First Read on Linear Algebra, Preliminaries

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0 Prerequisites (sort of)

0.1 Equivalence Relation

Definition 0.1.1. *Given a set X, A binary relation $\sim \subseteq X \times X$ is said to be an* ***equivalence relation*** *if it is reflexive, symmetric, and transitive. That is, for any $a, b, c \in X$, we have*

- *Reflexivity: $a \sim a$*
- *Symmetry: $a \sim b$ implies $b \sim a$*
- *Transitivity: If $a \sim b$ and $b \sim c$ then $a \sim c$*

Definition 0.1.2. *A subset Y of X such that for any $a, b \in Y$, $a \sim b$ but for any $c \in X \setminus Y$, $a \not\sim c$ is called an* ***equivalence class*** *of X by $\sim$.*

We can denote the equivalence class to which a belongs to by $[a] := \{x \in X \mid a \sim x\}$.

An equivalence relation is a way of partioning a set into its equivalence classes. That is, elements of X equivalent to each other are elements of the same equivalent class and only when they are equivalent. For instance, the modulo operation on integers for a fixed n yields an equivalence relation by $a \equiv b$ if $a - b$ is divisible by n.

In the context of algebra, we often consider **congruence relations**, which is an equivalence relation whose domain X is also the underlying set of some algebraic structure, and the relation respects this structure. For instance, taking the previous example, the modulo equivalence class respects addition and multiplication in the sense that given $\alpha_1 \equiv \alpha_2$ and $\beta_1 = \beta_2$, we have

$$\alpha_1 + \beta_1 \equiv \alpha_2 + \beta_2 \qquad \alpha_1\beta_1 \equiv \alpha_2\beta_2$$

More generally, when $f : X \to Y$, if $x_1 \sim x_2$ implies $f(x_1) = f(x_2)$, then f is said to be a **morphism for** $\sim$, or simply **invariant under** $\sim$. On the other hand, given a function $f : X \to Y$, we can define an equivalence relation $x \sim y$ if and only if $f(x) = f(y)$, which is known as the **equivalence kernel**.

Definition 0.1.3. *The set of all equivalence classes of X by $\sim$ denoted $X/\sim := \{[x] \mid x \in X\}$ is the* ***quotient set*** *of X by the relation $\sim$.*

Definition 0.1.4. *The* ***projection*** *of $\sim$ is the function $\pi : X \to X/\sim$ defined by $\pi(x) = [x]$ which maps elements of X into their respective equivalence classes by $\sim$.*

Proposition 0.1.5. *Let $f : X \to B$ be a function such that $a \sim b$ implies $f(a) = f(b)$. Then there is a unique function $g : X/\sim \to B$ such that $f = g \circ \pi$.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & B \\ X/\sim \arrow[ur, dashed, "\exists!\ g"'] & \end{tikzcd}$$