First Read on Linear Algebra

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1 Introduction

This note will aim to provide a quick introduction on Linear Algebra, covering the key concepts that one would come across in an introductory course. This is aimed to condense Oxford's Prelims Linear Algebra I, Linear Algebra II, and Part A Linear Algebra, which is about 40 lectures of material down to it's key concepts so that one could get an intuition for what main theorems the course is getting to.

We first start with two independent sections, covering Vector Spaces in Section 2 and Matrices in Section 3.

As a disclaimer, these notes are compressed to note the key concepts and hence does not have many numerical / computation examples. I have tried my best to give construction when possible, but please note it may not be as comprehensive as other introductory material.

2 Vector Spaces

2.1 Fields and Vector Spaces

Definition 2.1.1. *A **Field**, denoted $\mathbb{F}$ is a tuple $(X, +, \times, 0, 1)$, on some set X with binary operators such that*

2.2 Basis

Definition 2.2.1. *Let V be a vector space over $\mathbb{F}$. We say that the set $S \subseteq V$ (where S may be finite or infinite) has a **spanning set** generated over $\mathbb{F}$ is*

$$\langle S \rangle_{\mathbb{F}} = \{\sum_{i=0}^{n} a_i v_i \mid n \geq 0, a_1, \ldots, a_n \in \mathbb{F}, v_1, \ldots, v_n \in S\}$$

*This is the set of all finite linear combinations of elements in S. We say that S **spans** V if $V = \langle S \rangle_{\mathbb{F}}$.*

Notation 2.2.2. When we are working with a finite $S := \{v_1, \ldots, v_n\}$, we often omit the set notation, writing $\langle v_1, \ldots, v_n \rangle$ to mean $\langle S \rangle$. The base field subscript is omitted when it is clear what field we are working under.

Lemma 2.2.3 (Steinitz Exchange Lemma)**.** *Let V be a vector space over a field $\mathbb{F}$. Take any finite $X := \{v_1, \ldots, v_n\} \subseteq V$. Suppose now we have a $u \in \langle X \rangle$ but $u \notin \langle X \backslash \{v_i\} \rangle$ for some i. Let*

$$Y := (X \backslash \{v_i\}) \cup \{u\}$$

Then, $\langle Y \rangle = \langle X \rangle$.

Proof. Let $u \in \langle X \rangle$. Then, we can find $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that

$$u = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

By assumption, there is some $v_i \in X$ such that $u \notin \langle X \backslash \{v_i\} \rangle$. Without loss of generality, take $i = n$. As $u \notin \langle X \backslash \{v_n\} \rangle$, $\alpha_n \neq 0$. This gives

$$v_n = \frac{1}{\alpha_n}(u - \alpha_1 v_1 - \cdots - \alpha_{n-1} v_{n-1})$$

Taking any $w \in \langle Y \rangle$, then we can write w as a linear combination of elements in Y. Replacing u with $\alpha_1 v_1 + \cdots + \alpha_n v_n$, we can write w as linear combination of elements of X. This gives $\langle Y \rangle \subseteq \langle X \rangle$. Noting that $w \in \langle X \rangle$, we can write w as a linear combination of elements of X. Replacing v_n by the equality above, we can write w as a linear combination of elements of Y. This gives $\langle X \rangle \subseteq \langle Y \rangle$. $\square$

Definition 2.2.4. *Let V be a vector space over $\mathbb{F}$. The set $S := \{v_1, \ldots, v_n\}$ is **linearly independent** over $\mathbb{F}$ if*

$$\sum_{i=1}^{n} a_i v_i = 0$$

*implies that $a_i = 0$ for all $i = 1, \ldots, n$. If we can find a nontrivial set that sums to 0, we call the set **linearly dependent**.*

Definition 2.2.5. *A **basis** for a vector space V over $\mathbb{F}$ is a set $\mathcal{B}$ who is both spanning and linearly independent. If V has a finite basis, we call V **finite dimensional**.*

2.3 Dimension

A nice property about finite dimensional vector spaces is that we can always find a basis, and that we can extend any linearly independent set to a basis.

2.4 Mapping Between Vector Spaces

A linear map is a structure preserving morphism between vector spaces.

Definition 2.4.1. *Let $T : V \to W$ be a linear map between vector spaces V and W. We define the **kernel of** T to be*

$$\ker T := \{v \in V \mid T(v) = 0\}$$

*Similarly, we define the **image of** T to be*

$$\operatorname{Im} T = \{T(v) \mid v \in V\}$$

Proposition 2.4.2. *Let $T : V \to W$ be a linear map between vector spaces. We note some basic facts about kernels and images:*

1. $\ker T$ *is a subspace of* V
2. $\operatorname{Im} T$ *is a subspace of* W

2.4.1 Quotient Spaces

Given a vector space, we can construct new spaces from the original. Here we introduce quotient spaces, which acts by 'killing' by a subspace.

Definition 2.4.3. *Let U be a subspace of a vector space V. For any $v \in V$, define*

$$v + U = \{v + u \mid u \in U\}$$

We define addition and scalar multiplication on these cosets by

1. *For $\lambda \in \mathbb{F}, v \in V$, $\lambda(v + U) := \lambda v + U$*
2. *For any $v, w \in V$, $(v + U) + (w + U) := (v + w) + U$*

Proposition 2.4.4. *The operations defined above are well-defined. That is, the outcome of the operation does not depend on the choice of representative.*

Proof. Suppose that $v + U = v' + U$ and $w + U = w' + U$. Then we have $v = v' + u, w = w' + \tilde{u}$ for some $u, \tilde{u} \in U$.

Now,

$$(v + U) + (w + U) = (v + w) + U = (v' + u + w' + \tilde{u}) + U = (v' + w') + U = (v' + U) + (w' + U)$$

and

$$\lambda(v + U) = \lambda v + U = \lambda v' + \lambda u + U = \lambda v' + U = \lambda(v' + U)$$

where we used the fact $\lambda u \in U$. $\square$

Definition 2.4.5. *Given a subspace U of a vector space V over $\mathbb{F}$, define the* ***quotient space*** *to be the space*

$$V/U := \{v + U \mid v \in V\}$$

where scalar multiplication is given by $\lambda(v+U) = \lambda v + U$ and addition by $(v+U)+(w+U) = (v+w)+U$.

Notation 2.4.6. As notation, we may write v_U or $[v]$ to mean the set $v + U$, whenever it is clear what we are representing. In this case, $\lambda[v] = [\lambda v]$ and $[v] + [w] = [v + w]$. This definition is well defined and does not depend on the choice of representative due to Proposition 2.4.4.

Remark 2.4.7. Quotienting by a subspace U groups elements that are different by an element of U. Addition and scalar multiplication works by inheriting it's structure from V. There is a natural linear map from V to V/U by

$$q_U : V \to V/U \qquad v \mapsto v + U$$

We will call this map the **quotient map**.

Theorem 2.4.8 (First Isomorphism Theorem for Vector Spaces)**.** *Let V and W be vector spaces over a field $\mathbb{F}$ and $T : V \to W$ be a linear map. Then we have an isomorphism of vector spaces*

$$\tilde{T} : V/\ker T \overset{\cong}{\to} \operatorname{Im} T$$

by the map $v + \ker T \mapsto T(v)$.

Proof. We first show that this map is well defined. Choose two $v, w \in V$ such that $v + \ker T = w + \ker T$. In particular, we can find $v' \in \ker T$ such that $v + v' = w$. Then, applying T,

$$\tilde{T}(w + \ker T) = T(w) = T(v + v') = T(v) + T(v') = T(v) = \tilde{T}(v + \ker T)$$

As $T(v) = T(w)$, the map does not depend on the choice of representative, and thus well defined. Linearity is inherited from T, and explicitly checking is routine. By construction, the map is surjective onto the image of T. Suppose $\tilde{T}(v + \ker T) = 0$. Then by definition, $v \in \ker T$. Hence, $v + \ker T = \ker T$, meaning the map is injective. Thus $\tilde{T}$ is a bijective linear map, hence an isomorphism. □

Proposition 2.4.9 (Universal Property of Quotients)**.** *Let $T : V \to W$ be a map between vector spaces. Let U be a subspace of V such that $U \subseteq \ker T$. Then, there exists a unique ψ such that the following diagram commutes.*

Remark 2.4.10. The first isomorphism theorem at it's heart says that when collapse everything on the kernel onto a point, the resulting quotient embeds exactly onto the image of T in W. In particular, when we have an injective map, V embeds into W, essentially being able to act like a subspace of W. When the map is surjective, W is structurally the same as $V/\ker T$.

This intertwines nicely with the universal property of quotients, because this shows that we can always factor any map that kills the kernel of T uniquely through $V/\ker T$.

Theorem 2.4.11 (Rank Nullity)**.** *Let $T : V \to W$ be a linear map between vector spaces, where V is finite dimensional. Then,*

$$\dim(V) = \dim(\ker T) + \dim(\operatorname{Im} T)$$

Proof. Noting that $\dim(V) = \dim(\ker T) + \dim(V/\ker T)$, it then follows immediately from the First Isomorphism Theorem, which gives $\dim(V/\ker T) = \dim(\operatorname{Im} T)$. □

2.5 Direct Sums and Decomposition

Definition 2.5.1. *Let V be a vector space and U, W be subspaces of V. We write*

$$V = U \oplus W$$

*if $V = U + W$ and $U \cap W = \{0\}$. This is called the **inner direct product**.*

Proposition 2.5.2. *Let V be a vector space and U, W be subspaces of V. The following are equivalent:*

1. *$V = U \oplus W$*
2. *for any $v \in V$, we can uniquely write $v = u + w$ for some $u \in U$ and $w \in W$.*

3 Matrices

4 Spectral Theorem

4.1 Eigenvectors and Eigenvalues

Definition 4.1.1. *Let $T : V \to V$ be a linear map between a vector space. A nonzero vector $v \in V$ is called a **eigenvector** if $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$. The λ is called the **eigenvalue**.*

Eigenstuff
Characteristic, norm, trace