PORTFOLIO THEORY

LECTURE 2

QUADRATIC FORMULA AND COMPLETING SQUARE

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

FIRST, SECOND DERIVATIVES AND OPTIMIZATION

- 1^{st} derivative is the slope, 2^{nd} derivative is the change of slope
- When 1^{st} derivative is 0, $f'(x^*) = 0$, (also called the First Order Condition, FOC). x^* can be a maximum, a minimum or an inflection point
- x^* is a (local) maximum if $f'(x_-^*) > 0$ and $f'(x_+^*) < 0$ or $f''(x^*) < 0$ (f is concave)
- x^* is a (local) minimum if $f'(x_-^*) < 0$ and $f'(x_+^*) > 0$ or $f''(x^*) > 0$ (f is convex)
- x^* is an inflection point if $\mathrm{sign}(f'(x_-^*)) = sign\left(f'(x_+^*)\right)$

OPTIMIZATION WITH CONSTRAINTS

- First Order Condition (FOC): $f'(x^*) = 0$ (necessary condition)
- Second Order Condition (SOC): max if $f''(x^*) < 0$; min if $f''(x^*) > 0$

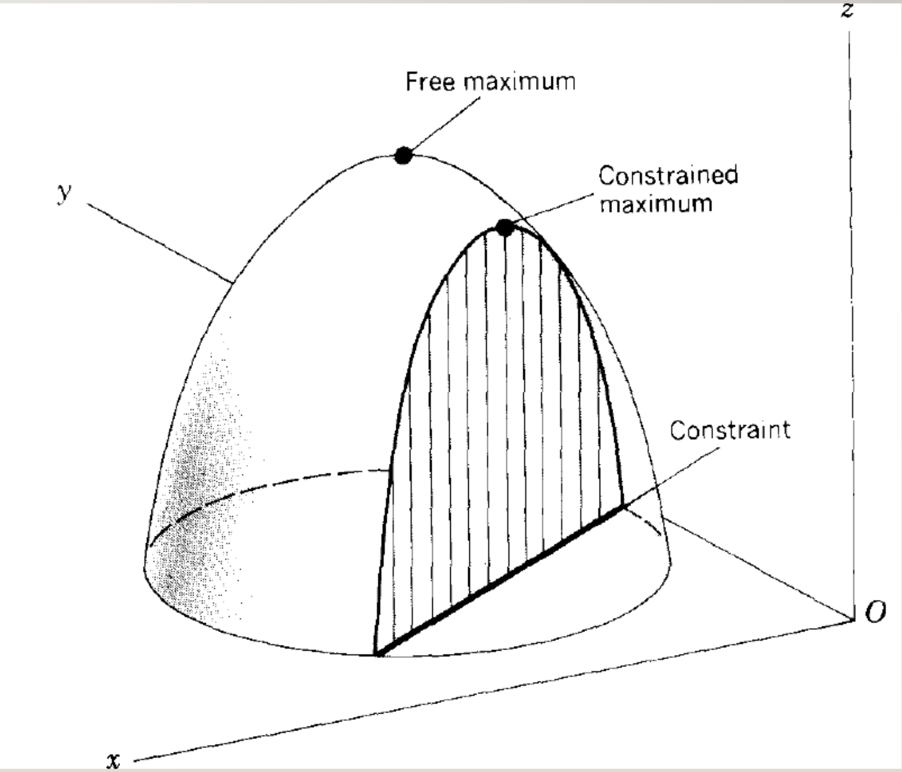
Lagrange Multiplier Method:
Given $z = f(x, y)$ subject to $g(x, y) = c$

Langrangian function:

$$Z = f(x,y) + \lambda[c - g(x,y)]$$

λ is called the Lagrange multiplier
The stationary values of Z is regarded as a function of $\lambda, x,$ $and\ y$. The FOC are:

$$Z_\lambda = c - g(x,y) = 0$$
$$Z_x = f_x - \lambda g_x = 0$$
$$Z_y = f_y - \lambda g_y = 0$$



LAGRANGE MULTIPLIER METHOD- EXAMPLE

$$z = x_1^2 + x_2^2 \qquad \text{subject to} \qquad x_1 + 4x_2 = 2$$

The Lagrangian function is

$$Z = x_1^2 + x_2^2 + \lambda(2 - x_1 - 4x_2)$$

for which the necessary condition for a stationary value is

$$\left.\begin{aligned} Z_\lambda &= 2 - x_1 - 4x_2 = 0 \\ Z_1 &= 2x_1 - \lambda = 0 \\ Z_2 &= 2x_2 - 4\lambda = 0 \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{array}{rrrl} & x_1 & +\,4x_2 & = 2 \\ -\lambda & +\,2x_1 & & = 0 \\ -4\lambda & & +\,2x_2 & = 0 \end{array}\right.$$

The stationary value of Z, defined by the solution

$$\bar{\lambda} = \tfrac{4}{17} \qquad \bar{x}_1 = \tfrac{2}{17} \qquad \bar{x}_2 = \tfrac{8}{17}$$

MATRIX REVIEW

- Students are expected to have elementary knowledge of matrix
- Let A and B are matrices, matrix addition and subtraction are operated element wise, i.e. $A_{ij} \pm B_{ij}$ where the subscript correspond to the element at the i^{th} row and j^{th} column
- For $A \pm B$ to be defined (or we sometimes call them 'conformable', they must have the same dimension, say $m \times n$, where m is the dimension of row and n is the dimension of column. If $m = n$, we call the matrix a square matrix. If $m = 1$, we usually call the matrix a row vector, if $n = 1$, we usually call it a column vector. In case $m = n = 1$, it is a number and we call it a 'scalar'
- Multiplication of a scalar c with a matrix A is just performed by multiplying the element each A_{ij} by c for $i = 1, \dots, m; j = 1, \dots, n$.
- Matrix transpose, usually denoted as A' or A^T, is to switch the row and column dimension, i.e. $A'_{ij} = A_{ji}$ and so the dimension of A' will be $n \times m$.

MATRIX REVIEW

- Matrix multiplication is defined in a special way. For the product AB to be defined (or conformable), the column dimension by A needs to equal the row dimension of B, that is, if the dimension of A is $m \times n$, then the row dimension of B needs to be n. Suppose B has dimension $n \times p$, AB will then have dimension $m \times p$. The product is defined as $(AB)_{ik} = \sum_j A_{ij}B_{jk}$, where $i = 1, \dots, m; j = 1, \dots, n; k = 1, \dots, p$. We note that the dimension n no longer appears in the product AB as it is being 'summed'.
- Matrix does not have 'division', so we cannot write A/B. However, we know for two scalars a and b, division is defined as a/b. We can find the 'inverse' of b by defining $b^{-1} = 1/b$ and so $a/b = b^{-1}a$. The concept of matrix inverse comes similarly. Nevertheless, the role of '1' (identity) in scalar is represented by an identity matrix which is a square matrix I with '1's on the diagonal but zero otherwise. The inverse of matrix A will is defined as matrix A^{-1} such that $A^{-1}A = I$.

MATRIX REVIEW

- Matrix A has inverse only if it is a square matrix. The additional condition is that A needs to be linearly independent, i.e. we cannot find row (or column) being a linear combination other row(s) (or column(s)).
- Formally speaking, a matrix A is linearly independent if there exists no vector v, other than $v = (0, \dots, 0)'$ such that $Av = (0, \dots, 0)'$
- If a matrix A is linearly independent, we say it is non-singular, or it has full rank; else if it is linearly dependent, or singular, with the rank less than full dimension. To check whether a matrix is non-singular, we need the concept of 'determinant'. The determinant of a matrix is non-zero if and only if it is non-singular. Thus, to say that a matrix has full-rank, or non-singular, or has non-zero determinant or invertible means the same thing.

MATRIX REVIEW

- The calculation of determinant is rather tedious, and we will only review the $2x2$ and 3×3 cases. The general $n \times n$ case will then be straight forward extension of the 3×3 case. For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A, denoted $|A|$ is calculated as $ad - bc$.

- For a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, we need to pick a row or column (any row or column will work and will yield the same result), say we pick the first row. For a_{11}, we define the 'minor' of a_{11}as $|M_{11}| \coloneqq \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, which is essentially the sub-matrix removing the corresponding row and column of a_{ij}. Similarly, we define $|M_{12}| \coloneqq \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and $|M_{13}| \coloneqq \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

MATRIX REVIEW

- We further define a 'cofactor' as $\left|C_{ij}\right| := (-1)^{i+j}\left|M_{ij}\right|$, then the determinant (if we expand at the i^{th} row) is calculated as

$$|A| = \sum_j a_{ij}\left|C_{ij}\right|$$

- The result will be the same if we expand at the j^{th} column, but the summation will be over the rows in this case.
- There are couple of properties about matrix transpose and inverse:
  - $(AB)' = B'A'$ (prove by checking the conformability)
  - $(AB)^{-1} = B^{-1}A^{-1}$ (proof: Let $C = (AB)^{-1}$, $CABB^{-1}A^{-1} = IB^{-1}A^{-1} = B^{-1}A^{-1}$. Note that $CABB^{-1}A^{-1} = CAIA^{-1} = C$, so $C = B^{-1}A^{-1}$
  - $(A')^{-1} = (A^{-1})'$ (proof: Let $D = (A')^{-1}$, $DA' = I = I' = (AA^{-1})' = (A^{-1})'A'$)

MATRIX REVIEW

- There are some properties of determinant:

1. The determinant of a matrix has the same value to that of its transpose
2. Interchange of any 2 rows or columns will alter the sign but not the value of the determinant
3. Multiplication of any one row (or one column) by a scalar k will change the determinant k-fold
4. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the value of the determinant unaltered
5. If one row (or column) is a linear combination of another row(s) (or column(s)), the value of the determinant will be zero
6. Expansion of a determinant by alien cofactors (cofactors of a different row or column) will be zero

MATRIX REVIEW

- Property 6 is important because it provides a method for finding the inverse of a matrix. Consider a matrix A (with elements a_{ij}) and another matrix C with each element being the cofactor of each a_{ij}. Then we have:

$$AC' = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}|C_{1j}| & \sum_{j=1}^{n} a_{1j}|C_{2j}| & \dots & \sum_{j=1}^{n} a_{1j}|C_{nj}| \\ \sum_{j=1}^{n} a_{2j}|C_{1j}| & \sum_{j=1}^{n} a_{2j}|C_{2j}| & \dots & \sum_{j=1}^{n} a_{2j}|C_{nj}| \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{n} a_{nj}|C_{1j}| & \sum_{j=1}^{n} a_{nj}|C_{2j}| & \dots & \sum_{j=1}^{n} a_{nj}|C_{nj}| \end{bmatrix} = \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & |A| \end{bmatrix} = I|A|$$

$$AC' = I|A|$$

$$A^{-1}AC' = A^{-1}|A|$$

$$A^{-1} = \frac{C'}{|A|}$$

MATRIX REVIEW

- One of the main uses of matrix inverse is to solve a system of equation of the form:

$$\boldsymbol{Ax} = \boldsymbol{d}$$

where $\boldsymbol{x}$ is an $n \times 1$ vector of unknown variables and d is an $n \times 1$ vector of constants. Pre-multiplying both sides by A^{-1} we can solve for solution x^*:

$$x^* = A^{-1}$$

- We can replace A^{-1} by $C'/|A|$ and we get:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d_1|C_{11}| + d_2|C_{21}| + \cdots + d_n|C_{n1}| \\ d_1|C_{12}| + d_2|C_{22}| + \cdots + d_n|C_{n2}| \\ \vdots \\ \vdots \\ d_1|C_{1n}| + d_2|C_{2n}| + \cdots + d_n|C_{nn}| \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} \sum_{i=1}^n d_i|C_{i1}| \\ \sum_{i=1}^n d_i|C_{i2}| \\ \vdots \\ \vdots \\ \sum_{i=1}^n d_i|C_{in}| \end{bmatrix}$$

- This is known as the Cramer's rule. It tells us that we do not need to find the inverse explicitly to solve for the system of equations. In fact, the solution for x_j^* would be replacing the j^{th} column of matrix A by d and evaluate the determinant, to be divided by the determinant of A

MATRIX DIFFERENTIATION

- Given $\boldsymbol{y} = \boldsymbol{\psi}(\boldsymbol{x})$ where $\boldsymbol{y}$ is an m-element vector and **x** is an n-element vector and $\boldsymbol{\psi}(\boldsymbol{x})$ is the linear transformation.

- $\frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ . & & & \\ . & & & \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$ is called the Jacobian matrix of the transformation

- If y is a scalar, then $\frac{\partial y}{\partial x}$ will be a vector of $\left(\frac{\partial y}{\partial x_1} \ \ldots \ \frac{\partial y}{\partial x_n}\right)$

LAGRANGE APPROACH TO MEAN-VARIANCE FRONTIER

- All investors are mean-variance optimizers
- Theorem: As long as the covariance matrix of returns is non-singular, there is a mean-variance frontier
- *Notations:*

$$\alpha = \begin{pmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{pmatrix}, \tilde{R} = \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{r}_n \end{pmatrix}, R = E(\tilde{R}) = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \mu_n \end{pmatrix}, \Omega = cov(\tilde{R}, \tilde{R}') \text{ (write out as classwork)}$$

- $\min_{\alpha} \alpha' \Omega \alpha \;\; s.t. \, \alpha' R = \mu_p; \;\; \alpha' \mathbb{I} = 1$
- $\mathcal{L} = \frac{1}{2} \alpha' \Omega \alpha + \lambda(\mu_p - \alpha' R) + \gamma(1 - \alpha' \mathbb{I})$, solving for optimal α_p, the first order condition is:
- $\Omega \alpha_p - \lambda R - \gamma \mathbb{I} = 0 \Rightarrow \alpha_p = \lambda \Omega^{-1} R + \gamma \Omega^{-1} \mathbb{I}$. With the 2 constraints, we have

$$\lambda R' \Omega^{-1} R + \gamma R' \Omega^{-1} \mathbb{I} = \mu_p$$
$$\lambda \mathbb{I}' \Omega^{-1} R + \gamma \mathbb{I}' \Omega^{-1} \mathbb{I} = 1$$

LAGRANGE APPROACH TO MEAN-VARIANCE FRONTIER

Define $A = R'\Omega^{-1}R, B = R'\Omega^{-1}\mathbb{I} = \mathbb{I}'\Omega^{-1}R, C = \mathbb{I}'\Omega^{-1}\mathbb{I}$

- $\lambda = \frac{\mu_p C - B}{AC - B^2}, \gamma = \frac{A - B\mu_p}{AC - B^2}$
- $\alpha_p = \frac{\mu_p C - B}{AC - B^2}\Omega^{-1}R + \frac{A - B\mu_p}{AC - B^2}\Omega^{-1}\mathbb{I}$
- $\alpha_p = \frac{\mu_p C - B}{AC - B^2}\Omega^{-1}R + \frac{A - B\mu_p}{AC - B^2}\Omega^{-1}\mathbb{I} = \frac{B(\mu_p C - B)}{AC - B^2}\frac{\Omega^{-1}R}{B} + \frac{C(A - B\mu_p)}{AC - B^2}\frac{\Omega^{-1}\mathbb{I}}{C}$
- $= x\frac{\Omega^{-1}R}{B} + (1 - x)\frac{\Omega^{-1}\mathbb{I}}{C}$
- Note that $\frac{\mathbb{I}'\Omega^{-1}R}{B} = 1 = \frac{\mathbb{I}'\Omega^{-1}\mathbb{I}}{C}$, i.e. $\alpha_d = \frac{\Omega^{-1}R}{B}$ and $\alpha_g = \frac{\Omega^{-1}\mathbb{I}}{C}$ are portfolio weights!

TWO-FUND SEPARATION

- Every optimal portfolio is spanned by α_d and α_g - Two-Fund Separation
- Also, any two distinct efficient portfolios can act as separating portfolio:
- $\alpha_1 = x_1\alpha_d + (1 - x_1)\alpha_g, \alpha_2 = x_2\alpha_d + (1 - x_2)\alpha_g$
- $\psi\alpha_1 + (1 - \psi)\alpha_2 = [\psi x_1 + (1 - \psi)x_2]\alpha_d + [\psi(1 - x_1) + (1 - \psi)(1 - x_2)]\alpha_g = y\alpha_d + (1 - y)\alpha_g$
- Thus, a portfolio of MV optimal portfolios is also MV optimal

MEAN-VARIANCE FRONTIER - CONTINUE

- $\Omega\alpha_p = \lambda R + \gamma\mathbb{1} = \frac{\mu_p C - B}{AC - B^2} R + \frac{A - B\mu_p}{AC - B^2}\mathbb{1}$

- $var(Port_p) = \alpha'\Omega\alpha = \frac{C\mu_p^2 - B\mu_p}{AC - B^2} + \frac{A - B\mu_p}{AC - B^2} = \frac{C\mu_p^2 - 2B\mu_p + A}{AC - B^2} = \frac{C\left(\mu_p - \frac{B}{C}\right)^2 - \frac{B^2}{C} + A}{AC - B^2} =$
$\frac{C\left(\mu_p - \frac{B}{C}\right)^2 + \frac{AC - B^2}{C}}{AC - B^2} = \frac{C\left(\mu_p - \frac{B}{C}\right)^2}{AC - B^2} + \frac{1}{C}$

- At $\mu_p = \frac{B}{C}$, minimum variance is attained at $\frac{1}{C}$, the corresponding weight $\alpha_{minvar} = \frac{\Omega^{-1}\mathbb{1}}{\mathbb{1}'\Omega^{-1}\mathbb{1}} = \alpha_g$. Let's call the minimum weight portfolio as $\tilde{R}_g$ and $\mu_g = \frac{B}{C}$

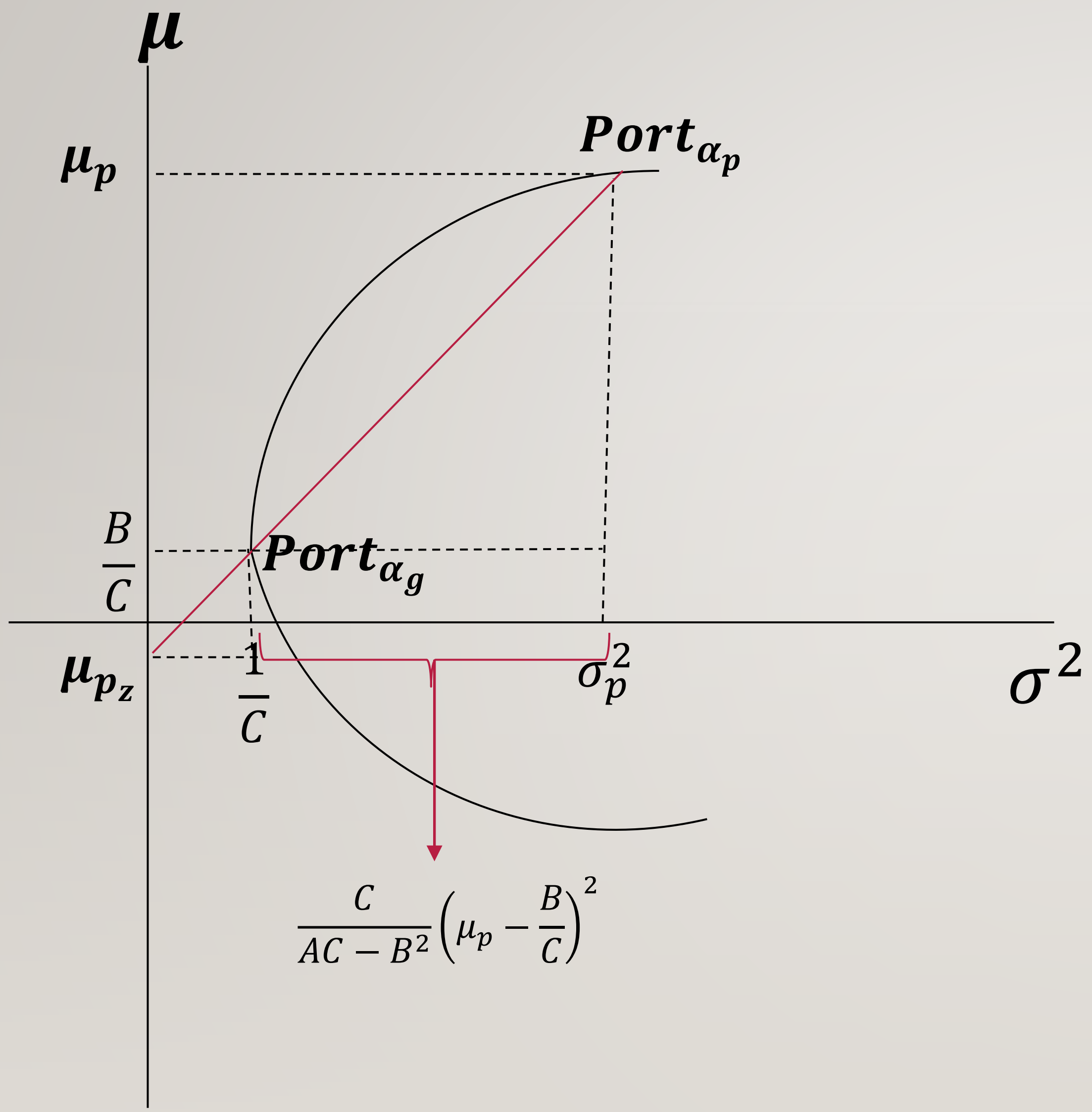
- **Classwork:** Draw the graph and locate μ_p and σ_p^2

MEAN-VARIANCE FRONTIER - CONTINUE

- For any portfolio A, $Cov\left(\tilde{R}_A,\ \tilde{R}_p{}'\right) = Cov\left(\alpha_A'\tilde{R}, \tilde{R}'\alpha_p\right) = \alpha_A'\Omega\alpha_p = \alpha_A'(\lambda R + \gamma\mathbb{I}) = \gamma + \lambda\mu_A$

$$\gamma + \lambda\mu_A = \frac{A - B\mu_p}{AC - B^2} + \frac{\mu_p C - B}{AC - B^2}\mu_A = \frac{C}{AC - B^2}\left(\mu_p - \frac{B}{C}\right)\left(\mu_A - \frac{B}{C}\right) + \frac{1}{C} \qquad (*)$$

- If $\mu_A = \mu_p$, then it is the variance of the MV portfolio
- $Cov\left(\tilde{R}_A, \tilde{R}_g\right) = \frac{1}{C}$ and it holds for all assets!
- Theorem: For every MV portfolio, there exists some 'sister' portfolio such that their covariance is zero. If $Port_p$ is MV optimal and $\mu_p \neq \frac{B}{C}$, there exist $Port_{p_z}$ such that $Cov\left(\tilde{R}_p, \tilde{R}_{p_z}\right) = 0$, where if $\mu_p > \frac{B}{C}, \mu_{p_z} < \frac{B}{C}$ and if $\mu_p < \frac{B}{C}, \mu_{p_z} > \frac{B}{C}$



In mean-variance space:

To show μ_{p_z} is straight line from Port_{α_p} through $Port_{\alpha_g}$ to the μ axis:

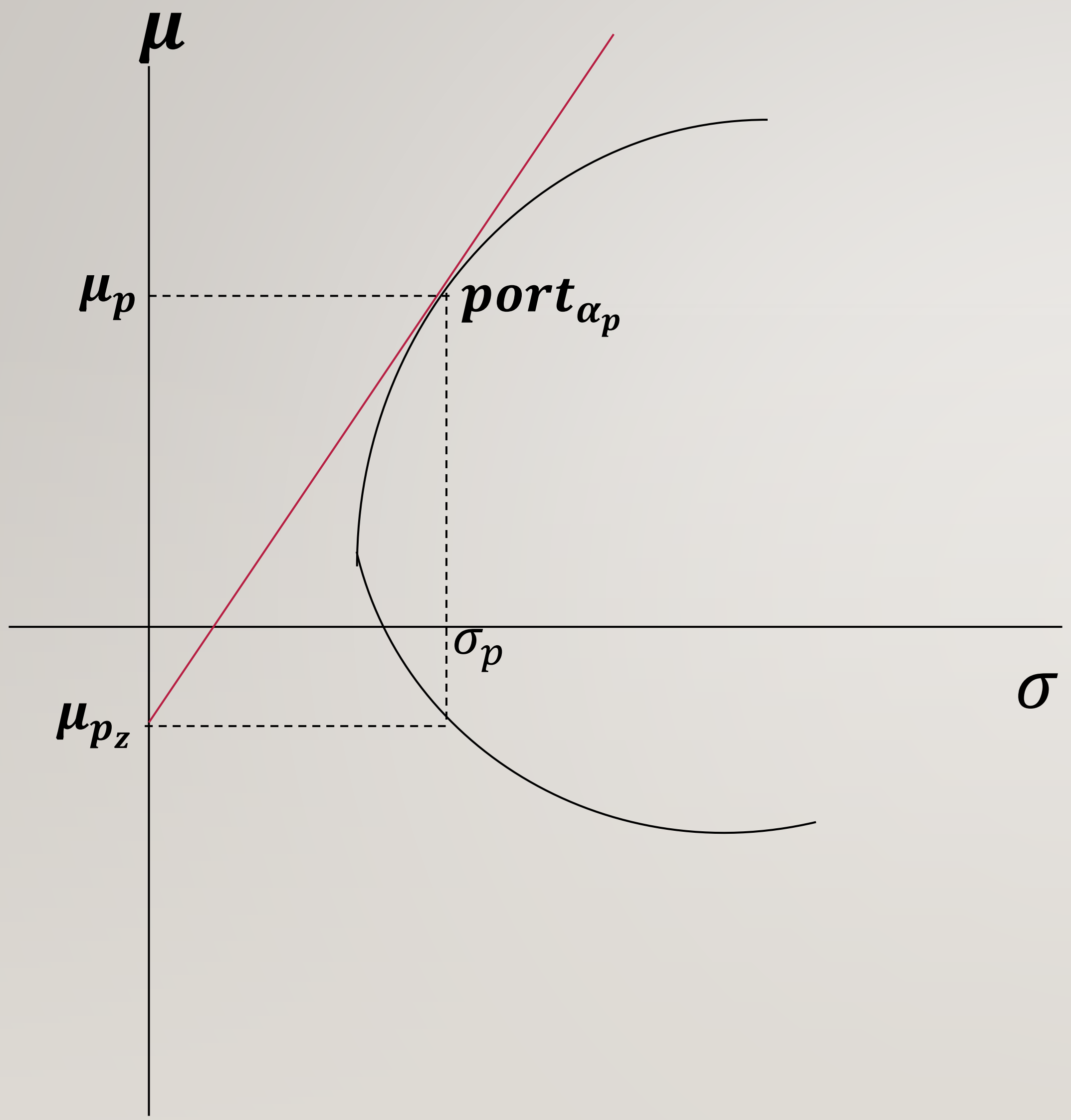
- First need to find μ_{pz} such that

$$\frac{\mu_p-\frac{B}{C}}{\frac{C}{AC-B^2}\left(\mu_p-\frac{B}{C}\right)^2}=\frac{\frac{B}{C}-\mu_{pz}}{\frac{1}{C}}$$

$$\mu_{p_z}=\frac{A-B\mu_p}{B-C\mu_p}$$

- Then, we need to show $cov\left(\tilde{R}_p,\tilde{R}_{p_z}\right)=0$, i.e.

$$\frac{C}{AC-B^2}\left(\mu_p-\frac{B}{C}\right)\left(\mu_{p_z}-\frac{B}{C}\right)\quad+\frac{1}{C}=0$$



In mean-std space:

To show μ_{pz} is the tangent line of Port_{α_p} to the μ axis:

- First need to find μ_p such that $\mathrm{f}(\mu_p) = \frac{\mu_p - \mu_{p_z}}{\sigma_p}$ is maximized, where

$$\sigma_p = \sqrt{\frac{C}{AC - B^2}\left(\mu_p - \frac{B}{C}\right)^2 + \frac{1}{C}}$$

- Set $f'(\mu_p) = 0$ we can find that

$$\mu_p = \frac{A - B\mu_z}{B - C\mu_z} \text{ or } \mu_{p_z} = \frac{A - B\mu_p}{B - C\mu_p}$$

- Then, we need to show $cov(\tilde{R}_p, \tilde{R}_{p_z}) = 0$, i.e.

$$\frac{C}{AC - B^2}\left(\mu_p - \frac{B}{C}\right)\left(\mu_{p_z} - \frac{B}{C}\right) + \frac{1}{C} = 0$$

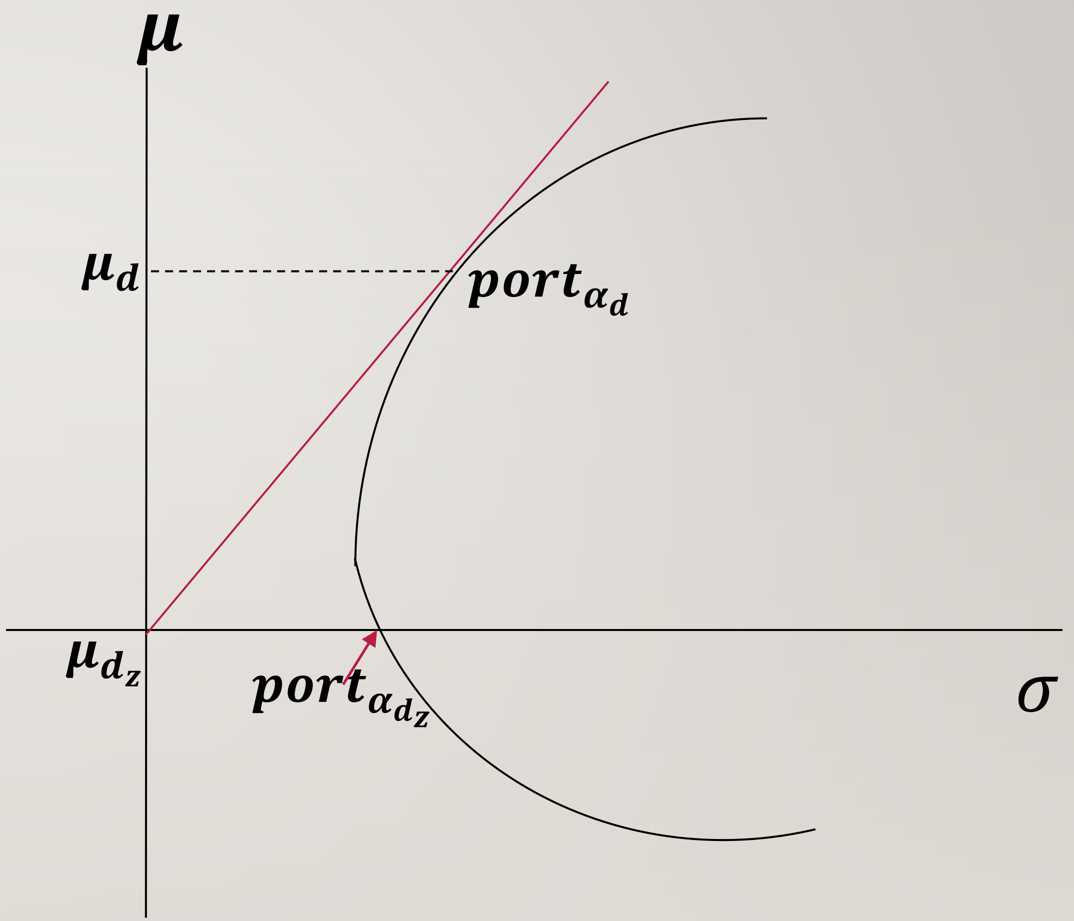
BACK TO $Port_d$

- Remember $\alpha_d = \frac{\Omega^{-1}R}{B} \Rightarrow \mu_d = \frac{A}{B}$
- Find $\tilde{R}_{d_z}$:

$$0 = Cov(\tilde{R}_d, \tilde{R}_{d_z})$$

$$= \frac{C}{AC - B^2}\left(\frac{A}{B} - \frac{B}{C}\right)\left(\mu_{d_z} - \frac{B}{C}\right) + \frac{1}{C} \Leftrightarrow \mu_{d_z} = 0$$

- $Port_d$ is the sister portfolio (uncorrelated) with the portfolio of zero expected return, i.e. the efficient frontier crossing the x-axis



BETA PRICING

- Regress returns of arbitrary asset i on $\tilde{R}_d$ and $\tilde{R}_{d_z}$, find β_1, β_2 such that:

$$\tilde{R}_i = \beta_1 \tilde{R}_d + \beta_2 \tilde{R}_{d_z} + \beta_0 + \epsilon$$

- With $Cov\left(\tilde{R}_d, \epsilon\right) = Cov\left(\tilde{R}_{d_z}, \epsilon\right) = E(\epsilon) = 0$

$$Cov\left(\tilde{R}_i, \tilde{R}_d\right) = \beta_1 \sigma_d^2, Cov\left(\tilde{R}_i, \tilde{R}_{d_z}\right) = \beta_2 \sigma_{d_z}^2$$

- $\beta_1 = \frac{Cov(\tilde{R}_i, \tilde{R}_d)}{\sigma_d^2} = \frac{\mu_i / B}{A / B^2} = \frac{B}{A} \mu_i$ (prove it using (*))

- $\beta_2 = \frac{Cov\left(\tilde{R}_i, \tilde{R}_{d_z}\right)}{\sigma_{d_z}^2} = \frac{A - B\mu_i}{A} = 1 - \frac{B}{A} \mu_i = 1 - \beta_1$ (prove it using (*))

BETA PRICING

- $\tilde{R}_i = \beta_1 \tilde{R}_d + (1 - \beta_1)\tilde{R}_{d_z} + \beta_0 + \epsilon$
- Take expectation on both sides: $\mu_i = \beta_1 \mu_d + \beta_0 = \frac{B}{A}\mu_i \frac{A}{B} + \beta_0$=> β_0=0
- Thus $\tilde{R}_i - \tilde{R}_{d_z} = \beta(\tilde{R}_d - \tilde{R}_{d_z})\ + \epsilon$
- Taking expectation on both sides, we have

$$\mu_i = \beta \mu_d$$

$$\beta = \frac{Cov\left(\tilde{R}_i, \tilde{R}_d\right)}{\sigma_d^2}$$

BETA PRICING

- One can show that the result applies to any MV efficient portfolio p with μ_p,

$$\beta_1 = \frac{Cov(\tilde{R}_i, \tilde{R}_p)}{\sigma_p^2} = \frac{\frac{C}{AC - B^2}\left(\mu_i - \frac{B}{C}\right)\left(\mu_p - \frac{B}{C}\right) + \frac{1}{C}}{\frac{C}{AC - B^2}\left(\mu_p - \frac{B}{C}\right)^2 + \frac{1}{C}}$$

$$\beta_2 = \frac{Cov(\tilde{R}_i, \tilde{R}_{p_z})}{\sigma_{p_z}^2} = \frac{\frac{C}{AC - B^2}\left(\mu_i - \frac{B}{C}\right)\left(\frac{A - B\mu_p}{B - C\mu_p} - \frac{B}{C}\right) + \frac{1}{C}}{\frac{C}{AC - B^2}\left(\frac{A - B\mu_p}{B - C\mu_p} - \frac{B}{C}\right)^2 + \frac{1}{C}} = 1 - \beta_1$$

- Thus we have:

$$\tilde{R}_i - \tilde{R}_{MV_z} = \beta(\tilde{R}_{MV} - \tilde{R}_{MV_z}) + \epsilon$$

- Taking expectation on both sides, we have $\mu_i - \mu_{MV_z} = \beta\left(\mu_{MV} - \mu_{MV_z}\right)$

WITH RISK-FREE ASSET

- Now denote risk-free asset return as R_f and risky asset portfolio return as $\tilde{R}_s$, x as the proportion of an investor's wealth invested in the risky asset portfolio. The mixed portfolio of risk-free and risky asset as $\tilde{R}_p$.

$$\mu_p = x\mu_s + (1-x)R_f = R_f + x(\mu_s - R_f)$$

- $\sigma^2 = var\left(\tilde{R}_p\right) = x^2\sigma_s^2 \Rightarrow \sigma = \pm x\sigma_s \Rightarrow \mu_p = R_f \pm \frac{\sigma}{\sigma_s}(\mu_s - R_f)$

- Best trade-off is the tangency portfolio, i.e. maximizing the slope (Sharpe ratio), which is $\frac{\mu_s - Rf}{\sigma_s}$

- Denote the tangency portfolio as $\tilde{R}_T$, we have

$$\mu_i - R_f = \beta(\mu_T - R_f)$$

$$\beta = \frac{Cov\left(\tilde{R}_i, \tilde{R}_T\right)}{\sigma_T^2}$$

TOWARDS THE CAPM

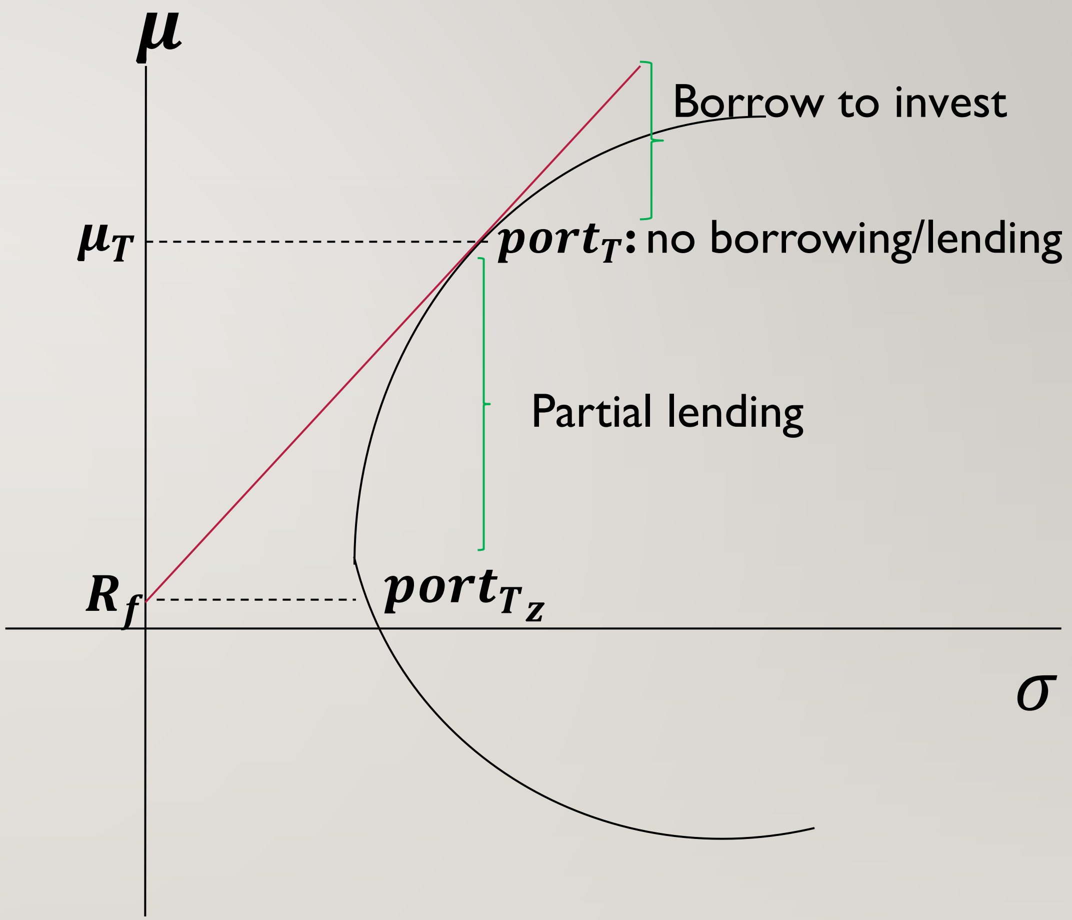
- Assumptions:
  1. All agents are MV optimizers
  2. All have homogeneous beliefs
  3. All have same trading opportunities
  4. Risk-free Asset is ZERO net supply
  5. No trading frictions (zero cost and limit)

Every agent holds a MV optimal portfolio $x_i \tilde{R}_T + (1 - x_i) R_f$ (Two fund separation)

Total wealth:

$W = \sum_i w_i \left(x_i \tilde{R}_T + (1 - x_i) R_f \right) = \left(\sum_i w_i x_i \right) \tilde{R}_T$

=> $\tilde{R}_T$ is the total wealth, i.e. the market portfolio



LAGRANGE APPROACH WITH RISK-FREE ASSET

$$\omega = \begin{pmatrix} \omega_1 \\ . \\ . \\ . \\ \omega_n \end{pmatrix}, R = \begin{pmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ . \\ \mu_n \end{pmatrix}, \tilde{R} = \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ . \\ . \\ . \\ \tilde{r}_n \end{pmatrix}, \Omega = cov(\tilde{R}, \tilde{R}')$$

$$L = \frac{1}{2}\omega'\Omega\omega + \delta\left(\mu_p - \omega' R - (1 - \omega'\mathbb{I})R_f\right)$$

- FOC: $\Omega\omega_p - \delta\left(R - R_f\mathbb{I}\right) = 0 \Rightarrow \omega_p = \delta\Omega^{-1}\left(R - R_f\mathbb{I}\right)$
- $\omega_p' R + \left(1 - \omega_p'\mathbb{I}\right)R_f = \mu_p \Rightarrow \delta R'\Omega^{-1}\left(R - R_f\mathbb{I}\right) + \left[1 - \delta\mathbb{I}'\Omega^{-1}\left(R - R_f\mathbb{I}\right)\right]\mathrm{R_f} = \mu_p$
- $\delta\left[\left(R - R_f\mathbb{I}\right)'\Omega^{-1}\left(R - R_f\mathbb{I}\right)\right] = \mu_p - R_f \Rightarrow \delta = \frac{\mu_p - R_f}{\left(R - R_f\mathbb{I}\right)'\Omega^{-1}\left(R - R_f\mathbb{I}\right)}$

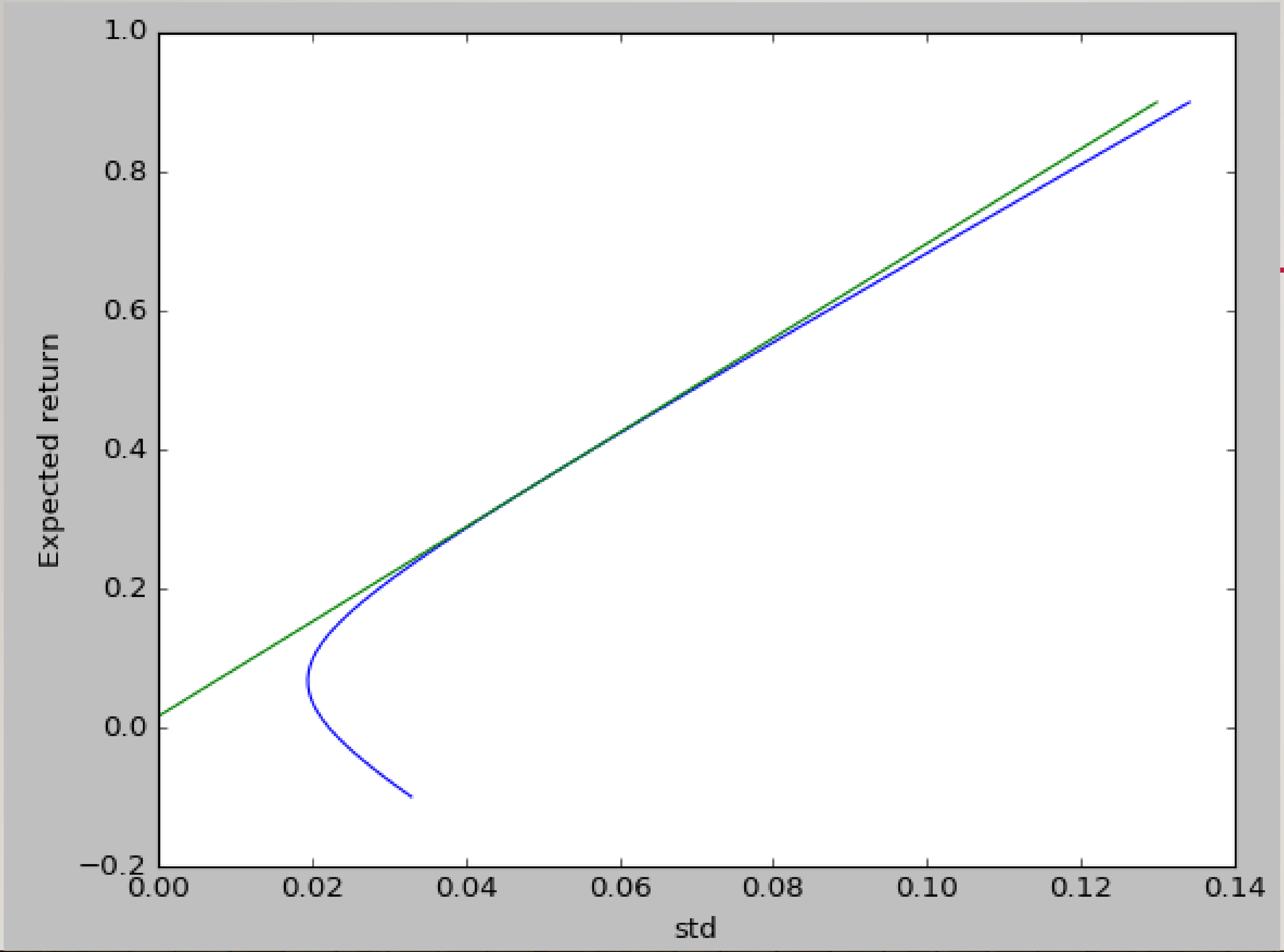
$$\omega_p = \frac{\mu_p - R_f}{\left(R - R_f\mathbb{I}\right)'\Omega^{-1}\left(R - R_f\mathbb{I}\right)}\Omega^{-1}\left(R - R_f\mathbb{I}\right)$$

FRONTIER WITH RISK-FREE RATE

- $\omega_p = \frac{\mu_p - R_f}{(R - R_f \mathbb{I})' \Omega^{-1} (R - R_f \mathbb{I})} \Omega^{-1} (R - R_f \mathbb{I})$
- Thus with a risk-free asset all minimum-variance portfolios are a combination of a given risky asset portfolio and the risk-free asset
- The portfolio of risky assets is the tangency portfolio and has the weight vector $\mathrm{w_q} = \frac{1}{\mathbb{I}' \Omega^{-1} (R - R_f \mathbb{I})} \Omega^{-1} (R - R_f \mathbb{I})$ as $\mathbb{I}' w_q = 1$, i.e. when all weight is assigned to the risky portfolio q with zero position in risk-free asset
- All efficient portfolios lie along the line from the risk-free asset through the tangency portfolio and the Sharpe ratio is maximized

EXAMPLE: A SMALL HK PORTFOLIO

- A simple portfolio of 5 Hong Kong names, 10 HK (Hang Lung Properties), 16 HK (Sun Hung Kai Properties), 2 HK (CLP), 4 HK (Wharf), 941 HK (China Mobile)
- Say we calculate the covariance matrix simply based on the weekly returns. You can do this by first getting all Friday observation (you can do this by Pandas), calculate the weekly returns, then the covariance matrix (numpy.cov)
- For the expected returns, lets simply plug in the consensus analyst price target from current price, respectively 10%, 15%, 3%, 1% and 25%
- Follow the Lagrangian approach, we get the efficient frontier. Functions used are numpy.linalg.inv, numpy.dot
- The corresponding minimum standard is 0.01948 with expected return 0.0669



Min Std: 1.945%
Min ER: 6.69%

Tangent port mean: 35.775%
Tangent port std: 5.0%