First Read on Linear Algebra, Lecture 4

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4 Matrices

4.1 Basic Definitions

Definition 4.1.1. *Let m nand n be nonnegative integers and $\mathbb{F}$ be a field. A **m-by$n$ matrix** A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns, written*

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

where we write $A_{i,j}$ to denote the entry in row i, column j of A. When it is not clear, we write $A = (a_{i,j})$ to represent that A is a matrix with entries given by $a_{i,j}$. The set of all $m \times n$ matrices given by entries in $\mathbb{F}$ is given by $\mathcal{M}_{m\times n}(\mathbb{F})$.

Definition 4.1.2. *Given $A, B \in \mathcal{M}_{m\times n}(\mathbb{F})$, we define $A + B$ to be the matrix whose entries are given by $(A + B)_{i,j} = A_{i,j} + B_{i,j}$.*

Proposition 4.1.3. *Matrices addition is commutative and associative. That is, given matrices L, M, N where addition is well-defined, we have*

$$M + N = N + M$$

and

$$L + (M + N) = (L + M) + N$$

Proof. This is a consequence of the fact that addition over a field is commutative and associative. □

Remark 4.1.4. The $m \times n$ **zero matrix** is the matrix with m rows and n columns with entries all 0. We write 0 to denote this matrix or $0_{m,n}$ when the dimension is not clear. Then, the zero matrix is the additive identity on matrices.

Definition 4.1.5. *Given $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ and $\lambda \in \mathbb{F}$, we define λA to be the matrix whose entries are given by $(\lambda A)_{i,j} = \lambda A_{i,j}$.*

Proposition 4.1.6. *Given a matrix A and $\lambda, \mu \in \mathbb{F}$, We have the following identities:*

$$0A = 0_{m,n} \qquad A + (-A) = 0_{m,n} \qquad 1A = A$$

$$(\lambda + \mu)A = \lambda A + \mu A \qquad \lambda(A + B) = \lambda A + \lambda B \qquad \lambda(\mu A) = (\lambda\mu)A$$

Proof. Straightforward by checking what happens to each entry. □

Remark 4.1.7. The above proposition shows that $\mathcal{M}_{m,n}(\mathbb{F})$ is a vector space over $\mathbb{F}$.

Definition 4.1.8. *Given $A \in \mathcal{M}_{m\times k}(\mathbb{F})$ and $B \in \mathcal{M}_{k\times n}(\mathbb{F})$, we define AB to be the matrix whose entries are given by $(AB)_{i,j} = \sum_{\ell=1}^{k} a_{i,\ell} b_{\ell,j}$.*

Definition 4.1.9. *The $n \times n$ **identity matrix** I_n is the matrix with entries*

$$\delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

*We write I to denote the identity matrix when the dimension is clear. The entries $\delta_{i,j}$ is also called the **Kronecker delta**.*

Proposition 4.1.10. *Let A be an $m \times n$ matrix. Then,*

$$A0_{n,p} = 0_{n,p} \qquad 0_{l,m}A = 0_{l,n} \qquad AI_n = A \qquad I_M A = A$$

Proof. Is a straightforward check by seeing what the entries are. □

Proposition 4.1.11. *Matrix multiplication is associative and distributive. That is, for matrix A, B, C where the operations are defined, we have*

$$A(BC) = (AB)C$$

and

$$A(B + C) = AB + AC \qquad (A + B)C = AC + BC$$

Proof. Is a straightforward check by seeing what the entries are. For associativity we note that finite sums can be swapped. □

Remark 4.1.12. Matrix multiplication is in general not commutative, and cannot deduce from $AB = 0$ that $A = 0$ or $B = 0$. For instance,

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition 4.1.13. *For a square matrix A, we write A^2 for the product AA and A^n for*

$$\underbrace{AA \cdots A}_{n \text{ times}}$$

where $A^0 = I$. Then, given a polynomial $p(x) = a_k x^k + \cdots + a_0$, define

$$p(A) = a_k A^k + \cdots + a_0 I$$

Definition 4.1.14. *Let A be a square matrix. We say that B is an **inverse** of A if $BA = AB = I$, and we write A^{-1} to represent such inverse. A matrix with an inverse is **invertible** and otherwise is called **singular**.*

Proposition 4.1.15. *We have the following properties about invertible matrices:*

1. *Matrix inverses are unique*

2. *If A, B are invertible matrices, then AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$*

3. *If A is invertible, then so is A^{-1} with $(A^{-1})^{-1} = A$*

Proof. (i) Suppose B and C are inverses for A. Then,

$$C = I_n C = (BA)C = B(AC) = BI_n = B$$

(ii) We have

$$AB(B^{-1}A^{-1}) = AA^{-1} = I \qquad (B^{-1}A^{-1})AB = B^{-1}B = I$$

(iii) We note that

$$(A^{-1})A = A(A^{-1}) = I$$

and so $(A^{-1})^{-1} = A$ by uniqueness. □

Definition 4.1.16. *Given $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, we define the* ***transpose*** *of A, written A^T to be the $n \times m$ matrix given by $(A^T)_{i,j} = A_{j,i}$.*

Proposition 4.1.17. *Given matrices A, B and $\lambda \in \mathbb{F}$, we have the following properties (when operations make sense):*

- $(A + B)^T = A^T + B^T$ *and* $(\lambda A)^T = \lambda A^T$
- $(AB)^T = B^T A^T$
- $(A^T)^T = A$
- *A square matrix A is invertible if and only if A^T is invertible. Then, we have $\left(A^T\right)^1 = (A^{-1})^T$.*

4.2 ERO Decomposition

Definition 4.2.1. *A* ***elementary row operation*** *or* ***ERO*** *is an operation that is of the following form:*

- $S_{i,j}$ *: swapping rows i and j*
- $M_i(\lambda)$ *: multiplies row i by $\lambda \neq 0$*
- *For $i \neq j$, $A_{i,j}(\lambda)$: adds λ times row i to row j.*

Proposition 4.2.2. *On a $m \times n$ matrix A, applying EROs is equivalent to premultiplying A by certain matrices, which we give by $S_{I,J}$, $M_I(\lambda)$, and $A_{I,J}(\lambda)$ by:*

$$\text{the } (i,j)\text{th entry of } S_{I,J} = \begin{cases} 1 & i = j \neq I, J, \\ 1 & i = J,\ j = I, \\ 1 & i = I,\ j = J, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{the } (i,j)\text{th entry of } M_I(\lambda) = \begin{cases} 1 & i = j \neq I, \\ \lambda & i = j = I, \\ 0 & \text{otherwise.} \end{cases}$$

$$\textit{the } (i,j)\textit{th entry of } A_{I,J}(\lambda) = \begin{cases} 1 & i = j, \\ \lambda & i = J,\ j = I, \\ 0 & \textit{otherwise.} \end{cases}$$

These matrices are known as ***elementary matrices****.*

Proof. By explicitly calculating what happens to each entry after the premultiplication. □

Proposition 4.2.3. *Elementary matrices are invertible.*

Proof. Follows from the fact that

$$(S_{i,j})^{-1} = S_{j,i} = S_{i,j} \qquad (A_{i,j}(\lambda))^{-1} = A_{i,j}(-\lambda) \qquad (M_i(\lambda))^{-1} = M_i(\lambda^{-1})$$

□