PageRank algorithm
- Image compression

## 7 Conclusion

Eigenvalues and eigenvectors provide deep insights into the structure and behavior of linear transformations.