PELEG BAR SAPIR

PHYSICS SIMULATIONS (LECTURE NOTES)

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Introduction

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1.4 Some Mathematical Background

1.4.1 Radians

In the context of trigonometry in mathematics and physics, we most commonly use **radians** in place of **degrees**. This sometimes causes some confusion with newer students, so I find it appropriate to briefly discuss the what and why of radians and their usage.

Degrees are defined as a measurement of **angles**. That is, for historical and practical reasons we define a full rotation as $360°$. The main problem with using angles is that they are units by themselves, i.e. they differ from lengths. This can cause some consistency issues with regards to units. Consider for example the following equation for a **centripetal force**:

$$F = m\frac{v^2}{r}, \tag{1.1}$$

where m is the mass of the rotationg object, v^2 is the square of the linear velocity the object experiences in each time instance, and r is the distance of the rotating object from the center of rotation (Figure 1.1).

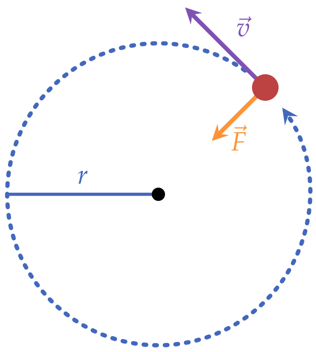

Figure 1.1: Object with mass m in perfect cicular motion with distance r around a central point. The force acting on the object is directed towards the center and has value $F = m\frac{v^2}{r}$.

In the standard SI scheme, F has units of $[\mathrm{N}]$ ("Newtons"), i.e. $\left[\mathrm{kg\,m\,s^{-1}}\right]$. We know that m has units of $[\mathrm{kg}]$, v of $\left[\mathrm{m\,s^{-1}}\right]$ and r units of $[\mathrm{m}]$ - so both sides of the equation have the same units.

However, nothing prevents us from measuring the rotation associated with $\frac{v^2}{r}$ directly using angular velocity ω - which can be defined as the amounth of degrees of rotation per time T if the rotation is uniform:

$$\omega = \frac{\theta}{T}, \tag{1.2}$$

or more generally as the time derivative of the angle,

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}. \tag{1.3}$$

If we measure the angle θ in degrees, we get that the units of angular velocity is $\left[{}^\circ\,\mathrm{s}^{-1}\right]$. If we then want to use ω in the centripetal force equation, we must somehow cancel the degrees unit to get the proper units of $[\mathrm{N}]$. This obviously leads to a somewhat cumbersome equation.

A much better approach is to measure the *ratio* between the length of an arc created by our angle at a radius R and the same radius itself (Figure 1.2). This gives a unitless measure which we call *radians*, and can be either ignored in the unit calculation (since it is unitless), or simply denoted $[\mathrm{rad}]$.

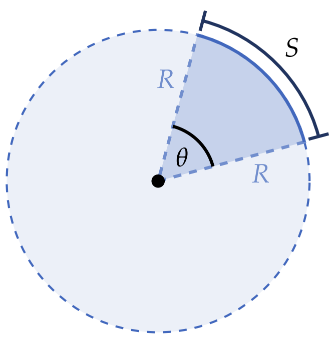

Figure 1.2: Measuring angles using radians. We express the length of S in units of R, e.g. if $S = 1$ then the arc length is equal to R, if $S = \pi$ the arc length is half that of the entire circumference of the circle, etc. The ratio S/R is constant for any real positive value of R, and thus can be used to uniquely describe the angle θ.

Another way to view radians is that they measure an arc length in units of the radius (not necessarily a unit radius). If we scale the radius by any scalar, an arc length of the same angle will scale by exactly the same scalar, and so the amount of radians we measure for the arc length won't change (the ratio stays the same, as both the numerator and denominator are scaled by the same number).

We can easily replace any measurement of angles in degrees by a corresponding measurement in radians, since any amount of degrees has a 1-to-1 correspondence with a single value in radians. Let us find this correspondence: since there are 2π radii in a full circle, and a full angle is $360°$, we see that each radii length of an arc corresponds to $\frac{360°}{2\pi} = \frac{180°}{\pi} \approx 57.2958°$.

Now, using the radian unit to measure an angle θ, we get that ω has units of $\left[\mathrm{rad\,s^{-1}}\right]$, or simply $\left[\mathrm{s^{-1}}\right]$. In turn, if we raise ω to the 2nd power and multiply it by the distance r between the rotating object and center of rotation, and the mass m of the object, we get the units $\left[\mathrm{kg\,m\,s^{-1}}\right]$, which is exactly the excplicit form of Newtons. Indeed, using this perspective, Equation 1.1 can be written as

$$F = m\omega^2 r. \tag{1.4}$$

In my mind, this is simply beautiful.

Let's finish with some example correspondences between degrees and radians, which can be seen in Table 1.1 below.

[°]	[rad]
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
120	$\frac{2\pi}{3}$
180	π
270	$\frac{3\pi}{2}$
360	2π

Table 1.1: Example correspondences between measurements in degrees and radians.

1.4.2 *Taylor Series*

For some functions, it is rather easy to calculate their values at some point x_0: for example, given a polynomial function

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n, \tag{1.5}$$

it is rather easy to calculate its value at any real point x: all the operations that we need to use are addition and multiplication of real numbers, and raising real numbers to an integer power (which in principle can be implemented as repeated multiplications). Modern computers calculate such operations at the rate of billions of times a second.

Note 1.1 Floating point representation

For the sake of simplicity, I'm ignoring the entire topic of floating point numbers and relevant questions of precision.

Example 1.1 Calculating a value of a simple polynomial

Given the polynomial $P(x) = 3x^2 - 2x + 7$, we can easily calculate its value at, say, $x = 5$:

$$P(x = 5) = 3 \cdot 5^2 - 2 \cdot 5 + 7 = 3 \cdot 25 - 10 + 7 = 72.$$

For other functions, on the other hand, it is not that easy to calculate their values at most points. For example, consider the exponential function $f(x) = \mathrm{e}^x$. We know precisely exactly one real value of the function: at $x = 0$, the function returns 1. But for any other value of x, we don't really know the value of the function. That is, we know all values *symbolicaly* (for example, $f(1) = \mathrm{e}$, $f(2) = \mathrm{e}^2$, etc.) - but not in explicit form.

However, given that the function behaves nicely enough (we'll discuss in a moment what that means), we can *approximate* its value to whatever percision we wish, using a method called the **Taylor series** of the function. The "price" we pay for greater percision is simply performing more calculations.

The basic idea of a Taylor series of a function $f(x)$ is that we approximate the function by adding higher and higher derivatives of the same function, at some point a for which we know precisely the value of the function and its derivatives to any order. This might sound a bit abstract, so let's explore this process using an example function - again, the exponential $f(x) = \mathrm{e}^x$. As mentioned, we only really know one value of the function precisely, namely at $a = 0$: $\mathrm{e}^0 = 1$.

We can therefore start approximating e^x simply as e^0. This is obviously a very impercise approximation, but the important thing is that if we look at a very close neighborhood of $x = 0$, the approximation is not *that* bad: consider, for example $x_0 = 0.0001$. Our approximation gives $\mathrm{e}^{x_0} \approx 1$, which is not far from the more precise value $\mathrm{e}^{x_0} = 1.000100005$ (the value here is shown up to the ninth decimal). Of course, the closer we get to $a = 0$, the better our approximation gets: for example, with $x_1 = 0.00001$, we get $\mathrm{e}^{x_1} \approx 1$ again, where in reality $\mathrm{e}^{x_1} = 1.000010000$ (also shown here up to the ninth decimal). On the other hand, if we as we get farther away from $x = 0$, the approximation becomes worse and worse, as seen in Table 1.2 below.

x	e^x (exact)	Δ (error)
0.000	1.000000000	0.000000000
0.001	1.001000500	0.001000500
0.010	1.010050167	0.010050167
0.100	1.105170918	0.105170918
0.500	1.648721271	0.648721271
0.510	1.665291195	0.665291195
0.520	1.682027650	0.682027650
1.000	2.718281828	1.718281828
1.100	3.004166024	2.004166024
1.500	4.481689070	3.481689070
2.000	7.389056099	6.389056099

Table 1.2: Zero order Taylor series approximation of e^x.

Now, let's take this a step further: since the derivative of a function at a point tells us how the functions changes close to the point, we can use this information to improve our approximation by adding the value of the first derivative of e^x at $x = 0$, which is also 1. In fact, this is true for any order derivative of e^x, since $\frac{d^n}{dx^n}e^x = e^x$ for any $n \in \mathbb{N}$. Since the derivative changes with the value of x, we will multiply it by x in the approximation. Thus we get

$$e^x \approx 1 + 1 \cdot x = 1 + x, \tag{1.6}$$

which we call the **first order** approximation of the function e^x. Table 1.3 below shows the same values from Table 1.2 for x, but using the first order approximation for e^x.

x	$1 + x$	e^x (exact)	Δ (error)
0.000	1.000000000	1.000000000	0.000000000
0.001	1.001000000	1.001000500	0.000000500
0.010	1.010000000	1.010050167	0.000050167
0.100	1.100000000	1.105170918	0.005170918
0.500	1.500000000	1.648721271	0.148721271
0.510	1.510000000	1.665291195	0.155291195
0.520	1.520000000	1.682027650	0.162027650
1.000	2.000000000	2.718281828	0.718281828
1.100	2.100000000	3.004166024	0.904166024
1.500	2.500000000	4.481689070	1.981689070
2.000	3.000000000	7.389056099	4.389056099

Table 1.3: First order Taylor series approximation of e^x.

By examining Table 1.3 it's clear that $1 + x$ is a much better approximation for e^x than just 1. Of course, we can take a further step: the second derivative of a function at a point tells us how the change in the function itself changes around that point. We can take the second derivative of e^x (which is by itslef e^x), again substitute $x = 0$, get $\mathrm{e}^0 = 1$, and use this with x^2 to apperoximate e^x even better. In fact, let's already take this idea to its logical conclusion: we'll simply use the infinite power series in x, i.e. $1, x, x^2, x^3, \ldots, x^n, \ldots$ and attach to each x^k the value given by $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^x\Big|_{x=0} = 1$, so we get

$$\mathrm{e}^x \approx 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n. \tag{1.7}$$

This, however doesn't quite work. The coefficients of each term have to be adjusted. By assuming that at the limit where $n \to \infty$ the approximation should be with zero error, we get that for each n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^x = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(1 + x + x^2 + \ldots\right) \equiv s(x). \tag{1.8}$$

Calculating the general n-th derivative of Equation 1.7 is rather easy. Let's examine the case for $n = 3$: the first three derivatives of Equation 1.7 are

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}s(x) &= 1 + 2x + 3x^2 + 4x^3 + \ldots,\\
\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}s(x) &= 2 + 6x + 12x^2 + \ldots,\\
\tfrac{\mathrm{d}^3}{\mathrm{d}x^3}s(x) &= 6 + 24x + \ldots,
\end{aligned}$$

1.5 Harmonic Oscillator

1.5.1 Simple Harmonic Oscillator

Many systems in physics present a simple, periodic (repeating) motion. One such system is a simple mass-less spring connected to a mass m and allowed to move in a single dimension only. If we ignore the effects of gravity, the only force acting on the mass arises from the spring itself: the more we pull or push the spring, the stronger it will resist to that change. This resistant force is given by

$$F = -kx, \tag{1.9}$$

where k is the **spring constant**, and x is the amount by which the spring contracts or expands relative to its rest position x_0. In Figure 1.3 we show three cases: when the mass is at the springs rest position, $x_m = x_0$, the spring applies no force on it. When the mass is displaced by a positive amount $\Delta x > 0$, the spring applied a *negative* force on it: $F = -k\Delta x < 0$ (see note 1.2). And when the mass is displaced such that it contracts the spring, $\Delta x < 0$ and thus the force applied by the spring is positive, $F = k\Delta x > 0$.

Note 1.2 Negative and positive forces

Recall that in this context, negative force means a force in the negative x direction.

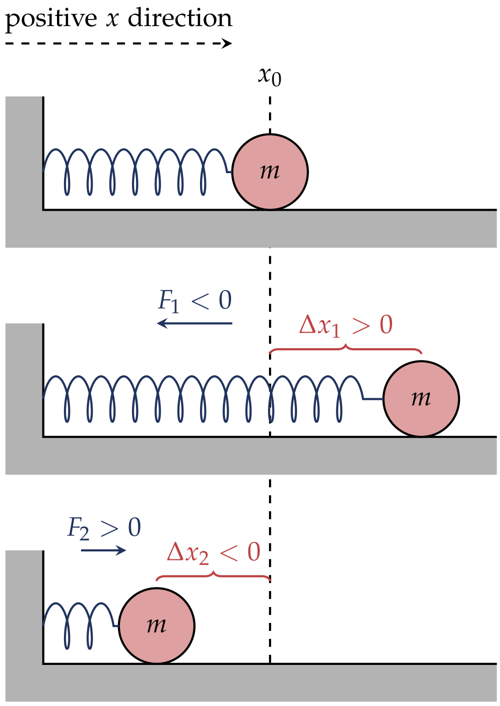

Figure 1.3: A simple spring-mass system with spring constant k and a mass m. The top figure shows the spring at rest - i.e. when the mass is located at position x_0 the spring applies no force on the mass (since $\Delta x = x_m - x_0 = 0$). The middle figure show the spring being at a *positive* displacement $\Delta x_1 > 0$, causing the spring to pull back with a negative force $F_1 = -k\Delta x_1$. The bottom picture shows the spring contracting by $\Delta x_2 < 0$, casing the spring to apply a positive force $F_2 = -k\Delta x_2$ on the mass.

Using Newtons second law of motion (REF?) with x being the displacement from the rest position x_0, we get the relation

$$a = -kx, \tag{1.10}$$

but since $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$, we can re-write Equation 1.10 as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx, \tag{1.11}$$

or even more succinctly as

$$\ddot{x} = -kx. \tag{1.12}$$

Equation 1.12 is one of the simplest possible 2nd-order *ordinary* differential equations. Its solution is a combination of the two basic trigonometric equations:

$$x(t) = c_1 \sin(\alpha t) + c_2 \cos(\beta t), \tag{1.13}$$

where c_1 and c_2 are constants which we can find using the *starting conditions* (see note 1.3), and α, β are parameters of the motion. Since function arguments must be unitless, these two parameters also cause the total quantity inside the trigonometric functions to be unitless by having units corresponding to $\left[\frac{1}{\text{time}}\right]$. For example, if we measure the time in $[\mathrm{s}]$, then the units of α and β are $\left[\mathrm{s}^{-1}\right] = [\mathrm{Hz}]$.

Note 1.3 Starting conditions for solving ODEs

Recall that in order to completely solve an ordinary differential equation of order n we must have n starting conditions.

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In the case where the initial position of the mass is x_0 and the initial velocity is $\dot{x}_0 = 0$ we get the simplified solution

$$x(t) = x_0 \cos(\omega t), \tag{1.14}$$

where $\omega = \sqrt{\frac{k}{m}}$. The plot of the position $x(t)$ of the mass is given in Figure 1.4. Note how, when displayed side-by-side using the same time axes, the position plot is "lagging behind" the velocity plot by $\phi = \frac{\pi}{2}$: when we start the motion, the velocity is 0 and then starts to increase in the negative direction (i.e. the mass is moving to the left). The position is at its maximum at that time, and decreases from $x_{\max}$ at $t = 0$ to $x = 0$ at $t = \frac{\pi}{2}$, while still staying positive. At $t = \frac{\pi}{2}$ the velocity reaches its maximum negative value, $\dot{x} = -v_{\max}$ (which is 1 in units of $\frac{1}{v_{\max}}$), and the position becomes negative (since it is to the left of the rest position of the spring).

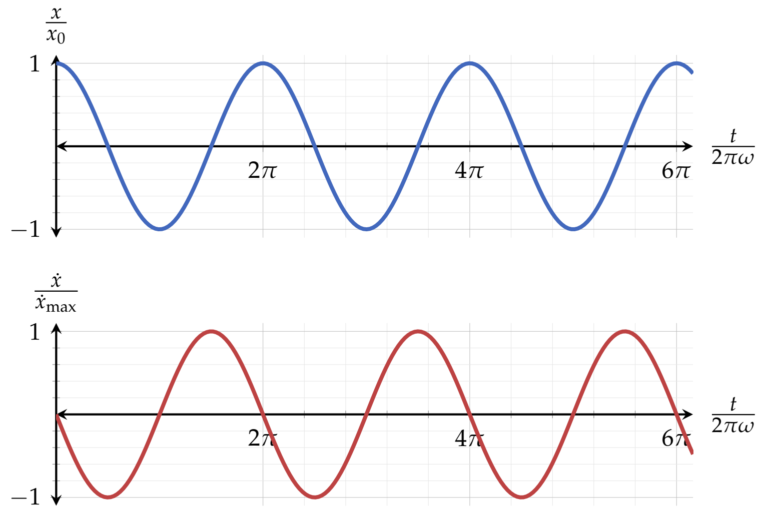

Figure 1.4: Simple harmonic oscillator, top (blue): position x vs. time t. The axes units are such that a full period of the oscillation takes $\Delta t = 2\pi$, and that the minimum and maximum values of the position are ± 1, respectively. Bottom (red): velocity vs. time on the same time axes, and a velocity axis which is scaled such that $v_{\max} = 1$.

It would be useful to understand how does the spring-mass system evolve given a specific combination of position and velocity. This can be done by plotting the **phase space** of the system (Figure 1.5): on the horizontal axis we specify the position x of the mass, and on the vertical axis the velocity $\dot{x}$.

In the case of a simple harmonic oscialltor, the evolution of the system is shown on the phase space plot as ellipses - and when we use normalized coordinates ($\tilde{x} = \frac{x}{x_0}$ and $\tilde{\dot{x}} = \frac{\dot{x}}{\dot{X}_{\max}}$), the ellipses turn into perfect circles - just like we see in Figure 1.5. These circles or ellipses are paths of constant energy: this can be seen when we "translate" the plot to have the potential energy and kinetic energy as horizontal and vertical axes, respectively.

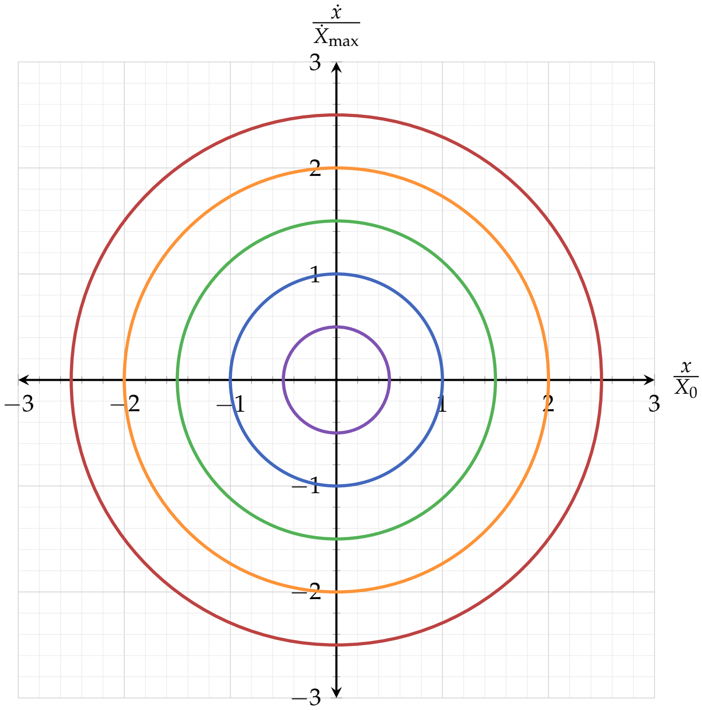

Figure 1.5: Phase space plot of simple harmonic oscillators with different momenta (either their masses are different, or their initial distance are different). The axes are scaled such that their units are the initial position X_0 and maximum velocity $\dot{X}_0$, respectively, of the second oscillator (trajectory drawn in blue).

Recall that for close mechanical systems, the total energy E is given by the sum of the sum of the *kinetic energy* K and the *potential energy* U:

$$E = K + U. \tag{1.15}$$

The kinetic energy is a function of the velocity $v = \dot{x}$:

$$K = \frac{1}{2}\dot{x}^2 = \frac{p^2}{2m}, \tag{1.16}$$

and the potential energy is a function of the position x via the force F:

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x}. \tag{1.17}$$

In the case of an harmonic oscillator $F = -kx$, and therefore

$$U = \int F\,\mathrm{d}x = -k\int x\,\mathrm{d}x = -\frac{1}{2}kx^2 + c. \tag{1.18}$$

Since we can add to the potential any constant with no change to the force derived from it (remember that the derivative of a constant is zero), we can simply set the potential at $x = 0$ to $U(x=0) = 0$, meaning that the integral constant is $c = 0$. Altogether, we get the following system energy:

$$E = \frac{p}{2m} - \frac{1}{2}kx^2, \tag{1.19}$$

Essentially, this is equal to applying the following transformations to the phase space plot:

$$x \to -\frac{1}{2}kx^2, \tag{1.20}$$

$$\dot{x} \to \frac{p^2}{2m} = \frac{1}{2m}m^2\dot{x}^2 = \frac{1}{2}m\dot{x}^2. \tag{1.21}$$

Since the transformation “streches” both axes by the same power (up to the constants m and k), the shapes remain the same as in the original plot.

The total energy of the system can then be extracted from the radius of each circle path:

$$E = \sqrt{K^2 + U^2} = \sqrt{\frac{p^2}{4m^2} + \frac{k^2x^4}{4}} = \frac{1}{2}\left(\frac{p^2}{m^2} + k^2x^4\right). \tag{1.22}$$

When $K = 0$ (i.e. $p = 0$) the entire energy is stored as potential energy:

$$E = \frac{1}{2}\sqrt{k^2x^4} = \frac{1}{2}kx^2. \tag{1.23}$$

And when $U = 0$ (i.e. $x = 0$) the entire energy is stored as kinetic energy:

$$E = \frac{1}{2}\sqrt{\frac{p^2}{m^2}} = \frac{p}{2m} = \frac{1}{2}m\dot{x}^2. \tag{1.24}$$

1.5.2 *Damped Harmonic Oscillator*

We can make the harmonic oscillator model a bit more realistic if we add a damping force proportional to the velocity of the mass. This corresponds e.g. to introducing friction into the model. The form of the damping force is $F_{\text{damp}}(t) = -cv(t) = -c\dot{x}$, for some real damping coefficient $c \geq 0$ (in the case where $c = 0$ we get back the simple harmonic oscillator model). Thus, the overall Newton 2nd law equation of the system has the following form:

$$m\ddot{x} = -c\dot{x} - kx. \tag{1.25}$$

The term $-c\dot{x}$ always acts in the opposite direction to the velocity due to the minus sign. Re-arranging Equation 1.25 we get the more “canonical” form

$$m\ddot{x} + c\dot{x} + kx = 0, \tag{1.26}$$

We then use the solution $x(t) = e^{\lambda t}$, subtituting it into Equation 1.26:

$$\begin{aligned} F_{\text{total}}(t) &= \lambda^2 m e^{\lambda t} + c e^{\lambda t} + k e^{\lambda t} && (1.27)\\ &= e^{\lambda t}\left(m\lambda^2 + c\lambda + k\right) && (1.28)\\ &= 0. && (1.29) \end{aligned}$$

Since Equation 1.29 is true for all t (and in any case $e^{\lambda t} \neq 0$ for all t), for the equation to be true it must be that the quadratic eqaution in λ equals zero. Using the quadratic formula we get that

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}. \tag{1.30}$$

For the solutions to λ to be real numbers, $c^2 \geq 4km$ - otherwise the term in the square root is negative. When $c^2 > 4km$ we call the system **overdamped**, and the case where $c^2 = 4km$ is the **critical damping** value. MORE TEXT.

The overall position vs. time relationship of the underdamped system is given by

$$x(t) = x_0 e^{-\frac{c}{2m}t} \cos(\omega t), \tag{1.31}$$

where as before $\omega = \sqrt{\frac{k}{m}}$ (see Figure 1.6).

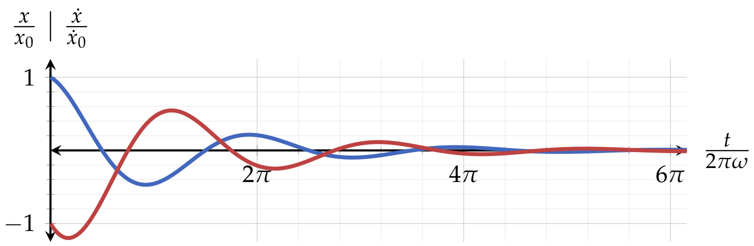

Figure 1.6: Underdamped harmonic oscillation...

The phase space plot of the damped system shows...

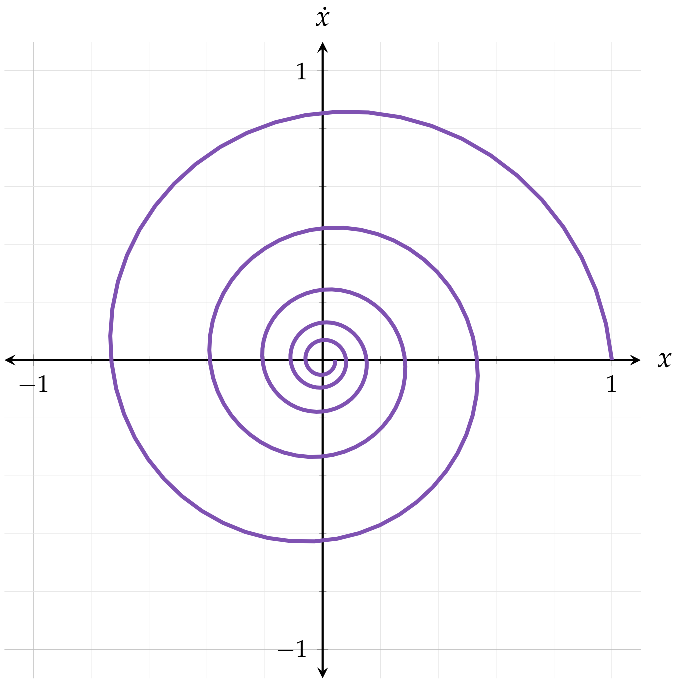

Figure 1.7: Phase space plot of an underdamped harmonic oscillator...

1.5.3 *Simulating Harmonic Oscillators in Python*

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2

Simulating Orbital Mechanics

2.1 Preface

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2.2 Relevant Physical and Mathetmatical Background

2.2.1 Classic Gravitational Force

Already in the 17th century, *Isaac Newton* formulated the gravitational force existing between any two objects with masses greater than zero. The strength of the force is given by the equation

$$F = G\frac{m_1 m_2}{r^2}, \tag{2.1}$$

where m_1 and m_2 are the respective masses of the two objects, r is the distance between them, and G is a the *universal gravitational constant*,

$$G = (6.6743 \pm 0.0015) \times 10^{-11} \left[\mathrm{N\,m^2\,kg^{-2}}\right] \tag{2.2}$$

The direction of the force is the line connecting the centers of mass of the two objects. Due to Newton's third law, the forces acting on the two objects are equal and opposite: the force applied by m_1 on m_2, $F_{1\to2}$, is pointing **from** m_2 **onto** m_1, and the force applied by m_2 on m_1, $F_{2\to1}$ is pointing **from** m_1 **onto** m_2 - and is exactly opposite to $F_{1\to2}$, i.e. in vector notation

$$\vec{F}_{1\to2} = -\vec{F}_{2\to1}. \tag{2.3}$$

If the two objects have positions $\vec{r}_1$ and $\vec{r}_2$, the vector pointing from object 1 to object 2 is

$$\vec{r}_{1\to2} = \vec{r}_2 - \vec{r}_1, \tag{2.4}$$

with the vector pointing from object 2 to object 1 having the exact opposite components, i.e. $\vec{r}_{2\to1} = -\vec{r}_{1\to2}$. The norms of $\vec{r}_{1\to2}$

and $\vec{r}_{2\to1}$ are simply r (the distance between the objects), and their directions are the unit vectors in the direction of $\vec{r}_{1\to2}$ and $\vec{r}_{2\to1}$, respectively:

$$\begin{aligned}\hat{r}_{1\to2} &= \frac{\vec{r}_{1\to2}}{\|\vec{r}\|_{1\to2}} = \frac{\vec{r}_{1\to2}}{r},\\ \hat{r}_{2\to1} &= \frac{\vec{r}_{2\to1}}{\|\vec{r}\|_{2\to1}} = \frac{\vec{r}_{2\to1}}{r} = -\hat{r}_{1\to2}.\end{aligned} \tag{2.5}$$

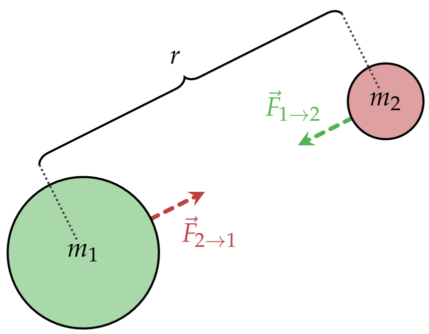

Figure 2.1: Gravitational force between two objects with masses m_1 and m_2. Each object applies an attrctive force on the other object, with norm $F = G\frac{m_1m_2}{r^2}$ (where r is the distance between the objects) and in the direction pointing from each object to the other object.

In total, the vector notation of the gravitational force applied by the objects on each other are

$$\begin{aligned}\vec{F}_{1\to2} &= Gm_1m_2\frac{\hat{r}_{1\to2}}{r^2},\\ \vec{F}_{2\to1} &= Gm_1m_2\frac{\hat{r}_{2\to1}}{r^2} = -\vec{F}_{1\to2}.\end{aligned} \tag{2.6}$$

Note 2.1 Another gravity force vector notation

In some textbooks, Equation 2.6 are written without the unit vectors $\hat{r}_{1\to2}$ and $\hat{r}_{2\to1}$, instead using the distance vectors and dividing by r^3, i.e.

$$\begin{aligned}\vec{F}_{1\to2} &= Gm_1m_2\frac{\vec{r}_{1\to2}}{r^3},\\ \vec{F}_{2\to1} &= Gm_2m_1\frac{\vec{r}_{2\to1}}{r^3}.\end{aligned}$$

The result is of course the same as in Equation 2.6, since for any non zero vector $\vec{v}$,

$$\frac{\vec{v}}{\|\vec{v}\|^3} = \frac{1}{\|\vec{v}\|^2}\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|^2}\hat{v}.$$

!

Let us look at an example of calculating the gravitational forces between two objects.

Example 2.1 Calculating a gravitational force

Let us calculate the gravitational forces between two objects

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continuing from the previous page

A and B, using the following parameters:

$$\vec{r}_A = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},\ m_A = 1,$$

$$\vec{r}_B = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix},\ m_B = 2.$$

For the sake of simplicity, we use $G = 1$ and don't consider units with this example.
The vector pointing from A to B is

$$\vec{r}_{A\to B} = \vec{B} - \vec{A} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix},$$

and the vector pointing from B to A is

$$\vec{r}_{B\to A} = -\vec{r}_{A\to B} = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}.$$

The disntace r between the objects is the norm of either of the above vectors, so we'll use $\vec{r}_{A\to B}$:

$$r = \|\vec{r}\|_{A\to B} = \sqrt{1^2 + 7^2 + 2^2} = \sqrt{19} \approx 7.3485.$$

The direction vectors are therefore

$$\hat{r}_{A\to B} = \frac{1}{7.3485} \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.1361 \\ 0.9526 \\ -0.2722 \end{bmatrix},$$

$$\hat{r}_{B\to A} = -\hat{r}_{A\to B} = \begin{bmatrix} -0.1361 \\ -0.9526 \\ 0.2722 \end{bmatrix}.$$

The gravity force which A applies onto B is then

$$\vec{F}_{A\to B} = \cancel{G}\frac{\overbrace{m_1 m_2}^{=2\times 1}}{r^2}\hat{r}_{A\to B} = \frac{2}{54} \begin{bmatrix} 0.1361 \\ 0.9526 \\ -0.2722 \end{bmatrix} = \begin{bmatrix} 0.0050 \\ 0.0353 \\ -0.101 \end{bmatrix}.$$

and similarily,

$$\vec{F}_{B\to A} = -\vec{F}_{A\to B} = \begin{bmatrix} -0.0050 \\ -0.0353 \\ 0.101 \end{bmatrix}.$$

In the case where we only consider two objects, and choose our frame of reference such that one of the objects is stationary - an analytical solution to the spatial trajectory taken by the second object is known and well studied. It is called a **Keplerian orbit**, and it always takes the form of a conic section. Let us take a short detour to discuss conic sections.

2.2.2 *Conic Sections*

A conic section (sometimes simply just called "a conic") is a 2-dimensional shape resulting from the intersection of a plane and a cone (see Figure 2.2). Depending on the angle α by which the plane intersects the cone relative to the the cone's side, the resulting shape can be one of 3 general types (here θ is the cone's angle):

1. If $\alpha > \theta$ the intersection is an **ellipse**. If in addition $\alpha = 90°$ the ellipse becomes a **circle**.

2. If $\alpha = \theta$ the intersection is a **parabola**.

3. If $\alpha < \theta$ the intersection is a **hyperbola**.

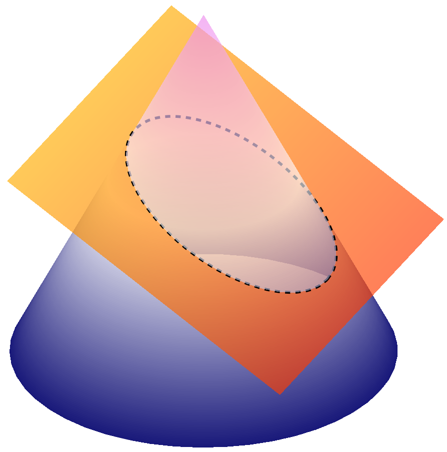

Figure 2.2: An intersection of a cone and a plane. Both the cone and plane are infinite - the cone extends infinitely "down", but also has a second "inverted" part on the top, also extending to infinity. In the case here shown, the intersection is an ellipse. Image reproduced with modifications from !SOURCE!

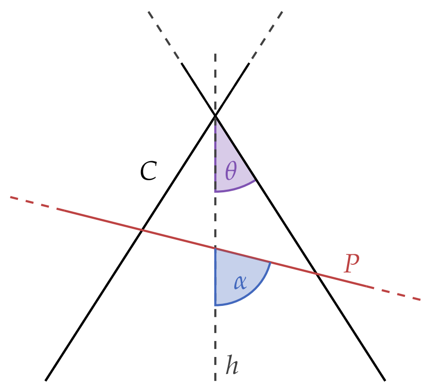

Figure 2.3: Side view of an infinite cone C and an infinite plane P intersecting it. The angle between P and the cone's height line h is α, and the angle between the cone's surface and h is θ. In this figure $0 < \theta < \alpha < 90°$, and thus the shape formed by the intersection of C and P is an ellipse.

Depending on the exact parameters of both C and P, the resulting conic section can be **degenerate** - either a point, a line or two intersection lines. This happens if P goes throgh the vertex point of C: if $\alpha = 90°$ the result is a single point [1], if $\alpha = \theta$ the result is a

[1] one can understand this as being a circle with radius $r = 0$.

single line, and if $\alpha > \theta$ the result is two intersecting lines.

Geometric Properties of Conic Sections

Of the non-degenerate conic sections, the ellipse is the only closed curve. Both the parabola and hyperbola are open: in essence, this means that they diverge to infinity. A common geometric definition for all conic sections is the following: given a line L (called the **directrix**) and a point F (called the **locus**), a conic section is the set C of all points $\{p\}$ for which the distance Fp is equal to a constant multiple of the distance Lp:

$$C = \left\{ p \mid |Fp| = e\,|Lp| \right\}. \tag{2.7}$$

The constant e is called the **eccentricity** of the conic section.

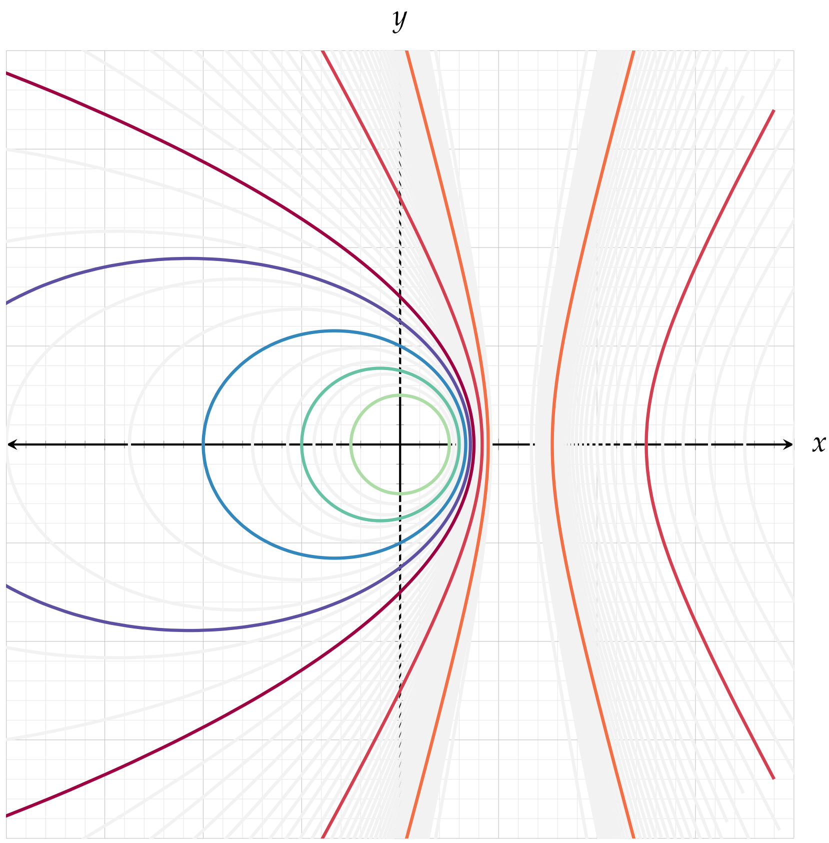

Figure 2.4: Several conic sections with different eccentricities $\{e_i\}$ and the same locus F and directrix L. (NOTE: figure still WIP)

MORE TEXT HERE

Cartesian coefficients

All conic sections can be expressed as the solutions to the following general equation in $\mathbb{R}^2$:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{2.8}$$

where A, B, C, D, E, F are all real coefficients such that A, B and C are all nonzero. The above equation can be written in matrix form as

$$\begin{bmatrix} x, & y, & 1 \end{bmatrix} \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0. \tag{2.9}$$

In this form, the different typs of conic sections arise from the sign of the term

$$\Delta = B^2 - 4AC, \tag{2.10}$$

called the **discriminent** of the conic equation, as following:

1. If $B^2 - 4AC < 0$ the equation represents an ellipse. If in addition $A = C$ and $B = 0$ the discriminent collapses to $-A^2$ - which represents a circle.

2. If $B^2 - 4AC = 0$, the equation represents a parabola.

3. If $B^2 - 4AC > 0$, the equation represents a hyperbola.

Note that the discriminent can be represented as a 2×2 determinant derived from the matrix form of the Cartesian coefficients:

$$\Delta = -4 \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix}. \tag{2.11}$$

Conic Section From 5 Points

If we know 5 points lying on a conic,

$$\begin{cases} \boldsymbol{p}_1 = (x_1, y_1) \\ \boldsymbol{p}_2 = (x_2, y_2) \\ \vdots \\ \boldsymbol{p}_5 = (x_5, y_5)\,, \end{cases} \tag{2.12}$$

we can determine all the conic Cartesian coefficients by solving the equation

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.13}$$

which can be done by finding the null space of the matrix in the above equation.

Some More About Ellipses

Unlike the other conic sections, an ellipse has two focus points (simply called its **foci**): F_1 and F_2. One of these always corresponds to the conic section definition of the focus. In an ellipse, the sum of the distances from any point to the two foci is always constant. In a sense, an ellipse is an elognated circle: instead of having a single radius, it has two orthogonal **axes**: the semi-major axis a and the semi-minor axis b (such that $a \geq b$).

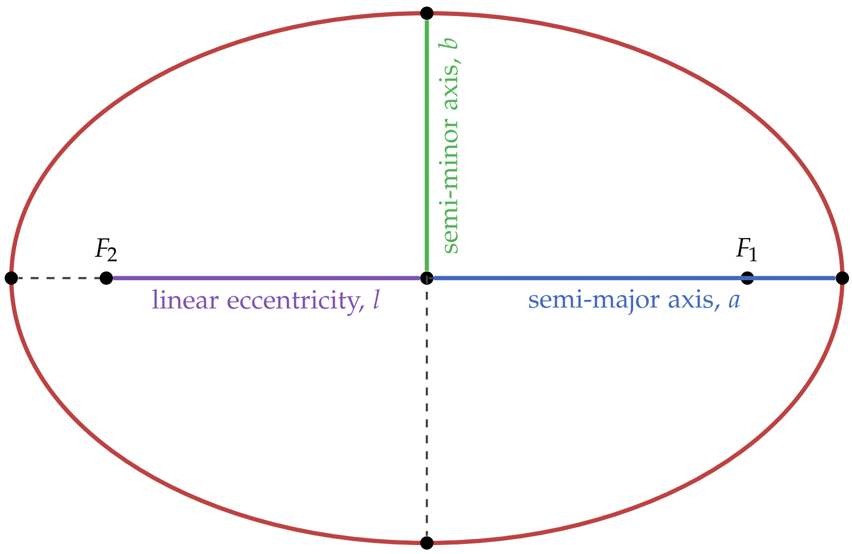

Figure 2.5: Some common geometric properties of an ellipse. NOTE: figure still WIP

The two foci are at a distance of $c = \sqrt{a^2 - b^2}$ away from the center of the ellipse, a measure that also called **linear eccentricity**. The eccentricity of the ellipse is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}}. \tag{2.14}$$

As mentioned already, in the case of a circle $e = 0$, and we get that $c = 0$ and that $\sqrt{1 - \frac{b^2}{a^2}} = 0$, i.e. $a = b$. The first equality means that the foci are located at the center of the ellipse, and the second equality means that the semi-major and semi-minor axes of the ellipse are the same. This is exactly what we excpect for a circle.

2.2.3 *Orbital Shapes*

Back from our short detour, we can now discuss the Keplerian orbits in more details. Suppose there are two objects: the first has mass m_1, and second mass m_2. For simplicity we will assume that $m_1 \gg m_2$, such that we can assume it is stationary, while the second object experiences the Keplerian orbit - consider for example a satellite orbiting the earth. We will indeed from now on refer to the first (more massive) object simply as the "massive object", and the second object as the "satellite".

Although space is 3-dimensional, a Keplerian orbit is always 2-dimensional - since as mentioned before, it is always a conic section. The plane on which the orbit takes place, the **orbital plane**, is determined by the direction of velocity of the satellite and the direction connecting the centers of mass of the massive object and the satellite. The massive object is found at the locus of the conic section. The relative values of m_1 and the angular momentum of the satellite determine the eccentricity of the orbit.

For now, let us concentrate on an elliptic orbit, i.e. where $0 \leq e < 1$. As mentioned, in such an orbit the massive object is at the locus of the ellipse, which is one of its foci - we will call it F_1 here. The point of closest to F_1 on the elliptical orbit is called the **periapsis** and denoted r_p. The point directly opposite r_p, i.e. the point closest to the second focus F_2 of the ellipse and furthest away from F_1 is called the **apoapsis**, denoted r_a (Figure 2.6). The direction from F_1 to the periapsis point is (unsuprisingly) called the **direction of periapsis**.

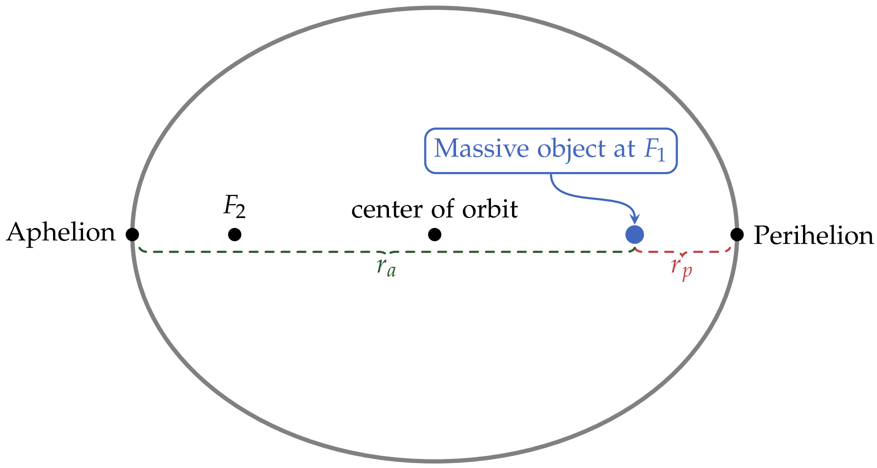

Figure 2.6: Common terms in elliptical orbits.

The relation between the eccentricity of the orbit and the two distances r_p, r_a is

$$e = \frac{r_a - r_p}{r_a + r_p} = 1 - \frac{2}{\frac{r_a}{r_p} + 1}. \tag{2.15}$$

Conversly, we can write the above relation as

$$\frac{r_a}{r_p} = \frac{1+e}{1-e}. \tag{2.16}$$

Example 2.2 Earth's orbit around the sun

In the sun-earth system, where the sun is the massive object and the earth is the satellite, the periapsis distance is $r_p = 1.470\,984\,50 \times 10^{11}$ [m], and the apoapsis distance is $r_a = 1.520\,975\,97 \times 10^{11}$ [m]. Therefore, the orbital eccentric-

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ity of Earth's orbit around the sun is about

$$\begin{aligned} e &= 1 - \frac{2}{\frac{r_a}{r_p} + 1} = 1 - \frac{2}{\frac{1.52097597}{1.47098450} + 1} \\ &= 1 - \frac{2}{1.033985 + 1} = 1 - \frac{2}{2.033985} \\ &= 1 - 0.983291 = 0.016709. \end{aligned}$$

This is a pretty round orbit (not a scientific term).

At any point in a spatial trajectory, the velocity of the object in motion is always tangent to the trajectory it traces. In the case of a Keplerian orbit, the angle between the direction of periapsis and the line connecting F_1 and the position of the satellite $r(t)$ at some given time t is called the **true anomaly**, denoted as θ.

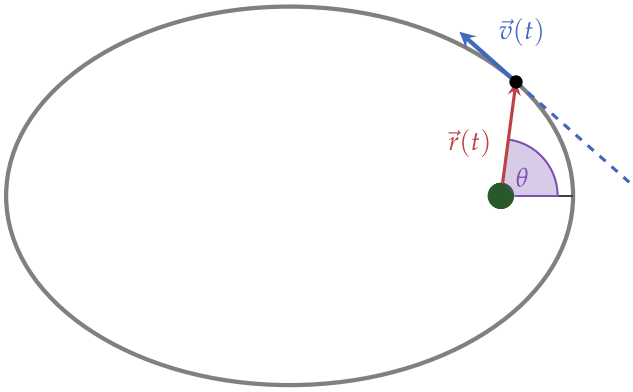

Figure 2.7: Position $\vec{r}$, velocity $\vec{v}$ and true anomaly θ in an elliptic orbit. Notice how the velocity is tangent to the trajectory.

In the 17th century, **Johannes Kepler** formalized his **laws of planetary motion**. There are three of them, and we already met the first one: in a 2-body system, the trajectory of the satellite object under gravity is a conic section. His second law states that in any time period Δt along its path, the satellite will sweep a constant area A in relation to the massive object (see Figure 2.8). A direct result of this law is that the satellite moves faster when close to the massive object, and slower when it is far away.

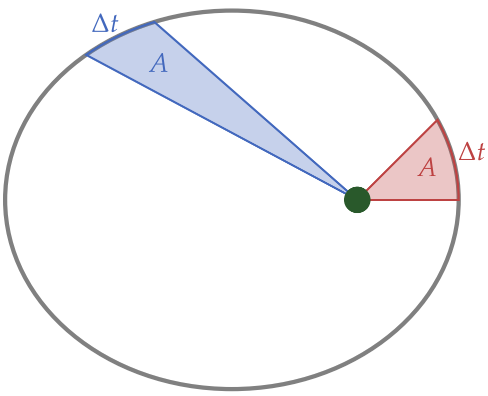

Figure 2.8: Kepler's second law: at any constant time period Δt, the satellite sweeps a constant area A defined between its two positions at times t and $t + \Delta t$ and the massive object.

Kepler's third law states that for a given massive object, there is a constant relation between a^3, the semi-major axis of the orbit to third power - and T^2, the orbital period (the time it takes the satellite to complete a single orbit), given by

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2}. \tag{2.17}$$

2.2.4 *Multiple Objects*

We can extend the 2-body problem into an n-body problem: we define a system with n objects of masses $m_1, m_2, \ldots, m_n$, positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ and velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. The total gravitational force experienced by the i-th object is the sum of all of the gravitational forces from the other objects:

$$\vec{F}_i = \sum_{j \neq i} \vec{F}_{j\to i} = G \sum_{j \neq i} \frac{m_i m_j}{r_{ij}^2} \hat{r}_{ij}. \tag{2.18}$$

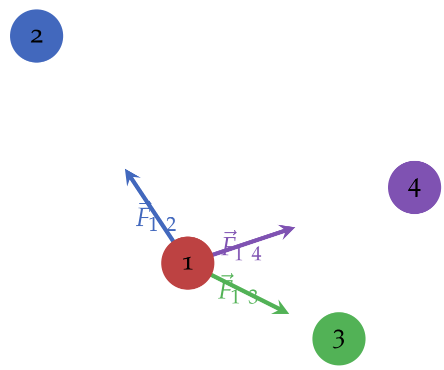

Figure 2.9: A 4-body system, showing the total gravitational force acting on one of the objects as the sum of the gravitational forces exerted on it by all other objects.

Unfortunately, there is no general analytical solution to this problem. We can either use approximations and special cases, or use numerical integration to simulate such a system. The next section deals exactly with these kinds of simulations.

2.3 *Integration Methods*

The term **integration** in the context of numerical simulations means the discrete propagation of a system. In the case of the n-body problem, an integration step includes for each object the following: first finding all the forces acting on the object, adding up all these forces, calculating the object's accelaration (according to Newton's second law) and thus its new velocity (since $a = \dot{v}$) - and finally using this velocity to calculate its new position (since $v = \dot{x}$).

The reason such a scheme is refered to as an "integration" is that we can reverse the derivative-connection between acceleration, velocity and position and write it as an integral:

$$x(t) = x(0) + \int_0^t v(t)\,\mathrm{d}t, \tag{2.19}$$

and in turn

$$v(t) = v(0) + \int_0^t a(t)\,\mathrm{d}t. \tag{2.20}$$

In the case of a constant acceleration, the above two equations add up together to give

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2, \tag{2.21}$$

where $x_0 = x(0)$ and $v_0 = v(0)$.

2.3.1 *Euler Method*

2.3.2 *Verlet Integration*

2.3.3 *Runge-Kutta Method*