

PELEG BAR SAPIR

# PHYSICS SIMULATIONS (LECTURE NOTES)



# 1

## Introduction

### 1.1 *Why are Simulations Used?*

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#### 1.4.1 *Radians*

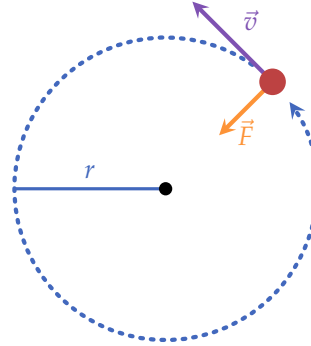
In the context of trigonometry in mathematics and physics, we most commonly use **radians** in place of **degrees**. This sometimes causes some confusion with newer students, so I find it appropriate to briefly discuss the what and why of radians and their usage.

Degrees are defined as a measurement of **angles**. That is, for historical and practical reasons we define a full rotation as  $360^\circ$ . The main problem with using angles is that they are units by themselves, i.e. they differ from lengths. This can cause some consistency issues with regards to units. Consider for example the following equation for a **centripetal force**:

$$F = m \frac{v^2}{r}, \quad (1.1)$$

where  $m$  is the mass of the rotating object,  $v^2$  is the square of the linear velocity the object experiences in each time instance, and  $r$  is the distance of the rotating object from the center of rotation (Figure 1.1).

Figure 1.1: Object with mass  $m$  in perfect circular motion with distance  $r$  around a central point. The force acting on the object is directed towards the center and has value  $F = m \frac{v^2}{r}$ .



In the standard SI scheme,  $F$  has units of  $[\text{N}]$  ("Newtons"), i.e.  $[\text{kg m s}^{-1}]$ . We know that  $m$  has units of  $[\text{kg}]$ ,  $v$  of  $[\text{m s}^{-1}]$  and  $r$  units of  $[\text{m}]$  - so both sides of the equation have the same units.

However, nothing prevents us from measuring the rotation associated with  $\frac{v^2}{r}$  directly using angular velocity  $\omega$  - which can be defined as the amount of degrees of rotation per time  $T$  if the rotation is uniform:

$$\omega = \frac{\theta}{T}, \quad (1.2)$$

or more generally as the time derivative of the angle,

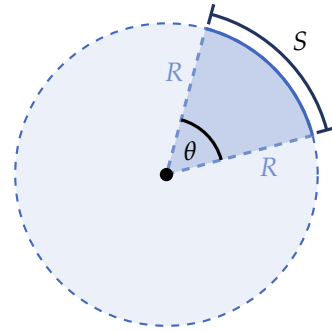
$$\omega = \frac{d\theta}{dt} = \dot{\theta}. \quad (1.3)$$

If we measure the angle  $\theta$  in degrees, we get that the units of angular velocity is  $[\text{° s}^{-1}]$ . If we then want to use  $\omega$  in the centripetal force equation, we must somehow cancel the degrees unit to get the proper units of  $[\text{N}]$ . This obviously leads to a somewhat cumbersome equation.

A much better approach is to measure the *ratio* between the length of an arc created by our angle at a radius  $R$  and the same radius itself (Figure 1.2). This gives a unitless measure which we call *radians*, and can be either ignored in the unit calculation (since it is unitless), or simply denoted  $[\text{rad}]$ .

Another way to view radians is that they measure an arc length in units of the radius (not necessarily a unit radius). If we scale the radius by any scalar, an arc length of the same angle will scale by exactly the same scalar, and so the amount of radians we measure for the arc length won't change (the ratio stays the same, as both the numerator and denominator are scaled by the same number).

Figure 1.2: Measuring angles using radians. We express the length of  $S$  in units of  $R$ , e.g. if  $S = 1$  then the arc length is equal to  $R$ , if  $S = \pi$  the arc length is half that of the entire circumference of the circle, etc. The ratio  $S/R$  is constant for any real positive value of



$R$ , and thus can be used to uniquely describe the angle  $\theta$ . We can easily replace any measurement of angles in degrees by a corresponding measurement in radians, since any amount of degrees has a 1-to-1 correspondence with a single value in radians. Let us find this correspondence: since there are  $2\pi$  radii in a full circle, and a full angle is  $360^\circ$ , we see that each radii length of an arc corresponds to  $\frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} \approx 57.2958^\circ$ .

Now, using the radian unit to measure an angle  $\theta$ , we get that  $\omega$  has units of  $[\text{rad s}^{-1}]$ , or simply  $[\text{s}^{-1}]$ . In turn, if we raise  $\omega$  to the 2nd power and multiply it by the distance  $r$  between the rotating object and center of rotation, and the mass  $m$  of the object, we get the units  $[\text{kg m s}^{-1}]$ , which is exactly the explicit form of Newtons. Indeed, using this perspective, Equation 1.1 can be written as

$$F = m\omega^2 r. \quad (1.4)$$

In my mind, this is simply beautiful.

Let's finish with some example correspondences between degrees and radians, which can be seen in Table 1.1 below.

$[\circ]$	$[\text{rad}]$
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
120	$\frac{2\pi}{3}$
180	$\pi$
270	$\frac{3\pi}{2}$
360	$2\pi$

Table 1.1: Example correspondences between measurements in degrees and radians.

### 1.4.2 Taylor Series

For some functions, it is rather easy to calculate their values at some point  $x_0$ : for example, given a polynomial function

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n, \quad (1.5)$$

it is rather easy to calculate its value at any real point  $x$ : all the operations that we need to use are addition and multiplication of real numbers, and raising real numbers to an integer power (which in principle can be implemented as repeated multiplications). Modern computers calculate such operations at the rate of billions of times a second.

#### Note 1.1 Floating point representation

For the sake of simplicity, I'm ignoring the entire topic of floating point numbers and relevant questions of precision.



#### Example 1.1 Calculating a value of a simple polynomial

Given the polynomial  $P(x) = 3x^2 - 2x + 7$ , we can easily calculate its value at, say,  $x = 5$ :

$$P(x = 5) = 3 \cdot 5^2 - 2 \cdot 5 + 7 = 3 \cdot 25 - 10 + 7 = 72.$$



For other functions, on the other hand, it is not that easy to calculate their values at most points. For example, consider the exponential function  $f(x) = e^x$ . We know precisely exactly one real value of the function: at  $x = 0$ , the function returns 1. But for any other value of  $x$ , we don't really know the value of the function. That is, we know all values *symbolically* (for example,  $f(1) = e$ ,  $f(2) = e^2$ , etc.) - but not in explicit form.

However, given that the function behaves nicely enough (we'll discuss in a moment what that means), we can *approximate* its value to whatever precision we wish, using a method called the **Taylor series** of the function. The "price" we pay for greater precision is simply performing more calculations.

The basic idea of a Taylor series of a function  $f(x)$  is that we approximate the function by adding higher and higher derivatives of the same function, at some point  $a$  for which we know precisely the value of the function and its derivatives to any order. This might sound a bit abstract, so let's explore this process using an example function - again, the exponential  $f(x) = e^x$ . As mentioned, we only really know one value of the function precisely, namely at  $a = 0$ :  $e^0 = 1$ .

We can therefore start approximating  $e^x$  simply as  $e^0$ . This is obviously a very imprecise approximation, but the important thing

is that if we look at a very close neighborhood of  $x = 0$ , the approximation is not *that* bad: consider, for example  $x_0 = 0.0001$ . Our approximation gives  $e^{x_0} \approx 1$ , which is not far from the more precise value  $e^{x_0} = 1.000100005$  (the value here is shown up to the ninth decimal). Of course, the closer we get to  $a = 0$ , the better our approximation gets: for example, with  $x_1 = 0.00001$ , we get  $e^{x_1} \approx 1$  again, where in reality  $e^{x_1} = 1.000010000$  (also shown here up to the ninth decimal). On the other hand, if we as we get farther away from  $x = 0$ , the approximation becomes worse and worse, as seen in Table 1.2 below.

$x$	$e^x$ (exact)	$\Delta$ (error)
0.000	1.000000000	0.000000000
0.001	1.001000500	0.001000500
0.010	1.010050167	0.010050167
0.100	1.105170918	0.105170918
0.500	1.648721271	0.648721271
0.510	1.665291195	0.665291195
0.520	1.682027650	0.682027650
1.000	2.718281828	1.718281828
1.100	3.004166024	2.004166024
1.500	4.481689070	3.481689070
2.000	7.389056099	6.389056099

Table 1.2: Zero order Taylor series approximation of  $e^x$ .

Now, let's take this a step further: since the derivative of a function at a point tells us how the functions changes close to the point, we can use this information to improve our approximation by adding the value of the first derivative of  $e^x$  at  $x = 0$ , which is also 1. In fact, this is true for any order derivative of  $e^x$ , since  $\frac{d^n}{dx^n} e^x = e^x$  for any  $n \in \mathbb{N}$ . Since the derivative changes with the value of  $x$ , we will multiply it by  $x$  in the approximation. Thus we get

$$e^x \approx 1 + 1 \cdot x = 1 + x, \quad (1.6)$$

which we call the **first order** approximation of the function  $e^x$ .

Table 1.3 below shows the same values from Table 1.2 for  $x$ , but using the first order approximation for  $e^x$ .

By examining Table 1.3 it's clear that  $1 + x$  is a much better approximation for  $e^x$  than just 1. Of course, we can take a further step: the second derivative of a function at a point tells us how the change in the function itself changes around that point. We can take the second derivative of  $e^x$  (which is by itself  $e^x$ ), again substitute  $x = 0$ , get  $e^0 = 1$ , and use this with  $x^2$  to apperoximate  $e^x$  even better. In fact, let's already take this idea to its logical conclusion: we'll simply use the infinite power series in  $x$ , i.e.  $1, x, x^2, x^3, \dots, x^n, \dots$

$x$	$1 + x$	$e^x$ (exact)	$\Delta$ (error)
0.000	1.000000000	1.000000000	0.000000000
0.001	1.001000000	1.001000500	0.000000500
0.010	1.010000000	1.010050167	0.000050167
0.100	1.100000000	1.105170918	0.005170918
0.500	1.500000000	1.648721271	0.148721271
0.510	1.510000000	1.665291195	0.155291195
0.520	1.520000000	1.682027650	0.162027650
1.000	2.000000000	2.718281828	0.718281828
1.100	2.100000000	3.004166024	0.904166024
1.500	2.500000000	4.481689070	1.981689070
2.000	3.000000000	7.389056099	4.389056099

Table 1.3: First order Taylor series approximation of  $e^x$ .

and attach to each  $x^k$  the value given by  $\frac{d^n}{dx^n} e^x \Big|_{x=0} = 1$ , so we get

$$e^x \approx 1 + x + x^2 + x^3 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n. \quad (1.7)$$

This, however doesn't quite work. The coefficients of each term have to be adjusted. By assuming that at the limit where  $n \rightarrow \infty$  the approximation should be with zero error, we get that for each  $n$ ,

$$\frac{d^n}{dx^n} e^x = \frac{d^n}{dx^n} (1 + x + x^2 + \dots) \equiv s(x). \quad (1.8)$$

Calculating the general  $n$ -th derivative of Equation 1.7 is rather easy. Let's examine the case for  $n = 3$ : the first three derivatives of Equation 1.7 are

$$\begin{aligned} \frac{d}{dx} s(x) &= 1 + 2x + 3x^2 + 4x^3 + \dots, \\ \frac{d^2}{dx^2} s(x) &= 2 + 6x + 12x^2 + \dots, \\ \frac{d^3}{dx^3} s(x) &= 6 + 24x + \dots, \end{aligned}$$

and if we substitute  $x = 0$  we get



## 2

# Mechanics

### 2.1 Preface

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### 2.2 Pendulum

#### 2.2.1 Simple pendulum

Text text text.

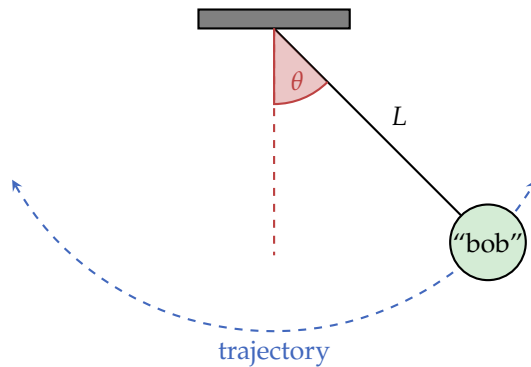


Figure 2.1: A simple pendulum. TBW: add more info.

Using force analysis we can derive an equation of motion for the bob (see Figure 2.2): since the rod can't change its length (it's always  $L$ ), the only variable quantity is the angle  $\theta$ , and the bob's trajectory is a circle. Any force acting in a radial direction to the trajectory must be counter-balanced (otherwise there will be some acceleration - and therefore motion - in that direction). We are therefore left with only a tangential force, with magnitude

$$F = -mg \sin(\theta), \quad (2.1)$$

the minus sign here is chosen to represent that gravity acts in the negative  $y$  direction (i.e. "down").

Applying Newton's second law we get that

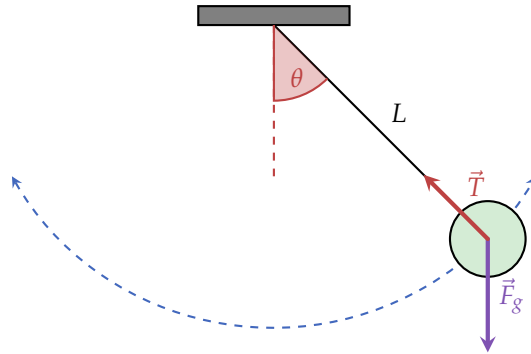
$$F = ma = -mg \sin(\theta), \quad (2.2)$$

i.e.

$$a = -g \sin(\theta). \quad (2.3)$$

Now we see that the minus sign also makes sense physically, as it shows that the acceleration is always in the opposite direction to the angle (which is negative to the left and positive to the right).

Figure 2.2: Forces acting on a simple pendulum. TBW: Force compnents.



The tangential position  $s$  of the bob can be calculated from the angle  $\theta$  by

$$s = L\theta \quad (2.4)$$

(recall that  $\theta$  is given in radians), and therefore the tangential velocity is

$$v = \dot{s} = L\dot{\theta}, \quad (2.5)$$

and the acceleration is therefore

$$a = \dot{v} = \ddot{s} = L\ddot{\theta}. \quad (2.6)$$

Since we know that  $a = -g \sin(\theta)$ , we get

$$L\ddot{\theta} = -g \sin(\theta), \quad (2.7)$$

and by moving the rhs term to the left and divide by  $l$  we get

$$\ddot{\theta} + \frac{g}{L} \sin(\theta) = 0. \quad (2.8)$$

This is a differential equation without analytical solution. We will therefore take two approaches: (1) use an approximation to yield an analytical solution, and (2) solve the equation numerically.

### 2.2.2 Small-angle approximation

The Taylor series expansion of  $\sin(x)$  around  $x = 0$  is

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad (2.9)$$

We can therefore approximate  $\sin(x)$  as  $x$  for small values of  $x$ :

$$\sin(x) \approx x. \quad (2.10)$$

This is known as the “small-angle approximation” of the sine function. By using this approximation we get that the (analytically) unsolvable Equation 2.8 reduces to

$$\ddot{\theta} + \frac{g}{L}\theta = 0, \quad (2.11)$$

for which we have an exact solution:

$$\theta(t) = A \cos(\omega t + \phi), \quad (2.12)$$

where  $\omega = \sqrt{\frac{g}{L}}$ . The parameters  $A$  and  $\phi$  depend on the initial conditions (i.e. angle and tangential velocity).

TBW: resonance freq, how it looks like (+phase space), python plot?

### 2.2.3 Numerical integration

Another approach to solve Equation 2.8 is doing so numerically, i.e. essentially running a computer simulation. While computers can carry out many calculations per second (in the order of billions, in fact) - they are limited to performing discrete calculations. That is to say, a numerical calculation is also an approximation. However, unlike the small-angle approximation, in principle we can improve the approximation indefinitely, although in practice this is of course impossible.

Let us use a rather naive approach to numerically approximating Equation 2.8: instead of viewing  $\theta$  as a continuous function of time  $t$ , we instead define  $t$  to only have equally spaced discrete values  $t_0, t_1, t_2, \dots$  such that

$$t_n = t_0 + n\Delta t, \quad (2.13)$$

or phrased differently: we look at the values  $t_i$  of  $t$  at intervals  $\Delta t$  starting from  $t_0$  (see Figure 2.3).

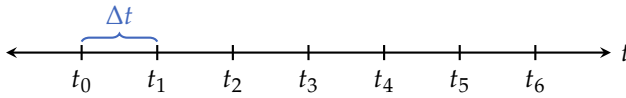


Figure 2.3: Discrete time steps.

In this discrete view, a function of  $t$  can also only be evaluated at discrete points. For example, the angle  $\theta$  of the bob as a function of time is a discrete function taking the values  $\theta_0, \theta_1, \theta_2, \dots$ , where each  $\theta_i$  corresponds to the time  $t_i$ .

How does the discrete angular velocity look like in this scheme? It's worth looking at the definition of angular velocity in the continuous case:

$$\omega(t) = \dot{\theta}(t) = \frac{d\theta(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\theta(t + \Delta t) - \theta(t)}{\Delta t}. \quad (2.14)$$

To discretize, we can replace  $\theta(t)$  with  $\theta_{t_i} = \theta_i$ , and therefore  $\theta(t + \Delta t)$  with  $\theta_{t_i + \Delta t} = \theta_{i+1}$ . The limit  $\lim_{\Delta t \rightarrow 0}$  is pretty meaningless, since the smallest  $\Delta t$  possible in the discrete case is the  $\Delta t$  we chose to discretize out time steps. Therefore, we get that the discrete version of  $\omega(t)$  is

$$\omega_i = \frac{\theta_{i+1} - \theta_i}{\Delta t}, \quad (2.15)$$

which is nothing more than saying that the angular velocity at time  $t_i$  is simply the difference between the angle at time  $t_{i+1}$  and the angle at time  $t_i$ , divided by the time difference  $\Delta t$ .

Equation 2.15 gives us a powerful tool: if we only know the current angle  $\theta_i$  of the bob and its current velocity  $\omega_i$ , then its next position  $\theta_{i+1}$  is a simple rearrangement of the equation:

$$\theta_{i+1} = \theta_i + \omega_i \Delta t. \quad (2.16)$$

The same logic can be applied to the angular acceleration  $\alpha(t)$ : in the continuous version its given by

$$\alpha(t) = \dot{\omega}(t) = \frac{d\omega(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\omega(t + \Delta t) - \omega(t)}{\Delta t}, \quad (2.17)$$

and thus the discrete version is

$$\alpha_i = \frac{\omega_{i+1} - \omega_i}{\Delta t}, \quad (2.18)$$

and recovering the angular velocity  $\omega_{i+1}$  from the angular velocity  $\omega_i$  and the angular acceleration  $\alpha_i$  is done by

$$\omega_{i+1} = \omega_i + \alpha_i \Delta t. \quad (2.19)$$

Since we use the inverse of the differentiation operation (in the discrete sense) to recover the quantity we're after, this scheme is known as a **numerical integration**. Specifically, this kind of numerical integration is called the **Forward-Euler** method, and it is generally considered unfavourable due to the fast rate with which it gains errors, and is rarely used in practice.

However, for the sake of simplicity of writing our first simulations, we will not discuss these issues now, nor will we generalize the method and present better ones - both of which we will do later in the course. Instead, for now we will continue with using this method to devise a numerical integration scheme for a simple pendulum.

In the case of the pendulum, recall that the acceleration at time  $t$  is  $\ddot{\theta}(t) = -\frac{g}{L} \sin(\theta)$  (Equation 2.8), and therefore we can discretize it as

$$\alpha_{i+1} = -\frac{g}{L} \sin(\theta_i). \quad (2.20)$$

We then use the forward Euler method to get the angular velocity and angle, as seen in Equation 2.19 and Equation 2.16, respectively.

#### 2.2.4 Damped oscillation

A slightly more realistic model of a pendulum also considers the way the pendulum loses energy over time (e.g. via friction). This can be modelled by adding a force which resists the angular velocity (see also Figure 2.4):

$$F_d = -bL\dot{\theta}, \quad (2.21)$$

where  $b$  is simply a parameter which adjusts how strong the damping is.

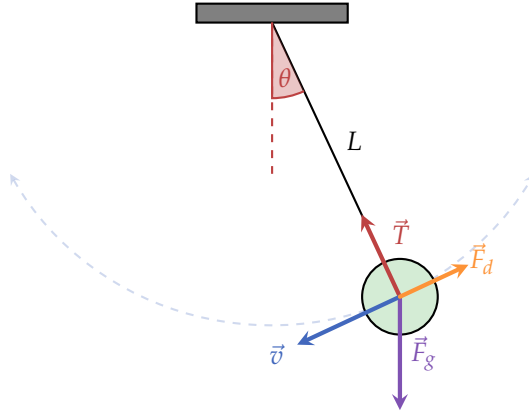


Figure 2.4: Forces on a pendulum including a damping force.

Using Newton's second law, recalling that  $a = L\ddot{\theta}$  (Equation 2.6), we get

$$mL\ddot{\theta} = F_g + F_d = -mg \sin(\theta) - bL\dot{\theta}. \quad (2.22)$$

It's common to use  $\beta = \frac{b}{2m}$  and of course  $\omega_0 = \sqrt{\frac{g}{L}}$ , which together with some rearrangement gives us

$$\ddot{\theta} = -2\beta\dot{\theta} - \omega_0^2 \sin(\theta). \quad (2.23)$$

#### 2.2.5 Double pendulum

An interesting system arises when we take a simple pendulum and attach another simple pendulum to it (as seen in Figure 2.5): unlike a simple pendulum, this system exhibits very complex dynamics which is sensitive to the initial conditions - i.e. it is **chaotic**, meaning that small changes in the initial conditions evolve into big differences in the dynamics of the system.

To analyze the behaviour of the system we again seek to find a differential equation which will describe the system completely. Unfortunately, this differential equation is rather complicated and is composed of two coupled differential equations. We will only present the equation here but not how to derive it - this will be added in an appendix.

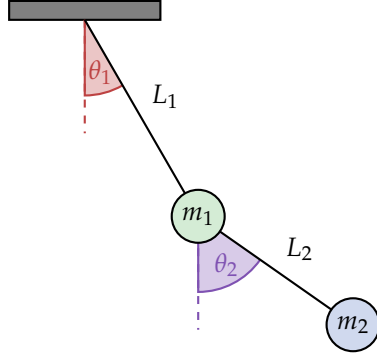


Figure 2.5: A double pendulum. TBW: add more info.

The differential equations governing the dynamics of the system are:

$$0 = (m_1 + m_2) L_1 \ddot{\theta}_1 + m_2 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin(\theta_1), \quad (2.24)$$

and

$$0 = L_2 \ddot{\theta}_2 + L_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - L_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin(\theta_2). \quad (2.25)$$

To help us get explicit expressions for  $\ddot{\theta}_1$  and  $\ddot{\theta}_2$  we can define the following quantities:

$$\alpha = \frac{m_2}{m_1}, \quad \beta = \frac{L_2}{L_1}, \quad \gamma = \frac{g}{L_1}, \quad \Delta\theta = \theta_1 - \theta_2. \quad (2.26)$$

we then get the following "beautiful" expressions for  $\ddot{\theta}_1$  and  $\ddot{\theta}_2$ :

$$\begin{aligned} \ddot{\theta}_1 &= - \frac{(1 + \alpha) \gamma \sin(\theta_1) + \alpha \beta \dot{\theta}_2^2 \sin(\Delta\theta) + \alpha \cos(\Delta\theta) \left[ \dot{\theta}_1^2 \sin(\Delta\theta) - \gamma \sin(\theta_2) \right]}{1 + \alpha \sin^2(\Delta\theta)}, \\ \ddot{\theta}_2 &= \frac{(1 + \alpha) \left( \dot{\theta}_1^2 \sin(\Delta\theta) - \gamma \sin(\theta_2) + \cos(\Delta\theta) \left[ (1 + \alpha) \gamma \sin(\theta_1) + \alpha \beta \dot{\theta}_2^2 \sin(\Delta\theta) \right] \right)}{\beta \left[ 1 + \alpha \sin^2(\Delta\theta) \right]} \end{aligned} \quad (2.27)$$

Discretizing Equation 2.27 can then be done as following: first,

1.  $\alpha$ ,  $\beta$  and  $\gamma$  are defined as in Equation 2.26.
2. Set  $\Delta\theta_k = \theta_{1,k} - \theta_{2,k}$ .
3. For compactness, we set  $N_k = 1 + \alpha \sin^2(\Delta\theta_k)$ ,  $S(x) = \sin(x)$ ,  $C(x) = \cos(x)$  and  $A = 1 + \alpha$ .

Then, we get

$$\begin{aligned}\ddot{\theta}_{1,i} &= -\frac{A\gamma S(\theta_{1,i-1}) + \alpha\beta\dot{\theta}_{2,i-1}^2 S(\Delta\theta_{i-1}) + \alpha C(\Delta\theta_{i-1}) \left[ \dot{\theta}_{1,i-1}^2 S(\Delta\theta_{i-1}) - \gamma S(\theta_{2,i-1}) \right]}{N_{i-1}}, \\ \ddot{\theta}_{2,i} &= \frac{A \left( \dot{\theta}_{1,i-1}^2 S(\Delta\theta_{i-1}) - \gamma S(\theta_{2,i-1}) + C(\Delta\theta_{i-1}) \left[ A\gamma S(\theta_{1,i-1}) + \alpha\beta\dot{\theta}_{2,i-1}^2 S(\Delta\theta_{i-1}) \right] \right)}{\beta N_{i-1}}.\end{aligned}\tag{2.28}$$

(lovely, isn't it?)





# 3

## Thermodynamics

### 3.1 Preface

Text text text

### 3.2 Ideal gas

#### 3.2.1 Model: perfectly elastic spheres

Text text text

Consider two solid spheres which have a single point of contact  
A. Let  $m_1, r_1, \vec{x}_1$  and  $\vec{v}_1$  be the mass, radius, position and velocity of the first sphere, and  $m_2, r_2, \vec{x}_2$  and  $\vec{v}_2$  the respective quantities for the second sphere (Figure 3.1). The line connecting the centers of the two spheres is in the direction  $\hat{n}$  (without loss of generality let us assume that the normal vector  $\hat{n}$  points from  $\vec{x}_1$  towards  $\vec{x}_2$ ). The unit vector  $\hat{t}$  is orthogonal to  $\hat{n}$  (and without loss of generality we will assume that it is oriented counter-clockwise from  $\hat{n}$ ).

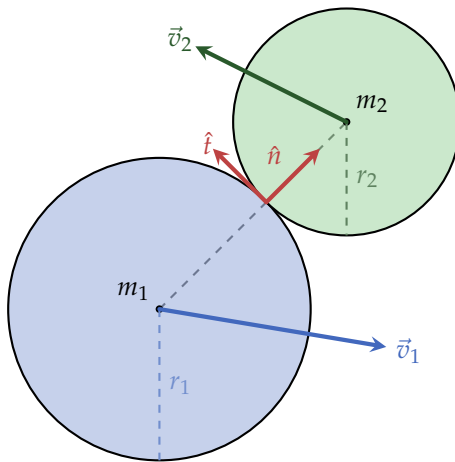


Figure 3.1: Text text text

Note that we did not define a coordinate system, nor the number of dimensions  $d$  for the problem. The only restriction is that  $d \geq 2$ .

Conservation of momentum means that the velocities of the spheres following the collision,  $\vec{u}_1, \vec{u}_2$ , are related to their velocities before the collision by

$$\begin{aligned} m_1 \vec{v}_1 + m_2 \vec{v}_2 &= m_1 \vec{u}_1 + m_2 \vec{u}_2 \\ \Rightarrow m_1 (\vec{v}_1 - \vec{u}_1) &= m_2 (\vec{u}_2 - \vec{v}_2). \end{aligned} \quad (3.1)$$

Conservation of energy means that the velocities are also related by

$$\begin{aligned} \frac{1}{2} m_1 \|\vec{v}_1\|^2 + \frac{1}{2} m_2 \|\vec{v}_2\|^2 &= \frac{1}{2} m_1 \|\vec{u}_1\|^2 + \frac{1}{2} m_2 \|\vec{u}_2\|^2 \\ \Rightarrow m_1 (\|\vec{v}_1\|^2 - \|\vec{u}_1\|^2) &= m_2 (\|\vec{u}_2\|^2 - \|\vec{v}_2\|^2). \end{aligned} \quad (3.2)$$

However, the forces involved in the collision can not have a component in the  $\hat{t}$  direction, and are limited to only point in the  $\hat{n}$  direction. Therefore, we can reduce the problem to this direction only by projecting all velocities involved in the problem on  $\hat{n}$ , i.e. Equation 3.1 becomes

$$m_1 \langle \vec{v}_1 | \hat{n} \rangle + m_2 \langle \vec{v}_2 | \hat{n} \rangle = m_1 \langle \vec{u}_1 | \hat{n} \rangle + m_2 \langle \vec{u}_2 | \hat{n} \rangle. \quad (3.3)$$

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$$\begin{aligned} \vec{u}_1 &= \vec{v}_1 - \frac{2m_2}{m_1 + m_2} \langle \vec{v}_1 - \vec{v}_2 | \hat{n} \rangle \hat{n}, \\ \vec{u}_2 &= \vec{v}_2 + \frac{2m_1}{m_1 + m_2} \langle \vec{v}_1 - \vec{v}_2 | \hat{n} \rangle \hat{n}. \end{aligned} \quad (3.4)$$

To avoid redundant calculations, we can factor out the common quantity of both velocities:

$$K = \frac{2}{m_1 + m_2} \langle \vec{v}_1 - \vec{v}_2 | \hat{n} \rangle \hat{n}, \quad (3.5)$$

yielding

$$\begin{aligned} \vec{u}_1 &= \vec{v}_1 - Km_2, \\ \vec{u}_2 &= \vec{v}_2 + Km_1. \end{aligned} \quad (3.6)$$