First Read on Linear Algebra, Lecture 2

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Definition 2.3.10. *A* ***basis*** *for a vector space V over $\mathbb{F}$ is a set $\mathcal{B}$ who is both spanning and linearly independent. If V has a finite basis, we call V* ***finite dimensional****.*

Example 2.3.11. In $\mathbb{R}^n$ for $1 \leq i \leq n$, define $\mathbf{e}_i$ to be the row vector with coordinate 1 in the ith entry and 0 elsewhere. Then $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent (by checking ith entries), and as

$$(\alpha_1, \ldots, \alpha_n) = \alpha_1 \mathbf{e}_1 + \cdots + \alpha_n \mathbf{e}_n$$

$\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a basis of $\mathbb{R}^n$. We call this the **standard basis** or **canonical basis** of $\mathbb{R}^n$.

Example 2.3.12. Let $V = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\} \leq \mathbb{R}^2$. Then a basis for V is $\{(1, -1)\}$.

Proposition 2.3.13. *Let V be a vector space over $\mathbb{F}$ and let $S = \{v_1, \ldots, v_n\} \subseteq V$. Then S is a basis of V if and only if every vector in V has a unique expression as a linear combination of elements of S.*

Proof. ($\Rightarrow$) Let S be a basis of V. Take $v \in V$. as S spans V, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

As S is linearly independent, these scalars are unique.

($\Leftarrow$) Suppose that every vector in V has an unique expression as a linear combination of elements of S. Then clearly S is a spanning set. Now, given any $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, if $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0 = 0v_1 + \cdots + 0v_n$, by uniqueness we have $\alpha_i = 0$ for all $1 \leq i \leq n$. Hence S is linearly independent, thus is a basis for V. $\square$

Example 2.3.14. The space $\mathbb{F}[x]$ of polynomials over a field $\mathbb{F}$ has standard basis

$$\{1, x, x^2, \ldots\}$$

As every polynomial can be uniquely written as a finite linear combination of this basis.

Definition 2.3.15. *Given a basis $\{v_1, \ldots, v_n\}$ of V, then every $v \in V$ can be uniquely written as*

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

The scalars $\alpha_1, \ldots, \alpha_n$ are defined to be ***coordinates*** *of v with respect to the basis $\{v_1, \ldots, v_n\}$.*

2.4 Dimension

Theorem 2.4.1. *Let V be a vector space. Let S and T be finite subsets of V. If S is linearly idependent and T spans V, then $|S| \leq |T|$.*

Proof. Assume that S is linearly independent and that T spans V. Let $S = \{u_1, \dots, u_m\}$ and $T = \{v_1, \dots, v_n\}$, where $|S| = m$ and $|T| = n$.

Define $T_0 = \{v_1, \dots, v_n\}$. As $\langle T_0 \rangle = V$, we have $u_1 \in \langle v_1, \dots, v_i \rangle$ for some $1 \leq i \leq n$, and let i be the minimal such one. Then by minimality, we have $u_1 \notin \langle v_1, \dots, v_{i-1} \rangle$. By the Steinitz Exchange Lemma,

$$\langle v_1, \dots, v_i \rangle = \langle u_1, v_1, \dots, v_{i-1} \rangle$$

Now,

$$\begin{aligned} V &= \langle v_1, \dots, v_n \rangle \\ &= \langle v_1, \dots, v_i \rangle + \langle v_{i+1}, \dots, v_n \rangle \\ &= \langle u_1, v_1, \dots, v_{i-1} \rangle + \langle v_{i+1}, \dots, v_n \rangle \\ &= \langle u_1, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n \rangle \end{aligned}$$

By relabeling the elements of T, without loss of generality we may assume that u_1 has been exchanged for v_1 and set $T_1 = \{u_1, v_2, \dots, v_n\}$, with $\langle T_1 \rangle = V$. Inductively repeating this process gives sets $T_k = \{u_1, \dots, u_k, v_{k+1}, \dots, v_n\}$ such that $\langle T_k \rangle = V$. We note that at each step $u_{k+1} \in \langle T_k \rangle$ but $u_{k+1} \notin \langle u_1, \dots, u_k \rangle$ by linear independence of S. Hence we can repeatedly replace elements of T with elements of S. We terminate when S is exhausted, hence $m \leq n$. $\square$

Corollary 2.4.2. *Let V be a finite dimensional vector space. Then any bases of V are finite and of the same size.*

Proof. As V is finite dimensional, V has a finite basis $\mathcal{B}$. Given another basis $\mathcal{E}$, it is linearly independent, hence every finite subset of $\mathcal{E}$ is linearly independent. By the previous theorem, any such set has size at most $|\mathcal{B}|$. Hence $\mathcal{E}$ is finite, with $|\mathcal{E}| \leq |\mathcal{B}|$. On the other hand, $\mathcal{B}$ is linearly independent and $\mathcal{E}$ is spanning, so $|\mathcal{B}| \leq |\mathcal{E}|$. $\square$

Definition 2.4.3. *Let V be a finite dimensional vector space. The **dimension** of V, written $\dim V$ is the size of any basis of V.*

Proposition 2.4.4. *Let V be a vector space over $\mathbb{F}$ and let S be a finite spanning set. Then S contains a basis.*

Proof. Let S be a finite spanning set for V. Take $T \subseteq S$ such that T is linearly independent, and take the largest such set. We claim that $\langle T \rangle = V$. Else, as $\langle S \rangle = V$, we can find a $v \in S \backslash \langle T \rangle$. On the other hand, $T \cup \{v\} \subseteq S$ is linearly independent. This contradicts maximality of T. Hence T is a linearly independent set that spans V, thus a basis. $\square$

Proposition 2.4.5. *Let V be a fintie dimensional vector space and $U \leq V$. Then U is finite dimensional with $\dim U \leq \dim V$. Moreover, if $\dim U = \dim V$, then $U = V$.*

Proof. Let $n = \dim V$. Then by Theorem 2.4.1, every linearly independent subset of V has size at most n. Let S be a largest linearly independent set contained in U, such that $|S| \leq n$. We claim that $\langle S \rangle = U$. Else, there exists a $u \in U \backslash \langle S \rangle$, which means $S \cup \{u\}$ is linearly independent, contradicting maximality of S. Hence $U = \langle S \rangle$ and S is linearly independent, so S is a basis of U. Now $\dim U = |S| \leq n = \dim V$.

Suppose now that $\dim U = \dim V$ but $U \neq V$. Then we can find a $v \in V \setminus U$. Then $U \cup \{v\}$ is a linearly independent subset of V with dimension $\dim U + 1 = \dim V + 1$, which is a contradiction. Hence $U = V$ in this case. $\square$

Proposition 2.4.6. *Let V be a finite dimensional vector space over $\mathbb{F}$ and let S be a linearly independent set. Then there exists a basis $\mathcal{B}$ such that $S \subseteq \mathcal{B}$.*

Proof. If $\langle S \rangle = V$, then we are done. If $\langle S \rangle \neq V$, by choosing an element $v_1 \in V \setminus \langle S \rangle$, we can extend S to $S_1 = S \cup \{v_1\}$ to create a larger linearly independent set. If $\langle S_1 \rangle = V$, then S_1 is a basis so we are done. We inductively continue this process until S_k is a basis. This process terminates as every linearly independent subset of V must contain at most $\dim V$ elements. $\square$

Corollary 2.4.7. *A maximal linearly independent subset of a finite dimensional vector space is a basis.*

Proof. Let S be a maximal linearly independent subset of a finite dimensional vector space V. If $\langle S \rangle \neq V$, we can extend this to a linearly independent set $S_1 = S \cup \{v\}$ with $v \in V \setminus \langle S \rangle$ which contradicts maximality of S. Hence $\langle S \rangle = V$. $\square$

Corollary 2.4.8. *A minimal spanning subset of a finite dimensional vector space is a basis.*

Proof. Let S be a minimal spanning subset of a finite-dimensional vector space V. If S is not linearly independent, there exist a $v \in S$ which can be written as a linear combination of elements of $S \setminus \{v\}$. Then $S \setminus \{v\}$ is still spanning, which contradicts the minimality of S. Hence S is linearly independent, thus a basis. $\square$

Theorem 2.4.9 (Dimension Formula)**.** *Let U and W be subspaces of a finite-dimensional vector space V over $\mathbb{F}$. Then,*

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

Proof. Take a basis $v_1, \ldots, v_m$ of $U \cap W$. By Proposition 2.4.6, we can extend this set to a basis $v_1, \ldots, v_m, u_1, \ldots, u_n$ of U and a basis $v_1, \ldots, v_m, w_1, \ldots, w_l$ of W.

Then we have,

$$\dim(U \cap W) = m \qquad \dim U = m + n \qquad \dim W = m + l$$

We claim that $S := \{v_1, \ldots, v_m, u_1, \ldots, u_n, w_1, \ldots, w_l\}$ is a basis for $U + W$. This contains

$$m + n + l = (m + n) + (m + l) - m = \dim U + \dim W - \dim(U \cap W)$$

elements, so showing this suffices to prove our result.

We first show that S is spanning. Take and $x \in U + W$ such that $x = u + w$ for some $u \in U$ and $w \in W$. Then,

$$u = \alpha_1 v_1 + \cdots + \alpha_m v_m + \alpha'_1 u_1 + \cdots + \alpha'_n u'_n$$
$$w = \beta_1 v_1 + \cdots + \beta_m v_m + \beta'_1 w_1 + \cdots + \beta'_l w'_l$$

for some scalars $\alpha_i, \alpha'_i, \beta_i, \beta'_i \in \mathbb{F}$. Then,

$$x = u + w = (\alpha_1 + \beta_1)v_1 + \cdots + (\alpha_m + \beta_m)v_m + \alpha'_1 u_1 + \cdots + \alpha'_n u_n + \beta'_1 w_1 + \cdots + \beta'_l w_l \in \langle S \rangle$$

Hence $\langle S \rangle = U + W$.

We now show S is linearly independent. Take $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_l \in \mathbb{F}$ such that

$$\alpha_1 v_1 + \cdots + \alpha_m v_m + \beta_1 u_1 + \cdots + \beta_n u_n + \gamma_1 w_1 + \cdots + \gamma_l w_l = 0$$

Then we can rewrite this to

$$\alpha_1 v_1 + \cdots + \alpha_m v_m + \beta_1 u_1 + \cdots + \beta_n u_n = -(\gamma_1 w_1 + \cdots + \gamma_l w_l)$$

The vector on the left is in U and the vector on the right is in W, so they are both in $U \cap W>$ As $v_1, \ldots, v_m$ form a basis of $U \cap W$, we can find $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ such that

$$-(\gamma_1 w_1 + \cdots + \gamma_l w_l) = \lambda_1 v_1 + \cdots + \lambda_m v_m$$

Rearranging this gives

$$\gamma_1 w_1 + \cdots + \gamma_l w_l + \lambda_1 v_1 + \cdots + \lambda_m v_m = 0$$

These vectors are a basis for W, thus linearly independent, meaning each γ_i is 0. This then implies that

$$\alpha_1 v_1 + \cdots + \alpha_m v_m + \beta_1 u_1 + \cdots + \beta_n u_n = 0$$

But these vectors are a basis for V, hence linearly independent, so each α_i and β_i equals 0. In particular, S is linearly independent, and so the result follows. $\square$

2.5 Moving Forward

In these first two lectures we stayed within a single vector space V organizing its internal structure via subspaces and bases, and highlighted intrinsic invariants such as dimension. With that foundation in place, we now shift from the structure of one space to the relationships between spaces: linear maps. These structure-preserving maps capture how vector spaces interact, letting us compare them, and transport invariants.