11-08-20

Lecture - 18(a)

Diagonalization and Powers of A

Previously, we use $Ax = \lambda x$ to find the eigenvalues and eigenvectors.

Let, A be a $(n \times n)$ matrix. If it's a normal matrix (not a triangular matrix or rotational matrix), then it will produce n eigenvectors.

Let, the eigenvectors be $\longrightarrow x_1, x_2 \cdots x_n$

$$S = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1, & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

Here, each eigenvector is encompassing a single column.

Multiplying by A, we get,

$$AS = A \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

What if we want to extract, the $\begin{bmatrix} \uparrow & & \uparrow \\ x_1 & \cdots & x_n \\ \downarrow & & \downarrow \end{bmatrix}$ from this matrix. We will then think of a matrix which when multiplied to S will produce the above product. If we have a matrix with only $\lambda_1 \dots \lambda_n$ as the diagonal; it will, ~~pr~~ after multiplying with, produce $\lambda_1 x_1 \dots \lambda_n x_n$ terms in all the columns and the rest of the ~~ext~~ extra values in the columns will be neutralized by the zeroes.

$$\text{So, } AS = \begin{bmatrix} \uparrow & & \uparrow \\ x_1 & \cdots & x_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Diagonal Eigenvalue Matrix (denoted by Λ)

$$\therefore AS = S\Lambda$$

$$\Rightarrow \boxed{S^{-1}AS = \Lambda}$$ (Multiplying S^{-1} to both sides and $S \cdot S^{-1} = I$)

Again, $AS \times S^{-1} = S\Lambda S^{-1}$

$$\Rightarrow \boxed{A = S\Lambda S^{-1}}$$

$$\Rightarrow A^2 = A \cdot A = S\Lambda S^{-1} \cdot S\Lambda S^{-1}$$

$$= S\Lambda^2 S^{-1}$$

$$\Rightarrow A^3 = A \cdot A^2 = S\Lambda S^{-1} \cdot S\Lambda^2 S^{-1}$$

$$= S\Lambda^3 S^{-1}$$

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Hence, we can generalize it as

$$\boxed{A^k = S\Lambda^k S^{-1}}$$

✱ has to be invertible

Now, if we want to diagonalize a matrix, A, we need to have ~~to diagonal~~ the eigenvector matrix invertible. i.e S^{-1} has to be valid.

We can easily find that if we see two same columns in the eigenvector matrix. If these two columns are same then S is dependent and ~~the~~ thus it won't be invertible. S will be invertible if and only if all the eigenvectors are different.

Ex: $A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$

$$\therefore \det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 5 \\ 0 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(6-\lambda) = 0$$

$$\lambda_s = 1, 6$$

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For $\lambda_1 = 1$, $\begin{vmatrix} 0 & 5 \\ 0 & 5 \end{vmatrix} \vec{x} = \vec{0}$

$$\therefore \vec{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 6$, $\begin{vmatrix} -5 & 5 \\ 0 & 0 \end{vmatrix} \vec{x} = \vec{0}$

$$\therefore \vec{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{So, } S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore S^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\text{And, } \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

We can confirm this by doing,

$$\Lambda = S^{-1} A S$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Similarly, $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\therefore A^k = S\Lambda^k S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}^K \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 6^k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6^k - 1 \\ 0 & 6^k \end{bmatrix}$$

If we have a high power vector multiplication, say, $A^{50} v = \begin{bmatrix} 1 & 6^{50} - 1 \\ 0 & 6^{50} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

→ From the discussion, we can see,

⊛ Due to the easiness of diagonal multiplication, the power gets distributed over the diagonals.

⊛ $A^k = S\Lambda^k S^{-1}$ form is convenient to easily find large powers of a matrix.

⊛ $A^k = S\Lambda^k S^{-1}$ is effecient for large scale calculations.

Another method to find the power of A:

Suppose, u_0 is a matrix which is to be multiplied to A^k which would produce the matrix product u_k

$$u_k \Rightarrow A^k \times u_0$$

→ u_0 has to be written as combination of eigenvectors.

$$u_0 = c_1 x_1 + c_2 x_2 + \cdots c_n x_n$$

$$\rightarrow A^k u_0 = c_1 A^k u_1 + \cdots + c_n A^k u_n$$

$$= c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$$

$$= \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} \uparrow & & \uparrow \\ x_1 & \cdots & x_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}$$

$$\therefore A^k u_0 = \Lambda^k S c$$

→ Combine all the eigenvectors

$$u_k = c_1 (\lambda_1)^k x_1 + \cdots c_n (\lambda_n)^k x_n$$

In this way higher power multiplication can be done using diagonalization of eigenvalues.

Things to remember during diagonalization:

(i) If all eigenvalues $(\lambda_1 ... \lambda_n)$ are different then all eigenvectors $(x_1 ... x_n)$ are different. Only then, the matrix can be diagonalized or else S^{-1} won't be invertible

(ii) Eigenvectors can be multiplied by non-zero constants to make them look more presentable. It doesn't invalidate the property of linearity.

(iii) Eigenvectors in S has to come in same order in Λ.
If $S = \begin{bmatrix} \uparrow & & \uparrow \\ x_1 & \cdots & x_n \\ \downarrow & & \downarrow \end{bmatrix}$, then Λ must also have eigenvalues in $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ order. Changing the order will change the result.

(iv) Invertibility and ~~diagonalization~~ diagonalizability are related but completely different.

Invertibility $\longrightarrow$ $\det(A) \neq 0$

Diagonalizability $\longrightarrow$ [No. of eigenvalues/vectors $< n$
if A is $(n \times n)$ matrix]

<u>Ex-2</u> Given, Markov matrix, $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$. Find the power of A for $k=2$, $k=100$ and $k \to \infty$.

<u>Ans:</u> We find the eigenvalues, $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (0.8-\lambda)(0.7-\lambda) - 0.06 = 0$$

$$\Rightarrow \lambda^2 - 1.5\lambda + 0.56 - 0.06 = 0$$

$$\Rightarrow \lambda^2 - 1.5\lambda + 0.5 = 0$$

$$\therefore \lambda = 1, 0.5$$

For $\lambda_1 = 1$, $\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \vec{x} = \vec{0}$

$$\cancel{\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}} \qquad \therefore \vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\cancel{\therefore \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

and, $\lambda_2 = 2$, $\begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix} \vec{x} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{[Row division]}$$

$$\therefore \vec{x}_2 = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

For, powers of A,

$$\cancel{A^k = S\Lambda^k S^{-1} = [\ 6}$$

$$S = \begin{bmatrix} 3 & +1 \\ 2 & -1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{5}\begin{bmatrix} +1 & +1 \\ 2 & -3 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

for $k=2$, $A^k = S\Lambda^k S^{-1} = S\Lambda^2 S^{-1}$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1^2 & 0 \\ 0 & 0.5^2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} +1 & +1 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} +0.2 & +0.2 \\ 0.4 & -0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & \cancel{-1} & -0.25 \\ 2 & \cancel{1} & 0.25 \end{bmatrix}\begin{bmatrix} +0.2 & +0.2 \\ 0.4 & -0.6 \end{bmatrix}$$

$$= \begin{bmatrix} +0.6-1 & +0.6+1.5 \\ +0.4+1 & +0.4-1.5 \end{bmatrix}$$

$$\cancel{= \begin{bmatrix} -1.6 & 0.9 \\ 0.6 & 1.9 \end{bmatrix} \text{(Ans.)}}$$

$$= \begin{bmatrix} -0.4 & 2.1 \\ 1.4 & -1.1 \end{bmatrix} \text{(Ans.)}$$

for $k=100$, $A^{100} = S\Lambda^{100}S^{-1}$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7.89\times10^{-31} \end{bmatrix} \begin{bmatrix} +0.2 & +0.2 \\ 0.4 & -0.6 \end{bmatrix}$$

As we can see, as the power, k, increases, λ_2 decrease

For $k \to \infty$, $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} +0.2 & +0.2 \\ 0.4 & -0.6 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} +0.2 & +0.2 \\ 0.4 & -0.6 \end{bmatrix}$$

$$= \begin{bmatrix} +0.6 & +0.6 \\ +0.4 & +0.4 \end{bmatrix} \text{ (Ans.)}$$

As we can see, as $k \to \infty$, the property of Markov matrix $(0.6+0.4)=1$ and $(0.6+0.4)=1$ is still valid. [Summation of column $=1$]

Q: When does $A^k \to$ zero matrix

A: When all $|\lambda| < 1$.