Notes on Topology and Groups

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1 Introduction

1.1 Definitions

Definition 1.1.1. *Let X be a space and let u and v be paths such that $u(1) = v(0)$. The **composite path** $u.v$ is given by*

$$u.v(t) = \begin{cases} u(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ v(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

2 Graph

Note to self: we only contain notes about graphs that are important from a topological perspective, and less from a number theory / algorithm perspective.

2.1 Definitions

Definition 2.1.1. *A **countable graph** Γ is specified by*

- *A finite or countable set V of **vertices***
- *A finite or countable set E of **edges***
- *A function δ which sends an edge e to a subset of V with either 1 or 2 elements. $\delta(e)$ is known as **endpoints** of e.*

We can construct an associated topological space, or the **graph** Γ as follows. Take a disjoint union of points corresponding to vertices, and a disjoint union of copies of the interval I corresponding to edges. For each $e \in E$, identity 0 in the associated copy of I with one vertex in $\delta(e)$ and 1 with the other vertex of $\delta(e)$.

Definition 2.1.2. *An **orientation** on the graph Γ is a choice of functions $\iota : E \to V$ and $\pi : E \to V$ such that for each $e \in E$, $\delta(e) = \{\iota(e), \pi(e)\}$. We say that $\iota(e)$ and $\pi(e)$ are **intial** and **terminal** vertices of the edge e, and we view the edge as running from the intial vertex to the terminal vertex (in a directed sense).*

Definition 2.1.3. *Let Γ be a graph with vertex set V, edge set E, and endpoint function δ. A **subgraph** of Γ is the vertex set $V' \subseteq V$ and edge set $E' \subseteq E$ with the endpoint function being the restriction of δ. To be well-defined, we need for each $e \in E'$, $\delta(e) \subseteq V'$. If Γ is oriented, then the subgraph inherits the orientation.*

Definition 2.1.4. *An **edge path** in a graph Γ is a concatenation $u_1 \dots u_n$ where each u_i is either a path running along a single edge at unit speed, or a constant path based at a vertex.*

*A **edge loop** is an edge path $u : I \to \Gamma$ where $u(0) = u(1)$.*

*An edge path (respectively, edge loop) is said to be **embedded** if u is injective (respectively, if the only points in I with the same image under u are 0 and 1).*

2.2 Tree

Definition 2.2.1. *A **tree** is a connected graph that contains no embedded edge loops.*

Lemma 2.2.2. *In a tree, there is a unique embedded edge path between distinct vertices.*

Proof. Any two distinct vertices are connected by an edge path, since the tree is connected. A shortest such path is embedded. We wish to show that this is unique.

Suppose for a contradiction there are two distinct embedded edge paths $p = u_1 \dots u_n$ and $p' = u'_1 \dots u'_n$, between a distinct pair of vertices. Let $u_i(0)$ be the point on p where the paths first diverge. Let $u_j(1)$ be the next point on p which lies in the image of p'. Then the concatenation of $u_i \dots u_j$ with the sub-arc of p' between $u_j(1)$ and $u_i(0)$ form an embedded edge loop, a contradiction on the assumption that we have a tree. □

Definition 2.2.3. *A **maximal tree** in a connected graph Γ is a subgraph T that is a tree, but any addition of any edge $E(\Gamma) \setminus E(T)$ to T gives a graph that is not a tree.*

Lemma 2.2.4. *Let Γ be a connected graph and let T be a subgraph that is a tree. Then the following are equivalent :*

1. *$V(T) = V(\Gamma)$*
2. *T is maximal*

Proof. $(i) \Rightarrow (ii)$ Let e be an edge of $E(\Gamma) \setminus E(T)$. If the endpoints of e are the same vertex, then adding e to T gives a subgraph that is not a tree, as it contains an embedded edge loop. Without loss of generality, assume the endpoints of d are distinct. They lie in T, as $V(T) = V(\Gamma)$. They are connected by an embedded edge path p in T by Lemma 2.2.2. Now, $p \cup e$ is an embedded loop in $T \cup e$, thus is not a tree.

$(ii) \Rightarrow (i)$ Suppose that T is a maximal tree and there is a vertex v of Γ that is not in $V(T)$. Pick a shortest edge path from T to v, which exists as Γ is connected. The first edge of this path starts in $V(T)$ but cannot end in $V(T)$. We can therefore add this to T to create a larger tree, which contradicts maximality. □

Lemma 2.2.5. *Any connected graph Γ contains a maximal tree.*

Proof. By definition, $V(\Gamma)$ is finite or countable. We can therefore choose a total ordering on $V(\Gamma)$. Without loss of generality, we may assume that for each $i \geq 2$, the i-th vertex shares an edge with an earlier vertex. We construct a nested sequence of subgraphs $T_1 \subsetneq T_2 \subsetneq \cdots$ of trees where $V(T_i)$ is the first i vertices up to the ordering.

Set T_1 to be the first vertex. By assumption, there is an edge e joining the i-th vertex to one of the previous vertices, so we can set $T_i = T_{i-1} \cup e$. There are no new embedded edge loops, so inductively any T_i is a tree.

We claim that $T = \bigcup_i T_i$ is a tree. Suppose that it contains an embedded edge loop ℓ. Then, as ℓ consists of finitely many edges, they must all appear in T_i, but then T_i is not a tree, a contradiction. As T contains all the vertices of Γ, it is maximal by Lemma 2.2.4. □

2.3 Cayley Graphs

Definition 2.3.1. *Let G be a group and let S be a set of generators for G. The associated **Cayley Graph** is an oriented graph with vertex set G and edge set $G \times S$. Eah edge is associated with a pair (g, s) where $g \in G$ and $s \in S$. The functions ι and π are specified by $\iota(g, s) = g$ and $\pi(g, s) = gs$. We say that this edge is **labelled** by the generator s.*

The Cayley graph of a group depends on a choice of generators. We also note that any two points in a Cayley graph can be joined by a path. Conversely, given any path from the identity to the g vertex, we can write g as a product of generators and their inverses. We therefore have a correspondence between closed loops starting at the identity and ways of writing the identity.

3 Topological Structures

3.1 Simplicial Complexes

Definition 3.1.1. *The* ***standard*** n***-simplex*** *is the set*

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \forall i \sum_i x_i = 1\}$$

The non-negative integer n is the ***dimension*** *of the simplex. The vertices denoted $V(\Delta^n)$ are points $(x_0, \ldots, x_n)$ in Δ^n such that $x_i = 1$ for some i. For each non-empty subset A of $\{0, \ldots, n\}$, there is a* ***face*** *of Δ^n which is*

$$\{(x_0, \ldots, x_n) \in \Delta^n \mid x_i = 0 \ \forall i \notin A\}$$

Note that Δ^n is a face of itself. The ***inside*** *of Δ^n is*

$$\text{inside}(\Delta^n) = \{(x_0, \ldots, x_n) \in \Delta^n \mid x_i > 0 \forall i\}$$

Note that the inside of Δ^0 is Δ^0.

We note that $V(\Delta^n)$ is a basis for $\mathbb{R}^{n+1}$. Thus any function $f : V(\Delta^n) \to \mathbb{R}^m$ extends to a unique linear map $\mathbb{R}^{n+1} \to \mathbb{R}^m$. The restriction of this to Δ^n is known as the **affine extension** of f, or just called affine.

Definition 3.1.2. *A* ***face inclusion*** *of a standard m-simplex into a standard n-simplex where $m < n$ is the affine extension of an injection $V(\Delta^m) \to V(\Delta^n)$.*

Definition 3.1.3. *An* ***abstract simplicial complex*** *is a pair (V, Σ) where V is a set of vertices and Σ is a set of non-empty finite subsets of V called simplices such that*

- *For each $v \in V$, $\{v\} \in \Sigma$*
- *If $\sigma \in \Sigma$, any nonempty subset of σ is also in Σ.*

We say that (V, Σ) is ***finite*** *if V is a finite set.*

Definition 3.1.4. *The* ***topological realisation*** *$|K|$ of an abstract simplicial complex $K = (V, \Sigma)$ is the space obtained by the following procedure:*

1. *For each $\sigma \in \Sigma$, take a copy of the standard n-simplex, where $n+1$ is the number of elements of σ. Denote this simplex Δ_σ, labelling its vertices with elements of σ*
2. *Whenever $\sigma \subsetneq \pi \in \Sigma$, identify Δ_σ with a subset of Δ_π via face inclusion that sends elements of σ to corresponding elements of π.*

Equivalently, it is the quotient space obtained by a disjoint union of simplices in (i) and imposing the equivalence in (ii).

Any point $x \in |K|$ lies inside a unique simplex $\sigma = (v_0, \ldots, v_n)$. Thus it can be expressed as

$$x = \sum_{i=0}^{n} \lambda_i v_i$$

for unique positive numbers $\lambda_0, \ldots, \lambda_n$ that sum to 1. If $V = \{w_0, \ldots, w_m\}$, we write $x = \sum \mu_i w_i$ taking $\mu_i = 0$ if $w_i \notin \{v_0, \ldots, v_n\}$. If $|K|$ is the topological realisation of an abstract simplicial complex K, we denote the images of the vertices in $|K|$ by $V(|K|)$.

Note that when we refer to a **simplicial complex**, we mean either the abstract simplicial complex or its topological realisation.

Definition 3.1.5. *A* ***triangulation*** *of a space X is a simplicial complex K with a choice of homeomorphism $|K| \to X$.*

Example 3.1.6. The torus $S^1 \times S^1$ has a triangulation using nine vertices, using the standard grid.

Definition 3.1.7. *A* ***subcomplex*** *of a simplicial complex (V, Σ) is a simplicial complex (V', Σ') such that $V' \subseteq V$ and $\Sigma' \subseteq \Sigma$.*

Definition 3.1.8. *A* ***simplicial map*** *between abstract simplicial complexes between (V_1, Σ_1) and (V_2, Σ_2) is a function $f : V_1 \to V_2$ such that for all $\sigma_1 \in \Sigma_1$, $f(\sigma_1) = \sigma_2$ for some $\sigma_2 \in \Sigma_2$. It is a* ***simplicial isomorphism*** *if it has a simplicial inverse.*

Note that this map need not be injective, thus may decrease the dimension of a simplex.

Proposition 3.1.9. *A simplicial map f between abstract simplicial complexes K_1 and K_2 induces a continuous map $|f| : |K_1| \to |K_2|$.*

Proof. Define $|f|$ on $V(|K_1|)$ according to f, extending to each simplex using the unique affine extension. □

This map is also called a simplicial map. Note also that this map is determined by the image of its vertices, and is uniquely determined from there.

Definition 3.1.10. *A* ***subdivision*** *of a simplicial complex K is a simplicial complex K' with a homeomorphism $h : |K'| \to |K|$ such that for any simplex σ' of K', $h(\sigma')$ lies entirely in a simplex of $|K|$ and the restriction of h to σ' is affine (linearity on convex combinations).*

Example 3.1.11. Let K be the triangulation of $I \times I$ with a single diagonal from the top left to bottom right. For any positive integer r, let K' be the triangulation of $I \times I$ by dividing $I \times I$ tinto a lattice of r^2 congruent squares, dividing each along the diagonal that runs from top left to bottom right. Then K' is a subdivision of K. We write $(I \times I)_{(r)}$ for this.

Definition 3.1.12. *Let K be a simplicial complex. An* **edge path** *is a finite sequence $(a_0, \dots, a_n)$ of vertices of K such that for each i, $\{a_{i-1}, a_i\}$ spans a simplex of K. The length of the path is n.*

An **edge loop** *is an edge path with $a_n = a_0$. We define concatenation of edge paths in the standard way.*

3.2 Cell complexes

Definition 3.2.1. *Let X be a space, and $f : S^{n-1} \to X$ be a map. The space obtained by attaching an n-cell to X along f is defined to be the quotient of the disjoint union $X \sqcup D^n$ such that for each point $x \in X$, $f^{-1}(x)$ and x are all identified to a point. We denote this by $X \cup_f D^n$.*

Remark 3.2.2. There is a homeomorphic image of both X and the interior of D^n in $X \cup_f D^n$ by the natural map. There is an induced map from $D^n \to X \cup_f D^n$ but this need not be injective, as the points in the boundary of D^n may be identified.

Definition 3.2.3. *A (finite)* **cell complex** *is a space X decomposed as*

$$K^0 \subsetneq K^1 \subsetneq \cdots \subsetneq K^n = X$$

where

1. K^0 *is a finite set of points*

2. K^i *is obtained from* K^{i-1} *by attaching a finite collection of* i*-cells.*

Example 3.2.4. A finite graph is precisely a finite cell complex that consists only of 0-cells and 1-cells.

Remark 3.2.5. Any finite simplicial complex is a finite cell complex by letting each n simplex be an n-cell.

Example 3.2.6. The torus $S^1 \times S^1$ has a cell structure with one 0-cell, two 1-cells and a single 2-cell. Viewing K^1 as a graph, give its two edges an orientation, labelling them a and b. The attaching map $f : S^1 \to K^1$ of the 2-cell sends the circle along the path $aba^{-1}b^{-1}$.

4 Homotopy

4.1 Basic Definitions and Properties

Definition 4.1.1. *A* ***homotopy*** *between two maps* $f, g : X \to Y$ *is a map* $H : X \times I \to Y$ *such that* $H(x, 0) = f(x)$ *and* $H(x, 1) = g(x)$ *for all* $x \in X$*. We say that* f, g *are* ***homotopic*** *and write* $f \simeq g$ *or* $H : f \simeq g$*, or* $f \overset{H}{\simeq} g$*.*

Example 4.1.2. Suppose that Y is a subset of $\mathbb{R}^n$ that is convex. Then for any two maps $f, g : X \to Y$ are homotopic by

$$(x, t) \mapsto (1 - t)f(x) + tg(x)$$

This is known as the straight-line homotopy.

Lemma 4.1.3 (Gluing Lemma)**.** *If* $\{C_1, \ldots, C_n\}$ *is a finite covering of a space* X *by closed subsets and* $f : X \to Y$ *is a function whose restriction to each* C_i *is continuous, then* f *is continuous.*

Proof. The map f is continuous if and only if $f^{-1}(C)$ is closed for each closed subset of Y. But $f^{-1}(C) = \bigcup_{i=1}^{n} f^{-1}(C) \cap C_i$, which is a finite union of closed sets, thus closed. □

Lemma 4.1.4. *For any two spaces* X *and* Y*, homotopy is an equivalence relation of continuous maps* $X \to Y$*.*

Proof. Reflexive: for any $f : X \to Y$, $H : f \simeq f$ by $H(x, t) = f(x)$.
Symmetric: if $H : f \simeq g$, then $\bar{H} : g \simeq f$ where $\bar{H}(x, t) = H(x, 1 - t)$.
Transitive: if $H : f \simeq g$ and $K : g \simeq h$, then $L : f \simeq h$ via

$$L(x, t) = \begin{cases} H(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ K(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

L is continuous by the gluing lemma. □

Remark 4.1.5. If we take X to be a single point, the continuous maps $X \to Y$ are points of Y, thus homotopies between them are paths. So the relation of being connected by a path is an equivalence relation on Y. These equivalence classes are called **path-components** of Y. If Y has a single path-component, we call is **path-connected**.

Lemma 4.1.6. *Given the following continuous maps :*

$$W \xrightarrow{f} X \underset{h}{\overset{g}{\rightrightarrows}} Y \xrightarrow{k} Z$$

If $g \simeq h$, then $gf \simeq hf$ and $kg \simeq kh$.

Proof. Let H be the homotopy between g and h. Then $k \circ H : X \times I \to Z$ is a homotopy between kg and kh.

Similarly, $H \circ (f \times \mathrm{id}_I) : W \times I \to Y$ is a homotopy between gf and hf. □

Definition 4.1.7. *Two spaces X and Y are* ***homotopy equivalent*** *written $X \simeq Y$ if there are maps*

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$.

Lemma 4.1.8. *Homotopy equivalence is an equivalence relation on spaces.*

Proof. Reflexivity and symmetry are straightforward. For transitivity, consider the following maps:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \underset{k}{\overset{h}{\rightleftarrows}} Z$$

where fg, gf, hk, kh are all homotopic to the relavant identity map. Then by Lemma 4.1.6, $gkhf \simeq g(\mathrm{id}_Y)f = gf \simeq \mathrm{id}_X$. So, $(gk)(hf) \simeq \mathrm{id}_X$, and similarly $(hf)(gk) \simeq \mathrm{id}_Z$. □

Definition 4.1.9. *A space X is* ***contractible*** *if it is homotopy equivalent to the space with one point.*

There is a unique map $X \to \{*\}$ and any map $\{*\} \to X$ sends $*$ to some point $x \in X$. Then $\{*\} \to X \to \{*\}$ is the identity, and $X \to \{*\} \to X$ is the constant map c_x. Hence X is contractible if and only if $\mathrm{id}_X \simeq c_x$ for some $x \in X$.

Example 4.1.10. If X is a convex subspace of $\mathbb{R}^n$, then for any $x \in X$, $c_x \simeq \mathrm{id}_X$ by the straight-line homotopy. Hence X is contractible. In particular, $\mathbb{R}^n$ and D^n are both contractible.

Definition 4.1.11. *When A is a subspace of a space X and $\iota : A \to X$ is the inclusion map, we say that a map $r : X \to A$ such that $ri = \mathrm{id}_A$ and $ir \simeq \mathrm{id}_X$ is a* ***homotopy retract****. In these circumstances, A and X are homotopy equivalent.*

Example 4.1.12. Let $\iota : S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ be the inclusion map, and define

$$r(x) = x/|x|$$

Then $ri = \mathrm{id}_{S^{n-1}}$ and $H : ir \simeq \mathrm{id}_{\mathbb{R}^n \backslash \{0\}}$ by

$$H(x,t) = tx + (1-t)x/|x|$$

This is well-defined as the straight line between x and $x/|x|$ does not go through the origin. Thus r is a homotopy retract and our equivalence follows.

Example 4.1.13. Let M denote the Möbius band. There is an inclusion map $\iota : S^1 \to M$ sending $e^{2\pi ix}$ to $(x, \frac{1}{2})$. There is a retraction map sending $(x, y) \mapsto (x, \frac{1}{2})$. Then r is a homotopy retract via the straight-line homotopy.

Similarly, $S^1 \times \{\frac{1}{2}\}$ is a homotopy retract of $S^1 \times I$. Hence $M \simeq S^1 \simeq S^1 \times I$.

Definition 4.1.14. *let X and Y be spaces and let A be a subspaces of X. Then two maps $f, g : X \to Y$ are **homotopic relative** to A if $f|_A = g|_A$ and there is a homotopy $H : f \simeq g$ such that $H(x, t) = f(x) = g(x)$ for all $x \in A$ and $t \in I$.*

Remark 4.1.15. With a similar notion to homotopy equivalence, there is closure under composition and is an equivalence relation. (Proof is similar.)

4.2 The Simplicial Approximation Theorem

Definition 4.2.1. *Let K be a simplicial complex, and let x be a point in $|K|$. The **star** of x in $|K|$ is the following subset of $|K|$,*

$$\mathrm{st}_K(x) = \bigcup\{\mathrm{inside}(\sigma) \mid \sigma \textit{ is a simplex of } |K| \textit{ and } x \in \sigma\}$$

Lemma 4.2.2. *For any $x \in |K|$, $\mathrm{st}_K(x)$ is open in $|K|$.*

Proof. Consider

$$\begin{aligned} |K| - \mathrm{st}_K(x) &= \bigcup\{\mathrm{inside}(\sigma) \mid \sigma \text{ is a simplex of } |K| \text{ and } x \notin \sigma\} \\ &= \bigcup\{\sigma \mid \sigma \text{ is a simplex of } |K| \text{ and } x \notin \sigma\} \end{aligned}$$

The second equality holds as any point lies in a simplex lies in the inside of some face τ of σ, and $x \notin \sigma$ implies $x \notin \tau$. Now the latter is a subcomplex of K. This is closed, thus $\mathrm{st}_K(x)$ is open. □

Proposition 4.2.3. *Let K and L be simplicial complexes, and let $f : |K| \to |L|$ be a continuous map. Suppose that for each vertex v of K, there is a vertex $g(v)$ of L such that $f(\mathrm{st}_K(v)) \subseteq \mathrm{st}_L(g(v))$. Then g is a simplicial map $V(K) \to V(L)$ and $|g| \simeq f$.*

Proof. First we claim the following. Let $\sigma = (v_0, \ldots, v_n)$ be a simplex of K and let $x \in \mathrm{inside}(\sigma)$. Let τ be the simplex of L such that $f(x)$ lies in the inside of τ. Then $g(v_0), \ldots, g(v_n)$ are vertices of τ.

Since x lies in the inside of τ, it is in $\mathrm{st}_K(v_i)$ for each i. So $f(x) \in f(\mathrm{st}_K(v_i)) \subseteq \mathrm{st}_L(g(v_i))$. Therefore the inside of τ lies in $\mathrm{st}_L(g(v_i))$. Thus $g(v_i)$ is a vertex of τ.

Now, as $g(v_0), \ldots, g(v_n)$ are vertices of τ, they span a simplex which is a face of τ and hence a member of L. Thus g is a simplicial map.

We show homotopy between f and $|g|$ as follows. First consider any $x \in K$. Let τ be a simplex of L that contains $f(x)$ in its inside. Write $x = \sum_{i=0}^n \lambda_i v_i$ where $v_0, \ldots, v_n$ are vertices of the same simplex with $\lambda_i \geq 0$, summing to 1. In particular, $|g|(x) = \sum_{i=0}^n \lambda_i g(v_i)$. The vertices $g(v_0), \ldots, g(v_n)$ are all vertices of τ. Thus, we may define a straight-line homotopy in τ that interpolates between $f(x)$ and $|g|(x)$. This is well-defined, as even though x may lie in several simplices, they all give the same point $H(x, t)$ for all $t \in I$.

H is continuous, as the map agrees on overlapping starts of simplices, and thus follows from the gluing lemma. □

Proposition 4.2.4. *Let K, L, f, g be as in the previous proposition. Let A be any subcomplex of K and let B be a subcomplex of L such that $f(|A|) \subseteq |B|$. Then g also maps A into B and the homotopy between $|g|$ and f sends $|A|$ to $|B|$ throughout.*

Proof. Let v be any vertex of A. Let τ be the simplex of L such that $f(v)$ lies in the inside of τ. Then by the claim above, $g(v)$ is a vertex of τ. Since $f(v) \in |B|$, we deduce that τ lies in $|B|$, and hence $g(v)$ is a vertex of B.

Now consider any point x in $|A|$. Let $(v_0, \ldots, v_n)$ be the simplex of K containing x in its inside. Let τ' be the simplex of L such that $f(x)$ lies in the inside of τ'. Then τ' lies in B as $f(x)$ lies in $|B|$. By the first claim in Proposition 4.2.3, $g(v_0), \ldots, g(v_n)$ must all be vertices of τ', and hence vertices of B. The straight-line homotopy between f and $|g|$ sends x into τ' throughout, and hence the image of x remains in $|B|$. □

Definition 4.2.5. *The **standard metric** d on a finite simplicial complex $|K|$ is defined to be*

$$d(\sum_i \lambda_i v_i, \sum_i \lambda_i' v_i) = \sum_i |\lambda_i - \lambda_i'|$$

Note this is an actual metric on $|K|$.

Definition 4.2.6. *Let K' be the subdivision on K, and let d be the standard metric on $|K|$. The **coarseness** of the subdivision is*

$$\sup\{d(x, y) \mid x \textit{ and } y \textit{ belong to the star of the same vertex of } K'\}$$

Example 4.2.7. The subdivision $(I \times I)_{(r)}$ has coarseness $4/r$ (by the standard metric).

Theorem 4.2.8 (Lebesgue Covering Theorem)**.** *Let X be a complact metric space, and let $\mathcal{U}$ be an open covering of X. Then there is a constant $\delta > 0$ such that every subset of X with diameter less than δ is entirely contained within some member of $\mathcal{U}$.*

Proof. For each $x \in X$, we can find an open $\mathcal{U}_x$ such that $x \in \mathcal{U}_x$. By openness of this, we can find a r_x such that $x \in B(x, r_x) \subseteq \mathcal{U}_x$.

Take the set $B(x, r_x/2)$, which covers X. By compactness, a finite set of balls with x_i that covers X. Take the minimum $r_{x_i}/2$ of this set and set it as δ.

Now, take any subset A of X with diameter less than δ. Pick any $a \in A$ and find the corresponding $B(x_i, r_{x_i}/2)$ such that $a \in B(x_i, r_{x_i}/2)$. By the triangle inequality, any $a' \in A$ is contained in $B(x_i, r_{x_i}) \subseteq \mathcal{U}_{x_i}$. □

Theorem 4.2.9 (Simplicial Approximation Theorem (Variant 1))**.** *Let K and L be simplicial complexes where K is finite, and let $f : |K| \to |L|$ be a continuous map. Then, there is a constant $\delta > 0$ with the following property. If K' is a subdivision of K with coarseness less than δ, then there is a simplicial map $g : K' \to L$ such that $|g| \simeq f$.*

Proof. The sets $\{\mathrm{st}_L(w) \mid w \text{ is a vetex of } L\}$ form an open covering of $|L|$, and so the sets $\{f^{-1}(\mathrm{st}_L(w))\}$ form an open covering of $|K|$. Let $\delta > 0$ be the constant from the Lebesgue Covering Theorem for this covering, and let K' be a subdivision of K with coarseness less than δ.

Then, for any vertex v of K', $\mathrm{diam}(\mathrm{st}'_K(v)) \leq \delta$. In particular, there is some vertex w of L such that $\mathrm{st}'_K(v) \subseteq f^{-1}(\mathrm{st}_L(w))$. Hence $f(\mathrm{st}'_K(v)) \subseteq \mathrm{st}_L(w)$. Setting $g(v) = w$ and applying Proposition 4.2.3 gives the claim. □

Proposition 4.2.10. *Let $A_1, \dots, A_n$ be subcomplexes of K and let $B_1, \dots, B_n$ be subcomplexes of L such that $f(A_i) \subseteq B_i$ for each i. Then the simplicial map $g : V(K') \to V(L)$ by the above sends A_i to B_i and the homotopy between f and $|g|$ sends A_i to B_i throughout.*

Proof. A simple consequence from Proposition 4.2.4. □

Definition 4.2.11. *Let $K = (V, \Sigma)$ be an abstract simplicial complex. Then its **barycentric subdivision** $K^{(1)} = (V', \Sigma')$ defined by $V' = \Sigma$ and Σ' specified by the following rule : $(\sigma_0, \dots, \sigma_n) \in \Sigma'$ if and only if (after possible reordering) $\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n$.*

For each $r \geq 2$, the subdivision $K^{(r)}$ is given by setting $(K^{(r-1)})^{(1)}$.

Proposition 4.2.12. *A finite simplicial complex K has subdivisions $K^{(r)}$ such that the coarseness of $K^{(r)}$ tends to 0 as $r \to \infty$.*

Proof. (Sketch) Without loss of generality, we may consider the K to be the standard n-simplex, as we can perform the operation on this simplex on all the simplices of K simultaneously.

In the reduced case, for each face F of Δ^n with vertices $v_1, \dots, v_r$, the barycenter of F is $(v_1 + \cdots + v_r)/r$. define a new simplicial complex K' with vertices precisely the barycenters of each of the faces. A set of vertices $w_1, \dots, w_s$ of K' corresponding to faces $F_1, \dots, F_s$ of Δ^n span a simplex of K' if and there are (up to re-ordering), there are inclusions $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_s$. This is a subdivision of Δ^n.

We finally note that the coarseness of this tends to 0 as r tends to infinity. □

Theorem 4.2.13 (Simplicial Approximation Theorem (Variant 2))**.** *Let K and L be simplicial complexes where K is finite, and let $f : |K| \to |L|$ be a continuous map. Then there is some subdivision K' of K and a simplicial map $g : K' \to L$ such that $|g|$ is homotopic to f.*

Proof. Follows from Theorem 4.2.9 and barycentric subdivision makes the coarseness of $K^{(r)}$ tend to 0 as $r \to \infty$. □

5 Groups

5.1 Free Group

Definition 5.1.1. *Given any set S, define S^{-1} to be a copy of S, where each element $x \in S$ is given a corresponding element of S^{-1} by x^{-1}. We note that $S \cap S^{-1} = \emptyset$, and that given $x^{-1} \in S^{-1}, (x^{-1})^{-1} = x$.*

Definition 5.1.2. *A **word** w is a finite sequence $x_1, \dots, x_m$ where $m \in \mathbb{Z}_{\geq 0}$ and each $x_i \in S \cup S^{-1}$. We write w as $x_1 x_2 \dots x_m$. The empty sequence given when $m = 0$ is denoted $\emptyset$.*

Definition 5.1.3. *The **concatenation** of two words $x_1 x_2 \dots x_m$ and $y_1 y_2 \dots y_n$ is the word $x_1 x_2 \dots x_m y_1 y_2 \dots y_n$.*

Definition 5.1.4. *A word w' is an **elementary contraction** of a word w, written $w \searrow w'$, if $w = y_1 x x^{-1} y_2$ and $w' = y_1 y_2$ for words y_1 and y_2, and $x \in S \cup S^{-1}$. We also write $w' \nearrow w$ and say that w is an **elementary expansion** of w'.*

Definition 5.1.5. *Two words w' and w are equivalent, written $w \sim w'$ if there are words $w_1, \dots, w_n$ where $w = w_1$ and $w' = w_n$ such that for each i, $w_i \nearrow w_{i+1}$ or $w_i \searrow w_{i+1}$. The equivalence class of a word is denoted $[w]$.*

Definition 5.1.6. *The* ***free group*** *on the set S, denoted $F(S)$ consists of equivalence classes of words in the alphabet S. The composition of two elements $[w]$ and $[w']$ is the class $[ww']$. The identity element is $[\emptyset]$, denoted e. The inverse of an element $[x_1x_2 \dots x_n]$ is $[x_n^{-1} \dots x_2^{-1}x_1^{-1}]$.*

Note that composition is well-defined, and is clear from definitions.

Definition 5.1.7. *A word is* ***reduced*** *if it does not admit an elementary contraction.*

Lemma 5.1.8. *Let w_1, w_2, w_3 be words such that $w_1 \nearrow w_2 \searrow w_3$. Then there is a word w_2' such that $w_1 \searrow w_2' \nearrow w_3$, or $w_1 = w_3$.*

Definition 5.1.9. *Since $w_1 \nearrow w_2$, we can write $w_1 = ab$ and $w_2 = axx^{-1}b$ for some $x \in S \cup S^{-1}$, and words a, b. As $w_2 \searrow w_3$, w_3 is obtained from w_2 by removing yy^{-1} for some $y \in S \cup S^{-1}$. The letters xx^{-1} and yy^{-1} intersect in either zero, one, or two letters. We do a case split.*

If they do not intersect, then we can remove yy^{-1} from w_1 Hence, denoting w_2' to be such a word, we have $w_1 \searrow w_2' \nearrow w_3$. If they intersect at a single letter, $x = y^{-1}$, so w_2 has a chain of letters $xx^{-1}x$ or $x^{-1}xx^{-1}$, and w_1, w_3 are obtained by performing an elementary contraction on these letters. Thus, $w_1 = w_3$. If they intersect in two letters, then we obviously have $w_1 = w_3$.

Proposition 5.1.10. *Any element of a free group $F(S)$ is represented by a unique reduced word.*

Proof. An elementary contraction to a word reduced the length by two. Thus, a shortest representative for an element of $F(S)$ must be reduced. We show that this representative is unique. Suppose there are two distinct words w and w' that are equivalent. Then by definition, we can find a sequence of words $w_1, \dots w_n$ such that $w = w_1$ and $w' = w_n$ and $w_i \nearrow w_{i+1}$ or $w_i \searrow w_{i+1}$ for all i. Consider a shortest such sequence. Then, we must have w_i distinct. Suppose that at some point we have $w_i \nearrow w_{i+1} \searrow w_{i+2}$. Then by Lemma 5.1.8, we can find a w_{i+1}' such that $w_i \searrow w_{i+1}' \nearrow w_{i+2}$. Repeating this, we can perform all $\searrow$ moves before $\nearrow$ ones. Thus, the sequence starts with $w_1 \searrow w_2$ or ends with $w_{n-1} \nearrow w_n$. This implies either w or w' was not reduced, a contradiction. ∎

Theorem 5.1.11 (Universal Property on Free Groups)**.** *Given any set S and group G and any function $f : S \to G$, there is a unique homomorphism $\phi : F(S) \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\iota"'] & G \\ F(S) \arrow[ur, "\phi"'] & \end{tikzcd}$$

where $\iota : S \to F(S)$ is the canonical inclusion.

Proof. We first show existence. Consider any word $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$, where $x_i \in S$ and $\epsilon_i \in \{-1, 1\}$. Define $\phi(w)$ to be $f(x_1)^{\epsilon_1} \dots f(x_n)^{\epsilon_n}$. To show that this is well-defined, given $w \sim w'$, they must have the same image under ϕ. It suffices to show that this is the case when w' is an elementary contraction of w, where $w = w_1xx^{-1}w_2$ or $w = w_1x^{-1}xw_2$ and $w' = w_1w_2$. In the case where $w = w_1xx^{-1}w_2$,

$$\phi(w) = \phi(w_1)f(x)f^{-1}(x)\phi(w_2) = \phi(w_1)\phi(w_2) = \phi(w')$$

The second case is similar. Thus ϕ is well-defined, and is clearly a homomorphism.

Note this map is unique, as if $x \in S$, $\phi(x) = f(x)$, and the fact ϕ is a homomorphism is determined by the map of the generators. ∎

5.2 Group Presentations

Definition 5.2.1. *Let B be a subset of a group G. The normal subgroup generated by B is the intersection of all normal subgroups of G that contain B. We write $\langle\langle B\rangle\rangle$ for this.*

Remark 5.2.2. The intersection of a collection of normal subgroups is a normal subgroup. Thus $\langle\langle B\rangle\rangle$ is normal in G. It is therefore also the smallest normal subgroup of G that contains B.

Proposition 5.2.3. *The subgroup $\langle\langle B\rangle\rangle$ consists of all expressions of the form*

$$\prod_{i=1}^{n} g_i b_i^{\epsilon_i} g_i^{-1}$$

where $n \in \mathbb{Z}_{\geq \not{\vdash}}, g_i \in G, b_i \in B$ and $\epsilon_i = \pm 1$ for all i.

Proof. Any normal subgroup containing B must contain all the elements of the form gbg^{-1} and $gb^{-1}g^{-1}$. Thus it must also contain a finite product of them. Taking N to be the set of all these finite products, we certainly have $N \subseteq \langle\langle B\rangle\rangle$. It remains to show that N is a normal subgroup, as we have $B \subseteq N$, giving $\langle\langle B\rangle\rangle \subseteq N$.

Identity, inverse, and closure are straightforward. To show normality, note that

$$g\left(\prod_{i=1}^{n} g_i b_i^{\epsilon_i} g_i^{-1}\right) g^{-1} = \prod_{i=1}^{n} g g_i b_i^{\epsilon_i} g_i^{-1} g^{-1} = \prod_{i=1}^{n} (g g_i) b_i^{\epsilon_i} (g g_i)^{-1}$$

which lies in N. □

Definition 5.2.4. *Let X be a set, and let R be a collection of elements of $F(X)$. The group with presentation $\langle X|R\rangle$ is defined to be $F(X)/\langle\langle R\rangle\rangle$. We slightly abuse notation by allowing relations of the form $w_1 = w_2$, which is shorthand for $w_1 w_2^{-1}$.*

Therefore, two words in the alphabet represent the same element of G precisely when there is an element $y \in \langle\langle R\rangle\rangle$ such that $w' = wy$.

Example 5.2.5. The dihedral group D_{2n} can be written as

$$\langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma \rangle$$

Proposition 5.2.6. *Let $G = \langle X|R\rangle$. Then two words w, w' in X represent the same element of G if and only if they differ by a finite sequence of the following moves*

1. *perform an elementary contraction or expansion*
2. *insert in the word one of the relations in R or its inverse*

Proof. Applying the moves does not change the element of G that it represents. To show that if w and w' represent the same elements of G, they differ by a finite sequence of moves. In particular, as elements of $F(X)$ have the equality $w' = wy$, we can write

$$w' = w \prod_{i=1}^{n} g_i r_i^{\epsilon_i} g_i^{-1}$$

We can obtain $w g_1 g_1^{-1}$ by the first move, then obtain $w g_1 r_1^{\epsilon_1} g_1^{-1}$ by the second move. Continuing, we can obtain w' from w. □

Example 5.2.7. We can turn $\tau\sigma^n\tau$ into e by the moves as follows :

$$\tau\sigma^n\tau \to \tau\sigma^n\sigma^{-n}\tau \to \tau\tau \to \tau^2\tau^{-2} \to e$$

Proposition 5.2.8. *Every group G has a presentation.*

Proof. Let $F(G)$ be the free group on the generating set G. Then $F(G)$ consists of all equivalence classes of words in G. Thus, if x_1 and x_2 are nontrivial elements of G and $x_3 = x_1x_2$ in G, $[x_3]$ and $[x_1][x_2]$ represent distinct elements of $F(G)$, as they are non-equivalent words in the alphabet G. We have a well-defined homomorphism from $F(G)$ to G, sending each generator of $F(G)$ to the corresponding element of G, which is clearly surjective. Let $R(G)$ be the kernel of this homomorphism. Then, by the first isomorphism theorem, we have $G \simeq F(G)/R(G)$. In particular G has presentation $\langle G|R(G)\rangle$. $\square$

Definition 5.2.9. *The canonical presentation for G is $\langle G|R(G)\rangle$.*

Definition 5.2.10. *A presentation $\langle X|R\rangle$ is **finite** if X and R are both finite sets. A group is **finitely presented** if it has a finite presentation.*

Lemma 5.2.11. *Let $\langle X|R\rangle$ and H both be groups. Let $f : X \to H$ induce a homomorphism $\phi : F(X) \to H$. This descends to a homomorphism $\langle X|R\rangle \to H$ if and only if $\phi(r) = e$ for all $r \in R$.*

Proof. Note that $\phi(r) = e$ is a necessary condition for ϕ to be a homomorphism, as any $r \in R$ represents the identity element of $\langle X|R\rangle$.

Conversely, if $\phi(r) = e$ for all $r \in R$, we note that any element w of $\langle\langle R\rangle\rangle$ can be written as

$$\prod_{i=1}^{n} w_i r_i^{\epsilon_i} w_i^{-1}$$

for $w_i \in F(X), r_i \in R$. As $\phi(r) = e$, we have $\phi(w) = e$. In particular, ϕ descends to a homomorphism $F(X)/\langle\langle R\rangle\rangle$. $\square$

Definition 5.2.12. *A **Tietze Transformation** is one of the following moves applied to a finite presentation $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_n\rangle$*

1. *Re-order generators or relations*
2. *Add or remove the relation e*
3. *Perform an elementary contraction or expansion to a relation r_i*
4. *Insert a relation r_i or its inverse into one of the other r_j or remove it*
5. *Add a generator x_{m+1} together with a relation $w(x_1, \ldots, x_m)x_{m+1}^{-1}$, which defines it as a word in the old generators, or perform the reverse of this operation*

Note first that these transformations don't alter the group, and are also invertible.

Lemma 5.2.13. *Let $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_n\rangle$ be a presentation for a group G. Suppose that a word w in the generators $x_1, \ldots, x_m$ is trivial in G. Then there is a sequence of (ii), (iii), and (iv) moves that transforms this presentation to $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_n, w\rangle$.*

Proof. Start by adding the relation e. As w is trivial in G, we can write get to this element via moves (iii) and (iv). $\square$

Theorem 5.2.14 (Tietze)**.** *Any two finite presentations of a group G are convertible into each other by a finite sequence of Tietze transformations.*

Proof. Let $\langle x_1, \dots, x_m \mid r_1, \dots, r_n\rangle$ and $\langle x'_1, \dots, x'_{m'} \mid r'_1, \dots, r'_{n'}\rangle$ be two presentations of G. Since each x'_i is an element of $\langle x_1, \dots, x_m \mid r_1, \dots, r_n\rangle$, it can be written as a word χ'_i in the generators $x_1, \dots, x_m$. Similarly each x_i can be written as a word χ_i in the generators $x'_1, \dots, x'_{m'}$.

We start by applying move (v) m' times to the first presentation to obtain

$$\langle x_1, \dots, x_m x'_1, \dots, x'_{m'} \mid r_1, \dots, r_n, \chi'_1(x'_1)^{-1}, \dots, \chi'_{m'}(x'_{m'})^{-1}\rangle$$

As the relation $x_i = \chi_i$ holds in the group, by Lemma 5.2.13, we can use moves $(ii) \sim (iv)$ to transform this into

$$\langle x_1, \dots, x_m x'_1, \dots, x'_{m'} \mid r_1, \dots, r_n, \chi'_1(x'_1)^{-1}, \dots, \chi'_{m'}(x'_{m'})^{-1}, \chi_1(x_1^{-1}), \dots, \chi_m(x_m^{-1})\rangle$$

Now, the relations $r'_1, \dots, r'_n$ also represent trivial words in the group. Thus by Lemma 5.2.13 again, we transform this into

$$\langle x_1, \dots, x_m x'_1, \dots, x'_{m'} \mid r_1, \dots, r_n, r'_1, \dots r'_{n'} \chi'_1(x'_1)^{-1}, \dots, \chi'_{m'}(x'_{m'})^{-1}, \chi_1(x_1^{-1}), \dots, \chi_m(x_m^{-1})\rangle$$

This presentation is symmetric with respect to primed and unprimed symbols up to reordering. Thus by applying (i) moves and then reversing the derivation, we can obtain the presentation

$$\langle x_1, \dots, x_m \mid r_1, \dots, r_n\rangle$$

$\square$

Proposition 5.2.15. *Given $G_1 = \langle X_1 \mid R_1\rangle$ and $G_2 = \langle X_2 \mid R_2\rangle$, then we have*

$$G_1 \times G_2 \simeq \langle X_1 \sqcup X_2 \mid R_1 \cup R_2 \cup \{xyx^{-1}y^{-1} \mid x \in X_1, y \in X_2\}\rangle$$

Proof. Let $H := F(X_1 \sqcup X_2)/\langle\langle R_1, R_2, [x, y]\rangle\rangle$. Take

$$\tilde{\phi}(x) = \begin{cases} (x, 1) & x \in X_1 \\ (1, x) & x \in X_2 \end{cases}$$

each relator is mapped to the identity, so $\tilde{\phi}$ descends to a well-defined homomorphism ϕ. This is by construction surjective, and is injective as if we take any x in the kernel of ϕ, projecting onto G_1 and G_2 shows they are trivial in each side. Thus x is trivial as we can permute letters inside H. $\square$

Example 5.2.16. D_{2n} has presentation $G := \langle a, b \mid a^n, b^2, abab\rangle$.

We give an explicit surjection from this group to D_{2n} and argue by size. specifically, note that sending $a \mapsto r$ and $b \mapsto s$, labelling rotation by r and reflection by s shows that each relator is sent to the identity by this map. In particular, we have a well-defined homomorphism to D_{2n}. We can reduce every symmetry of an n-gon to either r^k or sr^k, which is hit by words in G. By the same argument, this shows that $|G| \leq 2n$, thus the map is an isomorphism.

Proposition 5.2.17. *Every nontrivial word in the free group has infinite order.*

Proof. We first note that every nontrivial word is conjugate to a nontrivial reduced word such that the last letter is not the inverse of the first (by peeling off these elements, and this equals e if and only if the word itself is trivial). We call these words cyclically reduced for short. Now, conjugate words have the same order, and these words clearly have infinite order by a length argument. $\square$

Proposition 5.2.18. *If S has multiple elements, the center of $F(S)$ is the identity element.*

Proof. Let $|S| \geq 2$ and $a, b \in S$ be generators. Suppose $z \in Z(F(S))$. Then we have $za = az$ and $zb = bz$, each of which forces the first character of z to be a or a^{-1} and b or b^{-1}, which is impossible. $\square$

5.3 Push-out

Definition 5.3.1. *Let G_0, G_1, G_2 be groups and let $\phi_1 : G_0 \to G_1$ and $\phi_2 : G_0 \to G_2$ be homomorphisms. Let $\langle X_1 \mid R_1 \rangle$ and $\langle X_2 \mid R_2 \rangle$ be canonical presentations of G_1 and G_2, where $X_1 \cap X_2 = \emptyset$. We define the* ***push-out*** *$G_1 *_{G_0} G_2$ of*

$$G_1 \overset{\phi_1}{\leftarrow} G_0 \overset{\phi_2}{\rightarrow} G_2$$

to be the group

$$\langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\phi_1(g) = \phi_2(g) \mid g \in G_0\} \rangle$$

Note that the pushout depends on the homomorphism, but is not ambiguous up to isomorphism when ϕ_1 and ϕ_2 are injections (by viewing G_0 to be a subgroup of both G_1 and G_2).

Remark 5.3.2. The inclusions $X_1 \to X_1 \cup X_2$ and $X_2 \to X_1 \cup X_2$ induces canonical homomorphisms $\alpha_1 : G_1 \to G_1 *_{G_0} G_2$ and $\alpha_2 : G_2 \to G_1 *_{G_0} G_2$, such that the following diagrams commutes:

$$\begin{tikzcd}
 & G_0 \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\
G_1 \arrow[dr, "\alpha_1"'] & & G_2 \arrow[dl, "\alpha_2"] \\
 & G_1 *_{G_0} G_2 &
\end{tikzcd}$$

This is because the relation $\phi_1(g) = \phi_2(g)$ holds for each $g \in G_0$ holds in $G_1 *_{G_0} G_2$.

Proposition 5.3.3 (Universal Property of Pushouts)**.** *Let $G_1 *_{G_0} G_2$ be the pushout induced by $\phi_1 : G_0 \to G_1$ and $\phi_2 : G_0 \to G_2$. Let $\beta_1 : G_1 \to H$ and $\beta_2 : G_2 \to H$ be homomorphisms such that the following commutes:*

$$\begin{tikzcd}
 & G_0 \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\
G_1 \arrow[dr, "\beta_1"'] & & G_2 \arrow[dl, "\beta_2"] \\
 & H &
\end{tikzcd}$$

Then there exists a unique homomorphism $\beta : G_1 *_{G_0} G_2 \to H$ such that the following commutes:

$$\begin{tikzcd}
G_1 \arrow[r, "\alpha_1"] \arrow[dr, "\beta_1"'] & G_1 *_{G_0} G_2 \arrow[d, dashed, "\beta"] & G_2 \arrow[l, "\alpha_2"'] \arrow[dl, "\beta_2"] \\
& H &
\end{tikzcd}$$

Proof. The pushout $G_1 *_{G_0} G_2$ has generators $G_1 \cup G_2$. Define β on these generators by $\beta(g_i) = \beta_i(g_i)$ for all $g_i \in G_i$. Note that this is forced by the commutativity of the second diagram. Thus if the homomorphism exists, it is unique. To show this map is well-defined, it must send relations r in $G_1 *_{G_0} G_2$ with $\beta(r) = e$. This is true for relations in G_1 and G_2, as β_1 and β_2 are well-defined homomorphisms. The other type of relation is $\phi_1(g)(\phi_2(g))^{-1}$ for $g \in G_0$. But $(\beta(\phi_1(g)))(\beta(\phi_2(g)))^{-1} = e$ by the commutativity of the first diagram. $\square$

Lemma 5.3.4. *Let $G_0, G_1, G_2, \phi_1, \phi_2$ be as in the definition for pushouts. Let $\langle X'_1 \mid R'_1\rangle$ and $\langle X'_2 \mid R'_2\rangle$ be any presentations for G_1 and G_2 where $X'_1 \cap X'_2 = \emptyset$. Then the pushout is isomorphic to*

$$H := \langle X'_1 \cup X'_2 \mid R'_1 \cup R'_2 \cup \{\phi_1(g) = \phi_2(g) \mid g \in G_0\}\rangle$$

Proof. Let $G := G_1 *_{G_0} G_2$ be the pushout. Define $G_1 \to \langle X'_1 \mid R'_1\rangle$ and $G_2 \to \langle X'_2 \mid R'_2\rangle$ be the 'identity' maps. The inclusions of X'_1 and X'_2 induce homomorphisms $\beta_1 : G_1 \to H$ and $\beta_2 : G_2 \to H$. This gives a commutative diagram as in Proposition 5.3.3, as the relation $\phi_1(g) = \phi_2(g)$ holds in H. Thus by the same proposition, we have a homomorphism $\beta : G \to H$ with commutative properties.

Now note that there is a function $X'_i \to G_i$ sending each generator to the corresponding element of G_i. Composing this map with α_i to give a function $f : X'_1 \cup X'_2 \to G$. This induces a homomorphism $\phi : F(X'_1 \cup X'_2) \to G$, why descends to a homomorphism $\phi : H \to G$ as $\phi(r) = e$ for each relation r in the presentation of H. By construction, this is an inverse for β, thus $G \cong H$. $\square$

Definition 5.3.5. *When G_0 is the trivial group, the pushout $G_1 *_{G_0} G_2$ depends only on G_1 and G_2. This is known as the free product $G_1 * G_2$.*

Example 5.3.6. The free product $\mathbb{Z} * \mathbb{Z}$ is isomorphic to the free group on two generators, as we may take $\langle x \mid\rangle$ and $\langle y \mid\rangle$ for the two presentations, and we know by Lemma 5.3.4 that $\mathbb{Z} * \mathbb{Z}$ is isomorphic to $\langle x, y \mid\rangle$.

Definition 5.3.7. *When $\phi_1 : G_0 \hookrightarrow G_1$ and $\phi_2 : G_0 \hookrightarrow G_2$, the pushout $G_1 *_{G_0} G_2$ is known as the* ***amalgamated free product*** *of G_1 and G_2 along G_0.*

Example 5.3.8. Consider the pushout defined by

$$\mathbb{Z} \overset{\text{id}}{\leftarrow} \mathbb{Z} \overset{\times 2}{\rightarrow} \mathbb{Z}$$

Then taking the base set as $\langle t \mid\rangle$, the maps to $\langle x \mid\rangle$ and $\langle y \mid\rangle$ are given by $t \mapsto x$ and $t \mapsto y^2$. Then our imposed equality is $x^2 = y^{2n}$ for all $n \in \mathbb{Z}$. In particular $x = y^2$, so by eliminating the generator x, we are left with $\langle y \mid\rangle$ which is isomorphic to $\mathbb{Z}$

Example 5.3.9. Consider the pushout defined by

$$\mathbb{Z} \overset{\times 2}{\leftarrow} \mathbb{Z} \overset{\times 3}{\rightarrow} \mathbb{Z}$$

Then the push-out has presentation

$$\langle u, v \mid u^{2n} = v^{3n}, n \in \mathbb{Z}\rangle$$

which can be written simply as

$$\langle u, v \mid u^{2} = v^{3}, n \in \mathbb{Z}\rangle$$

We then consider the map $u \mapsto yxy$ and $v \mapsto xy$. Then taking $xyx = yxy$, we have

$$\phi(v) = (xy)^3 = xyxyxy = yxyyxy = (yxy)^2 = \phi(u)^2$$

ϕ is surjective as $\phi(uv^{-1}) = y$ and $\phi(v^2u) = x$. This also gives a straightforward inverse, which satisfies the braiding relation, so in particular give isomorphisms.

Proposition 5.3.10. *Let $\alpha : G \to G * H$ be the (left) canonical homomorphism. Then α is injective.*

Proof. We give an explicit π such that $\pi \circ \alpha = \mathrm{id}_G$. Specifically, we take

$$\pi(x) = \begin{cases} x & \text{if } x \in G \\ 1 & \text{if } x \in H \end{cases}$$

This satisfies $\pi\alpha = \mathrm{id}_G$, thus α is injective. □

Remark 5.3.11. Given any $H := G_1 *_{G_0} G_2$, we can reduce any word $w \in H$ to a word of the form

$$w = g_1 h_1 \dots g_n h_n$$

where $g_i \in G_1$ and $h_i \in G_2$, by amalgamating successive letters in G_1 (resp. G_2) and removing identities. By repeating this reduction sequence, we reach a **reduced** word such that g_i, h_i are all nontrivial except possibly g_1 and or h_n.

In the case the product is free, the reduced word is unique (we note that the reduction satisfies the diamond property).

6 Fundamental Group

6.1 Definitions

Definition 6.1.1. *A* ***loop based at a point*** *$b \in X$ is a path $\ell : I \to X$ such that $\ell(0) = \ell(1) = b$. The point b is knows as its* ***basepoint****.*

Definition 6.1.2. *The homotopy classes relative to ∂I of loops based at b form a group, called the* ***fundamental group*** *of (X, b), denoted $\pi_1(X, b)$. If ℓ and ℓ' are loops based at b, then $[\ell]$ and $[\ell']$ are their homotopy classes relative to ∂I, with their composition defined as $[\ell].[\ell'] = [\ell.\ell']$.*

Note the base-point is required as a consequence of making sure that two loops can always be composed. If we don't have the requirement that homotopies are relative to ∂I, then any two paths in the same path-component of X are homotopic, as I is contractible (intuitively, collapse f to the path connected node, and move along it, and uncollapse at g). Finally, note that composition of paths itself is not necessarily associative, as the images are equal, but the path traverses through them at different speeds.

We show this is well-defined, is associative, has an identity, and also have inverses.

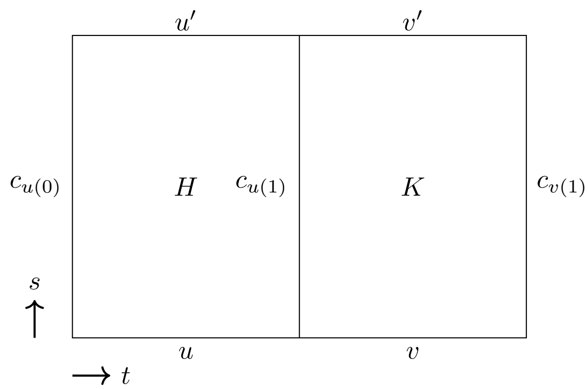
Lemma 6.1.3 (Well-definedness of Fundamental Groups)**.** *Suppose that u and v are paths in X such that $u(1) = v(0)$. Suppose also that u' (and respectively v') are paths with the same endpoints as u (respectively v). If $u \simeq u', v \simeq v'$ both relative to ∂I, then $u.v \simeq u'.v'$, relative to ∂I.*

Proof. Let $H : u \simeq u'$ and $K : v \simeq v'$ be the given homotopies. Then we can define $L : I \times I \to X$ by

$$L(t,s) = \begin{cases} H(2t, s) & \text{if } 0 \le t \le \frac{1}{2} \\ K(2t-1, s) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

This is continuous by the Gluing Lemma, thus we have $L : u.v \simeq u'.v'$, relative to ∂I.

Alternatively, the following diagram represents L.



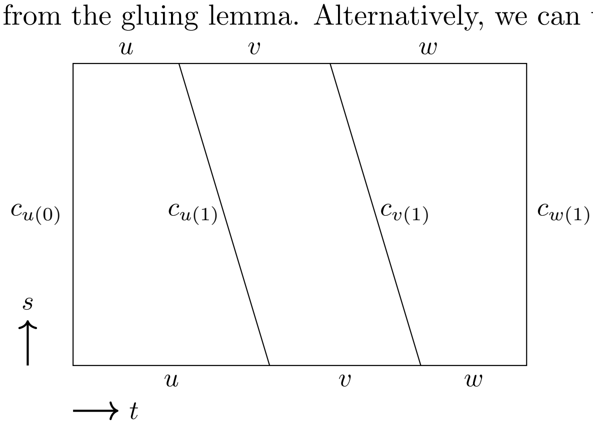
∎

Lemma 6.1.4 (Associativity of Fundamental Groups)**.** *Let u, v, w be paths in X such that $u(1) = v(0)$ and $v(1) = w(0)$. Then $u.(v.w) \simeq (u.v).w$ relative to ∂I.*

Proof. We give an explicit homotopy $H : I \times I \to X$ by

$$H(t,s) = \begin{cases} u(\frac{4t}{2-s}) & \text{if } 0 \le t \le \frac{1}{2} - \frac{1}{4}s \\ v(4t - 2 + s) & \text{if } \frac{1}{2} - \frac{1}{4}s \le t \le \frac{3}{4} - \frac{1}{4}s \\ w(\frac{4t-3+s}{1+s}) & \text{if } \frac{3}{4} - \frac{1}{4}s \le t \le 1 \end{cases}$$

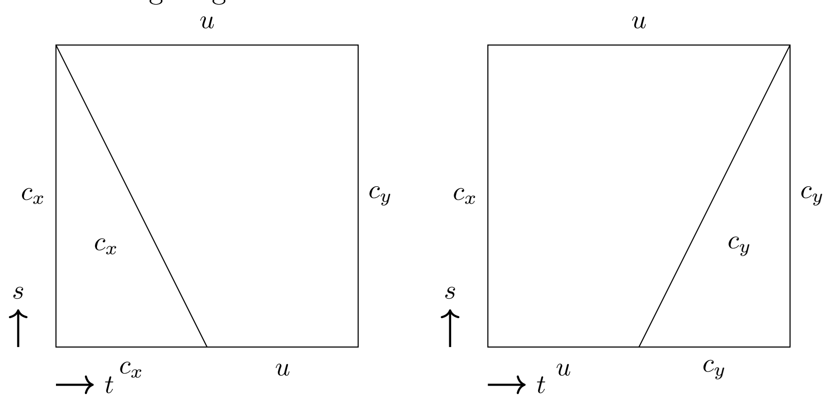
Again, continuity follows from the gluing lemma. Alternatively, we can use the following diagram:



∎

Lemma 6.1.5 (Identity of Fundamental Groups)**.** *Let u be a path in X with $u(0) = x$ and $u(1) = y$. Then $c_x.u \simeq u$ relative to ∂I and $u.c_y \simeq u$ relative to ∂I. In particular, $[c_b]$ is the identity element in $\pi_1(X, b)$.*

Proof. We note the following diagrams:



∎

Lemma 6.1.6 (Inverses in Fundamental Groups)**.** *Let u be a path in X with $u(0) = x$ and $u(1) = y$, and define u^{-1} as $u^{-1}(t) = u(1-t)$. Then $u.u^{-1} \simeq c_x$ relative to ∂I and $u^{-1}.u \simeq c_y$ relative to ∂I.*

Proof. We give an explicit homotopy between $u.u^{-1}$ and c_x:

$$H(t,s) = \begin{cases} u(2t(1-s)) & \text{if } 0 \leq t \leq \frac{1}{2} \\ u((2-2t)(1-s)) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

The idea is 'stopping' how far we go in u, and traversing back. A similar construction can be made for $u^{-1}.u$, by considering the inverse of their paths. ∎

Example 6.1.7. Let b be the origin in $\mathbb{R}^2$. Then $\pi_1(\mathbb{R}^2, b)$ is the trivial group. This is due to the fact every loop based at b is homotopic relative to ∂I to the constant loop c_b via straight-line homotopy.

Remark 6.1.8. We note that if X_0 is the path-component of X containing the basepoint b, then $\pi_1(X, b) = \pi_1(X_0, b)$. This is a simple consequence of the fact any loop in X based at b must lie entirely in X_0, and the homotopy between two such loops must also lie in X_0.

Proposition 6.1.9. *If b and b' lie in the same path-component of X, then $\pi_1(X, b) \simeq \pi_1(X, b')$.*

Proof. Let w be a path from b to b' in X. If ℓ is a loop based at b, then $w^{-1}.\ell.w$ is a loop based at b', and the function

$$\begin{gathered} w_\# : \pi_1(X, b) \to \pi_{(}X, b') \\ [\ell] \mapsto [w^{-1}.\ell.w] \end{gathered}$$

is well-defined. We also have

$$\begin{aligned} w_\#([\ell])w_\#([\ell']) &= [w^{-1}.\ell.w][w^{-1}.\ell'.w] \\ &= [w^{-1}.\ell.(w.w^{-1}).\ell'.w] \\ &= [w^{-1}.\ell.c_b.\ell'.w] \\ &= [w^{-1}.(\ell.\ell').w] \\ &= w_\#([\ell][\ell']) \end{aligned}$$

thus $w_\#$ is a homomorphism. Also, $w_\#$ has an inverse $(w^{-1})_\#$, since

$$(w^{-1})_\#(w_\#([\ell])) = (w^{-1})_\#([w^{-1}.\ell.w]) = [w.w^{-1}.\ell.w.w^{-1}] = [\ell]$$

∎

Remark 6.1.10. The isomorphism $w_\#$ depends on the choice of w. If u is another path from b to b', $u_\#^{-1} w_\#$ is the map $[\ell] \mapsto [u.w^{-1}.\ell.w.u^{-1}]$, which is the operation of conjugation by the element $[w.u^{-1}]$ of $\pi_1(X, b)$. As the fundamental group need not be abelian, this map need not be the identity.

Remark 6.1.11. There is a bijection between unbased loops in X in the component of b to conjugacy classes in $\pi_1(X, b)$. Let $\ell : S^1 \to X$.

Pick an abitrary path from b to $\ell(1)$. Then the loop $w.\ell.w^{-1}$ is a loop in X based at b. Applying a homotopy to ℓ does not change the homotopy class relative to ∂I of this loop.

Changing the choice to path w would alter this element by a conjugacy. In particular, we obtain a well-defined conjugacy class in $\pi(X, b)$ from any homotopy class of loop in X.

TODO: Show correspondence, have only shown well-definedness

Proposition 6.1.12. *Let (X, x) and (Y, y) be spaces with basepoints. Then, any continuous map $f : (X, x) \to (Y, y)$ induces a homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$. Further, we have*

1. $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x)}$
2. *if $g : (Y, y) \to (Z, z)$ is some map, $(gf)_* = g_* f_*$*
3. *if $f \simeq f'$ relative to $\{x\}$, then $f_* = f'_*$.*

Proof. Define $f_*([\ell]) = [f \circ \ell]$. Note this is well-defined by the version of Lemma 4.1.6 on homotopies relative to ∂I. Also, $f \circ (\ell.\ell') = (f \circ \ell).(f \circ \ell')$, thus f_* is a homomorphism.

The first two claims are straightforward, and the final one is a consequence of Lemma 4.1.6 for homotopies relative to a subspace (noting that $\ell(\partial I) \subseteq \{x\}$).

□

Proposition 6.1.13. *Let X and Y be path-connected spaces such that $X \simeq Y$. Then $\pi_1(X) \simeq \pi_1(Y)$.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ be homotopy equivalences.

Choose $x_0 \in X$ and let $y_0 \in f(x_0)$ and $x_1 = g(y_0)$, such that we have induced homomorphisms

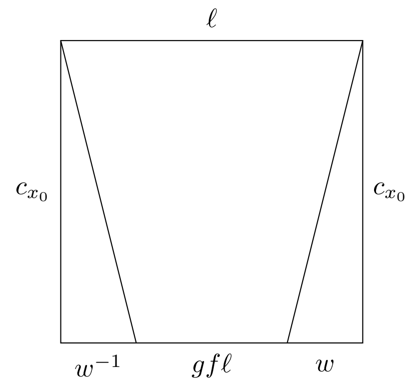
$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1)$$

Let H be the homotopy between gf and id_X. Then $w(t) = H(x_0, t)$ is a path from x_1 to x_0. Let ℓ be a loop in X based at x_0 and consider $K = H \circ (\ell \times \mathrm{id}_I) : I \times I \to X$.

We then rescale K to the trapezoid with maps that are constant on the first variable. This gives a homotopy relative to ∂I between $w^{-1}.(g \circ f \circ \ell).w$ and ℓ.

Thus, we have $w_\# g_* f_*[\ell] = [\ell]$. In particular, $w_\# g_* f_* = \mathrm{id}_{\pi_1(X,x_0)}$. In particular, f_* is injective, and as $w_\#$ is an isomorphism, g_* is surjective. By composing the other way around, we see that g_* is injective, and in particular an isomorphism.

Consequently, if X is a contractible space, $\pi_1(X)$ is the trivial group.



□

Definition 6.1.14. *A space is **simply-connected** if it is path-connected and has trivial fundamental group.*

Note that it need not be the case that simply-connected spaces are contractible. A counter-example is the 2-sphere.

6.2 Seifert Van Kampen

Theorem 6.2.1 (Seifert Van Kampen)**.** *Let K be a space which is a union of two path-connected open sets K_1 and K_2, where $K_1 \cap K_2$ is also path-connected. Let b be a point in $K_1 \cap K_2$ and let $\iota_i : K_1 \cap K_2 \to K_1$ and $\iota_2 : K_1 \cap K_2 \to K_2$ be the inclusion maps. Then $\pi_1(K, b)$ is isomorphic to the push-out of*

$$\pi_1(K_1, b) \overset{\iota_{1*}}{\leftarrow} \pi_1(K_1 \cap K_2, b) \overset{\iota_{2*}}{\rightarrow} \pi_1(K_2, b)$$

Moreover, the homomorphisms $\pi_1(K_1, b) \to \pi_1(K, b)$ and $\pi_1(K_2, b) \to \pi_1(K, b)$ which are the composition of the canonical homomorphisms to the pushout and the isomorphism to $\pi_1(K, b)$, are the maps induced by inclusion.

Explicitly, if $\langle X_1 \mid R_1 \rangle$ and $\langle X_2 \mid R_2 \rangle$ are presentations for $\pi_1(K_1, b)$ and $\pi_1(K_2, b)$ with $X_1 \cap X_2 = \emptyset$, then a presentation of $\pi_1(K, b)$ is given by

$$\langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\iota_{1*}(g) = \iota_{2*}(g) \mid g \in \pi_1(X_1 \cap X_2, b)\}\rangle$$

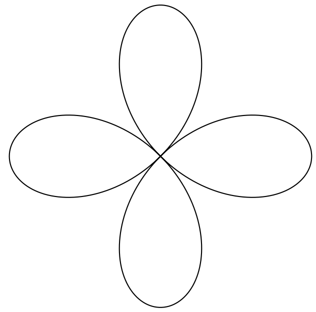
Moreover, the homomorphism $\langle X_i \mid R_i \rangle \to \pi_1(K, b)$ arising from the inclusion of generators $X_i \to X_1 \cup X_2$ is the map induced by the inclusion $K_i \to K$.

Proof. □

Definition 6.2.2. *The **wedge** $(X, x) \vee (Y, y)$ of two spaces with basepoints is the quotient of the disjoint union $X \sqcup Y$ with the identification $x \sim y$. It's basepoint is the image of x and y in the quotient.*

Example 6.2.3. By picking an arbitrary basepoint b in S^1 and wedging n copies of S^1, b together, we obtain the space $\bigvee^n S^1$ which is known as the **bouquet of circles**. For instance, the case with

$n = 4$ is the following:



Corollary 6.2.4. *the fundamental group of $\bigvee^n S^1$ is isomorphic to the free group on n generators.*

Proof. We apply induction on n. For the case $n = 1$, we have $\pi_1(S^1) \cong \mathbb{Z}$. For the inductive case, suppose that $\pi_1(\bigvee^{n-1} S^1)$ is the free group on $n-1$ generators. Let b be the vertex of this wedge, which we take to be the basepoint. Let N be a small open neighborhood of b. Decompose $\bigvee^n S^1$ as $K_1 = N \cup \bigvee^{n-1} S^1$ and $K_2 = N \cup S^1$. Then $\bigvee^{n-1} S^1$ is a homotopy retract of K_1 and S^1 is a homotopy retract of K_2. The intersection $K_1 \cap K_2$ is N, which is clearly contractible. So by Seifert Van Kampen, $\pi_1(\bigvee^n S^1)$ has a presentation with n generators and no relations. □

Theorem 6.2.5. *Let K be a connected cell complex, and let $\ell_i : S^1 \to K^1$ be the attaching maps of its 2-cells, where $1 \leq i \leq n$. Let b be a basepoint in K^0. Let $[\ell_i]$ be the conjugacy class of the loop ℓ_i in $\pi_1(K^1, b)$. Then $\pi_1(K, b)$ is isomorphic to $\pi_1(K^1, b)/\langle\langle [\ell_1], \ldots, [\ell_n] \rangle\rangle$.*

Theorem 6.2.6. *Let K be a connected cell complex, and let $\ell_i : S^1 \to K^1$ be the attaching maps of its 2-cells, where $1 \leq i \leq n$. Let b be a basepoint in K^0. Let $[\ell_i]$ be the conjugacy class of the loop ℓ_i in $\pi_1(K^1, b)$. Then*

$$\pi_1(K^1, b)/\langle\langle [\ell_1], \ldots, [\ell_n] \rangle\rangle \simeq \pi_1(K, b)$$

As $\pi_1(K^1, b)$ is free, then this also gives a presentation for $\pi_1(K, b)$.

Proof. Note first that ℓ_i need not be based at b, but give well-defined conjugacy classes. Picking out representatives ℓ_i' for $[\ell_i]$, we can write

$$\pi_1(K^1, b)/\langle\langle \ell_1', \ldots, \ell_n' \rangle\rangle \simeq \pi_1(K, b)$$

We split the space into open sets $K_1 = \{z \in D^n \mid |z| < \frac{2}{3}\}$ and $K_2 = \{z \in D^n \mid |z| > \frac{1}{3}\} \sqcup X/\sim$. Then, K_1 is homeomorphic to an open n-ball, and $K_1 \cap K_2$ is homeomorphic to $S^{n-1} \times (\frac{1}{3}, \frac{2}{3})$, which is homotopy equivalent to S^{n-1}. K_2 is homotopy eqiuvalent to X by homotopy retraction. We apply Seifert Van Kampen. When $n > 2$, $\pi_1(K_1 \cap K_2)$ and $\pi_1(K_1)$ are both trivial, so the attaching map has no effect on the fundamental group. When $n = 2$, $\pi_1(K_1 \cap K_2) \simeq \mathbb{Z}$ and $\pi_1(K_1)$ is trivial, so this has an effect of adding a relation to $\pi_1(X)$ that represents the loop $[f]$ (by construction). □

Corollary 6.2.7. *Any finitely presented group can be realised as the fundamental group of a finite connected cell-complex. Moreover, this may be given a triangulation.*

Proof. Given $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$, take $K^0 = *$, $K^1 = \bigvee_m S^1$. Give a triangulation on this by splitting the circle into three 1-simplices. Then $\pi_1(K^1)$ is afree group on m generators. Attach 2-cells along the words r_i, then by the previous theorem, this gives the required fundamental group. We note that 2-cells have a canonical simplicial structure, which gives the whole space a triangulation. □

Corollary 6.2.8. *The following are equivalent for a group G*

- *G is finitely presented*
- *G is isomorphic to the fundamental group of a finite simplicial complex*
- *G is isomorphic to the fundamental group of a finite cell complex*

Proof. $(i) \Rightarrow (ii)$ is from above. Note that any finite simplicial complex is a finite cell complex, so $(ii) \Rightarrow (iii)$. Finally, we note that attaching maps of $n > 2$ have no effect on the fundamental group, so in particular from Theorem 6.2.6 is finitely presented. $\square$

6.3 Classification of Fundamental Groups

6.3.1 Fundamental Group of Simplicial Complexes

Definition 6.3.1. *Let α be an edge path. An* ***elementary contraction*** *of α is an edge path obtained from α by performing one of the following :*

1. *removing a_i given $a_{i-1} = a_i$*
2. *replacing a_{i-1}, a_i, a_{i+1} with a_{i-1} given $a_{i-1} = a_{i+1}$*
3. *replacing a_{i-1}, a_i, a_{i+1} with a_{i-1}, a_{i+1} provided $\{a_{i-1}, a_i, a_{i+1}\}$ span a 2-simplex of K.*

α is an elementary expansion of β if β is an elementary expansion of α. We write $\alpha \sim \beta$ if we can pass from α to β. This gives an equivalence relation on edge paths.

Theorem 6.3.2. *Let K be a simplicial complex, and let b be a vertex of K. The equivalence classes of edge loops in K based at b form a group denoted $E(K, b)$, called the edge-loop group.*

Proof. The product is induced by the product of edge loops. This respects the equivalence relation. It is associative because the product of edge loops is associative. The identity is the equivalence class of (b). The inverse of $(b, b_1, \ldots, b_{n-1}, b)$ is $(b, b_{n-1}, \ldots, b_1, b)$. $\square$

Theorem 6.3.3. *For a simplicial complex K and vertex b, $E(K, b)$ is isomorphic to $\pi_1(|K|, b)$.*

Proof. Let $I_{(n)}$ be the triangulation of I with n 1-simplices each of length $\frac{1}{n}$. We can regard an edge path of length n as a simplicial map $I_{(n)} \to K$. This gives a mapping

$$\{\text{edge loops in } K \text{ based at } b\} \xrightarrow{\theta} \{\text{loops in } |K| \text{ based at } b\}$$

If α is obtained from β by an elementary contraction, $\theta(\alpha)$ and $\theta(\beta)$ are homotopic relative to ∂I. Thus, θ gives a well-defined mapping from $E(K, b) \to \pi_1(|K|, b)$. It remains to show that is is an isomorphism.

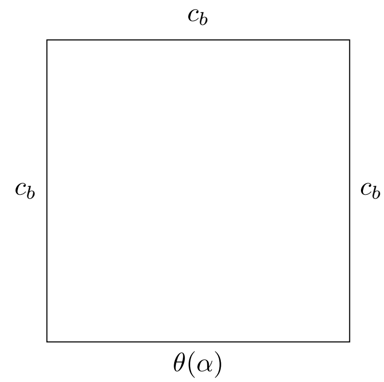
For edge loops α and β, we have $\theta(\alpha.\beta) \simeq \theta(\alpha).\theta(\beta)$, this is a homomorphism.

Surjectivity: Let $\ell : I \to |K|$ be any loop in $|K|$ based at b. Give I the triangulation $I_{(1)}$ and view $I_{(n)}$ as the subdivision. The coarseness of $I_{(n)}$ is $4/r$, which tends to 0 as $n \to \infty$, so by the Simplicial Approximation Theorem (Variant 1), there is a simplicial map $\alpha : I_{(n)} \to K$ for some n such that $\ell \simeq \theta(\alpha) = |\alpha|$ relative to ∂I. In particular, $\theta([\alpha]) = [\ell]$.

Injectivity: Let $\alpha = (b_0, \ldots, b_n)$ be an edge loop based at b. Suppose that $\theta([\alpha])$ is the identity in $\pi_1(|K|, b)$. Then $\theta(\alpha) \simeq c_b$ relative to ∂I via some homotopy $H : I \times I \to |K|$. Triangulate $I \times I$

using the triangulation $(I \times I)_{(r)}$. By The Simplicial Approximation Theorem, for a sufficiently large r, we have a simplicial map $G : (I \times I)_{(r)} \to K$ with $G \simeq H$.

By Proposition 4.2.10, we can ensure that G sends $\partial I \times I$ and $I \times \{1\}$ to b.



Using the same Proposition, when r is a multiple of n, we can ensure that $G(i/n, 0) = b_i$, sending the 1-simplices between $(i/n, 0)$ and $((i+1)/2, 0)$ to (b_i, b_{i+1}). Thus, the restriction of G to $I \times \{0\}$ is an edge path which contracts to α.

We can apply a sequence of elementary contractions and expansion that take this edge path to the edge path where every vertex is b. This is equivalent to (b). In particular, $[\alpha]$ is the identity element of $E(K, b)$ (as the map preserves fundamental groups).

□

Definition 6.3.4. *For any simplicial complex and non-negative integer n, define the n-**skeleton** of K, denoted $\operatorname{skel}^n(K)$ is the subcomplex of K consisting of simplices with dimension at most n.*

Corollary 6.3.5. *For any simplicial complex K and vertex b, $\pi_1(|K|, b)$ is isomorphic to $\pi_1(|\operatorname{skel}^2(K)|, b)$.*

Proof. $E(K, b)$ involves only simplices of dimension at most 2, and $E(K, b) \simeq \pi_1(|K|, b)$, □

Corollary 6.3.6. *For $n \geq 2$, $\pi_1(S^n)$ is trivial.*

Proof. Impose a triangulation on S^n, coming from the n-skeleton of Δ^{n+1}. Then S^n and Δ^{n+1} have the same 2-skeleton. But Δ^{n+1} is contractible, so has trivial fundamental group, so does S^n. □

6.3.2 Fundamental Group of the Circle

We view S^1 here as a circle in $\mathbb{C}$, taking $1 \in S^1$ to be the basepoint.

Theorem 6.3.7. *$\pi_1(S^1) \simeq \mathbb{Z}$.*

Proof. Impose a triangulation K on S^1 using three vertices and three 1-simplices. We aim to show that $E(K, 1)$ is isomorphic to $\mathbb{Z}$.

Consider a simplicial loop $\alpha = (b_0, \ldots, b_n)$ based at 1. If $b_i = b_{i+1}$ for some i, then we may preform some elementary contraction. If the loop traverses a 1-simplex and then in reverse, we may also perform an elementary contraction. Thus, a shortest loop equivalent to α traverses all the simplices with the same orientation. It is therefore equivalence to ℓ^n for some $n \in \mathbb{Z}$.

Define the winding number to be the time a simplicial path traverses the (1,2) simplex minus the times it traverses it in the backwards direction. Then, the winding number of ℓ^n is n, and any elementary contraction or expansion leaves the winding number unchanged.

Thus, we can set up a bijection $E(K, 1) \to \mathbb{Z}$ based on its winding number. This is an isomorphism, since $\ell^n . \ell^m = \ell^{n+m}$. □

Theorem 6.3.8 (Fundamental Theorem of Algebra)**.** *Any non-constant polynomial with complex coefficients has at least one root in $\mathbb{C}$.*

Proof. let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial where $a_n \neq 0$ and $n > 0$. Let $C_r = \{x \in \mathbb{C} \mid |x| < r\}$. Let $k = p(r)/r^n$ and $q(x) = kx^n$. Then $p(r) = q(r)$.

We claim that if r is sufficiently large, then $p|_{C_r}$ and $q|_{C_r}$ and the straight-line homotopy all miss 0. If not, then for some $x \in C_r$ and some $t \in [0,1]$,

$$(1-t)p(x) + tq(x) = 0$$

Equivalently,

$$(1-t)(a_n x^n + \cdots + a_0) + t(\frac{a_n|x|^n + \cdots + a_0}{|x|^n})x^n = 0$$

rearranging,

$$a_n x^n + \cdots + a_0 = t(a_{n-1}x^{n-1} + \cdots + a_0 - a_{n-1}\frac{x^n}{|x|} - \cdots - a_0\frac{x^n}{|x|^n})$$

The left side has order x^n, whereas the right is at most x^{n-1}. Hence as $|x| \to \infty$, $|t| \to \infty$. In particular, given r sufficiently large, there is no solution in the range $t \in [0,1]$.

So $p|_{C_r}$ and $q|_{C_r}$ are homotopic relative to $\{r\}$. Suppose that p has no root in $\mathbb{C}$. Then we have a commutative diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "p"] & \mathbb{C} - \{0\} \\ C_r \arrow[u, "\iota"] \arrow[ur, "p|_{C_r}"'] & \end{tikzcd}$$

This induces a function between fundamental groups

$$\begin{tikzcd} 0 = \pi_1(\mathbb{C}, r) \arrow[r, "p_*"] & \mathbb{Z} \simeq \pi_1(\mathbb{C} - \{0\}, r) \\ \mathbb{Z} \simeq \pi_1(C_r, r) \arrow[u, "\iota_*"] \arrow[ur, "(p|_{C_r})_*"'] & \end{tikzcd}$$

In particular, $(p|_{C_r})_*$ is the 0-homomorphism. But $(p|_{C_r})_* = (q|_{C_r})_*$, which sends a generator of $\pi_1(C_r)$ to n times the generator of $\pi_1(\mathbb{C} \setminus \{0\})$, which is a contradiction. $\square$

6.3.3 Fundamental Group of a Graph

Theorem 6.3.9. *The fundamental group of a connected graph is a free group.*

Proof. Let T be a maximal tree in Γ, which exists by Lemma 2.2.5. Let b be a vertex of Γ, which we take as the baespoint. For any vertex $v \in \Gamma$, let $\theta(v)$ be the unique embedded edge path from b to v in T. This exists as $V(T) = V(\Gamma)$ by Lemma 2.2.4. Set $E(\Gamma)$ and $E(T)$ to be the edges of Γ and T respectively. Assign an orientation to each edge $e \in E(\Gamma) \setminus E(T)$, taking $\iota(e), \pi(e)$ to be its initial and terminal vertices. We claim that the elements $\{\theta(\iota(e)).e.\theta(\pi(e))^{-1} \mid e \in E(\Gamma) \setminus E(T)\}$ form a free generating set for $\pi_1(\Gamma, b)$. $\square$

7 Covering Spaces

7.1 Basic Definitions

Definition 7.1.1. *A continuous map $p : \tilde{X} \to X$ is a* ***covering map*** *if X and $\tilde{X}$ are non-empty path connected spaces, and given any $x \in X$, there exists some open set U_x containing x such that $p^{-1}(U_x)$ is a disjoint union of open sets V_j such that $p|_{V_j} : V_j \to U_x$ is a homeomorphism for some indexing set J. The open sets U_x are are called* ***elementary open sets****. $\tilde{X}$ is a* ***covering space*** *of X.*

If we give basepoints $\tilde{b}$ and b such that $p(\tilde{b}) = b$, then $p : (\tilde{X}, \tilde{b}) \to (X, b)$ is a ***based covering map****.*

Example 7.1.2. There is a covering map $p : \mathbb{R} \to S^1$ with $t \mapsto \exp(2\pi i t)$.

Given $x \in S^1$, take U_x to be the open semi-circle with x as its midpoint. For instance, $p^{-1}(U_1) = \bigcup_{n \in \mathbb{Z}} (n - \frac{1}{4}, n + \frac{1}{4})$.

Example 7.1.3. For any nonzero integer n, the map $S^1 \to S^1$ by $z \mapsto z^n$ is also covering.

Example 7.1.4. Let $\mathbb{R}P^n$ be the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. Define $p : S^n \to \mathbb{R}P^n$ to be the map that sends a point $y \in S^n$ to the 1-dimensional subspace through y.

For each point $x \in \mathbb{R}P^n$, $p^{-1}(x)$ is two points. Take the quotient topology induced by p. Then, taking U_x sufficiently small, $p^{-1}(U_x)$ is two copies of U_x, and the restriction gives a homeomorphism onto U_x. Thus, p is a covering map.

Proposition 7.1.5. *Let $p : \tilde{X} \to X$ be a covering map. Then,*

1. *p is an open mapping*
2. *for $x_1, x_2 \in X$, $p^{-1}(x_1), p^{-1}(x_2)$ have the same cardinality on J*
3. *p is surjective*
4. *p is a quotient map*

Proof. (i) Let U be an open set in $\tilde{X}$. For any $y \in U$, we wish to find an open set $p(y)$ contained in $p(U)$. Let V_j be the copy of $U_{p(y)}$ in $p^{-1}(U_{p(y)})$ that contains y. As the restriction of p to V_j is a homeomorphism, $p(V_j \cap U)$ is open in X. This is an open set containing $p(y)$ in $p(U)$.

(ii) The cardinality of $p^{-1}(x)$ is locally constant on $\tilde{X}$. As $\tilde{X}$ is connected, it must be globally constant

(iii) As $\tilde{X}$ is nonempty, $p^{-1}(x)$ is nonempty for some $x \in X$. As the cardinality is constant, $p^{-1}(x)$ is nonempty for any $x \in X$, thus p is surjective.

(iv) A surjective open mapping is a quotient map. $\square$

Definition 7.1.6. *The* ***degree*** *of a covering map $p : \tilde{X} \to X$ is the cardinality of $p^{-1}(x)$ for any $x \in X$.*

Definition 7.1.7. *If $p : \tilde{X} \to X$ is a covering map and $f : Y \to X$ is a map, then a* ***lift*** *of f is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$. Equivalently, the following diagram commutes:*

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Example 7.1.8. Given a covering map $p : \mathbb{R} \to S^1$ from before, the map $f : I \to S^1$ sending $t \mapsto \exp(2\pi i t)$ lifts to $\tilde{f} : I \to \mathbb{R}$, where $\tilde{f}(t) = t$.

Conversely, the identity map from $S^1 \to S^1$ does not lift, as if a lift $\tilde{f} : S^1 \to \mathbb{R}$ existed, then by commutativity, $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$. This induces a commutative diagram,

$$\begin{tikzcd}
 & \pi_1(\mathbb{R}, n) \arrow[d, "p_*"] \\
\pi_1(S^1, 1) \arrow[ur, "\tilde{f}_*"] \arrow[r, "\mathrm{id}"'] & \pi_1(S^1, 1)
\end{tikzcd}$$

which is impossible, as $\pi_1(\mathbb{R})$ is trivial, whereas $\pi_1(S^1)$ is nontrivial.

Theorem 7.1.9 (Uniqueness of lifts)**.** *Let $p : \tilde{X} \to X$ be a covering map, and let $f : Y \to X$ be a map, where Y is connected. Suppose that g and h are lifts of f and that $g(y_0) = h(y_0)$ for some $y_0 \in Y$. Then $g = h$.*

Proof. Let $C = \{y \in Y \mid g(y) = h(y)\}$. By $y_0 \in C$, C is nonempty. We show that C is closed and open, and as Y is connected, it is the entirety of Y.

As p is a covering map, there is an elementary open set $U_{f(y)}$ containing $f(y)$ for any $y \in Y$, and open sets V_1, V_2 in $\tilde{X}$ such that $p|_{V_1}$ and $p|_{V_2}$ are homeomorphisms from V_1 and V_2 to $U_{f(y)}$ and $g(y) \in V_1$, $h(y) \in V_2$.

Now let $y \in Y - C$. Then $V_1 \cap V_2 = \emptyset$. Thus, $g^{-1}(V_1) \cap h^{-1}(V_2)$ is contained in $Y - C$. This is an open set containing y, so $Y - C$ is open.

Suppose that $y \in C$. Then $V_1 = V_2$. Taking $g^{-1}(V_1) \cap h^{-1}(V_2)$, we have $p \circ g = p \circ h$. As $p|_{V_1}$ is an injection, $g = h$ on this set. Thus it is in C. This is an open set containing y, so C is open. □

Theorem 7.1.10 (Path Lifting)**.** *Let $p : \tilde{X} \to X$ be a covering map. Let $\alpha : I \to X$ be a path with $\alpha(0) = x$. Given $\tilde{x} \in p^{-1}(x)$, α has a lift $\tilde{\alpha} : I \to \tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{x}$.*

Proof. Let $A = \{t \in I \mid \text{there exists a lift of } \alpha|_{[0,t]} \text{ starting at } \tilde{x}\}$. A is nonempty, as it contains 0. Take T to be the supremum of A. Pick an elementary open set $U_{\alpha(T)}$ around $\alpha(T)$.

Pick an $\epsilon > 0$ such that $(T - \epsilon, T + \epsilon) \cap [0,1]$ is mapping into $U_{\alpha(T)}$ by α. Let $t = \max\{0, T - \frac{\epsilon}{2}\}$. Let $\tilde{\alpha} : [0,t] \to \tilde{X}$ be a lift of $\alpha|_{[0,t]}$ starting at $\tilde{x}$.

Let V_j be the copy of $U_{\alpha(T)}$ in $p^{-1}(U_{\alpha(T)})$ that contains $\tilde{\alpha}(t)$. The homeomorphism $U_{\alpha(T)} \cong V_j$ specifies a way of extending $\tilde{\alpha}$ to a lift of $\alpha|_{[0,T+\epsilon] \cap [0,1]}$. This implies $T = 1$. Hence A is all of I, and thus $\tilde{\alpha}$ has been defined on all $[0,1]$. □

Theorem 7.1.11 (Homotopy Lifting)**.** *Let $p : \tilde{X} \to X$ be a covering map. Let Y be a space, and let $H : Y \times I \to X$ be a map. If h is a lift of $H_{Y \times \{0\}}$, then H has a unique lift $\tilde{H} : Y \times I \to \tilde{X}$ such that $\tilde{H}|_{Y \times \{0\}} = h$.*

Proof. TODO!! Omitted for revision sake □

Remark 7.1.12. When $Y = \{*\}$, then it always exists by Path lifting.

Corollary 7.1.13. *If $p : (\tilde{X}, \tilde{b}) \to (X, b)$ is a based covering map, then $p_* : \pi_1(\tilde{X}, \tilde{b}) \to \pi_1(X, b)$ is an injection.*

Proof. Let ℓ be a loop in $\tilde{X}$ based at $\tilde{b}$. Then $p \circ \ell$ is a loop in X based at b. Suppose that $p_*[\ell] = [p \circ \ell]$ is trivial in $\pi_1(X, b)$, and let $H : I \times I \to X$ be the homotopy relative to ∂I between

$p \circ \ell$ and c_b. Now ℓ is a lift of $H|_{I\times\{0\}}$. Thus by Homotopy Lifting, there is a lift $\tilde{H} : I \times I \to \tilde{X}$ of H such that $\tilde{H}|_{I\times\{0\}} = \ell$.

Now, $\tilde{H}_{\{0\}\times I}, \tilde{H}_{\{1\}\times I}, \tilde{H}_{I\times\{1\}}$ are all constant maps, as the lift of a constant map is constant, as $p^{-1}(b)$ is discrete, and continuous functions map path-connected sets to path-connected sets. Thus, they must all be $\tilde{b}$, as this is where ℓ sends ∂I. In particular, $\tilde{H}$ is a homotopy relative to ∂I between ℓ and $c_{\tilde{b}}$. Thus $[\ell]$ is trivial in $\pi_1(X, b)$, giving p_* to be an injection. □

Remark 7.1.14. Fix a based covering map $p : (\tilde{X}, \tilde{b}) \to (X, b)$. If two loops ℓ and ℓ' based at b are homotopic relative to ∂I, they can be lifted to paths $\tilde{\ell}$ and $\tilde{\ell}'$ starting at $\tilde{b}$. By the previous corollary, they are homotopic relative to ∂I.

Thus, $\tilde{\ell}(1) = \tilde{\ell}'(1)$

Definition 7.1.15. *Noting the above remark, define a function*

$$\pi_1(X, b) \xrightarrow{\lambda} p^{-1}(b)$$

by $[\ell] \mapsto \tilde{\ell}(1)$.

Proposition 7.1.16. *Fix a based covering map* $p : (\tilde{X}, \tilde{b}) \to (X, b)$. *Given elements* g_1, g_2 *of* $\pi_1(X, b)$, $\lambda(g_1) = \lambda(g_2)$ *if and only if* g_1 *and* g_2 *belong to the same right coset of* $p_*\pi_1(\tilde{X}, \tilde{b})$. *This induces a bijection between right cosets of* $p_*\pi_1(\tilde{X}, \tilde{b})$ *and points of* $p^{-1}(b)$.

$$\begin{tikzcd}
\tilde{X} \arrow[d, "p"] & p^{-1}(b) \arrow[l, "\supseteq"'] \arrow[d, Leftrightarrow] & \pi_1(\tilde{X}, \tilde{b}) \arrow[d, "p_*"] \\
X & \textit{right cosets of } p_*(\pi_1(\tilde{X}, \tilde{b})) & \pi_1(X, b) \arrow[ul, dashed, "\lambda"]
\end{tikzcd}$$

Proof. Let ℓ_1 and ℓ_2 be loops based at b such that $[\ell_i] = g_i$. Suppose that $\tilde{\ell}_1(1) = \tilde{\ell}_2(1)$. Then $\tilde{\ell}_1 \cdot \tilde{\ell}_2^{-1}$ is a loop based at $\tilde{b}$. The map p sends this to $\ell_1\dot{\ell}_2^{-1}$, so

$$[\ell_1][\ell_2]^{-1} = p_*[\tilde{\ell}_1\dot{\tilde{\ell}}_2^{-1}] \in p_*\pi_1(\tilde{X}, \tilde{b})$$

Thus g_1 and g_2 belong to the same right coset of $p_*\pi_1(\tilde{X}, \tilde{(b)})$.

Conversely, suppose that $[\ell_1]$ and $[\ell_2]$ belong to the same right coset of $p_*\pi_1(\tilde{X}, \tilde{b})$ such that $[\ell_1][\ell_2]^{-1} \in p_*\pi_1(\tilde{X}, \tilde{b})$. Then $\ell_1\dot{\ell}_2^{-1}$ is homotopic relative to ∂I to $p \circ \ell$ for some loop ℓ in $\tilde{X}$ based at $\tilde{b}$. This homotopy lifts to a homotopy relative to ∂I between ℓ and a lift of $\ell_1\dot{\ell}_2^{-1}$. Thus, $\ell_1\dot{\ell}_2^{-1}$ lifts to a loop based at $\tilde{b}$. The lift is $\tilde{e}ll_1.\tilde{\ell}_2^{-1}$. Thus $\tilde{\ell}_1(1) = \tilde{\ell}_2(1)$. □

Corollary 7.1.17. *A loop* ℓ *in* X *based at* b *lifts to a loop based at* $\tilde{b}$ *if and only if* $[\ell] \in p_*\pi_1(\tilde{X}, \tilde{b})$.

Proof. ℓ lifts to a loop based at $\tilde{b}$ if and only if $\lambda[\ell] = \tilde{b}$, but *tildeb* corresponds to the identity coset of $p_*\pi_1(\tilde{X}, \tilde{b})$, in particular $[\ell] \in p_*\pi_1(\tilde{X}, \tilde{b})$. □

Remark 7.1.18. When $p_*\pi_1(\tilde{X}, \tilde{b})$ is a normal subgroup of $\pi_1(X, b)$, the right cosets form a group in which we can quotient by, and is bijective with $p^{-1}(b)$. To see the group structure from the quotient, consider the following. Let $\tilde{\ell}_1$ and $\tilde{\ell}_2$ be paths from $\tilde{b}$ to b_1 and b_2 respectively. Then $\ell_1 = p \circ \tilde{\ell}_1$ and $\ell_2 = p \circ \tilde{\ell}_2$ are loops in X based at b such that $\lambda([\ell_i]) = b_i$. To compute $\lambda([\ell_1].[\ell_2])$, lift $\ell_1.\ell_2$ to a path based at b, and then $b_1.b_2$ is the endpoint. Alternatively, take the lift of ℓ_2 that starts at b_1.

Definition 7.1.19. *When $\tilde{X}$ is simply connected, a based covering map $p : (\tilde{X}, \tilde{b}) \to (X, b)$ is known as the **universal cover** of X.*

Remark 7.1.20. In this case, $p^{-1}(b)$ is bijective with $\pi_1(X, b)$, as $\pi_1(\tilde{X}, \tilde{b}) = \{1\}$.

Corollary 7.1.21. *The fundamental group of a circle is isomorphic to $\mathbb{Z}$.*

Proof. We give a uniersal cover $\mathbb{R} \to S^1$ in the usual sense, and then this bijectively corresponds to $p^{-1}(1) = \mathbb{Z}$. Using the procedure above, this gives an isomorphism $\pi_1(S^1, 1) = \mathbb{Z}$. $\square$

Remark 7.1.22. The above proof works for any $\prod S^1$ by taking the universal cover $\mathbb{R}^n$.

7.2 Uniqueness of Coverings

Definition 7.2.1. *A space Y is **locally path-connected** if for each point y of Y and each neighborhood V of y, there is an open neighborhood of y contained in V that is path-connected.*

Example 7.2.2. Any 2-manifold is locally path-connected. In particular, simplicial complex is locally path-connected.

Theorem 7.2.3 (Existence of Lifts)**.** *Let $p : (\tilde{X}, \tilde{b}) \to (X, b)$ be a based covering map. Let Y be a path-connected, locally path-connected space and let $f : (Y, y_0) \to (X, b)$ be some map. Then f has a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{b})$ if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{b})$.*

Definition 7.2.4. *Two based covering spaces $p : (\tilde{X}, \tilde{b}) \to (X, b)$ and $p' : (\tilde{X}', \tilde{b}') \to (X, b)$ are **equivalent** if there is a homeomorphism f such that the following commutes:*

$$\begin{tikzcd}
(\tilde{X}, \tilde{b}) \arrow[rr, "f"] \arrow[dr, "p"'] & & (\tilde{X}', \tilde{b}') \arrow[dl, "p'"] \\
& (X, b) &
\end{tikzcd}$$

Theorem 7.2.5 (Uniqueness of Covering Spaces)**.** *Let X be a path-connected, locally path-connected space, and let b be a basepoint in X. Then for any subgroup H of $\pi_1(X, b)$, there is at most one based covering space $p : (\tilde{X}, \tilde{b}) \to (X, b)$ up to equivalence such that $p_*\pi_1(\tilde{X}, \tilde{b}) = H$.*

Proof. Let $p' : (\tilde{X}', \tilde{b}') \to (X, b)$ be another another covering such that $p'_*\pi_1(\tilde{X}', \tilde{b}') = H$. Then by Theorem 7.2.3, p' admits a lift $\tilde{p}' : (\tilde{X}', \tilde{b}') \to (\tilde{X}, \tilde{b})$, and similarly for p, such that the following commutes:

$$\begin{tikzcd}
(\tilde{X}', \tilde{b}') \arrow[r, "\tilde{p}'"] \arrow[dr, "p'"'] & (\tilde{X}, \tilde{b}) \arrow[r, "\tilde{p}"] \arrow[d, "p"'] & (\tilde{X}', \tilde{b}') \arrow[dl, "p'"] \\
& (X, b) &
\end{tikzcd}$$

By uniqueness of lifts (as the basepoints agree), $\tilde{p}'\tilde{p} = \mathrm{id}_{\tilde{X}}$, thus $\tilde{p}'$ is a homeomorphism, and so the coverings are equivalent. $\square$

Theorem 7.2.6. *Let K be a path-connected simplicial complex, and let b be a vertex of K. Then for any subgroup H of $\pi_1(K, b)$ there is a based covering $p : (\tilde{K}, \tilde{b}) \to (K, b)$ such that $p_*\pi_1(\tilde{K}, \tilde{b}) = H$. Moreover, $\tilde{K}$ is a simplicial complex and p is a simplicial map.*

Corollary 7.2.7. *Let K be a path-connected simplicial complex, and let b be a vertex of K. Then there is precisely one based covering space up to equivalence for each subgroup of $\pi_1(K, b)$.*

Theorem 7.2.8 (Nielsen-Schreier)**.** *Any subgroup of a finitely generated free group is free.*

Proof. Let F be the free group on n generators. Let X be the wedge of n circles, and let b be the central vertex. Then $\pi_1(X, b) \simeq F$. Taking any subgroup H of F, there is a based covering map $p : (\tilde{X}, \tilde{b}) \to (X, b)$ such that $p_*\pi_1(\tilde{X}, \tilde{b}) = H$. As p_* is injective, so $\pi_1(\tilde{X}, \tilde{b}) \simeq H$. By Theorem 7.2.6, $\tilde{X}$ is a simplicial complex and p is a simplcial map. As p is a local homeomorphism, $\tilde{X}$ can contain only zero and one dimensional simplices. Hence $\tilde{X}$ is a graph, thus has free fundamental group. $\square$

Remark 7.2.9. To construct the free generating set given a graph, take a maximal tree and use those.

Remark 7.2.10. Lift exists iff image of lift is in image of covering map (as a fundamental group) (so if there is a lift, it is a subgroup)

Theorem 7.2.11. *Let G be a finitely generated group and let H be a finite index subgroup. Then H is finitely presented.*

Definition 7.2.12. *Let $p : \tilde{X} \to X$ be a covering map. Then a* ***covering transformation*** *is a homeomorphism $t : \tilde{X} \to \tilde{X}$ such that the following commutes:*

$$\begin{tikzcd} \tilde{X} \arrow[rr, "t"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Definition 7.2.13. *A covering map $p : (\tilde{X}, \tilde{b})$ is* ***regular*** *if any two points of $p^{-1}b$ differ by a covering transformation.*

Theorem 7.2.14. *Let $p : (\tilde{X}, \tilde{b}) \to (X, b)$ be a regular covering map. Then $p_*\pi_1(\tilde{X}, \tilde{b})$ is a normal subgroup of $\pi_1(X, b)$.*

Proof. Sketch, lifting $\alpha \in \pi_1(X, b)$ takes you to a point $\tilde{\alpha}(1)$. Use regularity to transform ℓ to $t\ell$, moving from a loop based at $\tilde{b}$ to $\tilde{\alpha}(1)$. $\square$

Theorem 7.2.15. *Let $p : (\tilde{X}, \tilde{b}) \to (X, b)$ be a covering map, where X is locally path-connected. Suppose that $p_*\pi_1(\tilde{X}, \tilde{b})$ is a normal subgroup of $\pi_1(X, b)$. Then p is regular.*

Proof. Sketch, use normality, transform path based at $\tilde{b}'$ to $\tilde{b}$, use subset argument to generate lift, uniqueness of lifts to argue $tt' = \mathrm{id}$. $\square$

8 Notes

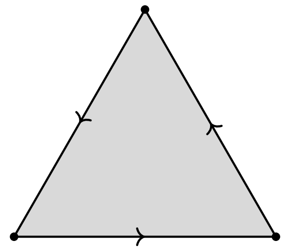
8.0.1 Cell attaching

We can view the attaching map as a pushout square:

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] & X \cup_f D^n \end{tikzcd}$$

When the attaching maps are homotopic, we can 'slide' along the attaching map on $\Phi : X \cup_f D^n \to X \cup_g D^n$ which is constant on X and the interior of the disk, and on the boundary, uses the homotopy H to carry the gluing f continuously over to the gluing g.

Example 8.0.1. The dunce-hat is contractible.



The attaching map onto the single loop 1-skeleton is homotopic to the identity on S^1, so the resulting space is homotopic to D^2, which is contractible.

8.0.2 Aside on Contractible Spaces

Proposition 8.0.2. *Let X be contractible and Y be any space. Then,*

- *X is path-connected*
- *$X \times Y \simeq Y$*
- *Any $f, g : Y \to X$ are homotopic*
- *If Y is path connected, any two maps $X \to Y$ are homotopic*

Proof. By contractibility, picking any basepoint $x_0 \in X$, we have a homotopy $H : X \times I \to X$ with $H(x, 0) = x$ and $H(x, 1) = x_0$. Then for any x, the path $\gamma(t) = H(x, t)$ runs from x to x_0. This proves (i).

The projector $p_Y : X \times Y \to Y$ is a deformation with homotopy

$$(x, y) \mapsto (H(x, t), y)$$

which in $t = 1$ collapses X to a single x_0. This shows (ii)

Consider the function $K_f : Y \times I \to X$ with $K_f(y, t) = H(f(y), t)$. This gives a homotopy $f \simeq c_{x_0}$. By considering K_g and with transitivity of homotopy, we have $f \simeq g$.

The important part about mapso from contractible domains is that it essentially only depends on a point it contracts to. That is, given $f, g : Y \to X$, build homotopies

$$f_t(x) = f(H(x, t))$$

such that at $t = 0$, we have $f_0(x) = f(x)$ and at $t = 1$, we have $f_1(0) = f(x_0)$. In particular, $f \simeq c_{f(x_0)}$, and similarly for g. Finally, we connect the two constant paths via the homotopy that path connects $f(x_0)$ and $g(x_0)$, as Y is path connected. □

8.0.3 Aside on higher-dimensional balls

We specifically consider functions about S^n and their behavior with antipodal points.

Proposition 8.0.3. *The antimpodal map $\alpha : S^n \to S^n$ where $\alpha(x) = -x$ is homotopic to id_{S^n} when n is odd.*

Proof. When $n = 2k+1$ for some integer k, we note that

$$S^{2k+1} \subseteq \mathbb{R}^{2k+2} = \underbrace{\mathbb{R}^2 \otimes \cdots \otimes \mathbb{R}^2}_{k+1 \text{ copies}}$$

Now we can view a point on $x \in S^{2k+1}$ as $(x_1, \ldots, x_{k+1})$ where $x_i \in \mathbb{R}^2$. Then we give explicit homotopy

$$H(x,0) = (x_1, \ldots, x_{k+1}) = x$$
$$H(x,1) = (R_\pi(x_1), \ldots, R_\pi(x_{k+1})) = (-x_1, \ldots, -x_{k+1}) = -x$$

where R_θ is the rotation in the plane by angle θ. Thus H is a homotopy id $\simeq \alpha$ □

Proposition 8.0.4. *If $f, g : X \to S^n$ never hit antipodes, they are homotopic. That is,*

$$f(x) \neq -g(x)$$

for all $x \in X$.

Proof. We consider the straightline homotopy in $\mathbb{R}^{n+1}$, and then normalize it back to the sphere, as they never pass through the origin. Thus, we take

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}$$

□

8.0.4 Homotopic Equivalent Spaces

Proposition 8.0.5. *The following are homotopy equivalent :*

1. $S^1 \vee S^1$
2. $S^1 \times S^1$ *with one point removed*
3. $\mathbb{R}^2$ *minus two distinct points*

Proof. $(i) \simeq (ii)$ is straightforward, by considering the cell complex of the torus, and noting that removing a point acts as removing the 2-cell.

$(i) \simeq (iii)$ by noting that removing two distinct points on $\mathbb{R}^2$ gives exactly the structure we expect from $S^1 \vee S^1$ up to retraction.

□

8.1 Additonal Properties about Spaces

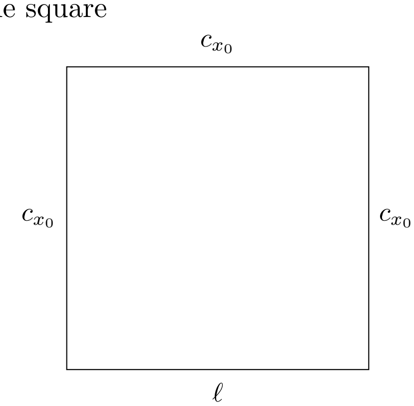
Proposition 8.1.1. *For any space X, the following conditions are equivalent:*

- *Every map $S^1 \to X$ is homotopic to a constant map*
- *Every map $S^1 \to X$ extends to a map $D^2 \to X$*
- $\pi_1(X, x_0) = 0$ *for all $x_0 \in X$.*

Proof. $(i) \Rightarrow (ii)$ Let $f : S^1 \to X$ be null-homotopic, such that $H : S^1 \times I \to X$ is the homotopy that takes f to c_x for some $x \in X$. Now note that $S^1 \times I$ is homeomorphic to the Annulus $\{w \in \mathbb{C} \mid 1 \leq |w| \leq 2\}$. The boundary components come from $S^1 \times \{0\}$ and $S^1 \times \{1\}$, which under H is sent to f and c_x respectively. Noting that $D^2 = (S^1 \times [0,1])/(S^1 \times \{1\})$, we see the natural extension that sends elements on the outside boundary to x.

$(ii) \Rightarrow (i)$ If $f : S^1 \to X$ extends to a $\tilde{f} : D^2 \to X$, then precomposing with the deformation $\rho : D^2 \to \{0\} \subseteq D^2$, gives a homotopy from f to the constant map at $\tilde{f}(0)$. Explicitly, we take the retraction $R : D^2 \times [0,1] \to D^2$ with $R(x,0) = x$ and $R(x,1) = 0$. Then, we construct a homotopy $H : S^1 \times [0,1] \to X$ with $H(z,t) = \tilde{f}(R(z,t))$.

$(ii) \Leftrightarrow (iii)$ If a loop $S^1 \to X$ extends to a map $D^2 \to X$, then this is null homotopic. This follows from the fact we have the square



which is homotopic to the disk. Thus extendable implies every loop is trivial in $\pi_1(X)$. In the other direction, we use the exact same argument in reverse to go from the square to the disk for an extension. $\square$

Proposition 8.1.2. *The fundamental group of a product splits. That is, given (X, x_0) and (Y, y_0) be based spaces,*

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Proof. We give an explicit group isomorphism $\phi : \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by

$$\phi([\ell_1], [\ell_2]) = [t \mapsto (\ell_1(t), \ell_2(t))]$$

This is well-defined (concatenation of loops in factors goes to concatenation in the product). It is injective by considering projections to X and Y, and is surjective as homotopies in the product are exactly the pairs of homotopies in each factor. $\square$

Corollary 8.1.3. *The torus has fundamental group $\mathbb{Z}^2$.*

Proof. Immediate, noting that $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$. $\square$

Definition 8.1.4. *A **retraction** of a space X onto a subspace A is a map $r : X \to A$ such that $ri = \mathrm{id}_A$ where $i : A \to X$ is the inclusion map.*

Example 8.1.5. There is no retraction map $r : D^2 \to S^1$.

We note the induced map from S^1 to D^2 then S^1 looks on the fundamental group looks like

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{r_*} \mathbb{Z}$$

But the composition is clearly not the identity.

Thus, every $f : D^2 \to D^2$ has a fixed point. If for every $x \in D^2$ we have $f(x) \neq x$, then we can draw a ray from $f(x)$ to x and define the map r to be the assignment of x to the point the ray hits on S^1. This is a continuous retraction, but no such retraction exists.

Proposition 8.1.6. *Given $n > 2$, none of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^n$ are homeomorphic.*

Proof. $\mathbb{R}$ is not homeomorphic to neither of these, as removing a point makes $\mathbb{R}$ disconnected but not the others. For the case $\mathbb{R}^2$ and $\mathbb{R}^n$, we note that removing a point from both gives S^1 and S^{n-1}, which have different fundamental groups, so are not homeomorphic. □

Proposition 8.1.7. *S^2 is not homeomorphic to S^n for any $n \neq 2$.*

Proof. Use the fact S^n minus two points is homotopic to S^{n-1}. □

Proposition 8.1.8. *There is no retraction of the Möbius band onto its boundary.*

Proof. Sketch: take the induced maps on fundamental groups, and note that the inclusion sends to $2\mathbb{Z}$. □

Remark 8.1.9. Considering the Torus obtained by side identifications of a square, we can consider the cell decomposition, such that removing a single disk about the center of the square is homotopic to $S^1 \vee S^1$. Reading back, the generators spell $xyx^{-1}y^{-1}$ on the generating set of $\pi_1(X, b)$.

Example 8.1.10. Let S be the two-holed torus. We obtain this via X, who is $S^1 \times S^1$ minus a disc D. Then we have

$$S = X_1 \cup_{\partial D} X_2$$

where $X_1 \cap X_2 \simeq S^1$.

obtained by taking two copies of the torus, cutting them out about a disc and identifying them. We will show that $\pi_1(S)$ is an amalgamated free product.

Then by Seifert Van Kampen, the fundamental group is simply

$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} = x_2 y_2 x_2^{-1} y_2^{-1} \rangle$$

And is an amalgamated free product induced by $X_1 \cup N *_N X_2 \cup N$, where N is a slight-extension of X_1 and X_2 who is homeomorphic to $S^1 \times (0, 1)$.

Example 8.1.11. Some examples of simply connected covering spaces:

- If we consider the square with side identifications which is homeomorphic to the Möbius band, the universal cover is the bundle of these over the real line in an infinite strip, with $\tilde{M} \simeq \mathbb{R} \times I$
- The wedge $S^2 \vee S^1$ has fundamental group $\mathbb{Z}$, its universal cover is an infinite string along $\mathbb{R}$ where we have a copy of S^2 along each integer $n \in \mathbb{Z}$
- The punctured plane $\mathbb{R}^2\{\text{pt}\}$ has universal cover $\tilde{U} = \mathbb{R}^2$ with covering map $p(u, v) = (e^u \cos v, e^u \sin v)$ (which comes from the mapping $z \mapsto e^z$).

8.2 Notes

- Cayley 2-skeleton of $\tilde{K}$.
- The 1-skeleton of $\tilde{K}$ is the Cayley grpah of G wrt to generators of presentation
- Seifert Van Kampen, path connected (else consider $S^1 \setminus \{1\}$ and $S^1 \setminus \{-1\}$)