Short Proof of The Genericity Lemma

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1 Proof Setup

We will assume a basic knowledge about Lambda Calculus and Topology, only refreshing on topics related to Genericity Lemma. We give some definitions to make notation clear.

1.1 Basic Definitions

Definition 1.1.1. *We write Lambda Terms to be defined by*

$$M, N := x \mid \lambda x.M \mid MN$$

where x is some variable, which we assume to have countably many of. The set of all terms is written Λ.

Definition 1.1.2. *A (unary) context is defined as*

$$C[X] := X \mid \lambda x.C[X] \mid C[X]\,M \mid M\,C[X]$$

where M is a lambda term.

1.2 Head Normal Form

Definition 1.2.1. *A closed term is in **head normal form (hnf)** if it has the shape*

$$\lambda x_1.\cdots\lambda x_n.gN_1\cdots N_k$$

for $k \geq 0$ and g is a variable with $N_i \in \Lambda$ for $i = 1, \ldots, k$.

Definition 1.2.2. *A closed term M is **solvable** if there is some closed context $C[X]$ and some term H in hnf such that*

$$C[M] \twoheadrightarrow_h H$$

*If no such context exists, M is **unsolvable**.*

1.3 The Topology

Definition 1.3.1. *Define the subbasis of opens for the topology τ of Λ to be*

$$U_{C,H} = \{M \in \Lambda \mid C[M] \twoheadrightarrow_h H\}$$

Lemma 1.3.2. *The map $C[\_] : \Lambda \to \Lambda$ by $M \mapsto C[M]$ is continuous.*

Proof. Take any open set from the subbasis, say $U_{D,H}$. The preimage of this under $C[\_]$ is $U_{C \circ D,H}$. □

Lemma 1.3.3. *The set of solvable terms is given by*

$$\bigcup_{C,H} U_{C,H}$$

In particular, this is an open set.

Lemma 1.3.4. *The set of unsolvable terms is a countable intersection of dense open sets.*

2 Genericity Lemma

Theorem 2.0.1. *Let M be an unsolvable term. Then for any context $\mathcal{C}$, then*