Notes on Categories

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Contents

1 Introduction

This note aims to put an uncondensed summary of concepts related to Category Theory I've come across. Hopefully full of examples, as this should have what I needed to gain an intuition of the concepts. The notes here are (so far) based off of:

1. C2.7: Category Theory by Pavel Safronov (Oxford Maths Cateogry Theory Course Notes)
2. Introduction to Categories and Categorical Logic by Samson Abramsky and Nikos Tzevelekos (Oxford CS Category Proofs and Processes Course Notes)
3. Categories for the Working Mathematician (Second Edition, only sections 1 to 6) by Saunders Mac Lane
4. Category Theory by Steve Awodey

2 Basics

2.1 Object Definitions

Definition 2.1.1. *A category $\mathcal{C}$ consists of the following data:*

- *A collection* $\mathrm{ob}\ \mathcal{C}$ *of objects of* $\mathcal{C}$
- *For every* $x, y \in \mathrm{ob}\ \mathcal{C}$ *a collection* $\mathrm{Hom}_{\mathcal{C}}(x, y)$ *of morphisms*
- *For every* $x \in \mathrm{ob}\ \mathcal{C}$*, the identity morphism* $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$
- *For every* $x, y, z \in \mathrm{ob}\ \mathcal{C}$*, the composition map*

$$\circ : \mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

These then must satisfy the following axioms:

- *For any two* $x, y \in \mathrm{ob}\ \mathcal{C}$ *and any morphism* $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ *we have*

$$f \circ \mathrm{id}_x = f \qquad \mathrm{id}_y \circ f = f$$

- *Morphisms under composition are associative*

Notation 2.1.2. We write $x \in \mathcal{C}$ for $x \in ob\ \mathcal{C}$ and omit the subscript in *Hom* when the category is clear. We may sometimes write $\mathcal{C}(x, y)$ for $Hom_{\mathcal{C}}(x, y)$. We also write $Hom_{\mathcal{C}}(x)$ for $Hom_{\mathcal{C}}(x, x)$ and call these endomorphisms.

2.1.1 Functor

Definition 2.1.3. *A functor $F : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}, \mathcal{D}$ consists of the following data:*

- *a map* $F : \mathrm{ob}\ \mathcal{C} \to \mathrm{ob}\ \mathcal{D}$
- *For any two objects* $x, y \in \mathcal{C}$*, a map of sets* $F : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$

Such that they satisfy

- *Unit: for any* $x \in \mathcal{C}$*,* $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
- *For any objects* $x, y, z \in \mathcal{C}$ *and morphisms* $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(y, z)$*, we have*

$$F(g \circ f) = F(g) \circ F(f)$$

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is ***faithful*** *if the map $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ is injective for any objects x and y. We say that F is* ***full*** *if this map is surjective, and that it is* ***fully faithful*** *if it is both full and faithful.*

We say that F is a ***contravariant functor*** *from $\mathcal{C}$ to $\mathcal{D}$ if it is a functor $\mathcal{C}^{\mathrm{op}} \to D$. A functor $\mathcal{C}$ to $\mathcal{D}$ is also referred to as a covariant functor.*

Note we use the same F to refer to the map between both objects and morphisms.

Definition 2.1.4. *A morphism $f \in \mathrm{Hom}(x, y)$ in a category is an* ***isomorphism*** *if there is a morphism $f^{-1} \in \mathrm{Hom}(y, x)$ such that $f^{-1} \circ f = \mathrm{id}_x$ and $f \circ f^{-1} = \mathrm{id}_y$. We also say that f is* ***invertible****. If two objects $x, y \in \mathcal{C}$ are isomorphic, we write $x \cong y$.*

Remark 2.1.5. As usual, inverses are unique. Suppose that $f \in Hom(x, y)$ and we have $g \circ f = id_x$ and $f \circ h = id_y$. Then,

$$g = g \circ id_y = g \circ (f \circ h) = (g \circ f) \circ h = id_x \circ h$$

Definition 2.1.6. *A category $\mathcal{C}$ is called a* ***groupoid*** *if every morphism is invertible. We say that a groupoid is* ***connected*** *if any two objects are isomorphic.*

Remark 2.1.7. Consider a groupoid with a single object. The data required to specify such a groupoid is the monoid of endomorphisms in which every object has an inverse. This is just a group.

2.1.2 Natural Transformation

Definition 2.1.8. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta : F \Rightarrow G$ consists of morphisms $\eta_x \in \mathrm{Hom}_{\mathcal{D}}(F(x), G(x))$ for every object $x \in \mathcal{C}$ such that the diagram*

$$\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \\
G(x) \arrow[r, "G(f)"'] & G(y)
\end{tikzcd}$$

commutes for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.

We say a natural transformation $\eta : F \Rightarrow G$ is a ***natural isomorphism*** *if the morphisms η_x are isomorphisms for any $x \in \mathcal{C}$.*

Equivalently, we can also write something like:

$$\begin{tikzcd}
F(x) \arrow[rr, "\eta_x"] \arrow[dr, "F(f)"] \arrow[dd, "F(g \circ f)"'] & & G(x) \arrow[dr, "G(f)"] \arrow[dd, "G(g \circ f)"' near start] & \\
& F(y) \arrow[rr, "\eta_y"] \arrow[dl, "F(g)"] & & G(y) \arrow[dl, "G(g)"] \\
F(z) \arrow[rr, "\eta_z"'] & & G(z) &
\end{tikzcd}$$

Given two categories $\mathcal{C}$ and $\mathcal{D}$, one can construct a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors between $\mathcal{C}$ and $\mathcal{D}$. It's objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are given by natural transformations. Then we can view natural isomorphisms as isomorphisms in the functor category.

Notation 2.1.9. We write natural transformations as

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D}
\end{tikzcd}$$

1. Consider

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] & \mathcal{D}
\end{tikzcd}$$

This gives *vertical composition* $F \Rightarrow H$ with components given by the composition $F(x) \xrightarrow{\eta_x} G(x) \xrightarrow{\epsilon_x} H(x)$ for all $x \in \mathcal{C}$.

2. Consider

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\epsilon"] & \mathcal{E} \end{tikzcd}$$

This gives *horizontal composition* $F_2F_1 \Rightarrow G_2G_1$ given by composition

$$F_2F_1(x) \overset{F_2(\eta_x)}{\to} F_2G_1(x) \overset{\epsilon_(G_1(x))}{\to} G_2G_1(x)$$

for every $x \in \mathcal{C}$. Picking η or ϵ to be the identity functor, we can compose by functors as well.

Example 2.1.10. Define $\det_K(M)$ be the determinant of the $n \times n$ matrix with entries in the commutative ring K, and let K^* denote the group of units of K. Thus M is non-singular when $\det_K(M)$ is a unit, so this gives a morphism $\mathrm{GL}_n(K) \to K^*$ of groups. Then, for any morphism $f : K \to K'$ this gives a commutative diagram:

$$\begin{tikzcd} \mathrm{GL}_n(K) \arrow[r, "\det_K"] \arrow[d, "\mathrm{GL}_n f"'] & K^* \arrow[d, "f^*"] \\ \mathrm{GL}_n(K') \arrow[r, "\det_{K'}"'] & K'^* \end{tikzcd}$$

Showing that the transformation $\det\{K : \mathbf{CRng}\} : \mathrm{GL}_n(K) \to K^*$ between the two functors $\mathbf{CRng} \to \mathbf{Grp}$.

Example 2.1.11. For each group G the projection $p_G : G \to G/[G,G]$ defines a transformation from the identity functor on **Grp** to the factor-commutator functor $\mathbf{Grp} \to \mathbf{Ab} \to \mathbf{Grp}$. Then p is natural, as for any group homomorphism $f : G \to H$ we have the evident homomorphism f' for which the following commutes:

$$\begin{tikzcd} G \arrow[r, "p_G"] \arrow[d, "f"'] & G/[G,G] \arrow[d, "f'"] \\ H \arrow[r, "p_H"'] & H/[H,H] \end{tikzcd}$$

2.1.3 Monos, Epis, Zeros

TODO!!!: S1.5

2.1.4 Equivalence

Definition 2.1.12. *An* ***equivalence of categories*** $\mathcal{C}, \mathcal{D}$ *is a pair of functors* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ *along with natural isomorphisms* $e : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ *and* $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.

Definition 2.1.13. *An* ***adjoing equivalence*** *of categories* $\mathcal{C}, \mathcal{D}$ *is an equivalence* (F, G, e, ϵ) *satisfying the following:*

1. *The composite natural transformation*

$$F \cong F \circ \mathrm{id}_{\mathcal{C}} \overset{\mathrm{id}_F \circ e}{\Rightarrow} FGF \overset{\epsilon \circ \mathrm{id}_F}{\Rightarrow} \mathrm{id}_{\mathcal{D}} \circ F \cong F$$

is the identity natural transformation on F.

2. *The composite natural transformation*

$$G \cong \mathrm{id}_{\mathcal{C}} \circ G \overset{e \circ \mathrm{id}_G}{\Rightarrow} GFG \overset{\mathrm{id}_G \circ \epsilon}{\Rightarrow} G \circ \mathrm{id}_{\mathcal{D}} \cong G$$

is the identity natural transformation on G

2.1.5 Examples to keep in mind

Example 2.1.14. Groups form a category **Grp** with morphisms by homomorphisms of groups. We can restrict groups to be abelian, which forms a category **Ab**.

We consider some functors:

- The forgetful functor from **Ab** to **Grp** is fully faithful.
- The abelianization $G \mapsto G/[G, G]$ gives a functor $\mathbf{Grp} \to \mathbf{Ab}$. It is neither full nor faithful.

Example 2.1.15. If k is a field, k-vector spaces form a category $\mathbf{Vect}_k$ with morphisms given by linear maps.

Example 2.1.16. (Small) categories form a category **Cat** where morphisms are given by functors.

Example 2.1.17. A set X can be regarded as a category $\mathcal{C}$ with $ob\ \mathcal{C} = X$ where $Hom_{\mathcal{C}}(x, y) = \emptyset$ for $x \neq y$ and $End_{\mathcal{C}}(x) = \{id_x\}$. These categories are called a **discrete category**.

Given a set and viewing it as a discrete category, this gives a fully faithful functor $\mathbf{Set} \to \mathbf{Cat}$.

2.2 Constructing Categories

2.2.1 Contravariance

Definition 2.2.1. *Given a category $\mathcal{C}$, we have the **opposite category** $\mathcal{C}^{\mathrm{op}}$, which has the same objects and* $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x)$.

3 Universals, Limits, and Adjoints

3.1 Universals

Definition 3.1.1. *If $F : \mathcal{D} \to \mathcal{C}$ is a functor and $c \in \mathcal{C}$, a **universal arrow** from c to F is a pair $\langle r, u\rangle$ consisting of an object $r \in \mathcal{D}$ and an arrow $u : c \to F(r)$ of $\mathcal{C}$ such that for every pair $\langle d, f\rangle$ with an object $d \in \mathcal{D}$ and an arrow $f : c \to F(d)$ of $\mathcal{C}$, there is a unique arrow $f' : r \to d$ of $\mathcal{D}$ with $F(f' \circ u) = f$. Alternatively, every arrow f to S factors uniquely through the universal arrow u:*

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, "u"'] & S(d) \\
S(r) \arrow[ur, dashed, "S(f')"'] &
\end{tikzcd}$$

3.2 Limits and Colimits