Line integral of a complex function

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Complex Line Integrals

Let $f(z)$ be continuous at all points of a curve C [Fig. 4-1], which we shall assume has a finite length, i.e., C is a *rectifiable curve*.

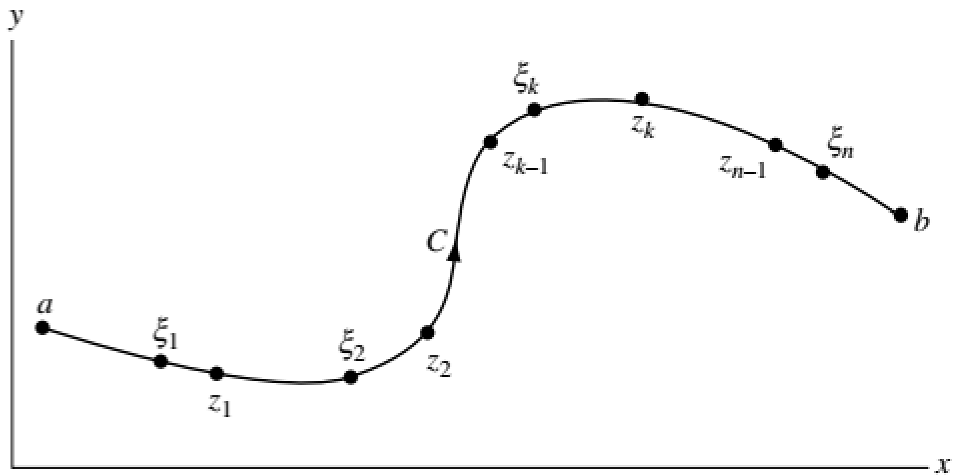

Fig. 4-1

Subdivide C into n parts by means of points $z_1, z_2, \ldots, z_{n-1}$, chosen arbitrarily, and call $a = z_0$, $b = z_n$. On each arc joining z_{k-1} to z_k [where k goes from 1 to n], choose a point ξ_k. Form the sum

$$S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(b - z_{n-1}) \tag{4.1}$$

On writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta z_k \tag{4.2}$$

Let the number of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then, since $f(z)$ is continuous, the sum S_n approaches a limit that does not depend on the mode of subdivision and we denote this limit by

$$\int_a^b f(z)\,dz \quad \text{or} \quad \int_C f(z)\,dz \tag{4.3}$$

called the *complex line integral* or simply *line integral* of $f(z)$ along curve C, or the *definite integral* of $f(z)$ from a to b along curve C. In such a case, $f(z)$ is said to be *integrable* along C. If $f(z)$ is analytic at all points of a region $\mathcal{R}$ and if C is a curve lying in $\mathcal{R}$, then $f(z)$ is continuous and therefore integrable along C.

Connection Between Real and Complex Line Integrals

Suppose $f(z) = u(x, y) + iv(x, y) = u + iv$. Then the complex line integral (3) can be expressed in terms of real line integrals as follows:

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy)$$
$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy \qquad (4.5)$$

For this reason, (4.5) is sometimes taken as a definition of a complex line integral.

Simply and Multiply Connected Regions

A region $\mathcal{R}$ is called *simply-connected* if any simple closed curve [Section 3.13], which lies in $\mathcal{R}$, can be shrunk to a point without leaving $\mathcal{R}$. A region $\mathcal{R}$, which is not simply-connected, is called *multiply-connected*.

For example, suppose $\mathcal{R}$ is the region defined by $|z| < 2$ shown shaded in Fig. 4-2. If Γ is any simple closed curve lying in $\mathcal{R}$ [i.e., whose points are in $\mathcal{R}$], we see that it can be shrunk to a point that lies in $\mathcal{R}$, and thus does not leave $\mathcal{R}$, so that $\mathcal{R}$ is simply-connected. On the other hand, if $\mathcal{R}$ is the region defined by $1 < |z| < 2$, shown shaded in Fig. 4-3, then there is a simple closed curve Γ lying in $\mathcal{R}$ that cannot possibly be shrunk to a point without leaving $\mathcal{R}$, so that $\mathcal{R}$ is multiply-connected.

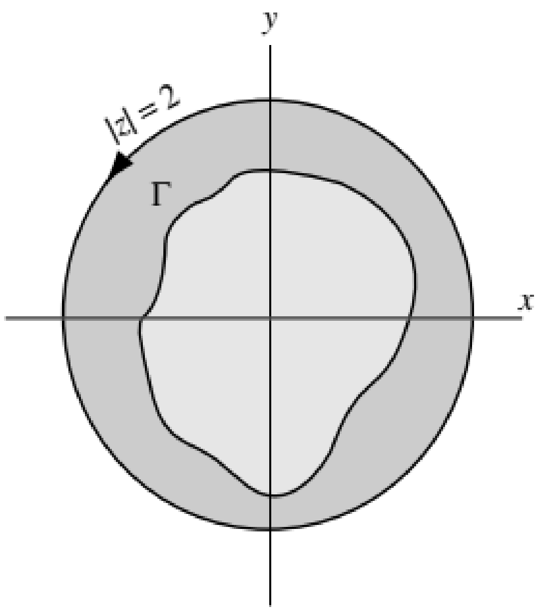

Fig. 4-2

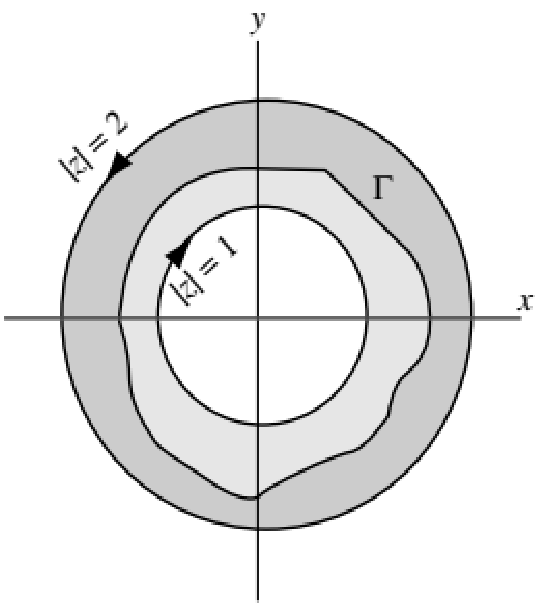

Fig. 4-3

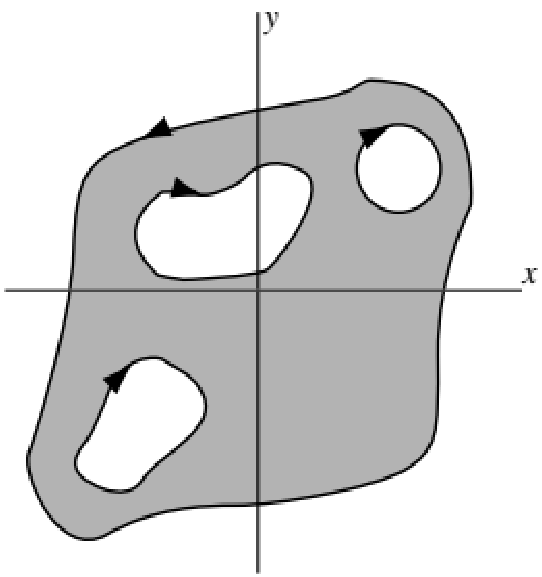

Fig. 4-4

Intuitively, a simply-connected region is one that does not have any "holes" in it, while a multiply-connected region is one that does. The multiply-connected regions of Figs. 4-3 and 4-4 have, respectively, one and three holes in them.

Ex-1: Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along: (a) the parabola $x = 2t$, $y = t^2 + 3$; (b) straight lines from (0, 3) to (2, 3) and then from (2, 3) to (2, 4); (c) a straight line from (0, 3) to (2, 4).

Solution

(a) The points (0, 3) and (2, 4) on the parabola correspond to $t = 0$ and $t = 1$, respectively. Then, the given integral equals

$$\int_{t=0}^{1} [2(t^2 + 3) + (2t)^2]2\,dt + [3(2t) - (t^2 + 3)]2t\,dt = \int_{0}^{1} (24t^2 + 12 - 2t^3 - 6t)\,dt = \frac{33}{2}$$

(b) Along the straight line from (0, 3) to (2, 3), $y = 3$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{2} (6 + x^2)\,dx + (3x - 3)0 = \int_{x=0}^{2} (6 + x^2)\,dx = \frac{44}{3}$$

Along the straight line from (2, 3) to (2, 4), $x = 2$, $dx = 0$ and the line integral equals

$$\int_{y=3}^{4} (2y + 4)0 + (6 - y)\,dy = \int_{y=3}^{4} (6 - y)\,dy = \frac{5}{2}$$

Then, the required value $= 44/3 + 5/2 = 103/6$.

(c) An equation for the line joining (0, 3) and (2, 4) is $2y - x = 6$. Solving for x, we have $x = 2y - 6$. Then, the line integral equals

$$\int_{y=3}^{4} [2y + (2y-6)^2]2\,dy + [3(2y-6) - y]\,dy = \int_{3}^{4} (8y^2 - 39y + 54)\,dy = \frac{97}{6}$$

The result can also be obtained by using $y = \frac{1}{2}(x+6)$.

Ex-2: Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by: (a) $z = t^2 + it$,

(b) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

Solution

(a) The points $z = 0$ and $z = 4 + 2i$ on C correspond to $t = 0$ and $t = 2$, respectively. Then, the line integral equals

$$\int_{t=0}^{2} (\overline{t^2 + it})\,d(t^2 + it) = \int_{0}^{2} (t^2 - it)(2t + i)\,dt = \int_{0}^{2} (2t^3 - it^2 + t)\,dt = 10 - \frac{8i}{3}$$

(b) The given line integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

The line from $z = 0$ to $z = 2i$ is the same as the line from (0, 0) to (0, 2) for which $x = 0$, $dx = 0$ and the line integral equals

$$\int_{y=0}^{2} (0)(0) + y\,dy + i\int_{y=0}^{2} (0)(dy) - y(0) = \int_{y=0}^{2} y\,dy = 2$$

The line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from (0, 2) to (4, 2) for which $y = 2$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{4} x\,dx + 2\cdot 0 + i\int_{x=0}^{4} x\cdot 0 - 2\,dx = \int_0^4 x\,dx + i\int_0^4 -2\,dx = 8 - 8i$$

Then, the required value $= 2 + (8 - 8i) = 10 - 8i$.

Ex-3: Let C be the curve $y = x^3 - 3x^2 + 4x - 1$ joining points (1, 1) and (2, 3). Find the value of $\int_C (12z^2 - 4iz)\, dz$.

Solution

the integral is independent of the path joining (1, 1) and (2, 3). Hence, any path can be chosen. In particular, let us choose the straight line paths from (1, 1) to (2, 1) and then from (2, 1) to (2, 3).

Case 1. Along the path from (1, 1) to (2, 1), $y = 1$, $dy = 0$ so that $z = x + iy = x + i$, $dz = dx$. Then, the integral equals

$$\int_{x=1}^{2} \left\{12(x+i)^2 - 4i(x+i)\right\} dx = \left\{4(x+i)^3 - 2i(x+i)^2\right\}\Big|_1^2 = 20 + 30i$$

Case 2. Along the path from (2, 1) to (2, 3), $x = 2$, $dx = 0$ so that $z = x + iy = 2 + iy$, $dz = i\,dy$. Then, the integral equals

$$\int_{y=1}^{3} \left\{12(2+iy)^2 - 4i(2+iy)\right\} i\,dy = \left\{4(2+iy)^3 - 2i(2+iy)^2\right\}\Big|_1^3 = -176 + 8i$$

Then, adding the required value $= (20 + 30i) + (-176 + 8i) = -156 + 38i$.

Ex-1: Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$ along (a) the curve $y = x^2 + 1$, (b) the straight line joining $(0, 1)$ and $(2, 5)$, (c) the straight lines from $(0, 1)$ to $(0, 5)$ and then from $(0, 5)$ to $(2, 5)$, (d) the straight lines from $(0, 1)$ to $(2, 1)$ and then from $(2, 1)$ to $(2, 5)$.

Ex-2: (a) Evaluate $\oint_C (x+2y)\,dx + (y-2x)\,dy$ around the ellipse C defined by $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \le \theta < 2\pi$ if C is described in a counterclockwise direction.

(b) What is the answer to (a) if C is described in a clockwise direction?

Ex-3: Evaluate $\int_C (x^2 - iy^2)\,dz$ along (a) the parabola $y = 2x^2$ from $(1, 2)$ to $(2, 8)$, (b) the straight lines from $(1, 1)$ to $(1, 8)$ and then from $(1, 8)$ to $(2, 8)$, (c) the straight line from $(1, 1)$ to $(2, 8)$.

Ex-4: Evaluate $\oint_C |z|^2\,dz$ around the square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$.

Ex-5: Evaluate $\int_C (z^2 + 3z)\,dz$ along (a) the circle $|z| = 2$ from $(2, 0)$ to $(0, 2)$ in a counterclockwise direction, (b) the straight line from $(2, 0)$ to $(0, 2)$, (c) the straight lines from $(2, 0)$ to $(2, 2)$ and then from $(2, 2)$ to $(0, 2)$.

Ex-6: Suppose $f(z)$ and $g(z)$ are integrable. Prove that

(a) $\displaystyle\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$, (b) $\displaystyle\int_C \{2f(z) - 3ig(z)\}\,dz = 2\int_C f(z)\,dz - 3i\int_C g(z)\,dz.$

Ex-7: Evaluate $\int_{i}^{2-i} (3xy + iy^2)\, dz$ (a) along the straight line joining $z = i$ and $z = 2 - i$, (b) along the curve $x = 2t - 2$, $y = 1 + t - t^2$.

Ex-8: Evaluate $\oint_C \bar{z}^2\, dz$ around the circles (a) $|z| = 1$, (b) $|z - 1| = 1$.

Ex-9: Evaluate $\oint_C (5z^4 - z^3 + 2)\, dz$ around (a) the circle $|z| = 1$, (b) the square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, (c) the curve consisting of the parabolas $y = x^2$ from $(0, 0)$ to $(1, 1)$ and $y^2 = x$ from $(1, 1)$ to $(0, 0)$.

Ex-10: Evaluate $\int_C (z^2 + 1)^2\, dz$ along the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from the point where $\theta = 0$ to the point where $\theta = 2\pi$.

Ex-11: Evaluate $\int_C \bar{z}^2\, dz + z^2\, d\bar{z}$ along the curve C defined by $z^2 + 2z\bar{z} + \bar{z}^2 = (2 - 2i)z + (2 + 2i)\bar{z}$ from the point $z = 1$ to $z = 2 + 2i$.

Ex-12: Evaluate $\oint_C dz/z - 2$ around
(a) the circle $|z - 2| = 4$, (b) the circle $|z - 1| = 5$, (c) the square with vertices at $3 \pm 3i$, $-3 \pm 3i$.

Ex-13: Evaluate $\oint_C (x^2 + iy^2)\, ds$ around the circle $|z| = 2$ where s is the arc length.

Thanks a lot …