PELEG BAR SAPIR

PHYSICS SIMULATIONS (LECTURE NOTES)

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1.5 Harmonic Oscillator

1.5.1 Simple Harmonic Oscillator

Many systems in physics present a simple, periodic (repeating) motion. One such system is a simple mass-less spring connected to a mass m and allowed to move in a single dimension only. If we ignore the effects of gravity, the only force acting on the mass arises from the spring itself: the more we pull or push the spring, the stronger it will resist to that change. This resistant force is given by

$$F = -kx, \tag{1.1}$$

where k is the **spring constant**, and x is the amount by which the spring contracts or expands relative to its rest position x_0. In Figure 1.1 we show three cases: when the mass is at the springs rest position, $x_m = x_0$, the spring applies no force on it. When the mass is displaced by a positive amount $\Delta x > 0$, the spring applied a *negative* force on it: $F = -k\Delta x < 0$ (see note 1.1). And when the mass is displaced such that it contracts the spring, $\Delta x < 0$ and thus the force applied by the spring is positive, $F = k\Delta x > 0$.

Note 1.1 Negative and positive forces

Recall that in this context, negative force means a force in the negative x direction.

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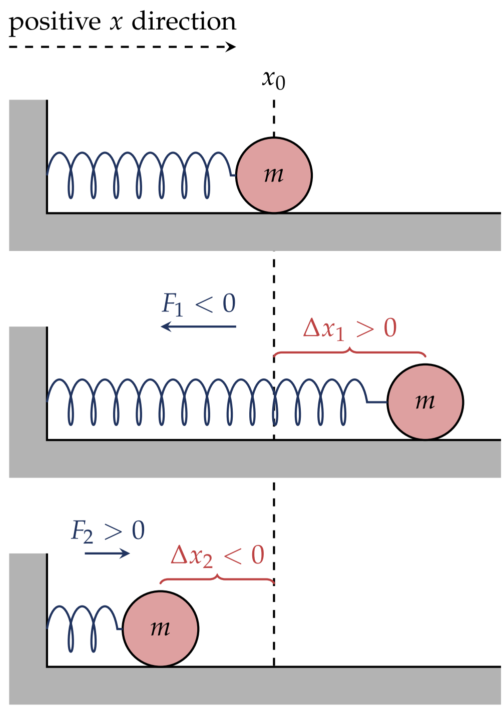

Figure 1.1: A simple spring-mass system with spring constant k and a mass m. The top figure shows the spring at rest - i.e. when the mass is located at position x_0 the spring applies no force on the mass (since $\Delta x = x_m - x_0 = 0$). The middle figure show the spring being at a *positive* displacement $\Delta x_1 > 0$, causing the spring to pull back with a negative force $F_1 = -k\Delta x_1$. The bottom picture shows the spring contracting by $\Delta x_2 < 0$, casing the spring to apply a positive force $F_2 = -k\Delta x_2$ on the mass.

Using Newtons second law of motion (REF?) with x being the displacement from the rest position x_0, we get the relation

$$a = -kx, \tag{1.2}$$

but since $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$, we can re-write Equation 1.2 as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx, \tag{1.3}$$

or even more succinctly as

$$\ddot{x} = -kx. \tag{1.4}$$

Equation 1.4 is one of the simplest possible 2nd-order *ordinary* differential equations. Its solution is a combination of the two basic trigonometric equations:

$$x(t) = c_1 \sin(\alpha t) + c_2 \cos(\beta t), \tag{1.5}$$

where c_1 and c_2 are constants which we can find using the *starting conditions* (see note 1.2), and α, β are parameters of the motion. Since function arguments must be unitless, these two parameters also cause the total quantity inside the trigonometric functions to be unitless by having units corresponding to $\left[\frac{1}{\text{time}}\right]$. For example, if we measure the time in $[\text{s}]$, then the units of α and β are $\left[\text{s}^{-1}\right]$ = $[\text{Hz}]$.

Note 1.2 Starting conditions for solving ODEs

Recall that in order to completely solve an ordinary differential equation of order n we must have n starting conditions.

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In the case where the initial position of the mass is x_0 and the initial velocity is $\dot{x}_0 = 0$ we get the simplified solution

$$x(t) = x_0 \cos(\omega t), \tag{1.6}$$

where $\omega = \sqrt{\frac{k}{m}}$. The plot of the position $x(t)$ of the mass is given in Figure 1.2. Note how, when displayed side-by-side using the same time axes, the position plot is "lagging behind" the velocity plot by $\phi = \frac{\pi}{2}$: when we start the motion, the velocity is 0 and then starts to increase in the negative direction (i.e. the mass is moving to the left). The position is at its maximum at that time, and decreases from x_{max} at $t = 0$ to $x = 0$ at $t = \frac{\pi}{2}$, while still staying positive. At $t = \frac{\pi}{2}$ the velocity reaches its maximum negative value, $\dot{x} = -v_{\text{max}}$ (which is 1 in units of $\frac{1}{v_{\text{max}}}$), and the position becomes negative (since it is to the left of the rest position of the spring).

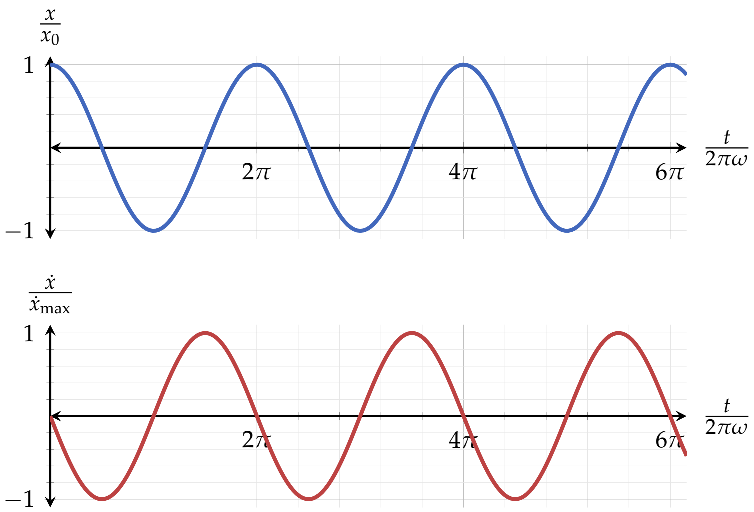

Figure 1.2: Simple harmonic oscillator, top (blue): position x vs. time t. The axes units are such that a full period of the oscillation takes $\Delta t = 2\pi$, and that the minimum and maximum values of the position are ± 1, respectively. Bottom (red): velocity vs. time on the same time axes, and a velocity axis which is scaled such that $v_{\text{max}} = 1$.

It would be useful to understand how does the spring-mass system evolve given a specific combination of position and velocity. This can be done by plotting the **phase space** of the system (Figure 1.3): on the horizontal axis we specify the position x of the mass, and on the vertical axis the velocity $\dot{x}$.

In the case of a simple harmonic oscialltor, the evolution of the system is shown on the phase space plot as ellipses - and when we use normalized coordinates ($\tilde{x} = \frac{x}{x_0}$ and $\tilde{\dot{x}} = \frac{\dot{x}}{\dot{X}_{\text{max}}}$), the ellipses turn into perfect circles - just like we see in Figure 1.3. These circles

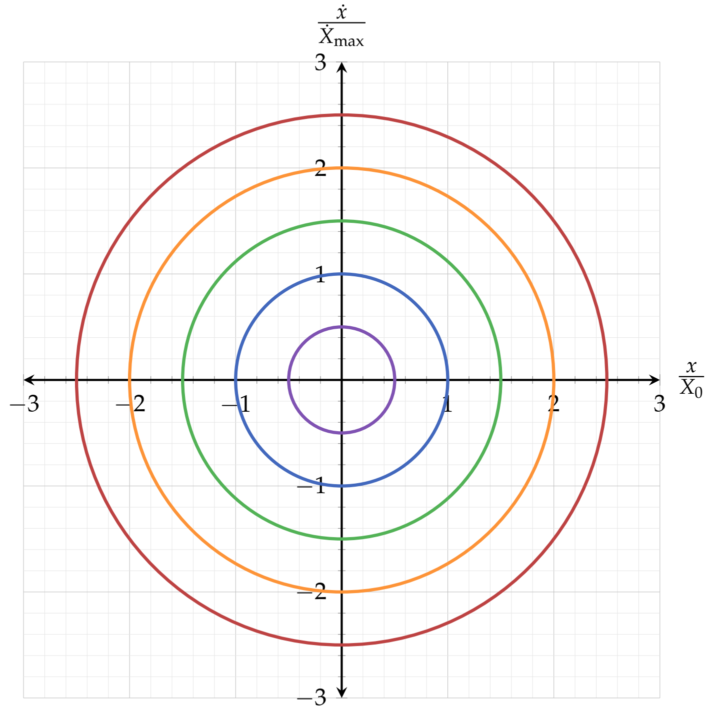

Figure 1.3: Phase space plot of simple harmonic oscillators with different momenta (either their masses are different, or their initial distance are different). The axes are scaled such that their units are the initial position X_0 and maximum velocity $\dot{X}_0$, respectively, of the second oscillator (trajectory drawn in blue).

or ellipses are paths of constant energy: this can be seen when we "translate" the plot to have the potential energy and kinetic energy as horizontal and vertical axes, respectively.

Recall that for close mechanical systems, the total energy E is given by the sum of the sum of the *kinetic energy* K and the *potential energy* U:

$$E = K + U. \tag{1.7}$$

The kinetic energy is a function of the velocity $v = \dot{x}$:

$$K = \frac{1}{2}\dot{x}^2 = \frac{p^2}{2m}, \tag{1.8}$$

and the potential energy is a function of the position x via the force F:

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x}. \tag{1.9}$$

In the case of an harmonic oscillator $F = -kx$, and therefore

$$U = \int F\,\mathrm{d}x = -k\int x\,\mathrm{d}x = -\frac{1}{2}kx^2 + c. \tag{1.10}$$

Since we can add to the potential any constant with no change to the force derived from it (remember that the derivative of a constant is zero), we can simply set the potential at $x = 0$ to $U(x=0) = 0$, meaning that the integral constant is $c = 0$. Altogether, we get the following system energy:

$$E = \frac{p}{2m} - \frac{1}{2}kx^2, \tag{1.11}$$

Essentially, this is equal to applying the following transformations to the phase space plot:

$$x \to -\frac{1}{2}kx^2, \tag{1.12}$$

$$\dot{x} \to \frac{p^2}{2m} = \frac{1}{2m}m^2\dot{x}^2 = \frac{1}{2}m\dot{x}^2. \tag{1.13}$$

Since the transformation "streches" both axes by the same power (up to the constants m and k), the shapes remain the same as in the original plot.

The total energy of the system can then be extracted from the radius of each circle path:

$$E = \sqrt{K^2 + U^2} = \sqrt{\frac{p^2}{4m^2} + \frac{k^2x^4}{4}} = \frac{1}{2}\left(\frac{p^2}{m^2} + k^2x^4\right). \tag{1.14}$$

When $K = 0$ (i.e. $p = 0$) the entire energy is stored as potential energy:

$$E = \frac{1}{2}\sqrt{k^2x^4} = \frac{1}{2}kx^2. \tag{1.15}$$

And when $U = 0$ (i.e. $x = 0$) the entire energy is stored as kinetic energy:

$$E = \frac{1}{2}\sqrt{\frac{p^2}{m^2}} = \frac{p}{2m} = \frac{1}{2}m\dot{x}^2. \tag{1.16}$$

1.5.2 Damped Harmonic Oscillator

We can make the harmonic oscillator model a bit more realistic if we add a damping force proportional to the velocity of the mass. This corresponds e.g. to introducing friction into the model. The form of the damping force is $F_{\text{damp}}(t) = -cv(t) = -c\dot{x}$, for some real damping coefficient $c \geq 0$ (in the case where $c = 0$ we get back the simple harmonic oscillator model). Thus, the overall Newton 2nd law equation of the system has the following form:

$$m\ddot{x} = -c\dot{x} - kx. \tag{1.17}$$

The term $-c\dot{x}$ always acts in the opposite direction to the velocity due to the minus sign. Re-arranging Equation 1.17 we get the more "canonical" form

$$m\ddot{x} + c\dot{x} + kx = 0, \tag{1.18}$$

We then use the solution $x(t) = \mathrm{e}^{\lambda t}$, subtituting it into Equation 1.18:

$$\begin{aligned} F_{\text{total}}(t) &= \lambda^2 m\mathrm{e}^{\lambda t} + c\mathrm{e}^{\lambda t} + k\mathrm{e}^{\lambda t} && (1.19)\\ &= \mathrm{e}^{\lambda t}\left(m\lambda^2 + c\lambda + k\right) && (1.20)\\ &= 0. && (1.21) \end{aligned}$$

Since Equation 1.21 is true for all t (and in any case $e^{\lambda t} \neq 0$ for all t), for the equation to be true it must be that the quadratic eqaution in λ equals zero. Using the quadratic formula we get that

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}. \tag{1.22}$$

For the solutions to λ to be real numbers, $c^2 \geq 4km$ - otherwise the term in the square root is negative. When $c^2 > 4km$ we call the system **overdamped**, and the case where $c^2 = 4km$ is the **critical damping** value. MORE TEXT.

The overall position vs. time relationship of the underdamped system is given by

$$x(t) = x_0 e^{-\frac{c}{2m}t} \cos(\omega t), \tag{1.23}$$

where as before $\omega = \sqrt{\frac{k}{m}}$ (see Figure 1.4).

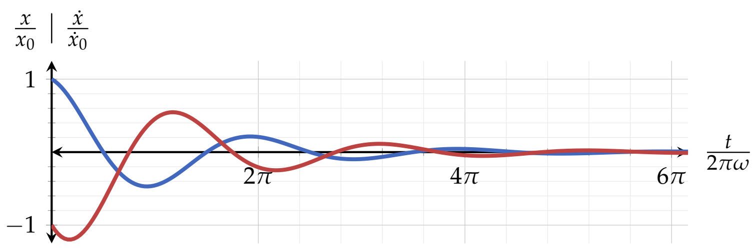

Figure 1.4: Underdamped harmonic oscillation...

The phase space plot of the damped system shows...

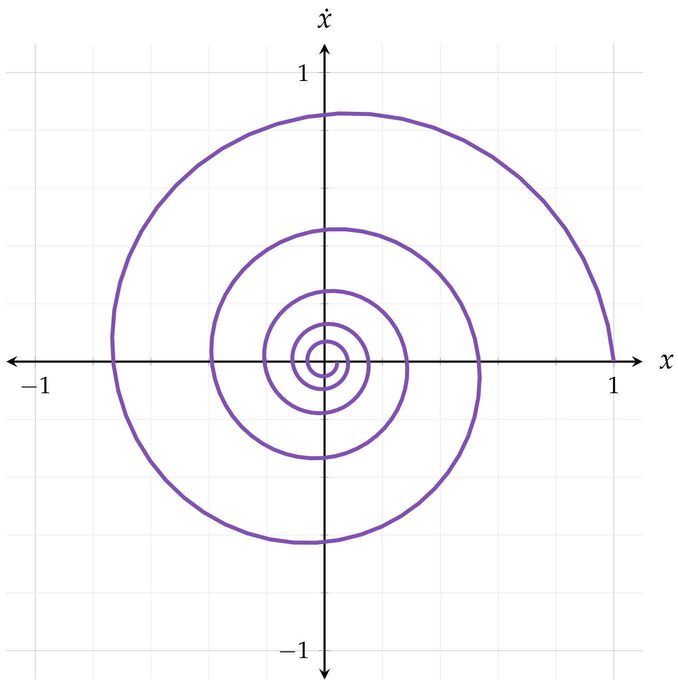

Figure 1.5: Phase space plot of an underdamped harmonic oscillator...

1.5.3 *Simulating Harmonic Oscillators in Python*

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Simulating Orbital Mechanics

2.1 Preface

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2.2 Relevant Physical and Mathetmatical Background

2.2.1 Classic Gravitational Force

Already in the 17th century, *Isaac Newton* formulated the gravitational force existing between any two objects with masses greater than zero. The strength of the force is given by the equation

$$F = G\frac{m_1 m_2}{r^2}, \tag{2.1}$$

where m_1 and m_2 are the respective masses of the two objects, r is the distance between them, and G is a the *universal gravitational constant*,

$$G = (6.6743 \pm 0.0015) \times 10^{-11} \left[\mathrm{N\,m^2\,kg^{-2}}\right] \tag{2.2}$$

The direction of the force is the line connecting the centers of mass of the two objects. Due to Newton's third law, the forces acting on the two objects are equal and opposite: the force applied by m_1 on m_2, $F_{1\to2}$, is pointing **from** m_2 **onto** m_1, and the force applied by m_2 on m_1, $F_{2\to1}$ is pointing **from** m_1 **onto** m_2 - and is exactly opposite to $F_{1\to2}$, i.e. in vector notation

$$\vec{F}_{1\to2} = -\vec{F}_{2\to1}. \tag{2.3}$$

If the two objects have positions $\vec{r}_1$ and $\vec{r}_2$, the vector pointing from object 1 to object 2 is

$$\vec{r}_{1\to2} = \vec{r}_2 - \vec{r}_1, \tag{2.4}$$

with the vector pointing from object 2 to object 1 having the exact opposite components, i.e. $\vec{r}_{2\to1} = -\vec{r}_{1\to2}$. The norms of $\vec{r}_{1\to2}$

and $\vec{r}_{2\to 1}$ are simply r (the distance between the objects), and their directions are the unit vectors in the direction of $\vec{r}_{1\to 2}$ and $\vec{r}_{2\to 1}$, respectively:

$$\begin{aligned}\hat{r}_{1\to 2} &= \frac{\vec{r}_{1\to 2}}{\|\vec{r}\|_{1\to 2}} = \frac{\vec{r}_{1\to 2}}{r},\\ \hat{r}_{2\to 1} &= \frac{\vec{r}_{2\to 1}}{\|\vec{r}\|_{2\to 1}} = \frac{\vec{r}_{2\to 1}}{r} = -\hat{r}_{1\to 2}.\end{aligned} \tag{2.5}$$

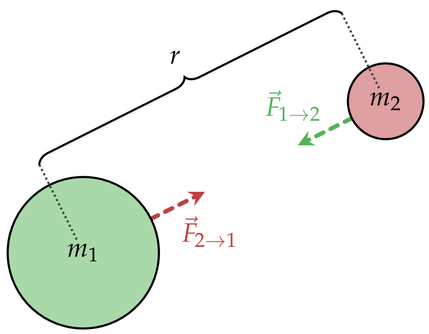

Figure 2.1: Gravitational force between two objects with masses m_1 and m_2. Each object applies an attrctive force on the other object, with norm $F = G\frac{m_1 m_2}{r^2}$ (where r is the distance between the objects) and in the direction pointing from each object to the other object.

In total, the vector notation of the gravitational force applied by the objects on each other are

$$\begin{aligned}\vec{F}_{1\to 2} &= Gm_1m_2\frac{\hat{r}_{1\to 2}}{r^2},\\ \vec{F}_{2\to 1} &= Gm_1m_2\frac{\hat{r}_{2\to 1}}{r^2} = -\vec{F}_{1\to 2}.\end{aligned} \tag{2.6}$$

Note 2.1 Another gravity force vector notation

In some textbooks, Equation 2.6 are written without the unit vectors $\hat{r}_{1\to 2}$ and $\hat{r}_{2\to 1}$, instead using the distance vectors and dividing by r^3, i.e.

$$\begin{aligned}\vec{F}_{1\to 2} &= Gm_1m_2\frac{\vec{r}_{1\to 2}}{r^3},\\ \vec{F}_{2\to 1} &= Gm_2m_1\frac{\vec{r}_{2\to 1}}{r^3}.\end{aligned}$$

The result is of course the same as in Equation 2.6, since for any non zero vector $\vec{v}$,

$$\frac{\vec{v}}{\|\vec{v}\|^3} = \frac{1}{\|\vec{v}\|^2}\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|^2}\hat{v}.$$

!

Let us look at an example of calculating the gravitational forces between two objects.

Example 2.1 Calculating a gravitational force

Let us calculate the gravitational forces between two objects

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A and B, using the following parameters:

$$\vec{r}_A = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},\ m_A = 1,$$

$$\vec{r}_B = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix},\ m_B = 2.$$

For the sake of simplicity, we use $G = 1$ and don't consider units with this example.
The vector pointing from A to B is

$$\vec{r}_{A\to B} = \vec{B} - \vec{A} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix},$$

and the vector pointing from B to A is

$$\vec{r}_{B\to A} = -\vec{r}_{A\to B} = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}.$$

The disntace r between the objects is the norm of either of the above vectors, so we'll use $\vec{r}_{A\to B}$:

$$r = \|\vec{r}\|_{A\to B} = \sqrt{1^2 + 7^2 + 2^2} = \sqrt{19} \approx 7.3485.$$

The direction vectors are therefore

$$\hat{r}_{A\to B} = \frac{1}{7.3485} \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.1361 \\ 0.9526 \\ -0.2722 \end{bmatrix},$$

$$\hat{r}_{B\to A} = -\hat{r}_{A\to B} = \begin{bmatrix} -0.1361 \\ -0.9526 \\ 0.2722 \end{bmatrix}.$$

The gravity force which A applies onto B is then

$$\vec{F}_{A\to B} = \cancel{G} \frac{\overbrace{m_1 m_2}^{=2\times 1}}{r^2} \hat{r}_{A\to B} = \frac{2}{54} \begin{bmatrix} 0.1361 \\ 0.9526 \\ -0.2722 \end{bmatrix} = \begin{bmatrix} 0.0050 \\ 0.0353 \\ -0.101 \end{bmatrix}.$$

and similarily,

$$\vec{F}_{B\to A} = -\vec{F}_{A\to B} = \begin{bmatrix} -0.0050 \\ -0.0353 \\ 0.101 \end{bmatrix}.$$

In the case where we only consider two objects, and choose our frame of reference such that one of the objects is stationary - an analytical solution to the spatial trajectory taken by the second object is known and well studied. It is called a **Keplerian orbit**, and it always takes the form of a conic section. Let us take a short detour to discuss conic sections.

2.2.2 *Conic Sections*

A conic section (sometimes simply just called "a conic") is a 2-dimensional shape resulting from the intersection of a plane and a cone (see Figure 2.2). Depending on the angle α by which the plane intersects the cone relative to the the cone's side, the resulting shape can be one of 3 general types (here θ is the cone's angle):

1. If $\alpha > \theta$ the intersection is an **ellipse**. If in addition $\alpha = 90°$ the ellipse becomes a **circle**.

2. If $\alpha = \theta$ the intersection is a **parabola**.

3. If $\alpha < \theta$ the intersection is a **hyperbola**.

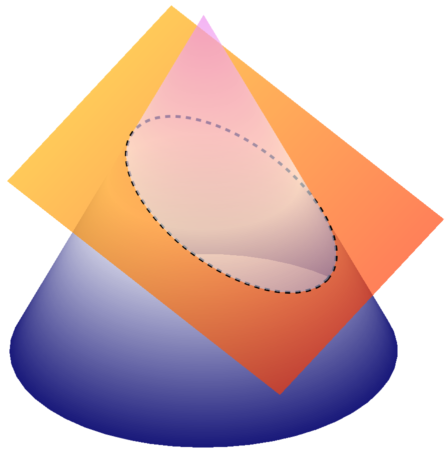

Figure 2.2: An intersection of a cone and a plane. Both the cone and plane are infinite - the cone extends infinitely "down", but also has a second "inverted" part on the top, also extending to infinity. In the case here shown, the intersection is an ellipse. Image reproduced with modifications from !SOURCE!

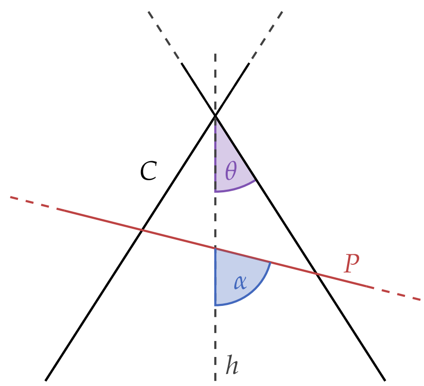

Figure 2.3: Side view of an infinite cone C and an infinite plane P intersecting it. The angle between P and the cone's height line h is α, and the angle between the cone's surface and h is θ. In this figure $0 < \theta < \alpha < 90°$, and thus the shape formed by the intersection of C and P is an ellipse.

Depending on the exact parameters of both C and P, the resulting conic section can be **degenerate** - either a point, a line or two intersection lines. This happens if P goes throgh the vertex point of C: if $\alpha = 90°$ the result is a single point [1], if $\alpha = \theta$ the result is a

[1] one can understand this as being a circle with radius $r = 0$.

single line, and if $\alpha > \theta$ the result is two intersecting lines.

Geometric Properties of Conic Sections

Of the non-degenerate conic sections, the ellipse is the only closed curve. Both the parabola and hyperbola are open: in essence, this means that they diverge to infinity. A common geometric definition for all conic sections is the following: given a line L (called the **directrix**) and a point F (called the **locus**), a conic section is the set C of all points $\{p\}$ for which the distance Fp is equal to a constant multiple of the distance Lp:

$$C = \left\{ p \mid |Fp| = e\,|Lp| \right\}. \tag{2.7}$$

The constant e is called the **eccentricity** of the conic section.

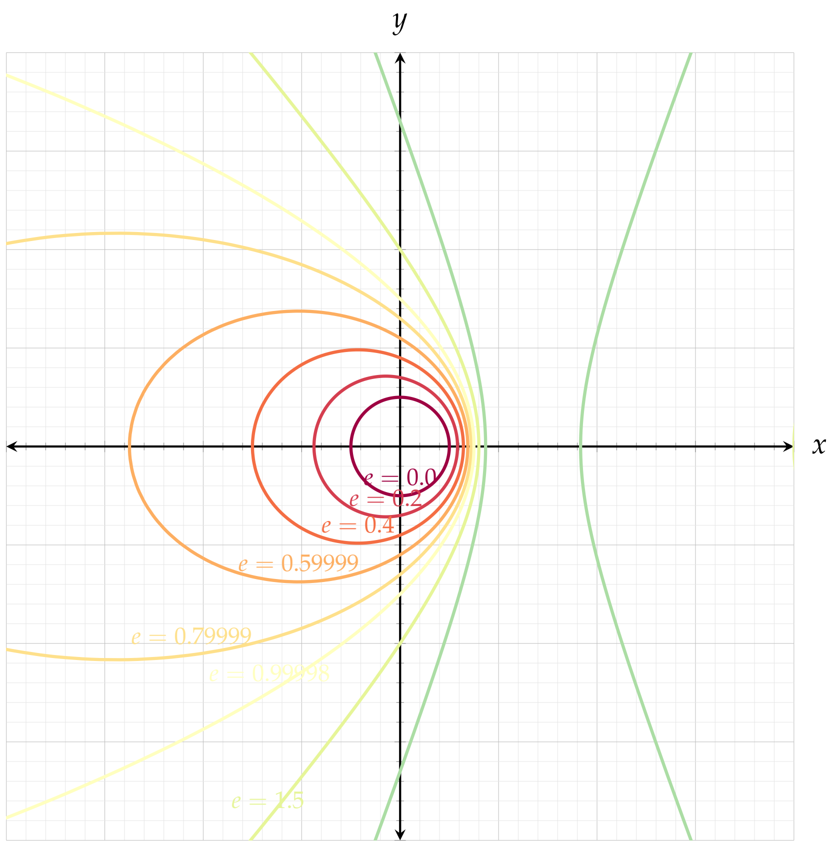

Figure 2.4: Several conic sections with different eccentricities $\{e_i\}$ and the same locus F and directrix L. (NOTE: figure still WIP)

MORE TEXT HERE

Cartesian coefficients

All conic sections can be expressed as the solutions to the following general equation in $\mathbb{R}^2$:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{2.8}$$

where A, B, C, D, E, F are all real coefficients such that A, B and C are all nonzero. The above equation can be written in matrix form as

$$\begin{bmatrix} x, & y, & 1 \end{bmatrix} \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0. \tag{2.9}$$

In this form, the different typs of conic sections arise from the sign of the term

$$\Delta = B^2 - 4AC, \tag{2.10}$$

called the **discriminent** of the conic equation, as following:

1. If $B^2 - 4AC < 0$ the equation represents an ellipse. If in addition $A = C$ and $B = 0$ the discriminent collapses to $-A^2$ - which represents a circle.

2. If $B^2 - 4AC = 0$, the equation represents a parabola.

3. If $B^2 - 4AC > 0$, the equation represents a hyperbola.

Note that the discriminent can be represented as a 2×2 determinant derived from the matrix form of the Cartesian coefficients:

$$\Delta = -4 \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix}. \tag{2.11}$$

Conic Section From 5 Points

If we know 5 points lying on a conic,

$$\begin{cases} \boldsymbol{p}_1 = (x_1, y_1) \\ \boldsymbol{p}_2 = (x_2, y_2) \\ \vdots \\ \boldsymbol{p}_5 = (x_5, y_5), \end{cases} \tag{2.12}$$

we can determine all the conic Cartesian coefficients by solving the equation

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.13}$$

which can be done by finding the null space of the matrix in the above equation.

Some More About Ellipses

Unlike the other conic sections, an ellipse has two focus points (simply called its **foci**): F_1 and F_2. One of these always corresponds to the conic section definition of the focus. In an ellipse, the sum of the distances from any point to the two foci is always constant. In a sense, an ellipse is an elognated circle: instead of having a single radius, it has two orthogonal **axes**: the semi-major axis a and the semi-minor axis b (such that $a \geq b$).

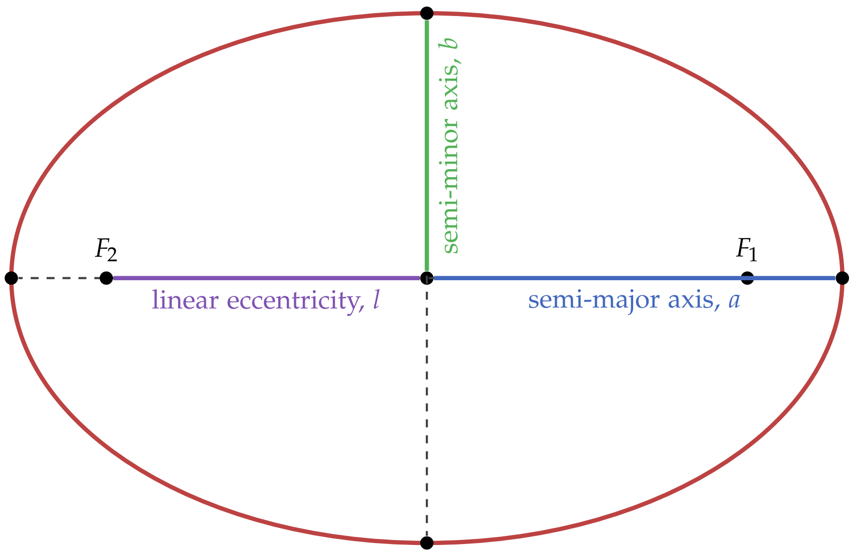

Figure 2.5: Some common geometric properties of an ellipse. NOTE: figure still WIP

The two foci are at a distance of $c = \sqrt{a^2 - b^2}$ away from the center of the ellipse, a measure that also called **linear eccentricity**. The eccentricity of the ellipse is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}}. \tag{2.14}$$

As mentioned already, in the case of a circle $e = 0$, and we get that $c = 0$ and that $\sqrt{1 - \frac{b^2}{a^2}} = 0$, i.e. $a = b$. The first equality means that the foci are located at the center of the ellipse, and the second equality means that the semi-major and semi-minor axes of the ellipse are the same. This is exactly what we excpect for a circle.

2.2.3 *Orbital Shapes*

Back from our short detour, we can now discuss the Keplerian orbits in more details. Suppose there are two objects: the first has mass m_a, and second mass m_b. For simplicity we will assume that $m_1 \gg m_2$, such that we can assume it is stationary, while the second object experiences the Keplerian orbit - consider for example a satellite orbiting the earth. We will indeed from now on refer to the first (more massive) object simply as the "massive object", and the second object as the "satellite".

Although space is 3-dimensional, a Keplerian orbit is always 2-dimensional - since as mentioned before, it is always a conic section. The plane on which the orbit takes place, the **orbital plane**, is determined by the direction of velocity of the satellite and the direction connecting the centers of mass of the massive object and the satellite. The massive object is found at the locus of the conic section. The relative values of m_1 and the angular momentum of the satellite determine the eccentricity of the orbit.

For now, let us concentrate on an elliptic orbit, i.e. where $0 \leq e < 1$. As mentioned, in such an orbit the massive object is at the locus of the ellipse, which is one of its foci - we will call it F_1 here. The point of closest to F_1 on the elliptical orbit is called the **periapsis** and denoted r_p. The point directly opposite r_p, i.e. the point closest to the second focus F_2 of the ellipse and furthest away from F_1 is called the **apoapsis**, denoted r_a (Figure 2.6).

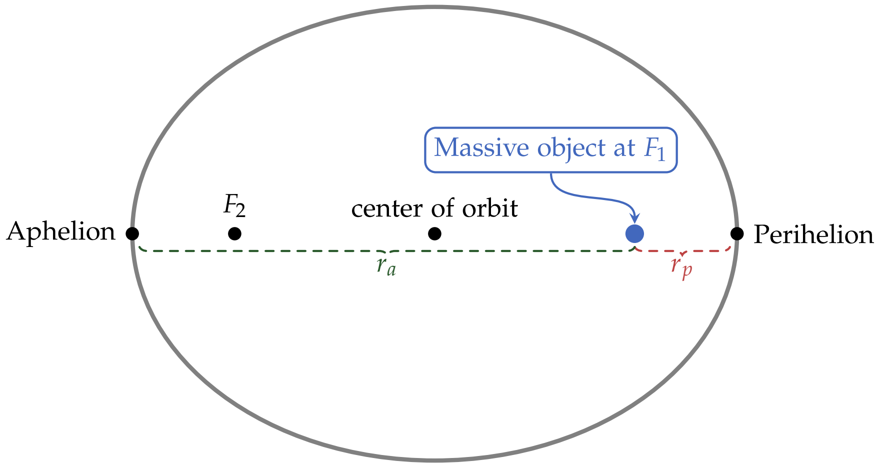

Figure 2.6: Common terms in elliptical orbits.

The relation between the eccentricity of the orbit and the two distances r_p, r_a is

$$e = \frac{r_a - r_p}{r_a + r_p} = 1 - \frac{2}{\frac{r_a}{r_p} + 1}. \tag{2.15}$$

Conversly, we can write the above relation as

$$\frac{r_a}{r_p} = \frac{1+e}{1-e}. \tag{2.16}$$

Example 2.2 Earth's orbit around the sun

In the sun-earth system, where the sun is the massive object and the earth is the satellite, the periapsis distance is $r_p = 1.470\,984\,50 \times 10^{11}$ [m], and the apoapsis distance is $r_a = 1.520\,975\,97 \times 10^{11}$ [m]. Therefore, the orbital eccentric-

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ity of Earth's orbit around the sun is about

$$
\begin{aligned}
e &= 1 - \frac{2}{\frac{r_a}{r_p} + 1} = 1 - \frac{2}{\frac{1.52097597}{1.47098450} + 1} \\
&= 1 - \frac{2}{1.033985 + 1} = 1 - \frac{2}{2.033985} \\
&= 1 - 0.983291 = 0.016709.
\end{aligned}
$$

This is a pretty round orbit (not a scientific term).

At any point in a Keplerian trajectory, the velocity of the satellite is always tangent to the trajectory. In the case...

2.2.4 *Multiple Objects*

2.3 *Integration Methods*

2.3.1 *Euler Method*

2.3.2 *Verlet Integration*

2.3.3 *Runge-Kutta Method*