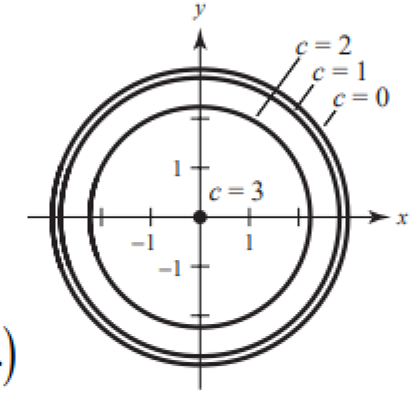
52. $f(x, y) = \sqrt{9 - x^2 - y^2}$

The level curves are of the form

$c = \sqrt{9 - x^2 - y^2}$

$x^2 + y^2 = 9 - c^2$, circles.

$\left(x^2 + y^2 = 0 \text{ is the point } (0, 0).\right)$



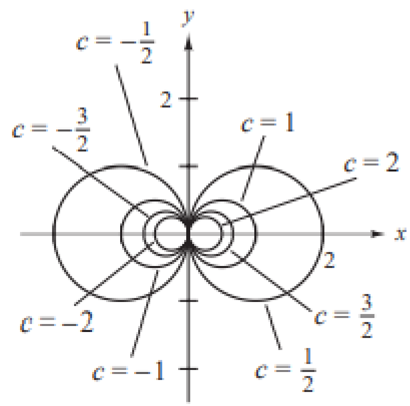
55. $f(x, y) = \dfrac{x}{x^2 + y^2}$

The level curves are of the form

$$c = \frac{x}{x^2 + y^2}$$

$$x^2 - \frac{x}{c} + y^2 = 0$$

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2.$$



So, the level curves are circles passing through the origin and centered at $(\pm 1/2c, 0)$.

9. $\lim_{(x,y)\to(2,1)} (2x^2 + y) = 8 + 1 = 9$

Continuous everywhere

10. $\lim_{(x,y)\to(0,0)} (x + 4y + 1) = 0 + 4(0) + 1 = 1$

Continuous everywhere

11. $\lim_{(x,y)\to(1,2)} e^{xy} = e^{1(2)} = e^2$

Continuous everywhere

12. $\lim_{(x,y)\to(2,4)} \dfrac{x+y}{x^2+1} = \dfrac{2+4}{2^2+1} = \dfrac{6}{5}$

Continuous everywhere

13. $\lim_{(x,y)\to(0,2)} \dfrac{x}{y} = \dfrac{0}{2} = 0$

Continuous for all $y \neq 0$

14. $\lim_{(x,y)\to(-1,2)} \dfrac{x+y}{x-y} = \dfrac{-1+2}{-1-2} = -\dfrac{1}{3}$

Continuous for all $x \neq y$.

15. $\lim_{(x,y)\to(1,1)} \dfrac{xy}{x^2+y^2} = \dfrac{1}{2}$

Continuous except at $(0, 0)$

16. $\lim_{(x,y)\to(1,1)} \dfrac{x}{\sqrt{x+y}} = \dfrac{1}{\sqrt{1+1}} = \dfrac{\sqrt{2}}{2}$

Continuous for $x + y > 0$

17. $\lim_{(x,y)\to(\pi/4,2)} y\cos(xy) = 2\cos\dfrac{\pi}{2} = 0$

Continuous everywhere

18. $\lim_{(x,y)\to(2\pi,4)} \sin\dfrac{x}{y} = \sin\dfrac{2\pi}{4} = 1$

Continuous for all $y \neq 0$

19. $\lim_{(x,y)\to(0,1)} \dfrac{\arcsin xy}{1-xy} = \dfrac{\arcsin 0}{1} = 0$

Continuous for $xy \neq 1$, $|xy| \leq 1$

20. $\lim_{(x,y)\to(0,1)} \dfrac{\arccos\left(\dfrac{x}{y}\right)}{1+xy} = \dfrac{\arccos 0}{1} = \dfrac{\pi}{2}$

Continuous for $xy \neq -1$, $y \neq 0$, $0 \leq \dfrac{x}{y} \leq \pi$

21. $\lim_{(x,y,z)\to(1,3,4)} \sqrt{x+y+z} = \sqrt{1+3+4} = 2\sqrt{2}$

Continuous for $x + y + z \geq 0$

22. $\lim_{(x,y,z)\to(-2,1,0)} xe^{yz} = (-2)e^{1(0)} = -2$

Continuous everywhere

23. $\lim_{(x,y)\to(1,1)} \dfrac{xy-1}{1+xy} = \dfrac{1-1}{1+1} = 0$

24. $\lim_{(x,y)\to(1,-1)} \dfrac{x^2y}{1+xy^2} = \dfrac{-1}{1+1} = -\dfrac{1}{2}$

25. $\lim_{(x,y)\to(0,0)} \dfrac{1}{x+y}$ does not exist

Because the denominator $x + y$ approaches 0 as $(x, y) \to (0, 0)$.

26. $\lim_{(x,y)\to(0,0)} \dfrac{1}{x^2y^2}$ does not exist because the denominator xy approaches 0 as $(x, y) \to (0, 0)$.

27. $\lim_{(x,y)\to(0,0)} \dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

does not exist because you can't approach $(0, 0)$ from negative values of x and y.

28. $\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1} \cdot \frac{\sqrt{x-y}+1}{\sqrt{x-y}+1}$

$= \lim_{(x,y)\to(2,1)} \frac{(x-y-1)\left(\sqrt{x-y}+1\right)}{(x-y)-1}$

$= \lim_{(x,y)\to(2,1)} \left(\sqrt{x-y}+1\right) = 2$

29. The limit does not exist because along the line $y = 0$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y} = \lim_{(x,0)\to(0,0)} \frac{x}{x^2} = \lim_{(x,0)\to(0,0)} \frac{1}{x}$$

which does not exist.

30. The limit does not exist because along the line $x = y$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2} = \lim_{(x,x)\to(0,0)} \frac{x}{x^2-x^2} = \lim_{(x,x)\to(0,0)} \frac{x}{0}.$$

Because the denominator is 0, the limit does not exist.

31. $\lim_{(x,y)\to(0,0)} \frac{x^2}{\left(x^2+1\right)\left(y^2+1\right)} = \frac{0}{(1)(1)} = 0$

32. $\lim_{(x,y)\to(0,0)} \ln\left(x^2+y^2\right)$ does not exist

because $\ln\left(x^2+y^2\right) \to -\infty$ as $(x,y) \to (0,0)$.

33. The limit does not exist because along the path $x = 0$, $y = 0$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(0,0,z)\to(0,0,0)} \frac{0}{z^2} = 0$$

whereas along the path $x = y = z$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(x,x,x)\to(0,0,0)} \frac{x^2+x^2+x^2}{x^2+x^2+x^2}$$

$$= 1$$

34. The limit does not exist because along the path $y = z = 0$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim_{(x,0,0)\to(0,0,0)} \frac{0}{x^2} = 0$$

However, along the path $z = 0$, $x = y$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim_{(x,x,0)\to(0,0,0)} \frac{x^2}{x^2+x^2}$$

$$= \frac{1}{2}$$

35. $\lim_{(x,y)\to(0,0)} e^{xy} = 1$

Continuous everywhere

36. $\lim_{(x,y)\to(0,0)} \left[1 - \frac{\cos\left(x^2+y^2\right)}{x^2+y^2}\right] = -\infty$

The limit does not exist.

Continuous except at $(0,0)$

41. $f(x, y) = \sqrt{x+y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+y} - \sqrt{x+y}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{x+\Delta x+y} - \sqrt{x+y}\right)\left(\sqrt{x+\Delta x+y} + \sqrt{x+y}\right)}{\Delta x\left(\sqrt{x+\Delta x+y} + \sqrt{x+y}\right)} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+y} + \sqrt{x+y}} = \frac{1}{2\sqrt{x+y}}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt{x+y+\Delta y} - \sqrt{x+y}}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{\left(\sqrt{x+y+\Delta y} - \sqrt{x+y}\right)\left(\sqrt{x+y+\Delta y} + \sqrt{x+y}\right)}{\Delta y\left(\sqrt{x+y+\Delta y} + \sqrt{x+y}\right)}$$
$$= \lim_{\Delta y \to 0} \frac{1}{\sqrt{x+y+\Delta y} + \sqrt{x+y}} = \frac{1}{2\sqrt{x+y}}$$

48. $f(x, y) = \arccos(xy)$

$$f_x(x, y) = \frac{-y}{\sqrt{1 - x^2y^2}}$$

At $(1, 1)$, f_x is undefined.

$$f_y(x, y) = \frac{-x}{\sqrt{1 - x^2y^2}}$$

At $(1, 1)$, f_y is undefined.

49. $f(x, y) = \dfrac{xy}{x - y}$

$$f_x(x, y) = \frac{y(x-y) - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

At $(2, -2)$: $f_x(2, -2) = -\dfrac{1}{4}$

$$f_y(x, y) = \frac{x(x-y) + xy}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

At $(2, -2)$: $f_y(2, -2) = \dfrac{1}{4}$

$$H_y(x, y, z) = 2\cos(x + 2y + 3z)$$
$$H_z(x, y, z) = 3\cos(x + 2y + 3z)$$

54. $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

$$f_x(x, y, z) = 6xy - 5yz$$
$$f_y(x, y, z) = 3x^2 - 5xz + 10z^2$$
$$f_z(x, y, z) = -5xy + 20yz$$

55. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$
$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$
$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

45. $f(x, y) = \cos(2x - y)$

$f_x(x, y) = -2\sin(2x - y)$

At $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$, $f_x\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = -2\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = -1.$

$f_y(x, y) = \sin(2x - y)$

At $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$, $f_y\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{2}.$

46. $f(x, y) = \sin xy$

$f_x(x, y) = y\cos xy$

At $\left(2, \frac{\pi}{4}\right)$, $f_x\left(2, \frac{\pi}{4}\right) = \frac{\pi}{4}\cos\frac{\pi}{2} = 0.$

$f_y(x, y) = x\cos xy$

At $\left(2, \frac{\pi}{4}\right)$, $f_y\left(2, \frac{\pi}{4}\right) = 2\cos\frac{\pi}{2} = 0.$

47. $f(x, y) = \arctan\frac{y}{x}$

$f_x(x, y) = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$

At $(2, -2)$: $f_x(2, -2) = \frac{1}{4}$

$f_y(x, y) = \frac{1}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$

At $(2, -2)$: $f_y(2, -2) = \frac{1}{4}$

50. $f(x, y) = \frac{2xy}{\sqrt{4x^2 + 5y^2}}$

$f_x(x, y) = \frac{10y^3}{\left(4x^2 + 5y^2\right)^{3/2}}$

At $(1, 1)$, $f_x(1, 1) = \frac{10}{9^{3/2}} = \frac{10}{27}.$

$f_y(x, y) = \frac{8x^3}{\left(4x^2 + 5y^2\right)^{3/2}}$

At $(1, 1)$, $f_y(1, 1) = \frac{8}{9^{3/2}} = \frac{8}{27}.$

51. $g(x, y) = 4 - x^2 - y^2$

$g_x(x, y) = -2x$

At $(1, 1)$: $g_x(1, 1) = -2$

$g_y(x, y) = -2y$

At $(1, 1)$: $g_y(1, 1) = -2$

52. $h(x, y) = x^2 - y^2$

$h_x(x, y) = 2x$

At $(-2, 1)$: $h_x(-2, 1) = -4$

$h_y(x, y) = -2y$

At $(-2, 1)$: $h_y(-2, 1) = -2$

53. $H(x, y, z) = \sin(x + 2y + 3z)$

57. $F(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\ln(x^2 + y^2 + z^2)$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

58. $G(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$G_x(x, y, z) = \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_y(x, y, z) = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_z(x, y, z) = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_z(x, y, z) = \frac{\quad}{yz^2}$$

$$f_z(1, -1, -1) = 1$$

62. $f(x, y, z) = \frac{xy}{x + y + z}$

$$f_x(x, y, z) = \frac{(x + y + z)y - xy}{(x + y + z)^2} = \frac{y^2 + yz}{(x + y + z)^2}$$

$$f_x(3, 1, -1) = \frac{1 - 1}{3^2} = 0$$

$$f_y(x, y, z) = \frac{(x + y + z)x - xy}{(x + y + z)^2} = \frac{x^2 + xz}{(x + y + z)^2}$$

$$f_y(3, 1, -1) = \frac{9 - 3}{3^2} = \frac{2}{3}$$

$$f_z(x, y, z) = \frac{(x + y + z)(0) - xy}{(x + y + z)^2} = \frac{-xy}{(x + y + z)^2}$$

$$f_z(3, 1, -1) = \frac{-3}{9} = \frac{-1}{3}$$

125. $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $$f_x(x, y) = \frac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

(b) $$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0/\left[(\Delta x)^2\right] - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0/\left[(\Delta y)^2\right] - 0}{\Delta y} = 0$$

(c) $$f_{xy}(0, 0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\bigg|_{(0,0)} = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y(-(\Delta y)^4)}{((\Delta y)^2)^2(\Delta y)} = \lim_{\Delta y \to 0}(-1) = -1$$

$$f_{yx}(0, 0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\bigg|_{(0,0)} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x((\Delta x)^4)}{((\Delta x)^2)^2(\Delta x)} = \lim_{\Delta x \to 0} 1 = 1$$

(d) f_{yx} or f_{xy} or both are not continuous at $(0, 0)$.

1. $z = 2x^2y^3$

$dz = 4xy^3\,dx + 6x^2y^2\,dy$

2. $z = 2x^4y - 8x^2y^3$

$dz = \left(8x^3y - 16xy^3\right)dx + \left(2x^4 - 24x^2y^2\right)dy$

3. $z = \dfrac{-1}{x^2 + y^2}$

$$dz = \frac{2x}{\left(x^2 + y^2\right)^2}\,dx + \frac{2y}{\left(x^2 + y^2\right)^2}\,dy$$

$$= \frac{2}{\left(x^2 + y^2\right)^2}(x\,dx + y\,dy)$$

4. $w = \dfrac{x + y}{z - 3y}$

$$dw = \frac{1}{z - 3y}\,dx + \frac{3x + z}{(z - 3y)^2}\,dy - \frac{x + y}{(z - 3y)^2}\,dz$$

5. $z = x\cos y - y\cos x$

$dz = (\cos y + y\sin x)\,dx + (-x\sin y - \cos x)\,dy$

$= (\cos y + y\sin x)\,dx - (x\sin y + \cos x)\,dy$

6. $z = \left(\dfrac{1}{2}\right)\left(e^{x^2+y^2} - e^{-x^2-y^2}\right)$

$$dz = 2x\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right)dx + 2y\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right)dy$$

$$= \left(e^{x^2+y^2} + e^{-x^2-y^2}\right)(x\,dx + y\,dy)$$

7. $z = e^x\sin y$

$dz = \left(e^x\sin y\right)dx + \left(e^x\cos y\right)dy$

8. $w = e^y\cos x + z^2$

$dw = -e^y\sin x\,dx + e^y\cos x\,dy + 2z\,dz$

9. $w = 2z^3y\sin x$

$dw = 2z^3y\cos x\,dx + 2z^3\sin x\,dy + 6z^2y\sin x\,dz$

10. $w = x^2yz^2 + \sin yz$

$dw = 2xyz^2\,dx + \left(x^2z^2 + z\cos yz\right)dy + \left(2x^2yz + y\cos yz\right)dz$

11. $f(x, y) = 2x - 3y$

(a) $f(2, 1) = 1$

$f(2.1, 1.05) = 1.05$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0.05$

(b) $dz = 2\,dx - 3\,dy = 2(0.1) - 3(0.05) = 0.05$

12. $f(x, y) = x^2 + y^2$

(a) $f(2, 1) = 5$

$f(2.1, 1.05) = 5.5125$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0.5125$

(b) $dz = 2x\,dx + 2y\,dy$

$= 2(2)(0.1) + 2(1)(0.05) = 0.5$

13. $f(x, y) = 16 - x^2 - y^2$

(a) $f(2, 1) = 11$

$f(2.1, 1.05) = 10.4875$

$\Delta z = f(2.1, 1.05) - f(2.1) = -0.5125$

(b) $dz = -2x\,dx - 2y\,dy$

$= -2(2)(0.1) - 2(1)(0.05) = -0.5$

14. $f(x, y) = \dfrac{y}{x}$

(a) $f(2, 1) = 0.5$

$f(2.1, 1.05) = 0.5$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0$

(b) $dz = \dfrac{-y}{x^2}\,dx + \dfrac{1}{x}\,dy = \dfrac{-1}{4}(0.1) + \dfrac{1}{2}(0.05) = 0$

15. $f(x, y) = ye^x$

(a) $f(2, 1) = e^2 \approx 7.3891$

$f(2.1, 1.05) = 1.05e^{2.1} \approx 8.5745$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 1.1854$

(b) $dz = ye^x\,dx + e^x\,dy$

$= e^2(0.1) + e^2(0.05) \approx 1.1084$

16. $f(x, y) = x \cos y$

(a) $f(2, 1) = 2 \cos 1 \approx 1.0806$

$f(2.1, 1.05) = 2.1 \cos 1.05 \approx 1.0449$

$\Delta z = f(2.1, 1.05) - f(2, 1) = -0.0357$

(b) $dz = \cos y\, dx - x \sin y\, dy$

$= \cos 1(0.1) - 2 \sin 1(0.05) \approx -0.0301$

17. Let $z = x^2 y$, $x = 2$, $y = 9$, $dx = 0.01$, $dy = 0.02$.

Then: $dz = 2xy\, dx + x^2\, dy$

$$(2.01)^2(9.02) - 2^2 \cdot 9 \approx 2(2)(9)(0.01) + 2^2(0.02) = 0.44$$

18. Let $z = (1 - x^2)/y^2$, $x = 3$, $y = 6$, $dx = 0.05$, $dy = -0.05$. Then:

$$dz = -\frac{2x}{y^2}\, dx + \frac{-2(1 - x^2)}{y^3}\, dy$$

$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

19. Let $z = \sqrt{x^2 + y^2}$, $x = 5$, $y = 3$, $dx = 0.05$, $dy = 0.1$.

Then:

$$dz = \frac{x}{\sqrt{x^2 + y^2}}\, dx + \frac{y}{\sqrt{x^2 + y^2}}\, dy$$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

20. Let $z = \sin(x^2 + y^2)$, $x = y = 1$, $dx = 0.05$, $dy = -0.05$. Then: $dz = 2x \cos(x^2 + y^2)\, dx + 2y \cos(x^2 + y^2)\, dy$

$$\sin\left[(1.05)^2 + (0.95)^2\right] - \sin 2 \approx 2(1) \cos(1^2 + 1^2)(0.05) + 2(1) \cos(1^2 + 1^2)(-0.05) = 0$$

29. $V = xyz$, $dV = yz\, dx + xz\, dy + xy\, dz$

Propagated error $= dV = 5(12)(\pm 0.02) + 8(12)(\pm 0.02) + 8(5)(\pm 0.02)$

$= (60 + 96 + 40)(\pm 0.02) = 196(\pm 0.02) = \pm 3.92 \text{ in.}^3$

The measured volume is $V = 8(5)(12) = 480 \text{ in.}^3$

$$\text{Relative error} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3.92}{480} \approx 0.008167 \approx 0.82\%$$

30. $V = \pi r^2 h$, $dV = 2\pi rh\, dr + \pi r^2\, dh$

Propagated error $= dV = 2\pi(3)(10)(\pm 0.05) + \pi(3)^2(\pm 0.05)$

$= (60\pi + 9\pi)(\pm 0.05) = \pm 3.45\pi \text{ cm}^3$

The measured volume is $V = \pi(3^2)(10) = 90\pi \text{ cm}^3$.

$$\text{Relative error} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3.45\pi}{90\pi} \approx 0.0383 = 3.83\%$$

9. $w = xy + xz + yz,\ x = t - 1,\ y = t^2 - 1,\ z = t$

(a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = (y + z) + (x + z)(2t) + (x + y)$

$= (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1)$

(b) $w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$

$\dfrac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1 = 3(2t^2 - 1)$

$w = \sin(2x + 3y)$

$x = s + t$

$y = s - t$

$\dfrac{\partial w}{\partial s} = 2\cos(2x + 3y) + 3\cos(2x + 3y)$

$= 5\cos(2x + 3y) = 5\cos(5s - t)$

$\dfrac{\partial w}{\partial t} = 2\cos(2x + 3y) - 3\cos(2x + 3y)$

$= -\cos(2x + 3y) = -\cos(5s - t)$

When $s = 0$ and $t = \dfrac{\pi}{2}$, $\dfrac{\partial w}{\partial s} = 0$ and $\dfrac{\partial w}{\partial t} = 0$.

20. (a) $w = x\cos yz,\ x = s^2,\ y = t^2,\ z = s - 2t$

$\dfrac{\partial w}{\partial s} = \cos(yz)(2s) - xz\sin(yz)(0) - xy\sin(yz)(1)$

$= \cos(st^2 - 2t^3)2s - s^2t^2\sin(st^2 - 2t^3)$

$\dfrac{\partial w}{\partial t} = \cos(yz)(0) - xz\sin(yz)(2t) - xy\sin(yz)(-2)$

$= -2s^2t(s - 2t)\sin(st^2 - 2t^3) + 2s^2t^2\sin(st^2 - 2t^3)$

$= (6s^2t^2 - 2s^3t)\sin(st^2 - 2t^3)$

(b) $w = x\cos yz = s^2\cos\left(t^2(s - 2t)\right) = s^2\cos\left(st^2 - 2t^3\right)$

$\dfrac{\partial w}{\partial s} = s^2\left(-\sin\left(st^2 - 2t^3\right)\right)(t^2) + 2s\cos\left(st^2 - 2t^3\right)$

$= 2s\cos\left(st^2 - 2t^3\right) - s^2t^2\sin\left(st^2 - 2t^3\right)$

$\dfrac{\partial w}{\partial t} = -s^2\sin\left(st^2 - 2t^3\right)(2st - 6t^2)$

$= (6t^2s^2 - 2s^3t)\sin\left(st^2 - 2t^3\right)$

23. $\ln\sqrt{x^2 + y^2} + x + y = 4$

$$\frac{1}{2}\ln\left(x^2 + y^2\right) + x + y - 4 = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{\dfrac{x}{x^2 + y^2} + 1}{\dfrac{y}{x^2 + y^2} + 1} = -\frac{x + x^2 + y^2}{y + x^2 + y^2}$$

25. $F(x, y, z) = x^2 + y^2 + z^2 - 1$

$$F_x = 2x, F_y = 2y, F_z = 2z$$

$$\frac{\partial_z}{\partial_x} = -\frac{F_x}{F_z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$$

1. $f(x, y) = x^2 + y^2$, $P(1, -2)$, $\theta = \pi/4$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= 2x\cos\theta + 2y\sin\theta$$

At $\theta = \pi/4$, $x = 1$, and $y = -2$,

$$D_{\mathbf{u}} f(1, -2) = 2(1)\cos\pi/4 + 2(-2)\sin\pi/4$$
$$= \sqrt{2} - 2\sqrt{2} = -\sqrt{2}.$$

2. $f(x, y) = \dfrac{y}{x + y}$, $P(3, 0)$, $\theta = -\pi/6$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= \frac{-y}{(x + y)^2}\cos\theta + \frac{x}{(x + y)^2}\sin\theta$$

At $\theta = -\pi/6$, $x = 3$, and $y = 0$,

$$D_{\mathbf{u}} f(3, 0) = \frac{3}{3^2}\sin\left(\frac{-\pi}{6}\right) = -\frac{1}{6}.$$

3. $f(x, y) = \sin(2x + y)$, $P(0, 0)$, $\theta = \pi/3$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= 2\cos(2x + y)\cos\theta + \cos(2x + y)\sin\theta$$

At $\theta = \pi/3$ and $x = y = 0$,

$$D_{\mathbf{u}} f(0, 0) = 2\cos\pi/3 + \sin\pi/3 = 1 + \sqrt{3}/2.$$

6. $f(x, y) = x^3 - y^3$, $P(4, 3)$, $\mathbf{v} = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = (3x^2)\left(\frac{\sqrt{2}}{2}\right) + (-3y^2)\left(\frac{\sqrt{2}}{2}\right)$$

$$D_{\mathbf{u}} f(4, 3) = 3(16)\frac{\sqrt{2}}{2} - 3(9)\frac{\sqrt{2}}{2}$$
$$= \frac{21\sqrt{2}}{2}$$

16. $z = \cos(x^2 + y^2)$

$$\nabla z(x, y) = -2x\sin(x^2 + y^2)\mathbf{i} - 2y\sin(x^2 + y^2)\mathbf{j}$$

$$\nabla z(3, -4) = -6\sin 25\mathbf{i} + 8\sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$$

21. $f(x, y, z) = x^2 + y^2 + z^2$

$$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k}$$

$$D_{\mathbf{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2}{3}\sqrt{3}$$

26. $h(x, y, z) = \ln(x + y + z)$

$$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\nabla h = \frac{1}{x + y + z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$$D_{\mathbf{u}}h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$$

33. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$\|\nabla f(1, 4, 2)\| = 1$$

4. $F(x, y, z) = 16x^2 - 9y^2 + 36z = 0$

$16x^2 - 9y^2 + 36z = 0$ Hyperbolic paraboloid

13. $g(x, y) = x^2 + y^2, (1, -1, 2)$

$G(x, y, z) = x^2 + y^2 - z$

$G_x(x, y, z) = 2x \quad G_y(x, y, z) = 2y \quad G_z(x, y, z) = -1$

$G_x(1, -1, 2) = 2 \quad G_y(1, -1, 2) = -2 \quad G_z(1, -1, 2) = -1$

$2(x - 1) - 2(y + 1) - 1(z - 2) = 0$

$2x - 2y - z = 2$

22. $x^2 + y^2 + z^2 = 9, (1, 2, 2)$

$F(x, y, z) = x^2 + y^2 + z^2 - 9$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 2z$

$F_x(1, 2, 2) = 2 \quad F_y(1, 2, 2) = 4 \quad F_z(1, 2, 2) = 4$

Direction numbers: 1, 2, 2

Plane: $(x - 1) + 2(y - 2) + 2(z - 2) = 0, x + 2y + 2z = 9$

Line: $\dfrac{x - 1}{1} = \dfrac{y - 2}{2} = \dfrac{z - 2}{2}$

26. $xy - z = 0, (-2, -3, 6)$

$$F(x, y, z) = xy - z$$

$$F_x(x, y, z) = y \qquad F_y(x, y, z) = x \qquad F_z(x, y, z) = -1$$

$$F_x(-2, -3, 6) = -3 \qquad F_y(-2, -3, 6) = -2 \qquad F_z(-2, -3, 6) = -1$$

Direction numbers: 3, 2, 1

Plane: $3(x + 2) + 2(y + 3) + (z - 6) = 0, 3x + 2y + z = -6$

Line: $\dfrac{x + 2}{3} = \dfrac{y + 3}{2} = \dfrac{z - 6}{1}$

30. $y \ln(xz^2) = 2, (e, 2, 1)$

$$F(x, y, z) = y[\ln x + 2 \ln z] - 2$$

$$F_x(x, y, z) = \frac{y}{x} \qquad F_y(x, y, z) = \ln x + 2 \ln z \qquad F_z(x, y, z) = \frac{2y}{z}$$

$$F_x(e, 2, 1) = \frac{2}{e} \qquad F_y(e, 2, 1) = 1 \qquad F_z(e, 2, 1) = 4$$

$$\frac{2}{e}(x - e) + (y - 2) + 4(z - 1) = 0$$

$$\frac{2}{e}x + y + 4z = 8$$

Direction numbers: $\dfrac{2}{e}, 1, 4$

$$\frac{x - e}{(2/e)} = \frac{y - 2}{1} = \frac{z - 1}{4}$$

34. $F(x, y, z) = \sqrt{x^2 + y^2} - z$ $\qquad G(x, y, z) = 5x - 2y + 3z - 22$

$\nabla F(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(x, y, z) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\nabla F(3, 4, 5) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(3, 4, 5) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix} = \dfrac{2}{5}\mathbf{i} - \dfrac{34}{5}\mathbf{j} - \dfrac{26}{5}\mathbf{k}$

Direction numbers: 1, –17, –13

$\dfrac{x - 3}{1} = \dfrac{y - 4}{-17} = \dfrac{z - 5}{-13}$; tangent line

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{-(8/5)}{\sqrt{2}\sqrt{38}} = \dfrac{-8}{5\sqrt{76}}$; not orthogonal

40. $F(x, y, z) = x^2 + y^2 - 5, (2, 1, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$

$\nabla F(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j}$

$$\cos\theta = \frac{|\nabla F(2, 1, 3) \cdot \mathbf{k}|}{\|\nabla F(2, 1, 3)\|} = 0$$

$\theta = \arccos 0 = 90°$

44. $F(x, y, z) = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4 - z$

$\nabla F(x, y, z) = (8x + 4y + 8)\mathbf{i} + (4x - 4y - 5)\mathbf{j} - \mathbf{k}$

$8x + 4y + 8 = 0$

$4x - 4y - 5 = 0$

Adding, $12x + 3 = 0 \Rightarrow x = -\frac{1}{4} \Rightarrow y = -\frac{3}{2}$, and

$z = -\frac{5}{4}$

Point: $\left(-\frac{1}{4}, -\frac{3}{2}, -\frac{5}{4}\right)$

6. $f(x, y) = -x^2 - y^2 + 10x + 12y - 64$

$$= -(x^2 - 10x + 25) - (y^2 - 12y + 36) + 25 + 36 - 64 = -(x - 5)^2 - (y - 6)^2 - 3 \le -3$$

Relative maximum: $(5, 6, -3)$

Check: $f_x = -2x + 10 = 0 \Rightarrow x = 5$

$f_y = -2y + 12 = 0 \Rightarrow y = 6$

$f_{xx} = -2, f_{yy} = -2, f_{xy} = 0, d = (-2)(-2) - 0 = 4 > 0$

At critical point $(5, 6)$, $d > 0$ and $f_{xx} < 0 \Rightarrow$ relative maximum at $(5, 6, -3)$.

11. $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$

$f_x = -6x + 3 = 0$ when $x = \frac{1}{2}$.

$f_y = -4y - 4 = 0$ when $y = -1$.

$f_{xx} = -6, f_{yy} = -4, f_{xy} = 0$

At the critical point $\left(\frac{1}{2}, -1\right)$, $f_{xx} < 0$

and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $\left(\frac{1}{2}, -1, \frac{31}{4}\right)$ is a relative maximum.

14. $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$$\left.\begin{aligned} f_x &= -10x + 4y + 16 = 0 \\ f_y &= 4x - 2y = 0 \end{aligned}\right\}$$ Solving simultaneously yields $x = 8$ and $y = 16$.

$f_{xx} = -10, f_{yy} = -2, f_{xy} = 4$

At the critical point $(8, 16)$, $f_{xx} < 0$

and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(8, 16, 74)$ is a relative maximum.

34. $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

(a) $\left.\begin{aligned} f_x &= 3x^2 - 12x + 12 = 0 \\ f_y &= 3y^2 + 18y + 27 = 0 \end{aligned}\right\}$ Solving yields $x = 2$ and $y = -3$.

(b) $f_{xx} = 6x - 12, f_{yy} = 6y + 18, f_{xy} = 0$

At $(2, -3)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$.

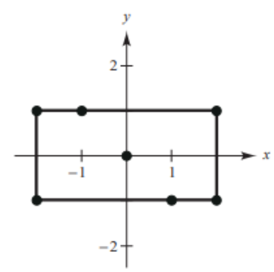
$(2, -3, 0)$ is a saddle point.

(c) Test fails at $(2, -3)$.

42. $f(x, y) = x^2 + xy, R = \{(x, y): |x| \leq 2, |y| \leq 1\}$

$\left.\begin{aligned} f_x &= 2x + y = 0 \\ f_y &= x = 0 \end{aligned}\right\} x = y = 0$

$f(0, 0) = 0$



Along $y = 1, -2 \leq x \leq 2, f = x^2 + x, f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2, f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f(2, 1) = 6$.

Along $y = -1, -2 \leq x \leq 2, f = x^2 - x, f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Thus, $f(-2, -1) = 6, f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}, f(2, -1) = 2$.

Along $x = 2, -1 \leq y \leq 1, f = 4 + 2y \Rightarrow f' = 2 \neq 0$.

Along $x = -2, -1 \leq y \leq 1, f = 4 - 2y \Rightarrow f' = -2 \neq 0$.

So, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$.

12. Maximum Volume The material for constructing the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. For a fixed amount of money C, find the dimensions of the box of largest volume that can be made.

Answer:

Let x, y, and z be the length, width, and height, respectively. Then $C_0 = 1.5xy + 2yz + 2xz$ and $z = \dfrac{C_0 - 1.5xy}{2(x + y)}$.

The volume is given by

$$V = xyz = \frac{C_0xy - 1.5x^2y^2}{2(x + y)}$$

$$V_x = \frac{y^2\left(2C_0 - 3x^2 - 6xy\right)}{4(x + y)^2}$$

$$V_y = \frac{x^2\left(2C_0 - 3y^2 - 6xy\right)}{4(x + y)^2}.$$

In solving the system $V_x = 0$ and $V_y = 0$, we note by the symmetry of the equations that $y = x$.

Substituting $y = x$ into $V_x = 0$ yields

$$\frac{x^2\left(2C_0 - 9x^2\right)}{16x^2} = 0,\ 2C_0 = 9x^2,\ x = \frac{1}{3}\sqrt{2C_0},\ y = \frac{1}{3}\sqrt{2C_0}, \text{ and } z = \frac{1}{4}\sqrt{2C_0}.$$

15. Maximum Revenue A company manufactures running shoes and basketball shoes. The total revenue (in thousands of dollars) from x_1 units of running shoes and x_2 units of basketball shoes is

$$R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$$

where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximize the revenue.

Answer:

$$R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$$

$$R_{x_1} = -10x_1 - 2x_2 + 42 = 0, 5x_1 + x_2 = 21$$

$$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, x_1 + 8x_2 = 51$$

Solving this system yields $x_1 = 3$ and $x_2 = 6$.

$$R_{x_1x_1} = -10$$

$$R_{x_1x_2} = -2$$

$$R_{x_2x_2} = -16$$

$$R_{x_1x_1} < 0 \text{ and } R_{x_1x_1}R_{x_2x_2} - \left(R_{x_1x_2}\right)^2 > 0$$

So, revenue is maximized when $x_1 = 3$ and $x_2 = 6$.

17. **Hardy-Weinberg Law** Common blood types are determined genetically by three alleles A, B, and O. (An allele is any of a group of possible mutational forms of a gene.) A person whose blood type is AA, BB, or OO is homozygous. A person whose blood type is AB, AO, or BO is heterozygous. The Hardy-Weinberg Law states that the proportion P of heterozygous individuals in any given population is

$$P(p, q, r) = 2pq + 2pr + 2qr$$

where p represents the percent of allele A in the population, q represents the percent of allele B in the population, and r represents the percent of allele O in the population. Use the fact that

$$p + q + r = 1$$

to show that the maximum proportion of heterozygous individuals in any population is $\frac{2}{3}$.

Answer:

$P(p, q, r) = 2pq + 2pr + 2qr.$

$p + q + r = 1$ implies that $r = 1 - p - q.$

$$\begin{aligned} P(p, q) &= 2pq + 2p(1 - p - q) + 2q(1 - p - q) \\ &= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2 = -2pq + 2p + 2q - 2p^2 - 2q^2 \end{aligned}$$

$$\frac{\partial P}{\partial p} = -2q + 2 - 4p; \frac{\partial P}{\partial q} = -2p + 2 - 4q$$

Solving $\frac{\partial P}{\partial p} = \frac{\partial P}{\partial q} = 0$ gives $\begin{aligned} q + 2p &= 1 \\ p + 2q &= 1 \end{aligned}$

and so $p = q = \frac{1}{3}$ and $P\left(\frac{1}{3}, \frac{1}{3}\right) = -2\left(\frac{1}{9}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{9}\right) - 2\left(\frac{1}{9}\right) = \frac{6}{9} = \frac{2}{3}.$

19. Minimum Cost A water line is to be built from point P to point S and must pass through regions where construction costs differ (see figure). The cost per kilometer (in dollars) is $3k$ from P to Q, $2k$ from Q to R, and k from R to S. Find x and y such that the total cost C will be minimized.

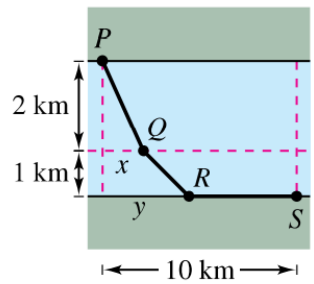


Figure for 19

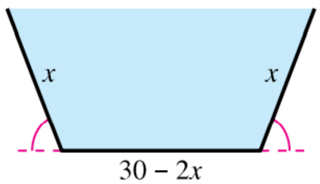


Figure for 20

Answer:

17. The distance from P to Q is $\sqrt{x^2+4}$. The distance from Q to R is $\sqrt{(y-x)^2+1}$. The distance from R to S is $10-y$.

$$C = 3k\sqrt{x^2+4} + 2k\sqrt{(y-x)^2+1} + k(10-y)$$

$$C_x = 3k\left(\frac{x}{\sqrt{x^2+4}}\right) + 2k\left(\frac{-(y-x)}{\sqrt{(y-x)^2+1}}\right) = 0$$

$$C_y = 2k\left(\frac{y-x}{\sqrt{(y-x)^2+1}}\right) - k = 0 \Rightarrow \frac{y-x}{\sqrt{(y-x)^2+1}} = \frac{1}{2}$$

$$3k\left(\frac{x}{\sqrt{x^2+4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$

$$\frac{x}{\sqrt{x^2+4}} = \frac{1}{3}$$

$$3x = \sqrt{x^2+4}$$

$$9x^2 = x^2+4$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

$$2(y-x) = \sqrt{(y-x)^2+1}$$

$$4(y-x)^2 = (y-x)^2+1$$

$$(y-x)^2 = \frac{1}{3}$$

$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3}+3\sqrt{2}}{6}$$

So, $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km and $y = \dfrac{2\sqrt{3}+3\sqrt{2}}{6} \approx 1.284$ km.

Finding the Least Squares Regression Line In Exercises 25–28, find the least squares regression line for the points. Use the regression capabilities of a graphing utility to verify your results. Use the graphing utility to plot the points and graph the regression line.

25. $(0, 0), \ (1, 1), \ (3, 6), \ (4, 8), \ (5, 9)$

26. $(0, 4), \ (4, 1), \ (7, -3)$

27. $(0, 6), \ (4, 3), \ (5, 0), \ (8, -4), \ (10, -5)$

28. $(6, 4), \ (1, 2), \ (3, 3), \ (8, 6), \ (11, 8), \ (13, 8)$

Answer:

28. $(6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8); n = 6$

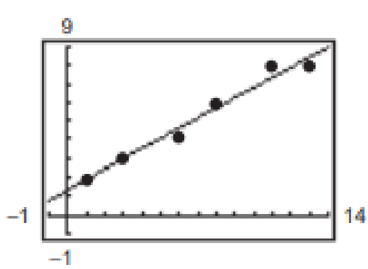
$\sum x_i = 42 \qquad \sum y_i = 31$

$\sum x_i y_i = 275 \qquad \sum x_i^2 = 400$

$$a = \frac{6(275) - (42)(31)}{6(400) - (42)^2} = \frac{29}{53} \approx 0.5472$$

$$b = \frac{1}{6}\left(31 - \frac{29}{53}42\right) = \frac{425}{318} \approx 1.3365$$

$$y = \frac{29}{53}x + \frac{425}{318}$$



29. **Modeling Data** The table shows the gross income tax collections (in billions of dollars) by the Internal Revenue Service for individuals x and businesses y for selected years. *(Source: U.S. Internal Revenue Service)*

Year	1980	1985	1990	1995
Individual, x	288	397	540	676
Business, y	72	77	110	174

Year	2000	2005	2010	2015
Individual, x	1137	1108	1164	1760
Business, y	236	307	278	390

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the model to estimate the business income taxes collected when the individual income taxes collected is \$1300 billion.

(c) In 1975, the individual income taxes collected was \$156 billion and the business income taxes collected was \$46 billion. Describe how including this information would affect the model.

Answer:

30. (a) Using a graphing utility, $y = 0.2x - 3$.

(b) When $x = 1300$, $y \approx \$257$ billion.

Answers will vary.

4. Maximize $f(x, y) = x^2 - y^2$.

Constraint: $2y - x^2 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = -2x\lambda\mathbf{i} + 2\lambda\mathbf{j}$

$2x = -2x\lambda \Rightarrow x = 0 \text{ or } \lambda = -1$

If $x = 0$, then $y = 0$ and $f(0, 0) = 0$.

If $\lambda = -1$,

$-2y = 2\lambda = -2 \Rightarrow y = 1 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}$.

$f(\sqrt{2}, 1) = 2 - 1 = 1$, Maximum

7. Note: $f(x, y) = \sqrt{6 - x^2 - y^2}$ is maximum when $g(x, y)$ is maximum.

Maximize $g(x, y) = 6 - x^2 - y^2$.

Constraint: $x + y - 2 = 0$

$$\left.\begin{aligned} -2x &= \lambda \\ -2y &= \lambda \end{aligned}\right\} x = y$$

$x + y = 2 \Rightarrow x = y = 1$

$f(1, 1) = \sqrt{g(1, 1)} = 2$

9. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z - 9 = 0$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 9 \Rightarrow x = y = z = 3$

$f(3, 3, 3) = 27$

13. Maximize or minimize $f(x, y) = x^2 + 3xy + y^2$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} 2x + 3y &= 2x\lambda \\ 3x + 2y &= 2y\lambda \end{aligned}\right\} x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}, y = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{5}{2}$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= 2x + 3y = 0 \\ f_y &= 3x + 2y = 0 \end{aligned}\right\} x = y = 0$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, f_{xx}f_{yy} - \left(f_{xy}\right)^2 \le 0$$

Saddle point: $f(0, 0) = 0$

By combining these two cases, we have a maximum of $\frac{5}{2}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ and a minimum of $-\frac{1}{2}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$.

14. Maximize or minimize $f(x, y) = e^{-xy/4}$.

Constraint: $x^2 + y^2 \leq 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} -(y/4)e^{-xy/4} &= 2x\lambda \\ -(x/4)e^{-xy/4} &= 2y\lambda \end{aligned}\right\} \Rightarrow x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = e^{1/8} \approx 1.1331$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = e^{-1/8} \approx 0.8825$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= -(y/4)e^{-xy/4} = 0 \\ f_y &= -(x/4)e^{-xy/4} = 0 \end{aligned}\right\} \Rightarrow x = y = 0$$

$$f_{xx} = \frac{y^2}{16}e^{-xy/4},\ f_{yy} = \frac{x^2}{16}e^{-xy/4},\ f_{xy} = e^{-xy}\left(\frac{1}{16}xy - \frac{1}{4}\right)$$

At $(0, 0)$, $f_{xx}f_{yy} - \left(f_{xy}\right)^2 < 0$.

Saddle point: $f(0, 0) = 1$

Combining the two cases, we have a maximum of $e^{1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$ and a minimum of $e^{-1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

16. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraints: $x + 2z = 6$

$x + y = 12$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j})$

$$\left.\begin{aligned} 2x &= \lambda + \mu \\ 2y &= \mu \\ 2z &= 2\lambda \end{aligned}\right\} 2x = 2y + z$$

$x + 2z = 6 \Rightarrow z = \dfrac{6 - x}{2} = 3 - \dfrac{x}{2}$

$x + y = 12 \Rightarrow y = 12 - x$

$2x = 2(12 - x) + \left(3 - \dfrac{x}{2}\right) \Rightarrow \dfrac{9}{2}x = 27 \Rightarrow x = 6$

$x = 6, z = 0$

$f(6, 6, 0) = 72$