Path and Cycles

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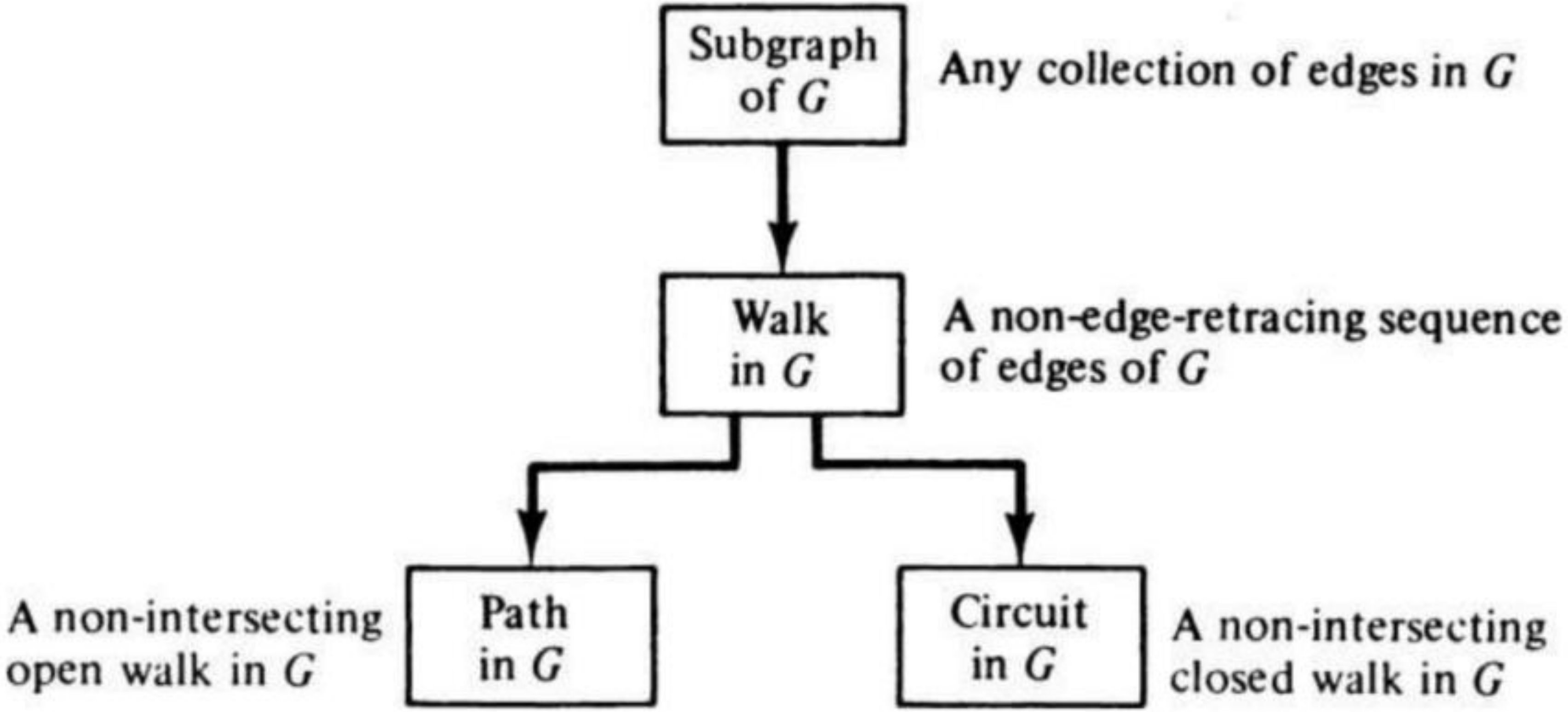
Walks, paths & circuits

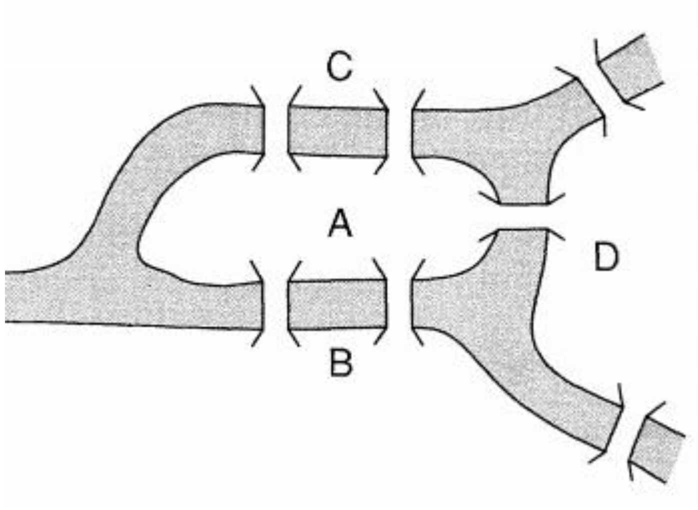
- *Walk/trail*: finite alternating sequence of vertices and edges, beginning and ending with vertices
- *Closed walk:* a walk to begin and end at the same vertex
- *Open walk*: the terminal vertices are distinct
- *Path*: open walk in which no vertex appears more than once
- *Circuit/Cycle*: closed walk in which no vertex (except the initial and the final vertex)appears more than once

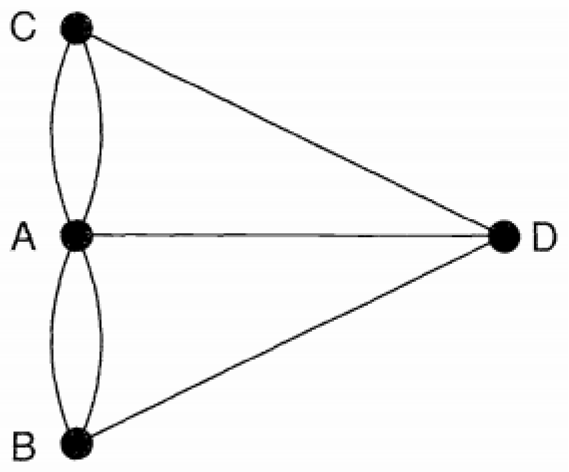
Walks, paths & circuits



Euler Graph

- Euler line: a closed walk running through every edge of *G* exactly once
- Looking for a Eulerian path in Konigsberg Bridge problem

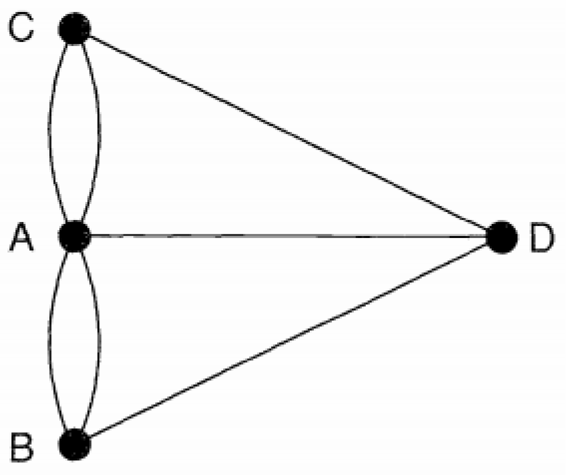




- Can one find necessary and sufficient conditions for a graph to be Eulerian?

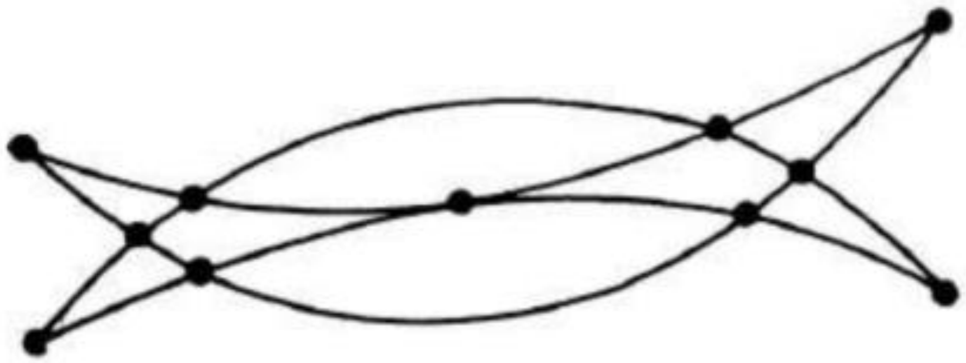
Euler Graph

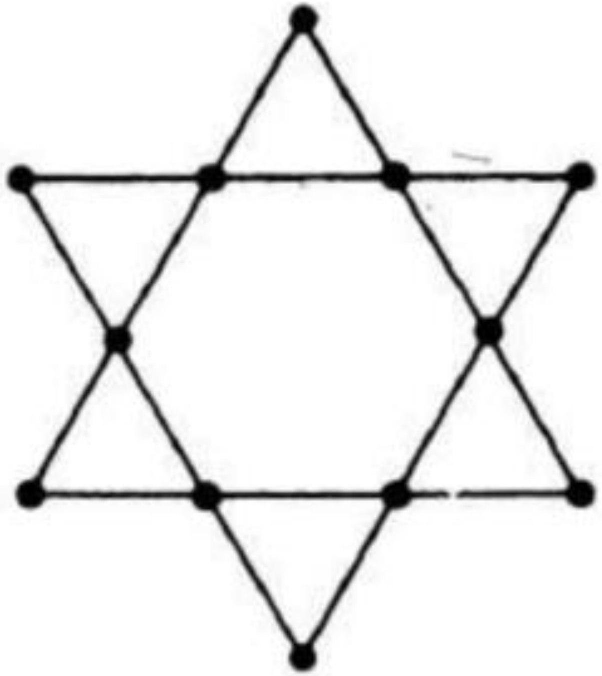
- THEOREM:
  A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.
- *Now check the* Konigsberg Bridge problem again.



Euler Graph

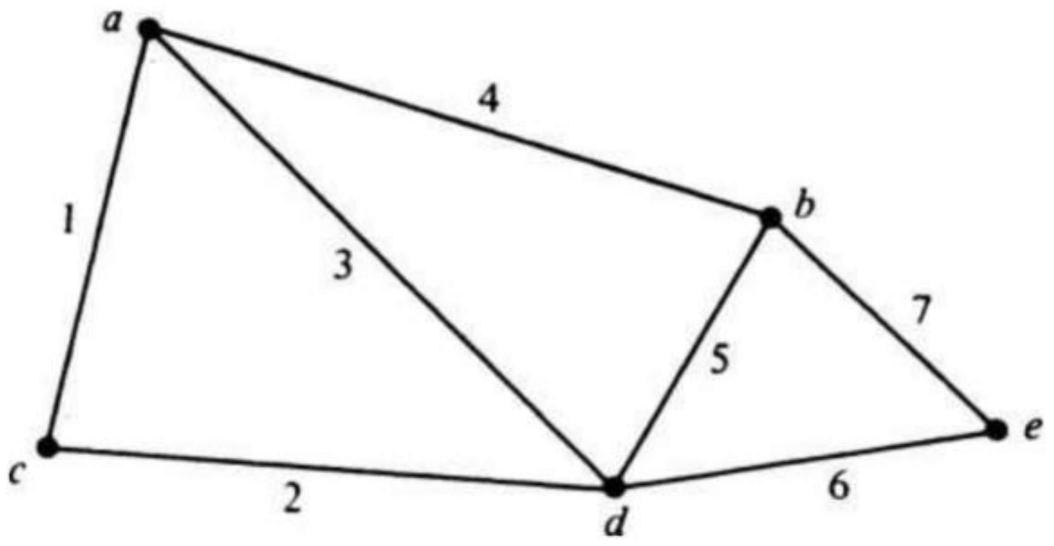
- Some more examples:





Euler graph

- Unicursal line: an open walk that includes (or traces or covers) all edges of a graph without retracing any edge a *unicursal line*
- Graph having unicursal line is unicursal graph (semi-Eulerian graph)



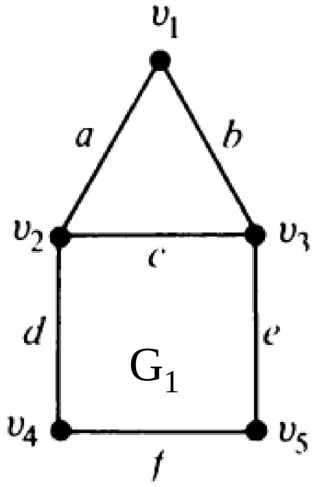
Operations on Graphs

Let’s assume: two graphs G1 = (V1, E1) and G2 = (V2, E2)

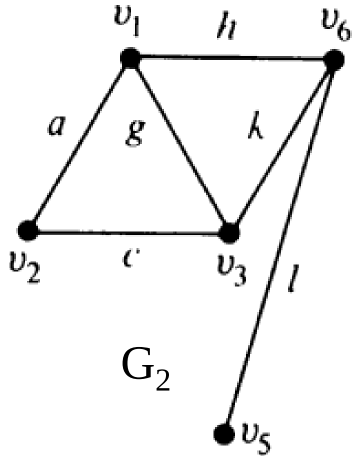
- Union - ∪ (G3 = G1 ⋃ G2)
  - vertex set V3 = V1 ⋃ V2 and the edge set E3 = E1 ⋃ E2
- Intersection - ⋂
  - Same as union
- Ring sum - ⨁ (G3 = G1 ⨁ G2)
  - graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but *not* in both.

Operations on Graphs

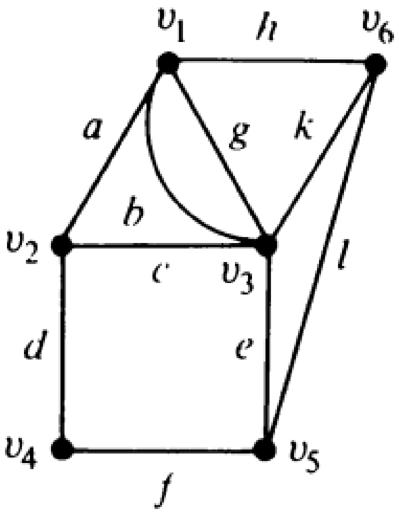
- Union - $\cup$
- Intersection - $\cap$
- Ring sum - $\oplus$



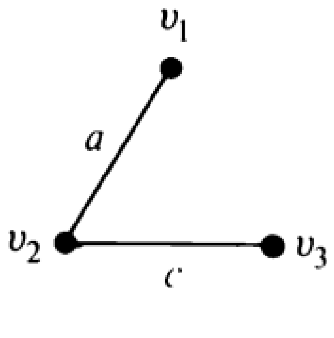
G_1



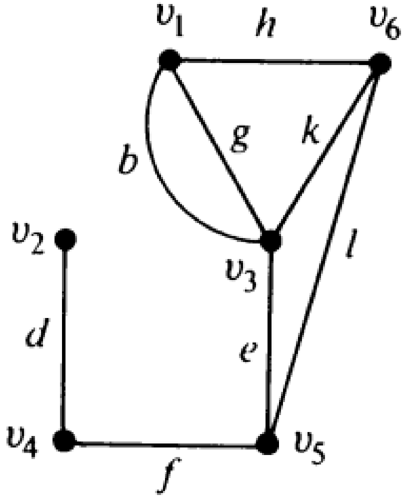
G_2



$G_1 \cup G_2$



$G_1 \cap G_2$



$G_1 \oplus G_2$

Operations on Graphs

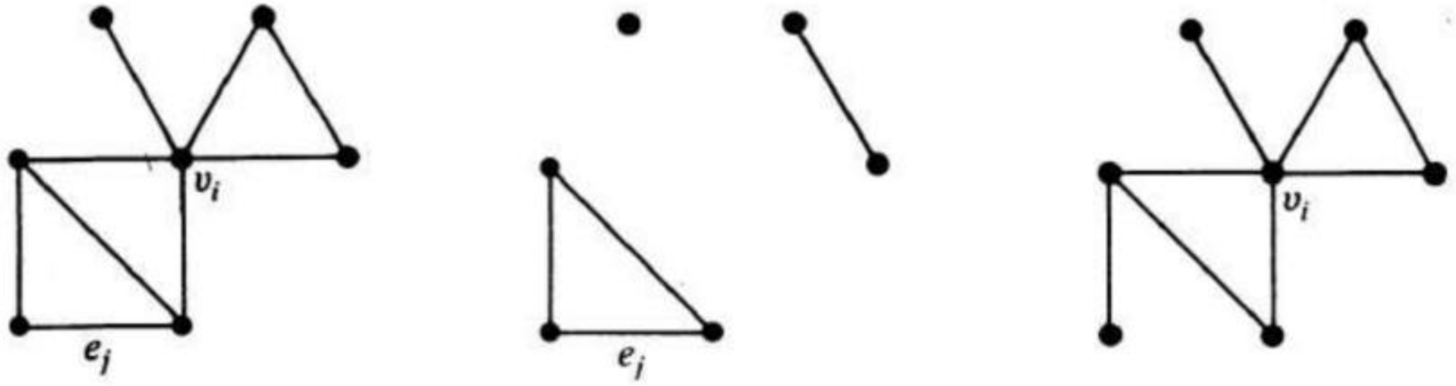
- If G_1 & G_2 are edge disjoint, then
  - $G_1 \cap G_2$ is a null graph
  - $G_1 \oplus G_2 = G_1 \cup G_2$
- If G_1 & G_2 are vertex disjoint, then
  - $G_1 \cap G_2$ is empty
- For any graph G
  - $G \cup G = G \cap G = G$
  - $G \oplus G$ is a null graph

Operations on Graphs

- If g is a subgraph of G, then
  - $G \oplus g = G - g$; is often called the complement of g in G
- Decomposition: a graph G is said to have decomposed into to subgraphs g_1 & g_2 if
  - $g_1 \cup g_2 = G$ and $g_1 \cap g_2$ is a null graph
- Deletion(of edge or vertex)
- Fusion (like contraction without edge deletion)

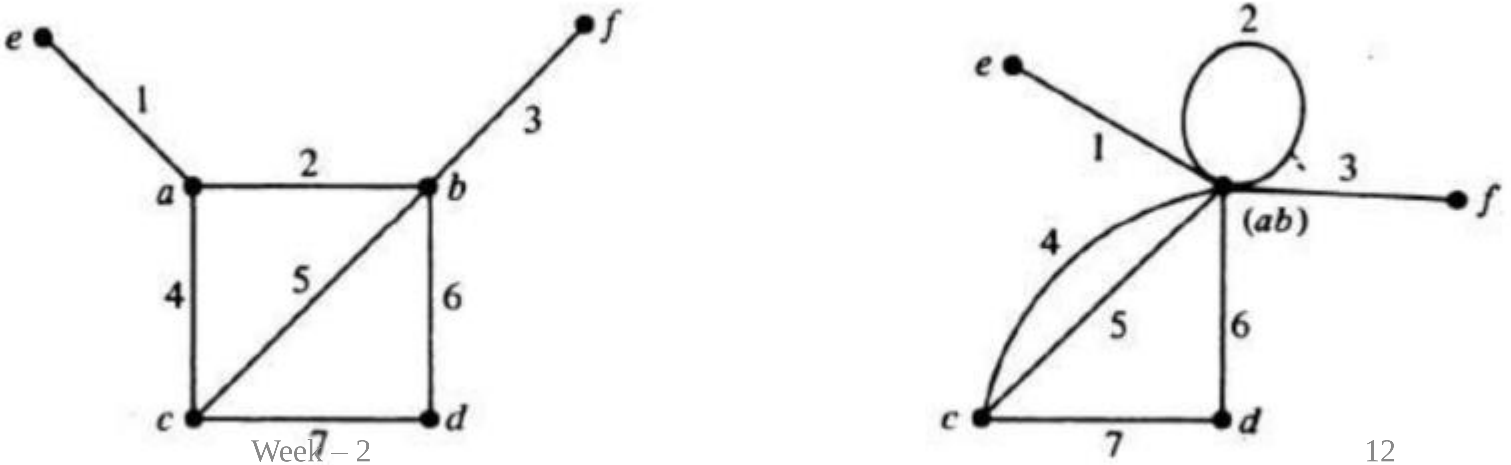
Operations on Graphs

Vertex deletion and edge deletion



G $(G - v_i)$ $(G - e_j)$

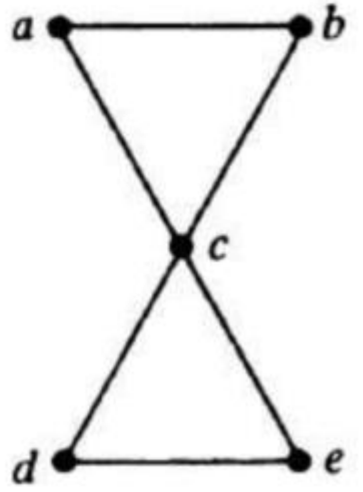
Fusion of vertices *a* and *b*



More on Euler graph

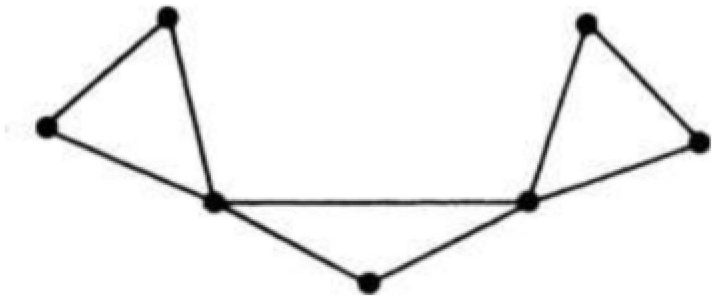
- Theorem:

A connected graph G is an Euler graph if and only if it can be decomposed into circuits.

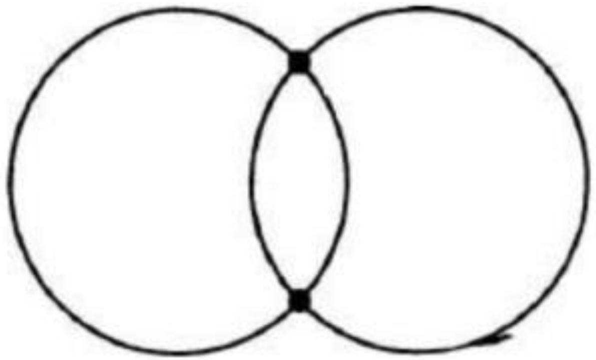


- *Arbitrarily Traceable Graphs from vertex v:*
  - *This graph is arbitrarily traceable from vertex c*

More on Euler graph



Not arbitrarily traceable from any vertex



arbitrarily traceable from all vertex

- Theorem:

An Euler graph G is arbitrarily traceable from vertex v in G if and only if every circuit in G contains v

Fleury’s Algorithms

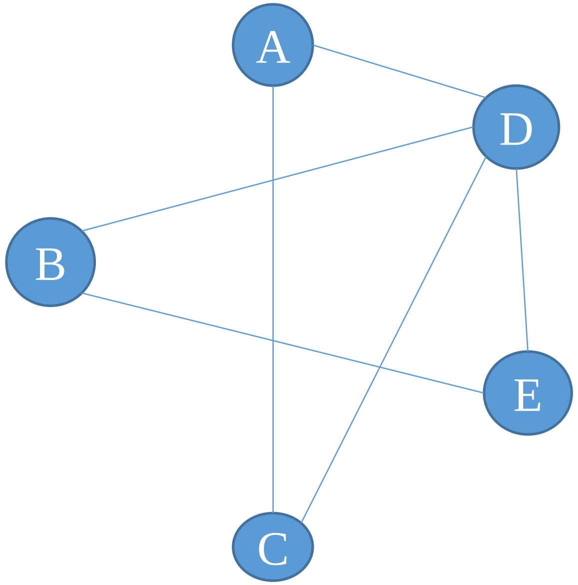
1. Make sure the graph has either 0 or 2 odd vertices

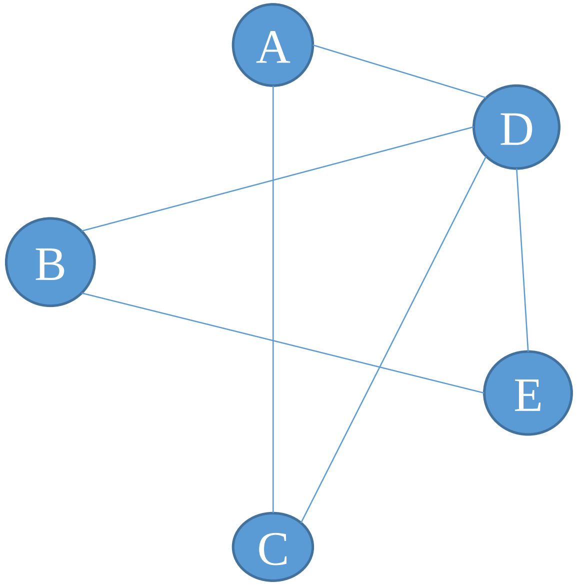
2. If there are 0 odd vertices, start anywhere. If there are 2 odd vertices, start at one of them.

3. Follow edges one at a time. If you have a choice between a bridge and a non-bridge, *always choose the non-bridge*.

4. Stop when you run out of edges.

Fleury's Algorithms

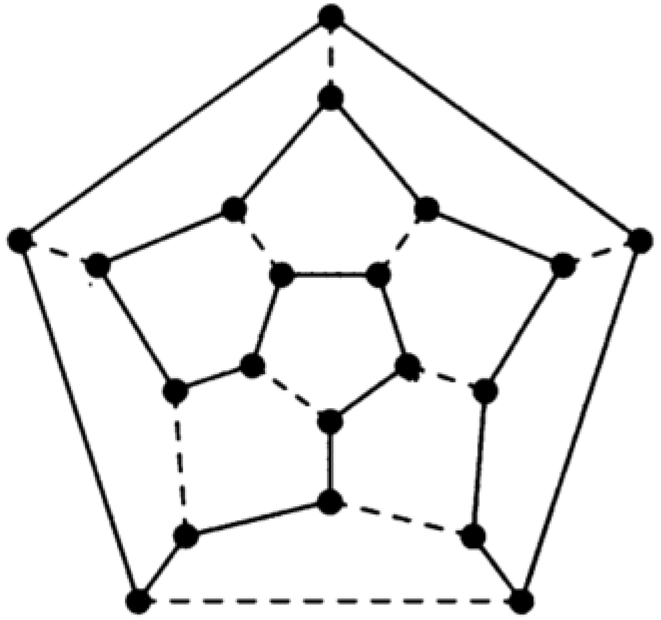




- Euler circuit is: A-D-E-B-D-C-A

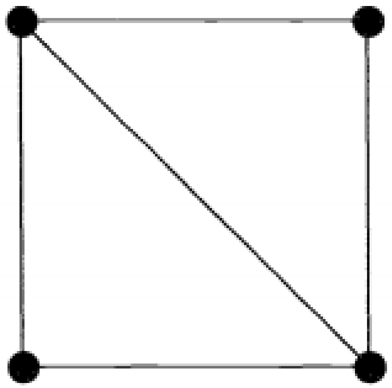
Hamiltonian graph

- Named after Sir William Hamilton
- Game called ‘Traveler's Dodecahedron’
- A player will choose 5-vertex path, the other player must extend it to a spanning cycle

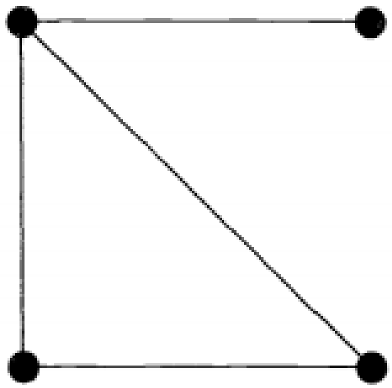


Hamiltonian Graphs

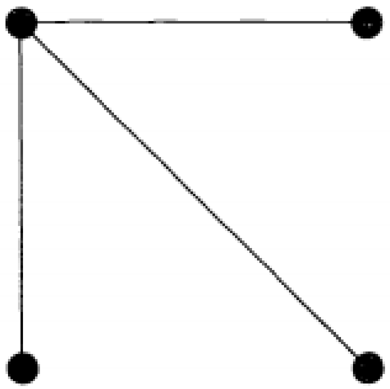
- Hamiltonian cycle – closed walk that visits every vertex exactly once
- Hamiltonian graph



Hamiltonian



Semi-Hamiltonian



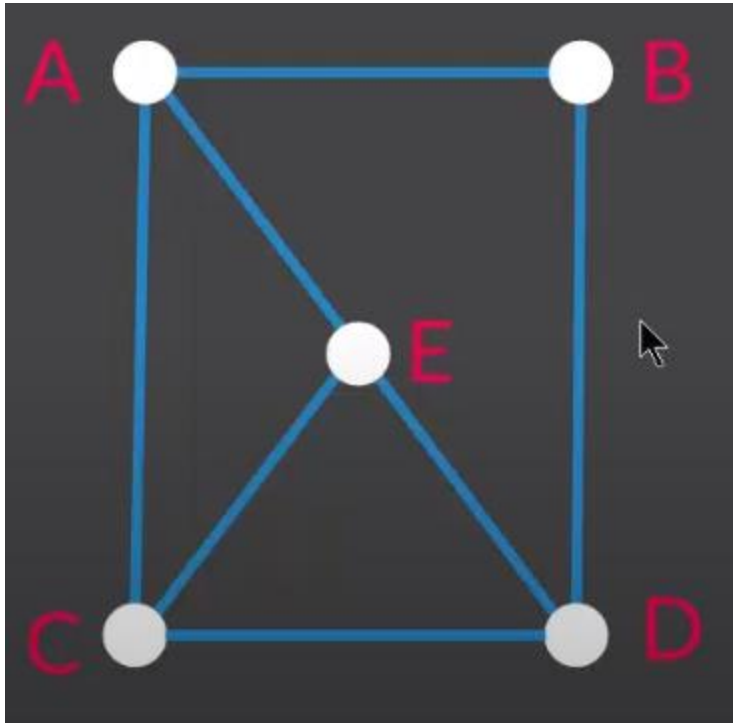
Non-Hamiltonian

Hamiltonian Graphs

- THEOREM 7.1 (Ore, 1960) – *If G is a simple graph with n* (> 3) *vertices, and if* **deg(v) + deg(w) ≥ n** *for each pair of non-adjacent vertices v and w, then G is Hamiltonian*.

AD: 6 BE: 5

BC: 5



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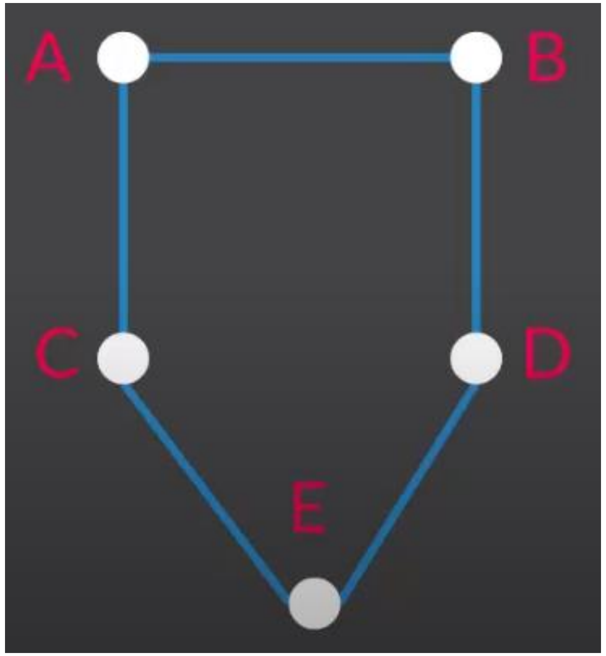


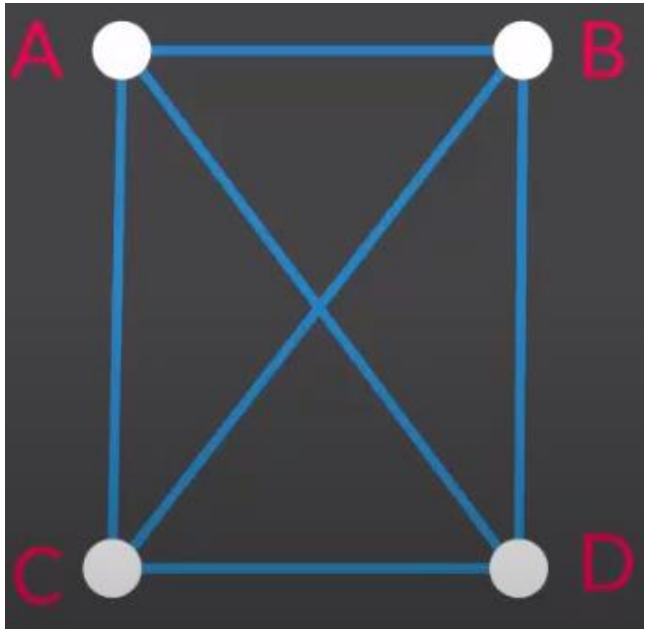
AD: 4 AE: 4

CD: 4 BE: 4

BC: 4

Hamiltonian Graphs

- COROLLARY 7.2 (Dirac, 1952) – *If G is a simple graph with n* (≥ 3) *vertices, and if* **deg(v) ≥ n/2** *for each vertex v, then G is Hamiltonian.*



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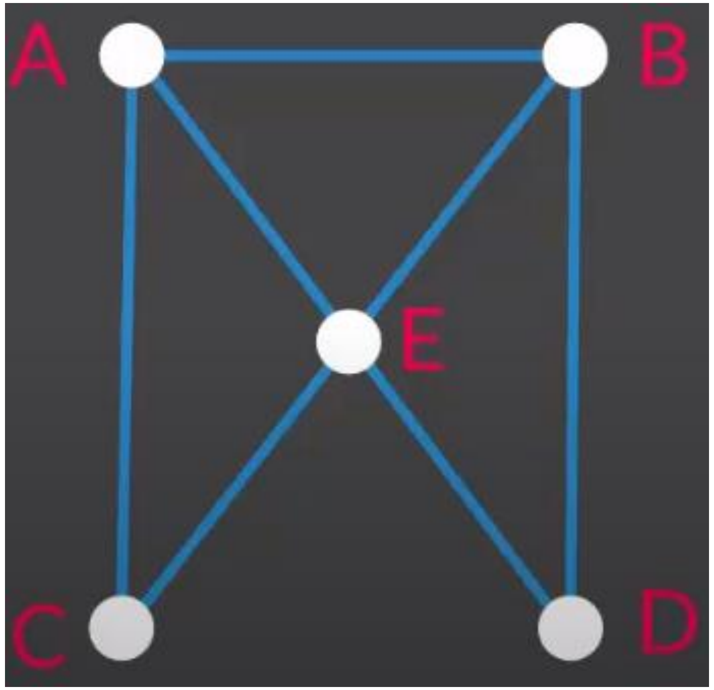

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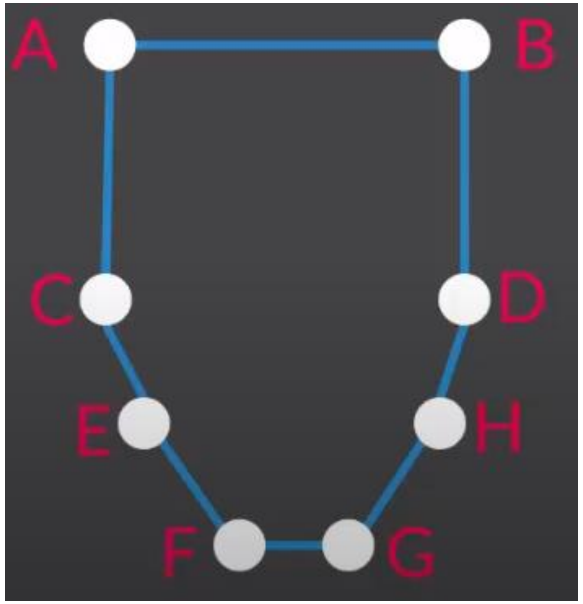


Hamiltonian Graphs

- Are complete graphs or cliques Hamiltonian?
- *Number of Hamiltonian Circuits in a Graph:*

Theorem:

In a complete graph with n vertices there are (n - l)/2 edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3.

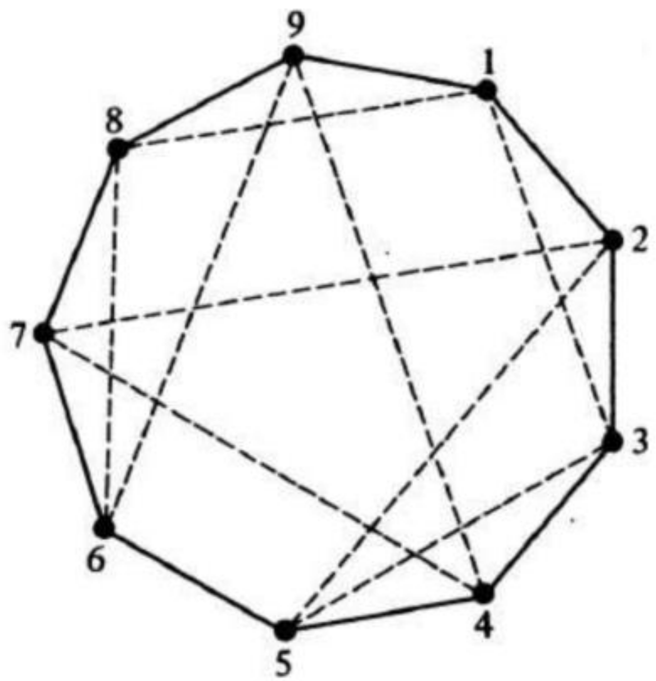
Seating arrangement

Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbors at each lunch. How many days can this arrangement last?

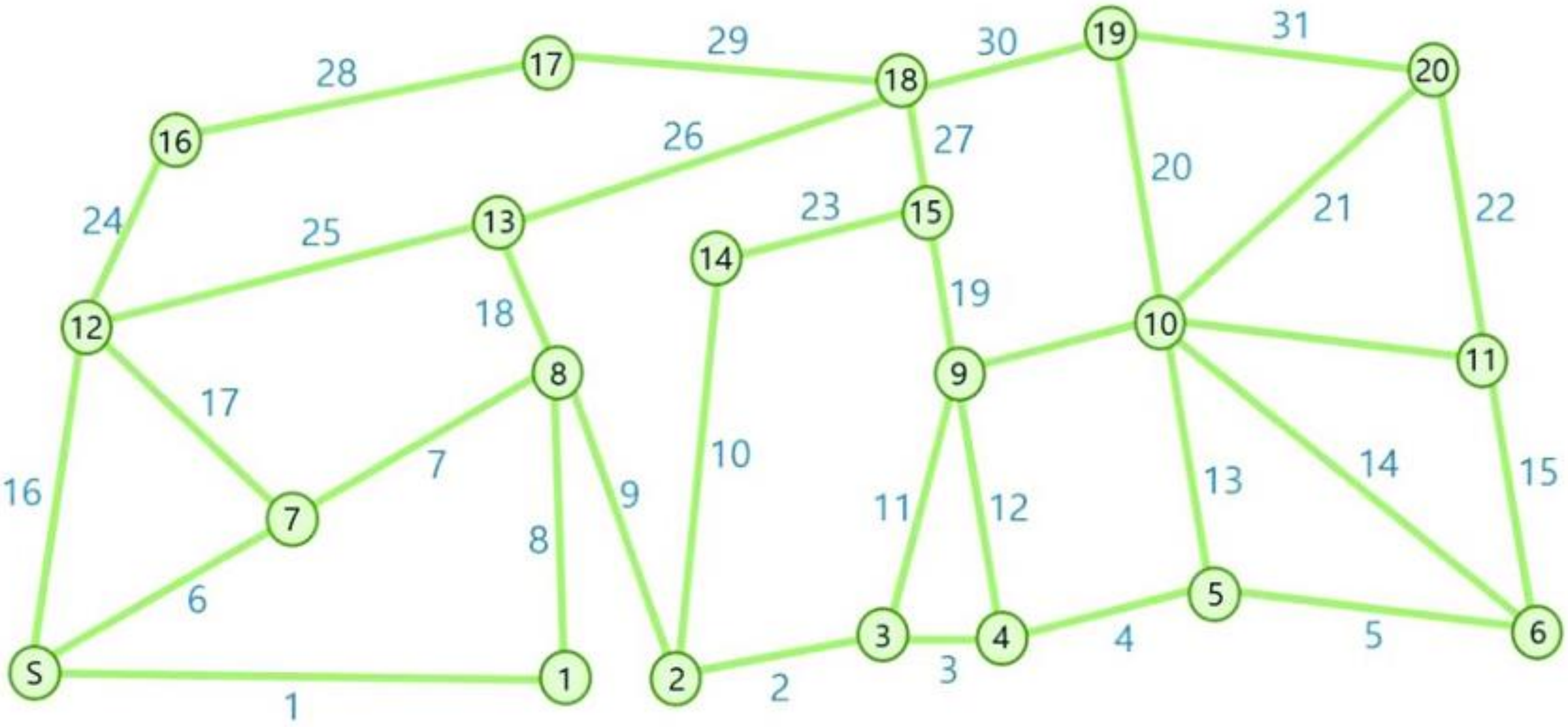
Solution:

Representing a member x by a vertex and the possibility of his sitting next to another member y by an edge between x and y, we construct a graph G

Let us find edge disjoint Hamiltonian cycles.

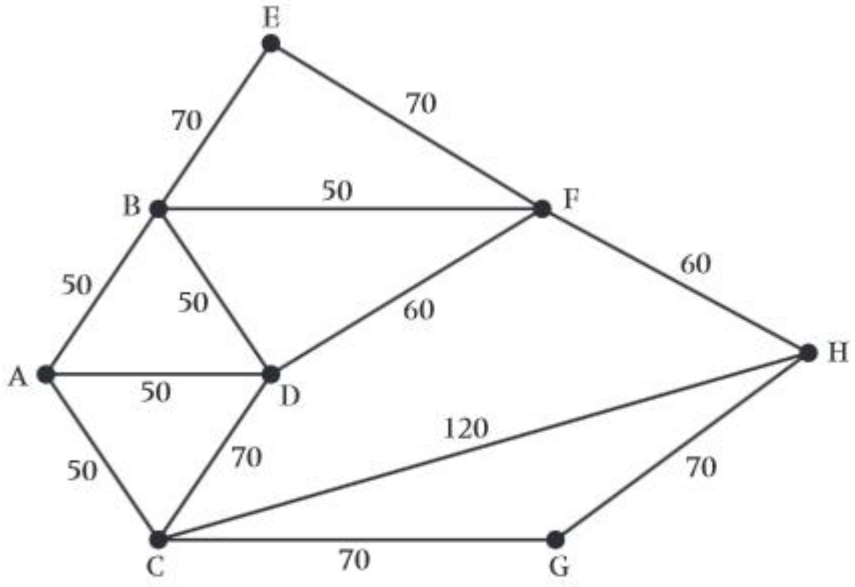


Travelling Salesman Problem



Chinese postman problem:

- It is the problem that the Chinese Postman faces: he wishes to travel along every road in a city in order to deliver letters, with the least possible distance. The problem is how to find a shortest closed walk of the graph in which each edge is traversed at least once, rather than exactly once.



Chinese postman problem: