Notes on Representation Theory

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Contents

1 Preliminaries

1.0.1 Linear Maps

Definition 1.0.1. *Let V be a vector space over a field k. Define $\mathrm{GL}(V)$ to be the set of invertible linear maps, with group operation defined by composition.*

Proposition 1.0.2. *Let $g \in \mathrm{GL}(V)$ be an element of finite order and suppose that k is algebraically closed with 0 characteristic. Then g is diagonalizable.*

Proof. Let n be the order of $g \in \mathrm{GL}(V)$. Then $g^n = 1$, so g is annihlated by the polynomial $f(x) := x^n - 1$. The $m_g | f$, but as f splits in k and has no repeated roots, $m_g(x)$ splits into distinct linear factors. Hence g is diagonalizable by the Primary Decomposition Theorem. $\square$

Remark 1.0.3. The converse does not hold in general. Consider $k = \overline{\mathbb{F}_2}$, $V = k\{e_1, e_2\}$ and $g \in \mathrm{GL}(V)$ be given by $g(e_1) = e_1, g(e_2) = e_1 + e_2$. Then $g^2 = 1$, and $m_g(x) = x^2 - 1 = (x-1)^2$. This has repeated roots, hence not diagonalizable.

1.0.2 Direct Sum

Definition 1.0.4. *Let V and W be vector spaces. The* ***external direct sum*** *is the vector space $V \oplus W := V \times W$.*

Remark 1.0.5. The external direct sum is consistent with the internal direct sum by identifying V and W with their images $\{(v,0) \mid v \in V\}$ and $\{(0,w) \mid w \in W\}$ inside $V \times W$. The sum of their images is all of $V \times W$ and the intersection is $\{(0,0)\}$.

Definition 1.0.6. *The* ***dual space*** *of a vector space V over $\mathbb{F}$ is $\mathrm{Hom}(V, \mathbb{F})$, where addition is defined pointwise and multiplication by composition.*

Remark 1.0.7. Idempotent actions decompose vector spaces. That is, if $P : V \to V$ with $P^2 = P$, then we can write

$$V = P(V) \oplus \ker P$$

noting that we can write $v = P(v) + (v - P(v))$.

1.0.3 Tensor Product

Definition 1.0.8. *Let V and W be two vectos spaces, with $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ as basis for V and W respectively. The* ***tensor product*** *of V and W, written $V \times W$ is the free vector space on the set of formal symbols*

$$\{v_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

If $v = \sum_{i=1}^m \lambda_i v_i$ and $w = \sum_{j=1}^n \mu_j w_j$ are elements of V and W respectively, we define the elementary tensor

$$v \otimes w := \sum_{i=1}^m \sum_{j=1}^n \lambda_i \mu_j (v_i \otimes w_j) \in V \otimes W$$

Remark 1.0.9. We note the following results, which are immediate from definition.

- $\dim V \otimes W = (\dim V)(\dim W)$
- The elementary tensors span $V \otimes W$

- Not every element of $V \otimes W$ is an elementary tensor $v \otimes w$.

The free vector space does not depend on the choice of basis.

Lemma 1.0.10. *Let $\{v'_1, \dots, v'_m\}$ and $\{w'_1, \dots, w'_n\}$ be any choice of basis for V and W. Then,*

$$X' := \{v'_i \otimes w'_j \mid 1 \le i \le m, 1 \le j \le n\}$$

is a basis for $V \otimes W$.

Proof. We note that elementary tensors in $V \otimes W$ distribute, in the sense that

$$(v + v') \otimes (w + w') = (v \otimes w) + (v \otimes w') + (v' \otimes w) + (v' \otimes w')$$

for all $v, v' \in V$, $w, w' \in W$, and hence

$$(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$$

for all $v \in V, w \in W, \lambda \in k$. Hence, we can write each v_i as a linear combination of $\{v'_1, \dots, v'_m\}$ and each w_j as a linear combination of $\{w'_1, \dots, w'_n\}$, and see that the original basis vectors $v_i \otimes w_j$ of $V \otimes W$ all lie in the span of X'. As the size of this set is at most mn, it must be linearly independent, thus a basis. $\square$

Remark 1.0.11. The above proof also shows that the canonical map $\otimes : V \times W \to V \otimes W$ by $(v, w) \mapsto v \otimes w$ is bilinear, such that

$$(\lambda v_1 + v_2) \otimes (\mu w_1 + w_2) = \lambda\mu(v_1 \otimes w_1) + \lambda(v_1 \otimes w) + \mu(v_2 \otimes w_1) + (v_2 \otimes w_2)$$

Lemma 1.0.12 (Universal Property of Tensor Product)**.** *Let V and W be vector spaces. Then for every blinear map $b : V \times W \to U$ for some third vector space U, we have a unique linear map $\tilde{b} : V \otimes W \to U$ such that the following commutes:*

$$\begin{tikzcd}
V \times W \arrow[r, "b"] \arrow[d] & U \\
V \otimes W \arrow[ur, "\tilde{b}"'] &
\end{tikzcd}$$

Proof. Fix bases $\{v_1, \dots, v_m\}$ for V and $\{w_1, \dots, w_n\}$ for W. Fixing a bilinear map $b : V \times W \to U$, this forces any $\tilde{b} : V \otimes W \to U$ to be the unique linear map that sends the basis vector $v_i \otimes w_j \in V \otimes W$ to $b(v_i, w_j)$, if it exists. It thus suffices to show this map commutes with any element in $V \times W$. Taking any $v = \sum_i \lambda_i v_i \in V$ and $w = \sum_j \mu_j w_j \in W$, $\tilde{b}$ sends the elementary tensor $v \otimes w = \sum_{i,j} \lambda_i \mu_j v_i \otimes w_j$ to

$$\sum_{i,j} \lambda_i \mu_j b(v_i, w_j) = b\left(\sum_i \lambda_i v_i, \sum_j \mu_j w_j\right) = b(v, w)$$

by using the bilinearity of b. $\square$

1.0.4 Module Endomorphisms

Definition 1.0.13. *The **center** of the ring A is*

$$Z(A) := \{z \in A \mid az = za, \textit{for all } a \in A\}$$

The center is a commutative unital subring of A.

Definition 1.0.14. *Let A be a ring and V be an A-module. The **endomorphism ring** of V, denoted $\operatorname{End}_A(V)$ is the set of all A-module homomorphisms $\psi : V \to V$ equipped with pointwise addition of homomorphisms and composition as multiplication.*

Remark 1.0.15. When V is an A-module, it is an $\operatorname{End}_A(V)$-module via evaluation, $f \cdot v := f(v)$, for $f \in \operatorname{End}_A(V)$ and $v \in V$. The two actions of A and $\operatorname{End}_A(V)$ on V commute pointwise by definition. In particular, the action of any central element $z \in Z(A)$ on V is by an A-module endomorphism.

Definition 1.0.16. *A ring A is an **k-algebra** if it contains k as a central subfield. If A is a semisimple ring, we say that A is a **semisimple k-algebra**. A **homomorphism** of k-algebras is a k-linear ring homomorphism.*

k being inside the center allows it to 'act' like the scalar, making the definition of homomorphism the way we think of it naturally.

1.0.5 Group and Module Notations

Definition 1.0.17. *Let G be a finite group and let $g \in G$. Define*

- *g^G to denote the **conjugacy class** of g in G,*

$$g^G := \{g^x \mid x \in G\} \textit{where } g^x := x^{-1}gx$$

- *$C_G(g)$ denotes the **centraliser** of g in G, with*

$$C_G(g) := \{x \in G \mid gx = xg\}$$

Remark 1.0.18. By The Orbit Stabilizer on the conjugation action, we have $|g^G| \cdot |C_G(g)| = |G|$ for any $g \in G$, where g^G is the conjugacy class, and the stabilizer of g is exactly the centraliser.

Definition 1.0.19. *Let V be a $\mathbb{C}G$-module. The **invariant submodule** of V is*

$$V^G := \{v \in V \mid g \cdot v = v, \textit{for all } g \in G\}$$

1.0.6 Algebraic Numbers

Notation 1.0.20. We write $\mathbb{A}$ for the set of algebraic integers over $\mathbb{Q}$. Note that the set of algebraic numbers is the union of all subfields of $\mathbb{C}$ of finite dimension as a $\mathbb{Q}$-vector space.

Remark 1.0.21. Note the following:

- Any integer is an algebraic integer by the linear function
- Any root of unity is an algebraic integer

- If z is an algebraic number, then mz is an algebraic integer for some integer m
- $\mathbb{Q} \cap \mathbb{A} = \mathbb{Z}$ (by taking any element of the form r/s, taking it's monic polynomial, clearing denominators to show that s divides r).

Proposition 1.0.22. *Let M be a finitely generated subgroup of $(\mathbb{C}, +)$. Then*

$$\{z \in \mathbb{C} \mid zM \subseteq M\} \subseteq \mathbb{A}$$

2 Definitions and Examples

Definition 2.0.1. *Let G be a finite group and let V be a finite dimensional vector space over k. A **representation** of G on V is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V)$$

*The **degree** of a representation is* $\dim V$.

Example 2.0.2. We record some examples of representations.

- The cyclic group $G = \langle g \rangle$ of order 2 acts on $V = k$ by negation taking $\rho(g) = -1$, giving a representation of G degree 1.
- If $G = D_6$ is the symmetry group of a triangle and $k = \mathbb{R}$, then G acts by $\mathbb{R}$-linear transformations on the plane $V = \mathbb{R}^2$, giving a representation. In general, the symmetry group of the regular n-gon $G = D_{2n}$ acts on $V = \mathbb{R}^2$ by $\mathbb{R}$-linear transformations, giving a natural representation of G of degree 2.
- Let $k = \mathbb{R}$ and let $X \subseteq \mathbb{R}^3$ be the se tof vertices of a cube centered at the origin, and let G be the stabilizer of X in the rotation group $SO_3(\mathbb{R})$. Then G is isomorphic to the symmetry group S_4, giving a degree 3 representation $S_4 \to \mathrm{GL}(\mathbb{R}^3)$.

Example 2.0.3. Ex 1.6 from RT, gal group into base field automorphisms

3 Representation of Finite Groups

Definition 3.0.1. *Let X be a finite set. The **free vector space on** X is the set*

$$kX := \left\{ \sum_{x \in X} a_x x \mid a_x \in k \right\}$$

*of **formal linear combinations** of members of X with coefficients a_x lying in k. Addition and scalar multiplication are taken as the natural ones.*

Remark 3.0.2. Note that X is naturally a basis for kX.

Let X be a finite set equipped with a left-action of the finite group G. Each $g \in G$ gives a permutation $\rho(g) : X \to X$ by $\rho(g)(x) = g \cdot x$. This permutation extends uniquely to an invertible linear map $\rho(g) : kX \to kX$ by

$$\rho(g) \left(\sum_{x \in X} a_x x \right) = \sum_{x \in X} a_x g \cdot x$$

Since $g \cdot (h \cdot x) = (gh) \cdot x$ for any $g, h \in G$ and $x \in X$, we have $\rho(g)\rho(h) = \rho(gh)$ in $\mathrm{GL}(kX)$ for all $g, h \in G$. Thus $\rho : G \to \mathrm{GL}(kX)$ is a representation.

Definition 3.0.3. *Noting the remark above, given X is a finite set equipped with a left action by a finite group G, $\rho : G \to \mathrm{GL}(kX)$ is a representation, called the* ***permutation representation associated with*** *X.*

Definition 3.0.4. *The representation $\rho : G \to \mathrm{GL}(V)$ is* ***faithful*** *if* $\ker \rho = \{1\}$.

Definition 3.0.5. *Let G be a finite group. A* ***matrix representation*** *is a group homomorphism $\rho : G \to \mathrm{GL}_n(k)$, where $\mathrm{GL}_n(k) = M_n(k)^\times$ is the group of invertible $n \times n$ matrices under matrix multiplication.*

Definition 3.0.6. *Let $\mathcal{B} := \{v_1, \ldots, v_n\}$ be a basis for V and let $\phi : V \to V$ be a linear map.The* ***matrix of ϕ with respect to $\mathcal{B}$*** *is ${}_{\mathcal{B}}[\phi]_{\mathcal{B}} = (a_{ij})_{i,j=1}^n$ where*

$$\phi(v_j) = \sum_{i=1}^n a_{ij} v_i$$

for all $j = 1, \ldots, n$.

Remark 3.0.7. Let V be a vector space V with basis $\mathcal{B}$. Then,

- The map $\phi \mapsto_{\mathcal{B}} [\phi]_{\mathcal{B}}$ gives an isomorphism of groups $\mathrm{GL}(V) \cong \mathrm{GL}_n(k)$.
- Every representation $\rho : G \to \mathrm{GL}(V)$ gives rise to a matrix representation

$$\rho_{\mathcal{B}}(g) :=_{\mathcal{B}} [\rho(g)]_{\mathcal{B}}$$

  for all $g \in G$.
- Every matrix representation $\sigma : G \to \mathrm{GL}_n(k)$ defines a representation $\underline{\sigma} : G \to \mathrm{GL}(k^n)$ on the space k^n of column vectors, taking $\underline{\sigma} : k^n \to k^n$ be the k-linear map

$$\underline{\sigma}(g)(v) = \sigma(g)v$$

  for all $g \in G, v \in k^n$ via matrix multiplication. By abuse of notation, the underline is sometimes omitted.

Example 3.0.8. Let $G = S_3$ act on $X = \{e_1, e_2, e_3\}$ by permutation of indices. This gives a degree 3 permutation representation $\rho : G \to \mathrm{GL}(kX)$ where for instance,

$$\rho_X((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Definition 3.0.9. *Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be two representations. A* ***homomorphism****, also known as the* ***intertwining operator*** *is a linear map*

$$\psi : V \to W$$

such that

$$\sigma(g) \circ \psi = \psi \circ \rho(g)$$

for all $g \in G$. We say that ψ is an ***isomorphism*** *if it is bijective.*

Definition 3.0.10. *Two matrix representations $\rho_1 : G \to \mathrm{GL}_n(k)$ and $\rho_2 : G \to \mathrm{GL}_n(k)$ are said to be* ***equivalent*** *if there exists $A \in \mathrm{GL}_n(k)$ such that*

$$\rho_2(g) = A\rho_1(g)A^{-1}$$

for all $g \in G$.

Remark 3.0.11. If ρ_1 and ρ_2 are equivalent matrix representations, then the equality of products of matrices $\rho_2(g)A = A\rho_1(g)$ translates to an equality of linear maps

$$\underline{\rho_2(g)} \circ \underline{A} = \underline{A} \circ \underline{\rho_1(g)}$$

in $\mathrm{GL}(k^n)$, meaning that representations $\underline{\rho_1}$ and $\underline{\rho_2}$ are isomorphic. The converse is also true.

Definition 3.0.12. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and let U be a linear subspace of V. Then,*

- *U is* ***G-stable*** *if $\rho(g)(u) \in U$ for all $g \in G$ and $u \in U$.*
- *Suppose that U is G-stable. Then the* ***subrepresentation of ρ afforded by*** *U is*

  $$\rho_U : G \to \mathrm{GL}(U)$$

  given by $\rho_U(g)(u) := \rho(g)(u)$ for all $g \in G$, $u \in U$.
- *Suppose that U is G-stable. The* ***quotient representation of ρ afforded by*** *U is*

  $$\rho_{V/U} : G \to \mathrm{GL}(V/U)$$

  given by $\rho_{V/U}(g)(v + U) := \rho(g)(v) + U$ for all $g \in G$ and $v + U \in V/U$.

  Note that the maps are well defined when U is G-stable.

Lemma 3.0.13 (First Isomorphism For Representations)**.** *Let $\psi : V \to W$ be a homomorphism between representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$. Then,*

1. $\ker\psi$ *is a G-stable is a subspace of V.*
2. $\mathrm{Im}\ \psi$ *is a G-stable subspace of W*
3. *There is a natural isomorphism*

   $$V/\ker\psi \cong \mathrm{Im}\ \psi$$

   between G-representations $\rho_{V/\ker\psi}$ and $\sigma_{\mathrm{Im}\ \psi}$

Proof. We note the commutative diagram

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\psi"'] & V \arrow[d, "\psi"] \\ W \arrow[r, "\sigma(g)"'] & W \end{tikzcd}$$

Then the first two cases are clear. We note the quotient representation of ρ afforded by $\ker\psi$ and the subrepresentation of σ afforded by Im ψ induces a map $\Psi : V/\ker(\psi) \to \text{Im } \psi$ with $v + \ker(\psi) \mapsto \psi(v)$ alongside a commutative diagram

$$\begin{tikzcd}
V/\ker\psi \arrow[r, "\rho_{V/\ker\psi}(g)"] \arrow[d, "\Psi"'] & V/\ker\psi \arrow[d, "\Psi"] \\
\mathrm{Im}\psi \arrow[r, "\sigma_{\mathrm{Im}\psi}(g)"'] & \mathrm{Im}\psi
\end{tikzcd}$$

In particular, $V/\ker\psi \cong \text{Im } \psi$. □

Definition 3.0.14. *Let G be a group. The* ***trivial representation*** *of G on a vector space V, $\mathbb{1} : G \to \mathrm{GL}(V)$ given by*

$$\mathbb{1}(g)(v) = v$$

for all $g \in G$, $v \in V$.

Example 3.0.15. A representation need not be trivial, even if the subrepresentation and quotient representation are both trivial.

Let $k = \mathbb{F}_p$ and $G = \langle g \rangle$ be the cyclic group of order p. Let $\rho : G \to \mathrm{GL}_2(k)$ be the matrix representation given by

$$\rho(g^i) = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

Let v_1, v_2 be the standard basis for $V = k^2$. Then $U := \langle v_1 \rangle$ is a G-stable subspace, as $\rho(g^i)(v_1) = v_1$ for all i. The subrepresentation $\rho_U : G \to \mathrm{GL}(U)$ and the quotient representation $\rho_{V/U} : G \to \mathrm{GL}(V/U)$ are both trivial, but ρ is clearly not trivial.

Definition 3.0.16. *the representation $\rho : G \to \mathrm{GL}(V)$ to a nonzero V is* ***irreducible*** *or* ***simple*** *if U being a G-stable subspace of V implies that either $U = \{0\}$ or $U = V$.*

Definition 3.0.17. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation and U be a G-stable subspace. A* ***G-stable complement*** *for U in V is a G-stable subspace W such that $V = U \oplus W$.*

Example 3.0.18. Consider the permutation representation of $G = S_3$ afforded by kX, where $X = \{e_1, e_2, e_3\}$. Then

$$U := \langle e_1 + e_2 + e_3 \rangle$$

is a G-stable subspace, with G fixing every vector in U. So U is a trivial subrepresentation of V. Now let

$$W := \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1 + a_2 + a_3 = 0\}$$

This is a G-stable complement to U in V, provided $\mathrm{char}(k) \neq 3$. Let $\mathcal{B} = \{v_1, v_2\}$ be the basis for W, where $v_1 := e_1 - e_2$ and $v_2 = e_1 - e_3$. Then the degree 2 matrix representation $\sigma := (\rho_W)_{\mathcal{B}} : G \to \mathrm{GL}_2(k)$ afforded by W is determined by

$$\sigma((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \sigma((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Proposition 3.0.19. *The nontrivial S_n-stable subspaces of the permutation representation $\rho : S_n \to \mathrm{GL}(kX)$ are $V = \langle \sum_{i=1}^n x_i \rangle, W = \{\sum_{i=1}^n a_ix_i \mid \sum_i a_i = 0\}$*

Proof. Sketch: $\dim V = 1$, so non nontrivial stable subspaces are contained in V. Taking any stable subspace U with nontrivial intersection with V, we show that this at least contains W. As $\dim W = n-1$, either this is W or kX. $\square$

Proposition 3.0.20. *Let X be a G-set and that the permutation representation $\rho : G \to \mathrm{GL}(kX)$ is irreducible. Then, the G-action on X is transitive. The converse need not hold.*

Proof. Suppose for a contradiction the G-action on X is not transitive. Then there is an $x \in X$ whose G-orbit $G \cdot x$ is a proper subset of X. In particular, $\emptyset \subsetneq \{x\} \subsetneq G \cdot x \subsetneq X$. Thus,

$$\{0\} \nleq k(G \cdot x) \nleq kX$$

Given any $g \in G$, $h \cdot x \in G \cdot x \subseteq k(G \cdot x)$,

$$\rho(g)(h \cdot x) = g \cdot (h \cdot x) = (gh) \cdot x \in G \cdot x \subseteq k(G \cdot x)$$

Thus as $G \cdot x$ is a natural basis for $k(G \cdot x)$, this is $\rho(g)$-invariant. This is a proper G-stable subspace, hence the permutation representation is reducible.

To see the converse is false, the permutation representation is reducible, but the S_n action on n elements is clearly transitive. $\square$

Theorem 3.0.21 (Maschke)**.** *Let G be a finite group and suppose that $|G| \neq 0$ in k. Let U be a G-stable subspace of a finite dimensional G-representation V. Then U admits at least one G-stable complement W in V.*

Proof. Pick a basis for U and extend it to a basis for V such that we find some Z with $V = U \oplus Z$. Z is not G-stable in general, but we will replace this with a stable one. Let $\rho : G \to \mathrm{GL}(V)$ be our representation, writing $g \cdot v := \rho(g)(v)$ as ρ is a group homomorphism.

Let $\pi : V \to V$ be the projection map along the decomposition $V = U \oplus Z$ such that $\pi(u+z) = u$ for all $u \in U, z \in Z$. Now define a new linear map $\psi : V \to V$ by

$$\psi(v) := \frac{1}{|G|} \sum_{x \in G} \rho(x)\pi(\rho(x)^{-1}(v))$$

for all $v \in V$. Note that this is well-defined as $|G|$ is invertible in the field k. Fixing a $g \in G$ and $v \in V$, we have

$$|G|\psi(g \cdot v) = \sum_{x \in G} x \cdot \pi(x^{-1} \cdot (g \cdot v))$$

Writing $y^{-1} = x^{-1}g$, noting that x runs over the entire group in the sum, we get

$$|G|\psi(g \cdot v) = \sum_{y \in G} (gy) \cdot \pi(y^{-1} \cdot v) = g \cdot \sum_{y \in G} y \cdot \pi(y^{-1} \cdot v) = g \cdot |G|\psi(v)$$

Cancelling $|G|$, we deduce that ψ is a homomorphism of representations. Also, for any $u \in U$,

$$\psi(u) = \frac{1}{|G|} \sum_{x \in G} x \cdot \pi(x^{-1} \cdot u) = \frac{1}{|G|} \sum_{x \in G} x \cdot (x^{-1} \cdot u) = u$$

noting that as U is G-stable, $\pi(x^{-1} \cdot u) = x^{-1} \cdot u$ for all $x \in G$. So the restriction of ψ to U is the identity map. As U is G-stable and $\pi(V) = U$, we have $\psi(V) \subseteq U$. As $\psi(U) = U$, we have $\psi(V) = U$. With $\mathrm{Im}(\psi) = U$, taking $W := \ker \psi$ gives a G-stable subspace of V. Noting that $\dim W + \dim U = \dim V$ by Rank Nullity and for any $v \in W \cap V$, $0 = \psi(v) = v$, we have $V = U \oplus W$, showing that W is a G-stable complement to U in V. $\square$

Remark 3.0.22. Maschke's Theorem fails if the characteristic of the ground field divides $|G|$, as in Example 3.0.15 (noting that the choice for W is $\langle v_2' \rangle$ where $v_2' = (a, b)$ for $b \neq 0$, which is never stable by ρ).

Definition 3.0.23. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation. ρ is* ***completely reducible*** *if $V = \{0\}$, or there exist G-stable subspaces $U_1, \ldots, U_m$ of V such that*

$$V = U_1 \oplus \cdots \oplus U_m$$

and the subrepresentation of G afforded by each U_i is irreducible.

Corollary 3.0.24. *Let G be a finite group and suppose that $\mathrm{char}(k) \nmid |G|$. Then every finite dimensional representation $\rho : G \to \mathrm{GL}(V)$ of G is completely reducible.*

Proof. We proceed by induction on $\dim V$, where the case $\dim V = 0$ is true by definition. Let U_1 be a G-stable non-zero subspace of V of smallest possible dimension (which exists, as V is one such space). By construction, U_1 is irreducible. Then U_1 admits a G-stable complement W by Maschke's Theorem. Now $\dim W < \dim V$, so by induction $W = U_2 \oplus \cdots \oplus U_m$ for some G-stable irreducible subspaces $U_2, \ldots, U_m$.. Hence $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$ is also completely reducible. $\square$

4 Decomposing Representations

Definition 4.0.1. *Let G be a finite group. The* ***group ring*** *of G (with coefficients in k) is the vector space kG with multiplication defined as*

$$\left(\sum_{x \in G} a_x x\right) \cdot \left(\sum_{y \in G} b_y y\right) = \sum_{g \in G} \left(\sum_{x \in G} a_x b_{x^{-1}g}\right) g$$

The identity element is the formal sum with identity coefficient on the identity of G.

Remark 4.0.2. The group G embeds into the group ring via the map $g \mapsto g$. This embedding respects multiplication, thus realises G as a subgroup of the group of units $kG^\times$ in the ring kG.

Example 4.0.3. Let $G = \langle x \rangle$ be a cyclic group of order n. Then kG has $G = \{1, x, \ldots, x^{n-1}\}$ as a basis, so it is generated by k and x as a ring, and k commutes with x. Define a ring homomorphism $\psi : k[T] \to kG$ by $\psi(f(T)) = f(x)$ for each $f(T) \in k[T]$. Then ψ is surjective and $\ker \psi = \langle T^n - 1 \rangle$. So, by the first isomorphism theorem for rings, we get

$$kG \cong k[T]/\langle T^n - 1 \rangle$$

If k contains a primitive n-th root of unity ζ (such that $T^n - 1$ splits), then it factors into a product of distinct linear factors $(T - 1)(T - \zeta) \cdots (T - \zeta^{n-1})$. By the Chinese Remainder Theorem, this implies

$$kG \cong k[T]/\langle T^n - 1 \rangle \cong \underbrace{k \times k \times \cdots \times k}_{n \text{ times}}$$

Proposition 4.0.4. *Let V be a vector space and G be a group.*

1. *Suppose that $\rho : G \to \mathrm{GL}(V)$ is a representation. Then V becomes a left kG-module via the action*

$$\left(\sum_{x \in G} a_x x\right) \cdot v = \sum_{x \in G} a_x \rho(x)(v)$$

for all $a_x \in k$, $v \in V$

2. *Suppose that V is a left kG-module. Then $\rho : G \to \mathrm{GL}(V)$ defined by*

$$\rho(g)(v) := g \cdot v$$

for all $g \in G$, $v \in V$ is a representation.

3. *This gives a bijection between the set of representations $\rho : G \to \mathrm{GL}(V)$ and the set of kG-module structures $kG \times V \to V$ on V.*

Remark 4.0.5. This correspondence of G-representations to kG-modules moves general theorems about modules into representations, including isomorphism theorems and correspondences.

Example 4.0.6. Let A be a ring. The **free A-module of rank 1** is the abelian group A equipped with the left-multiplication action of A by

$$a \cdot b = ab$$

for all $a, b \in A$. A-submodules of this A-module are called **left-ideals**.

Definition 4.0.7. *If $A = kG$, the representation $\rho : G \to \mathrm{GL}(kG)$ corresponding to the free kG-module of rank 1 is called the **left regular representation**.*

Remark 4.0.8. The left-regular representation coincides with the permutation representation of G on kG, where we extend from the natural action of G on G by left multiplication to kG.

Definition 4.0.9. *An A-module M is **irreducible** or **simple** if M is nonzero, and if N is an A-submodule of M, $N = \{0\}$ or $N = M$.*

*An A-module V is **completely reducible** if it is either the 0-module, or is equal to a direct sum of finitely many irreducible submodules.*

A homomorphism of representations is simply a map of kG-modules, known as a kG-linear map.

Definition 4.0.10. *Let A be a ring. We say that A is **semisimple** if the free A-module of rank 1 is completely reducible.*

Proposition 4.0.11. *Let G be a finite group such that $|G| \neq 0$ in k. Then the group ring kG is semisimple.*

Proof. Follows from correspondence and Maschke's Theorem. □

Definition 4.0.12. *Let V be an A-module. We say that V is **cyclic** if it can be generated by a single element $v : V = A \cdot v$. The **annihilator** of $v \in V$ is the left-ideal*

$$\mathrm{Ann}_A(v) := \{a \in A \mid av = 0\}$$

Example 4.0.13. Simple modules are cyclic. This is because any span of a single element produces a submodule which must be the entire thing.

Lemma 4.0.14. *Every cyclic A-module V is isomorphic to a quotient module of the free module of rank 1. If $V = A \cdot v$, then*

$$V \cong A/\mathrm{Ann}_A(v)$$

Proof. The map $\psi : A \to V$ given by $a \mapsto a \cdot v$ is an A-module homomorphism. This is surjective, so by the first isomorphism theorem, we have

$$V = \text{Im } \psi \cong A/\ker \psi$$

Now this follows as $\ker \psi = \text{Ann}_A(v)$. $\square$

Lemma 4.0.15. *Let V, W be simple A-modules. Then every non-zero A-linear map $\psi : V \to W$ is an isomorphism.*

Proof. We know $\ker \psi$ is an A-submodule of V and that $\text{Im } \psi$ is an A-submodule of W. As ψ is non-zero $\ker \psi$ is also not all of V and $\text{Im } \psi$ is nonzero. As V and W are both simple, it must be the case that $\ker \psi = 0$ and $\text{Im } \psi = W$. Hence ψ is bijective, and therefore is an isomorphism. $\square$

Proposition 4.0.16. *Let A be a semisimple ring. Then A has only finitely many simple A-modules up to isomorphism.*

Proof. Write $A = V_1 \oplus \cdots \oplus V_r$ for some simple A-submodules V_i of A. Let V be a simple A-module, pick a nonzero vector $v \in V$ and consider the A-module map $\psi : A \to V$ by $a \mapsto a \cdot v$. Let $\psi_i : V_i \to V$ be the restriction of ψ to V_i such that if $a = a_1 + \cdots + a_r$ is the decomposition of $a \in A$ with $a_i \in V_i$ for each i, then

$$\psi(a) = \psi_1(a_1) + \cdots + \psi_r(a_r)$$

If ψ_i is the zero-map for all i, then ψ is the zero map. Hence ψ_i is nonzero for some i. In particular, V is isomorphic to one of the irreducible representations in the list $V_1, \ldots, V_r$. $\square$

Theorem 4.0.17. *Let G be a finite group such that $|G| \neq 0$ in k. Then G has only finitely many irreducible representations up to isomorphism.*

Proof. The ring kG is semisimple by Maschke's Theorem. By Proposition 4.0.16 and correspondence, the proof follows. $\square$

Definition 4.0.18. *For a finite group G, we write $r_k(G)$ to denote the number of isomorphism classes of irreducible k-representations of G.*

Theorem 4.0.19 (Schur's Lemma)**.** *Suppose that k is algebraically closed. Let V be a simple module over a finite dimensional k-algebra A. Then every A-submodule endomorphism of V is given by the action of some scalar $\lambda \in k$ such that*

$$\text{End}_A(V) = k1_V$$

Proof. By Lemma 4.0.14, V is isomorphic to a quotient module of A, so V is itself finite dimensional as a k-vector space. Let $\psi : V \to V$ be an A-module endomorphism. Then it is a k-linear map, so has at least one eigenvalue $\lambda \in k$ (because the characteristic polynomial splits). Hence $\psi - \lambda 1_V : V \to V$ is a homomorphism with nonzero kernel, and as V is simple, is the zero map. Thus $\psi = \lambda 1_V$ is the action of $\lambda \in k$. $\square$

Definition 4.0.20. *Let A be a k-algebra and V be an A-module with $\text{End}_A(V) = k1_V$. Then by Schur's Lemma, every $z \in Z(A)$ acts on V by a scalar, which is denoted by z_V. The ring homomorphism $Z(A) \to k$ via $z \mapsto z_V$ is called the **central character** of V.*

4.1 Artin-Weddernburn

For this subsection, we fix A to be some semisimple ring, and $V_1, \ldots, V_r$ will denote the complete list of representatives for the isomorphism classes of simple A-modules. We also fix a decomposition

$$A = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i} L_{i,j}$$

of the A-module A into a direct sum of simple left ideals $L_{i,j}$ where $L_{i,j} \cong V_i$ for each i and j.

Note that we must have $n_1, \ldots, n_r \geq 1$, as each V_i occurs as a direct summand of A at least once. Also, the left-ideals are not unique in general.

Proposition 4.1.1. *Let A be a finite dimensional semisimple k-algebra and suppose that k is algebraically closed. Then* $\dim Z(A) \leq r$.

Proof. By Schur's Lemma, we have $\mathrm{End}_A(V_i) = k1_{V_i}$ for all i, so we can define a k-linear map $\psi : Z(A) \to k^r$ by $\psi(z) := (z_{V_1}, \ldots, z_{V_r})$.

Suppose now that $\psi(z) = 0$ for some $z \in Z(A)$, such that $z_{V_i} = 0$ for all i. We show $z = 0$.

By the decomposition of $1 \in A$ along the decomposition, we have

$$1 = \sum_{i=1}^{r} \sum_{j=1}^{n_i} e_{i,j}$$

for some $e_{i,j} \in L_{i,j}$.

Then we must have

$$z = z1 = \sum_{i=1}^{r} \sum_{j=1}^{n_i} z e_{i,j} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} z_{V_i} e_{i,j}$$

As $z_{V_i} = 0$ for all i, $z = 0$. Hence ψ is injective and so it follows that $\dim Z(A) \leq \dim k^r = r$. □

Lemma 4.1.2. *Each $B_i := \bigoplus_{j=1}^{n_i} L_{i,j}$ is a two-sided ideal of A, and $A = B_1 \oplus \cdots \oplus B_r$.*

Proof. The second part of the statement follows from definition, and each B_i is a left ideal of A. Hence it suffices to show it is a right ideal in A. Fix $a \in A$ and consider $L_{i,j} \subseteq B_i$. Let $i' \neq i$ and $1 \leq j' \leq n_{i'}$ be another pair of indices, and consider the projection $\psi \twoheadrightarrow L_{i',j'}$ along the decomposition.

The restriction of $\phi \circ r_a : A \to L_{i',j'}$ to $L_{i,j}$ by right multiplication is an A-module homomorphism from $L_{i,j}$ to $L_{i',j'}$. As $i' \neq i$, these modules are not isomorphic, so the restriction must be the zero map. Varying i' and j', the projection of $L_{i,j}$ onto each B_i' with $i' \neq i$ is zero. Hence $L_{i,j}a \subseteq B_i$. As B_i is equal to the sum of all $L_{i,j}$, we have $B_i a \subseteq B_i$ for all $a \in A$.

□

Lemma 4.1.3. *Let R be a k-algebra and suppose that $R = S_1 \oplus \cdots \oplus S_r$ for some non-zero two-sided ideals $S_1, \ldots, S_r$. Then* $\dim Z(R) \geq r$.

Proof. Write $1 = e_1 + \cdots + e_r$ for some $e_i \in S_i$. Let $a \in R$ and fix $i = 1, \ldots, r$. Since S_i is a left-ideal, $ae_i \in S_i$. As $a = ae_1 + \cdots + ae_r$, we see that ae_i is the component of a in S_i in the decomposition $R = S_1 \oplus \cdots \oplus S_r$. On the other hand, as S_i is a right ideal, $e_i a$ is the component of a in S_i by the same decomposition. Hence $ae_i = e_i a$ for all i and $a \in R$, thus e_i is central.

If $i \neq j$, then $e_i e_j \in S_i \cap S_j = \{0\}$, so $e_i e_j = 0$. Hence $e_i = e_i \cdot 1 = e_i \sum_{j=1}^{r} e_j = e_i^2$. In particular, the set $\{e_1, \ldots, e_r\}$ forms a set of pairwise orthogonal idempotents such that $e_i e_j = \delta_{i,j} e_i$.

Now suppose that $\sum_{i=1}^{r} \lambda_i e_i = 0$ for some $\lambda_i \in k$. Multiplying by e_j gives $\lambda_j e_j = 0$. If $e_j = 0$, then for all $a \in S_j$ we have $a = ae_j = 0$, contradicting the assumption that $S_j \neq \{0\}$. Thus $e_j \neq 0$ for all j, giving $\{e_1, \dots, e_r\}$ to be a linearly independent set over k. Thus $r \leq \dim Z(R)$. $\square$

Theorem 4.1.4. *Let A be a finite dimensional semisimple k-algebra and suppose that k is algebraically closed. Then $r = \dim Z(A)$.*

Proof. By Proposition 4.1.1, we have $r \geq \dim Z(A)$. By Lemma 4.1.2 $A = B_1 \oplus \cdots \oplus B_r$ for some two-sided ideals B_r, so by Lemma 4.1.3, $r \leq \dim Z(A)$, so the proof follows. $\square$

Definition 4.1.5. *For a finite group G, let $s(G)$ denote the number of conjugacy classes of G. Now let $C_1, \dots, C_s$ be the conjugacy classes of G. For each $i = 1, \dots, s$, define the **conjugacy class sum** of C_i to be*

$$\widehat{C_i} := \sum_{x \in C_i} x \in kG$$

Proposition 4.1.6. *$\{\widehat{C_1}, \dots, \widehat{C_s}\}$ is a basis for $Z(kG)$ as a vector space, thus*

$$\dim Z(kG) = s(G)$$

Proof. Let C_a be the conjugacy class of $a \in G$. Fix a $x \in G$, and define $\phi_x : G \to G$ by $y \mapsto x^{-1}yx$. This is a bijective function closed under conjugacy classes, we have

$$\widehat{C_a} = \sum_{y \in C_a} y = \sum_{z \in C_a} \phi_x(z) = \sum_{z \in C_a} x^{-1}zx = x^{-1}\left(\sum_{z \in C_a} z\right)x = x^{-1}\widehat{C_a}x$$

In particular, $\widehat{C_a}$ commutes with any $x \in G$. Hence it commutes with any element in kG. Thus $\widehat{C} \in Z(kG)$ for any choice of conjugacy class.

Fix $z := \sum_{x \in G} a_x x \in Z(kG)$. We have

$$\sum_{x \in G} a_x x = z = g^{-1}zg = g^{-1}\left(\sum_{x \in G} a_x x\right)g = \sum_{x \in G} a_x(g^{-1}xg)$$

In particular, coefficients agree within any conjugacy class. Write λ_i for the coefficient in the conjugacy class C_i. Then, particular, we can write

$$z = \sum_{i=1}^{s} \sum_{x \in C_i} a_x x = \sum_{i=1}^{s} \lambda_i \sum_{x \in C_i} x = \sum_{i=1}^{s} \lambda_i \widehat{C_i}$$

Thus $Z(kG)$ is spanned by the sums of conjugacy classes.

If $0 = \sum_{i=1}^{s} \lambda_i \widehat{C_i} = \sum_{i=1}^{s} \sum_{x \in C_i} \lambda_i x$. As G is a linearly independent set in kG, we have $\lambda_i = 0$ for all i. Thus the conjugacy classes are linearly independent. $\square$

Corollary 4.1.7. *Let G be a finite group and k be an algebraicaly closed field with $|G| \neq 0$ in k. Then $r_k(G) = s(G)$.*

Proof. By Maschke, kG is a semisimple k-algebra with $\dim Z(kG) = s(G)$ by Proposition 4.1.6. The proof follows from Theorem 4.1.4. $\square$

Corollary 4.1.8. *Suppose that $|G| \neq 0$ in k and take $e := \frac{1}{|G|}\sum_{g \in G} g \in kG$. Then e is a central idempotent.*

Proof. e is a kG-linear combination of all conjugacy class sums, so $e \in Z(kG)$ by Proposition 4.1.6. Now,

$$e^2 = \frac{1}{|G|}\sum_{g\in G}\left(\frac{1}{|G|}\sum_{h\in H} gh\right) = \frac{1}{|G|}\sum_{g\in G}\left(\frac{1}{|G|}\sum_{h\in G} h\right) = \frac{1}{|G|}\sum_{g\in G} e = e$$

□

Lemma 4.1.9. *We note the following properties about the ring decomposition:*

1. *Each B_i is a ring with identity element e_i*
2. *A is isomorphic to the product of rings (B_i, e_i)*

$$A \cong B_1 \times \cdots \times B_r$$

3. *Each B_i is a semisimple ring with unique simple module V_i*

Proof. (i) Lemma 4.1.2 shows that B_i is an additive subgroup of A stable under multiplication. In the proof of Lemma 4.1.3 we saw that for any $a \in A$, $ae_i = e_i a$ is the B_i component of a along the decomposition $A = B_1 \oplus \cdots \oplus B_r$. In particular, $ae_i = e_i a = a$ for all $a \in B_i$

(ii) The isomorphism sends $a \in A$ to $(ae_1, \ldots, ae_r) \in B_1 \times \cdots \times B_r$.

(iii) Fix $\ell = 1, \ldots, n_i$ and suppose that U is a B_i-submodule of $L_{i,\ell}$. Then,

$$A \cdot U = \left(\bigoplus_{j=1}^{r} B_j\right) \cdot U \leq U$$

where the last equality follows from the fact $B_j \cdot U \leq B_j \cdot B_i = B_j e_j \cdot e_i B_i = 0$ if $j \neq i$, and $B_i \cdot U \leq U$ as U is a B_i-submodule. In particular, U is an A-submodule of $L_{i,\ell}$, thus U is either zero of all of $L_{i,\ell}$ as it is a simple A-module. Thus $L_{i,\ell}$ hence V_i are all simple B_i-moduoes. As $B_i = \bigoplus_{j=1}^{n_i} L_{i,j}$, it is a semisimple ring. In particular, by Proposition 4.0.16, V_i is the only simple B_i-module up to isomorphism. □

Remark 4.1.10. Note that B_i is an additive subgroup of A stable under multiplication, but it is not a unital subring when $r \geq 2$, as the identity element e_i is not the identity element 1 in A.

Definition 4.1.11. *Let A be a ring. The* ***opposite ring*** *to A, written A^{op} has the same abelian group as A, but multiplication defined as*

$$a \star b = ba$$

for all $a, b \in A^{\mathrm{op}}$

Proposition 4.1.12. *For each $a \in A$, write $r_a : A \to A$ to be the left A-linear map given by $r_a(b) = ba$ for each $b \in A$. The map*

$$r : A^{\mathrm{op}} \to \mathrm{End}_A(A) \qquad a \mapsto r_a$$

is an isomorphism of rings.

Proof. We note that r is a ring homomorphism, taking the structure from A.

For $a \in A$, if $r_a = 0$, then we have $a = 1_A a = r_a(1_A) = 0(1_A) = 0$, so the kernel of r is trivial. Take any $\psi \in \mathrm{End}_A(A)$. For every $b \in A$, we have by A-linearity of ψ,

$$\psi(b) = \psi(b1) = b\psi(1) = r_{\psi(1)}(b)$$

In particular $\psi = r_{\psi(1)}$. So we have an isomorphism of rings. □

Proposition 4.1.13. *Let V be an A-module. Suppose $D := \mathrm{End}_A(V)$ and let $n \geq 1$.*

1. *The inclusion maps and projection maps to each coordinate give a ring homomorphism $M_n(D) \cong \mathrm{End}_A(V^n)$*
2. *For any ring S, $M_n(S)^{\mathrm{op}} \cong M_n(S^{\mathrm{op}})$.*

Proof. (i) Let $\sigma_j : V \to V^n$ be the inclusion maps and $\pi_j : V^n \to V$ be the projection maps. Define $\alpha : M_n(D) \to \mathrm{End}_A(V^n)$ by

$$(\phi_{i,j}) \mapsto \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i \circ \phi_{i,j} \circ \pi_j$$

and $\beta : \mathrm{End}_A(V^n) \to M_n(D)$ by

$$\phi \mapsto (\pi_i \circ \psi \circ \sigma_j)$$

Now,

$$(\beta \circ \alpha)(\phi_{i,j}) = \beta\left(\sum_i \sum_j \sigma_i \circ \phi_{i,j}\pi_j\right)$$

And the i', j'-th coordinate of the evaluation is

$$\pi_{i'} \circ \left(\sum_i \sum_j \sigma_i \circ \phi_{i,j}\pi_j\right)\sigma_{j'} = \sum_i \sum_j (\pi_{i'} \circ \sigma_i) \circ \phi_{i,j} \circ (\pi_j \circ \sigma_{j'}) = \phi_{i',j'}$$

Noting that $\pi_i \circ \sigma_j = \delta_{i,j}$, thus α is injective.

Also,

$$(\alpha \circ \beta)(\phi) = \alpha((\pi_1 \circ \phi \circ \sigma_j)) = \sum_i \sum_j \sigma_i \circ (\pi_i \circ \phi \circ \sigma_j) \circ \pi_j = \left(\sum_i \sigma_i \circ \pi_i\right) \circ \phi \circ \left(\sum_j \sigma_j \circ \pi_j\right) = \psi$$

Noting that $\sum_k \sigma_k \circ \pi_k = 1_{\mathrm{End}_A(V^n)}$. Finally, this is a unital ring homomorphism, induced by the linear structure of matrix multiplication and projection / inclusions.

(ii) This follows from the fact there is a natural isomorphism by transposing. □

Proposition 4.1.14. *Let B be a semisimple ring with exactly one simple module V up to isomorphism. Suppose that $B \cong \underbrace{V \oplus \cdots \oplus V}_{n \text{ times}}$ as a left B-module, and let $D := \mathrm{End}_B(V)$. Then there is a ring isomorphism*

$$B \cong M_n(D^{\mathrm{op}})$$

Proof. We know from Proposition 4.1.12 that $B \cong \mathrm{End}_B(B)^{\mathrm{op}}$. Since $B = V^n$ as a left B-module, we have $\mathrm{End}_B(B) \cong M_n(D)$. Hence

$$B \cong \mathrm{End}_B(B)^{\mathrm{op}} \cong M_n(D)^{\mathrm{op}} \cong M_n(D^{\mathrm{op}})$$

Noting Proposition 4.1.13. □

Theorem 4.1.15 (Artin-Weddernburn)**.** *Suppose that k is an algebraically closed field and that A is a finite dimensional semisimple k-algebra. Then there exist positive integers $n_1, \ldots, n_r$ and a k-algebra isomorphism*

$$A \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k)$$

Proof. By Lemma 4.1.9, without loss of generality, we may assume that $r = 1$, such that A has exactly one simple module V up to isomorphism. Then $A \cong M_n(D^{\mathrm{op}})$ where $D := \mathrm{End}_A(V)$ by Lemma 4.1.14. On the other hand, $D \cong k$ by Schur's Lemma. □

Proposition 4.1.16. *Suppose that $A = M_n(k)$ be the ring of $n \times n$ matrices with entires in k and let $V := k^n$ be the natural left A-module of $n \times 1$ column vectors. Then,*

1. *V is a simple A-module*
2. *A has no nonzero proper two sided ideals*
3. *$A = L_1 \oplus \cdots \oplus L_n$, where $L_j := M_n(k) \cdot E_{i,i}$ is a decomposition into simple left ideals.*

Proof. (i) First note that V is nonzero. Let W be an A-submodule of V and assume that $W \neq \{0\}$. Taking a nonzero $w \in W$, extend this to a basis for V. Consider the matrix that sends w to e_i a standard basis vector, and all other basis vectors to 0. Then we have $T(v) = e_i \in W$, so this forces $W = V$.

(ii) Pick any two-sided nonzero ideal $I \subseteq M_n(k)$. As I is nonzero, choose $A = (a_{i,j}) \in I$ that is nonzero. Let $a_{i,j} \neq 0$. Then,

$$E_{ri} A E_{js} = a_{ij} E_{rs} \in I$$

as I is two sided. Scaling, $E_{rs} \in I$. Hence every elementary matrix E_{rs} belongs in I, forcing $I = M_n(k)$.

(iii) Note first that the set $E_{i,i}$ over i is a pairwise orthogonal idempotent. The span of $E_{i,i}$ by left-multiplication gives elements of the form $E_{j,i}$. In particular, as $1 = \sum_i E_{i,i}$,

$$A = A \cdot 1 = A \sum_i E_{i,i} = \sum_i (A E_{i,i}) \in \sum_i I_i$$

If $\sum_i A_i E_{i,i} = 0$, then right multiplication by $E_{j,j}$ forces $A_j E_{j,j} = 0$, hence direct. Finally, $I_i \simeq k^n$, so is simple. □

Remark 4.1.17. Note that the decomposition above need not be unique. We can pick $E'_{i,i} = P E_{i,i} P^{-1}$ for some invertible matrix P (not diagonal, to change the matrix), then this is again a complete set of primitive orthogonal idempotent with a new decomposition $M_n(k) = \bigoplus_i M_n(k) E'_{i,i}$.

The main point is that $M_n(k)$ is semisimple, and the decomposition is unique up to isomorphism. Hence, permuting or conjugating maintains isomorphism.

Corollary 4.1.18. *Suppose that k is algebraically closed. Let G be a finite group such that $|G| \neq 0$ in k and let $V_1, \ldots, V_r$ be a complete list of pairwise non-isomorphic simple kG-modules. Then we have*

1. $kG \simeq V_1^{\dim V_1} \oplus \cdots \oplus V_r^{\dim V_r}$ *as a kG-module*

2. $|G| = \sum_{i=1}^r (\dim V_i)^2$

Proof. We know that kG is a semisimple ring by Maschke's Theorem, so by Artin Weddernburn, we can decompose

$$kG \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k)$$

The matrix algebra $M_n(k)$ acts on the space of column vectors k^n by left-multiplication, and this forces k^n to be a simple $M_n(k)$-module. On the other hand, $M_n(k)$ is isomorphic to a direct sum of n copies of k^n, so $n_i = \dim V_i$ for each $i = 1, \ldots, r$.

The second statement is then immediate from the first. □

Proposition 4.1.19. *Every representation $\rho : G \to \mathrm{GL}(V)$ extends to a k-algebra homomorphism $\tilde{\rho} : kG \to \mathrm{End}_k(V)$*

Proof. As $\mathrm{End}_k(V)$ is a k-vector space and G is a basis for kG, $\rho : G \to \mathrm{GL}(V) \subseteq \mathrm{End}_k(V)$ extends uniquely to a k-linear map $\tilde{\rho} : kG \to \mathrm{End}_k(V)$. Now this acts like a homomorphism for elements in G, so it follows that

$$\tilde{\rho}(xy) = \tilde{\rho}\left(\sum_{g\in G}\sum_{h\in H} a_g b_h gh\right) = \sum_{g\in G}\sum_{h\in H} a_g b_h \tilde{\rho}(gh) = \tilde{\rho}(g)\tilde{\rho}(h) = \left(\sum_{g\in G} a_g \tilde{\rho}(g)\right)\left(\sum_{h\in H} b_h \tilde{\rho}(h)\right)$$

Also, $\tilde{\rho}(1_{kG}) = \rho(1_G) = 1_{\mathrm{End}_k(V)}$. □

5 Constructing representations

Lemma 5.0.1. *Let V be a vector space and let $G \times V \to V$ be a G-action on the set V. This extends to a kG-module structure on V if and only if the G-action on V is linear, such that*

$$g \cdot (v + \lambda w) = (g \cdot v) + \lambda(g \cdot w)$$

for all $g \in G$, $v, w \in V$, $\lambda \in k$.

Definition 5.0.2. *Let V and W be G-repreesntations. The external direct sum $V \oplus W$ is a G-representation via*

$$g \cdot (v \cdot w) = (g \cdot v, g \cdot w)$$

for all $g \in G$, $v \in V$, $w \in W$.

Definition 5.0.3. *Let V be a G-representation. This induces a representation on the dual space V^* via*

$$(g \cdot f)(v) := f(g^{-1} \cdot v)$$

*for all $g \in G$, $f \in V^*$, $v \in V$. We call this the **dual representation**.*

Definition 5.0.4. *Let V, W be G-representations. The vector space $\mathrm{Hom}(V, W)$ of all linear maps from V to W admits a linear G-action by*

$$(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$$

for all $g \in G$, $f \in \mathrm{Hom}(V, W)$, $v \in V$. When W is the trivial 1-dimensional representation, we recover the dual space $\mathrm{Hom}(V, k) = V^*$.

Lemma 5.0.5. *Let V be a finite dimensional G-representation. The biduality isomorphism between vector spaces by*

$$\tau : V \to (V^*)^* \qquad \tau(v)(f) := f(v)$$

for all $f \in V^*, v \in V$ is an isomorphism of G-representations.

Proof. Noting that the map is a bijection, it suffices to check that this is indeed a homomorphism by the induced dual representations. Now,

$$(g \cdot \tau(v))(\psi) = \tau(v)(g^{-1} \cdot \psi) = (g^{-1} \cdot \psi)v = \psi(g \cdot v) = (\tau(g \cdot v))(\psi)$$

□

5.0.1 Actions on Tensor

Definition 5.0.6. *Let V and W be finite dimensional kG-modules. Define a G-action on $V \otimes W$ by setting*

$$g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$$

*for all $g \in G$, $v \in V$, $w \in W$. This is called the **tensor product representation** $V \otimes W$.*

This gives a well-defined G-representation as it is a linear G-action on $V \otimes W$.

Lemma 5.0.7. *Let V and W be finite dimensional kG-modules. Then there is an isomorphism of kG-modules*

$$V^* \otimes W \cong \mathrm{Hom}(V, W)$$

Proof. For every $f \in V^*$ and $w \in W$, we have a linear map $b(f, w) : V \to W$ given by $b(f, w)(v) := f(v)w$. The resulting map $b : V^* \times W \to \mathrm{Hom}(V, W)$ is blinear, so by the Universal Property on Tensors, extends to a linear map

$$\alpha : V^* \otimes W \to \mathrm{Hom}(V, W)$$

given by $\alpha(f \otimes w)(v) := f(v)w$ for all $f \in V^*, w \in W, v \in V$. Let $\{v_1, \ldots, v_n\}$ be a basis for V and let $\{v_1^*, \ldots, v_n^*\}$ be the corresponding dual basis for V^*. We define a linear map $\beta : \mathrm{Hom}(V, W) \to V^* \otimes W$ by

$$f \mapsto \sum_{i=1}^{n} v_i^* \otimes f(v_i)$$

We first show that these maps are mutual inverses.

Let $f \in \mathrm{Hom}(V, W)$ and $v \in V$. Then,

$$(\alpha \circ \beta)(f)(v) = \alpha(\beta(f))(v) = \sum_{i=1}^{n} \alpha(v_i^* \otimes f(v_i))(v) = \sum_{i=1}^{n} v_i^*(v) f(v_i) = f\left(\sum_{i=1}^{n} v_i^*(v) v_i\right) = f(v)$$

In particular, $\alpha \circ \beta = 1_{\mathrm{Hom}(V,W)}$. Also, for any $v \in V$,

$$(\beta \circ \alpha)(v_i^* \otimes w_j) = \beta(\alpha(v_i^* \otimes w_j)) = \sum_{k=1}^{n} v_k^* \otimes \alpha(v_i^* \otimes w_j)(v_k) = \sum_{k=1}^{n} v_k^* \otimes v_i^*(v_k) w_j = v_i^* \otimes w_j$$

So $\beta \circ \alpha = 1_{V^* \otimes W}$ as each basis element is sent to the identity. Finally, to show that α is a homomorphism of kG-modules, as we know α is k-linear, it suffices to show G-equivariance. Taking any $g \in G, f \in V^*, w \in W$, we have

$$\begin{aligned}
\alpha(g \cdot (f \otimes w))(v) &= \alpha((g \cdot f) \otimes (g \cdot w))(v) \\
&= (g \cdot f)(v)(g \cdot w) \\
&= f(g^{-1} \cdot v)(g \cdot w) \\
&= g \cdot (f(g^{-1} \cdot v)w) \\
&= g \cdot \alpha(f \otimes w)(g^{-1} \cdot v) \\
&= (g \cdot \alpha(f \otimes w))(v)
\end{aligned}$$

where the last line is an equality based on the homomorphism action induced by kG module actions on V and W. □

Definition 5.0.8. *Suppose that* $\operatorname{char}(k) \neq 2$ *and let* V *be a finite dimensional vector space.*

- *For each* $v, w \in V$*, define*

  $$vw := \frac{1}{2}(v \otimes w + w \otimes v) \in V \otimes V$$

  Then, the ***symmetric square*** *of* V *is the subspace of* $V \otimes V$ *given by*

  $$S^2V := \langle \{vw \mid v, w \in V\} \rangle$$

- *For each* $v, w \in V$*, define*

  $$v \wedge w := \frac{1}{2}(v \otimes w - w \otimes v) \in V \otimes V$$

  The ***alternating square*** *of* V *is the subspace of* $V \otimes V$ *defined by*

  $$\bigwedge^2 V := \langle \{v \wedge w \mid v, w \in V\} \rangle$$

Note that $vw = wv$ in S^2V and that $v \wedge w = -w \wedge v$ in $\bigwedge^2 V$ for all $v, w \in V$.

Lemma 5.0.9. *Let* $\dim V = n$ *and suppose that* $\operatorname{char}(k) \neq 2$*. Then,*

1. $V \otimes V = S^2V \oplus \bigwedge^2 V$
2. $\dim S^2V = \frac{n(n+1)}{2}$ *and* $\dim \bigwedge^2 V = \frac{n(n-1)}{2}$
3. *If* V *is a* G*-representation, then so are* S^2V *and* $\bigwedge^2 V$ *via the actions*

   $$g \cdot (vw) = (g \cdot v)(g \cdot w) g \cdot (v \wedge w) = (g \cdot v) \wedge (g \cdot w)$$

   for all $g \in G, v, w \in V$.

Proof. (*i*) Let $S_2 := \langle\sigma\rangle$ be the cyclic group of order 2. Since $\mathrm{char}(k) \neq 2$, the group ring kS_2 admits orthogonal idempotents $e_1 := \frac{1+\sigma}{2} \in kS_2$ and $e_2 := \frac{1-\sigma}{2} \in kS_2$, which gives rise to the decomposition

$$kS_2 = kS_2e_1 \oplus kS_2e_2 = ke_1 \oplus ke_2$$

by Lemma 4.1.2, where the last equality then follows from the fact $\sigma e_1 = e_1$ and $\sigma e_2 = -e_2$. Thus, every kS_2-module M admits an even-odd decomposition

$$M = e_1M \oplus e_2M = \{m \in M \mid \sigma m = m\} \oplus \{m \in M \mid \sigma m = -m\}$$

Now, S_2 acts linearly on $V \otimes V$ by

$$\sigma \cdot (v \otimes w) = w \otimes v$$

Then $S^2V = e_1 \cdot (V \otimes V)$ is the even part of $V \otimes V$ and $\bigwedge^2 V = e_2 \cdot (V \otimes V)$ is the odd part of $V \otimes V$. Then the even-odd decomposition gives $V \otimes V = S^2V \oplus \bigwedge^2 V$.

(*ii*) If $\{v_1, \dots, v_n\}$ is a basis for V, then $\{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ spans $V \otimes V$, so $\{e_1 \cdot (v_i \otimes v_j)\}$ spans S^2V. Now, $e_1 \cdot (v_i \otimes v_j) = v_iv_j = v_jv_i$, so $\{v_iv_j\}$ span S^2V. Hence,

$$\dim S^2V \leq \frac{n(n+1)}{2}$$

Similarly, $e_2 \cdot (v_i \otimes v_j) = v_i \wedge v_j$ spans $\bigwedge^2 V$, and therefore

$$\dim \bigwedge^2 V \leq \frac{n(n-1)}{2}$$

On the other hand, $\dim V \otimes V = n^2$, so decomposition implies the result.

(*iii*) We have two groups G and S_2 acting on $V \otimes V$. Now,

$$\sigma \cdot (g \cdot (v \otimes w)) = \sigma(g \cdot v \otimes g \cdot w) = g \cdot w \otimes g \cdot v = g \cdot (w \otimes v) = g \cdot (\sigma \cdot (v \otimes w))$$

for any $v, w \in V$ and $g \in G$. Hence, these actions commute pointwise, and so the actions preserve S^2V and $\bigwedge^2 V$. Hence these submodules inherit a linear G-action from $V \otimes V$ as claimed. ☐

Remark 5.0.10. This idea extends to finding proper kG-submodules to the tensor $V^{\otimes n}$ as a direct sum of kG-submodules $S^\lambda(V)$, one for each irreducible representation λ of the symmetric group S_n. This construction $V \mapsto S^\lambda(V)$ is called the **Schur Functor**.

Corollary 5.0.11. *Suppose* $\mathrm{char}(k) \neq 2$ *and V be a G-representation. The square tensor $V \otimes V$ is reducible when* $\dim V \geq 2$.

Proof. By Lemma 5.0.9, $V \otimes V$ decomposes as $S^2V \oplus \bigwedge^2 V$, and these are both nontrivial by part (*ii*) of the Lemma. S^2V and $\bigwedge^2 V$ are both G-representations by (*iii*). ☐

Lemma 5.0.12. *Let V be a finite dimensional kG-module. Let W be a one-dimensional kG-module. Then, $V \otimes W$ is simple if and only if V is simple.*

Proof. First note that $W \otimes W^* \cong k$, as

$$g \cdot (w \otimes w^*) = (g \cdot w) \otimes (g \cdot w^*) = \chi(g)w \otimes \chi(g)^{-1}w^* = 1 \cdot (w \otimes w^*)$$

Hence the group elements act trivially, and so in particular $W \otimes W^* \cong k$.

Hence it suffices to show one direction, as then we can use the congruence $V \cong V \otimes (W \otimes W^*)$.

Suppose that $V \otimes W$ is simple. Suppose for a contradiction that V is not simple. Then we have a nonzero proper kG-module U of V. Let $\mathcal{B}_U = \{u_1, \ldots, u_k\}$ be a basis for U and extend it to a basis $\mathcal{B}_V$ of V. Let $\{w\}$ be a basis for the one dimensional space W. Then,

$$g \cdot (u \otimes w) = \underbrace{g \cdot u}_{\in U} \otimes \underbrace{g \cdot w}_{\in U} \in U \otimes W$$

Hence $U \otimes W$ is a proper G-stable subspace of $V \otimes W$, hence a kG module, a contradiction. Thus V is simple. □

Proposition 5.0.13. *Let V be a finite dimensional kG-module. Then, V is simple if and only if V^* is simple.*

Proof. Note that it suffices to show $\Rightarrow$, as $V^{**} \cong V$. Suppose that V is simple. Take $U \subseteq V^*$ be a nonzero kG-module. Tqking the annihilator of U, this is a G-stable subspace of V. As V is simple, noting $\mathrm{Ann}(U) = V$, we have $U = 0$ or $\mathrm{Ann}(U) = 0$. The first case is ruled out by the fact $U \neq 0$, so $\mathrm{Ann}(U) = 0$.

Now, the proof follows from the fact

$$U \cong \mathrm{Hom}_k(V/\mathrm{Ann}(U), k) \cong \mathrm{Hom}_k(V, k) \cong V^*$$

□

6 Character Theory

6.1 Definitions and Basic Properties

Definition 6.1.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G. The **character** of ρ is the function*

$$\chi_\rho : G \to \mathbb{C} \qquad g \mapsto \mathrm{tr}(\rho(g))$$

*The **degree** of a character χ_ρ is the degree of the representation ρ.*

We write χ_V to denote the character of the representation afforded by a $\mathbb{C}G$-module V, when the $\mathbb{C}G$-module structure on V is understood.

Remark 6.1.2. Note that the character χ_V only depends on the isomorphism class of the $\mathbb{C}G$-module V, and the isomorphism class of the representation ρ. TODO!! more?

Definition 6.1.3. *A function $f : G \to \mathbb{C}$ is said to be a **class function** if it is constant on the conjugacy classes of G. That is,*

$$f(xgx^{-1}) = f(g)$$

for all $g, x \in G$. We denote the space of all class functions on G by $\mathcal{C}(G)$.

Note that $\mathcal{C}(G)$ is a commutative ring via pointwise multiplication of functions.

Lemma 6.1.4. *The character χ_V of any finite dimensional kG-module V is a class function.*

Proof. If $\rho : G \to \mathrm{GL}(V)$ is the corresponding representation, then the linear endomorphism $\rho(g)$ of V is conjugate to $\rho(xgx^{-1})$ in $\mathrm{GL}(V)$. But the conjugate linear maps have the same trace, as

$$\mathrm{tr}(ABA^{-1}) = \mathrm{tr}((AB)A^{-1}) = \mathrm{tr}(A^{-1}(AB)) = \mathrm{tr}(B)$$

for any $A, B \in \mathrm{GL}(V)$. □

Proposition 6.1.5. *Let G be a finite group and let V, W be finite dimensional $\mathbb{C}G$-modules. Then we have the following equalities in $\mathcal{C}(G)$:*

1. $\chi_{V^*} = \overline{\chi_V}$
2. $\chi_{V \oplus W} = \chi_V + \chi_W$
3. $\chi_{V \otimes W} = \chi_V \chi_W$
4. $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\chi_W$
5. $\chi_{S^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$ *for all* $g \in G$
6. $\chi_{\bigwedge^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$ *for all* $g \in G$.

Proof. Fix $g \in G$. The action $g_V \in \mathrm{GL}(V)$ of g on V is diagonalizable. Fix a basis of g_V-eigenvectors $\{v_1, \dots, v_n\}$ for V with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, and fix a basis of g_W-eigenvectors $\{w_1, \dots, w_m\}$ for W with eigenvalues $\mu_1, \dots, \mu_m$. Then,

$$\chi_V(g) = \mathrm{tr}(g_V) = \sum_{i=1}^{n} \lambda_i \qquad \chi_W(g) = \mathrm{tr}(g_W) = \sum_{j=1}^{m} \mu_j$$

(i) Let $\{v_1^*, \dots, v_n^*\}$ be the dual basis for V^* relative to $\{v_1, \dots, v_n\}$. Then,

$$(g \cdot v_i^*)(v_j) = v_i^*(g^{-1} \cdot v_j) = v_i^*(\lambda_j^{-1} v_j) = \lambda_j^{-1}\delta_{ij} = (\lambda_i^{-1} v_i^*)(v_j)$$

where the last line equality follows from the fact tht given equation is 0 unless $i = j$. Hence,

$$g \cdot v_i^* = \lambda_i^{-1} v_i^*$$

for all $i = 1, \dots, n$. On the other hand, as λ_i is a root of unity, we have $g \cdot v_i^* = \overline{\lambda_i} v_i^*$, thus

$$\chi_{V^*}(g) = \mathrm{tr}(g_{V^*}) = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\mathrm{tr}(g_V)} = \overline{\chi_V(g)}$$

(ii) The action is defined by

$$g_{V \oplus W} = \begin{pmatrix} g_V & 0 \\ 0 & g_W \end{pmatrix}$$

so the trace is exactly the sum of traces.

(iii) By definition, the elementary tensors form a basis for $V \otimes W$. We note

$$g \cdot (v_i \otimes w_j) = (g \cdot v_i) \otimes (g \cdot w_j) = (\lambda_i v_i) \otimes (\mu_j w_j) = \lambda_i \mu_j (v_i \otimes w_j)$$

In particular, the elementary tensors form a basis of eigenvectors for the g-action on $V \otimes W$ with eigenvalue $\lambda_i \mu_j$. Hence,

$$\chi_{V \otimes W}(g) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j = \left(\sum_{i=1}^{n} \lambda_i \right) \left(\sum_{j=1}^{m} \mu_j \right) = \chi_V(g)\chi_W(g)$$

(iv) We note the isomorphism $V^* \otimes W \cong \mathrm{Hom}(V, W)$, hence

$$\chi_{\mathrm{Hom}(V,W)} = \chi_{V^* \otimes W} = \chi_{V^*}\chi_W = \overline{\chi_V}\chi_W$$

(v) Noting that $v_i v_j$ is a basis for S^2V, we compute

$$g \cdot (v_i v_j) = (g \cdot v_i)(g \cdot v_j) = (\lambda_i v_i)(\lambda_j v_j) = \frac{1}{2}(\lambda_i v_i \otimes \lambda_j v_j + \lambda_j v_j \otimes \lambda_i v_i) = \lambda_i \lambda_j v_i v_j$$

In particular, the set $\{v_i v_j \mid 1 \le i \le j \le n\}$ form an eigenbasis for S^2V, thus

$$\chi_{S^2V}(g) = \sum_{1 \le i \le j \le n} \lambda_i \lambda_j = \sum_{1 \le i < j \le n} \lambda_i \lambda_j + \sum_i \lambda_i^2 = \frac{1}{2}\left(\sum_i \lambda_i\right)^2 + \frac{1}{2}\sum_i \lambda_i^2 = \frac{1}{2}\chi_V(g)^2 + \frac{1}{2}\chi_V(g^2)$$

(vi) By the same reasoning from (v), the set $\{v_i \wedge v_j \mid 1 \le i < j \le n\}$ form an eigenbasis for $\bigwedge^2 V$. Expanding,

$$\chi_{\bigwedge^2 V}(g) = \sum_{1 \le i < j \le n} \lambda_i \lambda_j = \frac{1}{2}\left(\sum_i \lambda_i\right)^2 - \frac{1}{2}\sum_i \lambda_i^2 = \chi_V(g)^2 - \chi_V(g^2)$$

□

Definition 6.1.6. *Let G be a finite group and let $\{g_1, \ldots, g_s\}$ be a set of representatives for the conjugacy classes of G. Let $V_1, \ldots, V_r$ be a complete list of representatives for the isomorphism classes of simple $\mathbb{C}G$-modules.*

*The **character table** of G is the $r \times s$ array with the (i,j)-th entry given by $\chi_{V_i}(g_j)$*

Remark 6.1.7. As $r = r_{\mathbb{C}}(G)$ and $s = s(G)$, the character table is always square. Also, $\chi(1) = \mathrm{tr}(\mathrm{Id}_V) = \dim V$.

Proposition 6.1.8. *Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional representation. Then,*

- $\chi_V(g) = \chi_V(1)$ *if and only if* $\rho(g) = 1$
- *If* $\dim V = 1$ *then* χ *is a group homomorphism.*

Proof. (i) $\Rightarrow$ If $\chi_V(g) = \chi_V(1) = \dim V$, then we note by the finite order of G that $\rho(g)^m = \rho(g^m) = \rho(1) = I_V$ for some m. Hence $\rho(g)$ satisfies the polynomial $X^m - I = 0$, hence the eigenvalues of $\rho(g)$ satisfy $\lambda^m = 1$. In particular, $|\lambda| = 1$. As the trace is then the sum of these eigenvalues which equals $\dim V$, by the triangle inequality we must have $\lambda = 1$ for any eigenvalue. In particular, $\rho(g) = I_V$.

$\Leftarrow$ If $\rho(g) = 1$, then we have

$$\chi_V(g) = \mathrm{tr}(1) = \dim V = \chi_V(1)$$

(ii) If $\dim V = 1$, then $\chi(g) = \rho(g)$, so the proof follows immediately from the fact ρ is a group homomorphism. □

Lemma 6.1.9. *Suppose that k is algebraically closed.*

1. *Suppose further that G is abelian. Every smiple kG-module is one-dimensional.*
2. *The converse of the above holds provided that $|G| \neq 0$ in k.*

Proposition 6.1.10. *Let χ be a character of G. Then $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$.*

Proposition 6.1.11. *Let $g \in G$. Then the following are equivalent:*

1. *g is conjugate to g^{-1}*

2. *for every character χ of G, we have $\chi(g) \in \mathbb{R}$.*

Definition 6.1.12. *Characters of degree 1 are called **linear characters**.*

Proposition 6.1.13. *Let $\chi_1, \dots, \chi_r$ be the complete list of characters of the irreducible complex representations of the finite group G. Then,*

$$\chi_1(1)^2 + \cdots + \chi_r(1)^2 = |G|$$

Proof. Suppose that the simple kG-module V_i affords the character χ_i. Then, $\chi_i(1) = \dim V_i$, thus the proof follows by Artin Weddernburn. $\square$

Definition 6.1.14. *Let N be a normal subgroup of the finite group G and let $\rho : G/N \to \mathrm{GL}(V)$ be a representation. The **inflated representation** of G,*

$$\dot{\rho} : G \to \mathrm{GL}(V)$$

is defined by $\dot{\rho} := \rho(gN)$ for all $g \in G$.

Definition 6.1.15. *Let G be a finite group. The **derived subgroup** G' is the subgroup of G generated by all **commutators** $[x, y] := xyx^{-1}y^{-1}$ in G, such that*

$$G' := \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$$

Remark 6.1.16. The derived subgroup is a normal subgroup of G. The commutator also satisfies $[g, h]^{-1} = [g^{-1}, h^{-1}]$ and $\phi([g, h]) = [\phi(g), \phi(h)]$ where $\phi : G \to H$ is a group homomorphism.

Proposition 6.1.17. *Suppose that N is a normal subgroup of G, and suppose further that G/N is abelian. Then G' is a normal subgroup of N.*

Proof. If G/N is abelian, then we know that for any $x, y \in G$, we have $xyN = (xN) \cdot (yN) = (yN) \cdot (xN) = yxN$. In particular, $xyx^{-1}y^{-1} \in N$. Hence we have $G' \subseteq N$, and as G' is normal in G, it is normal in N. $\square$

Proposition 6.1.18. *Every group homomorphism from G to an abelian group A is trivial on the commutator subgroup G' and hence factors through G/G'*

Proof. Let $\phi : G \to A$ be a group homomorphism where A is abelian. Then every commutator in A is equal to the identity, so every commutator $[g, h]$ of G lies in $\ker \phi$. In particular, $G' \leq \ker \phi$. Hence there is an induced map $\tilde{\phi} : G/G' \to A$ by sending $gG' \mapsto \phi(g)$, such that ϕ factors as $\tilde{\phi} \circ q_{G'}$ where $q_{G'}$ is the canonical quotient map. $\square$

Proposition 6.1.19. *Suppose that G is abelian. Then every simple kG-module is one-dimensional. The converse holds provided that $|G| \neq 0$ in k.*

Proof. Let V be a simple kG-module. Then we can find a nonzero vector $v \in V$. As k and G are abelian, so is kG.

The action of every $z \in Z(kG) = kG$ on V lies in $\mathrm{End}_{kG}(V)$. As k is algebraically closed and V is simple, by Schur's Lemma, we have $\mathrm{End}_{kG}(V) = k1_V$. As $k \cdot v \subseteq V$ is closed under actions by scalars, it follows that $k \cdot v$ is a nonzero kG-submodule of V. As V is simple, we must have $V = k \cdot v$, which is one dimensional over k.

If $|G| \neq 0$ in k, by Maschke's Theorem, kG is semisimple, and there is a complete list $V_1, \dots, V_r$ of representatives for the isomorphism classes of simple kG-modules. Hence by Artin Weddernburn, we have $kG \simeq M_{\dim V_1}(k) \times \cdots \times M_{\dim V_r(k)}$ as k-algebras. By assumption $\dim V_i = 1$ for all i, so we have $kG \simeq k^n$ as a commutative ring. In particular $G \leq kG^\times \simeq (k^n)^\times$ is abelian. $\square$

Lemma 6.1.20. *Let G be a finite group. There is a bijective correspondence between complex linear characters of G and irreducible complex characters of G/G'. Hence G has precisely $|G : G'|$ distinct complex linear characters.*

Proof. Let χ be a complex linear character of G afforded by the complex representation $\rho : G \to \mathrm{GL}(V)$. Then $\mathrm{GL}(V) \cong \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$, which is abelian, thus ρ induces a representation $\tilde{\rho} : G/G' \to \mathrm{GL}(V)$, such that $\tilde{\rho} \circ q_{G'} = \rho$. As $\dim V = 1$, we have $\tilde{\rho}$ is irreducible, and thus χ descends to an irreducible character $\tilde{\chi}$ of G/G' afforded by $\tilde{r}ho$ with $\tilde{\rho} \circ q_{G'} = \chi$.

On the other hand, suppose that ϕ is an irreducible complex character of G/G' afforded by the representation $\tau : G/G' \to \mathrm{GL}(V)$. As G/G' is abelian and $\mathbb{C}$ is algebraically closed, the simple $\mathbb{C}(G/G')$-module is 1-dimensional. Hence, the inflated representation $\dot{\tau} : G \to \mathrm{GL}(V)$ is also of degree 1, thus ϕ lifts to a linear character $\dot{\phi}$ of G afforded by $\cdot\tau$.

The bijection comes from the fact $\chi = \tilde{\chi} \circ \pi = \dot{\tilde{\chi}}$

As G/G' is abelian, $|G/G'| = |G : G'|$ is the number of conjugacy classes of G/G', which is the number of isomorphism classes of simple $\mathbb{C}(G/G')$-modules, which is the number of complex linear characters of G by correspondence.

□

Example 6.1.21. Let $G = A_4$ be the alternating group of order 12. We know that V_4 is a normal subgroup of order 4, written

$$V_4 := \{1, (12)(34), (14)(23), (13)(24)\}$$

Since A_4/V_4 has order 3, it must be a cyclic group of order 3, hence abelian. In particular, $A_4' \leq V_4$, forcing $|A_4'| \in \{1, 2, 4\}$. No subgroup of order 2 in V_4 is normal in A_4, and A_4' is nontrivial as it is not abelian, so it must be the case that $A_4' = V_4$. In particular, by Lemma 6.1.20, A_4 admits 3 distinct linear characters inflated from $A_4/V_4 \cong C_3$.

Definition 6.1.22. *Let G be a finite group. The* ***inner product on class functions***

$$\langle -, - \rangle : \mathcal{C}(G) \times \mathcal{C}(G) \to \mathbb{C}$$

is defined as

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi(g)$$

Remark 6.1.23. It is routine to verify that this is a complex inner product on $\mathcal{C}(G)$, satisfying our usual notions of sesquilinear, positive definite, and conjugate symmetry.

Proposition 6.1.24 (Fixed Point Formula)**.** *Let G be a finite group and let V be a finite dimensional $\mathbb{C}G$-module. Then*

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) = \langle \mathbb{1}, \chi_V \rangle$$

Proof. Let $e := \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}G$ Then $ge = eg = e$ fora all $g \in G$, so we have $e^2 = e$. We call e to be the **principal idempotent of** $\mathbb{C}G$.

Now we have a decomposition

$$V = e \cdot V \oplus (1 - e) \cdot V$$

If $g \in G$, then $g \cdot (e \cdot v) = (ge) \cdot v = e \cdot v$ so $e \cdot V \leq V^G$. On the other hand, if $v \in V^G$, then $g \cdot v = v$ for all $g \in G$, so $|G|e \cdot v = \sum_{g \in G} g \cdot v = |G|v$, giving $v = e \cdot v$, hence $v \in e \cdot V$. In particular, $e \cdot V = V^G$.

The action of $e \in \mathbb{C}G$ on V is a linear map $e_V : V \to V$ which is an idempotent with image $e \cdot V$. So, writing $\rho : G \to \mathrm{GL}(V)$ for the representation afforded by V, we have

$$\dim V^G = \dim e \cdot V = \mathrm{tr}(e_V) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}\rho(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

□

Proposition 6.1.25. *Let V and W be finite dimensional $\mathbb{C}G$-modules. Then,*

1. $\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}(V, W)^G$
2. $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_{\mathbb{C}G}(V, W)$

Proof. (i) Let $f \in \mathrm{Hom}(V, W)$. Then f is fixed by the G-action if and only if

$$g \cdot f(g^{-1} \cdot v) = f(v)$$

for all $g \in G$ and $v \in V$. In particular, we rewrite that

$$g_W \circ f = f \circ g_V$$

for all $g \in G$. By definition, this is exactly the functions $f in \mathrm{Hom}_{\mathbb{C}G}(V, W)$.

(ii) Noting the Fixed Point Formula, we have

$$\dim \mathrm{Hom}(V, W)^G = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,W)}(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g) = \langle \chi_V, \chi_W \rangle$$

□

Theorem 6.1.26 (Row Orthogonality)**.** *Let ϕ and ψ be irreducible characters of the finite group G. Then,*

$$\langle \phi, \psi \rangle = \begin{cases} 1 & \text{if } \phi = \psi \\ 0 & \text{if } \phi \neq \psi \end{cases}$$

Proof. Let V and W be the simple $\mathbb{C}G$-modules whose characters are $\phi = \chi_V$ and $\psi = \chi_W$. As V and W are simple, if they are not isomorphic, the only map is the 0 map. If the two are isomorphic, by Schur's Lemma we have

$$\dim \mathrm{Hom}_{\mathbb{C}G}(V, W) = \dim \mathrm{End}_{\mathbb{C}G}(V) = 1$$

Hence,

$$\dim \mathrm{Hom}_{\mathbb{C}G}(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases}$$

Hence by Proposition 6.1.25,

$$\langle \phi, \psi \rangle = \langle \chi_V, \chi_W \rangle = \dim_{\mathbb{C}G} \mathrm{Hom}(V, W) \in \{0, 1\}$$

Suppose that $\chi_V = \chi_W$. Then,

$$\langle \chi_V, \chi_W \rangle = ||\chi_V||^2 = \frac{1}{|G|}\sum_{g\in G} |\chi_V(g)|^2 \geq \frac{(\dim V)^2}{|G|} > 0$$

as $\chi_V(1) = \dim V$. Hence $\langle \chi_V, \chi_V \rangle = 1$. If $\chi_V \neq \chi_W$, then V cannot be isomorphic to W as isomorphic representations have the same characters, hence $\langle \phi, \psi \rangle = \dim \mathrm{Hom}_{\mathbb{C}G}(V, W) = 0$. $\square$

Remark 6.1.27. Let V be a finite dimensional kG-module,$\chi_1, \ldots, \chi_r$ be the complete list of characters of the irreducible complex representations of G, and suppose that V_i is the simple kG-module with character χ_i. By Maschke's Theorem, we know that V is a direct sum of simple kG-modules. Since $V_1, \ldots, V_r$ are the only possible simple kG-modules up to isomorphism, we can find non-negative integers $a_1, \ldots, a_r$ such that

$$V \cong V_1^{a_1} \oplus \cdots \oplus V_r^{a_r}$$

We call a_i the **multiplicity** of V_i in V.

Corollary 6.1.28. *Let V and W be two finite dimensional kG-modules. Then V is isomorphic to W if and only if $\chi_V = \chi_W$.*

Proof. Decompose V into simple kG-modules, such that

$$V \cong V_1^{a_1} \oplus \cdots \oplus V_r^{a_r}$$

Passing to characters, we have

$$\chi_V = a_1\chi_1 + \cdots + a_r\chi_r$$

Thus by row orthogonality, we can recover a_i from χ_V by

$$\langle \chi_i, \chi_V \rangle = \langle \chi_i, \sum_{j=}^{r} a_j\chi_j \rangle = \sum_{j=1}^{r} a_j\delta_{ij} = a_i$$

If $\chi_V = \chi_W$, decomposing $W = V_1^{b_1} \oplus \cdots \oplus V_r^{b_r}$ as a kG-module, then for any i, $a_i = \langle \chi_i, \chi_V \rangle = \langle \chi_i, \chi_W \rangle = b_i$. Hence $V \cong W$. The converse is straightforward. $\square$

Corollary 6.1.29. *The irreducible characters of G form an orthonormal basis for $\mathcal{C}(G)$.*

Proof. By row orthogonality, the characters are pairwise orthogonal elements in the inner product space $\mathcal{C}(G)$. On the other hand, $\dim \mathcal{C}(G) = s(G) = r_{\mathbb{C}} = r$, so $\{\chi_1, \ldots, \chi_r\}$ form a basis for $\mathcal{C}(G)$. $\square$

Theorem 6.1.30 (Column Orthogonality)**.** *Let G be a finite group. Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G and let $g, h \in G$. Then,*

$$\sum_{i=1}^{r} \overline{\chi_i(g)}\chi_i(h) = \begin{cases} |C_G(g)| & \text{if } g \text{ is conjugate to } h \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $\{g_1, \ldots, g_r\}$ be a complete list of representatives for the conjugacy classes of G. Suppose that $g \in g_j^G$ and $h \in g_k^G$ for some j, k. As the characters χ_i are class functions, we will assume without loss of generality that $g = g_j$ and $h = g_k$.

Define

$$x_{i,j} = \chi_i(g_j) \cdot c_j \qquad \text{where } c_j := \sqrt{|g_j^G|/|G|}$$

Then we can compute,

$$\begin{aligned}
\sum_{j=1}^{r} \overline{x_{i,j}} x_{k,j} &= \sum_{j=1}^{r} \overline{\chi_i(g_j)} c_j \chi_k(g_j) c_j \\
&= \sum_{j=1}^{r} \overline{\chi_i(g_j)} \chi_k(g_j) c_j^2 \\
&= \frac{1}{|G|} \sum_{j=1}^{r} |g_j^G| \overline{\chi_i(g_j)} \chi_k(g_j) \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(x)} \chi_k(x) \\
&= \langle \chi_i, \chi_k \rangle = \delta_{i,k}
\end{aligned}$$

Where the last line comes from row orthogonality. Hence the $r \times r$ matrix $X := (x_{i,j})$ is unitary:

$$\overline{X} \cdot X^T = I$$

So $\overline{X}$ is the left-inverse (and the right-inverse) of X^T in $\mathrm{GL}_r(\mathbb{C})$. Applying complex conjugation, we have

$$\overline{X}^T \cdot X = I$$

In particular,

$$(\overline{X}^T \cdot X)_{j,k} = \sum_{i=1}^{r} \overline{x_{i,j}} x_{i,k} = \sum_{i=1}^{r} \overline{\chi_i(g_j)} c_j \chi_i(g_k) c_k = \delta_{j,k}$$

Dividing both sides by $c_j c_k$ and taking $j = k$, we have $1/c_j^2 = |G|/|g_j^G| = |C_G(g_j)|$. □

6.2 Examples of Character Table Computation

When the explicit representations are known, computing the character tends to be straightforward.

Example 6.2.1. The character table for the cyclic group of order 3, $G = \{1, x, x^2\}$ is where $\omega :=$

	1	x	x^2
$\mathbb{1}$	1	1	1
χ	1	ω	ω^2
χ^2	1	ω^2	ω

$\exp(2\pi i/3)$ is a primitive cube root of unity. To see explicitly where these choices of representations came from, note that we have $\rho(g)^3 = \rho(g^3) = \rho(e) = 1$, so we must have $\rho(g)$ be sent to a primitive cube root of unity, and this determines all possible representations.

Example 6.2.2. Let $G = S_3$. Alongside the trivial character, we have a sign character $\epsilon : S_3 \to \{\pm 1\} \subseteq \mathbb{C}^\times$ by

$$\epsilon(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

We also have the two-dimensional irreducible representation W of S_3 from Example 3.0.18, we get the character table of S_3 as

	1	(123)	(12)
$\mathbb{1}$	1	1	1
ϵ	1	1	-1
χ_W	2	-1	0

We can use inflation to find character tables as well.

Example 6.2.3. Let $G = A_4$. Then $A_4' = V_4$, and G has 3 distinct linear characters. The representatives for the conjugacy classes in A_4 are 1, $g_2 := (12)(34)$, $g_3 := (123)$ and $g_4 := (132)$. Hence noting there are 4 conjugacy classes, our character table for A_4 looks as follows: where ω is

g	1	g_2	g_3	g_4
$\lvert g^G \rvert$	1	3	4	4
$\lvert C_G(g) \rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	d	a	b	c

a primitive third root of unity. As the sum of squares on the identity class is $|G| = 12$, we deduce that $d^2 = 3$. This equals $\dim V \in \mathbb{N}$, so we get $d = 3$.

Now by row orthogonality, we get

$$\begin{aligned}
0 &= |G|\langle \chi_1, \chi_4 \rangle = \sum_{g \in G} \overline{\chi_1(g)}\chi_4(g) = 1 \cdot 1 \cdot 3 + 3 \cdot 1 \cdot a + 4 \cdot 1 \cdot b + 4 \cdot 1 \cdot c = 3 + 3a + 4b + 4c \\
0 &= |G|\langle \chi_2, \chi_4 \rangle = 3 + 3a + 4b\omega + 4c\omega^2 \\
0 &= |G|\langle \chi_3, \chi_4 \rangle = 3 + 3a + 4b\omega^2 + 4c\omega
\end{aligned}$$

Solving this gives $a = -1$, $b = c = 0$, so the full character table follows by substituting these values.

Example 6.2.4. Let G be the symmetric group S_4. The conjugacy class representatives are $g_1 = 1$, $g_2 = (12)(34)$, $g_3 = (123)$, $g_4 = (12)$, $g_5 = (1234)$, with conjugacy classes of sizes 1, 3, 8, 6, 6 respectively. We know that $V_4 \leq S_4$ and $S_4/V_4 \cong S_3$. Hence, this gives irreducible characters $\tilde{\mathbb{1}}, \tilde{\epsilon}, \tilde{\chi_W}$ obtained by inflation from S_3.

This gives a partial table:

g	1	g_2	g_3	g_4	g_5
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
$\tilde{\mathbb{1}}$	1	1	1	1	1
$\tilde{\epsilon}$	1	1	1	-1	-1
$\tilde{\chi_W}$	2	2	-1	0	0
χ_4	d_4	α_4	β_4	γ_4	δ_4
χ_5	d_5	α_5	β_5	γ_5	δ_5

Now noting that $d_4^2 + d_5^2 = 24 - 1^2 - 1^2 - 2^2 = 18$, the only solutions with positive integers to this is $d_4 = d_5 = 3$ by column orthogonality. By applying this again to the first pair of columns and the second column, we obtain

$$1 + 1 + 4 + 3\alpha_4 + 3\alpha_5 = 0 \qquad 1^2 + 1^2 + 2^2 + |\alpha_4|^2 + |\alpha_5|^2 = 8$$

Hence $\alpha_4 + \alpha_5 = -2$ and $|\alpha_4|^2 + |\alpha_5|^2 = 2$, which solves to $\alpha_4 = \alpha_5 = -1$.

Applying this to the third column we get

$$1^2 + 1^2 + (-1)^2 + |\beta_4|^2 + |\beta_5|^2 = 3$$

Thus $\beta_4 = \beta_5 = 0$.

Similar considerations give $\gamma_5 = -\gamma_4$ and that $|\gamma_4| = 1$. As $g_4 = (12)$ has order 2, it acts with eigenvalues ± 1 in any representation. Hence γ_4 is the sum of these eigenvalues, and is a real number, so the choices are $\gamma_4 = \{1, -1\}$. Without loss of generality, we may assume that $\gamma_4 = 1$, otherwise swapping χ_4 with χ_5. Row orthogonality gives $\delta_4 = -1$ and $\delta_5 = 1$, completing our table.

6.3 Burnside's Theorem

Proposition 6.3.1. *$\chi(g)$ is an algebraic integer for all $g \in G$.*

Proof. $\chi(g)$ is a sum of ord-g-th roots of unity, why are all algebraic integers. These form a subring of $\mathbb{C}$ □

Lemma 6.3.2. *Let G be a finite group and let $C_1, \ldots, C_r$ be the conjugacy classes in G. Let S be the additive subgroup of $\mathbb{C}G$ generated by the conjugacy class sums. Then S is a subring of $\mathbb{Z}(\mathbb{C}G)$.*

Proof. Sketch. Show stability under multiplication (as we know it is stable under addition.) (show coef is the same in the same conjugacy class) □

Theorem 6.3.3. *Let V a simple $\mathbb{C}G$-module with $g \in G$.*

1. *The conjugacy class sum $\widehat{g^G}$ acts on V by the scalar $\frac{|g^G|\chi_V(g)}{\chi_V(1)} \in \mathbb{C}$*
2. *The scalar above is an algebraic integer*

Proof. As V is a simple $\mathbb{C}G$-module and the conjugacy class sum $z := \widehat{g^G}$ is central in $\mathbb{C}G$, it acts by a scalar $z_V \in \mathbb{C}$ on every simple $\mathbb{C}G$-module by Schur's Lemma. Taking the trace of this action, we get

$$z_V \dim V = |g^G|\chi_V(g)$$

Hence (i) follows from the fact $\dim V = \chi(1)$.

Now let $\rho : G \to \mathrm{GL}(V)$ be the representation afforded by V. Then ρ extends to a $\mathbb{C}$-algebra homomorphism $\tilde{\rho} : \mathbb{C}G \to \mathrm{End}(V)$. The restriction of this homomorphism to the center is the central character of V, so $\tilde{\rho}(\mathbb{C}G) \subseteq \mathbb{C}$. Hence $\tilde{\rho}(S)$ is a finitely generated abelian subgroup of $\mathbb{C}$. It is also a subring of $\mathbb{C}$ as $\tilde{\rho}$ is a ring homomorphism and S is a subring of $Z(\mathbb{C}G)$. Hence $z_V \cdot \tilde{\rho}(S) \subseteq \tilde{\rho}(S)$. □

Corollary 6.3.4. *If V is a simple $\mathbb{C}G$-module, then $\dim V$ divides $|G|$.*

Proof. By row orthogonality,

$$\sum_i \chi_V(g^{-1})\frac{|g_i^G|\chi_V(g_i)}{\chi_V(1)} = \frac{|G|}{\chi_V(1)}$$

Now the left side is an algebraic integer, and the right side shows it is a rational number. Hence this is an integer. ☐

Definition 6.3.5. *Let G be a finite group and let p be a prime. Write $|G| = p^\alpha m$ where $p \nmid m$ A **sylow p-subgroup** of G is a subgroup P of G order p^α.*

Theorem 6.3.6 (Sylow)**.** *Let G be a finite group.*

1. *G contains at least one Sylow p-subgroup.*
2. *Any two sylow p-subgroups are conjugate in G*
3. *The number of Sylow p-subgroups of G is congruent to 1 mod p, and this number divides $m = |G|/p^\alpha$.*

Lemma 6.3.7. *Let G be a group of order $p^\alpha q^\beta$ where p and q are distinct primes with $\alpha, \beta \geq 1$. Let g be a central element of a Sylow p-subgroup P of G. Then $|g^G|$ is a power of q.*

Proof. As P centralises g, we have $P \leq C_G(g)$. Hence $|G : C_G(g)|$ divides $|G|/|P| = q^\beta$. However, this index equals $|g^G|$. ☐

Lemma 6.3.8. *Let $\alpha = \frac{\zeta_1+\cdots+\zeta_n}{n}$ be sums of roots of unity, and α is an algebraic integer. Then either $\alpha = 0$ or $\alpha = \zeta_1 = \cdots = \zeta_n$.*

Theorem 6.3.9. *Let G be a finite group and suppose that the size of a conjugacy class of a non central element $g \in g$ is a power of q. Then G is not a simple group.*

Corollary 6.3.10 (Burnside)**.** *Let G be a non-abelian group of order $p^\alpha q^\beta$ where p, q are primes. Then G is not a simple group.*

6.4 Module vs Representation

Example 6.4.1. Consider the representation $\rho : \mathbb{Z} \to \mathrm{GL}(V)$ via the extension of $n \mapsto (M \mapsto M^n)$. Then,

$$k\mathbb{Z} = \oplus_{n\in\mathbb{Z}} kT^n = k[T, T^{-1}]$$

- $\rho : G \to \mathrm{GL}(V) \leftrightarrow kG$-module V
- ρ completely reducible $\leftrightarrow kG$ is completely reducible
- subrepresentations correspond exactly to submodules
- If $V = U \oplus W$ (subreps) correspond
- semisimple (kG as a kG module is completely reducible)
- Noting that $V^* \otimes V \cong \mathrm{End}(V)$, the left ideals of $\mathrm{End}(V)$ (are of the form $U \otimes V$, $U \subseteq V^*$) correspond to subspaces of V^*.

Example 6.4.2. Let $G = C_3 = \langle x \rangle$. Suppose that $\mathrm{char}(k) \neq 3$ and that k contains a primitive cube root of unity.

Then, kG is generated by kx, with the two commuting with each other. Thus, we have a surjective evaluation homomorphism from $k[t] \twoheadrightarrow kG$ sending $t \mapsto x$. Now by the first isomorphism theorem, $\ker \phi = (t^3 - 1)$. This induces an isomorphism

$$kC_3 \cong k[t]/(t^3 - 1) \cong k[t]/(t-1) \times k[t]/(t-\omega) \times k[t]/(t-\omega^2)$$

by the chinese remainder theorem.

Lemma 6.4.3. *Let χ be a character of G. The set $N := \{g \in G \mid \chi(g) = \chi(1)\}$ is a normal subgroup of G, and is exactly the kernel of the representation.*

Proof. Let χ be a character of the complex representation $\rho : G \to \mathrm{GL}(V)$. If $g \in N$, $\rho(g)$ is diagonalizable, so we can find a basis of $\rho(g)$-eigenvectors $\{v_1, \dots, v_n\}$ for V with eigenvalues $\lambda_1, \dots, \lambda_n$ where each λ_i is a order-g-th root of unity. Now,

$$\chi(g) = \mathrm{tr}\rho(g) = \lambda_1 + \cdots + \lambda_n = n$$

The argument then follows by Cauchy-Schwarz. □

Proposition 6.4.4. *G is simple if and only if $\chi(g) \neq \chi(1)$ for all $g \neq 1$ and every irreducible $\chi \neq \mathbb{1}$.*

Proof. $\Rightarrow$ □

Proposition 6.4.5. *Let χ be a character of G. $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$.*

Proof. Sketch. Take the eigenvalues, these are on roots of unity, the inverse forms a set of $\chi(g^{-1})$, the inverse is the conjugate. □

Proposition 6.4.6. *$g \in G$ is conjugate to g^{-1} if and only if $\chi(g) \in \mathbb{R}$ for every character χ of G.*

Proof. ($\Rightarrow$) Characters are class functions so $\chi(g^{-1}) = \chi(g) = \overline{\chi(g)} \in \mathbb{R}$.

($\Leftarrow$) If all $\chi_i \in \mathbb{R}$ for irreducible characters but g is not conjugate to g^{-1}, by column orthogonality, we have

$$0 = \sum_i \overline{\chi_i(g^{-1})}\chi_i(g) = \sum_i \chi_i(g)^2$$

As each component is real, this forces $\chi_i(g) = 0$. On the other hand by column orthogonality on itself,

$$0 = \sum_i \overline{\chi_i(g)}\chi_i(g) = |C_G(g)| \geq 1$$

Hence a contradiction, showing that g is conjugate to g^{-1}. □

Proposition 6.4.7. *The following things can be found from the character table:*

- $|G|$
- $|G : G'|$ *thus also* $|G'|$
- $|Z(G)|$

Proof. Note first that $|G| = \sum_i \chi_i(1)^2$ and that $|G : G'| = |\{i \mid \chi_i(1) = 1\}|$, and we use these to find $|G'| = |G|/|G : G'|$.

Finally, elements in the center are exactly those with trivial conjugacy classes, hence g is in the center if and only if $C_G(g) = G$. By column orthogonality, this is exactly when $\sum_i |\chi_i(g)|^2 = |C_G(g)| = |G|$ by column orthogonality. This gives an explicit method to compute the size of the center. $\square$

7 Techniques

7.0.1 Computing Conjugacy Classes

1. By the orbit stabilizer, we always have $|g^G||C_G(g)| = |G|$
2. The elements in the center each form their own conjugacy class. Then we can use the fact $|G| = |Z(G)| + \sum |C_i|$
3. For S_n, use cycle-type analysis
4. Normal subgroups are unions of conjugacy classes

7.0.2 Finding Characters

1.