# "By Chain Completeness"...? Proofs on Infinite Lists Week 2 MT25

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Oxford Compsoc

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# Outline

1 Introduction

Chains

3 Admissible Predicates





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Aim: Give a better background of how the proof works!





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 $\rightarrow$  Lists form a ccpo with bottom.



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because finite lists contain the information about termination.





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#### Aside

This is a continuity based on a topology on posets. In the scott topology,  $C \subseteq P$  is closed if

- C is lower:  $y \in C$  and  $x \sqsubseteq y$  implies  $x \in C$
- closed under directed (chain) suprema: when  $D \subseteq C$  is directed and  $\bigsqcup D$  exists,  $\mid D \in C$





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## Theorem (Kleene)

Given a chain complete poset P with bottom and a continuous function  $F:P\to P$ ,

$$\mathsf{lfp}(F) = \bigsqcup_{n \in \mathbb{N}} F^n(\bot)$$





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To do this, we introduce the notion of admissible predicates.





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Intuition: if every finite approximation satisfies P, then the limit (possibly infinite) also satisfies P.





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- Is the proposition I want to prove admissible?





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These are examples of 'limit-fragile' propositions.





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For example, where we ban certain patterns:

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- We never see "010"





A safety property say that 'bad things never happen'. A property is **safe** if having all finite prefixes of a list x lie in P implies that  $x \in P$ . Equivalently, if  $x \notin P$ , there exists a finite prefix  $y \sqsubseteq x$  with  $y \notin P$  (a finite counterexample).

There are many examples of safety properties on lists.

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Or when the proposition is prefix-invariant (bad patterns can be checked with a finite prefix):

- At most 1 in any prefix
- Nondecreasing list of numbers
- Every 1 is immediately followed by a 0





Recall,

#### Definition

 $C \subseteq P$  is scott-closed if

- C is lower:  $y \in C$  and  $x \sqsubseteq y$  implies  $x \in C$
- closed under directed (chain) suprema: when  $D \subseteq C$  is directed and  $\bigsqcup D$  exists,  $\mid D \in C$





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To then find continuous maps f, we note that

- Composition of continuous constructors
- Composition of continuous folds
- Products
- Evaluation of definable expressions

are all Scott-continuous.



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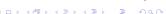
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- ∃i.P<sub>i</sub> corresponds to ⋃<sub>i</sub> C<sub>Pi</sub>, but unions of Scott-closed sets need not be Scott-closed.
- When we have P(xs) 
   ⇔ ∃n.drop n xs = ⊥, the basic disjunct is closed, but the union is not closed under limits.





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#### Further...

 We can generalize this to other algebraic objects, not just lists (and we can generate the inductive scheme given a suitable functor that describes it)





Questions?



