Solutions to Problems in Chapter 7 of
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7.1. $Z_0 = Z_{16} = Z_{32} = Z_{48} = \cdots = 7$, i.e., for $j = 0, 1, 2, ...,$ $Z_{16j} = 7$. Thus, for $j = 31$, $Z_{496} = Z_{16j} = 7$, so $Z_{500} = Z_{496+4} = Z_4 = 11$.

7.2. The following are $Z_1, Z_2, \ldots$ in each case:

(*a*) 11, 9, 3, 1, 11, 9, ... (period = $4 = m/4$).

(*b*) 6, 2, 6, 2, ... (period = $2 = m/8$).

(*c*) 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, 2, 4, ... (period = $12 = m - 1$).

(*d*) 3, 9, 1, 3, 9, ... (period = 3).

7.3. Checking Theorem 7.1 in Sec. 7.2, we get the following:

(*a*) Full period

(*b*) Parts (*b*) (2 divides 16, but does not divide 11 = a -1) and (*c*) (4 divides 16 but not 11) are not satisfied.

(*c*) Part (*a*) is violated, since 2 divides 16 and 12. The same is true for 4.

(*d*) Full period

7.4. The following are $Z_1, Z_2, \ldots$ in each case:

(*a*) 10, 15, 0, 13, 6, 11, 12, 9, 2, 7, 8, 5, 14, 3, 4, 1, 10, 15, ... (*appears* in reasonably random order, but m is obviously ridiculously small).

(*b*) 9, 9, 9, ... (terrible).

(*c*) 9, 1, 9, 1, ... (terrible).

(*d*) 0, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 12, ... (clearly awful, even though the period is full).

7.6. Here are the shuffled results, expressed in fractions rather than decimals so that the behavior of the Z_i's can be seen as well:

	V_1	V_2	I	V_I		V_1	V_2	I	V_I		V_1	V_2	I	V_I
1	0/13	12/13	2	12/13	34	6/13	5/13	2	5/13	67	0/13	11/13	1	0/13
2	0/13	11/13	2	11/13	35	6/13	4/13	1	6/13	68	10/13	11/13	1	10/13
3	0/13	10/13	1	0/13	36	3/13	4/13	1	3/13	69	9/13	11/13	2	11/13
4	9/13	10/13	1	9/13	37	2/13	4/13	2	4/13	70	9/13	8/13	2	8/13
5	8/13	10/13	2	10/13	38	2/13	1/13	2	1/13	71	9/13	7/13	1	9/13
6	8/13	7/13	2	7/13	39	2/13	0/13	1	2/13	72	6/13	7/13	1	6/13
7	8/13	6/13	1	8/13	40	12/13	0/13	1	12/13	73	5/13	7/13	2	7/13
8	5/13	6/13	1	5/13	41	11/13	0/13	2	0/13	74	5/13	4/13	2	4/13
9	4/13	6/13	2	6/13	42	11/13	10/13	2	10/13	75	5/13	3/13	1	5/13
10	4/13	3/13	2	3/13	43	11/13	9/13	1	11/13	76	2/13	3/13	1	2/13
11	4/13	2/13	1	4/13	44	8/13	9/13	1	8/13	77	1/13	3/13	2	3/13
12	1/13	2/13	1	1/13	45	7/13	9/13	2	9/13	78	1/13	0/13	2	0/13
13	0/13	2/13	2	2/13	46	7/13	6/13	2	6/13	79	1/13	12/13	1	1/13
14	0/13	12/13	2	12/13	47	7/13	5/13	1	7/13	80	11/13	12/13	1	11/13
15	0/13	11/13	1	0/13	48	4/13	5/13	1	4/13	81	10/13	12/13	2	12/13
16	10/13	11/13	1	10/13	49	3/13	5/13	2	5/13	82	10/13	9/13	2	9/13
17	9/13	11/13	2	11/13	50	3/13	2/13	2	2/13	83	10/13	8/13	1	10/13
18	9/13	8/13	2	8/13	51	3/13	1/13	1	3/13	84	7/13	8/13	1	7/13
19	9/13	7/13	1	9/13	52	0/13	1/13	1	0/13	85	6/13	8/13	2	8/13
20	6/13	7/13	1	6/13	53	12/13	1/13	2	1/13	86	6/13	5/13	2	5/13
21	5/13	7/13	2	7/13	54	12/13	11/13	2	11/13	87	6/13	4/13	1	6/13
22	5/13	4/13	2	4/13	55	12/13	10/13	1	12/13	88	3/13	4/13	1	3/13
23	5/13	3/13	1	5/13	56	9/13	10/13	1	9/13	89	2/13	4/13	2	4/13
24	2/13	3/13	1	2/13	57	8/13	10/13	2	10/13	90	2/13	1/13	2	1/13
25	1/13	3/13	2	3/13	58	8/13	7/13	2	7/13	91	2/13	0/13	1	2/13
26	1/13	0/13	2	0/13	59	8/13	6/13	1	8/13	92	12/13	0/13	1	12/13
27	1/13	12/13	1	1/13	60	5/13	6/13	1	5/13	93	11/13	0/13	2	0/13
28	11/13	12/13	1	11/13	61	4/13	6/13	2	6/13	94	11/13	10/13	2	10/13
29	10/13	12/13	2	12/13	62	4/13	3/13	2	3/13	95	11/13	9/13	1	11/13
30	10/13	9/13	2	9/13	63	4/13	2/13	1	4/13	96	8/13	9/13	1	8/13
31	10/13	8/13	1	10/13	64	1/13	2/13	1	1/13	97	7/13	9/13	2	9/13
32	7/13	8/13	1	7/13	65	0/13	2/13	2	2/13	98	7/13	6/13	2	6/13
33	6/13	8/13	2	8/13	66	0/13	12/13	2	12/13	99	7/13	5/13	1	7/13
										100	4/13	5/13	1	4/13

The period of the V_I's (which are returned for use in the simulation) is 26; the V_I's repeat themselves at their 4th, 30th, 56th, 82nd, ... value. This period is indeed much longer that that of the unshuffled generator [Prob. 7.3(*d*)], and the severe autocorrelation over an entire cycle of the unshuffled generator [Prob. 7.4(*d*)] appears to have been broken up. However, there is still unacceptable correlation in pairs of adjacent V_I's; for example, look at V_I's number 4 and 5, then 6 and 7, then 8 and 9, etc.

7.7. The generalization for the d-dimensional serial test is

```
Set f_{j_1 j_2 ... j_d} = 0 for j_p = 1,2,...,k and p = 1,2,...,d
For i = 1,...,n do
   For p = 1,...,d do
      Generate U_{(i-1)d+p}      (just the next random number from the generator)
      Set J_p = ⌈kU_{(i-1)d+p}⌉
   End do
   Replace f_{J_1 J_2 ... J_d} by f_{J_1 J_2 ... J_d} + 1
End do
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7.10. We used stream 1 of the generator in App. 7A and obtained the following results (to four decimals):

Sum of the dice	Known probabilities	Observed proportions
2	0.0278	0.0220
3	0.0556	0.0640
4	0.0833	0.0750
5	0.1111	0.1050
6	0.1389	0.1380
7	0.1667	0.1560
8	0.1389	0.1440
9	0.1111	0.1120
10	0.0833	0.0910
11	0.0556	0.0540
12	0.0278	0.0390

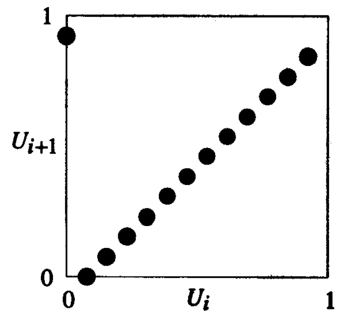
Using the chi-square goodness-of-fit test (see Sec. 6.6.2), we lumped together the outcomes 2 and 3, as well as the outcomes 11 and 12, in order to make the category probabilities closer to each other; thus, the $k = 9$ categories were $\{2, 3\}$, $\{4\}$, $\{5\}$, ..., $\{10\}$, and $\{11, 12\}$. The test statistic is thus

$$\frac{[(0.0220+0.0640)-(0.0278+0.0556)]^2}{0.0278+0.0556}+\frac{(0.0750-0.0833)^2}{0.0833}+\cdots$$
$$+\frac{(0.0910-0.0833)^2}{0.0833}+\frac{[(0.0540+0.0390)-(0.0556+0.0278)]^2}{0.0556+0.0278}$$

which works out to be 0.0039, a completely insignificant chi-square statistic with $k - 1 = 8$ df. In other words, there is no reason from this experiment to suspect any problems with this generator.

We repeated the above experiment, but used instead the "toy" generator described in Example 7.2, and obtained a chi-square statistic of 1.5053, which is also insignificant with 8 df. Thus, it would appear at first that this (ridiculous) generator is also "acceptable"; rather, the correct conclusion is that this test is very weak, i.e., has extremely low discriminative power to detect poor generators.

7.11.



7.12. (*a*) By definition, $(Z_i = Z_{i-1} + Z_{i-2}) (\text{mod } m)$, so $km + Z_i = Z_{i-1} + Z_{i-2}$ for some $k \in \{0, 1, 2, ...\}$. But since each $Z_j \in \{0, 1, ..., m-1\}$, $Z_{i-1} + Z_{i-2}$ is at most $2m - 2$, so k is at most 1, i.e., k must be either 0 or 1. If $k = 0$, then $Z_i = Z_{i-1} + Z_{i-2} \geq Z_{i-1}$ (since $Z_{i-2} \geq 0$), so $U_i = Z_i/m \geq Z_{i-1}/m = U_{i-1}$, contradicting the second inequality in the problem statement. On the other hand, if $k = 1$, we get

$$
\begin{aligned}
m + Z_i &= Z_{i-1} + Z_{i-2} \\
1 + U_i &= U_{i-1} + U_{i-2} \qquad &\text{(divide by } m\text{)} \\
1 - U_{i-1} &= U_{i-2} - U_i \qquad &\text{(rearrange)}
\end{aligned}
$$

Since $U_{i-1} \leq 1$, the left-hand side of the last equation above is nonnegative, implying that $U_i \leq U_{i-2}$, which contradicts the first inequality in the problem statement.

(*b*) In theory, the U_i's are IID, so each of the $3! = 6$ permutations of U_i, U_{i-1}, and U_{i-2} should be equally likely, i.e., should occur w.p. 1/6.

7.13. For any positive real number z, let $d(z) = z - \lfloor z \rfloor$, i.e., $d(z)$ is the decimal part of z. Thus, we are to show that $d(U_1 + U_2 + \cdots + U_k) \sim \mathrm{U}(0, 1)$ for all $k \geq 1$; the proof is by induction on k.

Anchor step ($k = 1$): Since $0 \leq U_1 < 1$ (w.p. 1), $d(U_1) = U_1 \sim \mathrm{U}(0, 1)$.

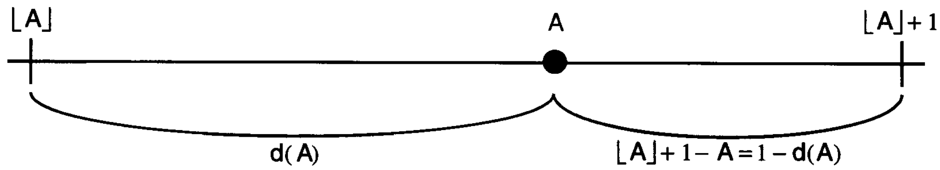
Induction step (Assume the conclusion for k, and prove it for $k + 1$): Let $A = U_1 + U_2 + \cdots + U_k$, so the induction hypothesis is that $d(A) \sim \mathrm{U}(0, 1)$, and we are to show that $d(A + U_{k+1}) \sim \mathrm{U}(0, 1)$ as well. This will be done by showing that the distribution function of $d(A + U_{k+1})$ is the U(0, 1) distribution function,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leq x \end{cases}$$

Since $0 \leq d(A + U_{k+1}) \leq 1$, it is clearly the case that $P[d(A + U_{k+1}) \leq x] = 0$ if $x \leq 0$, and that $P[d(A + U_{k+1}) \leq x] = 1$ if $1 \leq x$. The more interesting case is for $0 < x < 1$, where the distribution function of $d(A + U_{k+1})$ is

$$\begin{aligned} P\left[d(A + U_{k+1}) \leq x\right] &= P\left(A + U_{k+1} - \lfloor A + U_{k+1} \rfloor \leq x\right) \\ &= \int_0^1 P\left(A + U_{k+1} - \lfloor A + U_{k+1} \rfloor \leq x \,\middle|\, U_{k+1} = y\right) dy \qquad \left[\text{condition on } U_{k+1} \sim \mathrm{U}(0, 1)\right] \\ &= \int_0^1 P\left(A + y - \lfloor A + y \rfloor \leq x\right) dy \qquad \left(\text{since } U_{k+1} \text{ is independent of } A\right) \quad (*) \end{aligned}$$

Thus, we must evaluate the probability under the integral in (*). Consider the following diagram:



Thus, for a fixed $y \in [0, 1]$, $A + y < \lfloor A \rfloor + 1$ if $y < 1 - d(A)$, i.e., $\lfloor A + y \rfloor = \lfloor A \rfloor$ if $d(A) < 1 - y$. In other words,

$$\lfloor A + y \rfloor = \begin{cases} \lfloor A \rfloor & \text{if } d(A) < 1 - y \\ \lfloor A \rfloor + 1 & \text{if } d(A) \geq 1 - y \end{cases}$$

Intersecting the mutually exclusive and exhaustive events $d(A) < 1 - y$ and $d(A) \geq 1 - y$ in the probability under the integral in (*),

$$\begin{aligned} P\left(A + y - \lfloor A + y \rfloor \leq x\right) &= P\left[A + y - \lfloor A + y \rfloor \leq x, d(A) < 1 - y\right] + P\left[A + y - \lfloor A + y \rfloor \leq x, d(A) \geq 1 - y\right] \\ &= P\left[A + y - \lfloor A \rfloor \leq x, d(A) < 1 - y\right] + P\left[A + y - \left(\lfloor A \rfloor + 1\right) \leq x, d(A) \geq 1 - y\right] \\ &= P\left[d(A) \leq x - y, d(A) < 1 - y\right] + P\left[d(A) \leq x - y + 1, d(A) \geq 1 - y\right] \\ &= P\left[d(A) \leq x - y\right] + P\left[1 - y \leq d(A) \leq x - y + 1\right] \qquad (**) \end{aligned}$$

The first term in (**) is

$$P\left[d(A) \leq x - y\right] = \begin{Bmatrix} x - y & \text{if } x - y \geq 0 \\ 0 & \text{otherwise} \end{Bmatrix} = \begin{Bmatrix} x - y & \text{if } y \leq x \\ 0 & \text{otherwise} \end{Bmatrix}$$

since both x and y are on [0, 1]. The second term in (**) is

$$P\left[1 - y \leq d(A) \leq 1 - y + x\right] = \begin{Bmatrix} x - y + 1 - (1 - y) & \text{if } 1 - y + x \leq 1 \\ 1 - (1 - y) & \text{otherwise} \end{Bmatrix} = \begin{Bmatrix} x & \text{if } y \geq x \\ y & \text{otherwise} \end{Bmatrix}$$

Putting the above two probabilities back into (*) and splitting up its range of integration appropriately, we get

$$\int_0^x (x - y)\,dy + \int_x^1 0\,dy + \int_0^x y\,dy + \int_x^1 x\,dy$$

which works out by simple calculus to be x, as desired.

7.14. Ignoring the order, the values of the U_i's are $0/m$, $1/m$, ..., $(m-1)/m$, and so their average is

$$\frac{0/m + 1/m + 2/m + \cdots + (m-1)/m}{m} = \frac{0 + [1 + 2 + \cdots + (m-1)]}{m^2} = \frac{(m-1)m/2}{m^2} = \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m}$$

using Euler's formula with $k = m - 1$ to get from the second quantity to the third.

7.15. The first 36 W_i's are (every tenth W_i is in boldface)

15, 8, 13, 13, 4, 2, 5, 9, 15, **1**, 11, 10, 8, 4, 11, 3, 14, 3, 7, **5**,
0, 9, 6, 7, 12, 6, 14, 10, 1, **2**, 12, 15, 8, 13, 13, 4, …

Since W_{32} through W_{36} are the same as W_1 through W_5 (and the latter five numbers are not repeated earlier in the sequence), it follows that the period is 31.