First Read on Linear Algebra, Lecture 3

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3 Linear Maps

3.1 Basic Definitions

Definition 3.1.1. *Let V, W be vector spaces over $\mathbb{F}$. A map $T : V \to W$ is **linear** if*

1. *For all $v_1, v_2 \in V$, we have $T(v_1 + v_2) = T(v_1) + T(v_2)$*
2. *For all $v \in V$ and $\lambda \in \mathbb{F}$, $T(\lambda v) = \lambda T(v)$*

*We call T a **linear transformation**, **linear map**, or a **morphism** between vector spaces.*

That is, linear maps preserves the additive structure and scalar multiplication.

Proposition 3.1.2. *Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear map. We have $T(0_V) = 0_W$.*

Proof. We have

$$T(0_V) + T(0_V) = T(0_V + 0_V) = T(0_V)$$

□

Proposition 3.1.3. *Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$. The following are equivalent:*

1. *T is linear*
2. *For all $u, v \in V$ and $\lambda, \mu \in \mathbb{F}$, $T(\lambda u + \mu v) = \lambda T(u) + \mu T(v)$*
3. *For any $n \geq 1$, if $v_1, \ldots, v_n \in V$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$,*

$$T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \cdots \alpha_n T(v_n)$$

Proof. $(i) \Rightarrow (ii)$ We simply have

$$T(\lambda u + \mu v) = T(\lambda u) + T(\mu v) = \lambda T(u) + \mu T(v)$$

by linearity.

$(ii) \Rightarrow (iii)$ We note that

$$T(\alpha_1 v_1 + (\alpha_2 v_2 + \cdots + \alpha_n v_n)) = \alpha T(v_1) + T(\alpha_2 v_2 + \cdots + \alpha_n v_n)$$

and the rest follows by induction.

$(iii) \Rightarrow (i)$ Taking the case $n = 2$ and fixing $\alpha_1 = \alpha_2 = 1$, we have $T(v_1 + v_2) = T(v_1) + T(v_2)$. Taking $n = 1$ gives $T(\alpha_1 v_1) = \alpha_1 T(v_1)$. □

Example 3.1.4. We outline some examples of linear maps:

- Let V be a vector space. The **identity map** $\mathrm{id}_V : V \to V$ by $\mathrm{id}_V(v) = v$ for $v \in V$ is a linear map.
- Let V, W be vector spaces. The **zero map** sending $v \in V$ to 0_W is a linear map.
- Let $\mathbb{R}_n[x]$ be the vector space of polynomials degree at most n. Define $D : \mathbb{R}_n(x) \to \mathbb{R}_n[x]$ by $p(x) \mapsto p'(x)$. This is a linear map from $\mathbb{R}_n[x]$ to $\mathbb{R}_n[x]$. Alternatively, if $n > 0$, it is a linear map from $\mathbb{R}_n[x]$ to $\mathbb{R}_{n-1}[x]$.
- Let X be a set and take $V = \mathbb{R}^X$. For any $a \in X$, the **evaluation map** $E_a : V \to \mathbb{R}$ by $f \mapsto f(a)$ is a linear map.

Proposition 3.1.5. *Let V, W be vector spaces over a field $\mathbb{F}$. For $S, T : V \to W$ and $\lambda \in \mathbb{F}$, define $S + T$ by*

$$(S+T)(v) = S(v) + T(v)$$

for every $v \in V$ and λS by

$$(\lambda S)(v) = \lambda S(v)$$

for every $v \in V$.

Proof. straightforward exercise. $\square$

Definition 3.1.6. *Given vector spaces V, W over $\mathbb{F}$, the set of linear transformations over $\mathbb{F}$ with the above operations gives a vector space denoted $\mathrm{Hom}_{\mathbb{F}}(V, W)$.*

Proposition 3.1.7. *Let U, V, W be vector spaces over $\mathbb{F}$. Let $S : U \to V$ and $T : V \to W$ be linear maps. Then the composed map $T \circ S : U \to W$ is linear.*

Proof. Take any $u_1, u_2 \in U$ and $\lambda_1, \lambda_2 \in \mathbb{F}$. We have

$$\begin{aligned}
(T \circ S)(\lambda_1 u_1 + \lambda_2 u_2) &= T(S(\lambda_1 u_1 + \lambda_2 u_2)) \\
&= T(\lambda_1 S(u_1) + \lambda_2 S(u_2)) \\
&= \lambda_1 T(S(u_1)) + \lambda_2 T(S(u_2)) \\
&= \lambda_1 (T \circ S)(u_1) + \lambda_2 (T \circ S)(u_2)
\end{aligned}$$

Hence $T \circ S$ is linear. $\square$

Notation 3.1.8. We often omit the $\circ$, writing TS to mean $T \circ S$.

Definition 3.1.9. *Let V, W be vector spaces and let $T : V \to W$ be linear. T is **invertible** if there is a linear transformation $S : W \to V$ such that $ST = \mathrm{id}_V$ and $TS = \mathrm{id}_W$. Then, S is the **inverse** of T, with notation T^{-1}. An invertible map is called an **isomorphism**, and we say that V and W are **isomorphic**, written $V \cong W$.*

Remark 3.1.10. Note that we need the two-sided inverse, as we can define a map from $T : \mathbb{Z} \to \mathbb{R}$ with an inverse $S : \mathbb{R} \to \mathbb{Z}$ such that $ST = \mathrm{id}_{\mathbb{Z}}$, but not the other way around (by a countability argument).

Proposition 3.1.11. *Let V, W be vector spaces. Let $T : V \to W$ be an invertible linear map. The inverse of T is unique.*

Proof. Let $S_1, S_2 : W \to V$ be inverses for T. We have

$$S_1 = S_1 \circ \mathrm{id}_W = S_1 \circ (T \circ S_2) = (S_1 \circ T) \circ S_2 = \mathrm{id}_V \circ S_2 = S_2$$

$\square$

Proposition 3.1.12. *Let V, W be vector spaces. Let $T : V \to W$ be a linear map. Then T is invertible if and only if T is bijective.*

Proof. ($\Rightarrow$) Suppose that T is invertible. Now let $T(v_1) = T(v_2)$. Applying S gives

$$v_1 = S(T(v_1)) = S(T(v_2)) = v_2$$

hence T is injective. Now fix any $w \in W$, and define $v := S(w)$. Then

$$T(v) = T(S(w)) = (T \circ S)(w) = \mathrm{id}_W(w) = w$$

so T is surjective, hence bijective.

($\Leftarrow$) Assume that T is bijective. Then T has an inverse $S : W \to V$, so it suffices to show that S is linear. Take $w_1, w_2 \in W$ and $\lambda_1, \lambda_2 \in \mathbb{F}$, and define $v_1 := S(w_1)$ and $v_2 := S(w_2)$. Then, we have

$$\begin{aligned}
S(\lambda_1 w_1 + \lambda_2 w_2) &= S(\lambda_1 T(v_1) + \lambda_2 T(v_2)) \\
&= S(T(\lambda_1 v_1 + \lambda_2 v_2)) \\
&= \lambda_1 v_1 + \lambda_2 v_2 \\
&= \lambda_1 S(w_1) + \lambda_2 S(w_2)
\end{aligned}$$

$\square$

Proposition 3.1.13. *Let U, V, W be vector spaces. Let $S : U \to V$ and $T : V \to W$ be invertible linear maps. Then $TS : U \to W$ is invertible with $(TS)^{-1} = S^{-1}T^{-1}$.*

Proof. It suffices to show that $S^{-1}T^{-1}$ is indeed the inverse. Now,

$$(S^{-1}T^{-1}) \circ (TS) = S^{-1} \circ (T^{-1} \circ T) \circ S = S^{-1} \circ \mathrm{id}_V \circ S = S^{-1} \circ S = \mathrm{id}_U$$

and

$$(TS) \circ (S^{-1}T^{-1}) = T \circ (S \circ S^{-1}) \circ T^{-1} = T \circ \mathrm{id}_V \circ T^{-1} = T \circ T^{-1} = \mathrm{id}_W$$

$\square$

Proposition 3.1.14. *Let V, W be vector spaces where V is finite-dimensional. Let $v_1, \ldots, v_n$ be a basis for V. If $T : V \to W$ is a linear map that is injective, then $T(v_1), \ldots, T(v_n)$ is linearly independent in W. If T is instead surjective, then $T(v_1), \ldots, T(v_n)$ spans W.*

Proof. Suppose that T is injective. Now further suppose that $\sum_i \alpha_i T(v_i) = 0$. Linearity gives $T(\sum_i \alpha_i v_i) = 0$. Injectivity gives $\sum_i \alpha_i v_i = 0$. Linear independence on V gives $\alpha_i = 0$.

Suppose now that T is surjective. For any $w \in W$, choose $v \in V$ with $T(v) = w$, and write $v = \sum_i \alpha_i v_i$. Then $w = T(v) = T(\sum_i \alpha_i v_i) = \sum_i \alpha_i T(v_i)$, hence $\{T(v_1), \ldots, T(v_n)\}$ spans W. $\square$

Proposition 3.1.15. *Let V, W be vector spaces where V is finite-dimensional. If there exists some $T : V \to W$ which is an invertible linear map, then $\dim V = \dim W$. Conversely, if W is also finite dimensional with $\dim V = \dim W$, then there exists an invertible linear map.*

Proof. Let $v_1, \ldots, v_n$ be a basis for V. Suppose that there exists some invertible linear map. Then this is a bijective map, so by Proposition 3.1.14, $T(v_1), \ldots, T(v_n)$ is a basis for W. Hence $\dim V = \dim W$.

On the other hand, suppose that $\dim V = \dim W = n$. Pick a basis $w_1, \ldots, w_n$ of W. Define a map $T : V \to W$ by $T(v_i) = w_i$ and extend linearly. A simple check shows that T is a linear map. We claim T is bijective, hence invertible.

Injective: if $T(\sum_i \alpha_i v_i) = \sum_i \alpha_i w_i = 0$, then as $w_1, \ldots, w_n$ form a basis $\alpha_i = 0$

Surjective: Fix a $w \in W$. Writing $w = \sum_i \alpha_i w_i$ for some α_i, we note that $w = \sum_i \alpha_i w_i = T(\sum_i \alpha_i v_i)$. □

Remark 3.1.16. Noting the previous proposition, given that V and W are finite dimensional, V and W are isomorphic if and only if $\dim V = \dim W$.

3.2 Quotient Spaces

Definition 3.2.1. *Let V be a vector space over a field $\mathbb{F}$ and let U be a subspace. Define*

$$V/U := \{v + U \mid v \in V\}$$

Proposition 3.2.2. *Define operations on V/U by*

$$\begin{aligned}(v + U) + (w + U) &:= v + w + U \\ \alpha(v + U) &:= \alpha v + U\end{aligned}$$

for $v, w \in V$ and $\alpha \in \mathbb{F}$ is well-defined.

Proof. To show that these operations are well-defined, we must show that the operations behave equality regardless of the choice of the representative.

Assume that $v + U = v' + U$ and $w + U = w' + U$. That is, $v = v' + u$ and $w = w' + \tilde{u}$ for some $u, \tilde{u} \in U$. Now,

$$\begin{aligned}(v + U) + (w + U) &= v + w + U \\ &= v' + u + w' + \tilde{u} + U \\ &= v' + w' + U \\ &= (v' + U) + (w' + U)\end{aligned}$$

Similarly,

$$\begin{aligned}\alpha(v + U) &= \alpha v + U \\ &= \alpha v' + \alpha u + U \\ &= \alpha v' + U \\ &= \alpha(v' + U)\end{aligned}$$

□

Definition 3.2.3. *The above operations satisfy the vector space axioms (induced from V), and we call this vector space the* ***quotient space****.*

Definition 3.2.4. *Let V be a finite dimensonal vector space and U be a subspace of V. Let $\mathcal{E}$ be a basis of U, and extend $\mathcal{E}$ to a basis $\mathcal{B}$ of V. Define*

$$\overline{\mathcal{B}} := \{e + U \mid e \in \mathcal{B} \setminus \mathcal{E}\} \subseteq V/U$$

Proposition 3.2.5. *The set $\overline{\mathcal{B}}$ is a basis for V/U.*

Proof. Let $\mathcal{E} = \{u_1, \ldots, u_k\}$ and $\mathcal{B} = \{u_1, \ldots, u_k, w_1, \ldots, w_m\}$. Take any $v + U \in V/U$ with $v = \sum_i \alpha_i u_i + \sum_j \beta_j w_j$. Then,

$$v + U = \sum_i \alpha_i u_i + \sum_j \beta_j w_j + U = \sum_j \beta_j w_j + U = \sum_{j=1}^{m} b_j(w_j + U)$$

Hence $\overline{\mathcal{B}}$ is spanning. On the other hand, suppose that $\sum_j \beta_j(w_j + U) = 0 + U$. Then, $\sum_j \beta_j w_j \in U$, so we can write

$$\sum_j \beta_j w_j = \sum_i \alpha_i u_i$$

for some $\alpha_i \in \mathbb{F}$. Thus,

$$\sum_i (-\alpha_i) u_i + \sum_j \beta_j w_j = 0$$

As $\{u_1, \ldots, u_k, w_1, \ldots, w_m\}$ is a basis for V, we get $\alpha_i = \beta_j = 0$ for all i and j, and thus $\overline{\mathcal{B}}$ is linearly independent. □

Corollary 3.2.6. *If V is finite dimensional, then*

$$\dim V = \dim U + \dim(V/U)$$

Proof. Let $u_1, \ldots, u_k$ be a basis for U, and extend this to a basis $u_1, \ldots, u_k, w_1, \ldots, w_m$ of V. Then, the set $\{w_1 + U, \ldots, w_m + U\}$ is a basis for V/U. There are no duplicates as w_i are linearly independent, so $\dim(V/U) = |\overline{\mathcal{B}}| = m$. Now,

$$\dim V = |\mathcal{B}| = k + m = |\mathcal{E}| + |\overline{\mathcal{B}}| = \dim U + \dim(V/U)$$

□

3.3 Rank and Nullity

Definition 3.3.1. *Let V and W be vector spaces. Let $T : V \to W$ be linear. Define the* ***kernel*** *to be*

$$\ker T := \{v \in V \mid T(v) = 0_W\}$$

Define the ***image*** *of T to be*

$$\mathrm{Im}\ T := \{T(v) \mid v \in V\}$$

Proposition 3.3.2. *Let V and W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be linear. Then $\ker T$ is subspace of V and $\mathrm{Im}\ T$ is a subspace of W.*

Proposition 3.3.3. *Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be linear. Then T is injective if and only if $\ker T = \{0_V\}$*

Proof. ($\Rightarrow$) Take $v \in \ker T$. As $T(v) = 0 = T(0)$, injectivity gives $v = 0$.

($\Leftarrow$) Suppose that $T(v_1) = T(v_2)$. Then,

$$T(v_1 - v_2) = T(v_1) - T(v_2) = 0$$

so $v_1 - v_2 \in \ker T$. As the kernel only contains 0, we have $v_1 = v_2$. □

Theorem 3.3.4 (First Isomorphism Theorem)**.** *Let $T : V \to W$ be a linear map of vector spaces over $\mathbb{F}$. Then the induced map $\bar{T} : V/\ker T \to \text{Im } T$ given by $v + \ker T \mapsto T(v)$ is an isomorphism of vector spaces.*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "q"'] & \text{Im } T \\ V/\ker T \arrow[ur, dashed, "\cong"'] & \end{tikzcd}$$

Definition 3.3.5. *Let V, W be vector spaces with V finite dimensional. Let $T : v \to W$ be linear. Define the* ***nullity*** *of T to be* $\text{nullity}(T) := \dim(\ker T)$ *and the* ***rank*** *of T to be* $\text{rank}(T) := \dim(\text{Im } T)$

Theorem 3.3.6 (Rank-Nullity)**.** *Let V, W be vector spaces with V fintie-dimensional. Let $T : V \to W$ be linear maps. Then*

$$\dim V = \text{rank}(T) + \text{nullity}(T)$$

Proof. We first note that

$$\dim V = \dim(\ker T) + \dim(V/\ker T)$$

By the First Isomorphism Theorem, we have

$$\dim(V/\ker T) = \dim(\text{Im } T)$$

so the proof follows. □

Corollary 3.3.7. *Let V be a finite dimensiomal vector space. Let $T : V \to V$ be linear. The following are equivalent:*

1. *T is invertible*
2. rank $T = \dim V$
3. nullity $T = 0$

Corollary 3.3.8. *Let V be a fintie-dimensional vector space. Let $T : V \to V$ be linear. Then every one-sided inverse of T is a two-sided inverse.*