Notes on Programming Language Concepts and More

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1 Denotational Semantics

This section is concerned about the approach of giving meaning to programming languages by constructing objects (called denotations) that describe meanings of expressions.

Definition 1.1 *Given a partially ordered set D, we define a **directed set** to be a $S \subseteq D$ such that every pair of elements of S has an upper bound.*

Definition 1.2 *Let D be a partially ordered set with order $\sqsubseteq$. We define the **least upper bound** (lub) of a subset $S \subseteq D$ denoted $\bigsqcup S$ if it satisfies:*

- *$\bigsqcup S$ is greater than or equal to every element in S; $\forall x \in S,\ x \sqsubseteq \bigsqcup S$*
- *$\bigsqcup S$ is the smallest such element; $\forall y \in D, (\forall x \in S, x \sqsubseteq y) \implies (\bigsqcup S \sqsubseteq y)$*

Definition 1.3 *Let D be a partially ordered set over $\sqsubseteq$. We say that a function $f : D \to E$ is **continuous** (or Scott continuous) if it preserves least upper bounds for directed sets. That is, for every directed set S,*

$$f(\bigsqcup S) = \bigsqcup f(S)$$

Definition 1.4 *We say that a partial ordered set $(D, \sqsubseteq)$ is an ω**-complete partial order** (ω-cpo) if every countable ascending chain ($d_0 \sqsubseteq d_1 \sqsubseteq \dots$) has a least upper bound, written $\bigsqcup_{n \geq 0} d_n$. Additionally, we say $(D, \sqsubseteq)$ is a cpo with bottom, if it has a least element $\bot \in D$ (over $\sqsubseteq$).*

Remark 1.5 *Note the distinction between complete partial orders (including dcpo and ccpo) and complete lattices. Notably, we don't force every subset to have a suprema.*

Definition 1.6 *Let D be an ω-cpo and $f : D \to D$. We say that $d \in D$ is a **fixed point** of f if $f(d) = d$. We say it is a **prefixed point** if $f(d) \sqsubseteq d$*

Definition 1.7 *Given ω-cpos D, D', we say that a function $f : D \to D'$ is* ***monotonic*** *if*

$$\forall d, d' \in D, d \sqsubseteq d' \implies f(d) \sqsubseteq f(d')$$

Proposition 1.8 *Scott continuous functions are monotonic.*

Proof. Let $f : D \to D'$ over ω-cpos D, D'. Suppose we take $d, d' \in D$ such that $d \sqsubseteq d'$. Then, by continuity,

$$f(d') = f(d \sqcup d') = f(d) \sqcup f(d')$$

It therefore follows from the definition of directed suprema that $f(d) \sqsubseteq f(d')$. ■

Theorem 1.9 (Kleene's Fixed Point Theorem for ω-cpos) *Let D be a ω-cpo with bottom, and $f : D \to D$ be a continuous function. Define*

$$\mathrm{lfp}(f) = \bigsqcup_{n \in \mathbb{N}} f^n(\bot)$$

Then $\mathrm{lfp}(f)$ *is the least fixed point of f.*

Proof. We first show that $\mathrm{lfp}(f)$ is a fixed point of f. Noting that f is continuous, we have

$$\begin{aligned} f(\mathrm{lfp}(f)) &= f(\bigsqcup_{n \in \mathbb{N}} f^n(\bot)) \\ &= \bigsqcup_{n \in \mathbb{N}} f^{n+1}(\bot) \\ &= \bigsqcup_{n \in \mathbb{N}} f^{n+1}(\bot) \sqcup \{\bot\} \\ &= \bigsqcup_{n \in \mathbb{N}} f^n(\bot) \\ &= \mathrm{lfp}(f) \end{aligned}$$

This shows $\mathrm{lfp}(f)$ is a fixed point.

Let d be any prefixed point. Noting that $\bot \sqsubseteq d$, as scott-continuous functions are monotone, we have $f(\bot) \sqsubseteq f(d)$. As d is a prefixed point, $f(\bot) \sqsubseteq d$, and inductively $f^n(\bot) \sqsubseteq d$. This gives

$$\mathrm{lfp}(f) = \bigsqcup_{n \in \mathbb{N}} f^n(\bot) \sqsubseteq d$$

As all fixed points are prefixed points, this shows $\mathrm{lfp}(f)$ is the least fixed point of f. ■

1.1 An Aside on Continuity

Scott continuity has an interpretation under continuity in the topological sense, which we will discuss in this subsection.

Notation 1.10 *Given a partially ordered set $(D, \sqsubseteq)$ and a set $X \subseteq D$, we write*

- $\uparrow X = \{d \in D : \exists x \in X, x \sqsubseteq d\}$
- $\downarrow X = \{d \in D : \exists x \in X, d \sqsubseteq x\}$

Where it is clear, we may write $\uparrow x$ to represent $\uparrow \{x\}$ and similarly for $\downarrow x$.

Definition 1.11 *We call a set to be an* ***upper set*** *if $\uparrow X = X$. If $\downarrow X = X$, then we say that X is a* ***lower set****.*

Remark 1.12 *As $\sqsubseteq$ is reflexive, we have $X \subseteq \uparrow X$. Therefore, to prove X is an upper set, it suffices to show that $\uparrow X \subseteq X$.*

Definition 1.13 *Let $(D, \sqsubseteq)$ be a partially ordered set. A subset $O \subseteq D$ is called* ***Scott-open*** *if is an upper set that is inaccessible by directed suprema. That is, all directed sets S with a supremum in O have a non-empty intersection with O.*

Proposition 1.14 *The Scott-open subsets of a partially ordered set $(D, \sqsubseteq)$ form a topology on D, which is called the* ***Scott topology*** *and written as (D, τ).*

Proof. Start by noting that $\emptyset$ and D are both trivially Scott open.

Consider a family of Scott open sets $\mathcal{U} = \{U_i\}_{i \in I}$. Take any $x \in \uparrow (\bigcup \mathcal{U})$. Then, there exists an $i \in I$ such that $y \in U_i$ and $y \sqsubseteq x$. Now,

$$x \in \uparrow U_i = U_i \subseteq \bigcup \mathcal{U}$$

Henceforth $\uparrow (\bigcup \mathcal{U}) \subseteq \bigcup \mathcal{U}$ and equality follows. Now, let $X \subseteq D$ be a directed subset such that $\bigsqcup X \in \bigcup \mathcal{U}$. Then, there exists a $i \in I$ such that $\bigsqcup X \in U_i$. Then as U_i is Scott open, $X \cap U_i \neq \emptyset$. Using

$$X \cap U_i \subseteq \bigcup_{i \in I} (X \cap U_i) = X \cap \bigcup_{i \in I} U_i = X \cap (\bigcup \mathcal{U})$$

We see that $X \cap (\bigcup \mathcal{U}) \neq \emptyset$. Thus $\bigcup \mathcal{U}$ is Scott open.

Take any $U, V \in \tau$ and let $x \in \uparrow (U \cap V)$. Then, there exists a $y \in U \cap V$ with $y \sqsubseteq x$. Now, as U and V are both upper sets, we have

$$x \in (\uparrow U) \cap (\uparrow V) = U \cap V$$

This gives $\uparrow (U \cap V) \subseteq U \cap V$. Now, let $X \subseteq D$ be a directed subset such that $\bigsqcup X \in U \cap V$. As U and V are Scott open, there exists a $u \in X \cap U$ and $v \in X \cap V$. As X is directed, there exists a $x \in X$ such that $u \sqsubseteq x$ and $v \sqsubseteq x$. Now,

$$x \in X \cap ((\uparrow U) \cap (\uparrow V)) = X \cap (U \cap V)$$

Hence, $D \cap (U \cap V) \neq \emptyset$. Thus $U \cap V$ is Scott open. ■

Proposition 1.15 *Let $(D, \sqsubseteq)$ be a partially ordered set. For any $d \in D$, $D \backslash (\downarrow d)$ is Scott open.*

Proof. Let $x \in \uparrow (D \backslash \downarrow d)$. Then there exists a $y \in D \backslash \downarrow d$ such that $y \sqsubseteq x$. Suppose for a contradiction that $x \in \downarrow d$. That is, $x \sqsubseteq d$. By transitivity of $\sqsubseteq$, $y \sqsubseteq p$. This contradicts $y \in D \backslash \downarrow d$. Therefore $x \in D \backslash \downarrow d$. Thus, $D \backslash \downarrow d$ is an upper set.

Now consider a directed set $X \subseteq D$ such that $\bigsqcup X \in D \backslash \downarrow d$. Then, $\bigsqcup X \not\sqsubseteq d$. Suppose for a contradiction that $X \cap (D \backslash \downarrow d) = \emptyset$. This gives $X \subseteq \downarrow d$. This means that d is an upper bound for X, giving $\bigsqcup X \sqsubseteq d$, which is a contradiction. Thus, $X \cap (D \backslash \downarrow d) \neq \emptyset$. This shows $D \backslash \downarrow d$ is inaccessible by directed suprema, meaning it is Scott open. ■

Lemma 1.16 *If f is continuous under the Scott topology, it is monotonic.*

Proof. Let $f : D \to E$ be continuous under the Scott topology. Take $x, x' \in D$ such that $x \sqsubseteq x'$. Supose for a contradiction that $f(x) \not\sqsubseteq f(x')$. Then, $f(x) \in E \backslash \downarrow f(x')$. Noting this set is Scott open, we have $x \in f^{-1}(E \backslash \downarrow f(x'))$ which is also Scott open by continuity. As this set is upper closed, it follows that $x' \in f^{-1}(E \backslash \downarrow f(x'))$. Now,

$$\begin{aligned} x' \in f^{-1}(E \backslash \downarrow f(x')) &\implies f(x) \in E \backslash \downarrow f(x') \\ &\implies f(x') \sqsubseteq f(x') \end{aligned}$$

which is a contradiction. Hence, it follows that $f(x) \sqsubseteq f(x')$. ■

Theorem 1.17 *A function between partially ordered sets $(D, \sqsubseteq)$ is Scott continuous if and only if it is continuous with respect to the Scott topology.*

Proof. $(\Rightarrow)$ Suppose that $f : D \to E$ is Scott continuous. Take any Scott open set U in E. We want to show that $f^{-1}(U)$ is Scott open. Specifically, we wish to show that U is (i) an upper set and (ii) all directed sets with a supremum in $f^{-1}(U)$ has a non-empty intersection with $f^{-1}(U)$.

- For (i), take any $x \in f^{-1}(U)$ such that $f(x) \in U$. Given $x \sqsubseteq x'$, by monotonicity of Scott continuous functions we have $f(x) \sqsubseteq f(x')$. As U is an upper set, we have $f(x') \in U$. It follows that $x' \in f^{-1}(U)$.
- For (ii), Take $X \subseteq D$ be any directed set such that $\bigsqcup X \in f^{-1}(U)$. That is, $f(\bigsqcup X) \in U$. By Scott continuity, $\bigsqcup f(X) \in U$. As U is Scott open, we have $f(X) \cap U \neq \emptyset$. Equivalently, there is a $x \in X$ such that $f(x) \in U$. Thus, $x \in f^{-1}(U)$, therefore it is inaccessible by a directed suprema.

$(\Leftarrow)$ Suppose that f is continuous in the Scott topology, such that for any Scott open set $U \in E$, $f^{-1}(U)$ is Scott open in D. We wish to show that f is Scott continuous, such that for any directed set $X \subseteq D$ with a supremum,

$$f(\bigsqcup X) = \bigsqcup f(X)$$

We prove this by showing that $f(\bigsqcup X)$ is (i) an upper bound and (ii) the least upper bound with respect to $f(X)$.

- For (i), note that as f is monotone from Lemma 1.16, given any $x \in X$, we have $x \sqsubseteq \bigsqcup X$, meaning $f(x) \sqsubseteq f(\bigsqcup X)$. This shows $\bigsqcup f(X) \sqsubseteq f(\bigsqcup X)$.
- For (ii), Suppose for a contradiction that $f(\bigsqcup X) \not\sqsubseteq \bigsqcup f(X)$. Then, $f(\bigsqcup X) \in E \backslash \downarrow \bigsqcup f(X)$. It follows that $\bigsqcup X \in f^{-1}(E \backslash \downarrow \bigsqcup f(X))$. As this is Scott open, $X \cap f^{-1}(E \backslash \downarrow \bigsqcup f(X)) \neq \emptyset$. Taking x to be an element in this, we have $f(x) \in f(X)$ and $f(x) \in E \backslash \downarrow \bigsqcup f(X)$. The latter transforms into $f(x) \not\sqsubseteq \bigsqcup f(X)$, contradicting with $f(x) \in f(X)$. We therefore have $f(\bigsqcup X) \sqsubseteq \bigsqcup f(X)$.

■

1.2 Fixed Point Theorems

We write a brief section on some theorems regarding fixed points.

Theorem 1.18 (Bourbaki-Witt Theorem)

Theorem 1.19 (Knaster-Tarski Theorem)

1.3 The Factorial Function

Definition 1.20 *Given two sets A and B, an* ***empty function*** *is a partial function $A \rightharpoonup B$ that has no defined maps from elements of A. Alternatively, if we interpret this as a function from A to* Option *B that takes $a \mapsto$* Nothing*. We often write $\perp$ to represent this function.*

Proposition 1.21 *The set of partial functions from $\mathbb{N}$ to $\mathbb{N}$ with the standard $\sqsubseteq$ on partial functions is a ω-cpo.*

Example 1.22

Consider the factorial function which might be recursively defined as

```
int factorial(int n) {
    if (n == 0)
        then 1;
    else
        n * factorial(n-1);
}
```

under a programming context. We will write f to represent this function.

To give meaning to this f, we model this through an approximation as a partial function $\mathbb{N} \rightharpoonup \mathbb{N}$ starting with the empty function. We introduce a function $F : (\mathbb{N} \rightharpoonup \mathbb{N}) \to (\mathbb{N} \rightharpoonup \mathbb{N})$ defined by the map

$$f \mapsto \{0 \mapsto 1\} \ \oplus \ \{n \mapsto n * f(n-1) : n \in \mathbb{N} \backslash \{0\}\}$$

where $\oplus$ represents an overloading of function maps. Then we define $F^0(\perp) = \perp$ and $F^{n+1}(\perp) = F(F^n(\perp))$. This process builds a sequence of $\mathbb{N} \rightharpoonup \mathbb{N}$. Note that this sequence satisfies $F^n \sqsubseteq F^{n+1}$. As $\mathbb{N} \rightharpoonup \mathbb{N}$ is an ω-cpo, by Kleene's Fixed Point Theorem, setting

$$f := \bigsqcup_{n \in \mathbb{N}} F^n(\perp)$$

this define a suitable interpretation of the recursive function defined above. Notice that we take the least fixed point on F, as we are talking about functions that terminate.

2 Basics of Category Theory

3 Monads

3.1 In Category Theory

A monad is an endofunctor with additional structure (two natural transformations). In pop culture, we often say that a monad is a "monoid in the category of endofunctors" (for some fixed category).

Definition 3.1 (Monad) *Let C be a category. A* ***monad*** *on C is a triple (T, η, μ) where $T : C \to C$ is an endofunctor, $\eta : 1_C \to T$ (where 1_C is the identity functor on C), $\mu : T^2 \to T$, satisfying the following coherence conditions:*

- *$\mu \circ T\mu = \mu \circ \mu T$ where $T\mu$ and μT are formed by horizontal composition*
- *$\mu \circ T\eta = \mu \circ \eta T = 1_T$ where 1_T is the identity transformation from T to T.*

Alternatively, we can write these using two commutative diagrams

$$(i)\quad \begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad\qquad (ii)\quad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$