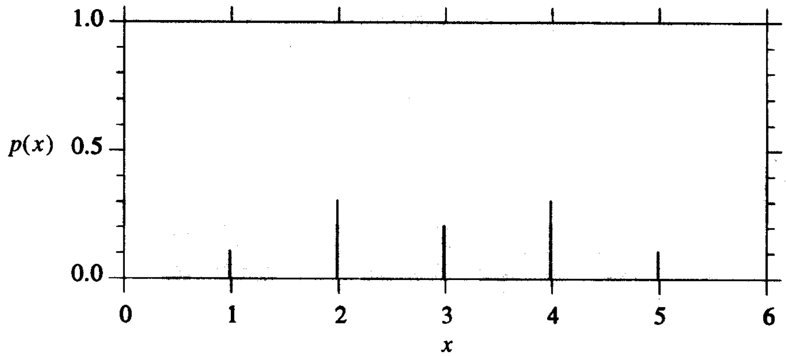
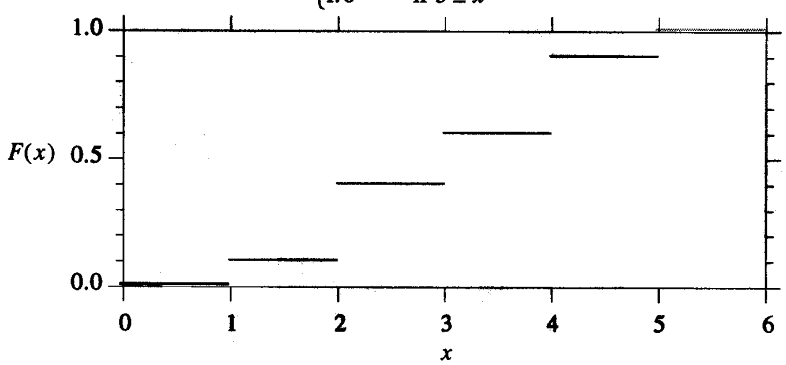
Solutions to Problems in Chapter 4 of
Simulation Modeling and Analysis, 5th ed., 2015, McGraw-Hill, New York
by Averill M. Law

4.1. *(a)*



(b)

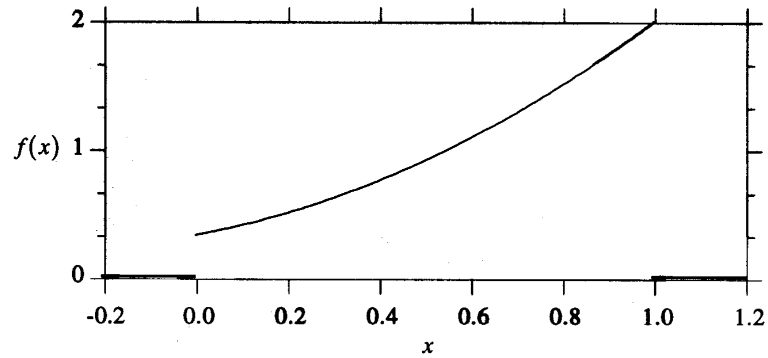
$$F(x) = \begin{cases} 0.0 & \text{if } x < 1 \\ 0.1 & \text{if } 1 \le x < 2 \\ 0.4 & \text{if } 2 \le x < 3 \\ 0.6 & \text{if } 3 \le x < 4 \\ 0.9 & \text{if } 4 \le x < 5 \\ 1.0 & \text{if } 5 \le x \end{cases}$$



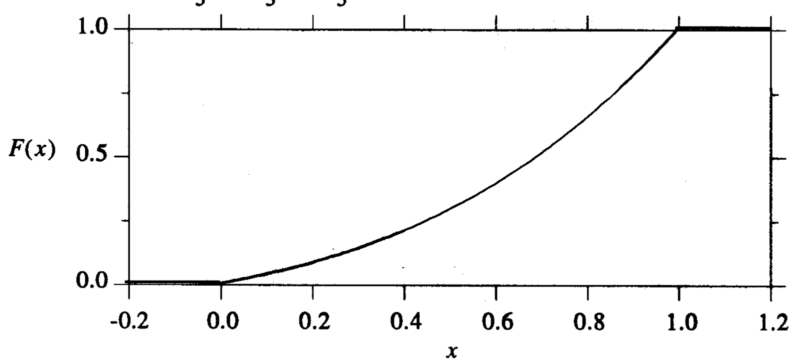
(c) $P(1.4 \le X \le 4.2) = 3/10 + 2/10 + 3/10 = 4/5$.
$E(X) = 1(1/10) + 2(3/10) + 3(2/10) + 4(3/10) + 5(1/10) = 3$.
$E(X^2) = 1^2(1/10) + 2^2(3/10) + 3^2(2/10) + 4^2(3/10) + 5^2(1/10) = 52/5$, so
$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 52/5 - 3^2 = 7/5$.

4.2. (*a*) $1=\int_0^c f(x)\,dx=\int_0^c\left(x^2+\frac{2}{3}x+\frac{1}{3}\right)dx=\left(\frac{1}{3}x^3+\frac{1}{3}x^2+\frac{1}{3}x\right)\Big|_{x=0}^{x=c}=\frac{1}{3}c\left(c^2+c+1\right)$. The solution to this equation is $c = 1$, by inspection.

(*b*)



(*c*) From the above derivation, $F(x)=\frac{1}{3}x^3+\frac{1}{3}x^2+\frac{1}{3}x$ for $0 \le x \le 1$.



(*d*) $P(1/3 \le X \le 2/3) = F(2/3) - F(1/3) = 38/81 - 13/81 = 25/81$.

$E(X)=\int_0^1 xf(x)\,dx=\int_0^1\left(x^3+\frac{2}{3}x^2+\frac{1}{3}x\right)dx=\left(\frac{1}{4}x^4+\frac{2}{9}x^3+\frac{1}{6}x^2\right)\Big|_{x=0}^{x=1}=\frac{23}{36}$.

$E(X^2)=\int_0^1 x^2f(x)\,dx=\int_0^1\left(x^4+\frac{2}{3}x^3+\frac{1}{3}x^2\right)dx=\left(\frac{1}{5}x^5+\frac{1}{6}x^4+\frac{1}{9}x^3\right)\Big|_{x=0}^{x=1}=\frac{43}{90}$, so

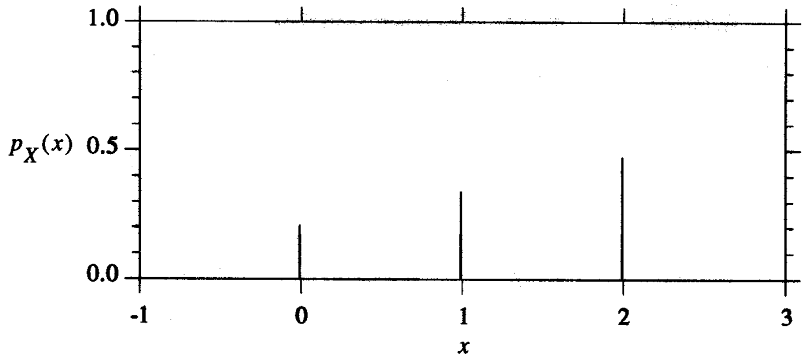
$\text{Var}(X) = E(X^2) - [E(X)]^2 = 43/90 - (23/36)^2 = 0.07$.

4.3. $p_X(x) = \sum_{y=1}^{x} p(x, y) = \sum_{y=1}^{x} 2/[n(n+1)] = 2x/[n(n+1)]$.

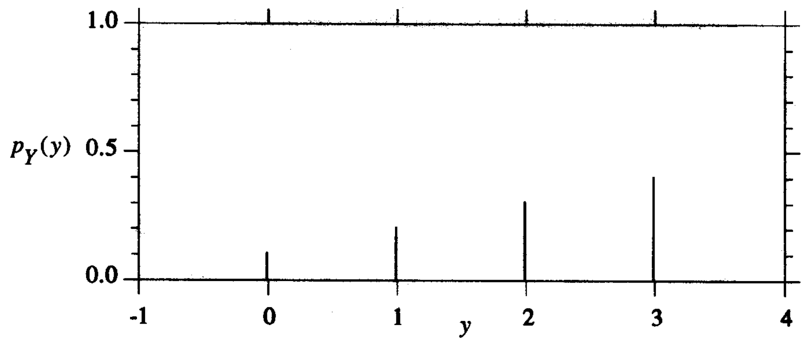
$p_Y(y) = \sum_{x=y}^{n} p(x, y) = \sum_{x=y}^{n} 2/[n(n+1)] = 2(n - y + 1)/[n(n+1)]$.

If $n = 1$, then $x = 1$, $y = 1$, and $p(1, 1) = 1 = p_X(1)p_Y(1)$ so X and Y are independent. For $n = 2, 3, \ldots, X$ and Y are not independent, since $p(1, 1) \neq p_X(1)p_Y(1)$.

4.4. (a) $p_X(x) = \sum_{y=0}^{3}(x+y)/30 = (2x+3)/15.$



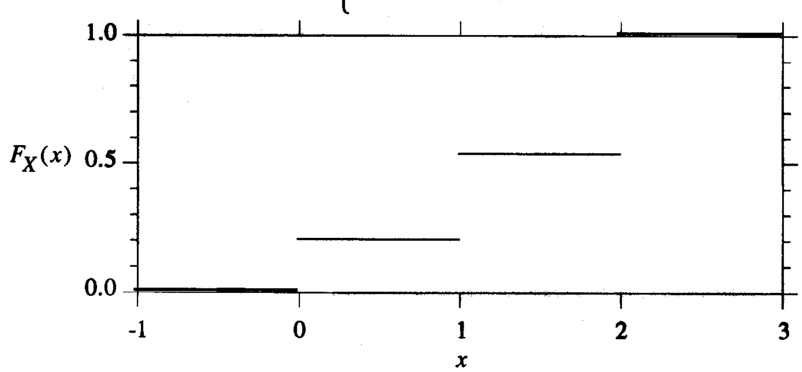
$p_Y(y) = \sum_{x=0}^{2}(x+y)/30 = (1+y)/10$



(b) $p_X(x)p_Y(y) = [(2x+3)/15][(1+y)/10] = (2x+2xy+3+3y)/150 \neq (x+y)/30 = p(x,y)$ in general. Therefore, X and Y are not independent.

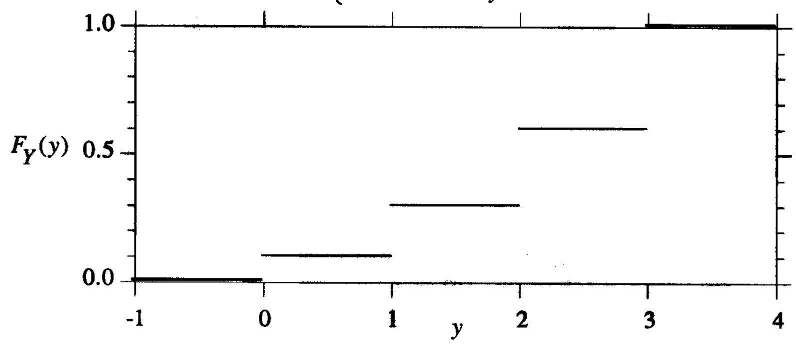
(c)

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{5} & \text{if } 0 \le x < 1 \\ \frac{8}{15} & \text{if } 1 \le x < 2 \\ 1 & \text{if } 2 \le x \end{cases}$$



(Problem 4.4, continued from previous page)

$$F_Y(y) = \begin{cases} 0.0 & \text{if } y < 0 \\ 0.1 & \text{if } 0 \le y < 1 \\ 0.3 & \text{if } 1 \le y < 2 \\ 0.6 & \text{if } 2 \le y < 3 \\ 1.0 & \text{if } 3 \le y \end{cases}$$

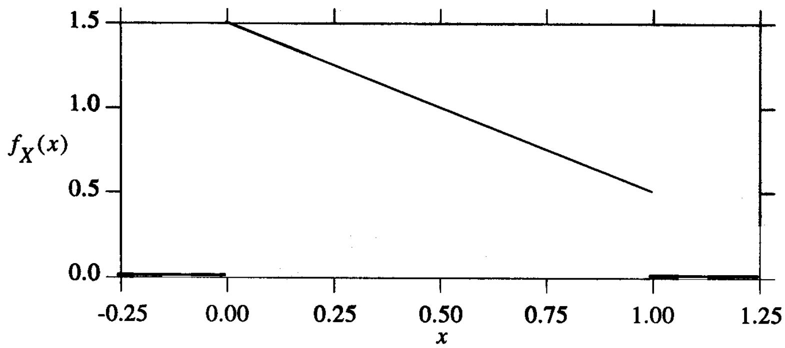


(*d*) $E(X) = 0(3/15) + 1(5/15) + 2(7/15) = 19/15$.
$E(X^2) = 0^2(3/15) + 1^2(5/15) + 2^2(7/15) = 33/15$, so
$\text{Var}(X) = E(X^2) - [E(X)]^2 = 33/15 - (19/15)^2 = 134/225$.
$E(Y) = 0(1/10) + 1(2/10) + 2(3/10) + 3(4/10) = 2$.
$E(Y^2) = 0^2(1/10) + 1^2(2/10) + 2^2(3/10) + 3^2(4/10) = 5$, so
$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 5 - 2^2 = 1$.
$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{3} xy\,p(x,y) = \frac{1}{30}\sum_{x=0}^{2}\sum_{y=0}^{3}\left(x^2 + xy^2\right) = \frac{1}{30}72 = \frac{12}{5}$, so
$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 12/5 - (19/15)(2) = -2/15$.
$\text{Cor}(X, Y) = \text{Cov}(X, Y)/\sqrt{\text{Var}(X)\text{Var}(Y)} = (-2/15)/\sqrt{(134/225)(1)} = -2/\sqrt{134} = -0.17$.

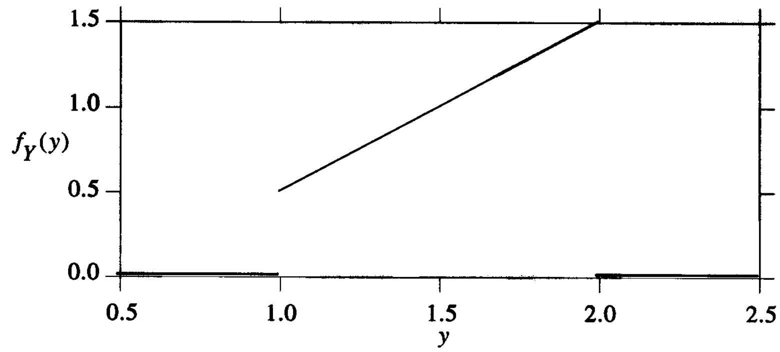
4.5. $p_X(1) = 2(4/52)(48/52) = 2(1/13)(12/13) = p_Y(1)$, and $p(1, 1) = 2(4/52)(4/52) = 2(1/13)^2$. Therefore, $p(1, 1) = 2(1/13)^2 \neq 2^2(1/13)^2(12/13)^2 = p_X(1)p_Y(1)$, and X and Y are *not* independent.

4.6. $f_X(x) = \int_0^1 f(x, y)\,dy = \int_0^1 32x^3y^7\,dy = 4x^3$, and $f_Y(y) = \int_0^1 f(x, y)\,dx = \int_0^1 32x^3y^7\,dx = 8y^7$. Since $f(x, y) = f_X(x) f_Y(y)$ for all x and y, X and Y are independent.

4.7. (a) $f_X(x)=\int_1^2(y-x)\,dy=\left(y^2/2-xy\right)\Big|_{y=1}^{y=2}=(2-2x)-(1/2-x)=3/2-x$ for $0<x<1$.



$f_Y(y)=\int_0^1(y-x)\,dx=\left(xy-x^2/2\right)\Big|_{x=0}^{x=1}=y-1/2$ for $1<y<2$.



(b) $f(x, y) = y - x \neq (3/2 - x)(y - 1/2) = f_X(x)f_Y(y)$. Therefore, X and Y are not independent.

(c) $F_X(x)=\int_0^x(3/2-z)\,dz=\left(3z/2-z^2/2\right)\Big|_{z=0}^{z=x}=3x/2-x^2/2=\frac{1}{2}x(3-x)$ for $0\le x\le 1$.

$F_Y(y)=\int_1^y(z-1/2)\,dz=\left(z^2/2-z/2\right)\Big|_{z=1}^{z=y}=\left(y^2/2-y/2\right)-(1/2-1/2)=\frac{1}{2}y(y-1)$ for $1\le y\le 2$.

(d) $E(X)=\int_0^1x(3/2-x)\,dx=\left(3x^2/4-x^3/3\right)\Big|_{x=0}^{x=1}=3/4-1/3=5/12$.

$E(X^2)=\int_0^1x^2(3/2-x)\,dx=\left(x^3/2-x^4/4\right)\Big|_{x=0}^{x=1}=1/2-1/4=1/4$, so

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 1/4 - (5/12)^2 = 11/144$.

$E(Y)=\int_1^2y(y-1/2)\,dy=\left(y^3/3-y^2/4\right)\Big|_{y=1}^{y=2}=(8/3-1)-(1/3-1/4)=19/12$.

$E(Y^2)=\int_1^2y^2(y-1/2)\,dy=\left(y^4/4-y^3/6\right)\Big|_{y=1}^{y=2}=(4-4/3)-(1/4-1/6)=31/12$, so

$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = 31/12 - (19/12)^2 = 11/144$.

$E(XY)=\int_0^1\int_1^2xy(y-x)\,dy\,dx=\int_0^1\left(xy^3/3-x^2y^2/2\right)\Big|_{y=1}^{y=2}dx=\int_0^1\left[\left(8x/3-2x^2\right)-\left(x/3-x^2/2\right)\right]dx$

$=\int_0^1\left(7x/3-3x^2/2\right)dx=\left(7x^2/6-x^3/2\right)\Big|_{x=0}^{x=1}=\frac{2}{3}$, so

$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 2/3 - (5/12)(19/12) = 1/144$, and

$\mathrm{Cor}(X, Y) = \mathrm{Cov}(X, Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = (1/144)/\sqrt{(11/144)(11/144)} = 1/11$.

4.8.

$$
\begin{aligned}
E(XY) &= \int\int xyf(x,y)\,dxdy \\
&= \int\int xyf_X(x)f_Y(y)\,dxdy \quad \text{(by independence)} \\
&= \int yf_Y(y)\left[\int xf_X(x)\,dx\right]dy \\
&= \left[\int xf_X(x)\,dx\right]\left[\int yf_Y(y)\,dy\right] \\
&= E(X)E(Y)
\end{aligned}
$$

Therefore, $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$.

4.9. $E(XY) = [(-2)(4) + (-1)(1) + (1)(1) + (2)(4)]/4 = 0$, and $E(X) = (-2 - 1 + 1 + 2)/4 = 0$. Therefore, $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$.

4.10. If $\rho_{12} = 0$, then

$$f_{X_1,X_2}(x_1,x_2) = \frac{e^{-\left[(x_1-\mu_1)^2/(2\sigma_1^2)+(x_2-\mu_2)^2/(2\sigma_2^2)\right]}}{2\pi\sqrt{\sigma_1^2\sigma_2^2}}$$

$$= \left[\frac{e^{-\left[(x_1-\mu_1)^2/(2\sigma_1^2)\right]}}{\sqrt{2\pi\sigma_1^2}}\right]\left[\frac{e^{-\left[(x_2-\mu_2)^2/(2\sigma_2^2)\right]}}{\sqrt{2\pi\sigma_2^2}}\right]$$

Thus, $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_2$

$$= \frac{e^{-\left[(x_1-\mu_1)^2/(2\sigma_1^2)\right]}}{\sqrt{2\pi\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\left[(x_2-\mu_2)^2/(2\sigma_2^2)\right]}\,dx_2$$

$$= \frac{e^{-\left[(x_1-\mu_1)^2/(2\sigma_1^2)\right]}}{\sqrt{2\pi\sigma_1^2}}$$

and $f_{X_2}(x_2) = \dfrac{e^{-\left[(x_2-\mu_2)^2/(2\sigma_2^2)\right]}}{\sqrt{2\pi\sigma_2^2}}$ (by symmetry)

Therefore, $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$ for all x_1, x_2.

4.11. $\text{Cov}(X, Y) = \text{Cov}(X, aX + b) = a\,\text{Var}(X)$. $\text{Var}(Y) = \text{Var}(aX + b) = a^2\,\text{Var}(X)$. Therefore,

$$\text{Cor}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{a\text{Var}(X)}{\sqrt{\text{Var}(X)a^2\text{Var}(X)}} = \frac{a}{\sqrt{a^2}} = \begin{cases} +1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}$$

4.12. Let $X_1 = Y - E(Y)$ and $X_2 = Z - E(Z)$.

$$E(X_1^2)E(X_2^2) \geq [E(X_1X_2)]^2 \qquad \text{implies}$$

$$\mathrm{Var}(Y)\mathrm{Var}(Z) \geq [\mathrm{Cov}(Y,Z)]^2 \qquad \text{implies}$$

$$\mathrm{Var}(X_1)\mathrm{Var}(X_2) \geq [\mathrm{Cov}(X_1,X_2)]^2 \qquad \text{implies}$$

$$1 \geq \frac{\mathrm{Cov}(X_1,X_2)}{\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)}} \geq -1 \qquad \text{implies}$$

$$1 \geq \rho_{12} \geq -1$$

4.13.

$$
\begin{aligned}
E\left[(a_1X_1 + a_2X_2)^2\right] &= E\left(a_1^2X_1^2 + 2a_1a_2X_1X_2 + a_2^2X_2^2\right) \\
&= a_1^2E\left(X_1^2\right) + 2a_1a_2E(X_1X_2) + a_2^2E\left(X_2^2\right) \\
E(a_1X_1 + a_2X_2) &= a_1E(X_1) + a_2E(X_2) \\
\text{Var}(a_1X_1 + a_2X_2) &= a_1^2E\left(X_1^2\right) + 2a_1a_2E(X_1X_2) + a_2^2E\left(X_2^2\right) - \left[a_1E(X_1) + a_2E(X_2)\right]^2 \\
&= a_1^2E\left(X_1^2\right) + 2a_1a_2E(X_1X_2) + a_2^2E\left(X_2^2\right) - a_1^2\left[E(X_1)\right]^2 - 2a_1a_2E(X_1)E(X_2) - a_2^2\left[E(X_2)\right]^2 \\
&= a_1^2\left\{E\left(X_1^2\right) - \left[E(X_1)\right]^2\right\} + 2a_1a_2\left[E(X_1X_2) - E(X_1)E(X_2)\right] + a_2^2\left\{E\left(X_2^2\right) - \left[E(X_2)\right]^2\right\} \\
&= a_1^2\text{Var}(X_1) + 2a_1a_2\text{Cov}(X_1, X_2) + a_2^2\text{Var}(X_2)
\end{aligned}
$$

4.14. If the $(i + 1)$st customer arrives after the ith customer departs, then $D_{i+1} = 0$. If the $(i + 1)$st customer arrives before the ith customer departs, then $D_{i+1} = t_i + D_i + S_i - (t_i + A_{i+1}) = D_i + S_i - A_{i+1}$. Therefore, $D_{i+1} = \max\{D_i + S_i - A_{i+1}, 0\}$.

4.15. A FORTRAN program is as follows:

```
      INTEGER I, N
      REAL MARRVT,MSERVT,DELAY,SUMDEL,U1,U2,ST,AT,AVGDEL
      MARRVT = 1.0
      MSERVT = 0.5
      N      = 1000
      DELAY  = -1.0E+30
      SUMDEL = 0.0
      DO 10 I = 1, 1000
         U1     = RAND(1)
         ST     = -MSERVT * LOG(U1)
         U2     = RAND(1)
         AT     = -MARRVT * LOG(U2)
         DELAY  = DELAY + ST - AT
         DELAY  = MAX(DELAY, 0.0)
         SUMDEL = SUMDEL + DELAY
   10 CONTINUE
      AVGDEL = SUMDEL / N
      WRITE(*, 2010) AVGDEL
 2010 FORMAT(F10.3)
      STOP
      END
```

Using the random-number generator from App. 7A, the result was 0.441.

4.16. $E\left[\overline{X}(n)\right] = E\left[\sum_{i=1}^{n} X_i/n\right] = \sum_{i=1}^{n} E(X_i)/n = n\mu/n = \mu$; this did not use the assumption that the X_i's are independent.

$$\begin{aligned}
E\left[S^2(n)\right] &= E\left\{\sum_{i=1}^{n}\left[X_i - \overline{X}(n)\right]^2 \Big/ (n-1)\right\} \\
&= E\left\{\sum_{i=1}^{n}\left[X_i^2 - 2\overline{X}(n)X_i + \left(\overline{X}(n)\right)^2\right]\right\}\Big/(n-1) \\
&= E\left\{\sum_{i=1}^{n} X_i^2 - 2n\left(\overline{X}(n)\right)^2 + n\left(\overline{X}(n)\right)^2\right\}\Big/(n-1) \\
&= \left\{\sum_{i=1}^{n} E\left(X_i^2\right) - nE\left[\left(\overline{X}(n)\right)^2\right]\right\}\Big/(n-1) \\
&= \left\{nE\left(X_1^2\right) - n\left[\operatorname{Var}\left(\overline{X}(n)\right) + \left(E\left(\overline{X}(n)\right)\right)^2\right]\right\}\Big/(n-1) \\
&= \left[n\left(\sigma^2 + \mu^2\right) - n\left(\sigma^2/n + \mu^2\right)\right]\Big/(n-1) \\
&= \sigma^2
\end{aligned}$$

4.17.

$$
\begin{aligned}
\operatorname{Var}\left[\overline{X}(n)\right] &= \operatorname{Cov}\left[\overline{X}(n), \overline{X}(n)\right] \\
&= \operatorname{Cov}\left[\textstyle\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right]\Big/ n^2 \\
&= \left[n\sigma^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \operatorname{Cov}(X_i, X_j) \right] \Bigg/ n^2 \\
&= \frac{n\sigma^2 + 2\sum_{j=1}^{n-1} (n-j)\operatorname{Cov}(X_1, X_{1+j})}{n^2} \\
&= \frac{\sigma^2 + 2\sum_{j=1}^{n-1} (1 - j/n) C_j}{n} \\
&= \sigma^2 \frac{1 + 2\sum_{j=1}^{n-1} (1 - j/n) \rho_j}{n}
\end{aligned}
$$

4.19. The event

$$\left\{-z_{1-\alpha/2} \le \frac{\overline{X}(n)-\mu}{\sqrt{S^2(n)/n}} \le z_{1-\alpha/2}\right\}$$

is equivalent to the event

$$\left\{-z_{1-\alpha/2}\sqrt{S^2(n)/n} \le \overline{X}(n)-\mu \le z_{1-\alpha/2}\sqrt{S^2(n)/n}\right\}$$

(multiplication by $\sqrt{S^2(n)/n}$ throughout), which is in turn equivalent to the event

$$\left\{\overline{X}(n) - z_{1-\alpha/2}\sqrt{S^2(n)/n} \le \mu \le \overline{X}(n) + z_{1-\alpha/2}\sqrt{S^2(n)/n}\right\}$$

(algebra applied to each of the two inequalities). Since the first and last events are equivalent, they have the same probabilities of occurrence.

4.20. The half-length is $t_{n-1,1-\alpha/2}\sqrt{S^2(n)/n}$. The critical point $t_{n-1,1-\alpha/2}$ will not change appreciably beyond a certain sample size n, since $t_{n-1,1-\alpha/2} \to z_{1-\alpha/2}$ as $n \to \infty$. Also, $S^2(n)$ will not change appreciably beyond a certain value of n, since $S^2(n) \to \sigma^2$ (w.p. 1) as $n \to \infty$. Therefore, if the sample size is increased from n to $4n$, then the half-length should decrease by a factor of approximately 2.

4.21. We have constructed a 90 percent confidence interval for μ, not an interval that will contain 90 percent of the observations themselves.

4.22.

$$\begin{aligned} 1-\alpha &= \lim_{n\to\infty} P(-z_{1-\alpha/2} \le Z_n \le z_{1-\alpha/2}) \\ &= \lim_{n\to\infty} P(-z_{1-\alpha/2} \le t_n \le z_{1-\alpha/2}) \\ &= \lim_{n\to\infty} P(-t_{n-1,1-\alpha/2} \le t_n \le t_{n-1,1-\alpha/2}) \end{aligned}$$

since $t_{n-1,1-\alpha/2} \to z_{1-\alpha/2}$ as $n \to \infty$.

4.23. $\overline{X}(10) = 6.38$, $S^2(10) = 2.16$, and an approximate 95 percent confidence interval for μ is given by $6.38 \pm 2.262\sqrt{2.16/10}$, or 6.38 ± 1.05.

4.24. $t_{10} = [\overline{X}(10) - 6]/\sqrt{S^2(10)/10} = 0.38/0.46 = 0.82 < 2.262 = t_{9,0.975}$. Therefore, we fail to reject the null hypothesis that $\mu = 6$ at level $\alpha = 0.05$.

4.25. As n gets large, the numerator approaches 0.5 and the denominator goes to zero $[S^2(n)$ converges to the constant $\sigma^2]$, making the quotient arbitrarily large. Therefore, for large n, t_n will exceed 1.833.

4.26. (*a*) For $n = 10$, the power will decrease from 0.433 to approximately 0.165, because it is harder to distinguish between $\mu = 1.25$ and $\mu_0 = 1$ than it is to distinguish between $\mu = 1.5$ and $\mu_0 = 1$.

(*b*) For $n = 10$, the power will increase from 0.433 to approximately 0.620, because it will easier to get a good estimate of the true mean μ due to the smaller variability of the data.

4.27. Since

$$t_{50} = \frac{\bar{X}(50) - 0.5}{\sqrt{S^2(50)/50}} = \frac{-0.05}{\sqrt{0.06/50}} = -1.443 > -2.010 = -t_{49,0.975}$$

we *fail to reject* H_0. (The value $t_{49,0.975} = 2.010$ was obtained from Excel.) This does not necessarily mean that H_0 is true, but rather that based on this test at level $\alpha = 0.05$, there is no evidence to reject it. A 95 percent confidence interval for μ is 0.45 ± 0.07, which contains 0.5.

4.28. Rejecting H_0 at level α is equivalent to

$$\frac{\bar{X}(n)-\mu_0}{\sqrt{S^2(n)/n}} > t_{n-1,1-\alpha/2} \quad \text{or} \quad \frac{\bar{X}(n)-\mu_0}{\sqrt{S^2(n)/n}} < -t_{n-1,1-\alpha/2}$$

which is equivalent to

$$\bar{X}(n) - t_{n-1,1-\alpha/2} > \mu_0 \quad \text{or} \quad \bar{X}(n) + t_{n-1,1-\alpha/2} < \mu_0$$

which is equivalent to μ_0 not being in the confidence interval. In summary, rejecting H_0 at level α is the same as a 100(1-α) percent confidence interval for μ not containing μ_0. Also, not rejecting H_0 at level α is the same as a 100(1-α) percent confidence interval for μ containing μ_0.

4.29. $$E\left[\widehat{\text{Cov}}(X,Y)\right] = \frac{\sum_{i=1}^{n} E\left\{\left[X_i - \bar{X}(n)\right]\left[Y_i - \bar{Y}(n)\right]\right\}}{n-1}$$

$$= \frac{\sum_{i=1}^{n} \left\{E(X_i Y_i) - E\left[X_i \bar{Y}(n)\right] - E\left[Y_i \bar{X}(n)\right] + E\left[\bar{X}(n)\bar{Y}(n)\right]\right\}}{n-1}$$

$$= \frac{\sum_{i=1}^{n} \left\{E(XY) - 2E(XY)/n - 2(n-1)E(X)E(Y)/n + E(XY)/n + (n-1)E(X)E(Y)\right\}/n}{n-1}$$

$$= \sum_{i=1}^{n} \left[E(XY) - E(X)E(Y)\right]/n$$

$$= E(XY) - E(X)E(Y)$$

$$= \text{Cov}(X,Y)$$

4.30. For an exponential distribution with mean β,

$$P(X > t + s \mid X > t) = \frac{P(X > t + s)}{P(X > t)} = \frac{e^{-(t+s)/\beta}}{e^{-t/\beta}} = e^{-s/\beta} = P(X > s)$$

Therefore, the exponential distribution is memoryless.

4.31. Let m and n be nonnegative integers. Then

$$P(X \geq n+m \mid X \geq m) = \frac{P(X \geq n+m)}{P(X \geq m)} = \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = P(X \geq n)$$

Therefore, the geometric distribution has the memoryless property.

4.32. Let N = number of keys required to open the door (a random variable).

(*a*) $E(N) = 1\left(\frac{1}{k}\right) + 2\left(\frac{k-1}{k}\right)\left(\frac{1}{k-1}\right) + 3\left(\frac{k-1}{k}\right)\left(\frac{k-2}{k-1}\right)\left(\frac{1}{k-2}\right) + \cdots$

$$= \frac{1}{k}\sum_{i=1}^{k} i$$

$$= \left(\frac{1}{k}\right)\left(\frac{k(k+1)}{2}\right)$$

$$= \frac{k+1}{2}$$

(*b*) Let $X = \begin{cases} 1 & \text{if first key is successful} \\ 0 & \text{otherwise} \end{cases}$

$$E(N) = E(N \mid X = 1)(1/k) + E(N \mid X = 0)[(k-1)/k]$$

$$= 1(1/k) + [1 + E(N)][(k-1)/k]$$

$$= 1 + E(N)[(k-1)/k]$$

or

$$(1/k)E(N) = 1$$

or

$$E(N) = k$$

This answer can also be obtained by summing an infinite series.

4.33. No. See, for example, the beta distribution with $\alpha_1 = \alpha_2 = 0.5$ in Sec. 6.2.2.

4.34. The long-run throughput (departure rate) is equal to the arrival rate in both cases. Clearly, in both cases the long-run departure rate cannot be greater than the arrival rate. It also cannot be less than the arrival rate, because then the queue length would become infinitely large.