Problem

Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), find a quadratic interpolant that passes through the data. Noting $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$, assume the quadratic interpolant $f_2(x)$ is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

While deriving the above Newton's divided difference second order polynomial, several students will claim that the expression for b_2 which is presented to them is *wrong* as it does not match theirs. They correctly do get

$$b_2 = \frac{\dfrac{f(x_2) - f(x_0)}{x_2 - x_0} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_1}$$

but the expression presented to them instead is

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

What gives and why do we write the latter expression?

Solution

Both expressions for b_2 are correct, but we choose the latter because it denotes a finite divided difference form. Writing it like this becomes the basis for a general form of Newton's divided difference polynomial for any order and becomes conducive to a computer algorithm. Many books, however, simply skip the steps to show how they got the second expression for b_2, while students think that there are typos in the text. So in this blog, we show these steps.

Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), pass a quadratic interpolant through the data. Noting $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$, assume the quadratic interpolant $f_2(x)$ is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \tag{1}$$

At $x = x_0$,

$$f_2(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1)$$

$$= b_0$$

$$\begin{aligned} b_0 &= f(x_0) \\ &= f[x_0] \end{aligned} \tag{2}$$

$f[x_0]$ is called a bracketed function for the zeroth divided difference and is given by $f(x_0)$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad (3)$$
$$= f[x_1, x_0]$$

$f[x_1, x_0]$ is a bracketed function for the first divided difference and is given by $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$b_2 = \frac{f(x_2) - f(x_0) - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \qquad (4)$$

$$= \frac{\dfrac{f(x_2) - f(x_0)}{x_2 - x_0} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_1}$$

But if we want to write this in the form where $(x_2 - x_0)$ is in the denominator so as to express it in the divided difference form of $f[x_2, x_1, x_0]$, we need to do the following manipulations.

Add 0 in the form of $\{-f(x_1) + f(x_1)\}$ to the numerator of equation (4)

$$b_2 = \frac{f(x_2) + \{-f(x_1) + f(x_1)\} - f(x_0) - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

Collecting $\{f(x_1) - f(x_0)\}$ terms together

$$b_2 = \frac{f(x_2) - f(x_1) + \{f(x_1) - f(x_0)\}(1 - \dfrac{x_2 - x_0}{x_1 - x_0})}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_1) + \{f(x_1) - f(x_0)\}(\frac{x_1 - x_0 - x_2 + x_0}{x_1 - x_0})}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_1) + \{f(x_1) - f(x_0)\}(\frac{x_1 - x_2}{x_1 - x_0})}{(x_2 - x_0)(x_2 - x_1)}$$

Dividing the numerator and denominator by $(x_2 - x_1)$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} + \frac{\{f(x_1) - f(x_0)\}(x_1 - x_2)}{(x_1 - x_0)(x_2 - x_1)}}{x_2 - x_0}$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \tag{4}$$

$$= f[x_2, x_1, x_0]$$

$f[x_2, x_1, x_0]$ is a bracketed function for the second divided difference and is given by

$$\frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

The Newton's divided difference second order polynomial is

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

and from equations (1)-(3),

$$f_2(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

$$= f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$