有效前沿：给定预期回报率，使不含无风险资产组合方差最小

Theorem: As long as the covariance matrix of returns is non-singular, there is a mean-variance frontier

定理：只要收益率的协方差矩阵是非奇异的，就存在均值方差前沿

Notations:

$$\alpha = \begin{pmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{pmatrix}, \quad R = E(\tilde{R}) = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \mu_n \end{pmatrix}, \quad \Omega = cov(\tilde{R}, \tilde{R}'), \quad \mathrm{I} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}_{(n,1)}$$

Model Construction:

假定有n个风险资产，给定投资组合的期望收益率为μ_p。

$$\min_{\alpha} \alpha'\Omega\alpha$$

$$s.t.$$

$$\begin{cases} \alpha' R = \mu_p \\ \alpha'\mathrm{I} = 1 \end{cases}$$

Model Solving:

$$\mathcal{L} = \frac{1}{2}\alpha'\Omega\alpha + \lambda(\mu_p - \alpha' R) + \gamma(1 - \alpha'\mathrm{I})$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \Omega\alpha - \lambda R - \gamma\mathrm{I} = 0$$

$$\Longrightarrow \alpha = \lambda\Omega^{-1}R + \gamma\Omega^{-1}\mathrm{I}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mu_p - \alpha' R = 0$$

$$\Longrightarrow \alpha' R = \mu_p \Longrightarrow R'\alpha = \mu_p$$

$$\Longrightarrow R'(\lambda\Omega^{-1}R + \gamma\Omega^{-1}\mathrm{I}) = \mu_p$$

$$\Longrightarrow \lambda R'\Omega^{-1}R + \gamma R'\Omega^{-1}\mathrm{I} = \mu_p$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = 1 - \alpha'\mathrm{I} = 0$$

$$\Longrightarrow \alpha'\mathrm{I} = 1 \Longrightarrow \mathrm{I}'\alpha = 1$$

$$\Longrightarrow \mathrm{I}'(\lambda\Omega^{-1}R + \gamma\Omega^{-1}\mathrm{I}) = 1$$

$$\Longrightarrow \lambda\mathrm{I}'\Omega^{-1}R + \gamma\mathrm{I}'\Omega^{-1}\mathrm{I} = 1$$

Define:

$$A = R'\Omega^{-1}R,\quad B = \mathrm{I}'\Omega^{-1}R,\quad C = \mathrm{I}'\Omega^{-1}\mathrm{I}$$

We can get:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \alpha} = \lambda A + \gamma B = \mu_p \\ \dfrac{\partial \mathcal{L}}{\partial \gamma} = \lambda B + \gamma C = 1 \end{cases} \Rightarrow \begin{cases} B\dfrac{\partial \mathcal{L}}{\partial \alpha} = \lambda AB + \gamma B^2 = B\mu_p \\ \mathrm{A}\dfrac{\partial \mathcal{L}}{\partial \gamma} = \lambda AB + \gamma AC = A \end{cases} \Rightarrow \begin{cases} \lambda = \dfrac{\mu_p C - B}{AC - B^2} \\ \gamma = \dfrac{A - B\mu_p}{AC - B^2} \end{cases}$$

$$\alpha_p = \alpha = \lambda\Omega^{-1}R + \gamma\Omega^{-1}\mathrm{I} = \frac{\mu_p C - B}{AC - B^2}\Omega^{-1}R + \frac{A - B\mu_p}{AC - B^2}\Omega^{-1}\mathrm{I}$$

由于 A、B、C 都可以通过历史数据获取，因此给定很多个投资组合的期望收益率为μ_p，可以求出与此一一对应的最佳投资组合配置α_p，然后求得投资组合的标准差，即可绘制有效前沿。但很多时候，我们好难把握期望收益率μ_p的范围，导致有效前沿曲线不优雅。下面继续简化最佳投资组合配置α_p。

$$\alpha_p = \frac{B(\mu_p C - B)}{AC - B^2}\frac{\Omega^{-1}R}{B} + \frac{C(A - B\mu_p)}{AC - B^2}\frac{\Omega^{-1}\mathrm{I}}{C} = x\frac{\Omega^{-1}R}{B} + (1 - x)\frac{\Omega^{-1}\mathrm{I}}{C}$$

通过改变 x 的值，就可以求到最佳投资组合配置α_p，继而求出投资组合的期望回报以及标准差。

在最佳投资组合配置α_p下，

$$\Omega\alpha_p = \frac{\mu_p C - B}{AC - B^2}R + \frac{A - B\mu_p}{AC - B^2}\mathrm{I}$$

投资组合方差：

$$var = \alpha_p{}'\Omega\alpha_p = \alpha'\left(\frac{\mu_p C - B}{AC - B^2}R + \frac{A - B\mu_p}{AC - B^2}\mathrm{I}\right) = \frac{\mu_p C - B}{AC - B^2}\alpha' R + \frac{A - B\mu_p}{AC - B^2}\alpha'\mathrm{I}$$

$$= \frac{\mu_p C - B}{AC - B^2}\mu_p + \frac{A - B\mu_p}{AC - B^2}\cdot 1 = \frac{C\mu_p^2 - B\mu_p}{AC - B^2} + \frac{A - B\mu_p}{AC - B^2}$$

$$= \frac{C\mu_p^2 - 2B\mu_p + A}{AC - B^2} = \frac{C\left(\mu_p - \frac{B}{C}\right)^2 - \frac{B^2}{C} + A}{AC - B^2}$$

$$= \frac{C\left(\mu_p - \frac{B}{C}\right)^2 + \frac{AC - B^2}{C}}{AC - B^2} = \frac{C\left(\mu_p - \frac{B}{C}\right)^2}{AC - B^2} + \frac{1}{C}$$

上式中，当$\mu_p=\frac{B}{C}$时，投资组合方差最小，最小方差为$\frac{1}{C}$。

$$\begin{aligned}\alpha_{minvar} &= \frac{\mu_p C-B}{AC-B^2}\Omega^{-1}R+\frac{A-B\mu_p}{AC-B^2}\Omega^{-1}\mathrm{I}=\frac{\frac{B}{C}C-B}{AC-B^2}\Omega^{-1}R+\frac{A-B\frac{B}{C}}{AC-B^2}\Omega^{-1}\mathrm{I}\\ &=0+\frac{\frac{1}{C}(AC-B^2)}{AC-B^2}\Omega^{-1}\mathrm{I}=\frac{\Omega^{-1}\mathrm{I}}{C}=\frac{\Omega^{-1}\mathrm{I}}{\mathrm{I}'\Omega^{-1}\mathrm{I}}\end{aligned}$$

资本市场线(Capital Market Line)：给定预期回报率，使含无风险资产组合方差最小(夏普比率最大)

Model Construction:

假定有n个风险资产以及 1 个无风险资产（无风险收益率为R_f）组成的投资组合，给定投资组合的期望收益率为μ_p。

$$\min_{\alpha} \alpha'\Omega\alpha$$

$$s.t.$$

$$\alpha'R + (1-\alpha'\mathrm{I})R_f = \mu_p$$

Model Solving:

$$\mathcal{L} = \frac{1}{2}\alpha'\Omega\alpha + \delta\left(\mu_p - \alpha'R - (1-\alpha'\mathrm{I})R_f\right)$$

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \Omega\alpha - \delta\left(R - R_f\mathrm{I}\right) = 0$$

$$\Longrightarrow \alpha = \delta\Omega^{-1}\left(R - R_f\mathrm{I}\right)$$

$$\frac{\partial\mathcal{L}}{\partial\delta} = \mu_p - \alpha'R - (1-\alpha'\mathrm{I})R_f = 0$$

$$\Longrightarrow \alpha'R + (1-\alpha'\mathrm{I})R_f = \mu_p$$

$$\Longrightarrow R'\alpha + (1-\mathrm{I}'\alpha)R_f = \mu_p$$

$$\Longrightarrow R'\delta\Omega^{-1}\left(R - R_f\mathrm{I}\right) + \left[1 - \mathrm{I}'\delta\Omega^{-1}\left(R - R_f\mathrm{I}\right)\right]R_f = \mu_p$$

$$\Longrightarrow \delta R'\Omega^{-1}\left(R - R_f\mathrm{I}\right) + R_f - \delta R_f\mathrm{I}'\Omega^{-1}\left(R - R_f\mathrm{I}\right) = \mu_p$$

$$\Longrightarrow \delta R'\Omega^{-1}\left(R - R_f\mathrm{I}\right) - \delta R_f\mathrm{I}'\Omega^{-1}\left(R - R_f\mathrm{I}\right) = \mu_p - R_f$$

$$\Longrightarrow \delta\left[\left(R' - R_f\mathrm{I}'\right)\Omega^{-1}\left(R - R_f\mathrm{I}\right)\right] = \mu_p - R_f$$

$$\Longrightarrow \delta\left[\left(R - R_f\mathrm{I}\right)'\,\Omega^{-1}\left(R - R_f\mathrm{I}\right)\right] = \mu_p - R_f$$

$$\Longrightarrow \delta = \frac{\mu_p - R_f}{\left[\left(R - R_f\mathrm{I}\right)'\,\Omega^{-1}\left(R - R_f\mathrm{I}\right)\right]}$$

$$\alpha_p = \alpha = \delta\Omega^{-1}\left(R - R_f\mathrm{I}\right) = \frac{\mu_p - R_f}{\left[\left(R - R_f\mathrm{I}\right)'\,\Omega^{-1}\left(R - R_f\mathrm{I}\right)\right]}\Omega^{-1}\left(R - R_f\mathrm{I}\right)$$