Notes on Quantum Information

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Contents

1 Introduction

2 Classical vs Quantum

2.1 Basic Definitions

In QI, the general notion of bit is replaced by a **qubit**, which is a quantum system with $|0\rangle$ and $|1\rangle$.

Definition 2.1.1. *A (pure) state of a quantum bit is represented by a linear combination of* $|0\rangle$ *and* $|1\rangle$ *of the form,*

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1$$

*. These linear combinations are called **quantum superpositions**.*

So the states of a qubit are unit vectors in $\mathbb{C}^2$ over $\mathbb{C}$. In actual use, this is then an equivalence modulo global phase, where if $|\psi\rangle$ and $|\psi'\rangle$ are different states and

$$|\psi'\rangle = e^{i\gamma}|\psi\rangle$$

for some $\gamma \in [0, 2\pi)$, they represent the same phase.

Definition 2.1.2 (Durak Notation)**.** *Let*

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2 \tag{1}$$

Then, representing the standard basis vectors of $\mathbb{C}^2$ *as* $|0\rangle$ *and* $|1\rangle$*, we have the mapping,*

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which has a natural correspondence to $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$*.*

Row vectors have a similar correspondence, where

$$\psi^\dagger = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \end{pmatrix} = \overline{\alpha} \begin{pmatrix} 1 & 0 \end{pmatrix} + \overline{\beta} \begin{pmatrix} 0 & 1 \end{pmatrix}$$

has a correspondence with $\langle\psi| = \overline{\alpha}\langle 0| + \overline{\beta}\langle 1|$*.*

The scalar product between $|\psi\rangle$ and $\langle\psi'|$ is called a **bracket** and is denoted $\langle\psi|\psi'\rangle$ where

$$\langle\psi|\psi'\rangle := \psi^\dagger\psi' = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \overline{\alpha}\alpha' + \overline{\beta}\beta'$$

Using this, we can write

$$||\psi|| = \sqrt{\psi^\dagger\psi} = \sqrt{\langle\psi|\psi\rangle}$$

Definition 2.1.3 (Extended Durak Notation)**.** *We can extend our durak notation from* $\mathbb{C}^2$ *to* $\mathbb{C}^d$ *for any* $d \in \mathbb{N}^+$*. We denote the standard basis for* $\mathbb{C}^d$ *using* $|0\rangle, \ldots, |d-1\rangle$ *and similarly for* $\langle i|$*. The natural correspondence follows. In particular,*

$$\langle\phi|\psi\rangle = \sum_{n=0}^{d-1} \overline{\phi}_n \psi_n$$

and the norm of $|\psi\rangle$ *is* $||\psi|| := \sqrt{\langle\psi|\psi\rangle}$*.*

We call the system with d perfectly distinguishable states a **qudit**, where states are of the form

$$|\psi\rangle = \psi_0|0\rangle + \dots \psi_{d-1}|d-1\rangle, \qquad \psi_0, \dots \psi_{d-1} \in \mathbb{C}, \sum_{n=0}^{d-1} |\psi_n|^2 = 1$$

and equivalence up to phrase.

In particular, there is a natural correspondence with a quantum system with $d < \infty$ perfectly distinguishable states is associated to the (Hilbert) vector space $\mathcal{H} = \mathbb{C}^d$.

Definition 2.1.4 (Basic Measurements: Born Rule)**.** *Basic measurements on a d-dimensional quantum system are represented by ON-basis in $\mathbb{C}^d$ up to global phases. Specifically, given a state $|\psi\rangle$, and ON-basis $\{|\psi_n\rangle \mid n = 0, \dots, d-1\}$,*

$$p_n = |\langle\psi_n|\psi\rangle|^2$$

If we obtain outcome n, then the state of the system immediately after the measurement is $|\psi_n\rangle$.

Proposition 2.1.5. *Suppose we have two ON-basis $\{|\psi_n\rangle \mid n = 0, \dots, d-1\}$ and $\{|\psi'_n\rangle \mid n = 0, \dots, d-1\}$ which are equal up to global phases. Then, the two basis assign the same probability for every possible state, and assign the same post-measurement state.*

Proof. Is immediate after substituting and rewriting both sides. □

Definition 2.1.6. *The standard basis $\{|n\rangle \mid n = 0, \dots d-1\}$ is called the **computation basis**. We define the **Fourier Basis** consisting of the **Fourier vectors** given by,*

$$|e_n\rangle := \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp\left[\frac{2\pi imn}{d}\right] |m\rangle, \qquad n = 0, \dots, d-1.$$

The fourier basis vectors are uniform superpositions of the vectors of the computational basis, where each $|e_n\rangle$ represented by the computational basis have the same coefficients on each basis vector.

Notation 2.1.7. *Taking the fourier basis in $d = 2$, we write*

$$|+\rangle := |e_0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle := |e_1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Remark 2.1.8. *Every basic measurement has the same number of outcomes equal to the dimension of the system's Hilbert space, meaning we can only extract finite information.*

Further, note that

$$||\psi||^2 = \sum_{n=0}^{d-1} |\psi_n|^2 = \sum_{n=0}^{d-1} |\langle n|\psi\rangle|^2 = \sum_{n=0}^{d-1} p_n = 1$$

Also, for any $n \in \{0, \dots, d-1\}$, $|\psi'\rangle = e^{i\gamma}|\psi\rangle$

$$|\langle\psi_n|\psi'\rangle|^2 = |e^{i\gamma}\langle\psi_n|\psi\rangle|^2 = |\langle\psi_n|\psi\rangle|^2$$

Specifically, global phases do not change probabilities.

In fact this condition is an if and only if. Suppose we have two states such that

$$|\langle\psi_n|\psi\rangle|^2 = |\langle\psi_n|\psi'\rangle|^2$$

for any choice of ON-basis $\{|\psi_n\rangle \mid n = 0, \dots, d-1\}$. Then, pick an ON-basis with vector $|\psi\rangle$. Then in particular, $|\langle\psi|\psi\rangle|^2 = |\langle\psi|\psi'\rangle|^2 = 1$. By CS inequality, $|\psi\rangle$ is a constant factor of $|\psi'\rangle$, and as they are both unit vectors, are phase apart.

2.2 Reversible Processes

Definition 2.2.1. *We say that $\mathcal{P}$ is a* ***reversible process*** *if there exists another process $\mathcal{P}'$ such that*

1. *applying $\mathcal{P}'$ after $\mathcal{P}$ brings the system back to the initial state*
2. *applying $\mathcal{P}$ after $\mathcal{P}'$ brings the system back to the initial state*

In this case, we say that $\mathcal{P}'$ is the ***inverse*** *of $\mathcal{P}$.*

The main idea of this subsection is that every unitary matrix represents a reversible process (Where a unitary matrix U is a matrix such that $U^\dagger U = I$).

Proposition 2.2.2. *Let $|\psi'\rangle = U|\psi\rangle$, where U is unitary. Then, $|||\psi'\rangle||^2 = |||\psi\rangle||^2$.*

Proof. Taking the complex conjugate on both sides, we have $\langle\psi'| = \langle\psi|U^\dagger$. Taking the product of both,

$$\begin{aligned} |||\psi'\rangle||^2 &= \langle\psi'|\psi'\rangle \\ &= |\psi\rangle U^\dagger U \langle\psi| \\ &= \langle\psi|\psi\rangle \\ &= |||\psi\rangle||^2 \end{aligned}$$

□

In particular, unit vectors are transformed into unit vectors after multiplication by unitary matricies. Thus, pure states are mapped into pure states. Note also that as matrix multiplication is a linear transformation, it can be thought of as mapping basis vectors by U.

Definition 2.2.3. *We define* ***Pauli Matrices*** *as*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We sometimes denote $\sigma_x, \sigma_y, \sigma_z$ as X, Y, Z respectively.

Pauli X is called the ***bit flip gate*** *as it flips the computational basis.*

Pauli Z is called the ***phase flip gate*** *as it flips the phase in front of $|1\rangle$ while leaving $|0\rangle$ unchanged. By flipping the phase, it flips $|+\rangle$ into $|-\rangle$ and vice versa.*

Proposition 2.2.4. *We have the following :*

1. *Pauli matrices are self-adjoint ($U^\dagger = U$)*
2. $X^2 = Y^2 = Z^2 = I$
3. *Pauli matrices are unitary*
4. *Distinct Pauli matrices anticommute: $U_1 U_2 = -U_2 U_1$.*

Proof. Immediate. One needs to work through (4). □

Pauli matrices give rise to a unitary matrix,

$$\boldsymbol{n} \cdot \boldsymbol{\sigma} := n_x\sigma_x + n_y\sigma_y + n_z\sigma_z = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

where $\boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ is a unit vector in $\mathbb{R}^3$.

This is self-adjoint and unitary for any unit vector $\boldsymbol{n}$.

Remark 2.2.5. *As usual, unitary matrices equal up to global phases are seen as equivalent, as there is no way to distinguish between the two processes.*

Remark 2.2.6. *Note the usual identities when dealing with unitary matrices:*

- $U^\dagger U = I$
- $U^\dagger = U^{-1}$
- $UU^\dagger = I$

Definition 2.2.7 (Dirac Notation for Matrices)**.** *Using two vectors $|\alpha\rangle, |\beta\rangle$ in $\mathbb{C}^d$, writing*

$$|\alpha\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{d-1} \end{pmatrix} \qquad |\beta\rangle = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{pmatrix}$$

we write $|\alpha\rangle\langle\beta|$ to represent a d-by-d matrix. Specifically,

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{d-1} \end{pmatrix} \begin{pmatrix} \overline{\beta}_0 & \overline{\beta}_1 & \cdots & \overline{\beta}_{d-1} \end{pmatrix} = \begin{pmatrix} \alpha_0\overline{\beta}_0 & \alpha_0\overline{\beta}_1 & \cdots \\ \alpha_1\overline{\beta}_0 & \alpha_1\overline{\beta}_1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

In particular, we have very natural identities with this notation.

Proposition 2.2.8. *We have the identity,*

$$|\alpha\rangle\langle\beta||\psi\rangle = |\alpha\rangle\langle\beta|\psi\rangle = \langle\beta|\psi\rangle|\alpha\rangle$$

Proof. Immediate, either with matrix expansion or using the fact that $\langle\beta|\psi\rangle$ is a scalar in $\mathbb{C}$ (thus commutes). □

Proposition 2.2.9. *Given $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, |\delta\rangle$ in $\mathbb{C}^d$, taking $A := |\alpha\rangle\langle\beta|$ and $B := |\gamma\rangle\langle\delta|$, $AB = \langle\beta|\gamma\rangle|\alpha\rangle\langle\delta|$.*

Proof. Immediate. □

Proposition 2.2.10. *Any matrix A can be written as a linear sum of matrices of the form $|\alpha_i\rangle\langle\beta_i|$.*

Proof. The standard basis for a matrix can be written by $|i\rangle\langle j|$. □

Proposition 2.2.11 (Adjoint Matrices with Dirac Notation)**.** *We have the following identities:*

1. *Given* $A = |\alpha\rangle\langle\beta|$, *we have* $A^\dagger = |\beta\rangle\langle\alpha|$
2. *Given a finite index* I, $A = \sum_{m\in I} c_m|\alpha_m\rangle\langle\beta_m|$, *we have* $A^\dagger = \sum_{m\in I} \overline{c}_m|\beta_m\rangle\langle\alpha_m|$

Proof. Immediate after expansion. □

Proposition 2.2.12. *Given any ON basis* $\{|\psi_m\rangle \mid m = 0, \dots, d-1\}$,

$$\sum_{m=0}^{d-1} |\psi_m\rangle\langle\psi_m| = I_d$$

Proof. Noting that $|\psi_m\rangle\langle\psi_m||\psi_n\rangle = |\psi_m\rangle\delta_{mn}$, we can show that post-multiplying our matrix with any ψ keeps the vector unchanged. □

Proposition 2.2.13. *For any choice of ON-basis* $\{|\psi_m\rangle \mid m = 0, \dots, d-1\}$ *and* $\{\delta_m \in \mathbb{R} \mid m = 0, \dots, d-1\}$,

$$U = \sum_{m=0}^{d-1} e^{i\gamma_m}|\psi_m\rangle\langle\psi_m|$$

is unitary.

Proof. Expanding $U^\dagger U$,

$$\begin{aligned}
(\sum_{n=0}^{d-1} e^{-i\gamma_n}|\psi_n\rangle\langle\psi_n|)(\sum_{m=0}^{d-1} e^{i\gamma_m}|\psi_m\rangle\langle\psi_m|) &= \sum_{n=0}^{d-1}\sum_{m=0}^{d-1}(e^{-i\gamma_n}|\psi_n\rangle\langle\psi_n|)(e^{i\gamma_m}|\psi_m\rangle\langle\psi_m|) \\
&= \sum_{n=0}^{d-1}\sum_{m=0}^{d-1}(e^{i(\gamma_m-\gamma_n)}|\psi_n\rangle\langle\psi_n||\psi_m\rangle\langle\psi_m|) \\
&= \sum_{n=0}^{d-1}\sum_{m=0}^{d-1}(e^{i(\gamma_m-\gamma_n)}\delta_{nm}|\psi_n\rangle\langle\psi_m|) \\
&= \sum_{m=0}^{d-1}(|\psi_m\rangle\langle\psi_m|) = I
\end{aligned}$$

□

Remark 2.2.14. *In fact, every unitary basis can be written in the above form, using the Spectral Theorem for unitary matrices in* $\mathbb{C}$*. Then, every vector* $|\psi_m\rangle$ *is an eigenvector of* U *with eigenvalue* $e^{i\delta_m}$*.*

From a QI view, this means that reversible processes applied to $|\psi_m\rangle$ *will be unchanged up to a global phase after the proceess.*

Proposition 2.2.15. *Given two ON-basis* $\{|\psi_m\rangle \mid m = 0, \dots, d-1\}$ *and* $\{|\phi_m\rangle \mid m = 0, \dots, d-1\}$,

$$U = \sum_{m=0}^{d-1} |\psi_m\rangle\langle\phi_m|$$

is unitary. In fact, every unitary matrix can be written this way.

Proof. To show is unitary follows the same structure as Proposition 2.2.13. Given a unitary matrix U, fixing any ON basis $\{|\psi_m\rangle \mid m = 0, \dots, d-1\}$, we have

$$U = UI = U(\sum_{m=0}^{d-1} |\psi_m\rangle\langle\psi_m|) = \sum_{m=0}^{d-1} U|\psi_m\rangle\langle\psi_m|$$

We show that $\{U|\psi_m\rangle \mid m = 0, \dots, d-1\}$ is an ON-basis. This follows from

$$(U|\psi_m\rangle)^\dagger(U|\psi_n\rangle) = \langle\psi_m|U^\dagger U|\psi_n\rangle = \langle\psi_m|\psi_n\rangle = \delta_{mn}$$

□

Example 2.2.16. *The matrix*

$$F = \sum_{m=0}^{d-1} |e_m\rangle\langle m|$$

*known as the **quantum Fourier Transform** is a reversible gate. This transforms the computational basis into the Fourier basis. In the specific case for qubits, this is known as the **Hadamard Gate**, denoted*

$$H = |+\rangle\langle 0| + |-\rangle\langle 1|$$

Recalling the equation, $U|\phi_m\rangle = |\psi_m\rangle$. Consequently, we can use unitary matrices to change between matrices. Thus, we can reduce the problem of basic measurements to a measurement in the computational basis (by passing it through this change of basis).

Importantly, measurements are not reversible. However, we can gain knowledge about the state by utilizing reversible processes in between.

Example 2.2.17 (Elitzur-Vaidman Bomb Tester)**.** *Suppose we set the initial state of our qubit into $|0\rangle$. Consider applying H twice onto this state, then measuring. as $H^2 = I$, we will always measure $|0\rangle$. However, if a measurement with the computational basis takes place in between the two applications, it transforms the qubit, giving a 1/2 probability for measuring $|1\rangle$ after the second application. This way, we may be able to identify if a measurement took place in between.*

To give a physical example, imagine a beamsplitter H that splits $|0\rangle$ into equal probabilities $|0\rangle$ and $|1\rangle$. Suppose there is a 50% probability there is an object along the path of $|0\rangle$ that will absorb this. If we combine the split light and pass it through another beamsplitter H, we will measure $|1\rangle$ only if there is an object along the path.

We can make this probability arbitrarily small. Consider the matrix,

$$U_n := \begin{pmatrix} \cos(\frac{\pi}{2n}) & -\sin(\frac{\pi}{2n}) \\ \sin(\frac{\pi}{2n}) & \cos(\frac{\pi}{2n}) \end{pmatrix}$$

Applying U_n to the intial state $|0\rangle$, we get

$$U_n|0\rangle = \cos\left(\frac{\pi}{2n}\right)|0\rangle + \sin\left(\frac{\pi}{2n}\right)|1\rangle$$

By simple trigonometry identities, after application k times, we get

$$(U_n)^k|0\rangle = \cos\left(\frac{k\pi}{2n}\right)|0\rangle + \sin\left(\frac{k\pi}{2n}\right)|1\rangle$$

Taking $k = n$, the final state of the photon after n iterations is $|1\rangle$. If the bomb is present, it will explode with probability $p(1) = |\langle 1|U_n|0\rangle|^2 = \left(\sin\frac{\pi}{2n}\right)^2$, we can make this value arbitrarily small by choosing suitably large n. If the bomb is present, it will remain on $|0\rangle$, with application always bring the state to $U_n|0\rangle$. After n iterations, the probability the bomb is unexploded and is on $|0\rangle$ is

$$(p_0)^n = \left[1 - \left(\sin\frac{\pi}{2n}\right)^2\right]^n \geq 1 - n\left(\sin\frac{\pi}{2n}\right)^2 \geq 1 - \frac{\pi^2}{4n}$$

Choosing n large enough, we can make this probability as close to 1 as we want.

The essential idea is the **quantum Zeno effect**, where frequent measurements by a fixed basis can "freeze" the change by time, locking the system to one of the states of the basis.

3 Qubit Bijection with the 2-sphere

3.1 Bloch Sphere

We can remove ambiguity about phase shifts by giving a bijection between pure states up to phase shifts and the Block Sphere.

Suppose first we are given a unit vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. We can write the coefficients in polar form, with

$$\alpha = |\alpha|e^{i\gamma} \qquad \beta = |\beta|e^{i\delta}$$

for some suitable γ, δ. Using the condition that $|\alpha|^2 + |\beta|^2 = 1$, we can write these as,

$$|\alpha| = \cos\frac{\theta}{2} \qquad |\beta| = \sin\frac{\theta}{2}$$

Note the condition that $\theta \in [0, \pi]$ for them to both be non-negative. Consequently,

$$|\psi\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

where $\phi = \delta - \gamma$. We can remove the global phase, obtaining

$$|\psi'\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

We finally remove the case $\theta = 0$ and $\theta = \pi$ separately as for these θ we get the same state for any ϕ. This gives,

$$\{\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle \mid \theta \in (0, \pi), \phi \in [0, 2\pi)\} \cup \{|0\rangle, |1\rangle\}$$

This gives a natural map to the unit sphere via

$$\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle \mapsto \begin{pmatrix} \sin\theta\ \cos\phi \\ \sin\theta\ \sin\phi \\ \cos\theta \end{pmatrix}$$

It also maps $|0\rangle$ to $(0\ 0\ 1)^T$ and $|1\rangle$ to $(0\ 0\ -1)^T$

Intuitively the z coordinate on the sphere corresponds to the magnitude on $|0\rangle$ and $|1\rangle$, where ϕ corresponds to the phase coefficient on $|1\rangle$. Quotienting by global phase, there is a correspondence between circles to circles between the dimensions of $\mathbb{C}^2$. Parameterizing on this circle, we get a 2-sphere.

When we use the unit sphere to represent qubit states, we call this the **Bloch Sphere**. The vector it maps to is called the **Bloch vector**.

3.1.1 Bloch Sphere by Pauli Matrices

Proposition 3.1.1. *We can obtain the bloch vector of a unit vector in* $\mathbb{C}^2$ *via the map*

$$\psi \mapsto \begin{pmatrix} \langle\psi|X|\psi\rangle \\ \langle\psi|Y|\psi\rangle \\ \langle\psi|Z|\psi\rangle \end{pmatrix}$$

where X, Y, Z *are the Pauli Matrices.*

Proof. Follows via simple computation. □

Proposition 3.1.2. *Define measurements by ON-basis* $\mathbf{B}_x = \{|+\rangle, |-\rangle\}$, $\mathbf{B}_y = \{|+i\rangle, |-i\rangle\}$, $\mathbf{B}_z = \{|0\rangle, |1\rangle\}$ *where* $|+i\rangle := (|0\rangle + i|1\rangle)/\sqrt{2}, |-i\rangle := (|0\rangle - i|1\rangle)/\sqrt{2}$. *Then,*

1. $X = |+\rangle\langle+| - |-\rangle\langle-|$, $Y = |+i\rangle\langle+i| - |-i\rangle\langle-i|$, $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$
2. $|\psi\rangle X\langle\psi| = |\langle+|\psi\rangle|^2 - |\langle-|\psi\rangle|^2$, $|\psi\rangle Y\langle\psi| = |\langle+i|\psi\rangle|^2 - |\langle-i|\psi\rangle|^2$, $|\psi\rangle Z\langle\psi| = |\langle 0|\psi\rangle|^2 - |\langle 1|\psi\rangle|^2$

Proof. Follows by simple expansion. □

As a corollary, it follows from this that if two qubits give the same probabilities for the measurements on the bases $\mathbf{B}_x, \mathbf{B}_y, \mathbf{B}_z$, they have the same Bloch vector, implying they correspond to the same pure state.

3.1.2 Measuring probabilities

Proposition 3.1.3. *Given two vectors in* $\mathbb{C}^2$, *we have the relation*

$$|\langle\psi|\psi'\rangle|^2 = \frac{1 + \mathbf{r}\cdot\mathbf{r}'}{2}$$

where $\mathbf{r}$ *and* $\mathbf{r}'$ *correspond to Bloch vectors from* $|\psi\rangle$ *and* $|\psi'\rangle$ *respectively.*

Proof. Follows from writing the unit vectors in the canonical trigonometric form, then taking the norm by carefully expanding. □

Consequently, two vectors are othogonal if and only if $\mathbf{r}\cdot\mathbf{r}' = -1$. Note that this is only possible if and only if the two Bloch vectors are at antipodes (follows from the CS-inequality). To go to antipodes, we map $\theta \mapsto \pi - \theta$ and $\phi \mapsto \psi + \pi$.

3.1.3 Reversible processess = rotations

Proposition 3.1.4. *Every two-by-two unitary matrix can be parametrized as*

$$U = e^{i\gamma}\left(\cos\frac{\alpha}{2} I - i\,\sin\frac{\alpha}{2}\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)$$

where $\gamma \in [0, 2\pi), \alpha \in [0, \pi)$. *In fact every matrix of this form is unitary.*

Proof. Using Proposition 2.2.13, we can write

$$U = e^{i\lambda_0}|\psi_0\rangle\langle\psi_0| + e^{i\lambda_1}|\psi_1\rangle\langle\psi_1|$$

for real numbers λ_0, λ_1 and orthonormal vectors $|\psi_0\rangle$ and $|\psi_1\rangle$. Writing $\gamma = (\lambda_0+\lambda_1)/2$, $\alpha = \lambda_1-\lambda_0$,

$$\begin{aligned} U &= e^{i\gamma}(e^{-i\alpha/2}|\psi_0\rangle\langle\psi_1| + e^{i\alpha/2}|\psi_1\rangle\langle\psi_1|) \\ &= e^{i\gamma}\left[\cos\frac{\alpha}{2}\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|\right) - i\,\sin\frac{\alpha}{2}\left(|\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1|\right)\right] \\ &= e^{i\gamma}\left(\cos\frac{\alpha}{2}I - i\,\sin\frac{\alpha}{2}A\right) \end{aligned}$$

Where $A = |\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1|$. It suffices to show this can be written as $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ for some $\boldsymbol{n}$. Writing $|\psi_0\rangle$ canonically, noting that $|\psi_1\rangle$ is on the opposite point of the Bloch sphere, expansion shows this is satisfied with $\boldsymbol{n} = (\sin\theta\,\cos\phi \;\; \sin\theta\,\sin\phi \;\; \cos\theta)^T$. □

A quick check shows that every matrix of this form is indeed unitary. We use the fact that $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ is unitary.

Consequently, by removing global phase, we can write general reversible processes by

$$U_{\alpha,\boldsymbol{n}} = \cos\frac{\alpha}{2}I - i\,\sin\frac{\alpha}{2}\boldsymbol{n}\cdot\boldsymbol{\sigma}$$

The matrix $U_{\alpha,\boldsymbol{n}}$ corresponds to the rotation by an angle α by the axis $\boldsymbol{n}$.

The proof for the above statement follows in three steps.

1. Show that the map $|\phi\rangle \mapsto U_{\alpha,\boldsymbol{n}}|\phi\rangle$ as represented on the Bloch sphere corresponds to a map $\mathbf{r} \mapsto O\mathbf{r}$ where O is an orthogonal (3 by 3) matrix.

2. Show that $\boldsymbol{n}$ is unchanged by multiplication by O

3. O acts as a rotation in the plane orthogonal to $\boldsymbol{n}$

As a remark, note that 2-by-2 unitary matrices has a correspondence with rotations on the Bloch sphere. Reversible processes on the Bloch sphere are more expressive (such as reflections about a plane or inversions) but in general, this does not correspond to reversible processes on qubits.

Also note that this bijection to Bloch spheres do not match intuitions about directions in the physical sense, we only refer to the idea that qubits quotiented by global phases map bijectively to the bloch sphere via Pauli Matrices.

Finally, in higher dimensions, d-dimensional pure states correspond to a subset of points of a d^2-1 ball. There are some points which do not correspond to pure states, and the bijection is a nice consequence we have with $d=2$ (and trivially for $d=1$).

4 Composite Quantum System

Definition 4.0.1. *Given two Hilbert Spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of dimensions d_A and d_B, the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is the space $\mathbb{C}^{d_A d_B}$ equipped with a map on vectors*

$$|\alpha\rangle = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{d_A-1} \end{pmatrix} \qquad |\beta\rangle = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{d_B-1} \end{pmatrix}$$

to their tensor product

$$|\alpha\rangle \otimes |\beta\rangle = \begin{pmatrix} \alpha_0\beta_0 \\ \vdots \\ \alpha_0\beta_{d_B-1} \\ \vdots \\ \alpha_{d_A-1}\beta_0 \\ \vdots \\ \alpha_{d_A-1}\beta_{d_B-1} \end{pmatrix}$$

As per qubits, we can give the computational basis as

$$\{|m\rangle \times |n\rangle \mid m = 0, \ldots, d_A - 1, n = 0, d_B - 1\}$$

and we can represent a composite quantum system with the linear combinations

$$|\Psi\rangle = \sum_{m=0}^{d_A-1} \sum_{n=0}^{d_B-1} \Psi_{mn} |m\rangle \otimes |n\rangle$$

where Ψ_{mn} satisfy $\sum_{m,n} |\Psi_{mn}|^2 = 1$.

Note the computational basis extracts exactly the coordinates described by the vector.