

PELEG BAR SAPIR

# PHYSICS SIMULATIONS (LECTURE NOTES)



# 1

## Mechanics

### 1.1 Preface

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### 1.2 Pendulum

#### 1.2.1 Simple pendulum

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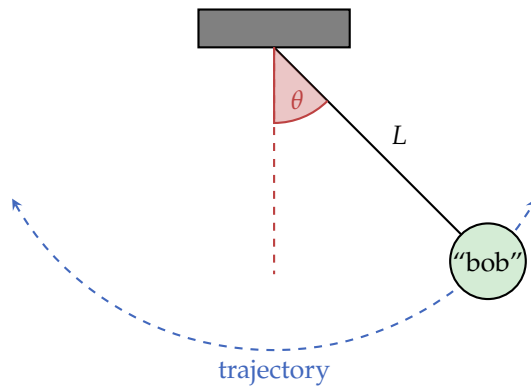


Figure 1.1: A simple pendulum. TBD: add more info.

Using force analysis we can derive an equation of motion for the bob (see Figure 1.2): since the rod can't change its length (it's always  $L$ ), the only variable quantity is the angle  $\theta$ , and the bob's trajectory is a circle. Any force acting in a radial direction to the trajectory must be counter-balanced (otherwise there will be some acceleration - and therefore motion - in that direction). We are therefore left with only a tangential force, with magnitude

$$F = -mg \sin(\theta), \quad (1.1)$$

the minus sign here is chosen to represent that gravity acts in the negative  $y$  direction (i.e. "down").

Applying Newton's second law we get that

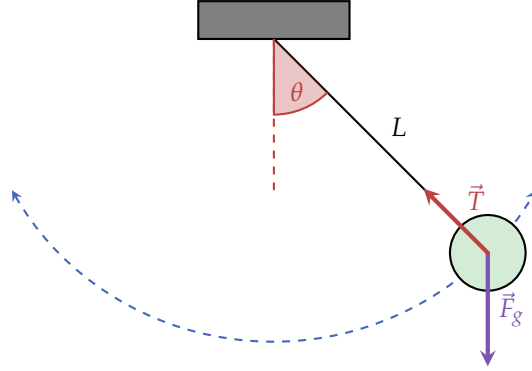
$$F = ma = -mg \sin(\theta), \quad (1.2)$$

i.e.

$$a = -g \sin(\theta). \quad (1.3)$$

Now we see that the minus sign also makes sense physically, as it shows that the acceleration is always in the opposite direction to the angle (which is negative to the left and positive to the right).

Figure 1.2: Forces acting on a simple pendulum. TBD: Force compnents.



The tangential position  $s$  of the bob can be calculated from the angle  $\theta$  by

$$s = L\theta \quad (1.4)$$

(recall that  $\theta$  is given in radians), and therefore the tangential velocity is

$$v = \dot{s} = L\dot{\theta}, \quad (1.5)$$

and the acceleration is therefore

$$a = \dot{v} = \ddot{s} = L\ddot{\theta}. \quad (1.6)$$

Since we know that  $a = -g \sin(\theta)$ , we get

$$L\ddot{\theta} = -g \sin(\theta), \quad (1.7)$$

and by moving the rhs term to the left and divide by  $l$  we get

$$\ddot{\theta} + \frac{g}{L} \sin(\theta) = 0. \quad (1.8)$$

This is a differential equation without analytical solution. We will therefore take two approaches: (1) use an approximation to yield an analytical solution, and (2) solve the equation numerically.

### 1.2.2 Small-angle approximation

The Taylor series expansion of  $\sin(x)$  around  $x = 0$  is

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad (1.9)$$

We can therefore approximate  $\sin(x)$  as  $x$  for small values of  $x$ :

$$\sin(x) \approx x. \quad (1.10)$$

This is known as the “small-angle approximation” of the sine function. By using this approximation we get that the (analytically) unsolvable Equation 1.8 reduces to

$$\ddot{\theta} + \frac{g}{L}\theta = 0, \quad (1.11)$$

for which we have an exact solution:

$$\theta(t) = A \cos(\omega t + \phi), \quad (1.12)$$

where  $\omega = \sqrt{\frac{g}{L}}$ . The parameters  $A$  and  $\phi$  depend on the initial conditions (i.e. angle and tangential velocity).

TBW: how it looks like (+phase space), python plot?

### 1.2.3 Numerical integration

Another approach to solve Equation 1.8 is doing so numerically, i.e. essentially running a computer simulation. While computers can carry out many calculations per second (in the order of billions, in fact) - they are limited to performing discrete calculations. That is to say, a numerical calculation is also an approximation. However, unlike the small-angle approximation, in principle we can improve the approximation indefinitely, although in practice this is of course impossible.

Let us use a rather naive approach to numerically approximating Equation 1.8: instead of viewing  $\theta$  as a continuous function of time  $t$ , we instead define  $t$  to only have equally spaced discrete values  $t_0, t_1, t_2, \dots$  such that

$$t_n = t_0 + n\Delta t, \quad (1.13)$$

or phrased differently: we look at the values  $t_i$  of  $t$  at intervals  $\Delta t$  starting from  $t_0$  (see Figure 1.3).

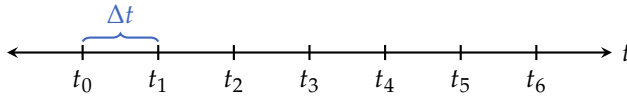


Figure 1.3: Discrete time steps.

In this discrete view, a function of  $t$  can also only be evaluated at discrete points. For example, the angle  $\theta$  of the bob as a function of time is a discrete function taking the values  $\theta_0, \theta_1, \theta_2, \dots$ , where each  $\theta_i$  corresponds to the time  $t_i$ .

How does the discrete angular velocity look like in this scheme? It's worth looking at the definition of angular velocity in the continuous case:

$$\omega(t) = \dot{\theta}(t) = \frac{d\theta(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\theta(t + \Delta t) - \theta(t)}{\Delta t}. \quad (1.14)$$

To discretize, we can replace  $\theta(t)$  with  $\theta_{t_i} = \theta_i$ , and therefore  $\theta(t + \Delta t)$  with  $\theta_{t_i + \Delta t} = \theta_{i+1}$ . The limit  $\lim_{\Delta t \rightarrow 0}$  is pretty meaningless, since the smallest  $\Delta t$  possible in the discrete case is the  $\Delta t$  we chose to discretize out time steps. Therefore, we get that the discrete version of  $\omega(t)$  is

$$\omega_i = \frac{\theta_{i+1} - \theta_i}{\Delta t}, \quad (1.15)$$

which is nothing more than saying that the angular velocity at time  $t_i$  is simply the difference between the angle at time  $t_{i+1}$  and the angle at time  $t_i$ , divided by the time difference  $\Delta t$ .

Equation 1.15 gives us a powerful tool: if we only know the current angle  $\theta_i$  of the bob and its current velocity  $\omega_i$ , then its next position  $\theta_{i+1}$  is a simple rearrangement of the equation:

$$\theta_{i+1} = \theta_i + \omega_i \Delta t. \quad (1.16)$$

The same logic can be applied to the angular acceleration  $\alpha(t)$ : in the continuous version its given by

$$\alpha(t) = \dot{\omega}(t) = \frac{d\omega(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\omega(t + \Delta t) - \omega(t)}{\Delta t}, \quad (1.17)$$

and thus the discrete version is

$$\alpha_i = \frac{\omega_{i+1} - \omega_i}{\Delta t}, \quad (1.18)$$

and recovering the angular velocity  $\omega_{i+1}$  from the angular velocity  $\omega_i$  and the angular acceleration  $\alpha_i$  is done by

$$\omega_{i+1} = \omega_i + \alpha_i \Delta t. \quad (1.19)$$

#### 1.2.4 Damped oscillation

#### 1.2.5 Double pendulum