CSE 4631

Digital Signal Processing

Reference: Chap 5, Smith

Requirement for Linearity

- Homogeneity
- Additivity
- Shift Invariance (not a strict requirement)

Homogeneity

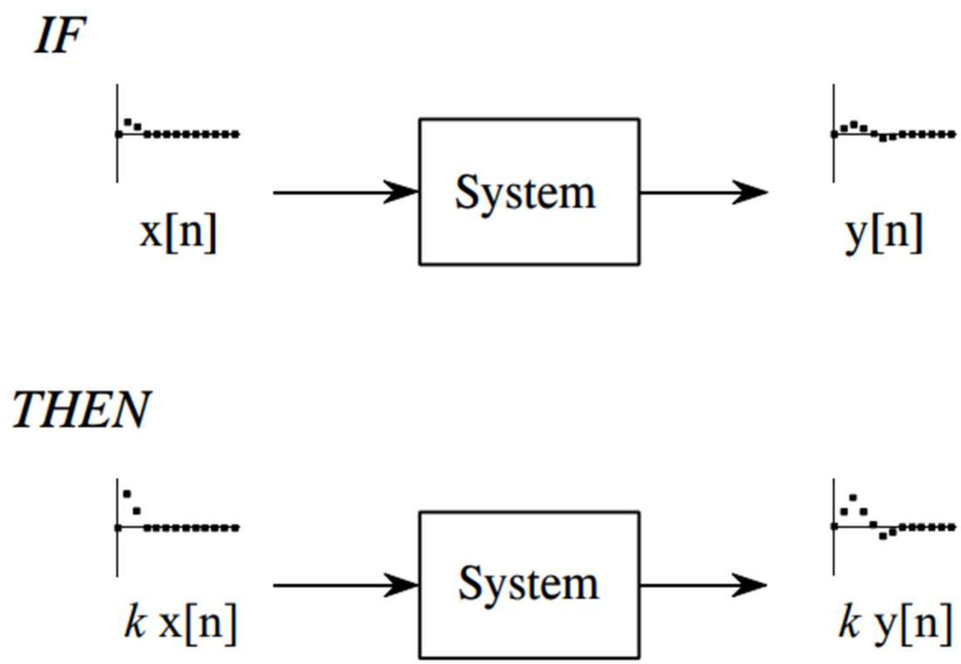


FIGURE 5-2
Definition of homogeneity. A system is said to be *homogeneous* if an amplitude change in the input results in an identical amplitude change in the output. That is, if $x[n]$ results in $y[n]$, then $kx[n]$ results in $ky[n]$, for any signal, $x[n]$, and any constant, k.

Additivity

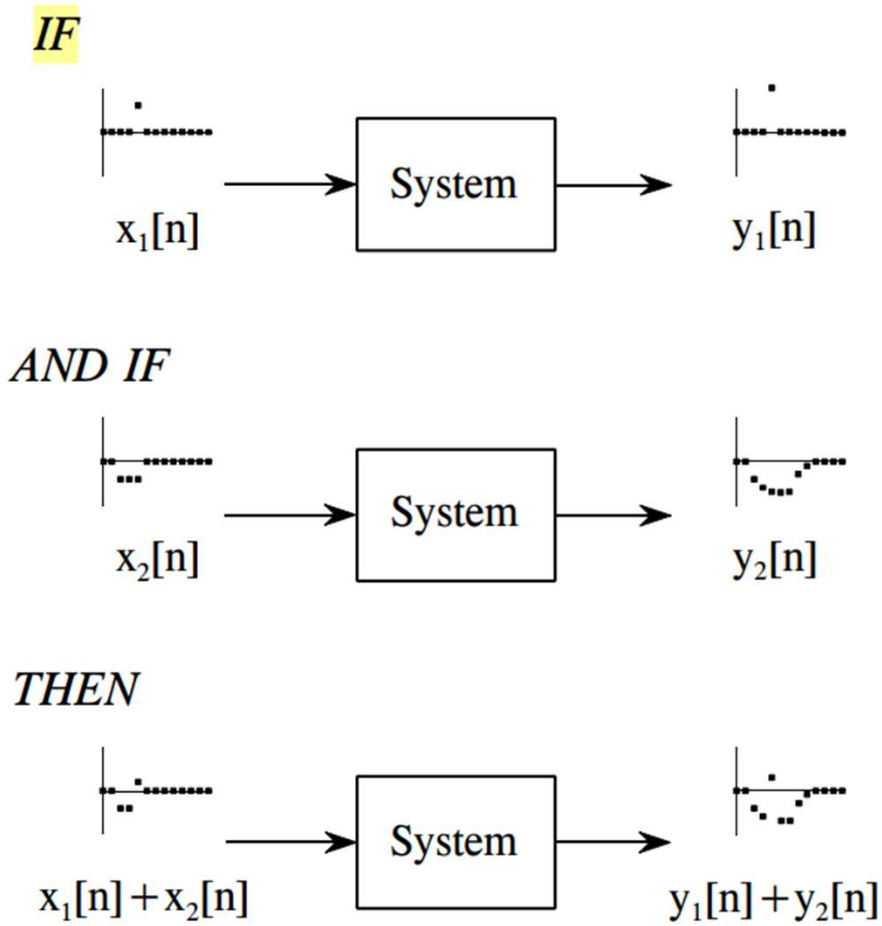


FIGURE 5-3
Definition of additivity. A system is said to be *additive* if added signals pass through it without interacting. Formally, if $x_1[n]$ results in $y_1[n]$, and if $x_2[n]$ results in $y_2[n]$, then $x_1[n]+x_2[n]$ results in $y_1[n]+y_2[n]$.

Shift Invariance

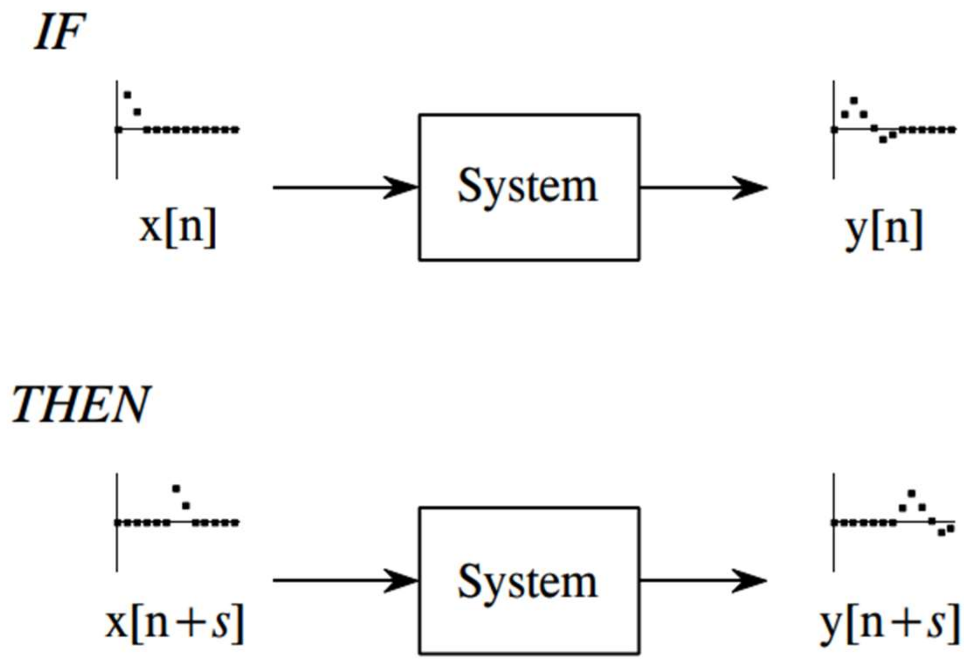


FIGURE 5-4
Definition of shift invariance. A system is said to be *shift invariant* if a shift in the input signal causes an identical shift in the output signal. In mathematical terms, if $x[n]$ produces $y[n]$, then $x[n+s]$ produces $y[n+s]$, for any signal, $x[n]$, and any constant, s.

Why shift invariance not strict requirement?

- Shift invariance does not have meaning outside the realm of signals and systems.

Why do homogeneity and additivity play a critical role in linearity, while shift invariance is something on the side? This is because linearity is a very broad concept, encompassing much more than just signals and systems. For example, consider a farmer selling oranges for \$2 per crate and apples for \$5 per crate. If the farmer sells only oranges, he will receive \$20 for 10 crates, and \$40 for 20 crates, making the exchange *homogenous*. If he sells 20 crates of oranges and 10 crates of apples, the farmer will receive: $20 \times \$2 + 10 \times \$5 = \$90$. This is the same amount as if the two had been sold individually, making the transaction *additive*. Being both homogenous and additive, this sale of goods is a linear process. However, since there are no signals involved, this is not a *system*, and *shift invariance* has no meaning. Shift invariance can be thought of as an additional aspect of linearity needed when signals and systems are involved.

Static Linearity

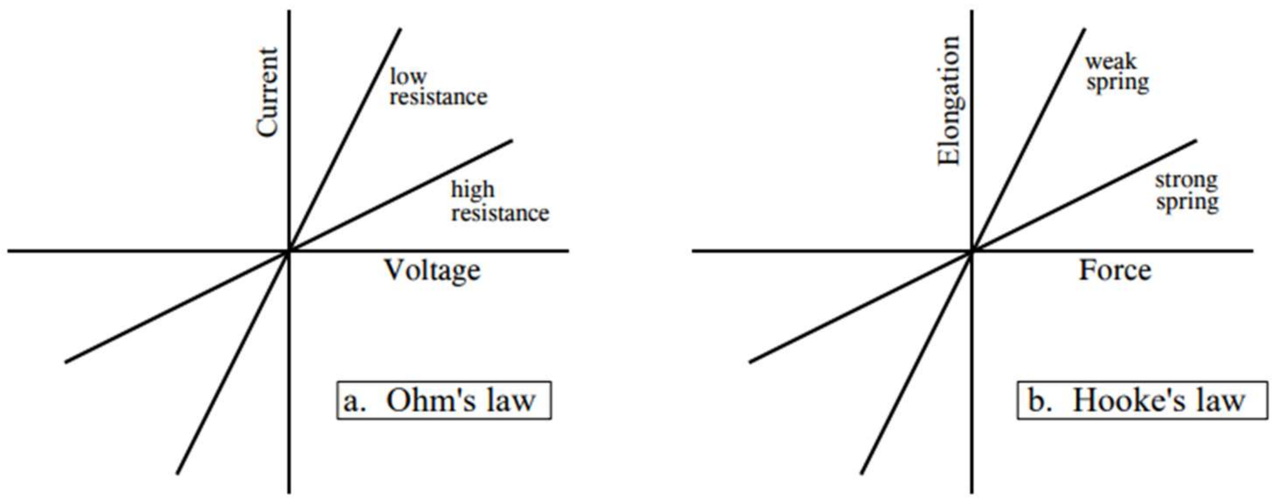


FIGURE 5-5
Two examples of static linearity. In (a), Ohm's law: the current through a resistor is equal to the voltage across the resistor divided by the resistance. In (b), Hooke's law: The elongation of a spring is equal to the applied force multiplied by the spring stiffness coefficient.

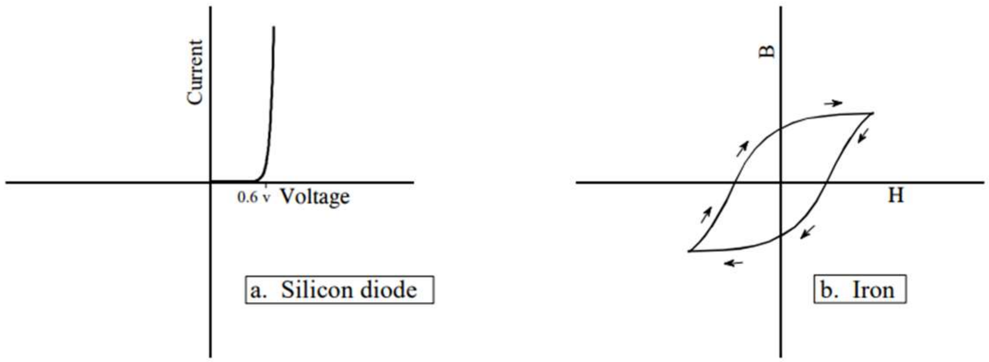


FIGURE 5-6
Two examples of DC nonlinearity. In (a), a silicon diode has an exponential relationship between voltage and current. In (b), the relationship between magnetic intensity, H, and flux density, B, in iron depends on the history of the sample, a behavior called *hysteresis*.

All linear systems have the property of *static linearity*. The opposite is usually true, but not always. There are systems that show static linearity, but are not linear with respect to changing signals. However, a very common class of systems can be completely understood with static linearity alone. In these systems it doesn't matter if the input signal is static or changing. These are called **memoryless** systems, because the output depends only on the present state of the input, and not on its history. For example, the instantaneous current in a resistor depends only on the instantaneous voltage across it, and not on how the signals came to be the value they are. If a system has static linearity, and is memoryless, then the system must be linear. This provides an important way to understand (and prove) the linearity of these simple systems.

Sinusoidal Fidelity

- Most linear systems have sinusoidal fidelity.

- But it does not guaranty that the system is linear. For example, there could be systems which have sinusoidal fidelity but don't have additivity.

An important characteristic of linear systems is how they behave with sinusoids, a property we will call **sinusoidal fidelity**: *If the input to a linear system is a sinusoidal wave, the output will also be a sinusoidal wave, and at exactly the same frequency as the input.* Sinusoids are the only waveform that have this property. For instance, there is no reason to expect that a square wave entering a linear system will produce a square wave on the output. Although a sinusoid on the input guarantees a sinusoid on the output, the two may be different in *amplitude* and *phase*. This should be familiar from your knowledge of electronics: a circuit can be described by its *frequency response*, graphs of how the circuit's gain and phase vary with frequency.

Examples of *Linear* Systems

Wave propagation such as sound and electromagnetic waves

Electrical circuits composed of resistors, capacitors, and inductors

Electronic circuits, such as amplifiers and filters

Mechanical motion from the interaction of masses, springs, and dashpots (dampeners)

Systems described by differential equations such as resistor-capacitor-inductor networks

Multiplication by a constant, that is, amplification or attenuation of the signal

Signal changes, such as echoes, resonances, and image blurring

The unity system where the output is always equal to the input

The null system where the output is always equal to the zero, regardless of the input

Differentiation and integration, and the analogous operations of *first difference* and *running sum* for discrete signals

Small perturbations in an otherwise nonlinear system, for instance, a small signal being amplified by a properly biased transistor

Convolution, a mathematical operation where each value in the output is expressed as the sum of values in the input multiplied by a set of weighing coefficients.

Recursion, a technique similar to convolution, except previously calculated values in the output are used in addition to values from the input

Examples of *Nonlinear* Systems

Systems that do not have static linearity, for instance, the voltage and power in a resistor: $P = V^2R$, the radiant energy emission of a hot object depending on its temperature: $R = kT^4$, the intensity of light transmitted through a thickness of translucent material: $I = e^{-\alpha T}$, etc.

Systems that do not have sinusoidal fidelity, such as electronics circuits for: peak detection, squaring, sine wave to square wave conversion, frequency doubling, etc.

Common electronic distortion, such as clipping, crossover distortion and slewing

Multiplication of one signal by another signal, such as in amplitude modulation and automatic gain controls

Hysteresis phenomena, such as magnetic flux density versus magnetic intensity in iron, or mechanical stress versus strain in vulcanized rubber

Saturation, such as electronic amplifiers and transformers driven too hard

Systems with a threshold, for example, digital logic gates, or seismic vibrations that are strong enough to pulverize the intervening rock

Commutative property of linear systems

FIGURE 5-7
The commutative property for linear systems. When two or more linear systems are arranged in a cascade, the order of the systems does not affect the characteristics of the overall combination.

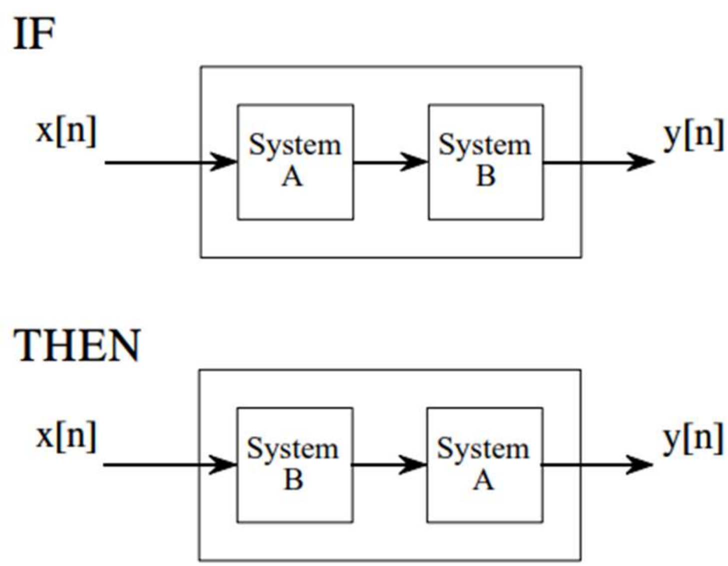
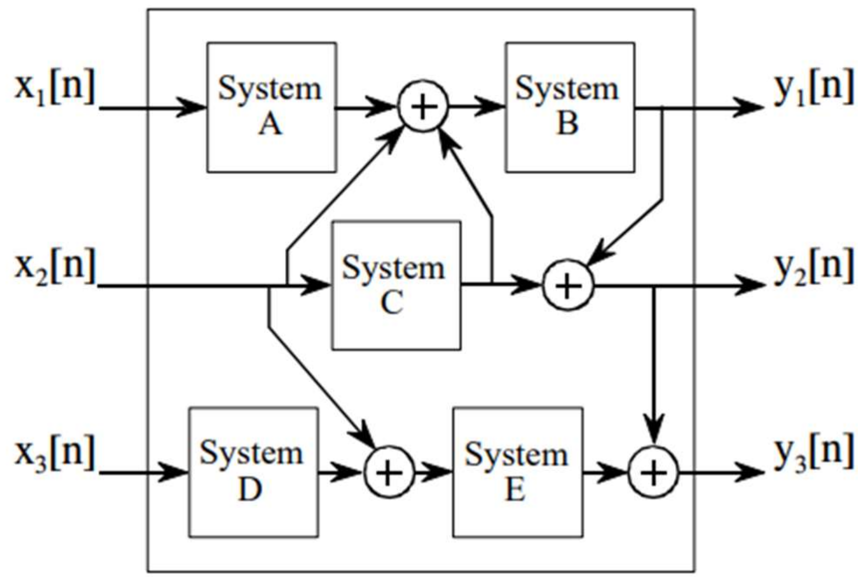


FIGURE 5-8
Any system with multiple inputs and/or outputs will be linear if it is composed of linear systems and signal additions.



Linearity of Multiplication

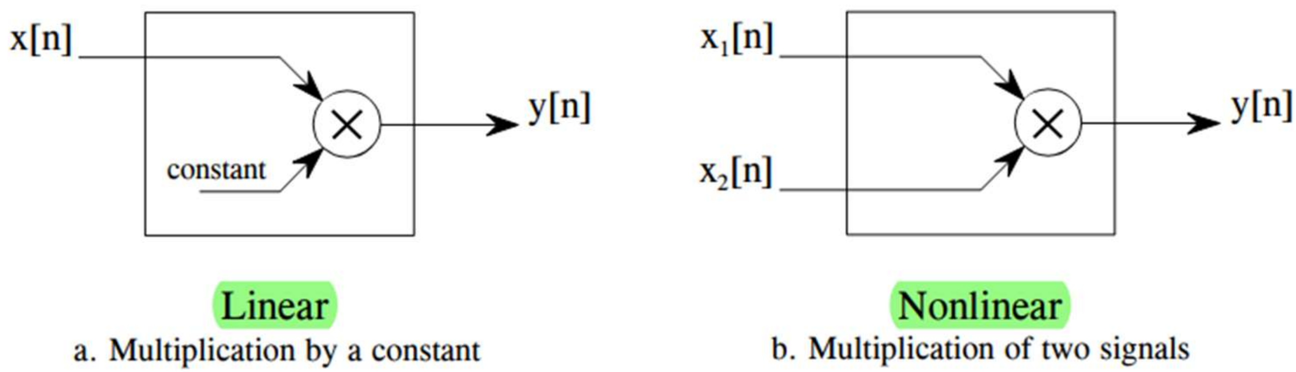


FIGURE 5-9
Linearity of multiplication. Multiplying a signal by a constant is a linear operation. In contrast, the multiplication of two signals is nonlinear.

Synthesis & Decomposition

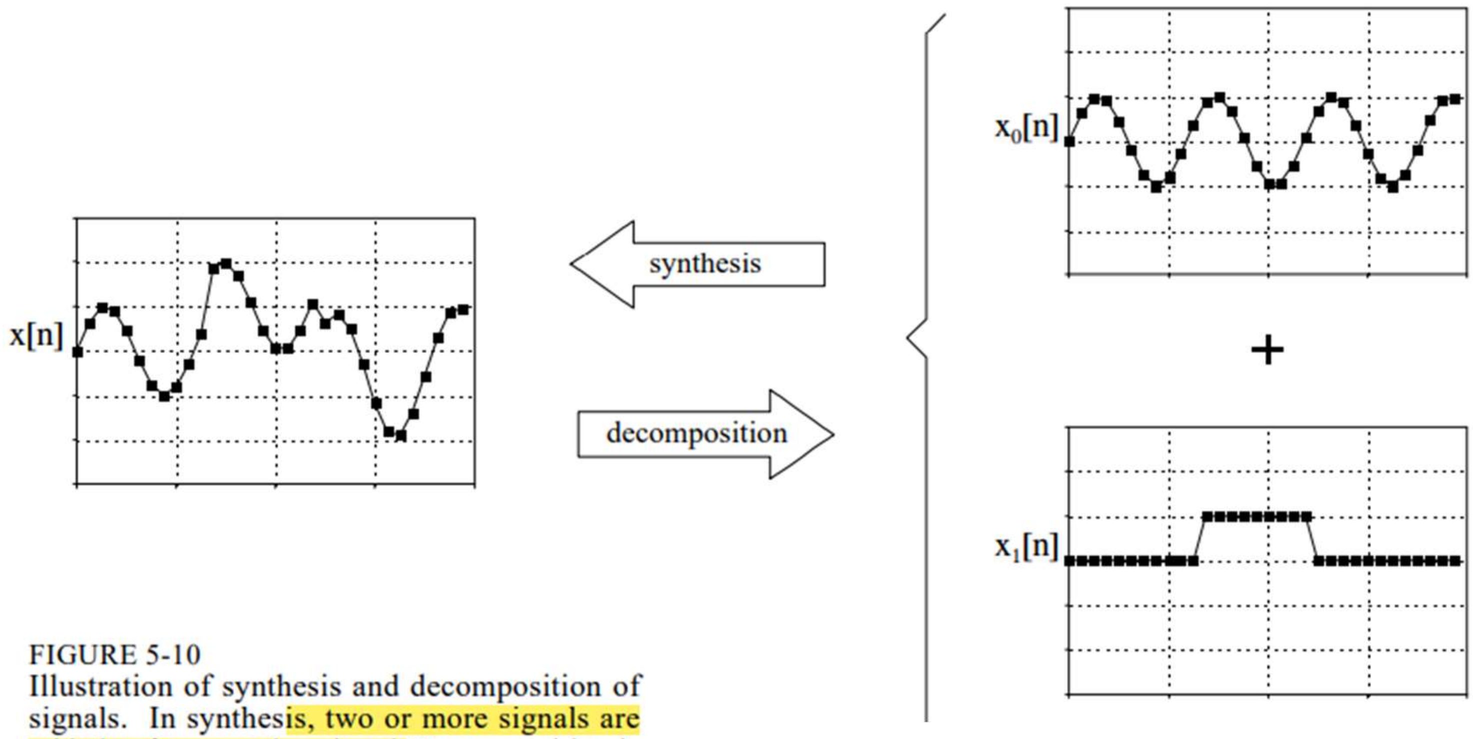


FIGURE 5-10
Illustration of synthesis and decomposition of signals. In synthesis, two or more signals are added to form another signal. Decomposition is the opposite process, breaking one signal into two or more additive component signals.

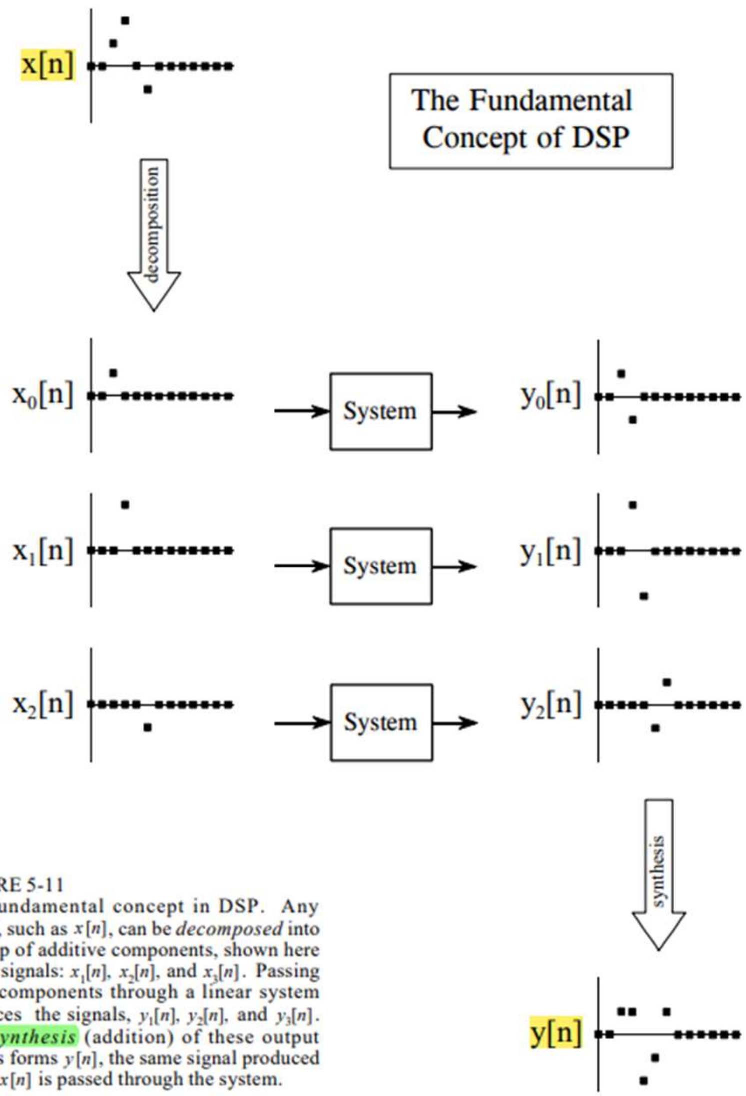
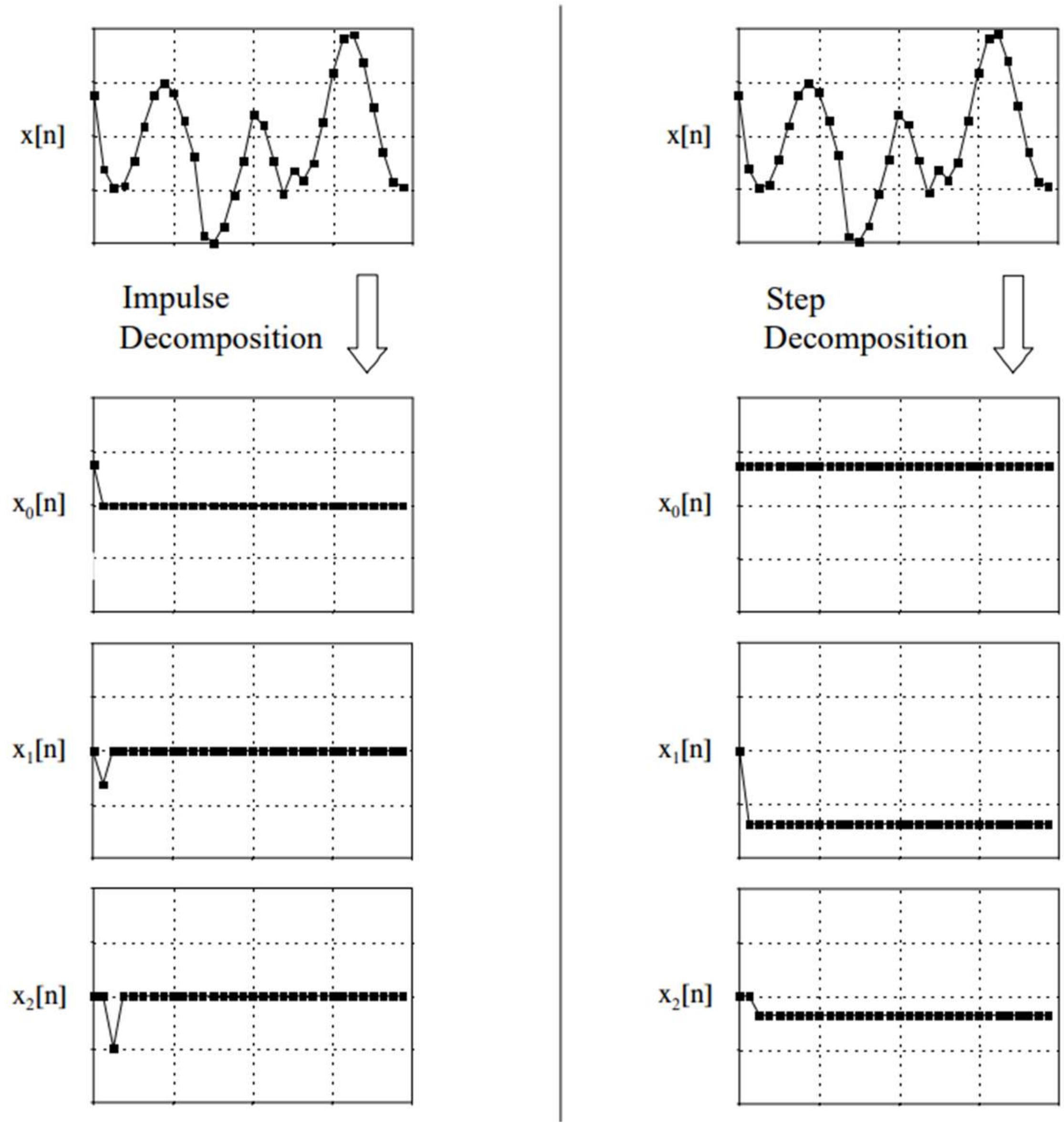


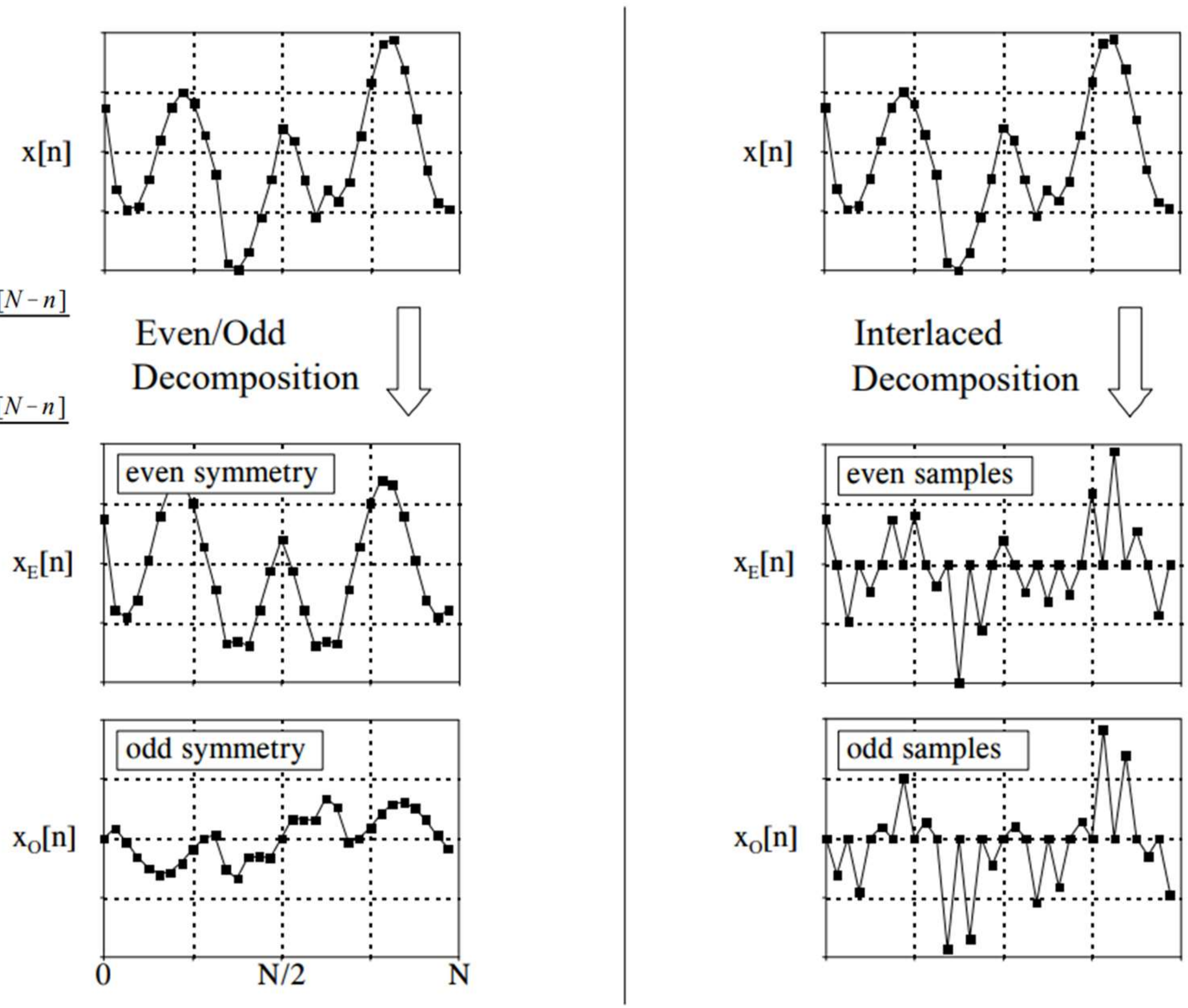
FIGURE 5-11
The fundamental concept in DSP. Any signal, such as $x[n]$, can be *decomposed* into a group of additive components, shown here by the signals: $x_1[n]$, $x_2[n]$, and $x_3[n]$. Passing these components through a linear system produces the signals, $y_1[n]$, $y_2[n]$, and $y_3[n]$. The *synthesis* (addition) of these output signals forms $y[n]$, the same signal produced when $x[n]$ is passed through the system.



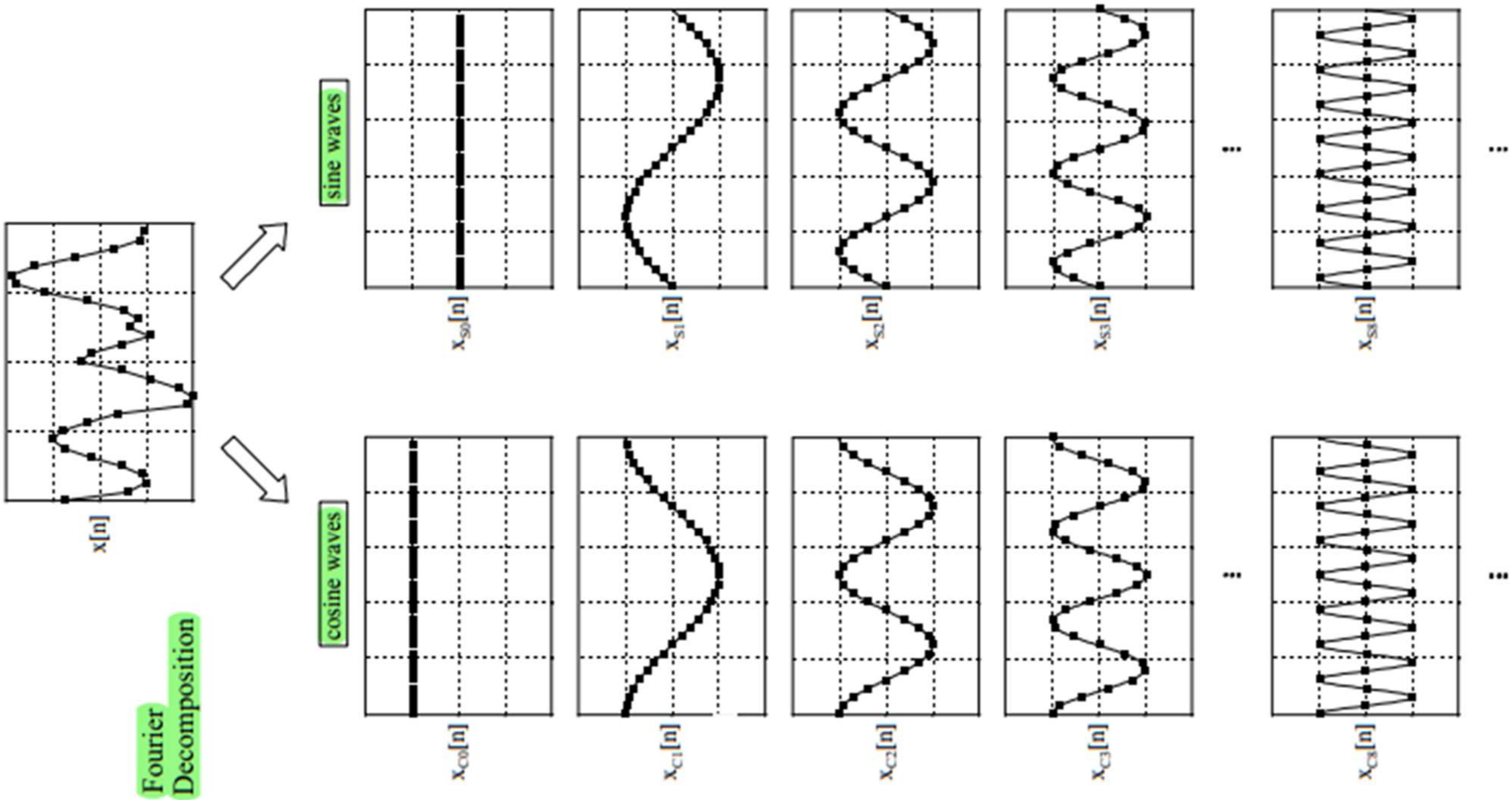
x[n]
Impulse
Decomposition
$x_0[n]$
$x_1[n]$
$x_2[n]$
x[n]
Step
Decomposition
$x_0[n]$
$x_1[n]$
$x_2[n]$

$$x_E[n] = \frac{x[n] + x[N-n]}{2}$$

$$x_O[n] = \frac{x[n] - x[N-n]}{2}$$



Fourier Decomposition



How to deal with nonlinearity

- Linear systems are so handy that we try to make nonlinear systems resemble linear systems –
  - If nonlinearity is small, ignore it
  - Keep the signal small; nonlinear system appear linear if signal amplitude is small
  - Apply a linear transformation such as logarithm
    - A = X*Y
    - Log(A) = Log(X) + Log(Y)