PELEG BAR SAPIR

PHYSICS SIMULATIONS (LECTURE NOTES)

1

Introduction

1.1 Why are Simulations Used?

Text

1.2 Python

1.2.1 General

Text

1.2.2 NumPy

Text

1.3 A Bit About Git

Text

1.4 Some Mathematical Background

1.4.1 Radians

In the context of trigonometry in mathematics and physics, we most commonly use **radians** in place of **degrees**. This sometimes causes some confusion with newer students, so I find it appropriate to briefly discuss the what and why of radians and their usage.

Degrees are defined as a measurement of **angles**. That is, for historical and practical reasons we define a full rotation as 360°. The main problem with using angles is that they are units by themselves, i.e. they differ from lengths. This can cause some consistency issues with regards to units. Consider for example the following equation for a **centripetal force**:

$$F = m\frac{v^2}{r}, \tag{1.1}$$

where m is the mass of the rotationg object, v^2 is the square of the linear velocity the object experiences in each time instance, and r is the distance of the rotating object from the center of rotation (Figure 1.1).

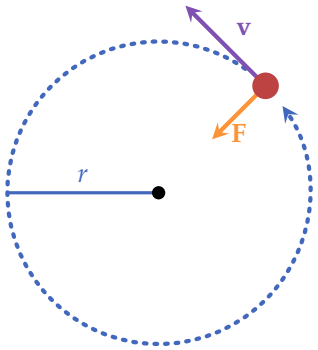

Figure 1.1: Object with mass m in perfect cicular motion with distance r around a central point. The force acting on the object is directed towards the center and has value $F = m\frac{v^2}{r}$.

In the standard SI scheme, F has units of $[\mathrm{N}]$ ("Newtons"), i.e. $\left[\mathrm{kg\,m\,s^{-1}}\right]$. We know that m has units of $[\mathrm{kg}]$, v of $\left[\mathrm{m\,s^{-1}}\right]$ and r units of $[\mathrm{m}]$ - so both sides of the equation have the same units.

However, nothing prevents us from measuring the rotation associated with $\frac{v^2}{r}$ directly using angular velocity ω - which can be defined as the amounth of degrees of rotation per time T if the rotation is uniform:

$$\omega = \frac{\theta}{T}, \tag{1.2}$$

or more generally as the time derivative of the angle,

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}. \tag{1.3}$$

If we measure the angle θ in degrees, we get that the units of angular velocity is $\left[^\circ\,\mathrm{s^{-1}}\right]$. If we then want to use ω in the centripetal force equation, we must somehow cancel the degrees unit to get the proper units of $[\mathrm{N}]$. This obviously leads to a somewhat cumbersome equation.

A much better approach is to measure the *ratio* between the length of an arc created by our angle at a radius R and the same radius itself (Figure 1.2). This gives a unitless measure which we call *radians*, and can be either ignored in the unit calculation (since it is unitless), or simply denoted $[\mathrm{rad}]$.

Another way to view radians is that they measure an arc length in units of the radius (not necessarily a unit radius). If we scale the radius by any scalar, an arc length of the same angle will scale by exactly the same scalar, and so the amount of radians we measure for the arc length won't change (the ratio stays the same, as both the numerator and denominator are scaled by the same number).

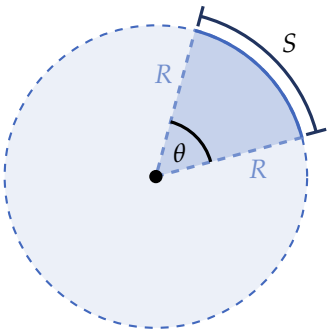

Figure 1.2: Measuring angles using radians. We express the length of S in units of R, e.g. if $S = 1$ then the arc length is equal to R, if $S = \pi$ the arc length is half that of the entire circumference of the circle, etc. The ratio S/R is constant for any real positive value of R, and thus can be used to uniquely describe the angle θ.

We can easily replace any measurement of angles in degrees by a corresponding measurement in radians, since any amount of degrees has a 1-to-1 correspondence with a single value in radians. Let us find this correspondence: since there are 2π radii in a full circle, and a full angle is $360°$, we see that each radii length of an arc corresponds to $\frac{360°}{2\pi} = \frac{180°}{\pi} \approx 57.2958°$.

Now, using the radian unit to measure an angle θ, we get that ω has units of $\left[\text{rad}\,\text{s}^{-1}\right]$, or simply $\left[\text{s}^{-1}\right]$. In turn, if we raise ω to the 2nd power and multiply it by the distance r between the rotating object and center of rotation, and the mass m of the object, we get the units $\left[\text{kg}\,\text{m}\,\text{s}^{-1}\right]$, which is exactly the excplicit form of Newtons. Indeed, using this perspective, Equation 1.1 can be written as

$$F = m\omega^2 r. \tag{1.4}$$

In my mind, this is simply beautiful.

Let's finish with some example correspondences between degrees and radians, which can be seen in Table 1.1 below.

[°]	[rad]
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
120	$\frac{2\pi}{3}$
180	π
270	$\frac{3\pi}{2}$
360	2π

Table 1.1: Example correspondences between measurements in degrees and radians.

1.4.2 *Taylor Series*

For some functions, it is rather easy to calculate their values at some point x_0: for example, given a polynomial function

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n, \tag{1.5}$$

it is rather easy to calculate its value at any real point x: all the operations that we need to use are addition and multiplication of real numbers, and raising real numbers to an integer power (which in principle can be implemented as repeated multiplications). Modern computers calculate such operations at the rate of billions of times a second.

Note 1.1 Floating point representation

For the sake of simplicity, I'm ignoring the entire topic of floating point numbers and relevant questions of precision.

Example 1.1 Calculating a value of a simple polynomial

Given the polynomial $P(x) = 3x^2 - 2x + 7$, we can easily calculate its value at, say, $x = 5$:

$$P(x = 5) = 3 \cdot 5^2 - 2 \cdot 5 + 7 = 3 \cdot 25 - 10 + 7 = 72.$$

For other functions, on the other hand, it is not that easy to calculate their values at most points. For example, consider the exponential function $f(x) = e^x$. We know precisely exactly one real value of the function: at $x = 0$, the function returns 1. But for any other value of x, we don't really know the value of the function. That is, we know all values *symbolicaly* (for example, $f(1) = e$, $f(2) = e^2$, etc.) - but not in explicit form.

However, given that the function behaves nicely enough (we'll discuss in a moment what that means), we can *approximate* its value to whatever percision we wish, using a method called the **Taylor series** of the function. The "price" we pay for greater percision is simply performing more calculations.

The basic idea of a Taylor series of a function $f(x)$ is that we approximate the function by adding higher and higher derivatives of the same function, at some point a for which we know precisely the value of the function and its derivatives to any order. This might sound a bit abstract, so let's explore this process using an example function - again, the exponential $f(x) = e^x$. As mentioned, we only really know one value of the function precisely, namely at $a = 0$: $e^0 = 1$.

We can therefore start approximating e^x simply as e^0. This is obviously a very impercise approximation, but the important thing

is that if we look at a very close neighborhood of $x = 0$, the approximation is not *that* bad: consider, for example $x_0 = 0.0001$. Our approximation gives $e^{x_0} \approx 1$, which is not far from the more precise value $e^{x_0} = 1.000100005$ (the value here is shown up to the ninth decimal). Of course, the closer we get to $a = 0$, the better our approximation gets: for example, with $x_1 = 0.00001$, we get $e^{x_1} \approx 1$ again, where in reality $e^{x_1} = 1.000010000$ (also shown here up to the ninth decimal). On the other hand, if we as we get farther away from $x = 0$, the approximation becomes worse and worse, as seen in Table 1.2 below.

x	e^x (exact)	Δ (error)
0.000	1.000000000	0.000000000
0.001	1.001000500	0.001000500
0.010	1.010050167	0.010050167
0.100	1.105170918	0.105170918
0.500	1.648721271	0.648721271
0.510	1.665291195	0.665291195
0.520	1.682027650	0.682027650
1.000	2.718281828	1.718281828
1.100	3.004166024	2.004166024
1.500	4.481689070	3.481689070
2.000	7.389056099	6.389056099

Table 1.2: Zero order Taylor series approximation of e^x.

Now, let's take this a step further: since the derivative of a function at a point tells us how the functions changes close to the point, we can use this information to improve our approximation by adding the value of the first derivative of e^x at $x = 0$, which is also 1. In fact, this is true for any order derivative of e^x, since $\frac{d^n}{dx^n}e^x = e^x$ for any $n \in \mathbb{N}$. Since the derivative changes with the value of x, we will multiply it by x in the approximation. Thus we get

$$e^x \approx 1 + 1 \cdot x = 1 + x, \tag{1.6}$$

which we call the **first order** approximation of the function e^x. Table 1.3 below shows the same values from Table 1.2 for x, but using the first order approximation for e^x.

By examining Table 1.3 it's clear that $1 + x$ is a much better approximation for e^x than just 1. Of course, we can take a further step: the second derivative of a function at a point tells us how the change in the function itself changes around that point. We can take the second derivative of e^x (which is by itslef e^x), again substitute $x = 0$, get $e^0 = 1$, and use this with x^2 to apperoximate e^x even better. In fact, let's already take this idea to its logical conclusion: we'll simply use the infinite power series in x, i.e. $1, x, x^2, x^3, \ldots, x^n, \ldots$ and

x	$1+x$	e^x (exact)	Δ (error)
0.000	1.000000000	1.000000000	0.000000000
0.001	1.001000000	1.001000500	0.000000500
0.010	1.010000000	1.010050167	0.000050167
0.100	1.100000000	1.105170918	0.005170918
0.500	1.500000000	1.648721271	0.148721271
0.510	1.510000000	1.665291195	0.155291195
0.520	1.520000000	1.682027650	0.162027650
1.000	2.000000000	2.718281828	0.718281828
1.100	2.100000000	3.004166024	0.904166024
1.500	2.500000000	4.481689070	1.981689070
2.000	3.000000000	7.389056099	4.389056099

Table 1.3: First order Taylor series approximation of e^x.

attach to each x^k the value given by $\frac{d^n}{dx^n}e^x\Big|_{x=0} = 1$, so we get

$$e^x \approx 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n. \tag{1.7}$$

This, however doesn't quite work. The coefficients of each term have to be adjusted. By assuming that at the limit where $n \to \infty$ the approximation should be with zero error, we get that for each n,

$$\frac{d^n}{dx^n}e^x = \frac{d^n}{dx^n}\left(1 + x + x^2 + \dots\right) \equiv s(x). \tag{1.8}$$

Calculating the general n-th derivative of Equation 1.7 is rather easy. Let's examine the case for $n = 3$: the first three derivatives of Equation 1.7 are

$$\begin{aligned}
\tfrac{d}{dx}s(x) &= 1 + 2x + 3x^2 + 4x^3 + \dots, \\
\tfrac{d^2}{dx^2}s(x) &= 2 + 6x + 12x^2 + \dots, \\
\tfrac{d^3}{dx^3}s(x) &= 6 + 24x + \dots,
\end{aligned}$$

and if we subtitute $x = 0$ we get... TBW!

The most general polynomial expansion of a smooth function $f(x)$ is an infinite polynomial in x:

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \tag{1.9}$$

If $P(x) = f(x)$ then all their derivatives are the equal, i.e.

$$\begin{aligned}
f'(x) &= P'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots \\
f''(x) &= P''(x) = 2a_2 + (2\cdot 3)a_3x + (4\cdot 3)a_4x^2 + \dots, \\
f'''(x) &= P'''(x) = (2\cdot 3)a_3 + (4\cdot 3\cdot 2)a_4x + \dots, \\
f''''(x) &= P''''(x) = (4\cdot 3\cdot 2)a_4 + \dots, \\
&\vdots \\
f^{(n)}(x) &= P^{(n)}(x) = n!a_n + \dots, \\
&\vdots
\end{aligned} \tag{1.10}$$

Subtituting $x = 0$ into the equality we get

$$\begin{aligned}
f'(0) &= P'(0) = a_1 + \cancel{2a_2x} + \cancel{3a_3x^2} + \cancel{4a_4x^3} + \cancel{\dots} \\
f''(0) &= P''(0) = 2a_2 + \cancel{(2\cdot 3)a_3x} + \cancel{(4\cdot 3)a_4x^2} + \cancel{\dots}, \\
f'''(0) &= P'''(0) = (2\cdot 3)a_3 + \cancel{(4\cdot 3\cdot 2)a_4x} + \cancel{\dots}, \\
f''''(0) &= P''''(0) = (4\cdot 3\cdot 2)a_4 + \cancel{\dots}, \\
&\vdots \\
f^{(n)}(0) &= P^{(n)}(x) = n!a_n + \cancel{\dots}, \\
&\vdots
\end{aligned} \tag{1.11}$$

or more succinctly:

$$a_n = \frac{f^n(x)}{n!}. \tag{1.12}$$

Therefore, the full taylor polynomial for $f(x)$ is

$$\begin{aligned}
P(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \dots \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.
\end{aligned} \tag{1.13}$$

If we take only the first N terms in $P(x)$ we get

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!}x^n. \tag{1.14}$$

Note 1.2 $(N+1)$-th derivative an N-th partial Taylor sum

Since the N-th partial sum of any taylor series $P(x)$ is

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!}x^n,$$

all terms except $\frac{a_N}{N!}$ cancel out by the N-derivative. And at the $(N+1)$-derivative this term cancels out as well, since its a constant. Therefore from its $(N+1)$-th term and on, the partial sum $P_N(x)$ reduces to zero:

$$P^{(k>N)}(x) = 0.$$

!

Example 1.2 Taylor series of $\exp(x)$

Any order derivative of $\exp(x)$ is equal to $\exp(x)$ itself, i.e. in more mathematical notation:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(x) = \exp(x), \ \forall n \in \mathbb{N}.$$

Thus, we know the value at $x = 0$ of any order derivative of $\exp(x)$:

$$\left.\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(x)\right|_{x=0} = \exp(0) = 1.$$

The Taylor series of $\exp(x)$ is therefore very easy to calculate around $x = 0$:

$$\begin{aligned}\exp(x) &= \sum_{k=0}^{\infty} \frac{\exp\left((n)\right)(0)}{n!}x^n \\ &= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots\end{aligned}$$

Example 1.3 Taylor series of $\sin x$ and $\cos(x)$

The derivatives of both $\sin(x)$ and $\cos(x)$ are periodic:

$$\begin{array}{ccc} \cos(x) & \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} & -\sin(x) \\ \uparrow{\scriptstyle \frac{\mathrm{d}}{\mathrm{d}x}} & & \downarrow{\scriptstyle \frac{\mathrm{d}}{\mathrm{d}x}} \\ \sin(x) & \xleftarrow[\frac{\mathrm{d}}{\mathrm{d}x}]{} & -\cos(x) \end{array}$$

And here it is summed as a table:

n	$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sin(x)$	$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\cos(x)$
0	$\sin(x)$	$\cos(x)$
1	$\cos(x)$	$-\sin(x)$
2	$-\sin(x)$	$-\cos(x)$
3	$-\cos(x)$	$\sin(x)$
4	$\sin(x)$	$\cos(x)$
5	$\cos(x)$	$-\sin(x)$
	$\vdots$	

And since

$$\begin{aligned}\sin(0) &= -\sin(0) = 0, \\ \cos(0) &= 1, \\ -\cos(0) &= -1,\end{aligned}$$

we can easily derive the Taylor series for both functions

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around $x = 0$:

$$\begin{aligned}
\sin(x) &= \sum_{k=0}^{\infty} \frac{\sin^{(n)}(0)}{n!} x^n \\
&= \frac{\sin(0)}{0!}x^0 + \frac{\cos(0)}{1!}x^1 + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3 + \dots \\
&= 0 + x + 0 - x^3 + 0 + x^5 + 0 - x^7 + 0 + \dots \\
&= x - x^3 + x^5 - x^7 + \dots \\
\cos(x) &= \sum_{k=0}^{\infty} \frac{\cos^{(n)}(0)}{n!} x^n \\
&= \frac{\cos(0)}{0!}x^0 + \frac{-\sin(0)}{1!}x^1 + \frac{-\cos(0)}{2!}x^2 + \frac{\sin(0)}{3!}x^3 + \dots \\
&= 1 + 0 - x^2 + 0 + x^4 + 0 - x^6 + \dots \\
&= 1 - x^2 + x^4 - x^6 + \dots
\end{aligned}$$

Looking at the above results for $\exp(x)$, $\sin(x)$ and $\cos(x)$, one can derive a nice connection: notice that

$$\mathrm{i}^0 = 1,\ \mathrm{i}^1 = \mathrm{i},\ \mathrm{i}^2 = -1,\ \mathrm{i}^3 = -\mathrm{i},\ \mathrm{i}^4 = 1,\ \mathrm{i}^5 = \mathrm{i},\ \dots \tag{1.15}$$

and if we plug $\mathrm{i}x$ into the Taylor expansion of $\exp(x)$ we get

$$\begin{aligned}
\exp(ix) &= 1 + \mathrm{i}x + \frac{1}{2!}(\mathrm{i}x)^2 + \frac{1}{3!}\mathrm{i}x^3 + \frac{1}{4!}\mathrm{i}x^4 + \frac{1}{5!}\mathrm{i}x^5 \\
&= \mathrm{i}^0 + \mathrm{i}^1 x + \frac{1}{2!}\mathrm{i}^2 x^2 + \frac{1}{3!}\mathrm{i}^3 x^3 + \frac{1}{4!}\mathrm{i}^4 x^4 + \frac{1}{5!}\mathrm{i}^5 x^5 + \dots \\
&= 1 + \mathrm{i}x - \frac{1}{2!}x^2 - \frac{1}{3!}\mathrm{i}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}\mathrm{i}x^5 + \dots \\
&= 1 + -\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots + \mathrm{i}x - \mathrm{i}\frac{1}{3!}x^3 + \mathrm{i}\frac{1}{5!}x^5 + \dots \\
&= \left(1 + -\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots\right) + \mathrm{i}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots\right) \\
&= \cos(x) + \mathrm{i}\sin(x).
\end{aligned} \tag{1.16}$$

Since in practice we can't use the full Taylor series to approximate a function at a point, we must intelligently choose how many terms we want to use to get a "good enough" approximation - that is, we want that the differemce between our polynomial approximation is the actual function to be no more than some value Δ:

$$\left|f(a) - P_n(a)\right| \leq \Delta. \tag{1.17}$$

We can construct a **remainder function** which gives the difference between the actual function value and out n-th degree approximation at every point on the interval $I = [0, a]$:

$$R_n(x) = f(x) - P_n(x). \tag{1.18}$$

Thanks to Lagrange and other great mathematicians, the following always holds for the remainder function: if the the $n+1$-th derivative of the original function is bounded on the open interval $(0, a)$, i.e.

$$\left|f^{(n+1)}(x)\right| \leq M, \tag{1.19}$$

then there's a finite value M such that

$$\left|R_n(x)\right| \leq \left|\frac{Mx^{n+1}}{(n+1)!}\right|. \tag{1.20}$$

This is called the **Lagrange error bound**. No proof is provided for it in this course (TBW: add reference to one?).

Example 1.4 Lagrange error bound

Let's calculate what is the minimum n needed to calculate $\cos(0.25)$ to within $\Delta = 10^{-3} = 0.001$ using a Taylor series: the first four derivatives of $\cos(x)$ are

$$\begin{array}{ccc} \cos(x) & \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} & -\sin(x) \\ \uparrow{\scriptstyle \frac{\mathrm{d}}{\mathrm{d}x}} & & \downarrow{\scriptstyle \frac{\mathrm{d}}{\mathrm{d}x}} \\ \sin(x) & \xleftarrow[\frac{\mathrm{d}}{\mathrm{d}x}]{} & -\cos(x) \end{array}$$

All of these functions are bounded to the interval $[-1, 1]$, and so their absolute value is bounded by 1. We can replace M in Equation 1.20 by 1 and we get

$$\left|R_n(0.25)\right| \leq \left|\frac{0.25^{n+1}}{(n+1)!}\right|.$$

It's rather difficult to isolate n from this relation, but we can simply substitute different values to it until we get a result smaller than 10^{-3}:

n	$\lvert R_n \rvert$
0	$\frac{1}{4^{1} \cdot 1!} = 0.25$
1	$\frac{1}{4^{2} \cdot 2!} = 0.03125$
2	$\frac{1}{4^{3} \cdot 3!} \approx 0.0026$
3	$\frac{1}{4^{4} \cdot 4!} \approx 0.0001627$

Meaning that we need at least the $n = 3$ term to reach our required precision.

1.4.3 *Projections, rejections and the dot product*

Every vector $\mathbf{a} \in \mathbb{R}^n$ can be decomposed into two components: one in the direction of another vector $\mathbf{b}$, and one orthogonal to $\mathbf{b}$. These components are called, respectively, the **projection** of $\mathbf{a}$ onto $\mathbf{b}$ and the **rejection** of $\mathbf{a}$ from $\mathbf{b}$. In $\mathbb{R}^3$ the same procedure can be

applied to projecting a vector onto- and rejecting it from a plane, since every plane in $\mathbb{R}^3$ has a single normal vector up to a sign (Figure 1.3).

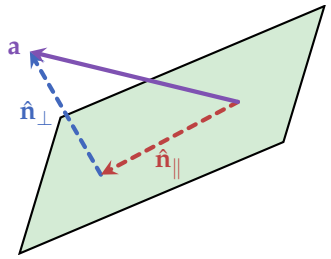

Figure 1.3: Decomposing the vector $\mathbf{a}$ into two components in respect to a plane: one parallel to the plane ($\mathbf{a}_\parallel$) and one orthogonal to it ($\mathbf{a}_\perp$). These are also called the projection and rejection of $\mathbf{a}$ on the plane, respectively.

The projection of a vector onto another vector gives rise to an important operation between two vectors: the **dot product**: given a vector $\mathbf{a}$, its projection on the vector $\mathbf{b}$ is

$$\text{proj}_\mathbf{b}\mathbf{a} = |\mathbf{a}|\cos(\theta), \tag{1.21}$$

where θ is the angle between the vectors. See Figure 1.4 for a visual representation. We then define the dot product between the two vectors as

$$\langle \mathbf{a}, \mathbf{b} \rangle = |\mathbf{b}|\,\text{proj}_\mathbf{b}\mathbf{a} = |\mathbf{b}|\,|\mathbf{a}|\cos(\theta). \tag{1.22}$$

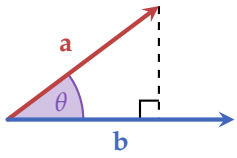

Figure 1.4: Projection of $\mathbf{a}$ onto $\mathbf{b}$: no matter how many dimensions we use, we can always rotate our view such that we look at the plane spanned by both vectors, and $\mathbf{b}$ lies horizontally. In this way it's easy to see why the projection of $\mathbf{a}$ onto $\mathbf{b}$ is $|\mathbf{a}|\cos(\theta)$: it's simply the definitions of the cosine function ("side next to the angle divided by the hypotenuse").

It is of course convenient to have a way to calculate the dot product component-wise. To find such form, we use the same two vectors $\mathbf{a}$ and $\mathbf{b}$ from before, and their difference $\mathbf{c} = \mathbf{a} - \mathbf{b}$ (Figure 1.5).

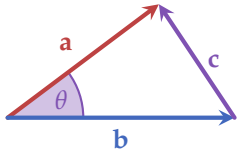

Figure 1.5: Text text text.

Thanks to trigonometry we know that the following relation holds ("the law of cosines"):

$$\begin{aligned} |\mathbf{c}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos(\theta) \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\langle \mathbf{a}, \mathbf{b} \rangle, \end{aligned} \tag{1.23}$$

which we can rearrange into

$$\langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2 \right). \tag{1.24}$$

In $\mathbb{R}^2$ the component version of the vectors are

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_x \\ c_y \end{bmatrix} = \begin{bmatrix} a_x - b_x \\ a_y - b_y \end{bmatrix}, \tag{1.25}$$

and Equation 1.24 becomes

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle &= \frac{1}{2}\left(a_x^2 + a_y^2 + b_x^2 + b_y^2 - (a_x - b_x)^2 - \left(a_y - b_y\right)^2\right) \\ &= \frac{1}{2}\left(a_x^2 + a_y^2 + b_x^2 + b_y^2 - a_x^2 + 2a_x b_x - b_x^2 - a_y^2 + 2a_y b_y + b_y^2\right) \\ &= a_x b_x + a_y b_y.\end{aligned} \tag{1.26}$$

In general, we can apply the same procedure in $\mathbb{R}^n$: the components of the vectors are

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{bmatrix}, \tag{1.27}$$

and Equation 1.24 becomes

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle &= \frac{1}{2}\Big(a_1^2 + a_2^2 + \cdots + a_n^2 + b_1^2 + b_2^2 + \cdots + b_n^2 - (a_1 - b_1)^2 \\ &\quad - (a_2 - b_2)^2 - \cdots - (a_n - b_n)^2\Big) \\ &= \frac{1}{2}\Big(a_1^2 + a_2^2 + \cdots + a_n^2 + b_1^2 + b_2^2 + \cdots + b_n^2 - a_1^2 + 2a_1 b_1 - \\ &\quad b_1^2 - a_2^2 + 2a_2 b_2 + b_2^2 + \cdots + a_n^2 + 2a_n b_n + b_n^2\Big) \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ &= \sum_{i=1}^{n} a_i b_i.\end{aligned} \tag{1.28}$$

Let us now explore some key properties of the dot product (the reader is encouraged to prove items 2 and 3):

1. Since the product of real numbers is commutative (i.e. $ab = ba$), the dot product is also commutative: if we exchange v_i and w_i in Equation 1.28 the result stays the same.

2. The dot product is distributive over vector addition: $\langle \mathbf{a}, \mathbf{b} + \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{a}, \mathbf{c} \rangle$.

3. We can take out scalars from within the product: given the scalars ϵ_1 and ϵ_2, $\langle \epsilon_1 \mathbf{a}, \epsilon_2 \mathbf{b} \rangle = \epsilon_1 \epsilon_2 \langle \mathbf{a}, \mathbf{b} \rangle$.

4. The angle between two non-zero orthogonal vectors $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \frac{\pi}{2}$, and therefore $\langle \mathbf{a}, \mathbf{b} \rangle = |\mathbf{a}|\,|\mathbf{b}| \cos\left(\frac{\pi}{2}\right) = 0$ (since $\cos\left(\frac{\pi}{2}\right) = 0$). This works in the other way around as well: if the dot product of two non-zero vectors is zero, then they are orthogonal.

5. We can't cancel vectors in the dot product the same we do with real numbers: consider three vectors $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b}$, $\mathbf{c}$ such that $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$. We can redistribute the equation to give $\langle \mathbf{a}, \mathbf{b} - \mathbf{c} \rangle = 0$, which means that $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$ are orthogonal, and doesn't imply that $\mathbf{b} - \mathbf{c} = \mathbf{0}$ (i.e. $\mathbf{b}$ and $\mathbf{c}$ might not be equal).