PELEG BAR SAPIR

PHYSICS SIMULATIONS (LECTURE NOTES)

1

Simulating Orbital Mechanics

1.1 Preface

Text text text

1.2 Relevant Physical and Mathetmatical Background

1.2.1 Classic Gravitational Force

Already in the 17th century, *Isaac Newton* formulated the gravitational force existing between any two objects with masses greater than zero. The strength of the force is given by the equation

$$F = G\frac{m_1 m_2}{r^2}, \tag{1.1}$$

where m_1 and m_2 are the respective masses of the two objects, r is the distance between them, and G is a the *universal gravitational constant*,

$$G = (6.6743 \pm 0.0015) \times 10^{-11} \left[\mathrm{N\,m^2\,kg^{-2}}\right] \tag{1.2}$$

The direction of the force is the line connecting the centers of mass of the two objects. Due to Newton's third law, the forces acting on the two objects are equal and opposite: the force applied by m_1 on m_2, $F_{1\to2}$, is pointing **from** m_2 **onto** m_1, and the force applied by m_2 on m_1, $F_{2\to1}$ is pointing **from** m_1 **onto** m_2 - and is exactly opposite to $F_{1\to2}$, i.e. in vector notation

$$\vec{F}_{1\to2} = -\vec{F}_{2\to1}. \tag{1.3}$$

If the two objects have positions $\vec{r}_1$ and $\vec{r}_2$, the vector pointing from object 1 to object 2 is

$$\vec{r}_{1\to2} = \vec{r}_2 - \vec{r}_1, \tag{1.4}$$

with the vector pointing from object 2 to object 1 having the exact opposite components, i.e. $\vec{r}_{2\to1} = -\vec{r}_{1\to2}$. The norms of $\vec{r}_{1\to2}$

and $\vec{r}_{2\to1}$ are simply r (the distance between the objects), and their directions are the unit vectors in the direction of $\vec{r}_{1\to2}$ and $\vec{r}_{2\to1}$, respectively:

$$\begin{aligned}\hat{r}_{1\to2} &= \frac{\vec{r}_{1\to2}}{\|\vec{r}\|_{1\to2}} = \frac{\vec{r}_{1\to2}}{r},\\ \hat{r}_{2\to1} &= \frac{\vec{r}_{2\to1}}{\|\vec{r}\|_{2\to1}} = \frac{\vec{r}_{2\to1}}{r} = -\hat{r}_{1\to2}.\end{aligned} \tag{1.5}$$

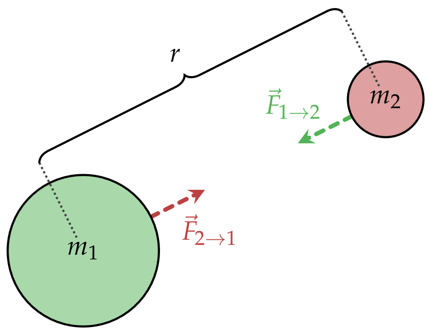

Figure 1.1: Gravitational force between two objects with masses m_1 and m_2. Each object applies an attrctive force on the other object, with norm $F = G\frac{m_1 m_2}{r^2}$ (where r is the distance between the objects) and in the direction pointing from each object to the other object.

In total, the vector notation of the gravitational force applied by the objects on each other are

$$\begin{aligned}\vec{F}_{1\to2} &= Gm_1m_2\frac{\hat{r}_{1\to2}}{r^2},\\ \vec{F}_{2\to1} &= Gm_1m_2\frac{\hat{r}_{2\to1}}{r^2} = -\vec{F}_{1\to2}.\end{aligned} \tag{1.6}$$

Note 1.1 Another gravity force vector notation

In some textbooks, Equation 1.6 are written without the unit vectors $\hat{r}_{1\to2}$ and $\hat{r}_{2\to1}$, instead using the distance vectors and dividing by r^3, i.e.

$$\begin{aligned}\vec{F}_{1\to2} &= Gm_1m_2\frac{\vec{r}_{1\to2}}{r^3},\\ \vec{F}_{2\to1} &= Gm_2m_1\frac{\vec{r}_{2\to1}}{r^3}.\end{aligned}$$

The result is of course the same as in Equation 1.6, since for any non zero vector $\vec{v}$,

$$\frac{\vec{v}}{\|\vec{v}\|^3} = \frac{1}{\|\vec{v}\|^2}\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|^2}\hat{v}.$$

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Let us look at an example of calculating the gravitational forces between two objects.

Example 1.1 Calculating a gravitational force

Let us calculate the gravitational forces between two objects

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A and B, using the following parameters:

$$\vec{r}_A = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},\ m_A = 1,$$
$$\vec{r}_B = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix},\ m_B = 2.$$

For the sake of simplicity, we use $G = 1$ and don't consider units with this example.
The vector pointing from A to B is

$$\vec{r}_{A\to B} = \vec{B} - \vec{A} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix},$$

and the vector pointing from B to A is

$$\vec{r}_{B\to A} = -\vec{r}_{A\to B} = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}.$$

The disntace r between the objects is the norm of either of the above vectors, so we'll use $\vec{r}_{A\to B}$:

$$r = \|\vec{r}\|_{A\to B} = \sqrt{1^2 + 7^2 + 2^2} = \sqrt{19} \approx 7.3485.$$

The direction vectors are therefore

$$\hat{r}_{A\to B} = \frac{1}{7.3485}\begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.1361 \\ 0.9526 \\ -0.2722 \end{bmatrix},$$
$$\hat{r}_{B\to A} = -\hat{r}_{A\to B} = \begin{bmatrix} -0.1361 \\ -0.9526 \\ 0.2722 \end{bmatrix}.$$

The gravity force which A applies onto B is then

$$\vec{F}_{A\to B} = \cancel{G}\frac{\overbrace{m_1 m_2}^{=2\times 1}}{r^2}\hat{r}_{A\to B} = \frac{2}{54}\begin{bmatrix} 0.1361 \\ 0.9526 \\ -0.2722 \end{bmatrix} = \begin{bmatrix} 0.0050 \\ 0.0353 \\ -0.101 \end{bmatrix}.$$

and similarily,

$$\vec{F}_{B\to A} = -\vec{F}_{A\to B} = \begin{bmatrix} -0.0050 \\ -0.0353 \\ 0.101 \end{bmatrix}.$$

In the case where we only consider two objects, and choose our frame of reference such that one of the objects is stationary - an analytical solution to the spatial trajectory taken by the second object is known and well studied. It is called a **Keplerian orbit**, and it always takes the form of a conic section.

1.2.2 *Conic Sections*

A conic section (sometimes simply just called “a conic”) is a 2-dimensional shape resulting from the intersection of a plane and a cone (see Figure 1.2). Depending on the angle α by which the plane intersects the cone relative to the the cone's side, the resulting shape can be one of 3 general types (here θ is the cone's angle):

1. If $\alpha > \theta$ the intersection is an **ellipse**. If in addition $\alpha = 90°$ the ellipse becomes a **circle**.

2. If $\alpha = \theta$ the intersection is a **parabola**.

3. If $\alpha < \theta$ the intersection is a **hyperbola**.

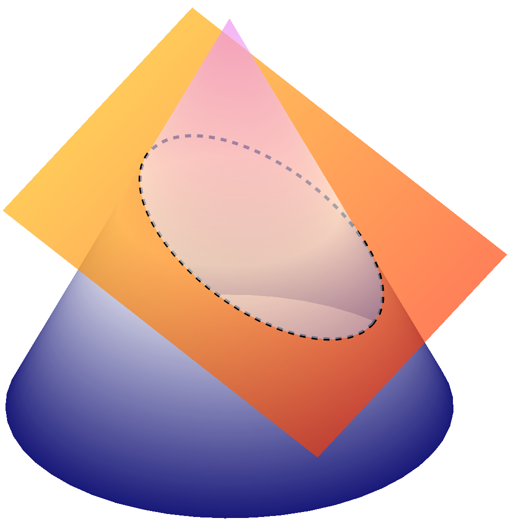

Figure 1.2: An intersection of a cone and a plane. Both the cone and plane are infinite - the cone extends infinitely “down”, but also has a second “inverted” part on the top, also extending to infinity. In the case here shown, the intersection is an ellipse. Image reproduced with modifications from !SOURCE!

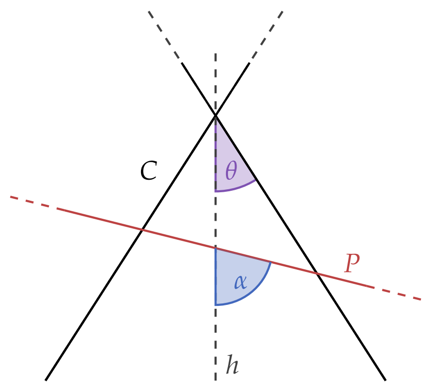

Figure 1.3: Side view of an infinite cone C and an infinite plane P intersecting it. The angle between P and the cone's height line h is α, and the angle between the cone's surface and h is θ. In this figure $0 < \theta < \alpha < 90°$, and thus the shape formed by the intersection of C and P is an ellipse.

1.3 Forward Euler Method

1.4 Backward Euler Method

1.5 Verlet Integration

1.6 Runge-Kutta Method