Linear Logic

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1 Preliminaries

1.1 Monoidal Category

Definition 1.1.1. *A **monoidal category** is a category $\mathcal{C}$ equipped with*

1. *A functor*

$$\otimes\, \mathcal{C} \times \mathcal{C} \to C$$

   *out of the product category of $\mathcal{C}$ with itself, called the **tensor product***

2. *An object //TODO!!! https://ncatlab.org/nlab/show/monoidal+category*

1.2 Positive and Negative Types

The idea of **polarity** classifies connectives / types by how they behave in proof search / program construction.

1.3 Linear Logic and S4

Linear exponentials behave like modalities in a variety of ways.

Proof theoretically, we have the following correspondences:

- Dereliction: $!A \multimap A$ mirrors $\mathbf{T} : \Box A \to A$
- Digging: $!A \multimap !!A$ mirrors $\mathbf{4} : \Box A \to \Box\Box A$
- K/Distribution: one can lift from $A \multimap B$ to $!A \multimap !B$, which echoes $\Box(A \to B) \to (\Box A \to \Box B)$

So the bang operator acts like an S4-necessity, but it controls structural rules rather than truth at worlds.

1.4 Weakening and Contraction

Without weakening and contraction, a context behaves like a *resource list*. In building a pair we have a rule like

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \otimes R$$

This gives a plain **symmetric monoidal product**, rather than a cartesian product.

- **Weakening**

The idea of weakening (dropping an assumption) has the proof-theoretic effect that a formula in a context does not have to be used. That is, we have

$$\frac{\Gamma \vdash C}{\Gamma, B \vdash C} \text{ weak}$$

This gives projections, where from a hypothesis $x : A \times B$, the derivation of A can ignore the B part via the map $\pi_1 : A \times B \to A$, and symmetrically for π_2.

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A & & B \end{tikzcd}$$

Categorically, weakening is having a **discard** (counit) $\epsilon_A : A \to 1$ that lets one forget components.

- **Contraction**

The idea of contraction (duplicating assumptions) has the proof theoretic effect that the same assumption can be used twice. That is, we have

$$\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\ \text{contr}$$

This lets us do pairing

$$\langle f, g\rangle : \Gamma \to A \times B$$

which requires both $f : \Gamma \to A$ and $g : \Gamma \to B$ to see the same context Γ.

Categorically, this is the duplicate (comultiplication) $\Delta_A : A \to A \times A$, which is what gives a diagonal on cartesian products.

The standard set interpretation on a context $\Gamma \vdash A$ is given by $[\![\Gamma]\!] \to [\![A]\!]$, so these notions are contravariant (precomposition) on the context to derivation. For instance, suppose that $f : [\![\Gamma]\!] \to [\![C]\!]$ interprets $\Gamma \vdash C$. The weakened judgement $\Gamma, B \vdash C$ is the composite

$$[\![\Gamma]\!] \otimes [\![B]\!] \xrightarrow{\mathrm{id}_{[\![\Gamma]\!]} \otimes \epsilon_{[\![B]\!]}} [\![\Gamma]\!] \xrightarrow{f} [\![C]\!]$$

Which justifies our notion of 'forgetting'.

Similarly, for the contraction rule, suppose that $g : [\![\Gamma]\!] \otimes [\![A]\!] \otimes [\![A]\!] \to [\![C]\!]$ interprets $\Gamma, A, A \vdash C$. The contracted judgement $\Gamma, A \vdash C$ is given by

$$[\![\Gamma]\!] \otimes [\![A]\!] \xrightarrow{\mathrm{id}_{[\![\Gamma]\!]} \otimes \Delta_{[\![A]\!]}} [\![\Gamma]\!] \otimes [\![A]\!] \otimes [\![A]\!] \xrightarrow{g} [\![C]\!]$$

These two rules gives a cartesian product as it lets us build $\langle f, g\rangle : [\![\Gamma]\!] \to [\![A]\!] \otimes [\![B]\!]$ from $f : [\![\Gamma]\!] \to [\![A]\!]$ and $g : [\![\Gamma]\!] \to [\![B]\!]$.

Then, using the context diagonal (with contraction), we can apply components in parallel:

$$[\![\Gamma]\!] \xrightarrow{\Delta_{[\![\Gamma]\!]}} [\![\Gamma]\!] \otimes [\![\Gamma]\!] \xrightarrow{f \otimes g} [\![A]\!] \otimes [\![B]\!]$$

We can also project components using discarding (via weakening):

$$\begin{tikzcd} & {[\![A]\!] \otimes [\![B]\!]} \arrow[dl, "\pi_1 := \mathrm{id}_{[\![A]\!]} \otimes \epsilon_{[\![B]\!]}"'] \arrow[dr, "\pi_2 := \epsilon_{[\![A]\!]} \otimes \mathrm{id}_{[\![B]\!]}"] & \\ {[\![A]\!]} & & {[\![B]\!]} \end{tikzcd}$$

Then, for any $h : [\![\Gamma]\!] \to [\![A]\!] \otimes [\![B]\!]$, we have the universal property that

$$h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$$

Hence $\otimes$ behaves exactly like the cartesian product.

2 Basic Definitions

Linear logic is typically given in terms of sequent calculus. There is a set of propositions (which intuitively is resources acquired, rather than statements to be proved). We also assume some set of propositional variables V.

Then, the set of propositions are defined as follows (by Girard's conventions):

$$\frac{p \in V}{p \text{ Prop}} \qquad \frac{A \text{ Prop}}{A^{\perp} \text{ Prop}} \qquad \frac{A \text{ Prop}}{!A \text{ Prop}} \qquad \frac{A \text{ Prop}}{?A \text{ Prop}}$$

$$\frac{A \text{ Prop} \quad B \text{ Prop}}{A \,\&\, B \text{ Prop}} \qquad \frac{A \text{ Prop} \quad B \text{ Prop}}{A \oplus B \text{ Prop}} \qquad \frac{A \text{ Prop} \quad B \text{ Prop}}{A \otimes B \text{ Prop}} \qquad \frac{A \text{ Prop} \quad B \text{ Prop}}{A \parr B \text{ Prop}}$$

There are also four constants $\top, \mathbf{0}, \mathbf{1}, \perp$ that go with this. The following is a short summary of the features:

Symbol	**Name**	**Fragment**	**Kind**	**Unit**	**Dual**	**Polarity**
$A \otimes B$	tensor (mult. $\wedge$)	multiplicative	conjunction	$\mathbf{1}$	$A^{\perp} \parr B^{\perp}$	positive
$A \parr B$	par (mult. $\vee$)	multiplicative	disjunction	$\perp$	$A^{\perp} \otimes B^{\perp}$	negative
$A \,\&\, B$	with (add. $\wedge$)	additive	conjunction	$\top$	$A^{\perp} \oplus B^{\perp}$	negative
$A \oplus B$	plus (add. $\vee$)	additive	disjunction	0	$A^{\perp} \,\&\, B^{\perp}$	positive
$!\,A$	"of course"	exponential	modality	—	$?(A^{\perp})$	positive
$?\,A$	"why not"	exponential	modality	—	$!(A^{\perp})$	negative

The sequent $\Gamma \vdash \Delta$ reads like "*using exactly the resources in* Γ *you can produce one of the goals in* Δ".

We also outline valid sequents, first via the structural rules:

$$\frac{\Gamma_1, A, \Gamma_2, B, \Gamma_3 \vdash \Sigma}{\Gamma_1, B, \Gamma_2, A, \Gamma_3 \vdash \Sigma} \qquad \frac{\Gamma \vdash \Sigma_1, A, \Sigma_2, B, \Sigma_3}{\Gamma \vdash \Sigma_1, B, \Sigma_2, A, \Sigma_3}$$

$$\frac{\Gamma, \Delta \vdash \Theta \quad A \text{ Prop}}{\Gamma, !A, \Delta \vdash \Theta} \qquad \frac{\Gamma \vdash \Delta, \Theta \quad A \text{ Prop}}{\Gamma \vdash \Delta, ?A, \Theta} \qquad \frac{\Gamma, !A, !A, \Delta, \vdash \Theta}{\Gamma, !A, \Delta \vdash \Theta} \qquad \frac{\Gamma \vdash \Delta, ?A, ?A, \Theta}{\Gamma \vdash \Delta, ?A, \Theta}$$

$$\frac{A \text{ Prop}}{A \vdash A} \qquad \frac{\Gamma \vdash A, \Phi \quad \Psi, A \vdash \Delta}{\Psi, \Gamma \vdash \Delta, \Phi}$$

which outlines the exchange, restricted weakening / contraction, variable, and cut rules.

The inference rules for each operation are given as follows:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta} \qquad \frac{}{\Gamma \vdash \Delta, \top, \Theta} \qquad \frac{}{\Gamma, \mathbf{0}, \Delta \vdash \Theta} \qquad \frac{\Gamma, \Delta \vdash \Theta}{\Gamma, \mathbf{1}, \Delta \vdash \Theta} \qquad \frac{}{\vdash \mathbf{1}} \qquad \frac{\Gamma \vdash \Delta, \Theta}{\Gamma \vdash \Delta, \perp, \Theta} \qquad \frac{}{\perp \vdash}$$

$$\frac{\Gamma, A, \Delta \vdash \Theta}{\Gamma, A \,\&\, B, \Delta, \vdash \Theta} \qquad \frac{\Gamma, B, \Delta \vdash \Theta}{\Gamma, A \,\&\, B, \Delta, \vdash \Theta} \qquad \frac{\Gamma \vdash \Delta, A, \Theta \quad \Gamma \vdash \Delta, B, \Theta}{\Gamma \vdash \Delta, A \,\&\, B, \Theta}$$

$$\frac{\Gamma \vdash \Delta, A, \Theta}{\Gamma \vdash \Delta, A \oplus B, \Theta} \qquad \frac{\Gamma \vdash \Delta, B, \Theta}{\Gamma \vdash \Delta, A \oplus B, \Theta} \qquad \frac{\Gamma, A, \Delta \vdash \Theta \quad \Gamma, B, \Delta, \vdash \Theta}{\Gamma, A \oplus B, \Delta \vdash \Theta}$$

$$\frac{\Gamma, A, B, \Delta \vdash \Theta}{\Gamma, A \otimes B, \Delta, \vdash \Theta} \qquad \frac{\Gamma \vdash \Delta, A \quad \Lambda \vdash B, \Theta}{\Gamma, \Lambda \vdash \Delta, A \otimes B, \Theta}$$

$$\frac{\gamma \vdash \Delta, A, B, \Theta}{\Gamma \vdash \Delta, A \parr B, \Theta} \qquad \frac{\Gamma, A \vdash \Delta \qquad B, \Theta \vdash \Lambda}{\Gamma, A \parr B, \Theta \vdash \Delta, \Lambda}$$

$$\frac{\Gamma, A, \Delta, \vdash \Theta}{\Gamma, !A, \Delta \vdash \Theta} \qquad \frac{\Gamma \vdash \Delta, A, \Theta \qquad \Gamma \text{ all } ! \qquad \Delta, \Theta \text{ all } ?}{\Gamma \vdash \Delta, !A, \Theta}$$

$$\frac{\Gamma \vdash \Delta, A, \Theta}{\Gamma \vdash \Delta, ?A, \Theta} \qquad \frac{\Gamma, A, \Delta \vdash \Theta \qquad \Gamma, \Delta \text{ all } ! \qquad \Theta \text{ all } ?}{\Gamma, ?A, \Delta \vdash \Theta}$$

Definition 2.0.1. *Two propositions A and B are **propositionally equivalent**, written $A \equiv B$ if $A \vdash B$ and $B \vdash A$.*

3 Monoidal Categories and Duality

A $*$-autonomous category may be seen as a symmetric monoidal closed category equipped with a dualizing object. This follows the following topography:

$$\begin{tikzcd}
& \text{Monoidal} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Left Closed} \arrow[dr] & \text{Symmetric Monoidal} \arrow[d] & \text{Right Closed} \arrow[dl] \\
& \text{Symmetric Monoidal Closed} \arrow[d] & \\
& *\text{-autonomous} &
\end{tikzcd}$$

On the other hand, it may also be seen as a symmetric linearly distributive category equipped with a duality. This is given by the following topography:

$$\begin{tikzcd}
& \text{Monoidal} \arrow[d] & \\
& \text{Linearly Distributive} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Left Duality} \arrow[dr] & \text{Symmetric Linearly Distributive} \arrow[d] & \text{Right Duality} \arrow[dl] \\
& *\text{-autonomous} &
\end{tikzcd}$$

Alternatively, it may also be seen as a dialogue category whose negation is involutive.

3.1 Monoidal Categories

Definition 3.1.1. *A **monoidal category** is a category $\mathcal{C}$ equipped with a bifunctor*

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

associative up to a natural isomorphism

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

alongside an object e unit up to natural isomorphisms

$$\lambda_A : e \otimes A \to A \qquad \rho_A : A \otimes e \to A$$

The structure maps α, λ, ρ must also satisfy the following two commutative diagrams:

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha"] \arrow[d, "\alpha \otimes id_D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id_A \otimes \alpha"']
\end{tikzcd}$$

for all objects A, B, C, D of the category, and

$$\begin{tikzcd}
(A \otimes e) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes id_B"'] & & A \otimes (e \otimes B) \arrow[dl, "id_A \otimes \lambda"] \\
& A \otimes B &
\end{tikzcd}$$

for all objects A and B of the category.

The pentagon and triangle axioms ensure that every diagram made of structure maps commutes in the category $\mathcal{C}$. This property is called the *coherence property* of monoidal categories. It implies many things, including that the structure morphisms $\lambda_e : e \otimes e \to e$ and $\rho_e : e \otimes e \to e$ coincide. This notion is important as this equality of maps is often given as a third axiom of monoidal categories.

Remark 3.1.2. Intuitively, a monoidal category is a setting where one can tensor things together (in some sense like parallel composition), with a base 'nothing' object. The coherence axioms ensures transformations we 'want equal' to be actually equal.

The pentagon is something like 'the order in which we move between associations doesn't matter', and the triangle is like 'the way in which we project away e doesn't matter'.

Then, we have the following:

Proposition 3.1.3. *The triangles*

$$\begin{tikzcd}
(e \otimes A) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\lambda \otimes id_B"'] & & e \otimes (A \otimes B) \arrow[dl, "\lambda"] \\
& A \otimes B &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes B) \otimes e \arrow[rr, "\alpha"] \arrow[dr, "\rho"'] & & A \otimes (B \otimes e) \arrow[dl, "id_A \otimes \rho"] \\
& A \otimes B &
\end{tikzcd}$$

commute in any monoidal category $\mathcal{C}$.

Proof. □

Proposition 3.1.4. *The two morphism λ_e and ρ_e coincide in any monoidal category $\mathcal{C}$.*

Proof. Naturality of λ implies that the diagram

$$\begin{tikzcd}
e \otimes (e \otimes B) \arrow[r, "\lambda"] \arrow[d, "e\otimes \mathrm{id}_B"'] & e \otimes B \arrow[d, "\lambda"] \\
e \otimes B \arrow[r, "\lambda"'] & B
\end{tikzcd}$$

commutes. Crucially, as $\lambda : e\otimes B \to B$ is an isomorphism, we have that the two structure morphisms

$$e \otimes (e \otimes B) \xrightarrow{\lambda} e \otimes B \qquad\qquad e \otimes (e \otimes B) \xrightarrow{e\otimes\lambda} e \otimes B$$

coincide. Then, noting the previous proposition, we see that the triangle

$$\begin{tikzcd}
(e \otimes e) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\lambda\otimes \mathrm{id}_B"'] & & e \otimes (e \otimes B) \arrow[dl, "\lambda = e\otimes\lambda"] \\
& e \otimes B &
\end{tikzcd}$$

commutes for every object B of the category $\mathcal{C}$. The triangular axiom of monoidal categories indicate then that the morphisms

$$e \otimes (e \otimes B) \xrightarrow{\lambda_e\otimes \mathrm{id}_B} e \otimes B \qquad\qquad e \otimes (e \otimes B) \xrightarrow{\rho_e\otimes \mathrm{id}_B} e \otimes B$$

coincide for every object B, in particular with $B = e$, such that $\lambda_e \otimes e$ and $\rho_e \otimes e$ coincide. Then, noting that the functor $- \otimes e : \mathcal{C} \to \mathcal{C}$ is full and faithful, we conclude that λ_e and ρ_e coincide. $\square$