Fundamental Theorem of Calculus

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1 Introduction

The Fundamental Theorem of Calculus connects differentiation and integration, showing that these two operations are essentially inverse processes.

2 The First Fundamental Theorem

If f is continuous on $[a, b]$ and F is defined by:

$$F(x) = \int_a^x f(t)\,dt \tag{1}$$

then F is differentiable on (a, b) and:

$$F'(x) = f(x) \tag{2}$$

3 The Second Fundamental Theorem

If f is continuous on $[a, b]$ and F is any antiderivative of f, then:

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{3}$$

4 Examples

4.1 Example 1

Find the derivative of:

$$g(x) = \int_0^x \sin(t^2)\,dt \tag{4}$$

By the First Fundamental Theorem:

$$g'(x) = \sin(x^2) \tag{5}$$

4.2 Example 2

Evaluate:

$$\int_1^4 (2x + 3)\,dx \tag{6}$$

An antiderivative is $F(x) = x^2 + 3x$, so:

$$\int_1^4 (2x + 3)\,dx = F(4) - F(1) = (16 + 12) - (1 + 3) = 24 \tag{7}$$

5 Applications

The Fundamental Theorem has many applications:

- Computing definite integrals
- Solving differential equations
- Physics problems involving rates of change
- Economics and optimization

6 Proof Sketch

The proof relies on the Mean Value Theorem and the definition of the derivative. The key insight is that:

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\, dt \tag{8}$$

As $h \to 0$, this approaches $f(x)$ by continuity.

7 Conclusion

The Fundamental Theorem of Calculus is one of the most important results in mathematics, providing the foundation for much of modern analysis and its applications.