Cauchy Integral Formula and Cauchy Theorem

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Cauchy Integral Formula (CIF)

If f(z) is analytic within and on a closed curve C and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

This CIF is specially helpful for evaluating the integral having finite poles within a specified domain.

Example 1: Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$, over $C:|z| = \frac{3}{2}$

Solution: The poles of the integrand are given by putting the denominator to zero. That is,

$$z(z-1)(z-2) = 0$$

$$\Rightarrow z = 0, z = 1, z = 2$$

The given circle $|z| = \frac{3}{2}$ with centre at the origin (z = 0) and radius $r = \frac{3}{2}$ encloses two poles z = 0, and z = =1 only and the other pole lies out side of the circle (domain). Now using CIF, we have

$$\int_C \frac{4-3z}{z(z-1)(z-2)}dz = \int_{C_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z}dz + \int_{C2} \frac{\frac{4-3z}{z(z-2)}}{z-1}dz$$

$$= 2\pi i\left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i\left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i\left[\frac{4}{2}\right] + 2\pi i\left[\frac{4-3}{1(-1)}\right]$$

$$= 4\pi i - 2\pi i = 2\pi i$$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)}dz = 2\pi i \quad \square$$

Example 2: Evaluate $\int_C \frac{dz}{\left(z^2-1\right)}$, where C is the circle given by $|z|=2$

Solution: The poles of the integrand are given by putting the denominator to zero. Therefore,

$$\left(z^2-1\right)=0$$
$$\Rightarrow (z-1)(z+1)=0$$
$$\Rightarrow z=1,\ z=-1$$

The given circle $|z|=2$ that is $x^2+y^2=4$ with centre at the origin having radius 2 encloses two simple poles at z = 12 and z = -1. Now by CIF, we have

$$\int_C \frac{dz}{\left(z^2-1\right)}=\int_{C_1} \frac{dz}{\left(z^2-1\right)}+\int_{C_2} \frac{dz}{\left(z^2-1\right)}$$
$$=\int_{C_1} \frac{\frac{1}{z+1}}{(z-1)}dz+\int_{C_2} \frac{\frac{1}{z-1}}{(z+1)}dz$$

$$= 2\pi i\left[\frac{1}{z+1}\right]_{z=1} + 2\pi i\left[\frac{1}{z-1}\right]_{z=-1}$$

$$= 2\pi i\left[\frac{1}{1+1}\right] + 2\pi i\left[\frac{1}{-1-1}\right]$$

$$= \pi i - \pi i = 0$$

$$\int_C \frac{dz}{\left(z^2-1\right)} = 0$$

Example 3: Evaluate the complex integral $\int_C \tan z dz$, where C is given by $|z| = 2$

Solution: We have $\int_C \tan z dz = \int_C \frac{\sin z}{\cos z} dz$

The poles of are given by putting the denominator to zero. Therefore,

$$\cos z = 0$$

$$\Rightarrow z = \frac{\pi}{2}, \frac{3\pi}{2}..............$$

The integral has a pole at $z = \frac{\pi}{2}$ inside the circle $|z| = 2$. Applying CIF, we have

$$\begin{aligned}\int_C \tan z dz &= \int_C \frac{\sin z}{\cos z} dz \\ &= 2\pi i \left[\sin z\right]_{z=\frac{\pi}{2}} \\ &= 2\pi i \left[\sin \frac{\pi}{2}\right] \\ &= 2\pi i \times 1 = 2\pi i\end{aligned}$$

Therefore, $\int_C \tan z dz = 2\pi i$ □

Cauchy Theorem (CT)

If a function *f(z)* is analytic and its derivative $f'(z)$ is continuous at all points inside and on a closed curve C and if a is any point within C, then

$$\int_C f(z)\,dz = \oint f(z)\,dz = 0.$$

This CIF is specially helpful for evaluating the integral having finite poles within a specified domain.

Example 1: Evaluate $\int_C \frac{z+4}{z^2+2z+5}dz$, if C is the circle $C:|z+1|=1$

Solution: The poles of the integrand are given by putting the denominator to zero. That is,

$$z^2+2z+5=0$$

$$\Rightarrow z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2 - 4\times1\times5}}{2\times1}$$
$$= \frac{-2 \pm \sqrt{-16}}{2}$$
$$= \frac{-2 \pm 4\sqrt{-1}}{2}$$
$$= \frac{2(-1 \pm 2i)}{2} = (-1 \pm 2i)$$
$$\therefore z = (-1+2i), (-1-2i)$$

The given circle $|z+1|=1$ with centre at z = -1 and radius unity ($r=1$) does not enclose any singularity of the function $\frac{z+4}{z^2+2z+5}$. Now by Cauchy Theorem, we have

$$\boxed{\int_C \frac{z+4}{z^2+2z+5}dz = 0 \quad \square}$$

Exercises :

$$(1)\int_C \frac{(z-4)\,dz}{z^2+2z+7},\quad C:|z-1|=1 \quad (2)\int_C \frac{dz}{z^2+2z+3},\quad C:|z|=\frac{1}{2}$$

Thanks a lot …