Notes on Commutative Algebra

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1 Introduction

We assume basic concepts like injectivity, surjectivity, bijectivity

2 Groups

2.1 Basic definitions

Definition 2.1.1. *A group G is a set with a binary operation $*$ such that the following hold*

1. *Closure by $*$. If $f, g \in G$, then $f * g \in G$*
2. *$*$ is associative*
3. *Exists $e \in G$ such that for all $g \in G$, $e * g = g * e = g$*
4. *For every $g \in G$, there exists a $h \in G$ such that $g * h = h * g = e$*

Notation 2.1.2. *The element h is called the **inverse** of g, and is typically denoted g^{-1}, noting that it is unique.*

When dealing with a group who uses addition as it's operator, we will write $-g$ to represent the inverse.

Definition 2.1.3. *A group G is **abelian** if $*$ is commutative.*

Definition 2.1.4. *A **group homomorphism** is a map $\phi : G_1 \to G_2$ such that for all $g, h \in G$, $\phi(g *_1 h) = \phi(g) *_2 \phi(h)$, where the operators $*_1$ and $*_2$ are those associated with G_1 and G_2 respectively.*

*A homomorphism from G to G is called an **endomorphism***

Definition 2.1.5. *The **kernel** of a homomorphism ϕ is defined by*

$$\operatorname{Ker}(\phi) = \{g \in G_1 : \phi(g) = e_2\}$$

*The **image** of ϕ is*

$$\operatorname{Im}(\phi) = \{\phi(g) : g \in G_1\}$$

Remark 2.1.6. *As with linear algebra, if ϕ has a trivial kernel, then it is injective.*

Definition 2.1.7. *Let $\phi : G_1 \to G_2$ be a bijective homomorphism. Then we call ϕ to be an **isomorphism**, and that G_1 and G_2 are **isomorphic**.*

In such case, we write $G_1 \cong G_2$.

Remark 2.1.8. *Isomorphic groups inherit all structural properties, as it can simply be seen as a group with elements relabelled. For instance, G_1 is abelian if and only if G_2 is abelian.*

Lemma 2.1.9. *If $m, n \in \mathbb{Z}$ and m, n are coprime, $C_m \times C_n \cong C_{mn}$.*

Proof. TODO!!! □

Lemma 2.1.10. *Any finite abelian group G is isomorphic to the product of cyclic groups. That is, there exists $m_1, \dots, m_k$ such that*

$$G \cong C_{m_1} \times \cdots \times C_{m_k}$$

Proof. TODO!!!! □

Definition 2.1.11. *A **subgroup** of a group G is a non-empty subset $H \subseteq G$ such that it forms a group with the operator on G restricted to H, which must be well-defined (must be closed by $*$).*

We write $H \leq G$ if H is a subgroup of G.

Definition 2.1.12. *Let $H \leq G$. The set $gH = \{g * h \mid h \in H\}$ is called the* ***left coset****, and the set $Hg = \{h * g \mid h \in H\}$ is called the* ***right coset****.*

Proposition 2.1.13. *When the number of distinct left cosets is finite, this equals the number of distinct right cosets.*

We will call this number the ***index of*** *H* ***in*** *G is denoted $[G : H]$.*

Proof. TODO!!! □

Notation 2.1.14. *When left / right cosets are written by addition, we may write $g + H$ or $H + g$ instead.*

Lemma 2.1.15. *Left cosets have the following property*

1. *Any two left cosets g_1H and g_2H are either equal or disjoint*
2. *If G is finite, g_1H and g_2H have the same number of elements*

Proof. TODO!!! □

Corollary 2.1.16. *The left cosets of H give a partition of G.*

Proof. We only need to show that the cosets cover G, but this is immediate from the fact that for any $g \in G$, $g \in gH$. □

Theorem 2.1.17 (Lagrange's Theorem)**.** *Let G be a finite group with $H \leq G$. then $|G| = |H||[G : H]|$*

Proof. TODO!! □

Definition 2.1.18. *Let $(G, *)$ be a group with $H \leq G$. We say that H is a* ***normal subgroup****, denoted $H \triangleleft G$ if for every $g \in G$, $gH = Hg$. Equivalently, forall $g \in G$ and $h \in H$, $g^{-1}hg \in H$.*

Definition 2.1.19. *Let $(G, *)$ be a group with $H \triangleleft G$. Then, the* ***quotient*** *of G is defined as $G/H = \{gH \mid g \in G\}$ under the operation $(g_1H)(g_2H) = (g_1g_2)H$*

Quotients by normal subgroups makes this operation well-defined for any choice of representatives. Also note that when G/H is finite, we have $\#G/H = [G : H]$.

Remark 2.1.20. *When $(G, *)$ is abelian, any subgroup is also normal, thus we can always quotient.*

Remark 2.1.21. *Equivalently, when defining quotients for a fixed $H \triangleleft G$, we can take $G/\sim$ where $\sim$ is the equivalence class defined by $g_1 \sim g_2$ if and only if there exists $h \in H$ such that $g_1 = g_2 * h$ (We can also say $g_1 g_2^{-1} \in H$).*

Proposition 2.1.22. *Given a homomorphism ϕ, $\mathrm{Ker}(\phi)$ is a normal subgroup.*

Proof. TODO!!! □

We can form the quotient $G/\mathrm{Ker}(\phi)$ which sends $g \mapsto g + \mathrm{Ker}(\phi)$.

Theorem 2.1.23 (First Isomorphism Theorem for Groups)**.** *Given a homomorphism $\phi : G \to H$, $G/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$. Note that $\mathrm{Im}(\phi) \leq H$.*

Proof. There is a natural map from $G/\mathrm{Ker}(\phi)$ to $\mathrm{Im}(\phi)$ via the map $g + \mathrm{Ker}(\phi) \mapsto \phi(g)$. This is a well-defined surjective map. It follows from the fact the map is injective (by showing if $\phi(g_1) = \phi(g_2)$, $g_1 - g_2 \in \mathrm{Ker}(\phi)$). □