Connectivity, Separability & Isomorphism

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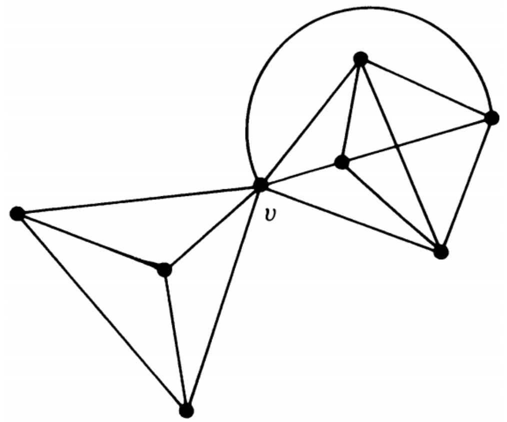
Connectivity & Separability

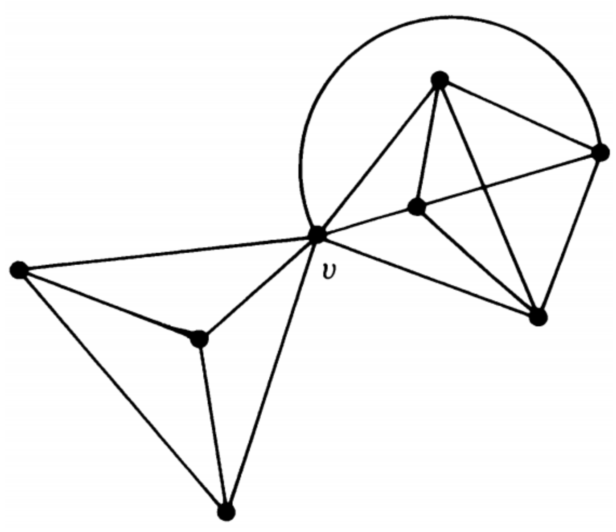
- Edge connectivity
- Vertex connectivity
- Separable graph is having vertex connectivity to be one
- Theorem 4.7 – A vertex *v* in a connected graph *G* is a cut-vertex if and only if there exist two vertices *x* and *y* in *G* such that every path between *x* and *y* passes through *v*.

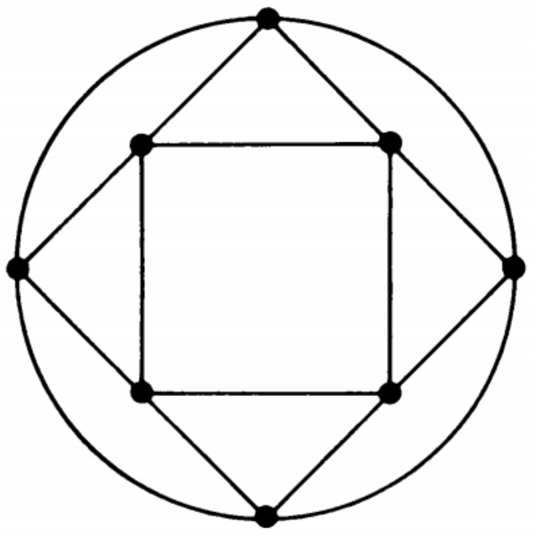


- *An Application:* Suppose we are given n stations that are to be connected by means of e lines (telephone lines, bridges, railroads, tunnels, or highways) where $e \geq n - 1$.
- What is the best way of connecting? By “best” we mean that the network should be as invulnerable to destruction of individual stations and individual lines as possible.
- Solution: construct a graph with n vertices and e edges that has the maximum possible edge connectivity and vertex connectivity.

Connectivity & Separability

- Graph with n=8, e=16





Edge Connectivity = 3

Vertex Connectivity = 1

Edge Connectivity = 4

Vertex Connectivity = 4

Connectivity & Separability

Theorem 4.8 – The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

Theorem 4.9 – The vertex connectivity of any graph G can never exceed the edge connectivity of G.

Connectivity & Separability

- Theorem 4.10 – The maximum vertex connectivity one can achieve with a graph *G* of *n* vertices and *e* edges($e \geq n - 1$) is the integral part of the number *2e/n*; that is, *floor(2e/n)*.
- Summary:
  - Vertex connectivity ≤ Edge connectivity ≤ $\frac{2e}{n}$
  - Maximum vertex connectivity possible = $floor(\frac{2e}{n})$

K-connected graph

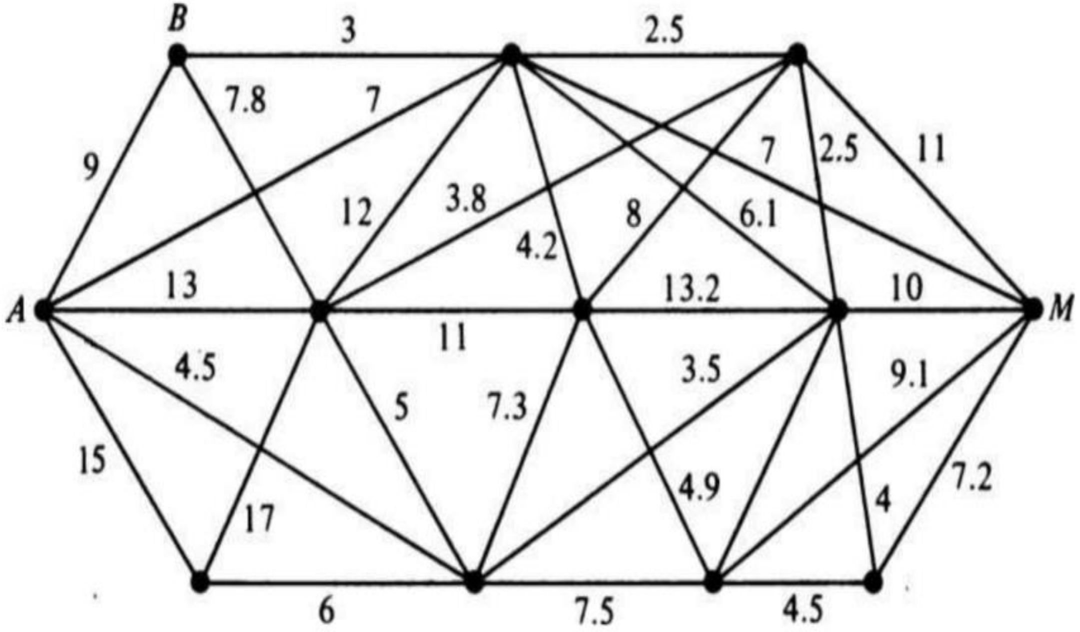
- *k-connected:* if the vertex connectivity of G is k
- *1-connected* graph is a separable graph

Theorem: 4.11- A connected graph G is k-connected if and only if every pair of vertices in G is joined by k or more paths that do not intersect, and at least one pair of vertices is joined by exactly k nonintersecting paths.

Theorem: 4.12- The edge connectivity of a graph G is k: if and only if every pair of vertices in G is joined by k or more edge-disjoint paths (i.e., paths that may intersect, but have no edges in common), and at least one pair of vertices is joined by exactly k edge-disjoint paths.

Network flows

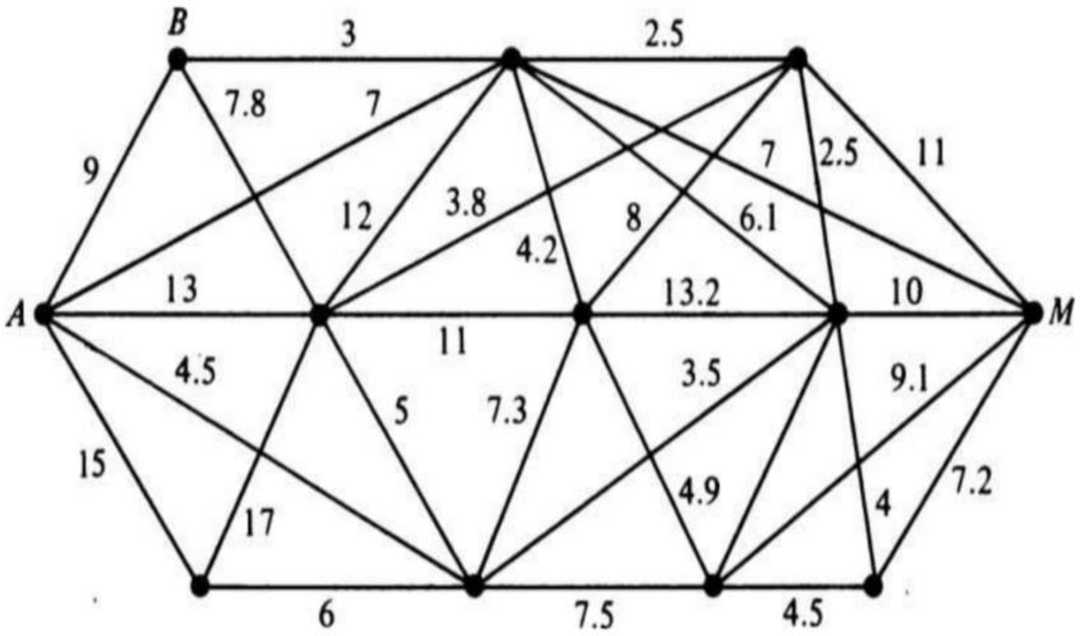
- flow network consisting of 12 stations and 31 lines
- Edge weights are capacity of the link
- For each vertex, rate of commodity entering and leaving are same
- Flow through each vertex is limited by the capacity of the edges
- Lines are lossless



Network flows

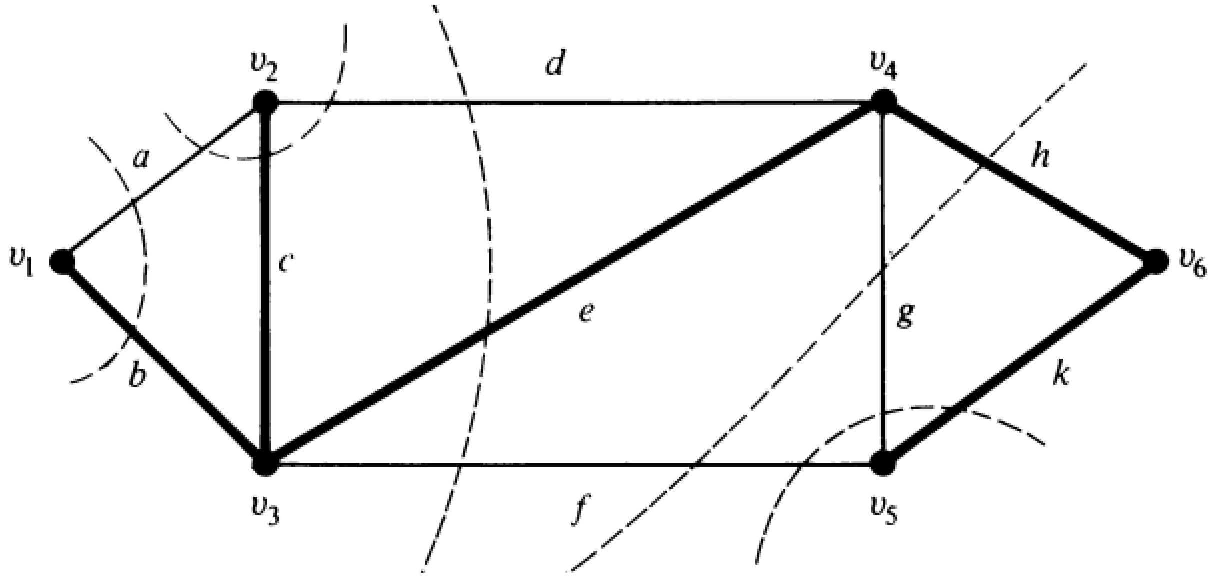
Questions:

- What is the maximum flow possible through the network between a specified pair of vertices?
- How do we achieve this flow?



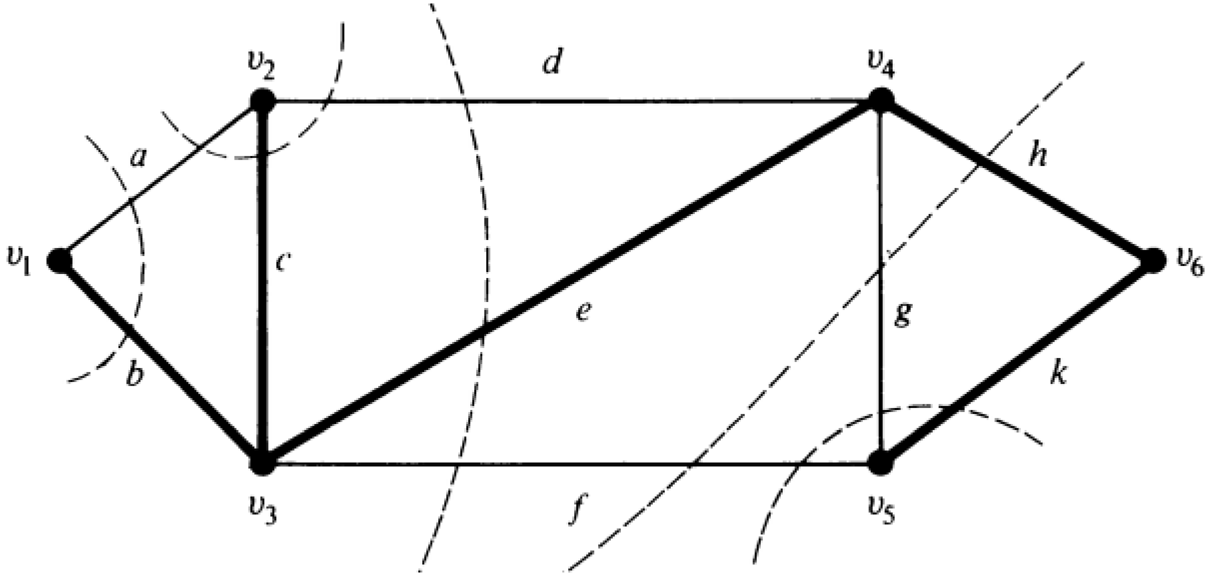
Network flows

- *cut-set with respect to a pair of vertices* v_1 *and* v_6
- *capacity of cut-set S* in a weighted connected graph



Network flows

Theorem 4.13- The maximum flow possible between two vertices *a* and *b* in a network is equal to the minimum of the capacities of all cut-sets with respect to *a* and *b*

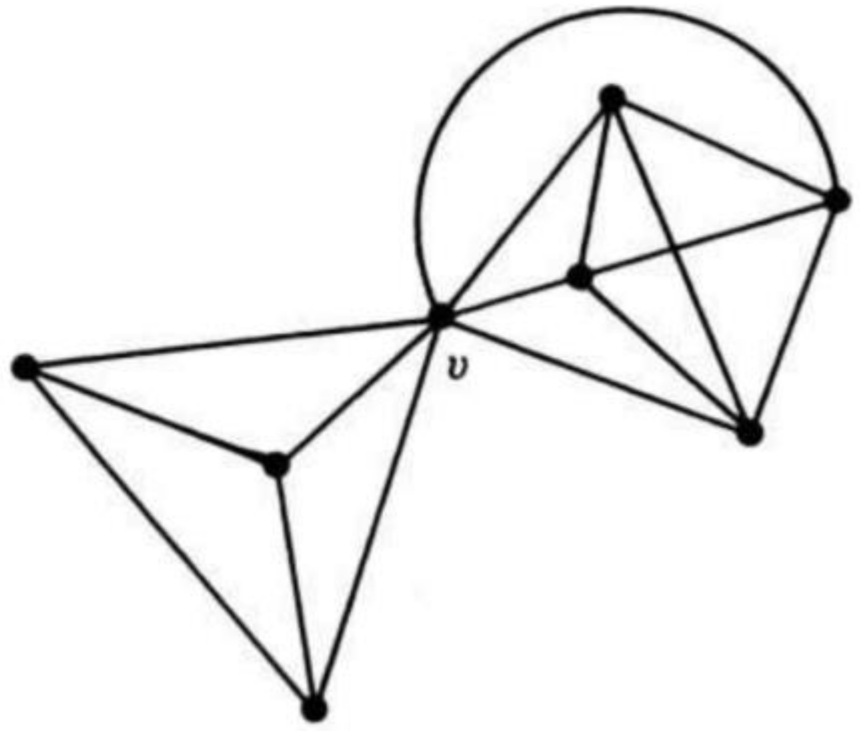


1-Isomorphism

• Blocks: largest nonseparable subgraphs in a graph

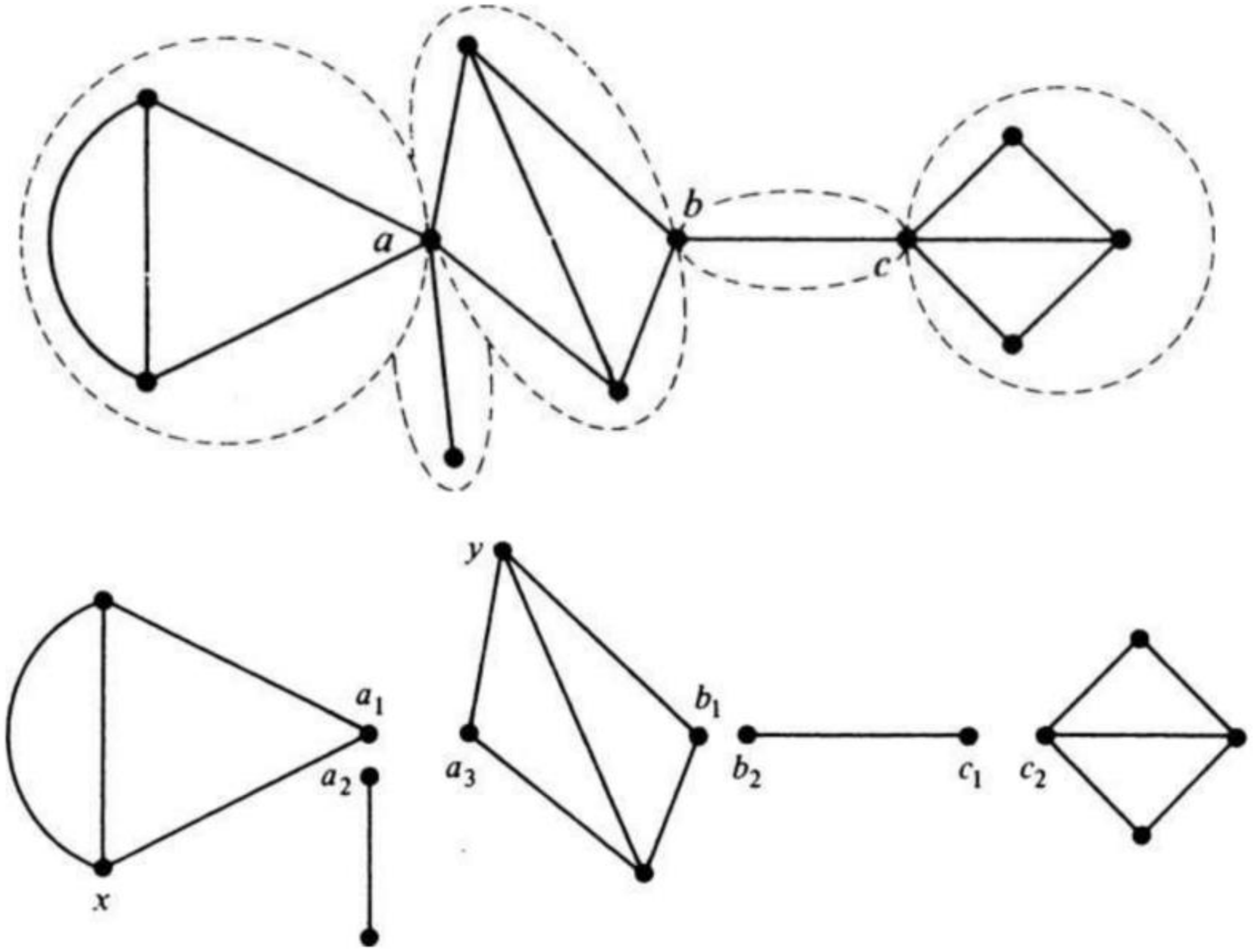
* Not to be confused with component

• If vertex v is removed this graph has two blocks



1-Isomorphism

- *Operation-1*: "Split" a cut-vertex into two vertices to produce two disjoint subgraphs
- Two graphs *G*1 and *G*2 are said to be *1-isomorphic* if they become isomorphic to each other under repeated application of the *Operation-1*.



1-Isomorphism

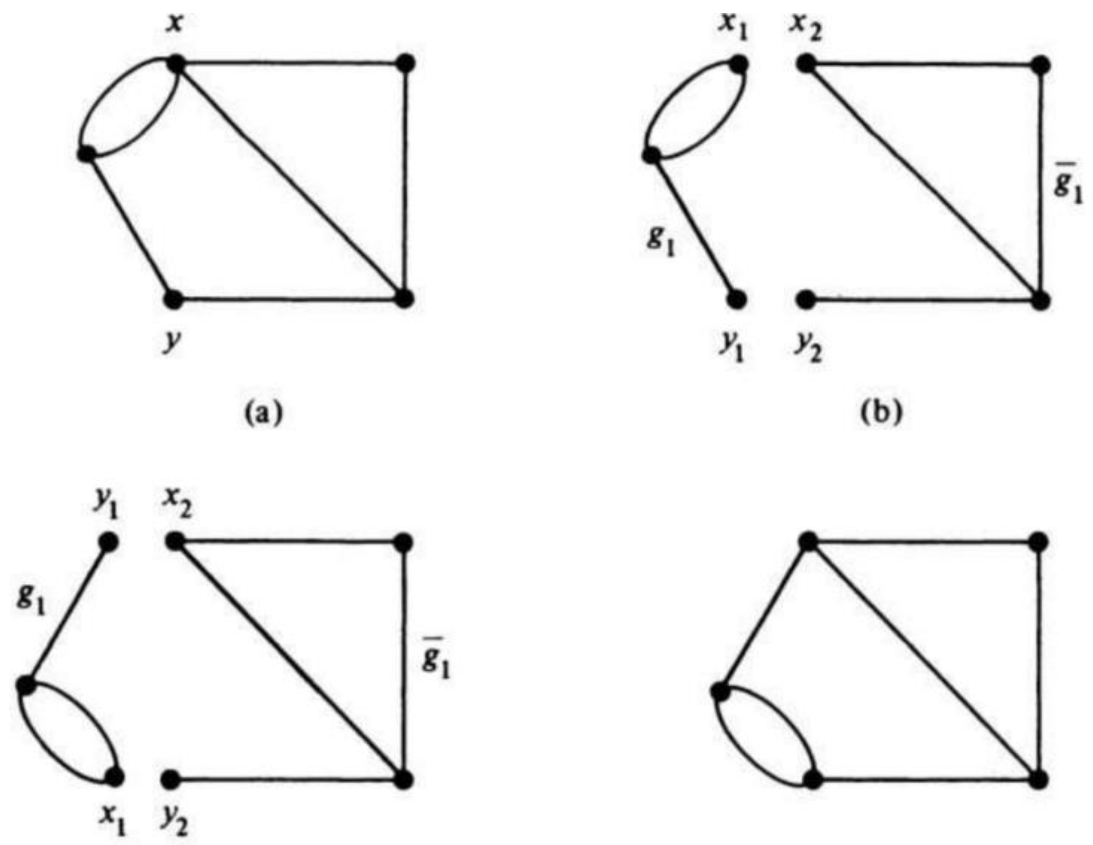
- Theorem 4.14 - If $G1$ and $G2$ are two 1-isomorphic graphs, the rank of $G1$ equals the rank of $G2$ and the nullity of $G1$ equals the nullity of $G2$.

- *Split* operation increase the number of vertices by 1. Increment of components is same as well.
- Rank remains invariant.
- nullity = number of edges - rank
- No edges are destroyed or no new edges are created.

2-Isomorphism

Operation 2:

- “Split” the vertex x into x_1 and x_2 and the vertex y into y_1 and y_2 such that G is split into g_1 and $\overline{g_1}$.
- Let vertices x_1 and y_1 go with g_1 and x_2 and y_2 with $\overline{g_1}$.
- Now rejoin the graphs g_1 and $\overline{g_1}$ by merging x_1 with y_2 and x_2 with y_1.



2-Isomorphism

- Two graphs are said to be *2-isomorphic* if they become isomorphic after undergoing *operation 1* or *operation 2*, or both operations any number of times.

Circuit correspondence

Circuit correspondence:

- One-to-one correspondence between edges
- One-to-one correspondence between circuits such that edges that forms the circuits has one-t-one correspondence

- Isomorphic graphs has circuit correspondence (obviously)
- Does 2-isomorphic graphs has circuit correspondence?

Circuit correspondence

-Yes, they have.

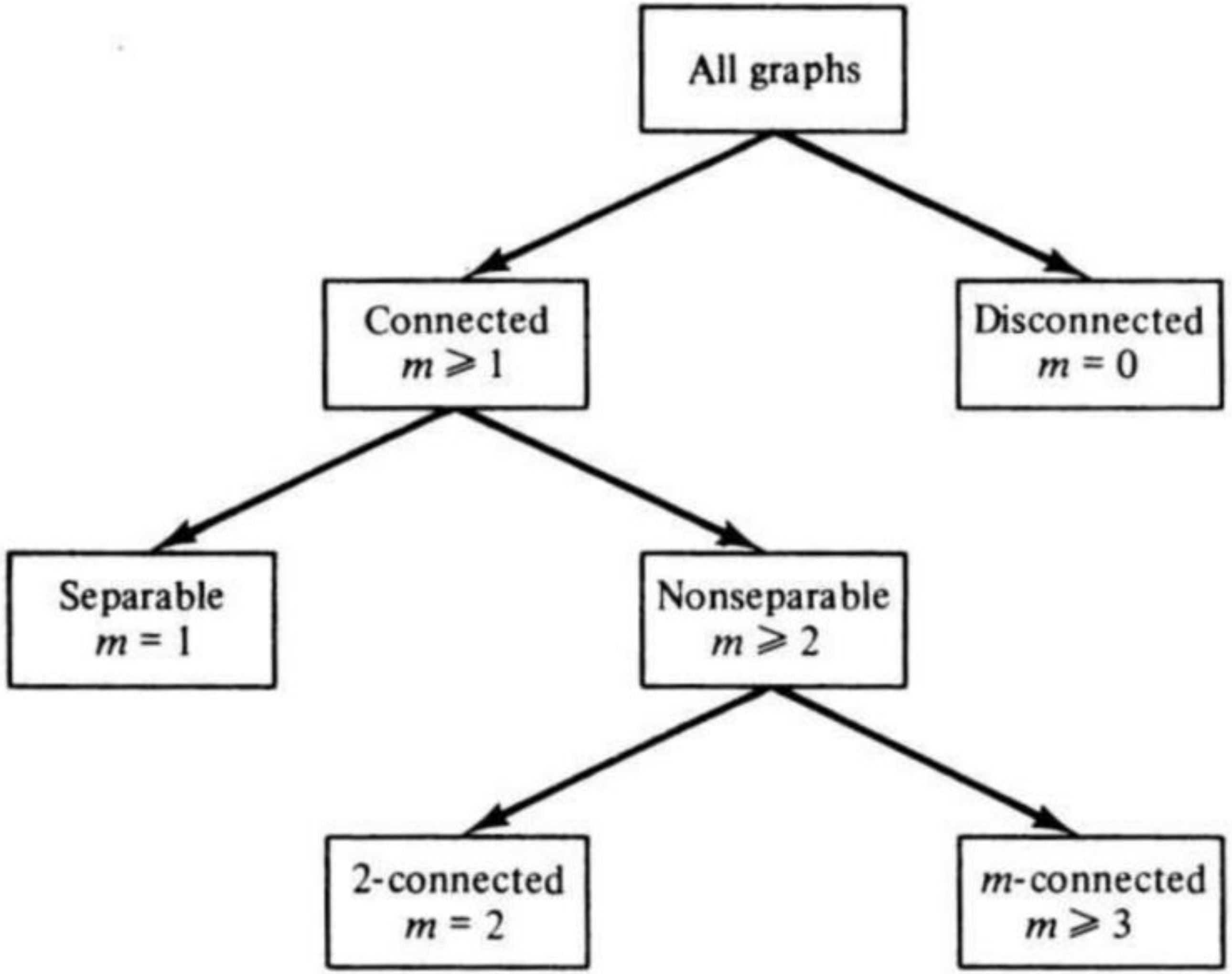
A circuit Γ in G will fall in one of three categories while undergoing *operation 2*:
1. Γ is made of edges all in g_1, or
2. Γ is made of edges all in $\overline{g_1}$, or
3. Γ is made of edges from both g_1 and $\overline{g_1}$, and in that case Γ must include both vertices x and y.

- In cases 1 and 2, Γ is unaffected by *operation 2*.
- In case 3, Γ still has the original edges forming the circuit

Circuit correspondence

Theorem 4.15 - Two graphs are 2-isomorphic if and only if they have circuit correspondence

Summary



All graphs
Connected
$m \geqslant 1$
Disconnected
$m = 0$
Separable
$m = 1$
Nonseparable
$m \geqslant 2$
2-connected
$m = 2$
m-connected
$m \geqslant 3$