Notes on Algebra

Apiros3

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1 Preliminaries

1.1 Algebras

1.1.1 Ambient Structure

Fix a base category $\mathcal{C}$, and a split fibration $p : \mathcal{E} \to \mathcal{C}$.

Definition 1.1.1. *Given an endofunctor $F : \mathcal{C} \to \mathcal{C}$, a* ***truth-preserving fibred lift*** *is a lift $\hat{F} : \mathcal{E} \to \mathcal{E}$ such that the following commutes:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{C} \\ \mathcal{E} \arrow[u, "p"] \arrow[r, "\hat{F}"'] & \mathcal{E} \arrow[u, "p"'] \end{tikzcd}$$

1.1.2 F-algebra

Definition 1.1.2. *An F-**algebra** is a pair (A, α) with*

$$\alpha : F(A) \to A$$

Definition 1.1.3. *A* ***morphism*** *of algebras*

$$f : (A, \alpha) \to (B, \beta)$$

is a map $f : A \to B$ in $\mathcal{C}$ such that the following square commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(B) \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

As notation, we write $\mathrm{Alg}(F)$ *for the category of such algebras/morphisms.*

Definition 1.1.4. $(\mu F, \mathrm{in})$ *is* ***initial*** *in* $\mathrm{Alg}(F)$ *if for every (B, β) there is a unique algebra morphism*

$$\mathrm{fold}_\beta : \mu F \to B$$

At the type theory level, $(\mu F, in)$ corresponds to the inductive type and its constructors. Then, fold_β is the non-dependent eliminator / recursor. The square is exactly the β-rule, and the uniqueness of fold is the η-rule.

1.1.3 Displayed (dependent) F-algebra

Fix a base algebra (A, α).

Definition 1.1.5. *A* ***displayed** F**-algebra*** *over (A, α) is*

- *an object $P \in \mathcal{E}_A$*

- *a **dependent structure map** lying over α with*

$$\hat{\alpha} : \hat{F}_A(P) \to P$$

with $p(\hat{\alpha}) = \alpha$, where p is the projection from $\hat{F}_A(P)$ to P.

That is, $\hat{\alpha}$ is a morphism in the fibre $\mathcal{E}_A$ lying over α.

Intuitively, P first states the goal to prove for each element of A. $\hat{F}_A(P)$ packages exactly the induction hypotheses needed at one constructor step, and $\hat{\alpha}$ is the method that consumes these hypotheses to produce a P-witness at the result of that constructor step.

From a type theory perspective $P : A \to \mathsf{Type}$ is the *motive*, $\hat{\alpha}$ packages one *method* per constructor, consuming inductive hypotheses ($\hat{F}_A(P)$).

1.1.4 Displayed Morphism over an Algebra

Fix an algebra $(A, \alpha), (B, \beta)$ and an algebra morphism $f : (A, \alpha) \to (B, \beta)$. Let $(P, \hat{\alpha})$ and $(Q, \hat{\beta})$ be the displayed algebras over (A, α) and (B, β) respectively.

Definition 1.1.6. *A **displayed morphism over** f is a fibre map*

$$\overline{f} : P \to f^*(Q)$$

*in $\mathcal{E}_A$, such that the following **displayed square** commutes:*

$$\begin{tikzcd}
\hat{F}_A(P) \arrow[r, "\hat{F}_A(\overline{f})"] \arrow[d, "\hat{\alpha}"'] & \hat{F}_A(f^*(Q)) \arrow[d, "f^*(\hat{\beta})"] \\
P \arrow[r, "\overline{f}"'] & f^*(Q)
\end{tikzcd}$$

We make sure this type checks by:

$$\begin{tikzcd}
\hat{F}_A(P) \arrow[r, "\hat{F}_A(\overline{f})"] \arrow[d, "\hat{\alpha}"'] & \hat{F}_A(f^*(Q)) \arrow[r, leftrightarrow, "\cong"] & (Ff)^*\hat{F}_B(Q) \arrow[d, "f^*(\hat{\beta})"] & \hat{F}_B(Q) \arrow[l, "(Ff)^*"'] \arrow[d, "\hat{\beta}"] \\
P \arrow[rr, "\overline{f}"'] & & f^*(Q) & Q \arrow[l, "f^*"]
\end{tikzcd}$$

Intuitively, $\overline{f}$ transports P-evidence on A to Q-evidence on B along f and does so constructor-by-constructor compatibly with the methods. This is the naturality / fusion condition for logical predicates.

Definition 1.1.7. *Let $\mathrm{DAlg}_A(F, \alpha)$ denote the category whose*

- *objects are displayed algebras $(P, \hat{\alpha})$*
- *morphisms $\theta : (P, \hat{\alpha}) \to (P', \hat{\alpha'})$ are maps $\theta : P \to P'$ with*

$$\begin{tikzcd}
\hat{F}_A(P) \arrow[r, "\hat{F}_A(\theta)"] \arrow[d, "\hat{\alpha}"'] & \hat{F}_A(P') \arrow[d, "\hat{\alpha}'"] \\
P \arrow[r, "\theta"'] & P'
\end{tikzcd}$$

Displayed morphisms compose over composition of base morphisms and identities are given by the fibre identities. This gives the projection

$$p^F : \mathrm{DAlg}(F) \to \mathrm{Alg}(F)$$

with the structure of a fibration, where objects over (A, α) are displayed algebras and arrows over f are displayed morphisms.