Lecture 14

Eigenvalues and Eigenvectors

Suppose that A is a square $(n \times n)$ matrix. We say that a nonzero vector $\mathbf{v}$ is an eigenvector and a number λ is its eigenvalue if

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{14.1}$$

Geometrically this means that $A\mathbf{v}$ is in the same direction as $\mathbf{v}$, since multiplying a vector by a number changes its length, but not its direction.

MATLAB has a built-in routine for finding eigenvalues and eigenvectors:

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≫  A = pascal(4)
≫  [v e] = eig(A)
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The results are a matrix v that contains eigenvectors as columns and a diagonal matrix e that contains eigenvalues on the diagonal. We can check this by

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≫  v1 = v(:,1)
≫  A*v1
≫  e(1,1)*v1
```

Finding Eigenvalues for 2×2 and 3×3

If A is 2×2 or 3×3 then we can find its eigenvalues and eigenvectors by hand. Notice that Equation (14.1) can be rewritten as

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}.$$

It would be nice to factor out the $\mathbf{v}$ from the right-hand side of this equation, but we can't because A is a matrix and λ is a number. However, since $I\mathbf{v} = \mathbf{v}$, we can do the following:

$$\begin{aligned} A\mathbf{v} - \lambda\mathbf{v} &= A\mathbf{v} - \lambda I\mathbf{v} \\ &= (A - \lambda I)\mathbf{v} \\ &= \mathbf{0} \end{aligned}$$

If $\mathbf{v}$ is nonzero, then by Theorem 3 in Lecture 10 the matrix $(A - \lambda I)$ must be singular. By the same theorem, we must have

$$\det(A - \lambda I) = 0.$$

This is called the *characteristic equation.*

For a 2×2 matrix, $A - \lambda I$ is calculated as in the following example:

$$A - \lambda I = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \lambda & 4 \\ 3 & 5 - \lambda \end{pmatrix}.$$

The determinant of $A - \lambda I$ is then

$$\det(A - \lambda I) = (1 - \lambda)(5 - \lambda) - 4 \cdot 3$$
$$= -7 - 6\lambda + \lambda^2.$$

The characteristic equation $\det(A - \lambda I) = 0$ is simply a quadratic equation:

$$\lambda^2 - 6\lambda - 7 = 0.$$

The roots of this equation are $\lambda_1 = 7$ and $\lambda_2 = -1$. These are the eigenvalues of the matrix A. Now to find the corresponding eigenvectors we return to the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$. For $\lambda_1 = 7$, the equation for the eigenvector $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is equivalent to the augmented matrix

$$\begin{pmatrix} -6 & 4 & | & 0 \\ 3 & -2 & | & 0 \end{pmatrix}. \tag{14.2}$$

Notice that the first and second rows of this matrix are multiples of one another. Thus Gaussian elimination would produce all zeros on the bottom row. Thus this equation has infinitely many solutions, i.e. infinitely many eigenvectors. Since only the direction of the eigenvector matters, this is okay, we only need to find one of the eigenvectors. Since the second row of the augmented matrix represents the equation

$$3x - 2y = 0,$$

we can let

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

This comes from noticing that $(x, y) = (2, 3)$ is a solution of $3x - 2y = 0$.

For $\lambda_2 = -1$, $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is equivalent to the augmented matrix

$$\begin{pmatrix} 2 & 4 & | & 0 \\ 3 & 6 & | & 0 \end{pmatrix}.$$

Once again the first and second rows of this matrix are multiples of one another. For simplicity we can let

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

One can always check an eigenvector and eigenvalue by multiplying:

$$A\mathbf{v}_1 = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 21 \end{pmatrix} = 7 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 7\mathbf{v}_1 \quad \text{and}$$
$$A\mathbf{v}_2 = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -1\mathbf{v}_2.$$

For a 3×3 matrix we could complete the same process. The $\det(A - \lambda I) = 0$ would be a cubic polynomial and we would expect to usually get 3 roots, which are the eigenvalues.

Larger Matrices

For a $n \times n$ matrix with $n \geq 4$ this process is too long and cumbersome to complete by hand. Further, this process is not well suited even to implementation on a computer program since it involves determinants and solving a n-degree polynomial. For $n \geq 4$ we need more ingenious methods. These methods rely on the geometric meaning of eigenvectors and eigenvalues rather than solving algebraic equations. We will overview these methods in Lecture 16.

Complex Eigenvalues

It turns out that the eigenvalues of some matrices are complex numbers, even when the matrix only contains real numbers. When this happens the complex eigenvalues must occur in conjugate pairs, i.e.

$$\lambda_{1,2} = \alpha \pm i\beta.$$

The corresponding eigenvectors must also come in conjugate pairs:

$$\mathbf{w} = \mathbf{u} \pm i\mathbf{v}.$$

In applications, the imaginary part of the eigenvalue, β, often is related to the frequency of an oscillation. This is because of Euler's formula

$$e^{\alpha + i\beta} = e^{\alpha}(\cos \beta + i \sin \beta).$$

Certain kinds of matrices that arise in applications can only have real eigenvalues and eigenvectors. The most common such type of matrix is the symmetric matrix. A matrix is symmetric if it is equal to its own transpose, i.e. it is symmetric across the diagonal. For example,

$$\begin{pmatrix} 1 & 3 \\ 3 & -5 \end{pmatrix}$$

is symmetric and so we know beforehand that its eigenvalues will be real, not complex.

Exercises

14.1 Find the eigenvalues and eigenvectors of the following matrix by hand:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

14.2 Find the eigenvalues and eigenvectors of the following matrix by hand:

$$B = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Can you guess the eigenvalues of the matrix

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}?$$