Probability

CSE 4711: Artificial Intelligence

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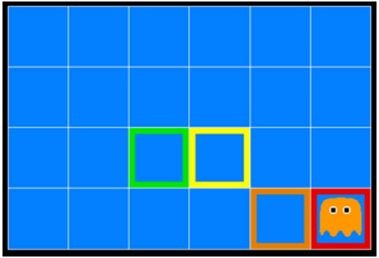
Inference in Ghostbusters

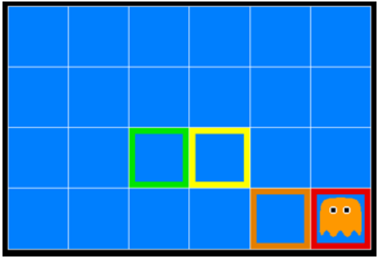
- A ghost is in the grid somewhere



Video: ghosts - manual

Inference in Ghostbusters

- A ghost is in the grid somewhere
- Two actions
  - Bust action to catch the ghost
  - Sensor readings tell how close a square is to the ghost
    - On the ghost: red
    - 1 or 2 away: orange
    - 3 or 4 away: yellow
    - 5+ away: green



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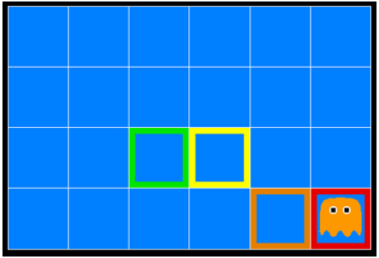


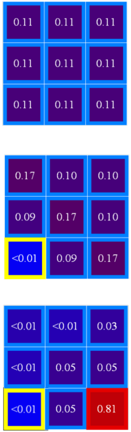
- Sensors are noisy, but we know $P(\text{Color}|\text{Distance})$

$P(\text{red}|3)$	$P(\text{orange}|3)$	$P(\text{yellow}|3)$	$P(\text{green}|3)$
0.05	0.15	0.50	0.30

Video: ghosts - manual

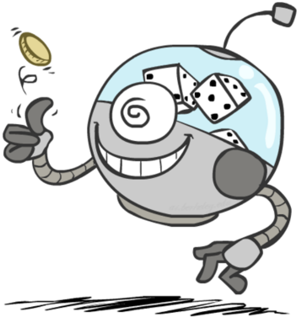
Uncertainty

- General situation:
  - **Observed variables (evidence):** Agent knows certain things about the state of the world (e.g., sensor readings or symptoms)
  - **Unobserved variables:** Agent needs to reason about other aspects (e.g. where an object is or what disease is present)
  - **Model:** Agent knows something about how the known variables relate to the unknown variables
- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge



Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- We denote random variables with capital letters



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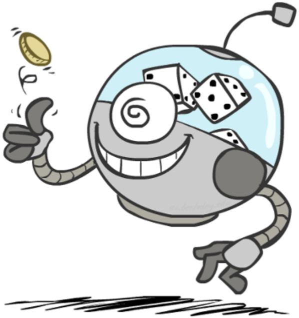
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- We denote random variables with capital letters
- Domains:
  - $R \in \{true, false\}$ (often written as $\{+r, -r\}$)
  - $T \in \{hot, cold\}$
  - $D \in [0, \infty)$
  - $L \in \{(0,0), (0,1), \ldots\}$



Probability Distributions

- Associate a probability with each value

Probability Distributions

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Temperature

$P(T)$	
T	P
hot	0.5
cold	0.5

Probability Distributions

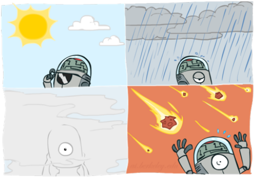
- Associate a probability with each value

Temperature

T	P
hot	0.5
cold	0.5

$P(T)$

Weather



$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

- A distribution is a TABLE of probabilities of values

Probability Distributions

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- A probability (lower case value) is a single number
  e.g.: $P(W = rain) = 0.1$

Probability Distributions

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$P(W)$

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fog	0.3
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- A distribution is a TABLE of probabilities of values
- A probability (lower case value) is a single number
  e.g.: $P(W = rain) = 0.1$
- Must have: $\forall x \; P(X = x) \geq 0$ and
  $\sum_x P(X = x) = 1$

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Shorthand notation:
$P(hot) = P(T = hot)$,
$P(cold) = P(T = cold)$,
$P(rain) = P(W = rain)$,
$\ldots$
OK *if* all domain entries are unique

- A distribution is a TABLE of probabilities of values
- A probability (lower case value) is a single number
  e.g.: $P(W = rain) = 0.1$
- Must have: $\forall x \; P(X = x) \geq 0$ and
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Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots, X_n$ specifies a real number for each assignment (or outcome):

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Joint Distributions

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$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

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$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

$$= P(x_1, x_2, \ldots, x_n)$$

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- Must obey:

  $$P(x_1, x_2, \ldots, x_n) \geq 0$$

  $$\sum_{(x_1, x_2, \ldots, x_n)} P(x_1, x_2, \ldots, x_n) = 1$$

$P(T, W)$

T	W	P
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- Size of distribution if n variables with domain sizes d?

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- Must obey:

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  $$\sum_{(x_1, x_2, \ldots, x_n)} P(x_1, x_2, \ldots, x_n) = 1$$

- Size of distribution if n variables with domain sizes d? $\rightarrow d^n$
  - Impractical to write out for large distributions

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Models

- A joint distribution over a set of random variables
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - Normalized: sum to 1.0
  - Ideally: only certain variables directly interact

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Events

- A set E of outcomes
  $P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$

$P(T, W)$

T	W	P
hot	sun	0.4
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Events

- A set E of outcomes
  $P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$
- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?

$P(T, W)$

T	W	P
hot	sun	0.4
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Events

- A set E of outcomes
  $P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$
- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?
- Typically, the events we care about are *partial assignments*, like $P(T = hot)$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz: Events

- $P(+x, +y)$?

$P(X, Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Events

- $P(+x, +y)?$
  $\rightarrow 0.2$
- $P(+x)?$

$P(X, Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Events

- $P(+x, +y)$?

  $\rightarrow 0.2$
- $P(+x)$?

  $P(+x, +y) + P(+x, -y)$

  $\rightarrow 0.5$
- $P(-y \text{ OR } +x)$?

$P(X, Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Events

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  $\rightarrow 0.2$
- $P(+x)$?

  $P(+x, +y) + P(+x, -y)$

  $\rightarrow 0.5$
- $P(-y \text{ OR } +x)$?

  $P(+x, -y) + P(-x, -y) + P(+x, +y)$

  $\rightarrow 0.6$

$P(X, Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Marginal Distributions

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

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T	W	P
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cold	rain	0.3

$$P(T) = ?$$

$$P(t) = \sum_s P(t, s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T,W)$

T	W	P
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$P(T) = ?$

$$P(t) = \sum_s P(t,s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W) = ?$

$$P(s) = \sum_t P(t,s)$$

$P(W)$

W	P
sun	0.6
rain	0.4

Marginal Distributions

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- Marginalization (summing out): Combine collapsed rows by adding

$P(T,W)$

T	W	P
hot	sun	0.4
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$P(T) = ?$

$$P(t) = \sum_s P(t,s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W) = ?$

$$P(s) = \sum_t P(t,s)$$

$P(W)$

W	P
sun	0.6
rain	0.4

- $P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$

Quiz: Marginal Distributions

$P(X, Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Marginal Distributions

- $P(x) = \sum_y P(x, y)$

$P(X)$

X	P
$+x$	
$-x$	

$P(X, Y)$

X	Y	P
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$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Marginal Distributions

- $P(x) = \sum_y P(x, y)$

$P(X)$

X	P
$+x$	0.5
$-x$	0.5

- $P(y) = \sum_x P(x, y)$

$P(Y)$

Y	P
$+y$	
$-y$	

$P(X, Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Marginal Distributions

- $P(x) = \sum_y P(x, y)$

$P(X)$

X	P
$+x$	0.5
$-x$	0.5

- $P(y) = \sum_x P(x, y)$

$P(Y)$

Y	P
$+y$	0.6
$-y$	0.4

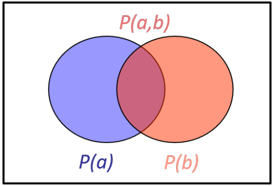
$P(X, Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Conditional Probabilities

- A simple relation between joint and conditional probabilities

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



$P(T,W)$

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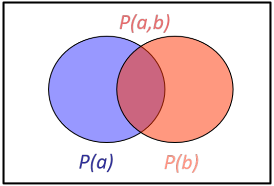

$P(T,W)$

T	W	P
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hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W = s|T = c) =$$

Conditional Probabilities

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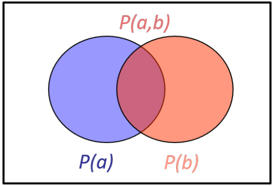

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

Quiz: Conditional Probabilities

- $P(+x | +y)$

$P(X,Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Conditional Probabilities

- $P(+x \mid +y)$

  $\rightarrow \frac{P(+x,+y)}{P(+y)}$

  $\rightarrow \frac{1}{3}$
- $P(-x \mid +y)$

$P(X, Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Conditional Probabilities

- $P(+x| + y)$

  $\rightarrow \frac{P(+x,+y)}{P(+y)}$

  $\rightarrow \frac{1}{3}$

- $P(-x| + y)$

  $\rightarrow \frac{P(-x,+y)}{P(-x,+y)+P(+x,+y)}$

  $\rightarrow \frac{2}{3}$

$P(X,Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Conditional Probabilities

- $P(+x|+y)$

  $\rightarrow \frac{P(+x,+y)}{P(+y)}$

  $\rightarrow \frac{1}{3}$

- $P(-x|+y)$

  $\rightarrow \frac{P(-x,+y)}{P(-x,+y)+P(+x,+y)}$

  $\rightarrow \frac{2}{3}$

- $P(-y|+x)$

$P(X,Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Quiz: Conditional Probabilities

- $P(+x|+y)$

  $\rightarrow \frac{P(+x,+y)}{P(+y)}$

  $\rightarrow \frac{1}{3}$

- $P(-x|+y)$

  $\rightarrow \frac{P(-x,+y)}{P(-x,+y)+P(+x,+y)}$

  $\rightarrow \frac{2}{3}$

- $P(-y|+x)$

  $\rightarrow \frac{P(+x,-y)}{P(+x,-y)+P(+x,+y)}$

  $\rightarrow \frac{3}{5}$

$P(X,Y)$

X	Y	P
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Conditional Distributions

- Probability distributions over some variables given fixed values of others

Joint Distribution
$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Distributions

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Joint Distribution
$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Distributions
$P(W|T = hot)$

W	P
sun	0.8
rain	0.2

Conditional Distributions

- Probability distributions over some variables given fixed values of others

Joint Distribution
$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Distributions
$P(W|T = hot)$

W	P
sun	0.8
rain	0.2

$P(W|T = cold)$

W	P
sun	0.4
rain	0.6

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(W \mid T = cold)$

W	P
sun	
rain	

Normalization Trick

$$P(W = s|T = c)$$

$P(T, W)$

T	W	P
hot	sun	0.4
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$P(W|T = cold)$

W	P
sun	
rain	

Normalization Trick

$$P(W = s|T = c)$$
$$= \frac{P(W = s, T = c)}{P(T = c)}$$

$P(T, W)$

T	W	P
hot	sun	0.4
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$P(W|T = cold)$

W	P
sun	
rain	

Normalization Trick

$P(T, W)$

T	W	P
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cold	sun	0.2
cold	rain	0.3

$$P(W = s|T = c)$$

$$= \frac{P(W = s, T = c)}{P(T = c)}$$

$$= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)}$$

$P(W|T = cold)$

W	P
sun	
rain	

Normalization Trick

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T	W	P
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$$= \frac{P(W = s, T = c)}{P(T = c)}$$

$$= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)}$$

$$= \frac{0.2}{0.2 + 0.3} = 0.4$$

$P(W|T = cold)$

W	P
sun	0.4
rain	

Normalization Trick

$P(T, W)$

T	W	P
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$$P(W = s | T = c)$$

$$= \frac{P(W = s, T = c)}{P(T = c)}$$

$$= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)}$$

$$= \frac{0.2}{0.2 + 0.3} = 0.4$$

$$P(W = r | T = c)$$

$P(W | T = cold)$

W	P
sun	0.4
rain	

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W = s|T = c)$$

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$$=\frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$

$P(W|T=cold)$	
W	P
sun	0.4
rain	

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
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$$= \frac{P(W = r, T = c)}{P(T = c)}$$

$$= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)}$$

$$= \frac{0.3}{0.2 + 0.3} = 0.6$$

$P(W|T = cold)$

W	P
sun	0.4
rain	0.6

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
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SELECT the joint probabilities matching the evidence

$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$P(W \mid T = c)$

W	P
sun	0.4
rain	0.6

Normalization Trick

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c,W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

- Why does this work?
  - Sum of selection is $P(evidence)$! ($P(T=c)$, here)

$$P(x_1|x_2) = \frac{P(x_1,x_2)}{P(x_2)} = \frac{P(x_1,x_2)}{\sum_{x_1} P(x_1,x_2)}$$

Quiz: Normalization Trick

Find $P(X|Y=-y)$

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence

NORMALIZE the selection (make it sum to one)

Quiz: Normalization Trick

Find $P(X|Y = -y)$

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence

X	Y	P
+x	-y	0.3
-x	-y	0.1

NORMALIZE the selection (make it sum to one)

Quiz: Normalization Trick

Find $P(X|Y = -y)$

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence

X	Y	P
+x	-y	0.3
-x	-y	0.1

NORMALIZE the selection (make it sum to one)

X	P
+x	0.75
-x	0.25

To Normalize

- (Dictionary) To bring or restore to a normal condition

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- Example 1

W	P
sun	0.2
rain	0.3

$\xrightarrow[Z = 0.5]{\text{Normalize}}$

W	P
sun	0.4
rain	0.6

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  - All entries sum to ONE
- Procedure:
  - Step 1: Compute Z = sum over all entries
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- Example 1

W	P
sun	0.2
rain	0.3

$\xrightarrow[Z = 0.5]{\text{Normalize}}$

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

To Normalize

- (Dictionary) To bring or restore to a normal condition
  - All entries sum to ONE
- Procedure:
  - Step 1: Compute Z = sum over all entries
  - Step 2: Divide every entry by Z

- Example 1

W	P
sun	0.2
rain	0.3

$\xrightarrow[Z = 0.5]{\text{Normalize}}$

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

$\xrightarrow[Z = 50]{\text{Normalize}}$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Inference

- Compute a desired probability from other known probabilities (e.g. conditional from joint)
- We generally compute conditional probabilities
  - $P(\text{on time}|\text{no accidents}) = 0.90$
  - These represent the agent's *beliefs* given the evidence
- Probabilities change with new evidence:
  - $P(\text{on time}|\text{no accidents, 5 a.m.}) = 0.95$
  - $P(\text{on time}|\text{no accidents, 5 a.m., raining}) = 0.80$
  - Observing new evidence causes *beliefs to be updated*

Inference by Enumeration

- General case $(X_1, X_2, \ldots, X_n)$
  - Evidence variables: $E_1 \ldots E_k = e_1 \ldots e_k$
  - Query* variable: Q

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- We want: $P(Q|e_1 \ldots e_k)$

  (works fine with multiple query variables, too)

Inference by Enumeration

- General case $(X_1, X_2, \ldots, X_n)$
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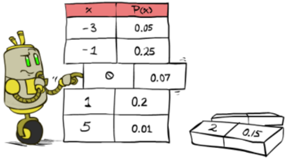
  - Query* variable: Q
  - Hidden variables: $H_1 \ldots H_r$
- We want: $P(Q|e_1 \ldots e_k)$

  (works fine with multiple query variables, too)

Inference by Enumeration

- Step 1: Select the entries consistent with the evidence



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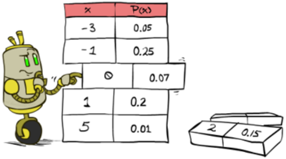
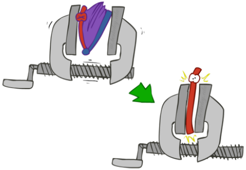


- Step 2: Sum out H to get joint of Query and evidence



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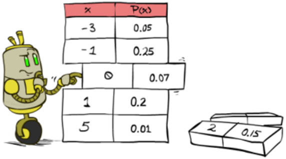

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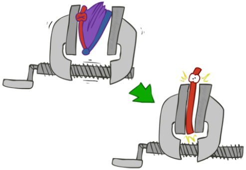

$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} P(\underbrace{Q, h_1 \dots h_r, e_1 \dots e_k}_{X_1, X_2, \dots, X_n})$$

Inference by Enumeration

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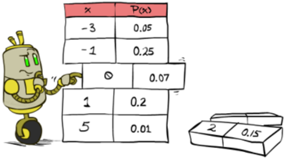

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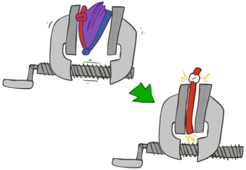

- Step 3: Normalize

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Inference by Enumeration

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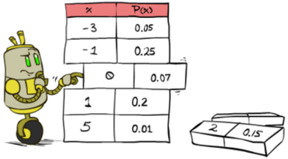

$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} P(\underbrace{Q, h_1 \dots h_r, e_1 \dots e_k}_{X_1, X_2, \dots, X_n})$$

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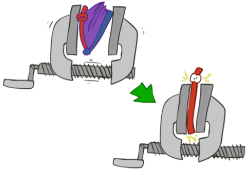

- Step 3: Normalize

$$Z = \sum_q P(Q, e_1 \dots e_k)$$

$$P(Q|e_1 \dots e_k) = \frac{1}{Z} P(Q, e_1 \dots e_k)$$

Example: Inference by Enumeration

- $P(W)$

- $P(W|winter)$

- $P(W|winter, hot)$

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Example: Inference by Enumeration

- $P(W)$

  $$P(W = sun) = 0.65$$
  $$P(W = rain) = 0.35$$

- $P(W|winter)$

- $P(W|winter, hot)$

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
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winter	hot	rain	0.05
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  $$P(W = sun|winter) = 0.50$$
  $$P(W = rain|winter) = 0.50$$

- $P(W|winter, hot)$

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
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- $P(W|winter)$

  $P(W = sun|winter) = 0.50$
  $P(W = rain|winter) = 0.50$
- $P(W|winter, hot)$

  $P(W = sun|winter, hot) = 0.666\ldots$
  $P(W = rain|winter, hot) = 0.333\ldots$

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
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Problems with Inference by Enumeration

- Obvious problems:
  - Worst-case time complexity: $O(d^n)$
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Problems with Inference by Enumeration

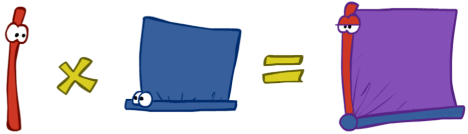
- Obvious problems:
  - Worst-case time complexity: $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution
- Availability of the joint distributions and evidence

The Product Rule

- Sometimes we have condition distributions, but want the joint distribution

$$P(x|y) = \frac{P(x,y)}{P(y)}$$

$$P(x,y) = P(y)P(x|y)$$



The Product Rule

$$P(x,y) = P(y)P(x|y)$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$\times$

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$=$

$P(D,W)$

D	W	P
wet	sun	
dry	sun	
wet	rain	
dry	rain	

The Product Rule

$$P(x, y) = P(y)P(x|y)$$

- Example:

$P(W)$

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sun	0.8
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$\times$

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$=$

$P(D, W)$

D	W	P
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06

The Chain Rule

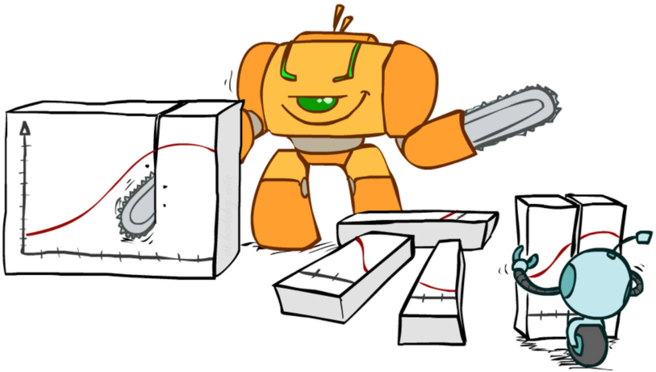
- More generally, we can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$
$$P(x_1, x_2, \ldots, x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Why is this always true?
  The following can be extended for any x_n:

$$\begin{aligned} P(x_2, x_1) &= P(x_1) \times P(x_2|x_1) \\ &= P(x_1) \times \frac{P(x_2, x_1)}{P(x_1)} \end{aligned}$$

Bayes' Rule



Bayes' Rule

- Two ways to factor a joint distribution over two variables:
  $P(x, y) = P(x|y)P(y) = P(y|x)P(x)$

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$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

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- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems (e.g. ASR, MT)

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- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems (e.g. ASR, MT)
- In the running for most important AI equation!

Inference with Bayes' Rule

- Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:
  - M: meningitis, S: stiff neck

$$P(+m) = 0.0001$$
$$P(+s|+m) = 0.8$$
$$P(+s|-m) = 0.01$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m)+P(|s|-m)P(-m)} = 0.00794$$

  - Note: posterior probability of meningitis still very small
  - Note: you should still get stiff necks checked out! Why?

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is $P(W|dry)$?

Ghostbusters (Revisited)

- Let's say we have two distributions:
  - **Prior distribution** over ghost location: $P(G)$
    - Let's say this is uniform
  - Sensor reading model: $P(R|G)$
    - Given: we know what our sensors do
    - R =reading color measured at $(1,1)$
    - e.g. $P(R = yellow|G = (1,1)) = 0.1$

0.11	0.11	0.11
0.11	0.11	0.11
0.11	0.11	0.11

Video: ghosts - with probability

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- Let's say we have two distributions:
  - **Prior distribution** over ghost location: $P(G)$
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  - Sensor reading model: $P(R|G)$
    - Given: we know what our sensors do
    - R =reading color measured at $(1,1)$
    - e.g. $P(R = yellow|G = (1,1)) = 0.1$
- We can calculate the **posterior distribution** $P(G|r)$ over ghost locations given a reading using Bayes' rule:

  $P(g|r) \inf P(r|g)P(g)$

0.11	0.11	0.11
0.11	0.11	0.11
0.11	0.11	0.11

0.17	0.10	0.10
0.09	0.17	0.10
<0.01	0.09	0.17

Video: ghosts - with probability

Suggested Reading

- Russell & Norvig: Chapter 13.1-13.5