CHAPTER 13
Functions of Several Variables

C H A P T E R 1 3
Functions of Several Variables

Section 13.1 Introduction to Functions of Several Variables

1. No, it is not the graph of a function. For some values of x and y (for example, $(x, y) = (0, 0)$), there are 2 z-values.

2. Yes, it is the graph of a function.

3. $$x^2z + 3y^2 - xy = 10$$
$$x^2z = 10 + xy - 3y^2$$
$$z = \frac{10 + xy - 3y^2}{x^2}$$

Yes, z is a function of x and y.

4. $xz^2 + 2xy - y^2 = 4$

No, z is not a function of x and y. For example, $(x, y) = (1, 0)$ corresponds to both $z = \pm 2$.

5. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

No, z is not a function of x and y. For example, $(x, y) = (0, 0)$ corresponds to both $z = \pm 1$.

6. $$z + x \ln y - 8yz = 0$$
$$z(1 - 8y) = -x \ln y$$
$$z = \frac{x \ln y}{8y - 1}$$

Yes, z is a function of x and y.

7. $f(x, y) = xy$

(a) $f(3, 2) = 3(2) = 6$

(b) $f(-1, 4) = -1(4) = -4$

(c) $f(30, 5) = 30(5) = 150$

(d) $f(5, y) = 5y$

(e) $f(x, 2) = 2x$

(f) $f(5, t) = 5t$

8. $f(x, y) = 4 - x^2 - 4y^2$

(a) $f(0, 0) = 4$

(b) $f(0, 1) = 4 - 0 - 4 = 0$

(c) $f(2, 3) = 4 - 4 - 36 = -36$

(d) $f(1, y) = 4 - 1 - 4y^2 = 3 - 4y^2$

(e) $f(x, 0) = 4 - x^2 - 0 = 4 - x^2$

(f) $f(t, 1) = 4 - t^2 - 4 = -t^2$

9. $f(x, y) = xe^y$

(a) $f(5, 0) = 5e^0 = 5$

(b) $f(3, 2) = 3e^2$

(c) $f(2, -1) = 2e^{-1} = \dfrac{2}{e}$

(d) $f(5, y) = 5e^y$

(e) $f(x, 2) = xe^2$

(f) $f(t, t) = te^t$

10. $g(x, y) = \ln|x + y|$

(a) $g(1, 0) = \ln|1 + 0| = 0$

(b) $g(0, -1) = \ln|0 - 1| = \ln 1 = 0$

(c) $g(0, e) = \ln|0 + e| = 1$

(d) $g(1, 1) = \ln|1 + 1| = \ln 2$

(e) $g\left(e, \dfrac{e}{2}\right) = \ln\left|e + \dfrac{e}{2}\right| = \ln\left(\dfrac{3e}{2}\right) = \ln 3 + \ln e - \ln 2$
$= 1 + \ln 3 - \ln 2$

(f) $g(2, 5) = \ln|2 + 5| = \ln 7$

11. $h(x, y, z) = \dfrac{xy}{z}$

(a) $h(2, 3, 9) = \dfrac{2(3)}{9} = \dfrac{2}{3}$

(b) $h(1, 0, 1) = \dfrac{1(0)}{1} = 0$

(c) $h(-2, 3, 4) = \dfrac{(-2)(3)}{4} = -\dfrac{3}{2}$

(d) $h(5, 4, -6) = \dfrac{5(4)}{-6} = -\dfrac{10}{3}$

12. $f(x, y, z) = \sqrt{x + y + z}$

(a) $f(0, 5, 4) = \sqrt{0 + 5 + 4} = 3$

(b) $f(6, 8, -3) = \sqrt{6 + 8 - 3} = \sqrt{11}$

(c) $f(4, 6, 2) = \sqrt{4 + 6 + 2} = \sqrt{12} = 2\sqrt{3}$

(d) $f(10, -4, -3) = \sqrt{10 - 4 - 3} = \sqrt{3}$

13. $f(x, y) = x \sin y$

(a) $f\left(2, \dfrac{\pi}{4}\right) = 2 \sin \dfrac{\pi}{4} = \sqrt{2}$

(b) $f(3, 1) = 3 \sin(1)$

(c) $f\left(-3, \dfrac{\pi}{3}\right) = -3 \sin \dfrac{\pi}{3} = -3\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{-3\sqrt{3}}{2}$

(d) $f\left(4, \dfrac{\pi}{2}\right) = 4 \sin \dfrac{\pi}{2} = 4$

14. $V(r, h) = \pi r^2 h$

(a) $V(3, 10) = \pi(3^2)10 = 90\pi$

(b) $V(5, 2) = \pi(5^2)2 = 50\pi$

(c) $V(4, 8) = \pi(4^2)8 = 128\pi$

(d) $V(6, 4) = \pi(6^2)4 = 144\pi$

15. $g(x, y) = \int_x^y (2t - 3)\, dt$

$$= \left[t^2 - 3t\right]_x^y = y^2 - 3y - x^2 + 3x$$

(a) $g(4, 0) = 0 - 16 + 12 = -4$

(b) $g(4, 1) = (1 - 3) - 16 + 12 = -6$

(c) $g\left(4, \frac{3}{2}\right) = \left(\frac{9}{4} - \frac{9}{2}\right) - 16 + 12 = -\frac{25}{4}$

(d) $g\left(\frac{3}{2}, 0\right) = 0 - \frac{9}{4} + \frac{9}{2} = \frac{9}{4}$

16. $g(x, y) = \int_x^y \dfrac{1}{t}\, dt = \ln|t|\Big]_x^y = \ln|y| - \ln|x| = \ln\left|\dfrac{y}{x}\right|$

(a) $g(4, 1) = \ln \dfrac{1}{4} = -\ln 4$

(b) $g(6, 3) = \ln \dfrac{3}{6} = -\ln 2$

(c) $g(2, 5) = \ln \dfrac{5}{2}$

(d) $g\left(\dfrac{1}{2}, 7\right) = \ln \dfrac{7}{\left(\frac{1}{2}\right)} = \ln 14$

17. $f(x, y) = 2x + y^2$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \dfrac{2(x + \Delta x) + y^2 - (2x + y^2)}{\Delta x} = \dfrac{2\Delta x}{\Delta x} = 2, \Delta x \neq 0$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \dfrac{2x + (y + \Delta y)^2 - 2x - y^2}{\Delta y} = \dfrac{2y\Delta y + (\Delta y^2)}{\Delta y} = 2y + \Delta y, \Delta y \neq 0$

18. $f(x, y) = 3x^2 - 2y$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \dfrac{3(x + \Delta x)^2 - 2y - (3x^2 - 2y)}{\Delta x} = \dfrac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = 6x + 3\Delta x,\ \Delta x \neq 0$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \dfrac{3x^2 - 2(y + \Delta y) - (3x^2 - 2y)}{\Delta y} = \dfrac{-2\Delta y}{\Delta y} = -2, \Delta y \neq 0$

19. $f(x, y) = x^2 + y^2$

Domain:
$\{(x, y)\colon\ x \text{ is any real number, } y \text{ is any real number}\}$

Range: $z \geq 0$

20. $f(x, y) = e^{xy}$

Domain: Entire xy-plane

Range: $z > 0$

21. $g(x, y) = x\sqrt{y}$

Domain: $\{(x, y): y \geq 0\}$

Range: all real numbers

22. $f(x, y) = \dfrac{y}{\sqrt{x}}$

Domain: $\{(x, y): x > 0\}$

Range: all real numbers

23. $z = \dfrac{x + y}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers

24. $z = \dfrac{xy}{x - y}$

Domain: $\{(x, y): x \neq y\}$

Range: all real numbers

25. $f(x, y) = \sqrt{4 - x^2 - y^2}$

Domain: $4 - x^2 - y^2 \geq 0$

$$x^2 + y^2 \leq 4$$

$$\{(x, y): x^2 + y^2 \leq 4\}$$

Range: $0 \leq z \leq 2$

26. $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

Domain: $4 - x^2 - 4y^2 \geq 0$

$$x^2 + 4y^2 \leq 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} \leq 1$$

$$\left\{(x, y): \frac{x^2}{4} + \frac{y^2}{1} \leq 1\right\}$$

Range: $0 \leq z \leq 2$

27. $f(x, y) = \arccos(x + y)$

Domain: $\{(x, y): -1 \leq x + y \leq 1\}$

Range: $0 \leq z \leq \pi$

28. $f(x, y) = \arcsin\left(\dfrac{y}{x}\right)$

Domain: $\left\{(x, y): -1 \leq \dfrac{y}{x} \leq 1\right\}$

Range: $-\dfrac{\pi}{2} \leq z \leq \dfrac{\pi}{2}$

29. $f(x, y) = \ln(4 - x - y)$

Domain: $4 - x - y > 0$

$$x + y < 4$$

$$\{(x, y): y < -x + 4\}$$

Range: all real numbers

30. $f(x, y) = \ln(xy - 6)$

Domain: $xy - 6 > 0$

$$xy > 6$$

$$\{(x, y): xy > 6\}$$

Range: all real numbers

31. $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$

(a) View from the positive x-axis: $(20, 0, 0)$

(b) View where x is negative, y and z are positive: $(-15, 10, 20)$

(c) View from the first octant: $(20, 15, 25)$

(d) View from the line $y = x$ in the xy-plane: $(20, 20, 0)$

32. (a) Domain:

$\{(x, y): x \text{ is any real number}, y \text{ is any real number}\}$

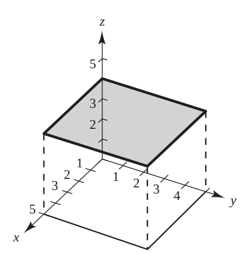
Range: $-2 \leq z \leq 2$

(b) $z = 0$ when $x = 0$ which represents points on the y-axis.

(c) No. When x is positive, z is negative. When x is negative, z is positive. The surface does not pass through the first octant, the octant where y is negative and x and z are positive, the octant where y is positive and x and z are negative, and the octant where x, y and z are all negative.

33. $f(x, y) = 4$

Plane: $z = 4$

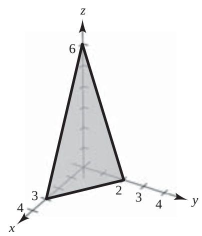


34. $f(x, y) = 6 - 2x - 3y$

Plane

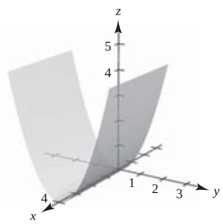
Domain: entire xy-plane

Range: $-\infty < z < \infty$



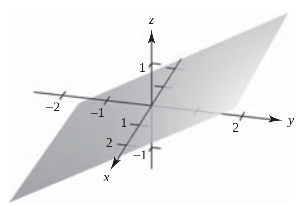
35. $f(x, y) = y^2$

Because the variable x is missing, the surface is a cylinder with rulings parallel to the x-axis. The generating curve is $z = y^2$. The domain is the entire xy-plane and the range is $z \geq 0$.



36. $g(x, y) = \frac{1}{2}y$

Plane: $z = \frac{1}{2}y$

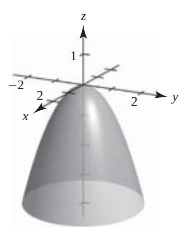


37. $z = -x^2 - y^2$

Paraboloid

Domain: entire xy-plane

Range: $z \leq 0$

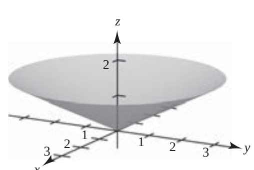


38. $z = \frac{1}{2}\sqrt{x^2 + y^2}$

Cone

Domain of f: entire xy-plane

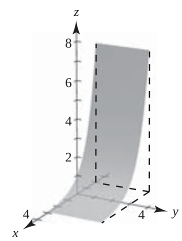
Range: $z \geq 0$



39. $f(x, y) = e^{-x}$

Because the variable y is missing, the surface is a cylinder with rulings parallel to the y-axis. The generating curve is $z = e^{-x}$.

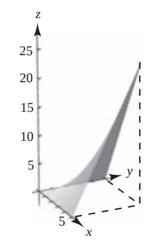
The domain is the entire xy-plane and the range is $z > 0$.



40. $f(x, y) = \begin{cases} xy, & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

Domain of f: entire xy-plane

Range: $z \geq 0$

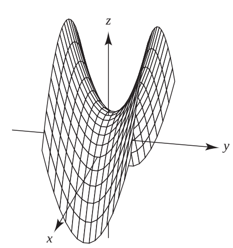


41. $z = y^2 - x^2 + 1$

Hyperbolic paraboloid

Domain: entire xy-plane

Range: $-\infty < z < \infty$



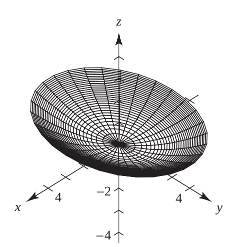
42. $f(x, y) = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

Semi-ellipsoid

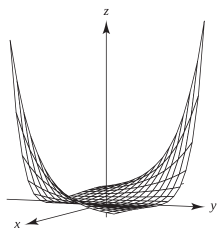
Domain: set of all points lying on or inside the ellipse

$\left(\frac{x^2}{9}\right) + \left(\frac{y^2}{16}\right) = 1$

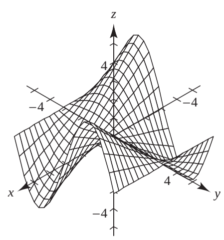
Range: $0 \leq z \leq 1$



43. $f(x, y) = x^2 e^{(-xy/2)}$



44. $f(x, y) = x \sin y$



45. $z = e^{1-x^2-y^2}$

Level curves:

$$c = e^{1-x^2-y^2}$$
$$\ln c = 1 - x^2 - y^2$$
$$x^2 + y^2 = 1 - \ln c$$

Circles centered at $(0, 0)$

Matches (c)

46. $z = e^{1-x^2+y^2}$

Level curves:

$$c = e^{1-x^2+y^2}$$
$$\ln c = 1 - x^2 + y^2$$
$$x^2 - y^2 = 1 - \ln c$$

Hyperbolas centered at $(0, 0)$

Matches (d)

47. $z = \ln\left|y - x^2\right|$

Level curves:

$$c = \ln\left|y - x^2\right|$$
$$\pm e^c = y - x^2$$
$$y = x^2 \pm e^c$$

Parabolas

Matches (b)

48. $z = \cos\left(\dfrac{x + 2y^2}{4}\right)$

Level curves:

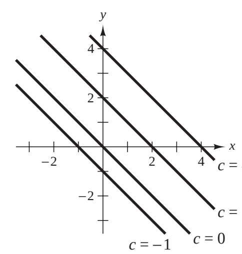
$$c = \cos\left(\frac{x^2 + 2y^2}{4}\right)$$
$$\cos^{-1} c = \frac{x^2 + 2y^2}{4}$$
$$x^2 + 2y^2 = 4\cos^{-1} c$$

Ellipses

Matches (a)

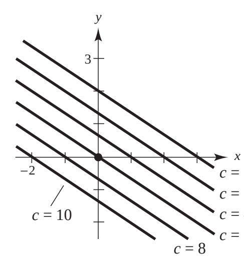
49. $z = x + y$

Level curves are parallel lines of the form $x + y = c$.



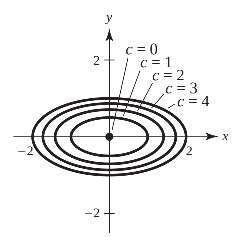
50. $f(x, y) = 6 - 2x - 3y$

The level curves are of the form $6 - 2x - 3y = c$ or $2x + 3y = 6 - c$. So, the level curves are straight lines with a slope of $-\frac{2}{3}$.



51. $z = x^2 + 4y^2$

The level curves are ellipses of the form $x^2 + 4y^2 = c$ (except $x^2 + 4y^2 = 0$ is the point $(0, 0)$).

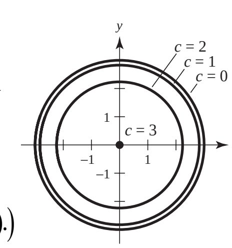


52. $f(x, y) = \sqrt{9 - x^2 - y^2}$

The level curves are of the form

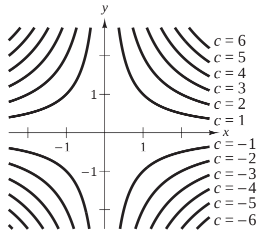
$$c = \sqrt{9 - x^2 - y^2}$$
$$x^2 + y^2 = 9 - c^2, \text{ circles.}$$

$\left(x^2 + y^2 = 0 \text{ is the point } (0, 0).\right)$



53. $f(x, y) = xy$

The level curves are hyperbolas of the form $xy = c$.

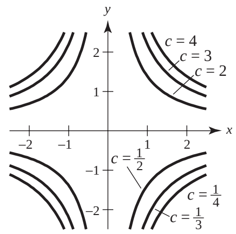


54. $f(x, y) = e^{xy/2}$

The level curves are of the form

$e^{xy/2} = c$, or $\ln c = \dfrac{xy}{2}$.

So, the level curves are hyperbolas.



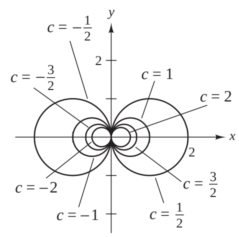
55. $f(x, y) = \dfrac{x}{x^2 + y^2}$

The level curves are of the form

$$c = \frac{x}{x^2 + y^2}$$

$$x^2 - \frac{x}{c} + y^2 = 0$$

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2.$$



So, the level curves are circles passing through the origin and centered at $(\pm 1/2c, 0)$.

56. $f(x, y) = \ln(x - y)$

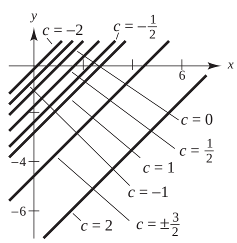
The level curves are of the form

$c = \ln(x - y)$

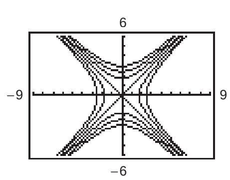
$e^c = x - y$

$y = x - e^c$.

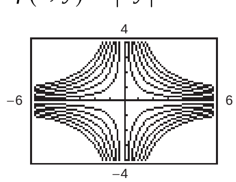
So, the level curves are parallel lines of slope 1 passing through the fourth quadrant.



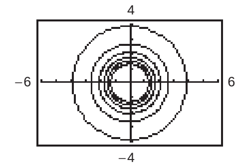
57. $f(x, y) = x^2 - y^2 + 2$



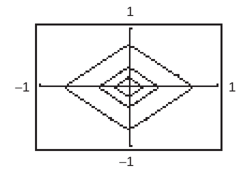
58. $f(x, y) = |xy|$



59. $g(x, y) = \dfrac{8}{1 + x^2 + y^2}$



60. $h(x, y) = 3\sin(|x| + |y|)$



61. The graph of a function of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. The graph can be interpreted as a surface in space. Level curves are the scalar fields $f(x, y) = c$, where c is a constant.

62. No, the following graphs are not hemispheres.

$z = e^{-(x^2 + y^2)}$

$z = x^2 + y^2$

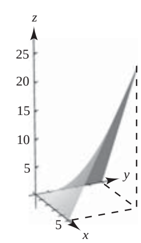
63. $f(x, y) = \dfrac{x}{y}$

The level curves are the lines $c = \dfrac{x}{y}$ or $y = \dfrac{1}{c}x$.

These lines all pass through the origin.

64. $f(x, y) = xy, x \ge 0, y \ge 0$

(a)

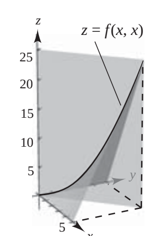


(b) g is a vertical translation of f three units downward.

(c) g is a reflection of f in the xy-plane.

(d) The graph of g is lower than the graph of f. If $z = f(x, y)$ is on the graph of f, then $\frac{1}{2}z$ is on the graph of g.

(e)



65. The surface is sloped like a saddle. The graph is not unique. Any vertical translation would have the same level curves.

One possible function is

$f(x, y) = |xy|$.

66. The surface could be an ellipsoid centered at $(0, 1, 0)$.

One possible function is

$$f(x, y) = x^2 + \frac{(y-1)^2}{4} - 1.$$

67. $V(I, R) = 1000\left[\dfrac{1 + 0.06(1 - R)}{1 + I}\right]^{10}$

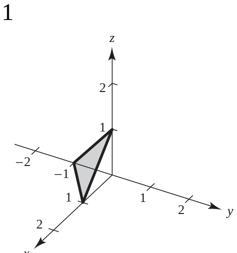
	Inflation Rate		
Tax Rate	0	0.03	0.05
0	1790.85	1332.56	1099.43
0.28	1526.43	1135.80	937.09
0.35	1466.07	1090.90	900.04

68. $A(r, t) = 5000e^{rt}$

	Number of Year			
Rate	5	10	15	20
0.02	5525.85	6107.01	6749.29	7459.12
0.03	5809.17	6749.29	7841.56	9110.59
0.04	6107.01	7459.12	9110.59	11,127.70
0.05	6420.13	8243.61	10,585.00	13,591.41

69. $f(x, y, z) = x - y + z, c = 1$

$1 = x - y + z$, Plane

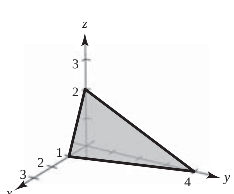


70. $f(x, y, z) = 4x + y + 2z$

$c = 4$

$4 = 4x + y + 2z$

Plane

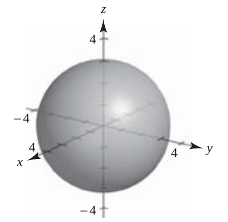


71. $f(x, y, z) = x^2 + y^2 + z^2$

$c = 9$

$9 = x^2 + y^2 + z^2$

Sphere



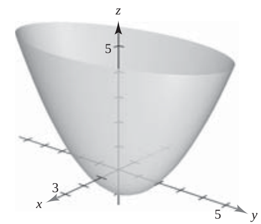
72. $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z$

$c = 1$

$1 = x^2 + \frac{1}{4}y^2 - z$

Elliptic paraboloid

Vertex: $(0, 0, -1)$

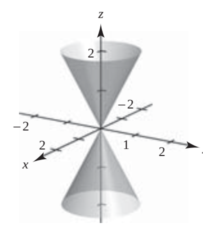


73. $f(x, y, z) = 4x^2 + 4y^2 - z^2$

$c = 0$

$0 = 4x^2 + 4y^2 - z^2$

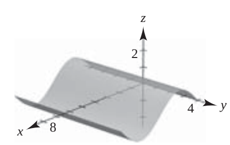
Elliptic cone



74. $f(x, y, z) = \sin x - z$

$c = 0$

$0 = \sin x - z$ or $z = \sin x$



75. $N(d, L) = \left(\dfrac{d-4}{4}\right)^2 L$

(a) $N(22, 12) = \left(\dfrac{22-4}{4}\right)^2(12) = 243$ board-feet

(b) $N(30, 12) = \left(\dfrac{30-4}{4}\right)^2(12) = 507$ board-feet

76. $w = \dfrac{1}{x - y}, \; y < x$

(a) $w(15, 9) = \dfrac{1}{15 - 9} = \dfrac{1}{6}\text{h} = 10$ min

(b) $w(15, 13) = \dfrac{1}{15 - 13} = \dfrac{1}{2}\text{h} = 30$ min

(c) $w(12, 7) = \dfrac{1}{12 - 7} = \dfrac{1}{5}\text{h} = 12$ min

(d) $w(5, 2) = \dfrac{1}{5 - 2} = \dfrac{1}{3}\text{h} = 20$ min

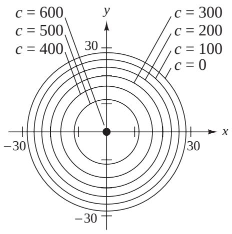
77. $T = 600 - 0.75x^2 - 0.75y^2$

The level curves are of the form

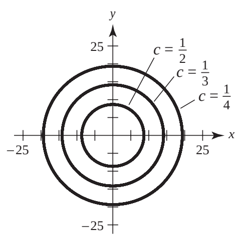
$$c = 600 - 0.75x^2 - 0.75y^2$$

$$x^2 + y^2 = \frac{600 - c}{0.75}.$$

The level curves are circles centered at the origin.



78. $V(x, y) = \dfrac{5}{\sqrt{25 + x^2 + y^2}}$



79.

$$\begin{aligned} f(x, y) &= 100x^{0.6}y^{0.4} \\ f(2x, 2y) &= 100(2x)^{0.6}(2y)^{0.4} \\ &= 100(2)^{0.6}x^{0.6}(2)^{0.4}y^{0.4} \\ &= 100(2)^{0.6}(2)^{0.4}x^{0.6}y^{0.4} \\ &= 2\left[100x^{0.6}y^{0.4}\right] = 2f(x, y) \end{aligned}$$

80. $z = Cx^a y^{1-a}$

$$\ln z = \ln C + a \ln x + (1 - a) \ln y$$

$$\ln z - \ln y = \ln C + a \ln x - a \ln y$$

$$\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}$$

81. $PV = kT$

(a) $26(2000) = k(300) \Rightarrow k = \dfrac{520}{3}$

(b) $P = \dfrac{kT}{V} = \dfrac{520}{3}\left(\dfrac{T}{V}\right)$

The level curves are of the form

$c = \dfrac{520}{3}\left(\dfrac{T}{V}\right)$, or $V = \dfrac{520}{3c}T$.

These are lines through the origin with slope $\dfrac{520}{3c}$.

82. $z = f(x, y) = 0.035x + 0.640y - 1.77$

(a)

Year	2006	2007	2008	2009	2010	2011
z	10.0	14.5	22.3	31.6	47.8	76.6
Model	9.9	15.0	22.7	30.1	48.6	76.5

(b) y has the greater influence because its coefficient (0.640) is greater than the coefficient of x (0.035).

(c) $$\begin{aligned} f(x, 150) &= 0.035x + 0.640(150) - 1.77 \\ &= 0.035x + 94.23 \end{aligned}$$

This gives the shareholder’s equity z in terms of net sales x, assuming total assets of \$150 billion.

83. (a) Highest pressure at C

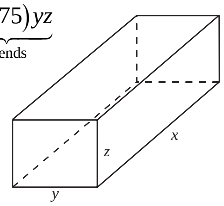
(b) Lowest pressure at A

(c) Highest wind velocity at B

84. Southwest

85. $C = \underbrace{1.20xy}_{\text{base}} + \underbrace{2(0.75)xz}_{\text{front and back}} + \underbrace{2(0.75)yz}_{\text{2 ends}}$

$= 1.20xy + 1.50(xz + yz)$



86. (a) No; the level curves are uneven and sporadically spaced.

(b) Use more colors.

87. False. Let

$f(x, y) = 2xy$

$f(1, 2) = f(2, 1)$, but $1 \neq 2$.

88. False. Let

$f(x, y) = 5$.

Then, $f(2x, 2y) = 5 \neq 2^2 f(x, y)$.

89. True

90. False. If there were a point (x, y) on the level curves $f(x, y) = C_1$ and $f(x, y) = C_2$, then $C_1 = C_2$.

91. We claim that $g(x) = f(x, 0)$. First note that $x = y = z = 0$ implies $3f(0, 0) = 0 \Rightarrow f(0, 0) = 0$.

Letting $y = z = 0$ implies $f(x, 0) + f(0, 0) + f(0, x) = 0 \Rightarrow -f(0, x) = f(x, 0)$.

Letting $z = 0$ implies $f(x, y) + f(y, 0) + f(0, x) = 0 \Rightarrow f(x, y) = -f(y, 0) - f(0, x) = f(x, 0) - f(y, 0)$.

Hence, $f(x, y) = g(x) - g(y)$, as desired.

Section 13.2 Limits and Continuity

1. $\lim_{(x, y)\to(1, 0)} x = 1$

$f(x, y) = x, L = 1$

We need to show that for all $\varepsilon > 0$, there exists a δ-neighborhood about $(1, 0)$ such that

$|f(x, y) - L| = |x - 1| < \varepsilon$

Whenever $(x, y) \neq (1, 0)$ lies in the neighborhood.

From $0 < \sqrt{(x-1)^2 + (y-0)^2} < \delta$, it follows that

$|x - 1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2 + (y-0)^2} < \delta$.

So, choose $\delta = \varepsilon$ and the limit is verified.

2. $\lim_{(x, y)\to(4, -1)} x = 4$

Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that

$|f(x, y) - L| = |x - 4| < \varepsilon$

whenever

$0 < \sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(x-4)^2 + (y+1)^2} < \delta$.

Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x-4)^2 + (y+1)^2} < \delta = \varepsilon$, we have

$\sqrt{(x-4)^2} < \varepsilon$

$|x - 4| < \varepsilon$.

3. $\lim_{(x, y)\to(1, -3)} y = -3$. $f(x, y) = y$, $L = -3$

We need to show that for all $\varepsilon > 0$, there exists a δ-neighborhood about $(1, -3)$ such that

$|f(x, y) - L| = |y + 3| < \varepsilon$

whenever $(x, y) \neq (1, -3)$ lies in the neighborhood.

From $0 < \sqrt{(x-1)^2 + (y+3)^2} < \delta$ it follows that

$|y + 3| = \sqrt{(y+3)^2} \leq \sqrt{(x-1)^2 + (y+3)^2} < \delta$.

So, choose $\delta = \varepsilon$ and the limit is verified.

4. $\lim_{(x, y)\to(a, b)} y = b$

Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |y - b| < \varepsilon$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$, we have

$\sqrt{(y-b)^2} < \varepsilon$

$|y - b| < \varepsilon$.

5. $\displaystyle\lim_{(x,y)\to(a,b)}\left[f(x,y)-g(x,y)\right] = \lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} g(x,y) = 4 - 3 = 1$

6. $\displaystyle\lim_{(x,y)\to(a,b)}\left[\frac{5f(x,y)}{g(x,y)}\right] = \frac{5\left[\lim_{(x,y)\to(a,b)} f(x,y)\right]}{\lim_{(x,y)\to(a,b)} g(x,y)} = \frac{5(4)}{3} = \frac{20}{3}$

7. $\displaystyle\lim_{(x,y)\to(a,b)}\left[f(x,y)g(x,y)\right] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim_{(x,y)\to(a,b)} g(x,y)\right] = 4(3) = 12$

8. $\displaystyle\lim_{(x,y)\to(a,b)}\left[\frac{f(x,y)+g(x,y)}{f(x,y)}\right] = \frac{\lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)}{\lim_{(x,y)\to(a,b)} f(x,y)} = \frac{4+3}{4} = \frac{7}{4}$

9. $\displaystyle\lim_{(x,y)\to(2,1)}\left(2x^2 + y\right) = 8 + 1 = 9$

Continuous everywhere

10. $\displaystyle\lim_{(x,y)\to(0,0)}\left(x + 4y + 1\right) = 0 + 4(0) + 1 = 1$

Continuous everywhere

11. $\displaystyle\lim_{(x,y)\to(1,2)} e^{xy} = e^{1(2)} = e^2$

Continuous everywhere

12. $\displaystyle\lim_{(x,y)\to(2,4)} \frac{x+y}{x^2+1} = \frac{2+4}{2^2+1} = \frac{6}{5}$

Continuous everywhere

13. $\displaystyle\lim_{(x,y)\to(0,2)} \frac{x}{y} = \frac{0}{2} = 0$

Continuous for all $y \neq 0$

14. $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{x+y}{x-y} = \frac{-1+2}{-1-2} = -\frac{1}{3}$

Continuous for all $x \neq y$.

15. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$

16. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x}{\sqrt{x+y}} = \frac{1}{\sqrt{1+1}} = \frac{\sqrt{2}}{2}$

Continuous for $x + y > 0$

17. $\displaystyle\lim_{(x,y)\to(\pi/4,2)} y\cos(xy) = 2\cos\frac{\pi}{2} = 0$

Continuous everywhere

18. $\displaystyle\lim_{(x,y)\to(2\pi,4)} \sin\frac{x}{y} = \sin\frac{2\pi}{4} = 1$

Continuous for all $y \neq 0$

19. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{\arcsin xy}{1-xy} = \frac{\arcsin 0}{1} = 0$

Continuous for $xy \neq 1, |xy| \leq 1$

20. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{\arccos\left(\frac{x}{y}\right)}{1+xy} = \frac{\arccos 0}{1} = \frac{\pi}{2}$

Continuous for $xy \neq -1,\ y \neq 0,\ 0 \leq \dfrac{x}{y} \leq \pi$

21. $\displaystyle\lim_{(x,y,z)\to(1,3,4)} \sqrt{x+y+z} = \sqrt{1+3+4} = 2\sqrt{2}$

Continuous for $x + y + z \geq 0$

22. $\displaystyle\lim_{(x,y,z)\to(-2,1,0)} xe^{yz} = (-2)e^{1(0)} = -2$

Continuous everywhere

23. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy-1}{1+xy} = \frac{1-1}{1+1} = 0$

24. $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2y}{1+xy^2} = \frac{-1}{1+1} = -\frac{1}{2}$

25. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x+y}$ does not exist

Because the denominator $x + y$ approaches 0 as $(x, y) \to (0, 0)$.

26. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x^2y^2}$ does not exist because the denominator xy approaches 0 as $(x, y) \to (0, 0)$.

27. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}}$

does not exist because you can't approach $(0, 0)$ from negative values of x and y.

28. $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}\cdot\frac{\sqrt{x-y}+1}{\sqrt{x-y}+1}$

$$= \lim_{(x,y)\to(2,1)} \frac{(x-y-1)\left(\sqrt{x-y}+1\right)}{(x-y)-1}$$

$$= \lim_{(x,y)\to(2,1)} \left(\sqrt{x-y}+1\right) = 2$$

29. The limit does not exist because along the line $y = 0$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y} = \lim_{(x,0)\to(0,0)} \frac{x}{x^2} = \lim_{(x,0)\to(0,0)} \frac{1}{x}$$

which does not exist.

30. The limit does not exist because along the line $x = y$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2} = \lim_{(x,x)\to(0,0)} \frac{x}{x^2-x^2} = \lim_{(x,x)\to(0,0)} \frac{x}{0}.$$

Because the denominator is 0, the limit does not exist.

31. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+1)(y^2+1)} = \frac{0}{(1)(1)} = 0$

32. $\displaystyle\lim_{(x,y)\to(0,0)} \ln(x^2+y^2)$ does not exist

because $\ln(x^2+y^2) \to -\infty$ as $(x,y) \to (0,0)$.

33. The limit does not exist because along the path $x = 0$, $y = 0$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(0,0,z)\to(0,0,0)} \frac{0}{z^2} = 0$$

whereas along the path $x = y = z$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(x,x,x)\to(0,0,0)} \frac{x^2+x^2+x^2}{x^2+x^2+x^2} = 1$$

34. The limit does not exist because along the path $y = z = 0$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim_{(x,0,0)\to(0,0,0)} \frac{0}{x^2} = 0$$

However, along the path $z = 0$, $x = y$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim_{(x,x,0)\to(0,0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

35. $\displaystyle\lim_{(x,y)\to(0,0)} e^{xy} = 1$

Continuous everywhere

36. $\displaystyle\lim_{(x,y)\to(0,0)} \left[1-\frac{\cos(x^2+y^2)}{x^2+y^2}\right] = -\infty$

The limit does not exist.

Continuous except at $(0,0)$

37. $f(x,y) = \dfrac{xy}{x^2+y^2}$

Continuous except at $(0,0)$

Path: $y = 0$

(x,y)	$(1,0)$	$(0.5,0)$	$(0.1,0)$	$(0.01,0)$	$(0.001,0)$
$f(x,y)$	0	0	0	0	0

Path: $y = x$

(x,y)	$(1,1)$	$(0.5,0.5)$	$(0.1,0.1)$	$(0.01,0.01)$	$(0.001,0.001)$
$f(x,y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function equals $\frac{1}{2}$.

38. $f(x, y) = -\dfrac{xy^2}{x^2 + y^4}$

Continuous except at $(0, 0)$

Path: $x = y^2$

(x, y)	$(1, 1)$	$(0.25, 0.5)$	$(0.01, 0.1)$	$(0.0001, 0.01)$	$(0.000001, 0.001)$
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Path: $x = -y^2$

(x, y)	$(-1, 1)$	$(-0.25, 0.5)$	$(-0.01, 0.1)$	$(-0.0001, 0.01)$	$(-0.000001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path $x = y^2$ the function equals $-\frac{1}{2}$, whereas along the path $x = -y^2$ the function equals $\frac{1}{2}$.

39. $f(x, y) = \dfrac{y}{x^2 + y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

Path: $y = x$

(x, y)	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	1	5	50	500

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function tends to infinity.

40. $f(x, y) = \dfrac{2x - y^2}{2x^2 + y}$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	$(1, 0)$	$(0.25, 0)$	$(0.01, 0)$	$(0.001, 0)$	$(0.000001, 0)$
$f(x, y)$	1	4	100	1000	1,000,000

Path: $y = x$

(x, y)	$(1, 1)$	$(0.25, 0.25)$	$(0.01, 0.01)$	$(0.001, 0.001)$	$(0.0001, 0.0001)$
$f(x, y)$	$\frac{1}{3}$	1.17	1.95	1.995	2.0

The limit does not exist because along the line $y = 0$ the function tends to infinity, whereas along the line $y = x$ the function tends to 2.

41. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2 + y^2)(x^2 - y^2)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} (x^2 - y^2) = 0$

So, f is continuous everywhere, whereas g is continuous everywhere except at $(0, 0)$. g has a removable discontinuity at $(0, 0)$.

42. $$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \left(\frac{x^2 + 2xy^2 + y^2}{x^2 + y^2}\right)$$
$$= \lim_{(x,y)\to(0,0)} \left(1 + \frac{2xy^2}{x^2 + y^2}\right) = 1$$

(same limit for g)

So, f is not continuous at $(0, 0)$, whereas g is continuous at $(0, 0)$.

43. $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{r\to 0} \frac{(r\cos\theta)(r^2\sin^2\theta)}{r^2}$$
$$= \lim_{r\to 0}(r\cos\theta\sin^2\theta) = 0$$

44. $$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2}$$
$$= \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$$

45. $$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2 + y^2} = \lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^2}$$
$$= \lim_{r\to 0} r^2\cos^2\theta\sin^2\theta = 0$$

46. $x = r\cos\theta,\ y = r\sin\theta,\ \sqrt{x^2 + y^2} = r,\ x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{r\to 0} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r} = \lim_{r\to 0} r(\cos^2\theta - \sin^2\theta) = 0$$

47. $\displaystyle\lim_{(x,y)\to(0,0)} \cos(x^2 + y^2) = \lim_{r\to 0}\cos(r^2) = \cos(0) = 1$

48. $\displaystyle\lim_{(x,y)\to(0,0)} \sin\sqrt{x^2 + y^2} = \lim_{r\to 0}\sin(r) = \sin(0) = 0$

49. $\sqrt{x^2 + y^2} = r$

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = \lim_{r\to 0^+} \frac{\sin(r)}{r} = 1$$

50. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r\to 0}\frac{\sin r^2}{r^2} = \lim_{r\to 0}\frac{2r\cos r^2}{2r} = \lim_{r\to 0}\cos r^2 = 1$

51. $x^2 + y^2 = r^2$

$$\lim_{(x,y)\to(0,0)} \frac{1 - \cos(x^2 + y^2)}{x^2 + y^2} = \lim_{x\to 0} \frac{1 - \cos(r^2)}{r^2} = 0$$

52. $x^2 + y^2 = r^2$

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2) = \lim_{r\to 0} r^2 \ln(r^2) = \lim_{r\to 0^+} 2r^2 \ln(r)$$

By L'Hôpital's Rule, $\lim_{r\to 0^+} 2r^2 \ln(r) = \lim_{r\to 0^+} \frac{2\ln(r)}{1/r^2} = \lim_{r\to 0^+} \frac{2/r}{-2/r^3} = \lim_{r\to 0^+}(-r^2) = 0$

53. $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

Continuous except at $(0, 0, 0)$

54. $f(x, y, z) = \dfrac{z}{x^2 + y^2 - 4}$

Continuous for $x^2 + y^2 \neq 4$.

55. $f(x, y, z) = \dfrac{\sin z}{e^x + e^y}$

Continuous everywhere

56. $f(x, y, z) = xy \sin z$

Continuous everywhere

57. For $xy \neq 0$, the function is clearly continuous.

For $xy \neq 0$, let $z = xy$. Then

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

implies that f is continuous for all x, y.

58. For $x^2 \neq y^2$, the function is clearly continuous.

For $x^2 \neq y^2$, let $z = x^2 - y^2$. Then

$$\lim_{z\to 0} \frac{\sin(z)}{z} = 1$$

implies that f is continuous for all x, y.

59. $f(t) = t^2,\ g(x, y) = 2x - 3y$

$f(g(x, y)) = f(2x - 3y) = (2x - 3y)^2$

Continuous everywhere

60. $f(t) = \dfrac{1}{t}$

$g(x, y) = x^2 + y^2$

$f(g(x, y)) = f(x^2 + y^2) = \dfrac{1}{x^2 + y^2}$

Continuous except at (0, 0)

61. $f(t) = \dfrac{1}{t}, g(x, y) = 2x - 3y$

$f(g(x, y)) = f(2x - 3y) = \dfrac{1}{2x - 3y}$

Continuous for all $y \neq \dfrac{2}{3}x$

62. $f(t) = \dfrac{1}{1 - t}, g(x, y) = x^2 + y^2$

$f(g(x, y)) = f(x^2 + y^2) = \dfrac{1}{1 - x^2 - y^2}$

Continuous for $x^2 + y^2 \neq 1$

63. $f(x, y) = x^2 - 4y$

(a) $\lim_{\Delta x\to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{\left[(x + \Delta x)^2 - 4y\right] - (x^2 - 4y)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0}(2x + \Delta x) = 2x$

(b) $\lim_{\Delta y\to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0} \dfrac{\left[x^2 - 4(y + \Delta y)\right] - (x^2 - 4y)}{\Delta y} = \lim_{\Delta y\to 0} \dfrac{-4\Delta y}{\Delta y} = \lim_{\Delta y\to 0}(-4) = -4$

64. $f(x, y) = x^2 + y^2$

(a) $\lim_{\Delta x\to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{\left[(x + \Delta x)^2 + y^2\right] - (x^2 + y^2)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0}(2x + \Delta x) = 2x$

(b) $\lim_{\Delta y\to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0} \dfrac{\left[x^2 + (y + \Delta y)^2\right] - (x^2 + y^2)}{\Delta y} = \lim_{\Delta y\to 0} \dfrac{2y\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y\to 0}(2y + \Delta y) = 2y$

65. $f(x, y) = \dfrac{x}{y}$

(a) $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{x + \Delta x}{y} - \dfrac{x}{y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{\Delta x}{y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{y} = \frac{1}{y}$$

(b) $$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{x}{y + \Delta y} - \dfrac{x}{y}}{\Delta y} = \lim_{\Delta y \to 0} \frac{xy - (xy + x\Delta y)}{(y + \Delta y)y\Delta y} = \lim_{\Delta y \to 0} \frac{-x\Delta y}{(y + \Delta y)y\Delta y} = \lim_{\Delta y \to 0} \frac{-x}{(y + \Delta y)y} = \frac{-x}{y^2}$$

66. $f(x, y) = \dfrac{1}{x + y}$

(a) $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + y} - \dfrac{1}{x + y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + y) - (x + \Delta x + y)}{(x + \Delta x + y)(x + y)\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + y)(x + y)\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

(b) By symmetry, $\displaystyle\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{-1}{(x + y)^2}$.

67. $f(x, y) = 3x + xy - 2y$

(a) $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x + \Delta x) + (x + \Delta x)y - 2y - (3x + xy - 2y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x + y\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (3 + y) = 3 + y$$

(b) $$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{3x + x(y + \Delta y) - 2(y + \Delta y) - (3x + xy - 2y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x\Delta y - 2\Delta y}{\Delta y} = \lim_{\Delta y \to 0} (x - 2) = x - 2$$

68. $f(x, y) = \sqrt{y}(y + 1)$

(a) $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{y}(y + 1) - \sqrt{y}(y + 1)}{\Delta x} = 0$$

(b) $$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{3/2} + (y + \Delta y)^{1/2} - (y^{3/2} + y^{1/2})}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{3/2} - y^{3/2}}{\Delta y} + \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{1/2} - y^{1/2}}{\Delta y}$$

$$= \frac{3}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \quad \text{(L'Hôpital's Rule)}$$

$$= \frac{3y + 1}{2\sqrt{y}}$$

69. True. Assuming $f(x, 0)$ exists for $x \neq 0$.

70. False. Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

See Exercise 37.

71. False. Let $f(x, y) = \begin{cases} \ln(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & x = 0, y = 0 \end{cases}$.

72. True

73. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{xy}$

(a) Along $y = ax$:

$$\lim_{(x,ax)\to(0,0)} \frac{x^2 + (ax)^2}{x(ax)} = \lim_{x\to 0} \frac{x^2(1 + a^2)}{ax^2} = \frac{1 + a^2}{a}, a \neq 0$$

If $a = 0$, then $y = 0$ and the limit does not exist.

(b) Along

$$y = x^2: \lim_{(x,x^2)\to(0,0)} \frac{x^2 + (x^2)^2}{x(x^2)} = \lim_{x\to 0} \frac{1 + x^2}{x}$$

Limit does not exist.

(c) No, the limit does not exist. Different paths result in different limits.

74. $f(x, y) = \dfrac{x^2 y}{x^4 + y^2}$

(a) $y = ax$: $f(x, ax) = \dfrac{x^2(ax)}{x^4 + (ax)^2} = \dfrac{ax}{x^2 + a^2}$

If $a \neq 0$, $\displaystyle\lim_{(x,ax)\to(0,0)} \frac{ax}{x^2 + a^2} = 0$.

(b) $y = x^2$: $f(x, x^2) = \dfrac{x^2(x^2)}{x^4 + (x^2)^2} = \dfrac{x^4}{2x^4}$

$$\lim_{(x,x^2)} \frac{x^4}{2x^4} = \frac{1}{2}$$

(c) No, the limit does not exist. f approaches different numbers along different paths.

75. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^2 + z^2} = \lim_{\rho\to 0^+} \frac{(\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi)}{\rho^2} = \lim_{\rho\to 0^+} \rho\left[\sin^2\phi \cos\theta \sin\theta \cos\phi\right] = 0$$

76. $$\lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\frac{1}{x^2 + y^2 + z^2}\right] = \lim_{\rho\to 0^+} \tan^{-1}\left[\frac{1}{\rho^2}\right] = \frac{\pi}{2}$$

77. As $(x, y) \to (0, 1)$, $x^2 + 1 \to 1$ and $x^2 + (y - 1)^2 \to 0$.

So, $\displaystyle\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{x^2 + 1}{x^2 + (y - 1)^2}\right] = \frac{\pi}{2}$.

78. $$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} (r\cos\theta)(r\sin\theta)\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} = \lim_{r\to 0} r^2\left[\cos\theta \sin\theta\left(\cos^2\theta - \sin^2\theta\right)\right] = 0$$

So, define $f(0, 0) = 0$.

79. Because $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L_1$, then for $\varepsilon/2 > 0$, there corresponds $\delta_1 > 0$ such that $\left|f(x, y) - L_1\right| < \varepsilon/2$ whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_1$.

Because $\displaystyle\lim_{(x,y)\to(a,b)} g(x, y) = L_2$, then for $\varepsilon/2 > 0$, there corresponds $\delta_2 > 0$ such that $\left|g(x, y) - L_2\right| < \varepsilon/2$ whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_2$.

Let δ be the smaller of δ_1 and δ_2. By the triangle inequality, whenever $\sqrt{(x - a)^2 + (y - b)^2} < \delta$, we have

$$\left|f(x, y) + g(x, y) - (L_1 + L_2)\right| = \left|\left(f(x, y) - L_1\right) + \left(g(x, y) - L_2\right)\right| \le \left|f(x, y) - L_1\right| + \left|g(x, y) - L_2\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $\displaystyle\lim_{(x,y)\to(a,b)} \left[f(x, y) + g(x, y)\right] = L_1 + L_2$.

80. Given that $f(x, y)$ is continuous, then $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) < 0$, which means that for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $|f(x, y) - f(a, b)| < \varepsilon$ whenever

$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$

Let $\varepsilon = |f(a, b)|/2$, then $f(x, y) < 0$ for every point in the corresponding δ neighborhood because

$$|f(x, y) - f(a, b)| < \frac{|f(a, b)|}{2} \Rightarrow -\frac{|f(a, b)|}{2} < f(x, y) - f(a, b) < \frac{|f(a, b)|}{2}$$
$$\Rightarrow \frac{3}{2}f(a, b) < f(x, y) < \frac{1}{2}f(a, b) < 0.$$

81. See the definition on page 881. Show that the value of $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ is not the same for two different paths to (x_0, y_0).

82. See the definition on page 884.

83. (a) No. The existence of $f(2, 3)$ has no bearing on the existence of the limit as $(x, y) \to (2, 3)$.

(b) No, $f(2, 3)$ can equal any number, or not even be defined.

84. The limit appears to exist at all the points except (c) $(0, 0)$. Near this point, the graph tends to $-\infty$.

Section 13.3 Partial Derivatives

1. No, x only occurs in the numerator.

2. Yes, y occurs in both the numerator and denominator.

3. No, y only occurs in the numerator.

4. Yes, x occurs in both the numerator and denominator.

5. Yes, x occurs in both the numerator and denominator.

6. No, y only occurs in the numerator.

7. $f(x, y) = 2x - 5y + 3$

$f_x(x, y) = 2$

$f_y(x, y) = -5$

8. $f(x, y) = x^2 - 2y^2 + 4$

$f_x(x, y) = 2x$

$f_y(x, y) = -4y$

9. $f(x, y) = x^2y^3$

$f_x(x, y) = 2xy^3$

$f_y(x, y) = 3x^2y^2$

10. $f(x, y) = 4x^3y^{-2}$

$f_x(x, y) = 12x^2y^{-2}$

$f_y(x, y) = -8x^3y^{-3}$

11. $z = x\sqrt{y}$

$\frac{\partial z}{\partial x} = \sqrt{y}$

$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}}$

12. $z = 2y^2\sqrt{x}$

$\frac{\partial z}{\partial x} = \frac{y^2}{\sqrt{x}}$

$\frac{\partial z}{\partial y} = 4y\sqrt{x}$

13. $z = x^2 - 4xy + 3y^2$

$\frac{\partial z}{\partial x} = 2x - 4y$

$\frac{\partial z}{\partial y} = -4x + 6y$

14. $z = y^3 - 2xy^2 - 1$

$\frac{\partial z}{\partial x} = -2y^2$

$\frac{\partial z}{\partial y} = 3y^2 - 4xy$

15. $z = e^{xy}$

$$\frac{\partial z}{\partial x} = ye^{xy}$$

$$\frac{\partial z}{\partial y} = xe^{xy}$$

16. $z = e^{x/y} = e^{xy^{-1}}$

$$\frac{\partial z}{\partial x} = \frac{1}{y}e^{x/y}$$

$$\frac{\partial z}{\partial y} = \frac{-x}{y^2}e^{x/y}$$

17. $z = x^2e^{2y}$

$$\frac{\partial z}{\partial x} = 2xe^{2y}$$

$$\frac{\partial z}{\partial y} = 2x^2e^{2y}$$

18. $z = ye^{y/x} = ye^{yx^{-1}}$

$$\frac{\partial z}{\partial x} = ye^{yx^{-1}}\left[-yx^{-2}\right] = \frac{-y^2}{x^2}e^{y/x}$$

$$\frac{\partial z}{\partial y} = e^{y/x} + \frac{1}{x}ye^{y/x} = e^{y/x}\left(1 + \frac{y}{x}\right)$$

19. $z = \ln \frac{x}{y} = \ln x - \ln y$

$$\frac{\partial z}{\partial x} = \frac{1}{x}$$

$$\frac{\partial z}{\partial y} = -\frac{1}{y}$$

20. $z = \ln\sqrt{xy} = \frac{1}{2}\ln(xy)$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{y}{xy} = \frac{1}{2x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\frac{x}{xy} = \frac{1}{2y}$$

21. $z = \ln\left(x^2 + y^2\right)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

22. $z = \ln \frac{x + y}{x - y} = \ln(x + y) - \ln(x - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x + y} - \frac{1}{x - y} = \frac{-2y}{(x + y)(x - y)}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x + y} + \frac{1}{x - y} = \frac{2x}{(x + y)(x - y)}$$

23. $z = \frac{x^2}{2y} + \frac{3y^2}{x}$

$$\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{3y^2}{x^2} = \frac{x^3 - 3y^3}{x^2y}$$

$$\frac{\partial z}{\partial y} = \frac{-x^2}{2y^2} + \frac{6y}{x} = \frac{12y^3 - x^3}{2xy^2}$$

24. $f(x, y) = \frac{xy}{x^2 + y^2}$

$$f_x(x, y) = \frac{\left(x^2 + y^2\right)(y) - (xy)(2x)}{\left(x^2 + y^2\right)^2} = \frac{y^3 - x^2y}{\left(x^2 + y^2\right)^2}$$

$$f_y(x, y) = \frac{\left(x^2 + y^2\right)(x) - (xy)(2y)}{\left(x^2 + y^2\right)^2} = \frac{x^3 - xy^2}{\left(x^2 + y^2\right)^2}$$

25. $h(x, y) = e^{-\left(x^2 + y^2\right)}$

$$h_x(x, y) = -2xe^{-\left(x^2 + y^2\right)}$$

$$h_y(x, y) = -2ye^{-\left(x^2 + y^2\right)}$$

26. $g(x, y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln\left(x^2 + y^2\right)$

$$g_x(x, y) = \frac{1}{2}\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$g_y(x, y) = \frac{1}{2}\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

27. $f(x, y) = \sqrt{x^2 + y^2}$

$$f_x(x, y) = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y(x, y) = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

28. $f(x, y) = \sqrt{2x + y^3}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(2x + y^3\right)^{-1/2}(2) = \frac{1}{\sqrt{2x + y^3}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\left(2x + y^3\right)^{-1/2}\left(3y^2\right) = \frac{3y^2}{2\sqrt{2x + y^3}}$$

29. $z = \cos xy$

$$\frac{\partial z}{\partial x} = -y \sin xy$$

$$\frac{\partial z}{\partial y} = -x \sin xy$$

30. $z = \sin(x + 2y)$

$$\frac{\partial z}{\partial x} = \cos(x + 2y)$$

$$\frac{\partial z}{\partial y} = 2 \cos(x + 2y)$$

31. $z = \tan(2x - y)$

$$\frac{\partial z}{\partial x} = 2 \sec^2(2x - y)$$

$$\frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

32. $z = \sin 5x \cos 5y$

$$\frac{\partial z}{\partial x} = 5 \cos 5x \cos 5y$$

$$\frac{\partial z}{\partial y} = -5 \sin 5x \sin 5y$$

33. $z = e^y \sin xy$

$$\frac{\partial z}{\partial x} = ye^y \cos xy$$

$$\frac{\partial z}{\partial y} = e^y \sin xy + xe^y \cos x$$

$$= e^y(x \cos xy + \sin xy)$$

34. $z = \cos(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = -2x \sin(x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = -2y \sin(x^2 + y^2)$$

35. $z = \sinh(2x + 3y)$

$$\frac{\partial z}{\partial x} = 2 \cosh(2x + 3y)$$

$$\frac{\partial z}{\partial y} = 3 \cosh(2x + 3y)$$

36. $z = \cosh xy^2$

$$\frac{\partial z}{\partial x} = y^2 \sinh xy^2$$

$$\frac{\partial z}{\partial y} = 2xy \sinh xy^2$$

37. $f(x, y) = \int_x^y (t^2 - 1)\, dt$

$$= \left[\frac{t^3}{3} - t\right]_x^y = \left(\frac{y^3}{3} - y\right) - \left(\frac{x^3}{3} - x\right)$$

$$f_x(x, y) = -x^2 + 1 = 1 - x^2$$

$$f_y(x, y) = y^2 - 1$$

[You could also use the Second Fundamental Theorem of Calculus.]

38. $f(x, y) = \int_x^y (2t + 1)\, dt + \int_y^x (2t - 1)\, dt$

$$= \int_x^y (2t + 1)\, dt - \int_x^y (2t - 1)\, dt$$

$$= \int_x^y 2\, dt = [2t]_x^y = 2y - 2x$$

$$f_x(x, y) = -2$$

$$f_x(x, y) = 2$$

39. $f(x, y) = 3x + 2y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x + \Delta x) + 2y - (3x + 2y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{3x + 2(y + \Delta y) - (3x + 2y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2\Delta y}{\Delta y} = 2$$

40. $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x)y + y^2 - x^2 + 2xy - y^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2y) = 2(x - y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2 - 2x(y + \Delta y) + (y + \Delta y)^2 - x^2 + 2xy - y^2}{\Delta y} = \lim_{\Delta y \to 0} (-2x + 2y + \Delta y) = 2(y - x)$$

41. $f(x, y) = \sqrt{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x + y} - \sqrt{x + y}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{x + \Delta x + y} - \sqrt{x + y}\right)\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)}{\Delta x\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x + y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt{x + y + \Delta y} - \sqrt{x + y}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\left(\sqrt{x + y + \Delta y} - \sqrt{x + y}\right)\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}{\Delta y\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\sqrt{x + y + \Delta y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

42. $f(x, y) = \dfrac{1}{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + y} - \dfrac{1}{x + y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{1}{x + y + \Delta} - \dfrac{1}{x + y}}{\Delta y} = \lim_{\Delta y \to 0} \frac{-1}{(x + y + \Delta y)(x + y)} = \frac{-1}{(x + y)^2}$$

43. $f(x, y) = e^y \sin x$

$f_x(x, y) = e^y \cos x$

At $(\pi, 0)$, $f_x(\pi, 0) = -1$.

$f_y(x, y) = e^y \sin x$

At $(\pi, 0)$, $f_y(\pi, 0) = 0$.

44. $f(x, y) = e^{-x} \cos y$

$f_x(x, y) = -e^{-x} \cos y$

At $(0, 0)$, $f_x(0, 0) = -1$.

$f_y(x, y) = -e^{-x} \sin y$

At $(0, 0)$, $f_y(0, 0) = 0$.

45. $f(x, y) = \cos(2x - y)$

$f_x(x, y) = -2\sin(2x - y)$

At $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$, $f_x\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = -2\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = -1.$

$f_y(x, y) = \sin(2x - y)$

At $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$, $f_y\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{2}.$

46. $f(x, y) = \sin xy$

$f_x(x, y) = y\cos xy$

At $\left(2, \frac{\pi}{4}\right)$, $f_x\left(2, \frac{\pi}{4}\right) = \frac{\pi}{4}\cos\frac{\pi}{2} = 0.$

$f_y(x, y) = x\cos xy$

At $\left(2, \frac{\pi}{4}\right)$, $f_y\left(2, \frac{\pi}{4}\right) = 2\cos\frac{\pi}{2} = 0.$

47. $f(x, y) = \arctan\frac{y}{x}$

$f_x(x, y) = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$

At $(2, -2)$: $f_x(2, -2) = \frac{1}{4}$

$f_y(x, y) = \frac{1}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$

At $(2, -2)$: $f_y(2, -2) = \frac{1}{4}$

48. $f(x, y) = \arccos(xy)$

$f_x(x, y) = \frac{-y}{\sqrt{1 - x^2y^2}}$

At $(1, 1)$, f_x is undefined.

$f_y(x, y) = \frac{-x}{\sqrt{1 - x^2y^2}}$

At $(1, 1)$, f_y is undefined.

49. $f(x, y) = \frac{xy}{x - y}$

$f_x(x, y) = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$

At $(2, -2)$: $f_x(2, -2) = -\frac{1}{4}$

$f_y(x, y) = \frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$

At $(2, -2)$: $f_y(2, -2) = \frac{1}{4}$

50. $f(x, y) = \frac{2xy}{\sqrt{4x^2 + 5y^2}}$

$f_x(x, y) = \frac{10y^3}{(4x^2 + 5y^2)^{3/2}}$

At $(1, 1)$, $f_x(1, 1) = \frac{10}{9^{3/2}} = \frac{10}{27}.$

$f_y(x, y) = \frac{8x^3}{(4x^2 + 5y^2)^{3/2}}$

At $(1, 1)$, $f_y(1, 1) = \frac{8}{9^{3/2}} = \frac{8}{27}.$

51. $g(x, y) = 4 - x^2 - y^2$

$g_x(x, y) = -2x$

At $(1, 1)$: $g_x(1, 1) = -2$

$g_y(x, y) = -2y$

At $(1, 1)$: $g_y(1, 1) = -2$

52. $h(x, y) = x^2 - y^2$

$h_x(x, y) = 2x$

At $(-2, 1)$: $h_x(-2, 1) = -4$

$h_y(x, y) = -2y$

At $(-2, 1)$: $h_y(-2, 1) = -2$

53. $H(x, y, z) = \sin(x + 2y + 3z)$

$H_x(x, y, z) = \cos(x + 2y + 3z)$

$H_y(x, y, z) = 2\cos(x + 2y + 3z)$

$H_z(x, y, z) = 3\cos(x + 2y + 3z)$

54. $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

$f_x(x, y, z) = 6xy - 5yz$

$f_y(x, y, z) = 3x^2 - 5xz + 10z^2$

$f_z(x, y, z) = -5xy + 20yz$

55. $w = \sqrt{x^2 + y^2 + z^2}$

$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$

$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$

$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

56. $w = \dfrac{7xz}{x + y} = 7xz(x + y)^{-1}$

$$\frac{\partial w}{\partial x} = \frac{(x + y)(7z) - 7xz}{(x + y)^2} = \frac{7yz}{(x + y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{-7xz}{(x + y)^2}$$

$$\frac{\partial w}{\partial z} = \frac{7x}{x + y}$$

57. $F(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2} = \dfrac{1}{2}\ln(x^2 + y^2 + z^2)$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

58. $G(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$G_x(x, y, z) = \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_y(x, y, z) = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_z(x, y, z) = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

59. $f(x, y, z) = x^3yz^2$

$$f_x(x, y, z) = 3x^2yz^2$$

$$f_x(1, 1, 1) = 3$$

$$f_y(x, y, z) = x^3z^2$$

$$f_y(1, 1, 1) = 1$$

$$f_z(x, y, z) = 2x^3yz$$

$$f_z(1, 1, 1) = 2$$

60. $f(x, y, z) = x^2y^3 + 2xyz - 3yz$

$$f_x(x, y, z) = 2xy^3 + 2yz$$

$$f_x(-2, 1, 2) = -4 + 4 = 0$$

$$f_y(x, y, z) = 3x^2y^2 + 2xz - 3z$$

$$f_y(-2, 1, 2) = 12 - 8 - 6 = -2$$

$$f_z(x, y, z) = 2xy - 3y$$

$$f_z(-2, 1, 2) = -4 - 3 = -7$$

61. $f(x, y, z) = \dfrac{x}{yz}$

$$f_x(x, y, z) = \frac{1}{yz}$$

$$f_x(1, -1, -1) = 1$$

$$f_y(x, y, z) = \frac{-x}{y^2z}$$

$$f_y(1, -1, -1) = 1$$

$$f_z(x, y, z) = \frac{-x}{yz^2}$$

$$f_z(1, -1, -1) = 1$$

62. $f(x, y, z) = \dfrac{xy}{x + y + z}$

$$f_x(x, y, z) = \frac{(x + y + z)y - xy}{(x + y + z)^2} = \frac{y^2 + yz}{(x + y + z)^2}$$

$$f_x(3, 1, -1) = \frac{1 - 1}{3^2} = 0$$

$$f_y(x, y, z) = \frac{(x + y + z)x - xy}{(x + y + z)^2} = \frac{x^2 + xz}{(x + y + z)^2}$$

$$f_y(3, 1, -1) = \frac{9 - 3}{3^2} = \frac{2}{3}$$

$$f_z(x, y, z) = \frac{(x + y + z)(0) - xy}{(x + y + z)^2} = \frac{-xy}{(x + y + z)^2}$$

$$f_z(3, 1, -1) = \frac{-3}{9} = \frac{-1}{3}$$

63. $f(x, y, z) = z\sin(x + y)$

$$f_x(x, y, z) = z\cos(x + y)$$

$$f_x\left(0, \frac{\pi}{2}, -4\right) = -4\cos\frac{\pi}{2} = 0$$

$$f_y(x, y, z) = z\cos(x + y)$$

$$f_y\left(0, \frac{\pi}{2}, -4\right) = -4\cos\frac{\pi}{2} = 0$$

$$f_z(x, y, z) = \sin(x + y)$$

$$f_z\left(0, \frac{\pi}{2}, -4\right) = \sin\frac{\pi}{2} = 1$$

64. $\sqrt{3x^2 + y^2 - 2z^2}$

$$f_x(x, y, z) = \frac{6x}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{3x}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_x(1, -2, 1) = \frac{6}{2\sqrt{3 + 4 - 2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$f_y(x, y, z) = \frac{2y}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{y}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_y(1, -2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

$$f_z(x, y, z) = \frac{-4z}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{-2z}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_z(1, -2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

65. $f_x(x, y) = 2x + y - 2 = 0$

$f_y(x, y) = x + 2y + 2 = 0$

$2x + y - 2 = 0 \Rightarrow y = 2 - 2x$

$x + 2(2 - 2x) + 2 = 0 \Rightarrow -3x + 6 = 0 \Rightarrow x = 2,$

$y = -2$

Point: $(2, -2)$

66. $f_x(x, y) = 2x - y - 5 = 0$

$f_y(x, y) = -x + 2y + 1 = 0$

$2x - y - 5 = 0 \Rightarrow y = 2x - 5$

$-x + 2(2x - 5) + 1 = 0 \Rightarrow 3x - 9 = 0 \Rightarrow x = 3,$

$y = 1$

Point: $(3, 1)$

67. $f_x(x, y) = 2x + 4y - 4,\ f_y(x, y) = 4x + 2y + 16$

$$f_x = f_y = 0:\ \begin{aligned} 2x + 4y &= 4 \\ 4x + 2y &= -16 \end{aligned}$$

Solving for x and y,

$x = -6$ and $y = 4$.

68. $f_x(x, y) = 2x - y = 0$

$f_y(x, y) = -x + 2y = 0$

$2x - y = 0 \Rightarrow y = 2x$

$-x + 2(2x) = 0 \Rightarrow x = 0, y = 0$

Point: $(0, 0)$

69. $f_x(x, y) = -\dfrac{1}{x^2} + y,\ f_y(x, y) = -\dfrac{1}{y^2} + x$

$f_x = f_y = 0:\ -\dfrac{1}{x^2} + y = 0$ and $-\dfrac{1}{y^2} + x = 0$

$$y = \frac{1}{x^2} \text{ and } x = \frac{1}{y^2}$$

$y = y^4 \Rightarrow y = 1 = x$

Points: $(1, 1)$

70. $f_x(x, y) = 9x^2 - 12y,\ f_y(x, y) = -12x + 3y^2$

$$f_x = f_y = 0:\ \begin{aligned} 9x^2 - 12y = 0 &\Rightarrow 3x^2 = 4y \\ 3y^2 - 12x = 0 &\Rightarrow y^2 = 4x \end{aligned}$$

Solving for x in the second equation, $x = y^2/4$, you obtain $3\left(y^2/4\right)^2 = 4y$.

$3y^4 = 64y \Rightarrow y = 0$ or $y = \dfrac{4}{3^{1/3}}$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{4}\left(\frac{16}{3^{2/3}}\right)$$

Points: $(0, 0), \left(\dfrac{4}{3^{2/3}}, \dfrac{4}{3^{1/3}}\right)$

71. $f_x(x, y) = (2x + y)e^{x^2 + xy + y^2} = 0$

$f_y(x, y) = (x + 2y)e^{x^2 + xy + y^2} = 0$

$2x + y = 0 \Rightarrow y = -2x$

$x + 2(-2x) = 0 \Rightarrow x = 0 \Rightarrow y = 0$

Point: $(0, 0)$

72. $f_x(x, y) = \dfrac{2x}{x^2 + y^2 + 1} = 0 \Rightarrow x = 0$

$f_y(x, y) = \dfrac{2y}{x^2 + y^2 + 1} = 0 \Rightarrow y = 0$

Points: $(0, 0)$

73. $z = 3xy^2$

$$\frac{\partial z}{\partial x} = 3y^2,\ \frac{\partial^2 z}{\partial x^2} = 0,\ \frac{\partial^2 z}{\partial y \partial x} = 6y$$

$$\frac{\partial z}{\partial y} = 6xy,\ \frac{\partial^2 z}{\partial y^2} = 6x,\ \frac{\partial^2 z}{\partial x \partial y} = 6y$$

74. $z = x^2 + 3y^2$

$$\frac{\partial z}{\partial x} = 2x, \frac{\partial^2 z}{\partial x^2} = 2, \frac{\partial^2 z}{\partial y \partial x} = 0$$

$$\frac{\partial z}{\partial y} = 6y, \frac{\partial^2 z}{\partial y^2} = 6, \frac{\partial^2 z}{\partial x \partial y} = 0$$

75. $z = x^2 - 2xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 2y$$
$$\frac{\partial^2 z}{\partial x^2} = 2$$
$$\frac{\partial^2 z}{\partial y \partial x} = -2$$
$$\frac{\partial z}{\partial y} = -2x + 6y$$
$$\frac{\partial^2 z}{\partial y^2} = 6$$
$$\frac{\partial^2 z}{\partial x \partial y} = -2$$

76. $z = x^4 - 3x^2y^2 + y^4$

$$\frac{\partial z}{\partial x} = 4x^3 - 6xy^2$$
$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 6y^2$$
$$\frac{\partial^2 z}{\partial y \partial x} = -12xy$$
$$\frac{\partial z}{\partial y} = -6x^2y + 4y^3$$
$$\frac{\partial^2 z}{\partial y^2} = -6x^2 + 12y^2$$
$$\frac{\partial^2 z}{\partial x \partial y} = -12xy$$

77. $z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{\left(x^2 + y^2\right)^{3/2}}$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{\left(x^2 + y^2\right)^{3/2}}$$
$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{\left(x^2 + y^2\right)^{3/2}}$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{\left(x^2 + y^2\right)^{3/2}}$$

78. $z = \ln(x - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x - y}$$
$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x - y)^2}$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{(x - y)^2}$$
$$\frac{\partial z}{\partial y} = \frac{-1}{x - y} = \frac{1}{y - x}$$
$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x - y)^2}$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(x - y)^2}$$

So, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

79. $z = e^x \tan y$

$$\frac{\partial z}{\partial x} = e^x \tan y$$
$$\frac{\partial^2 z}{\partial x^2} = e^x \tan y$$
$$\frac{\partial^2 z}{\partial y \partial x} = e^x \sec^2 y$$
$$\frac{\partial z}{\partial y} = e^x \sec^2 y$$
$$\frac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$$
$$\frac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$$

80. $z = 2xe^y - 3ye^{-x}$

$$\frac{\partial z}{\partial x} = 2e^y + 3ye^{-x}$$
$$\frac{\partial^2 z}{\partial x^2} = -3ye^{-x}$$
$$\frac{\partial^2 z}{\partial y \partial x} = 2e^y + 3ye^{-x}$$
$$\frac{\partial z}{\partial y} = 2xe^y - 3e^{-x}$$
$$\frac{\partial^2 z}{\partial y^2} = 2xe^y$$
$$\frac{\partial^2 z}{\partial x \partial y} = 2e^y + 3e^{-x}$$

81. $z = \cos xy$

$$\frac{\partial z}{\partial x} = -y \sin xy, \frac{\partial^2 z}{\partial x^2} = -y^2 \cos xy$$

$$\frac{\partial^2 z}{\partial y \partial x} = -yx \cos xy - \sin xy$$

$$\frac{\partial z}{\partial y} = -x \sin xy, \frac{\partial^2 z}{\partial y^2} = -x^2 \cos xy$$

$$\frac{\partial^2 z}{\partial x \partial y} = -xy \cos xy - \sin xy$$

82. $z = \arctan \dfrac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(y^2/x^2\right)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-\left(x^2 + y^2\right) + y(2y)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \left(y^2/x^2\right)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\left(x^2 + y^2\right) - x(2x)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

83. $z = x \sec y$

$$\frac{\partial z}{\partial x} = \sec y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \sec y \tan y$$

$$\frac{\partial z}{\partial y} = x \sec y \tan y$$

$$\frac{\partial^2 z}{\partial y^2} = x \sec y\left(\sec^2 y + \tan^2 y\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$$

So, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$

There are no points for which $z_x = 0 = z_y$, because

$$\frac{\partial z}{\partial x} = \sec y \neq 0.$$

84. $z = \sqrt{25 - x^2 - y^2}$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{25 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2 - 25}{\left(25 - x^2 - y^2\right)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{\left(25 - x^2 - y^2\right)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2 - 25}{\left(25 - x^2 - y^2\right)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{\left(25 - x^2 - y^2\right)^{3/2}}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \text{ if } x = y = 0$$

85. $z = \ln\left(\dfrac{x}{x^2 + y^2}\right) = \ln x - \ln\left(x^2 + y^2\right)$

$$\frac{\partial z}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} = \frac{y^2 - x^2}{x\left(x^2 + y^2\right)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^4 - 4x^2y^2 - y^4}{x^2\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{4xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{4xy}{\left(x^2 + y^2\right)^2}$$

There are no points for which $z_x = z_y = 0$.

86. $z = \dfrac{xy}{x-y}$

$$\frac{\partial z}{\partial x} = \frac{y(x-y) - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2y^2}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x-y)^2(-2y) + y^2(2)(x-y)(-1)}{(x-y)^4} = \frac{-2xy}{(x-y)^3}$$

$$\frac{\partial z}{\partial y} = -\frac{x(x-y) + xy}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x^2}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x-y)^2(2x) - x^2(2)(x-y)}{(x-y)^4} = \frac{-2xy}{(x-y)^3}$$

There are no points for which $z_x = z_y = 0$.

87. $f(x, y, z) = xyz$

$f_x(x, y, z) = yz$

$f_y(x, y, z) = xz$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = z$

$f_{yx}(x, y, z) = z$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

So, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

88. $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

$f_x(x, y, z) = 2x - 3y$

$f_y(x, y, z) = -3x + 4z$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = -3$

$f_{yx}(x, y, z) = -3$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

So, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

89. $f(x, y, z) = e^{-x} \sin yz$

$f_x(x, y, z) = -e^{-x} \sin yz$

$f_y(x, y, z) = ze^{-x} \cos yz$

$f_{yy}(x, y, z) = -z^2e^{-x} \sin yz$

$f_{xy}(x, y, z) = -ze^{-x} \cos yz$

$f_{yx}(x, y, z) = -ze^{-x} \cos yz$

$f_{yyx}(x, y, z) = z^2e^{-x} \sin yz$

$f_{xyy}(x, y, z) = z^2e^{-x} \sin yz$

$f_{yxy}(x, y, z) = z^2e^{-x} \sin yz$

So, $f_{xyy} = f_{yxy} = f_{yyz}$.

90. $f(x, y, z) = \dfrac{2z}{x+y}$

$$f_x(x, y, z) = \frac{-2z}{(x+y)^2}$$

$$f_y(x, y, z) = \frac{-2z}{(x+y)^2}$$

$$f_{yy}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{xy}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{yx}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{yyx}(x, y, z) = \frac{-12z}{(x+y)^4}$$

$$f_{xyy}(x, y, z) = \frac{-12z}{(x+y)^4}$$

$$f_{yxy}(x, y, z) = \frac{-12z}{(x+y)^4}$$

91. $z = 5xy$

$$\frac{\partial z}{\partial x} = 5y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial z}{\partial y} = 5x$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

So, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0 + 0 = 0.$

92. $z = \sin x\left(\dfrac{e^y - e^{-y}}{2}\right)$

$$\frac{\partial z}{\partial x} = \cos x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\frac{\partial z}{\partial y} = \sin x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \sin x\left(\frac{e^y - e^{-y}}{2}\right)$$

So,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right) + \sin x\left(\frac{e^y - e^{-y}}{2}\right) = 0.$$

93. $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

So, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$

94. $z = \arctan \dfrac{y}{x}$

From Exercise 82, we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{\left(x^2 + y^2\right)^2} + \frac{-2xy}{\left(x^2 + y^2\right)^2} = 0.$$

95. $z = \sin(x - ct)$

$$\frac{\partial z}{\partial t} = -c\cos(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct)$$

$$\frac{\partial z}{\partial x} = \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - ct)$$

So, $\dfrac{\partial^2 z}{\partial t^2} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right).$

96. $z = \cos(4x + 4ct)$

$$\frac{\partial z}{\partial t} = -4c\sin(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -16c^2 \cos(4x + 4ct)$$

$$\frac{\partial z}{\partial x} = -4\sin(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -16\cos(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(-16\cos(4x + 4ct)\right) = c^2\left(\frac{\partial^2 z}{\partial x^2}\right)$$

97. $z = \ln(x + ct)$

$$\frac{\partial z}{\partial t} = \frac{c}{x + ct}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{-c^2}{(x + ct)^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{x + ct}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-1}{(x + ct)^2}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{-c^2}{(x + ct)^2} = c^2\left(\frac{\partial^2 z}{\partial x^2}\right)$$

98. $z = \sin(\omega ct)\sin(\omega x)$

$$\frac{\partial z}{\partial t} = \omega c\cos(\omega ct)\sin(\omega x)$$

$$\frac{\partial^2 z}{\partial t^2} = -\omega^2 c^2 \sin(\omega ct)\sin(\omega x)$$

$$\frac{\partial z}{\partial x} = \omega\sin(\omega ct)\cos(\omega x)$$

$$\frac{\partial^2 z}{\partial x^2} = -\omega^2 \sin(\omega ct)\sin(\omega x)$$

So, $\dfrac{\partial^2 z}{\partial t^2} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right).$

99. $z = e^{-t}\cos\dfrac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t}\cos\frac{x}{c}$$

$$\frac{\partial z}{\partial x} = -\frac{1}{c}e^{-t}\sin\frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2}e^{-t}\cos\frac{x}{c}$$

So, $\dfrac{\partial z}{\partial t} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right).$

100. $$z = e^{-t}\sin\frac{x}{c}$$

$$\frac{\partial z}{\partial t} = -e^{-t}\sin\frac{x}{c}$$

$$\frac{\partial z}{\partial x} = \frac{1}{c}e^{-t}\cos\frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2}e^{-t}\sin\frac{x}{c}$$

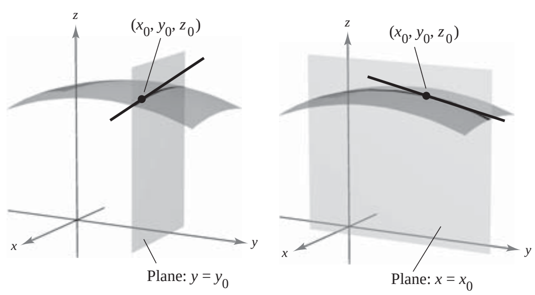
So, $\dfrac{\partial z}{\partial t} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right)$.

101. Yes. The function $f(x, y) = \cos(3x - 2y)$ satisfies both equations.

102. A function $f(x, y)$ with the given partial derivatives does not exist.

103. If $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x. Similarly, to find f_y, you consider x constant and differentiate with respect to y.

104.

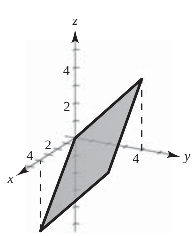


$\dfrac{\partial f}{\partial x}$ denotes the slope of surface in the x-direction.

$\dfrac{\partial f}{\partial y}$ denotes the slope of the surface in the y-direction.

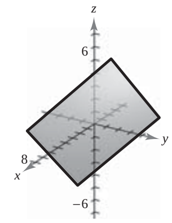
105. The plane $z = -x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} < 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



106. The plane $z = x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} > 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



107. In this case, the mixed partials are equal, $f_{xy} = f_{yx}$. See Theorem 13.3.

108. (a) $f_x(4, 1) < 0$

(b) $f_y(4, 1) > 0$

(c) $f_x(-1, -2) < 0$

(d) $f_y(-1, -2) > 0$

109. $R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2$

(a) $\dfrac{\partial r}{\partial x_1} = 200 - 8x_1 - 8x_2$

At $(x_1, x_2) = (4, 12)$, $\dfrac{\partial R}{\partial x_1} = 200 - 32 - 96 = 72.$

(b) $\dfrac{\partial R}{\partial x_2} = 200 - 8x_1 - 8x_2$

At $(x_1, x_2) = (4, 12)$, $\dfrac{\partial R}{\partial x^2} = 72.$

110. (a) $C = 32\sqrt{xy} + 175x + 205y + 1050$

$$\frac{\partial C}{\partial x} = 16\sqrt{\frac{y}{x}} + 175$$

$$\left.\frac{\partial C}{\partial x}\right]_{(80, 20)} = 16\sqrt{\frac{1}{4}} + 175 = 183$$

$$\frac{\partial C}{\partial y} = 16\sqrt{\frac{x}{y}} + 205$$

$$\left.\frac{\partial C}{\partial y}\right]_{(80, 20)} = 16\sqrt{4} + 205 = 237$$

(b) The fireplace-insert stove results in the cost increasing at a faster rate because $\dfrac{\partial C}{\partial y} > \dfrac{\partial C}{\partial x}$.

111. $IQ(M, C) = 100\dfrac{M}{C}$

$IQ_M = \dfrac{100}{C}, IQ_M(12, 10) = 10$

$IQ_c = \dfrac{-100M}{C^2}, IQ_c(12, 10) = -12$

When the chronological age is constant, IQ increases at a rate of 10 points per mental age year.

When the mental age is constant, IQ decreases at a rate of 12 points per chronological age year.

112. $f(x, y) = 200x^{0.7}y^{0.3}$

(a) $\dfrac{\partial f}{\partial x} = 140x^{-0.3}y^{0.3} = 140\left(\dfrac{y}{x}\right)^{0.3}$

At $(x, y) = (1000, 500)$,

$\dfrac{\partial f}{\partial x} = 140\left(\dfrac{500}{1000}\right)^{0.3} = 140\left(\dfrac{1}{2}\right)^{0.3} \approx 113.72.$

(b) $\dfrac{\partial f}{\partial y} = 60x^{0.7}y^{-0.7} = 60\left(\dfrac{x}{y}\right)^{0.7}$

At $(x, y) = (1000, 500)$,

$\dfrac{\partial f}{\partial y} = 60\left(\dfrac{1000}{500}\right)^{0.7} = 60(2)^{0.7} \approx 97.47.$

113. An increase in either price will cause a decrease in demand.

114. $V(I, R) = 1000\left[\dfrac{1 + 0.06(1 - R)}{1 + I}\right]^{10}$

$$V_I(I, R) = 10{,}000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^9\left[-\frac{1 + 0.06(1 - R)}{(1 + I)^2}\right] = -10{,}000\left[\frac{(1 + 0.06(1 - R))^{10}}{(1 + I)^{11}}\right]$$

$V_I(0.03, 0.28) = -11{,}027.20$

$$V_R(I, R) = 10{,}000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^9\left[-\frac{0.06}{1 + I}\right] = -600\left[\frac{(1 + 0.06(1 - R))^9}{(1 + I)^{10}}\right]$$

$V_R(0.03, 0.28) = -653.26$

The rate of inflation has the greater negative influence.

115. $T = 500 - 0.6x^2 - 1.5y^2$

$\dfrac{\partial T}{\partial x} = -1.2x, \dfrac{\partial T}{\partial x}(2, 3) = -2.4°/\text{m}$

$\dfrac{\partial T}{\partial y} = -3y = \dfrac{\partial T}{\partial y}(2, 3) = -9°/\text{m}$

116. $A = 0.885t - 22.4h + 1.20th - 0.544$

(a) $\dfrac{\partial A}{\partial t} = 0.885 + 1.20h$

$\dfrac{\partial A}{\partial t}(30°, 0.80) = 0.885 + 1.20(0.80) = 1.845$

$\dfrac{\partial A}{\partial h} = -22.4 + 1.20t$

$\dfrac{\partial A}{\partial h}(30°, 0.80) = -22.4 + 1.20(30°) = 13.6$

(b) The humidity has a greater effect on A because its coefficient -22.4 is larger than that of t.

117.

$$PV = \frac{n}{xB}RT$$

$$T = \frac{PV}{\frac{n}{xB}R} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{\frac{n}{xB}R}$$

$$P = \frac{\frac{n}{xB}RT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{\frac{n}{xB}RT}{V^2}$$

$$V = \frac{\frac{n}{xB}RT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{\frac{n}{xB}R}{P}$$

$$\frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} = \left(\frac{V}{\frac{n}{xB}R}\right)\left(-\frac{\frac{n}{xB}RT}{V^2}\right)\left(\frac{\frac{n}{xB}R}{P}\right)$$

$$= -\frac{\frac{n}{xB}RT}{VP} = -\frac{\frac{n}{xB}RT}{\frac{n}{xB}RT} = -1$$

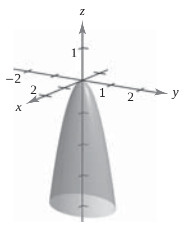
118. $U = -5x^2 + xy - 3y^2$

(a) $U_x = -10x + y$

(b) $U_y = x - 6y$

(c) $U_x(2, 3) = -17$ and $U_y(2, 3) = -16$. The person should consume one more unit of y because the rate of decrease of satisfaction is less for y.

(d)



119. $z = 0.461x + 0.301y - 494$

(a) $\dfrac{\partial z}{\partial x} = 0.461 \qquad \dfrac{\partial z}{\partial y} = 0.301$

(b) As the expenditures on amusement parks and campgrounds (x) increase, the expenditures on spectator sports (z) increase. As the expenditures on live entertainment (y) increase, the expenditures on spectator sports (z) increase.

120. $z = 11.734x^2 - 0.028y^2 - 888.24x + 23.09y + 12{,}573.9$

(a) $\dfrac{\partial z}{\partial x} = 23.468x - 888.24$

$\dfrac{\partial^2 z}{\partial x^2} = 23.468$

$\dfrac{\partial z}{\partial y} = -0.056y + 23.09$

$\dfrac{\partial^2 z}{\partial y^2} = -0.056$

(b) Traces parallel to the *xz*-plane are concave upward $\left(\dfrac{\partial^2 z}{\partial x^2} > 0\right)$. The rate of change of Medicare expenses is increasing with respect to worker's compensation (x).

(c) Traces parallel to the *yz*-plane are concave downward $\left(\dfrac{\partial^2 z}{\partial y^2} < 0\right)$. The rate of change of Medicare expenses is decreasing with respect to Medicaid (y).

121. False

Let $z = x + y + 1$.

122. True

123. True

124. True

125. $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(x, y) = \dfrac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

$f_y(x, y) = \dfrac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

(b) $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0/\left[(\Delta x)^2\right] - 0}{\Delta x} = 0$

$f_y(0, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{0/\left[(\Delta y)^2\right] - 0}{\Delta y} = 0$

(c) $f_{xy}(0, 0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)\Bigg|_{(0,0)} = \lim\limits_{\Delta y \to 0} \dfrac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{\Delta y\left(-(\Delta y)^4\right)}{\left((\Delta y)^2\right)^2(\Delta y)} = \lim\limits_{\Delta y \to 0} (-1) = -1$

$f_{yx}(0, 0) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)\Bigg|_{(0,0)} = \lim\limits_{\Delta x \to 0} \dfrac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x\left((\Delta x)^4\right)}{\left((\Delta x)^2\right)^2(\Delta x)} = \lim\limits_{\Delta x \to 0} 1 = 1$

(d) f_{yx} or f_{xy} or both are not continuous at $(0, 0)$.

126. $f(x, y) = (x^3 + y^3)^{1/3}$

(a) $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1$

$f_y(0, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y}$

$= \lim\limits_{\Delta y \to 0} \dfrac{\Delta y}{\Delta y} = 1$

(b) $f_x(x, y)$ and $f_y(x, y)$ fail to exist for $y = -x$, $x \neq 0$.

127. $f(x, y) = (x^2 + y^2)^{2/3}$

For $(x, y) \neq (0, 0)$, $f_x(x, y) = \dfrac{2}{3}(x^2 + y^2)^{-1/3}(2x) = \dfrac{4x}{3(x^2 + y^2)^{1/3}}$.

For $(x, y) = (0, 0)$, use the definition of partial derivative.

$f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(0 + \Delta x) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{(\Delta x)^{4/3}}{\Delta x} = \lim\limits_{\Delta x \to 0} (\Delta x)^{1/3} = 0$

Section 13.4 Differentials

1. $z = 2x^2y^3$

$dz = 4xy^3\,dx + 6x^2y^2\,dy$

2. $z = 2x^4y - 8x^2y^3$

$dz = (8x^3y - 16xy^3)\,dx + (2x^4 - 24x^2y^2)\,dy$

3. $z = \dfrac{-1}{x^2 + y^2}$

$$dz = \frac{2x}{(x^2 + y^2)^2}\,dx + \frac{2y}{(x^2 + y^2)^2}\,dy = \frac{2}{(x^2 + y^2)^2}(x\,dx + y\,dy)$$

4. $w = \dfrac{x + y}{z - 3y}$

$$dw = \frac{1}{z - 3y}\,dx + \frac{3x + z}{(z - 3y)^2}\,dy - \frac{x + y}{(z - 3y)^2}\,dz$$

5. $z = x\cos y - y\cos x$

$dz = (\cos y + y\sin x)\,dx + (-x\sin y - \cos x)\,dy$

$\quad = (\cos y + y\sin x)\,dx - (x\sin y + \cos x)\,dy$

6. $z = \left(\dfrac{1}{2}\right)\left(e^{x^2+y^2} - e^{-x^2-y^2}\right)$

$$dz = 2x\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right)dx + 2y\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right)dy = \left(e^{x^2+y^2} + e^{-x^2-y^2}\right)(x\,dx + y\,dy)$$

7. $z = e^x \sin y$

$dz = (e^x \sin y)\,dx + (e^x \cos y)\,dy$

8. $w = e^y \cos x + z^2$

$dw = -e^y \sin x\,dx + e^y \cos x\,dy + 2z\,dz$

9. $w = 2z^3y\sin x$

$dw = 2z^3y\cos x\,dx + 2z^3\sin x\,dy + 6z^2y\sin x\,dz$

10. $w = x^2yz^2 + \sin yz$

$$dw = 2xyz^2\,dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$$

11. $f(x, y) = 2x - 3y$

(a) $f(2, 1) = 1$

$f(2.1, 1.05) = 1.05$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0.05$

(b) $dz = 2\,dx - 3\,dy = 2(0.1) - 3(0.05) = 0.05$

12. $f(x, y) = x^2 + y^2$

(a) $f(2, 1) = 5$

$f(2.1, 1.05) = 5.5125$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0.5125$

(b) $dz = 2x\,dx + 2y\,dy$

$\quad = 2(2)(0.1) + 2(1)(0.05) = 0.5$

13. $f(x, y) = 16 - x^2 - y^2$

(a) $f(2, 1) = 11$

$f(2.1, 1.05) = 10.4875$

$\Delta z = f(2.1, 1.05) - f(2.1) = -0.5125$

(b) $dz = -2x\,dx - 2y\,dy$

$\quad = -2(2)(0.1) - 2(1)(0.05) = -0.5$

14. $f(x, y) = \dfrac{y}{x}$

(a) $f(2, 1) = 0.5$

$f(2.1, 1.05) = 0.5$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 0$

(b) $dz = \dfrac{-y}{x^2}\,dx + \dfrac{1}{x}\,dy = \dfrac{-1}{4}(0.1) + \dfrac{1}{2}(0.05) = 0$

15. $f(x, y) = ye^x$

(a) $f(2, 1) = e^2 \approx 7.3891$

$f(2.1, 1.05) = 1.05e^{2.1} \approx 8.5745$

$\Delta z = f(2.1, 1.05) - f(2, 1) = 1.1854$

(b) $dz = ye^x\,dx + e^x\,dy$

$\quad = e^2(0.1) + e^2(0.05) \approx 1.1084$

16. $f(x, y) = x \cos y$

(a) $f(2, 1) = 2 \cos 1 \approx 1.0806$

$f(2.1, 1.05) = 2.1 \cos 1.05 \approx 1.0449$

$\Delta z = f(2.1, 1.05) - f(2, 1) = -0.0357$

(b) $dz = \cos y\, dx - x \sin y\, dy$

$= \cos 1(0.1) - 2 \sin 1(0.05) \approx -0.0301$

17. Let $z = x^2 y,\ x = 2,\ y = 9,\ dx = 0.01,\ dy = 0.02.$

Then: $dz = 2xy\, dx + x^2\, dy$

$(2.01)^2(9.02) - 2^2 \cdot 9 \approx 2(2)(9)(0.01) + 2^2(0.02) = 0.44$

18. Let $z = (1 - x^2)/y^2,\ x = 3,\ y = 6,\ dx = 0.05,\ dy = -0.05.$ Then:

$$dz = -\frac{2x}{y^2}\, dx + \frac{-2(1 - x^2)}{y^3}\, dy$$

$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

19. Let $z = \sqrt{x^2 + y^2}, x = 5,\ y = 3,\ dx = 0.05,\ dy = 0.1.$

Then:

$$dz = \frac{x}{\sqrt{x^2 + y^2}}\, dx + \frac{y}{\sqrt{x^2 + y^2}}\, dy$$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

20. Let $z = \sin(x^2 + y^2),\ x = y = 1,\ dx = 0.05,\ dy = -0.05.$ Then: $dz = 2x \cos(x^2 + y^2)\, dx + 2y \cos(x^2 + y^2)\, dy$

$$\sin\left[(1.05)^2 + (0.95)^2\right] - \sin 2 \approx 2(1) \cos(1^2 + 1^2)(0.05) + 2(1) \cos(1^2 + 1^2)(-0.05) = 0$$

21. In general, the accuracy worsens as Δx and Δy increase.

22. The tangent plane to the surface $z = f(x, y)$ at the point P is a linear approximation of z.

23. If $z = f(x, y)$, then $\Delta z \approx dz$ is the propagated error, and $\dfrac{\Delta z}{z} \approx \dfrac{dz}{z}$ is the relative error.

24. The differential is greater at $\left(\frac{1}{2}, \frac{1}{2}\right)$ than at $(2, 2)$ because the surface is increasing faster there.

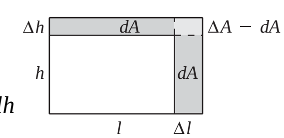
25. $A = lh$

$dA = l\, dh + h\, dl$

$\Delta A = (1 + dl)(h + dh) - lh$

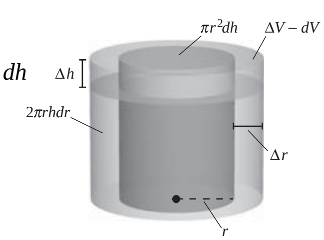
$= h\, dl + l\, dh + dl\, dh$

$\Delta A - dA = dl\, dh$



26. $V = \pi r^2 h$

$dV = 2\pi rh\, dr + \pi r^2\, dh$



27. $V = \dfrac{\pi r^2 h}{3}, r = 4, h = 8$

$$dV = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh = \frac{\pi r}{3}(2h\,dr + r\,dh) = \frac{4\pi}{3}(16\,dr + 4\,dh)$$

$$\Delta V = \frac{\pi}{3}\Big[(r + \Delta r)^2(h + \Delta h) - r^2 h\Big] = \frac{\pi}{3}\Big[(4 + \Delta r)^2(8 + \Delta h) - 128\Big]$$

Δr	Δh	dV	ΔV	$\Delta V - dV$
0.1	0.1	8.3776	8.5462	0.1686
0.1	–0.1	5.0265	5.0255	–0.0010
0.001	0.002	0.1005	0.1006	0.0001
–0.0001	0.0002	–0.0034	–0.0034	0.0000

28. $S = \pi r\sqrt{r^2 + h^2}, r = 6, h = 16$

$$\frac{dS}{dr} = \pi\left(r^2 + h^2\right)^{1/2} + \pi r^2\left(r^2 + h^2\right)^{-1/2} = \pi\frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}$$

$$\frac{dS}{dh} = \pi\frac{rh}{\sqrt{r^2 + h^2}}$$

$$dS = \frac{\pi}{\sqrt{r^2 + h^2}}\Big[\left(2r^2 + h^2\right)dr + (rh)dh\Big] = \frac{\pi}{\sqrt{292}}[328\,dr + 96\,dh]$$

$$S(6, 16) = 322.101353$$

$$\Delta S = \pi(r + \Delta r)\sqrt{(r + \Delta r)^2 + (h + \Delta h)^2} = \pi(6 + \Delta r)\sqrt{(6 + \Delta r)^2 + (16 + \Delta h)^2} - 322.101353$$

Δr	Δh	dS	ΔS	$\Delta S - dS$
0.1	0.1	7.7951	7.8375	0.0424
0.1	–0.1	4.2653	4.2562	–0.0091
0.001	0.002	0.0956	0.0956	0.0000
–0.0001	0.0002	–0.0025	–0.0025	–0.0000

29. $V = xyz, dV = yz\,dx + xz\,dy + xy\,dz$

$$\text{Propagated error} = dV = 5(12)(\pm 0.02) + 8(12)(\pm 0.02) + 8(5)(\pm 0.02)$$
$$= (60 + 96 + 40)(\pm 0.02) = 196(\pm 0.02) = \pm 3.92 \text{ in.}^3$$

The measured volume is $V = 8(5)(12) = 480 \text{ in.}^3$

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3.92}{480} \approx 0.008167 \approx 0.82\%$

30. $V = \pi r^2 h, dV = 2\pi rh\,dr + \pi r^2\,dh$

$$\text{Propagated error} = dV = 2\pi(3)(10)(\pm 0.05) + \pi(3)^2(\pm 0.05)$$
$$= (60\pi + 9\pi)(\pm 0.05) = \pm 3.45\pi \text{ cm}^3$$

The measured volume is $V = \pi\left(3^2\right)(10) = 90\pi \text{ cm}^3$.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3.45\pi}{90\pi} \approx 0.0383 = 3.83\%$

31. $C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$

$$\frac{\partial C}{\partial T} = 0.6215 + 0.4275v^{0.16}$$

$$\frac{\partial C}{\partial v} = -5.72v^{-0.84} + 0.0684Tv^{-0.84}$$

$$dC = \frac{\partial C}{\partial T}dT + \frac{\partial C}{\partial v}dv = \left(0.6215 + 0.4275(23)^{0.16}\right)(\pm 1) + \left(-5.72(23)^{-0.84} + 0.0684(8)(23)^{-0.84}\right)(\pm 3)$$

$$= \pm 1.3275 \pm 1.1143 = \pm 2.4418 \text{ Maximum propagated error}$$

$$\frac{dC}{C} = \frac{2.4418}{-12.6807} \approx 0.19 = 19\% \text{ Maximum relative error}$$

32. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

$$R = \frac{R_1R_2}{R_1 + R_2}$$

$$dR_1 = \Delta R_1 = 0.5$$

$$dR_2 = \Delta R_2 = -2$$

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}dR + \frac{\partial R}{\partial R_2}dR_2 = \frac{R_2^2}{(R_1 + R_2)^2}\Delta R_1 + \frac{R_1^2}{(R_1 + R_2)^2}\Delta R_2$$

When $R_1 = 10$ and $R_2 = 15$, we have $\Delta R \approx \dfrac{15^2}{(10 + 15)^2}(0.5) + \dfrac{10^2}{(10 + 15)^2}(-2) = -0.14$ ohm.

33. $P = \dfrac{E^2}{R}$, $\left|\dfrac{dE}{E}\right| = 3\% = 0.03$, $\left|\dfrac{dR}{R}\right| = 4\% = 0.04$

$$dP = \frac{2E}{R}dE - \frac{E^2}{R^2}dR$$

$$\frac{dP}{P} = \left[\frac{2E}{R}dE - \frac{E^2}{R^2}dR\right]\Big/P = \left[\frac{2E}{R}dE - \frac{E^2}{R^2}dR\right]\Big/\left(E^2/R\right) = \frac{2}{E}dE - \frac{1}{R}dR$$

Using the worst case scenario, $\dfrac{dE}{E} = 0.03$ and $\dfrac{dR}{R} = -0.04$: $\dfrac{dP}{P} \le 2(0.03) - (-0.04) = 0.10 = 10\%$.

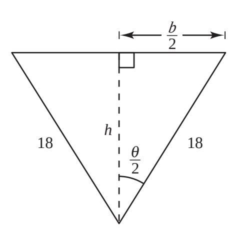
34. $a = \dfrac{v^2}{r}$

$$da = \frac{2v}{r}dv - \frac{v^2}{r^2}dr$$

$$\frac{da}{a} = 2\frac{dv}{v} - \frac{dr}{r} = 2(0.03) - (-0.02) = 0.08 = 8\%$$

Note: The maximum error will occur when *dv* and *dr* differ in signs.

35. (a) $V = \dfrac{1}{2}bhl = \left(18\sin\dfrac{\theta}{2}\right)\left(18\cos\dfrac{\theta}{2}\right)(16)(12) = 31{,}104\sin\theta \text{ in.}^3 = 18\sin\theta \text{ ft}^3$



V is maximum when $\sin\theta = 1$ or $\theta = \pi/2$.

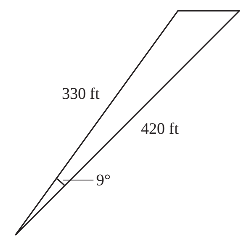
(b) $V = \dfrac{s^2}{2}(\sin\theta)l$

$$dV = s(\sin\theta)l\,ds + \frac{s^2}{2}l(\cos\theta)\,d\theta + \frac{s^2}{2}(\sin\theta)\,dl$$

$$= 18\left(\sin\frac{\pi}{2}\right)(16)(12)\left(\frac{1}{2}\right) + \frac{18^2}{2}(16)(12)\left(\cos\frac{\pi}{2}\right)\left(\frac{\pi}{90}\right) + \frac{18^2}{2}\left(\sin\frac{\pi}{2}\right)\left(\frac{1}{2}\right) = 1809 \text{ in.}^3 \approx 1.047 \text{ ft}^3$$

36. (a) Using the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A = 330^2 + 420^2 - 2(330)(420)\cos 9°$$
$$a \approx 107.3 \text{ ft.}$$



(b) $a = \sqrt{b^2 + 420^2 - 2b(420)\cos\theta}$

$$da = \frac{1}{2}\left[b^2 + 420^2 - 840b\cos\theta\right]^{-1/2}\left[(2b - 840\cos\theta)\,db + 840b\sin\theta\,d\theta\right]$$

$$= \frac{1}{2}\left[330^2 + 420^2 - 840(330)\left(\cos\frac{\pi}{20}\right)\right]^{-1/2}\left[\left(2(330) - 840\cos\frac{\pi}{20}\right)(6) + 840(330)\left(\sin\frac{\pi}{20}\right)\left(\frac{\pi}{180}\right)\right]$$

$$\approx \frac{1}{2}[11512.79]^{-1/2}[\pm 1774.79] \approx \pm 8.27 \text{ ft}$$

37. $L = 0.00021\left(\ln\dfrac{2h}{r} - 0.75\right)$

$$dL = 0.00021\left[\frac{dh}{h} - \frac{dr}{r}\right] = 0.00021\left[\frac{(\pm 1/100)}{100} - \frac{(\pm 1/16)}{2}\right] \approx (\pm 6.6) \times 10^{-6}$$

$$L = 0.00021(\ln 100 - 0.75) \pm dL \approx 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6} \text{ micro henrys}$$

38. $T = 2\pi\sqrt{\dfrac{L}{g}}$

$$dg = 32.23 - 32.09 = 0.14$$
$$dL = 2.48 - 2.50 = -0.02$$

$$\Delta T \approx dT = \frac{\partial T}{\partial g}dg + \frac{\partial T}{\partial L}dL = \frac{-\pi}{g}\sqrt{\frac{L}{g}}\,dg + \frac{\pi}{\sqrt{Lg}}\,dL$$

When $g = 32.09$ and $L = 2.50$, $\Delta T \approx \dfrac{-\pi}{32.09}\sqrt{\dfrac{2.5}{32.09}}(0.14) + \dfrac{\pi}{\sqrt{(2.5)(32.09)}}(-0.02) \approx -0.0108$ seconds.

39. $z = f(x, y) = x^2 - 2x + y$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = \left(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + y + (\Delta y)\right) - \left(x^2 - 2x + y\right)$$
$$= 2x(\Delta x) + (\Delta x)^2 - 2(\Delta x) + (\Delta y) = (2x - 2)\,\Delta x + \Delta y + \Delta x(\Delta x) + 0(\Delta y)$$
$$= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = \Delta x \text{ and } \varepsilon_2 = 0.$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

40. $z = f(x, y) = x^2 + y^2$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = x^2 + 2x(\Delta x) + (\Delta x)^2 + y^2 + 2y(\Delta y) + (\Delta y)^2 - (x^2 + y^2)$$
$$= 2x(\Delta x) + 2y(\Delta y) + \Delta x(\Delta x) + \Delta y(\Delta y) = f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = \Delta x \text{ and } \varepsilon_2 = \Delta y.$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

41. $z = f(x, y) = x^2 y$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = \left(x^2 + 2x(\Delta x) + (\Delta x)^2\right)(y + \Delta y) - x^2 y$$
$$= 2xy(\Delta x) + y(\Delta x)^2 + x^2\Delta y + 2x(\Delta x)(\Delta y) + (\Delta x)^2\Delta y = 2xy(\Delta x) + x^2\Delta y + (y\Delta x)\Delta x + \left[2x\Delta x + (\Delta x)^2\right]\Delta y$$
$$= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = y(\Delta x) \text{ and } \varepsilon_2 = 2x\Delta x + (\Delta x)^2.$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

42. $z = f(x, y) = 5x - 10y + y^3$

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= 5x + 5\Delta x - 10y - 10\Delta y + y^3 + 3y^2(\Delta y) + 3y(\Delta y)^2 + (\Delta y)^3 - (5x - 10y + y^3)\\ &= 5(\Delta x) + (3y^2 - 10)(\Delta y) + 0(\Delta x) + \left(3y(\Delta y) + (\Delta y)^2\right)\Delta y\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = 0 \text{ and } \varepsilon_2 = 3y(\Delta y) + (\Delta y)^2.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

43. $f(x, y) = \begin{cases} \dfrac{3x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0)\\ 0, & (x, y) = (0, 0)\end{cases}$

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{0}{(\Delta x)^4} - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{0}{(\Delta y)^2} - 0}{\Delta y} = 0$$

So, the partial derivatives exist at $(0, 0)$.

Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{3x^3}{x^4 + x^2} = \lim_{x \to 0} \frac{3x}{x^2 + 1} = 0$

Along the curve $y = x^2$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \frac{3x^4}{2x^4} = \frac{3}{2}$

f is not continuous at $(0, 0)$. So, f is not differentiable at $(0, 0)$. (See Theorem 12.5)

44. $f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0)\\ 0, & (x, y) = (0, 0)\end{cases}$

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

So, the partial derivatives exist at $(0, 0)$.

Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{5x^3}{2x^3} = \frac{5}{2}.$

Along the line $x = 0$, $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = 0.$

So, f is not continuous at $(0, 0)$. Therefore f is not differentiable at $(0, 0)$.

Section 13.5 Chain Rules for Functions of Several Variables

1. $w = x^2 + y^2$

$x = 2t,\ y = 3t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = (2x)(2) + (2y)(3)$$

$$= 4x + 6y = 8t + 18t = 26t$$

When $t = 2$, $\dfrac{dw}{dt} = 26(2) = 52$.

2. $w = \sqrt{x^2 + y^2}$

$x = \cos t,\ y = e^t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{x}{\sqrt{x^2 + y^2}}(-\sin t) + \frac{y}{\sqrt{x^2 + y^2}}e^t$$

$$= \frac{-x\sin t + ye^t}{\sqrt{x^2 + y^2}} = \frac{-\cos t\sin t + e^{2t}}{\sqrt{\cos^2 t + e^{2t}}}$$

When $t = 0$, $\dfrac{dw}{dt} = \dfrac{-(1)(0) + 1}{\sqrt{1^2 + 1}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

3. $w = x\sin y$

$x = e^t,\ y = \pi - t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{\partial t} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \sin y\left(e^t\right) + x\cos y(-1)$$

$$= \sin(\pi - t)e^t - e^t\cos(\pi - t) = e^t\sin t + e^t\cos t$$

When $t = 0$, $\dfrac{dw}{dt} = (1)(0) + (1)(1) = 0 + 1 = 1$.

4. $w = \ln\dfrac{y}{x}$

$x = \cos t$

$y = \sin t$

$$\frac{dw}{dt} = \left(\frac{-1}{x}\right)(-\sin t) + \left(\frac{1}{y}\right)(\cos t)$$

$$= \tan t + \cot t = \frac{1}{\sin t\cos t}$$

When $t = \dfrac{\pi}{4}$, $\dfrac{dw}{dt} = \dfrac{1}{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)} = \dfrac{1}{2}$.

5. $w = xy,\ x = e^t,\ y = e^{-2t}$

(a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

$$= y\left(e^t\right) + x\left(-2e^{-2t}\right) = e^{-2t}e^t - e^t2e^{-2t} = -e^{-t}$$

(b) $w = e^te^{-2t} = e^{-t}$

$$\frac{dw}{dt} = -e^{-t}$$

6. $w = \cos(x - y),\ x = t^2,\ y = 1$

(a) $\dfrac{dw}{dt} = -\sin(x - y)(2t) + \sin(x - y)(0)$

$$= -2t\sin(x - y) = -2t\sin\left(t^2 - 1\right)$$

(b) $w = \cos\left(t^2 - 1\right),\ \dfrac{dw}{dt} = -2t\sin\left(t^2 - 1\right)$

7. $w = x^2 + y^2 + z^2,\ x = \cos t,\ y = \sin t,\ z = e^t$

(a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$

$$= 2x(-\sin t) + 2y(\cos t) + 2z\left(e^t\right)$$

$$= -2\cos t\sin t + 2\sin t\cos t + 2e^{2t} = 2e^{2t}$$

(b) $w = \cos^2 t + \sin^2 t + e^{2t} = 1 + e^{2t}$

$$\frac{dw}{dt} = 2e^{2t}$$

8. $w = xy\cos z$

$x = t$

$y = t^2$

$z = \arccos t$

(a) $$\frac{dw}{dt} = (y\cos z)(1) + (x\cos z)(2t) + (-xy\sin z)\left(-\frac{1}{\sqrt{1-t^2}}\right) = t^2(t) + t(t)(2t) - t(t^2)\sqrt{1-t^2}\left(\frac{-1}{\sqrt{1-t^2}}\right)$$
$$= t^3 + 2t^3 + t^3 = 4t^3$$

(b) $w = t^4, \dfrac{dw}{dt} = 4t^3$

9. $w = xy + xz + yz,\ x = t - 1,\ y = t^2 - 1,\ z = t$

(a) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (y + z) + (x + z)(2t) + (x + y)$$
$$= (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1)$$

(b) $$w = (t-1)(t^2-1) + (t-1)t + (t^2-1)t$$
$$\frac{dw}{dt} = 2t(t-1) + (t^2-1) + 2t - 1 + 3t^2 - 1 = 3(2t^2-1)$$

10. $w = xy^2 + x^2z + yz^2,\ x = t^2, y = 2t, z = 2$

(a) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$
$$= (y^2 + 2xz)(2t) + (2xy + z^2)(2) + (x^2 + 2yz)(0) = (4t^2 + 4t^2)(2t) + (4t^3 + 4)(2) = 24t^3 + 8$$

(b) $$w = t^2(4t^2) + t^4(2) + 2t(4) = 6t^4 + 8t$$
$$\frac{dw}{dt} = 24t^3 + 8$$

11. Distance $= f(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(10\cos 2t - 7\cos t)^2 + (6\sin 2t - 4\sin t)^2}$

$$f'(t) = \frac{1}{2}\left[(10\cos 2t - 7\cos t)^2 + (6\sin 2t - 4\sin t)^2\right]^{-1/2}$$
$$\left[\left[2(10\cos 2t - 7\cos t)(-20\sin 2t + 7\sin t)\right] + \left[2(6\sin 2t - 4\sin t)(12\cos 2t - 4\cos t)\right]\right]$$

$$f'\left(\frac{\pi}{2}\right) = \frac{1}{2}\left[(-10)^2 + 4^2\right]^{-1/2}\left[\left[2(-10)(7)\right] + \left(2(-4)(-12)\right)\right] = \frac{1}{2}(116)^{-1/2}(-44) = \frac{-22}{2\sqrt{29}} = \frac{-11\sqrt{29}}{29} \approx -2.04$$

12. Distance $= f(t) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left[48t(\sqrt{3} - \sqrt{2})\right]^2 + \left[48t(1 - \sqrt{2})\right]^2} = 48t\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}}$

$$f'(t) = 48\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}} = f'(1)$$

13. $w = x^2 + y^2$

$x = s + t, y = s - t$

$$\frac{\partial w}{\partial s} = 2x(1) + 2y(1) = 2(s + t) + 2(s - t) = 4s$$

$$\frac{\partial w}{\partial t} = 2x(1) + 2y(-1) = 2(s + t) - 2(s - t) = 4t$$

When $s = 1$ and $t = 0$, $\dfrac{\partial w}{\partial s} = 4$ and $\dfrac{\partial w}{\partial t} = 0$.

14. $w = y^3 - 3x^2y$

$x = e^s, y = e^t$

$$\frac{\partial w}{\partial s} = -6xy(e^s) + (3y^2 - 3x^2)(0) = -6e^se^te^s = -6e^{2s+t}$$

$$\frac{\partial w}{\partial t} = (-6xy)(0) + (3y^2 - 3x^2)e^t = (3e^{2t} - 3e^{2s})e^t$$
$$= 3e^{3t} - 3e^{2s+t}$$

When $s = -1$ and $t = 2$, $\dfrac{\partial w}{\partial s} = -6$ and $\dfrac{\partial w}{\partial t} = 3e^6 - 3.$

15. $w = \sin(2x + 3y)$

$x = s + t$

$y = s - t$

$$\frac{\partial w}{\partial s} = 2\cos(2x + 3y) + 3\cos(2x + 3y)$$
$$= 5\cos(2x + 3y) = 5\cos(5s - t)$$

$$\frac{\partial w}{\partial t} = 2\cos(2x + 3y) - 3\cos(2x + 3y)$$
$$= -\cos(2x + 3y) = -\cos(5s - t)$$

When $s = 0$ and $t = \frac{\pi}{2}$, $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = 0$.

16. $w = x^2 - y^2$

$x = s\cos t$

$y = s\sin t$

$$\frac{\partial w}{\partial s} = 2x\cos t - 2y\sin t$$
$$= 2s\cos^2 t - 2s\sin^2 t = 2s\cos 2t$$

$$\frac{\partial w}{\partial t} = 2x(-s\sin t) - 2y(s\cos t) = -2s^2\sin 2t$$

When $s = 3$ and $t = \frac{\pi}{4}$, $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = -18$.

17. (a) $w = xyz$, $x = s + t$, $y = s - t$, $z = st^2$

$$\frac{\partial w}{\partial s} = yz(1) + xz(1) + xy(t^2)$$
$$= (s - t)st^2 + (s + t)st^2 + (s + t)(s - t)t^2 = 2s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4 = t^2(3s^2 - t^2)$$

$$\frac{\partial w}{\partial t} = yz(1) + xz(-1) + xy(2st) = (s - t)st^2 - (s + t)st^2 + (s + t)(s - t)(2st) = -2st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3$$
$$= 2st(s^2 - 2t^2)$$

(b) $w = xyz = (s + t)(s - t)st^2 = (s^2 - t^2)st^2 = s^3t^2 - st^4$

$$\frac{\partial w}{\partial s} = 3s^2t^2 - t^4 = t^2(3s^2 - t^2)$$

$$\frac{\partial w}{\partial t} = 2s^3t - 4st^3 = 2st(s^2 - 2t^2)$$

18. (a) $w = x^2 + y^2 + z^2$, $x = t\sin s$, $y = t\cos s$, $z = st^2$

$$\frac{\partial w}{\partial s} = 2x + \cos s + 2y(-t\sin s) + 2z(t^2)$$
$$= 2t^2\sin s\cos s - 2t^2\sin s\cos s + 2st^4 = 2st^4$$

$$\frac{\partial w}{\partial t} = 2x\sin s + 2y\cos s + 2z(2st)$$
$$= 2t\sin^2 s + 2t\cos^2 s + 4s^2t^3 = 2t + 4s^2t^3$$

(b) $w = x^2 + y^2 + z^2 = (t\sin s)^2 + (t\cos s)^2 + (st^2)^2$

$$= t^2(\sin^2 s + \cos^2 s) + s^2t^4$$
$$= t^2 + s^2t^4$$

$$\frac{\partial w}{\partial s} = 2st^4$$

$$\frac{\partial w}{\partial t} = 2t + 4s^2t^3$$

19. (a) $w = ze^{xy},\ x = s - t, y = s + t, z = st$

$$\begin{aligned}\frac{\partial w}{\partial s} &= yze^{xy}(1) + xze^{xy}(1) + e^{xy}(t)\\ &= e^{(s-t)(s+t)}\left[(s + t)st + (s - t)st + t\right]\\ &= e^{(s-t)(s+t)}\left[2s^2t + t\right] = te^{s^2-t^2}\left(2s^2 + 1\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= yze^{xy}(-1) + xze^{xy}(1) + e^{xy}(s)\\ &= e^{(s-t)(s+t)}\left[-(s + t)(st) + (s - t)st + s\right]\\ &= e^{(s-t)(s+t)}\left[-2st^2 + s\right] = se^{s^2-t^2}\left(1 - 2t^2\right)\end{aligned}$$

(b) $w = ze^{xy} = ste^{(s-t)(s+t)} = ste^{s^2-t^2}$

$$\frac{\partial w}{\partial s} = te^{s^2-t^2} + st(2s)e^{s^2-t^2} = te^{s^2-t^2}\left(1 + 2s^2\right)$$

$$\frac{\partial w}{\partial t} = se^{s^2-t^2} + st(-2t)e^{s^2-t^2} = se^{s^2-t^2}\left(1 - 2t^2\right)$$

20. (a) $w = x\cos yz,\ x = s^2,\ y = t^2,\ z = s - 2t$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \cos(yz)(2s) - xz\sin(yz)(0) - xy\sin(yz)(1)\\ &= \cos\left(st^2 - 2t^3\right)2s - s^2t^2\sin\left(st^2 - 2t^3\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= \cos(yz)(0) - xz\sin(yz)(2t) - xy\sin(yz)(-2)\\ &= -2s^2t(s - 2t)\sin\left(st^2 - 2t^3\right) + 2s^2t^2\sin\left(st^2 - 2t^3\right)\\ &= \left(6s^2t^2 - 2s^3t\right)\sin\left(st^2 - 2t^3\right)\end{aligned}$$

(b) $w = x\cos yz = s^2\cos\left(t^2(s - 2t)\right) = s^2\cos\left(st^2 - 2t^3\right)$

$$\begin{aligned}\frac{\partial w}{\partial s} &= s^2\left(-\sin\left(st^2 - 2t^3\right)\right)\left(t^2\right) + 2s\cos\left(st^2 - 2t^3\right)\\ &= 2s\cos\left(st^2 - 2t^3\right) - s^2t^2\sin\left(st^2 - 2t^3\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= -s^2\sin\left(st^2 - 2t^3\right)\left(2st - 6t^2\right)\\ &= \left(6t^2s^2 - 2s^3t\right)\sin\left(st^2 - 2t^3\right)\end{aligned}$$

21. $x^2 - xy + y^2 - x + y = 0$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{2x - y - 1}{-x + 2y + 1} = \frac{y - 2x + 1}{2y - x + 1}$$

22. $\sec xy + \tan xy + 5 = 0$

$$\begin{aligned}\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} &= -\frac{y\sec xy\tan xy + y\sec^2 xy}{x\sec xy\tan xy + x\sec^2 xy}\\ &= \frac{-y\left(\sec xy\tan xy + \sec^2 xy\right)}{x\left(\sec xy\tan xy + \sec^2 xy\right)} = -\frac{y}{x}\end{aligned}$$

23. $\ln\sqrt{x^2 + y^2} + x + y = 4$

$$\frac{1}{2}\ln\left(x^2 + y^2\right) + x + y - 4 = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{\dfrac{x}{x^2 + y^2} + 1}{\dfrac{y}{x^2 + y^2} + 1} = -\frac{x + x^2 + y^2}{y + x^2 + y^2}$$

24. $\dfrac{x}{x^2 + y^2} - y^2 - 6 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$
$$= -\frac{(y^2 - x^2)/(x^2 + y^2)^2}{(-2xy)/(x^2 + y^2)^2 - 2y}$$
$$= \frac{y^2 - x^2}{2xy + 2y(x^2 + y^2)^2}$$
$$= \frac{y^2 - x^2}{2xy + 2yx^4 + 4x^2y^3 + 2y^5}$$

25. $F(x, y, z) = x^2 + y^2 + z^2 - 1$

$F_x = 2x, F_y = 2y, F_z = 2z$

$$\frac{\partial_z}{\partial_x} = -\frac{F_x}{F_z} = -\frac{x}{z}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$$

26. $F(x, y, z) = xz + yz + xy$

$F_x = z + y$

$F_y = z + x$

$F_z = x + y$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y + z}{x + y}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x + z}{x + y}$$

27. $F(x, y, z) = x^2 + 2yz + z^2 - 1 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{-2x}{2y + 2z} = \frac{-x}{y + z}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-2z}{2y + 2z} = \frac{-z}{y + z}$$

28. $x + \sin(y + z) = 0$

(i) $1 + \dfrac{\partial z}{\partial x}\cos(y + z) = 0$ implies

$$\frac{\partial z}{\partial x} = -\frac{1}{\cos(y + z)} = -\sec(y + z).$$

(ii) $\left(1 + \dfrac{\partial z}{\partial y}\right)\cos(y + z) = 0$ implies $\dfrac{\partial z}{\partial y} = -1.$

29. $F(x, y, z) = \tan(x + y) + \tan(y + z) - 1$

$F_x = \sec^2(x + y)$

$F_y = \sec^2(x + y) + \sec^2(y + z)$

$F_z = \sec^2(y + z)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\sec^2(x + y)}{\sec^2(y + z)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\sec^2(x + y) + \sec^2(y + z)}{\sec^2(y + z)}$$
$$= -\left(\frac{\sec^2(x + y)}{\sec^2(y + z)} + 1\right)$$

30. $F(x, y, z) = e^x \sin(y + z) - z$

$F_x = e^x \sin(y + z)$

$F_y = e^x \cos(y + z)$

$F_z = e^x \cos(y + z) - 1$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{e^x \sin(y + z)}{1 - e^x \cos(y + x)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x \cos(y + z)}{1 - e^x \cos(y + z)}$$

31. $F(x, y, z) = e^{xz} + xy = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{ze^{xz} + y}{xe^{xz}}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-x}{xe^{xz}} = \frac{-1}{e^{xz}} = -e^{-xz}$$

32. $x \ln y + y^2z + z^2 - 8 = 0$

(i) $$\frac{\partial z}{\partial x} = \frac{-F_x(x, y, z)}{F_z(x, y, z)} = \frac{-\ln y}{y^2 + 2z}$$

(ii) $$\frac{\partial z}{\partial y} = \frac{-F_y(x, y, z)}{F_z(x, y, z)} = -\frac{\frac{x}{y} + 2yz}{y^2 + 2z} = -\frac{x + 2y^2z}{y^3 + 2yz}$$

33. $F(x, y, z, w) = xy + yz - wz + wx - s$

$F_x = y + w$

$F_y = x + z$

$F_z = y - w$

$F_w = -z + x$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{y + w}{-z + x} = \frac{y + w}{z - x}$$
$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{x + z}{-z + x} = \frac{x + z}{z - x}$$
$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{y - w}{-z + x} = \frac{y - w}{z - x}$$

34. $x^2 + y^2 - z^2 - 5yw + 10w^2 - 2 = F(x, y, z, w)$

$F_x = 2x, F_y = 2y - 5w,\ F_z = 2z, F_w = -5y + 20w$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \frac{-2x}{-5y + 20w} = \frac{2x}{5y - 20w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{5w - 2y}{20w - 5y}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = \frac{2z}{5y - 20w}$$

35. $F(x, y, z, w) = \cos xy + \sin yz + wz - 20$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{y \sin xy}{z}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{x \sin xy - z \cos yz}{z}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = -\frac{y \cos zy + w}{z}$$

36. $F(x, y, z, w) = w - \sqrt{x - y} - \sqrt{y - z} = 0$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{1}{2}\frac{(x - y)^{-1/2}}{1} = \frac{1}{2\sqrt{x - y}}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{-1}{2}(x - y)^{-1/2} + \frac{1}{2}(y - z)^{-1/2} = \frac{-1}{2\sqrt{x - y}} + \frac{1}{2\sqrt{y - z}}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y - z}}$$

37. (a) $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{xy}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

(b) $xf_x(x, y) + yf_y(x, y) = x\left(\dfrac{y^3}{\left(x^2 + y^2\right)^{3/2}}\right) + y\left(\dfrac{x^3}{\left(x^2 + y^2\right)^{3/2}}\right) = \dfrac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y)$

38. (a) $f(x, y) = x^3 - 3xy^2 + y^3$

$$f(tx, ty) = (tx)^3 - 3(tx)(ty)^2 + (ty)^3 = t^3\left(x^3 - 3xy^2 + y^3\right) = t^3 f(x, y)$$

Degree: 3

(b) $xf_x(x, y) + yf_y(x, y) = x\left(3x^2 - 3y^2\right) + y\left(-6xy + 3y^2\right) = 3x^3 - 9xy^2 + 3y^3 = 3f(x, y)$

39. (a) $f(x, y) = e^{x/y}$

$$f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y)$$

Degree: 0

(b) $xf_x(x, y) + yf_y(x, y) = x\left(\dfrac{1}{y}e^{x/y}\right) + y\left(-\dfrac{x}{y^2}e^{x/y}\right) = 0$

40. (a) $f(x, y) = \dfrac{x^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)^2}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{x^2}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

(b) $$xf_x(x, y) + yf_y(x, y) = x\left[\frac{x^3 + 2xy^2}{\left(x^2 + y^2\right)^{3/2}}\right] + y\left[\frac{-x^2y}{\left(x^2 + y^2\right)^{3/2}}\right] = \frac{x^4 + x^2y^2}{\left(x^2 + y^2\right)^{3/2}} = \frac{x^2\left(x^2 + y^2\right)}{\left(x^2 + y^2\right)^{3/2}} = \frac{x^2}{\sqrt{x^2 + y^2}} = f(x, y)$$

41. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dg}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dh}{dt}$

At $t = 2$, $x = 4$, $y = 3$, $f_x(4, 3) = -5$ and $f_y(4, 3) = 7$.

So, $\dfrac{dw}{dt} = (-5)(-1) + (7)(6) = 47$

42. $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial s} = (-5)(-3) + (7)(5) = 50$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial t} = (-5)(-2) + (7)(8) = 66$$

43. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$ (Page 907)

44. $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s}$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$ (Page 909)

45. $\dfrac{dy}{dx} = -\dfrac{f_x(x, y)}{f_y(x, y)}$

$\dfrac{\partial z}{\partial x} = -\dfrac{f_x(x, y, z)}{f_z(x, y, z)}$

$\dfrac{\partial z}{\partial y} = -\dfrac{f_y(x, y, z)}{f_z(x, y, z)}$ (page 912)

46. (a) $\dfrac{dw}{dr} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dr} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dr}$

(b) $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r}$

$\dfrac{\partial w}{\partial \theta} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial \theta}$

47. $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right) = \pi(12)\left[2(36)(6) + 12(-4)\right] = 4608\pi \text{ in.}^3/\text{min}$$

$$S = 2\pi r(r + h)$$

$$\frac{dS}{dt} = 2\pi\left[(2r + h)\frac{dr}{dt} + r\frac{dh}{dt}\right] = 2\pi\left[(24 + 36)(6) + 12(-4)\right] = 624\pi \text{ in.}^2/\text{min}$$

48. $pV = mRT$

$$T = \frac{1}{mR}(pV)$$

$$\frac{dT}{dt} = \frac{1}{mR}\left[V\frac{dp}{dt} + p\frac{dV}{dt}\right]$$

49. $I = \dfrac{1}{2}m\left(r_1^2 + r_2^2\right)$

$$\frac{dI}{dt} = \frac{1}{2}m\left[2r_1\frac{dr_1}{dt} + 2r_2\frac{dr_2}{dt}\right] = m\left[(6)(2) + (8)(2)\right] = 28m \text{ cm}^2/\text{sec}$$

50.
$$V = \frac{\pi}{3}\left(r^2 + rR + R^2\right)h$$
$$\frac{dV}{dt} = \frac{\pi}{3}\left[(2r + R)h\frac{dr}{dt} + (r + 2R)h\frac{dR}{dt} + \left(r^2 + rR + R^2\right)\frac{dh}{dt}\right]$$
$$= \frac{\pi}{3}\left[\left[2(15) + 25\right](10)(4) + \left[15 + 2(25)\right](10)(4) + \left[(15)^2 + (15)(25) + (25)^2\right](12)\right]$$
$$= \frac{\pi}{3}(19{,}500)$$
$$= 6{,}500\pi \text{ cm}^3/\text{min}$$
$$S = \pi(R + r)\sqrt{(R - r)^2 + h^2}$$
$$\frac{dS}{dt} = \pi\left\{\left[\sqrt{(R - r)^2 + h^2} - (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dr}{dt} + \left[\sqrt{(R - r)^2 + h^2} + (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dR}{dt}\right.$$
$$\left. + (R + r)\frac{h}{\sqrt{(R - r)^2 + h^2}}\frac{dh}{dt}\right\}$$
$$= \pi\left\{\left[\sqrt{(25 - 15)^2 + 10^2} - (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4)\right.$$
$$\left. + \left[\sqrt{(25 - 15)^2 + 10^2} + (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + (25 + 15)\left[\frac{10}{\sqrt{(25 - 15)^2 + 10^2}}(12)\right]\right\}$$
$$= 320\sqrt{2}\pi \text{ cm}^2/\text{min}$$

51.
$$w = f(x, y)$$
$$x = u - v$$
$$y = v - u$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$
$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

52.
$$w = (x - y)\sin(y - x)$$
$$\frac{\partial w}{\partial x} = -(x - y)\cos(y - x) + \sin(y - x)$$
$$\frac{\partial w}{\partial y} = (x - y)\cos(y - x) - \sin(y - x)$$
$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

53. Given $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) = r\left[\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta\right]$$

So, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = -r\left[-\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta\right]$$

So, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

54. Note first that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}.$$

$$\frac{\partial u}{\partial r} = \frac{x}{x^2 + y^2}\cos\theta + \frac{y}{x^2 + y^2}\sin\theta = \frac{r\cos^2\theta + r\sin^2\theta}{r^2} = \frac{1}{r}$$

$$\frac{\partial v}{\partial \theta} = \frac{-y}{x^2 + y^2}(-r\sin\theta) + \frac{x}{x^2 + y^2}(r\cos\theta) = \frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2} = 1$$

So, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{-y}{x^2 + y^2}\cos\theta + \frac{x}{x^2 + y^2}\sin\theta = \frac{-r\sin\theta\cos\theta + r\sin\theta\cos\theta}{r^2} = 0$$

$$\frac{\partial u}{\partial \theta} = \frac{x}{x^2 + y^2}(-r\sin\theta) + \frac{y}{x^2 + y^2}(r\cos\theta) = \frac{-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta}{r^2} = 0$$

So, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

55. $g(t) = f(xt, yt) = t^n f(x, y)$

Let $u = xt$, $v = yt$, then

$$g'(t) = \frac{\partial f}{\partial u}\cdot\frac{du}{dt} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dt} = \frac{\partial f}{\partial u}x + \frac{\partial f}{\partial v}y$$

and $g'(t) = nt^{n-1}f(x, y)$.

Now, let $t = 1$ and we have $u = x$, $v = y$. Thus,

$$\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y = nf(x, y).$$

Section 13.6 Directional Derivatives and Gradients

1. $f(x, y) = x^2 + y^2$, $P(1, -2)$, $\theta = \pi/4$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= 2x\cos\theta + 2y\sin\theta$$

At $\theta = \pi/4$, $x = 1$, and $y = -2$,

$$D_{\mathbf{u}} f(1, -2) = 2(1)\cos\pi/4 + 2(-2)\sin\pi/4$$
$$= \sqrt{2} - 2\sqrt{2} = -\sqrt{2}.$$

2. $f(x, y) = \dfrac{y}{x + y}$, $P(3, 0)$, $\theta = -\pi/6$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= \frac{-y}{(x + y)^2}\cos\theta + \frac{x}{(x + y)^2}\sin\theta$$

At $\theta = -\pi/6$, $x = 3$, and $y = 0$,

$$D_{\mathbf{u}} f(3, 0) = \frac{3}{3^2}\sin\left(\frac{-\pi}{6}\right) = -\frac{1}{6}.$$

3. $f(x, y) = \sin(2x + y)$, $P(0, 0)$, $\theta = \pi/3$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= 2\cos(2x + y)\cos\theta + \cos(2x + y)\sin\theta$$

At $\theta = \pi/3$ and $x = y = 0$,

$$D_{\mathbf{u}} f(0, 0) = 2\cos\pi/3 + \sin\pi/3 = 1 + \sqrt{3}/2.$$

4. $g(x, y) = xe^y,\ P(0, 2),\ \theta = \dfrac{2\pi}{3}$

$$D_{\mathbf{u}}g(x, y) = g_x(x, y)\cos\theta + g_y(x, y)\sin\theta$$
$$= e^y\cos\theta + xe^y\sin\theta$$

At $\theta = \dfrac{2\pi}{3}$, $x = 0$, and $y = 2$,

$$D_{\mathbf{u}}g(0, 2) = e^2\cos\frac{2\pi}{3} = -\frac{1}{2}e^2.$$

5. $f(x, y) = 3x - 4xy + 9y,\ P(1, 2),\ \mathbf{v} = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = (3 - 4y)\cos\theta + (-4x + 9)\sin\theta$$
$$D_{\mathbf{u}}(1, 2) = (3 - 4(2))\frac{3}{5} + (-4(1) + 9)\frac{4}{5}$$
$$= -3 + 4 = 1$$

6. $f(x, y) = x^3 - y^3,\ P(4, 3),\ \mathbf{v} = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = (3x^2)\left(\frac{\sqrt{2}}{2}\right) + (-3y^2)\left(\frac{\sqrt{2}}{2}\right)$$
$$D_{\mathbf{u}}f(4, 3) = 3(16)\frac{\sqrt{2}}{2} - 3(9)\frac{\sqrt{2}}{2}$$
$$= \frac{21\sqrt{2}}{2}$$

7. $g(x, y) = \sqrt{x^2 + y^2},\ P(3, 4),\ \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}g(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\left(\frac{3}{5}\right) + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{4}{5}\right)$$
$$D_{\mathbf{u}}g(3, 4) = \frac{3}{5}\left(\frac{3}{5}\right) + \frac{4}{5}\left(-\frac{4}{5}\right) = -\frac{7}{25}$$

8. $h(x, y) = e^{-(x^2 + y^2)},\ P(0, 0),\ \mathbf{v} = \mathbf{i} + \mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}h(x, y) = -2xe^{-(x^2+y^2)}\left(\frac{\sqrt{2}}{2}\right) + \left(-2ye^{-(x^2+y^2)}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$D_{\mathbf{u}}h(0, 0) = 0$$

9. $f(x, y) = x^2 + 3y^2,\ P(1, 1),\ Q(4, 5)$

$$\mathbf{v} = (4 - 1)\mathbf{i} + (5 - 1)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j}$$
$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = 2x\left(\frac{3}{5}\right) + 6y\left(\frac{4}{5}\right)$$
$$D_{\mathbf{u}}f(1, 1) = 2\left(\frac{3}{5}\right) + 6\left(\frac{4}{5}\right) = 6$$

10. $f(x, y) = \cos(x + y),\ P(0, \pi),\ Q\left(\dfrac{\pi}{2}, 0\right)$

$$\mathbf{v} = \left(\frac{\pi}{2} - 0\right)\mathbf{i} + (0 - \pi)\mathbf{j}$$
$$\mathbf{v} = \frac{\pi}{2}\mathbf{i} - \pi\mathbf{j}$$
$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = -\sin(x + y)\left(\frac{1}{\sqrt{5}}\right) - \sin(x + y)\left(\frac{-2}{\sqrt{5}}\right)$$
$$D_{\mathbf{u}}f(0, \pi) = 0$$

11. $f(x, y) = e^y \sin x,\ P(0, 0),\ Q(2, 1)$

$$\mathbf{v} = (2 - 0)\mathbf{i} + (1 - 0)\mathbf{j}$$

$$\mathbf{v} = 2\mathbf{i} + \mathbf{j},\ \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = e^y \cos x\left(\frac{2}{\sqrt{5}}\right) + e^y \sin x\left(\frac{1}{\sqrt{5}}\right)$$

$$D_{\mathbf{u}} f(0, 0) = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

12. $f(x, y) = \sin 2x \cos y,\ P(\pi, 0),\ Q\left(\frac{\pi}{2}, \pi\right)$

$$\mathbf{v} = \left(\frac{\pi}{2} - \pi\right)\mathbf{i} + (\pi - 0)\mathbf{j}$$

$$\mathbf{v} = -\frac{\pi}{2}\mathbf{i} + \pi\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = 2\cos 2x \cos y\left(-\frac{1}{\sqrt{5}}\right) + (-\sin 2x \sin y)\left(\frac{2}{\sqrt{5}}\right)$$

$$D_{\mathbf{u}} f(\pi, 0) = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

13. $f(x, y) = 3x + 5y^2 + 1$

$$\nabla f(x, y) = 3\mathbf{i} + 10y\mathbf{j}$$

$$\nabla f(2, 1) = 3\mathbf{i} + 10\mathbf{j}$$

14. $g(x, y) = 2xe^{y/x}$

$$\nabla g(x, y) = \left(-\frac{2y}{x}e^{y/x} + 2e^{y/x}\right)\mathbf{i} + 2e^{y/x}\mathbf{j}$$

$$\nabla g(2, 0) = 2\mathbf{i} + 2\mathbf{j}$$

15. $z = \ln(x^2 - y)$

$$\nabla z(x, y) = \frac{2x}{x^2 - y}\mathbf{i} - \frac{1}{x^2 - y}\mathbf{j}$$

$$\nabla z(2, 3) = 4\mathbf{i} - \mathbf{j}$$

16. $z = \cos(x^2 + y^2)$

$$\nabla z(x, y) = -2x\sin(x^2 + y^2)\mathbf{i} - 2y\sin(x^2 + y^2)\mathbf{j}$$

$$\nabla z(3, -4) = -6\sin 25\mathbf{i} + 8\sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$$

17. $w = 3x^2 - 5y^2 + 2z^2$

$$\nabla w(x, y, z) = 6x\mathbf{i} - 10y\mathbf{j} + 4z\mathbf{k}$$

$$\nabla w(1, 1, -2) = 6\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$$

18. $w = x\tan(y + z)$

$$\nabla w(x, y, z) = \tan(y + z)\mathbf{i} + x\sec^2(y + z)\mathbf{j} + x\sec^2(y + z)\mathbf{k}$$

$$\nabla w(4, 3, -1) = \tan 2\mathbf{i} + 4\sec^2 2\mathbf{j} + 4\sec^2 2\mathbf{k}$$

19. $f(x, y) = xy$

$$\mathbf{v} = \frac{1}{2}\left(\mathbf{i} + \sqrt{3}\mathbf{j}\right)$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\nabla f(0, -2) = -2\mathbf{i}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(0, -2) = \nabla f(0, -2) \cdot \mathbf{u} = -1$$

20. $h(x, y) = e^x \sin y$

$\mathbf{v} = -\mathbf{i}$

$\nabla h = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}$

$\nabla h\left(1, \frac{\pi}{2}\right) = e\mathbf{i}$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{i}$

$D_{\mathbf{u}}h\left(1, \frac{\pi}{2}\right) = \nabla h\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = -e$

21. $f(x, y, z) = x^2 + y^2 + z^2$

$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k}$

$D_{\mathbf{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2}{3}\sqrt{3}$

22. $f(x, y, z) = xy + yz + xz$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (y + x)\mathbf{k}$

$\nabla f(1, 2, -1) = \mathbf{i} + 3\mathbf{k}$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} - \mathbf{k})$

$D_{\mathbf{u}} f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u}$

$= \frac{2}{\sqrt{6}} - \frac{3}{\sqrt{6}} = \frac{-\sqrt{6}}{6}$

23. $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j},\ \mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j},\ \nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$

$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$

24. $\overrightarrow{PQ} = 4\mathbf{i} + 2\mathbf{j}, \mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$\nabla f = 6x\mathbf{i} - 2y\mathbf{j}, \nabla f(-1, 4) = -6\mathbf{i} - 8\mathbf{j}$

$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{12}{\sqrt{5}} - \frac{8}{\sqrt{5}} = -4\sqrt{5}$

25. $g(x, y, z) = xye^z$

$\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$

$\nabla g = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$

At $(2, 4, 0)$, $\nabla g = 4\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$.

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$

$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = -\frac{4}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{8}{\sqrt{5}}$

26. $h(x, y, z) = \ln(x + y + z)$

$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\nabla h = \frac{1}{x + y + z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

$D_{\mathbf{u}}h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$

27. $f(x, y) = x^2 + 2xy$

$\nabla f(x, y) = (2x + 2y)\mathbf{i} + 2x\mathbf{j}$

$\nabla f(1, 0) = 2\mathbf{i} + 2\mathbf{j}$

$\|\nabla f(1, 0)\| = 2\sqrt{2}$

28. $f(x, y) = \frac{x + y}{y + 1}$

$\nabla f(x, y) = \frac{1}{y + 1}\mathbf{i} + \frac{1 - x}{(y + 1)^2}\mathbf{j}$

$\nabla f(0, 1) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j}$

$\|\nabla f(0, 1)\| = \sqrt{\frac{1}{4} + \frac{1}{16}} = \frac{1}{4}\sqrt{5}$

29. $h(x, y) = x \tan y$

$\nabla h(x, y) = \tan y\mathbf{i} + x \sec^2 y\mathbf{j}$

$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$

$\left\|\nabla h\left(2, \frac{\pi}{4}\right)\right\| = \sqrt{17}$

30. $h(x, y) = y\cos(x - y)$

$\nabla h(x, y) = -y\sin(x - y)\mathbf{i} + \left[\cos(x - y) + y\sin(x - y)\right]\mathbf{j}$

$\nabla h\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{6}\mathbf{i} + \left(\frac{3 - \sqrt{3}\pi}{6}\right)\mathbf{j}$

$\left\|\nabla h\left(0, \frac{\pi}{3}\right)\right\| = \sqrt{\frac{3\pi^2}{36} + \frac{9 - 6\sqrt{3}\pi + 3\pi^2}{36}} = \frac{\sqrt{3\left(2\pi^2 - 2\sqrt{3}\pi + 3\right)}}{6}$

31. $g(x, y) = ye^{-x}$

$\nabla g(x, y) = -ye^{-x}\mathbf{i} + e^{-x}\mathbf{j}$

$\nabla g(0, 5) = -5\mathbf{i} + \mathbf{j}$

$\left\|\nabla g(0, 5)\right\| = \sqrt{26}$

32. $g(x, y) = \ln\sqrt[3]{x^2 + y^2} = \frac{1}{3}\ln\left(x^2 + y^2\right)$

$\nabla g(x, y) = \frac{1}{3}\left[\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right]$

$\nabla g(1, 2) = \frac{1}{3}\left(\frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$

$\left\|\nabla g(1, 2)\right\| = \frac{2\sqrt{5}}{15}$

33. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$

$\left\|\nabla f(1, 4, 2)\right\| = 1$

34. $w = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$\nabla w = \frac{1}{\left(\sqrt{1 - x^2 - y^2 - z^2}\right)^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$\nabla w(0, 0, 0) = \mathbf{0}$

$\left\|\nabla w(0, 0, 0)\right\| = 0$

35. $w = xy^2z^2$

$\nabla w = y^2z^2\mathbf{i} + 2xyz^2\mathbf{j} + 2xy^2z\mathbf{k}$

$\nabla w(2, 1, 1) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

$\left\|\nabla w(2, 1, 1)\right\| = \sqrt{33}$

36. $f(x, y, z) = xe^{yz}$

$\nabla f(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$

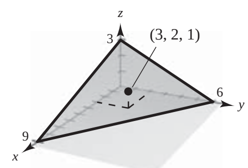
$\nabla f(2, 0, -4) = \mathbf{i} - 8\mathbf{j}$

$\left\|\nabla f(2, 0, -4)\right\| = \sqrt{65}$

For exercises 37–42, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ **and**

$D_{\mathbf{u}} f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta.$

37. $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$



38. (a) $D_{\mathbf{u}} f(3, 2) = -\left(\frac{1}{3}\right)\frac{\sqrt{2}}{2} - \left(\frac{1}{2}\right)\frac{\sqrt{2}}{2} = -\frac{5\sqrt{2}}{12}$

(b) $D_{\mathbf{u}} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{2 - 3\sqrt{3}}{12}$

(c) $D_{\mathbf{u}} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = \frac{2 + 3\sqrt{3}}{12}$

(d) $D_{\mathbf{u}} f(3, 2) = -\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{3 - 2\sqrt{3}}{12}$

39. (a) $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)(\mathbf{i} + \mathbf{j})$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\left(\frac{1}{3}\right)\frac{1}{\sqrt{2}} - \left(\frac{1}{2}\right)\frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{12}$$

(b) $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

(c) $\mathbf{v} - 3\mathbf{i} + 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$$

(d) $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{10}$$

$$\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{-11}{6\sqrt{10}} = -\frac{11\sqrt{10}}{60}$$

40. $\nabla f = -\left(\frac{1}{3}\right)\mathbf{i} - \left(\frac{1}{2}\right)\mathbf{j}$

41. $\|\nabla f\| = \sqrt{\frac{1}{9} + \frac{1}{4}} = \frac{1}{6}\sqrt{13}$

42. $\nabla f = -\frac{1}{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{13}}(-2\mathbf{i} - 3\mathbf{j})$$

So, $\mathbf{u} = \left(1/\sqrt{13}\right)(3\mathbf{i} - 2\mathbf{j})$ and

$D_{\mathbf{u}} f(3, 2) = \nabla f \cdot \mathbf{u} = 0.$ ∇f is the direction of greatest rate of change of f. So, in a direction orthogonal to ∇f, the rate of change of f is 0.

43. (a) In the direction of the vector $-4\mathbf{i} + \mathbf{j}$

(b) $\nabla f = \frac{1}{10}(2x - 3y)\mathbf{i} + \frac{1}{10}(-3x + 2y)\mathbf{j}$

$\nabla f(1, 2) = \frac{1}{10}(-4)\mathbf{i} + \frac{1}{10}(1)\mathbf{j} = -\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$

(Same direction as in part (a))

(c) $-\nabla f = \frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$, the direction opposite that of the gradient

44. (a) In the direction of the vector $\mathbf{i} + \mathbf{j}$

(b) $\nabla f = \frac{1}{2}y\frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j} = \frac{y}{4\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j}$

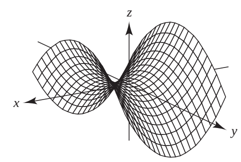
$\nabla f(1, 2) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

(Same direction as in part (a))

(c) $-\nabla f = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$, the direction opposite that of the gradient

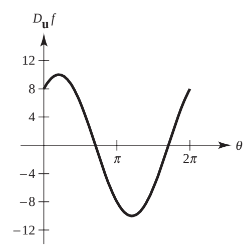
45. $f(x, y) = x^2 - y^2$, $(4, -3, 7)$

(a)



(b) $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = 2x\cos\theta - 2y\sin\theta$

$D_{\mathbf{u}} f(4, -3) = 8\cos\theta + 6\sin\theta$



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(c) Zeros: $\theta \approx 2.21, 5.36$

These are the angles θ for which $D_{\mathbf{u}} f(4, 3)$ equals zero.

(d) $g(\theta) = D_{\mathbf{u}} f(4, -3) = 8\cos\theta + 6\sin\theta$

$g'(\theta) = -8\sin\theta + 6\cos\theta$

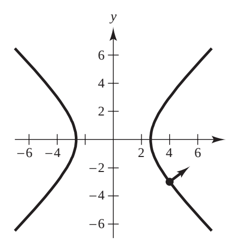
Critical numbers: $\theta \approx 0.64, 3.79$

These are the angels for which $D_{\mathbf{u}} f(4, -3)$ is a maximum (0.64) and minimum (3.79).

(e) $\|\nabla f(4, -3)\| = \|2(4)\mathbf{i} - 2(-3)\mathbf{j}\| = \sqrt{64 + 36} = 10$, the maximum value of $D_{\mathbf{u}} f(4, -3)$, at $\theta \approx 0.64$.

(f) $f(x, y) = x^2 - y^2 = 7$

$\nabla f(4, -3) = 8\mathbf{i} + 6\mathbf{j}$ is perpendicular to the level curve at $(4, -3)$.



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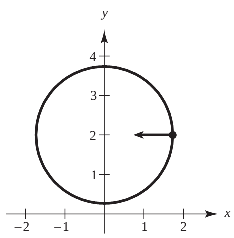
46. (a) $f(x, y) = \dfrac{8y}{1 + x^2 + y^2} = 2$

$$\Rightarrow 4y = 1 + x^2 + y^2$$
$$4 = y^2 - 4y + 4 + x^2 + 1$$
$$(y - 2)^2 + x^2 = 3$$

Circle: center: $(0, 2)$, radius: $\sqrt{3}$

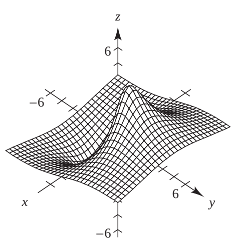
(b) $\nabla f = \dfrac{-16xy}{\left(1 + x^2 + y^2\right)^2}\mathbf{i} + \dfrac{8 + 8x^2 - 8y^2}{\left(1 + x^2 + y^2\right)^2}\mathbf{j}$

$$\nabla f\left(\sqrt{3}, 2\right) = \frac{-\sqrt{3}}{2}\mathbf{i}$$



(c) The directional derivative of f is 0 in the direction $\pm\mathbf{j}$.

(d)



47. $f(x, y) = 6 - 2x - 3y$

$c = 6, P = (0, 0)$

$\nabla f(x, y) = -2\mathbf{i} - 3\mathbf{j}$

$$6 - 2x - 3y = 6$$
$$0 = 2x + 3y$$

$\nabla f(0, 0) = -2\mathbf{i} - 3\mathbf{j}$

48. $f(x, y) = x^2 + y^2$

$c = 25, P = (3, 4)$

$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$

$x^2 + y^2 = 25$

$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}$

49. $f(x, y) = xy$

$c = -3, P = (-1, 3)$

$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$

$xy = -3$

$\nabla f(-1, 3) = 3\mathbf{i} - \mathbf{j}$

50. $f(x, y) = \dfrac{x}{x^2 + y^2}$

$c = \dfrac{1}{2}, P = (1, 1)$

$\nabla f(x, y) = \dfrac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\mathbf{i} - \dfrac{2xy}{\left(x^2 + y^2\right)^2}\mathbf{j}$

$\dfrac{x}{x^2 + y^2} = \dfrac{1}{2}$

$x^2 + y^2 - 2x = 0$

$\nabla f(1, 1) = -\dfrac{1}{2}\mathbf{j}$

51. $f(x, y) = 4x^2 - y$

(a) $\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$

$\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$

(b) $\|16\mathbf{i} - \mathbf{j}\| = \sqrt{257}$

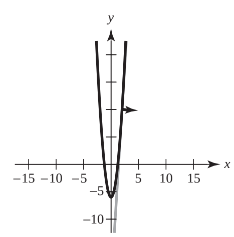
$\dfrac{1}{\sqrt{257}}(16\mathbf{i} - \mathbf{j})$ is a unit vector normal to the level curve $4x^2 - y = 6$ at $(2, 10)$.

(c) The vector $\mathbf{i} + 16\mathbf{j}$ is tangent to the level curve.

Slope $= \dfrac{16}{1} = 16$

$$y - 10 = 16(x - 2)$$
$$y = 16x - 22 \quad \text{Tangent line}$$

(d)



52. $f(x, y) = x - y^2$

(a) $\nabla f(x, y) = \mathbf{i} - 2y\mathbf{j}$

$\nabla f(4, -1) = \mathbf{i} + 2\mathbf{j}$

(b) $\|\nabla f(4, -1)\| = \sqrt{5}$

$\dfrac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$ is a unit vector normal to the level curve $x - y^2 = 3$ at $(4, -1)$.

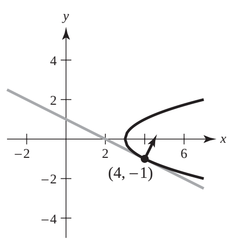
(c) The vector $2\mathbf{i} - \mathbf{j}$ is tangent to the level curve.

Slope $= -\dfrac{1}{2}$.

$$y + 1 = -\frac{1}{2}(x - 4)$$

$$y = -\frac{1}{2}x + 1 \qquad \text{Tangent line}$$

(d)



53. $f(x, y) = 3x^2 - 2y^2$

(a) $\nabla f = 6x\mathbf{i} - 4y\mathbf{j}$

$\nabla f(1, 1) = 6\mathbf{i} - 4\mathbf{j}$

(b) $\|\nabla f(1, 1)\| = \sqrt{36 + 16} = 2\sqrt{13}$

$\dfrac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$ is a unit vector normal to the level curve $3x^2 - 2y^2 = 1$ at $(1, 1)$.

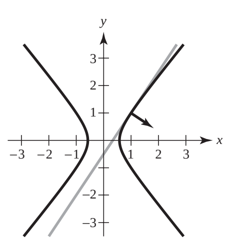
(c) The vector $2\mathbf{i} + 3\mathbf{j}$ is tangent to the level curve.

Slope $= \dfrac{3}{2}$.

$$y - 1 = \tfrac{3}{2}(x - 1)$$

$$y = \tfrac{3}{2}x - \tfrac{1}{2} \qquad \text{tangent line}$$

(d)



54. $f(x, y) = 9x^2 + 4y^2$

(a) $\nabla f = 18x\mathbf{i} + 8y\mathbf{j}$

$\nabla f(2, -1) = 36\mathbf{i} - 8\mathbf{j}$

(b) $\|\nabla f(2, -1)\| = \sqrt{1360} = 4\sqrt{85}$

$\dfrac{1}{\sqrt{85}}(9\mathbf{i} - 2\mathbf{j})$ is a unit vector normal to the level curve $9x^2 + 4y^2 = 40$ at $(2, -1)$.

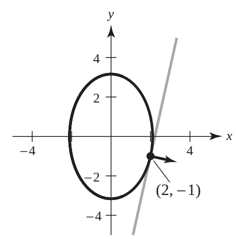
(c) The vector $2\mathbf{i} + 9\mathbf{j}$ is tangent to the level curve.

Slope $= \dfrac{9}{2}$.

$$y + 1 = \frac{9}{2}(x - 2)$$

$$y = \frac{9}{2}x - 10 \quad \text{Tangent line}$$

(d)



55. See the definition, page 916.

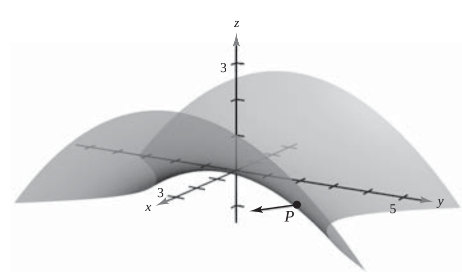
56. Let $f(x, y)$ be a function of two variables and $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ a unit vector.

(a) If $\theta = 0°$, then $D_{\mathbf{u}} f = \dfrac{\partial f}{\partial x}$.

(b) If $\theta = 90°$, then $D_{\mathbf{u}} f = \dfrac{\partial f}{\partial y}$.

57. See the definition, pages 918 and 919.

58.

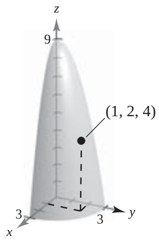


59. The gradient vector is normal to the level curves. See Theorem 13.12.

60. $f(x, y) = 9 - x^2 - y^2$ and

$$D_{\mathbf{u}} f(x, y) = -2x \cos\theta - 2y \sin\theta$$
$$= -2(x\cos\theta + y\sin\theta)$$

(a) $f(x, y) = 9 - x^2 - y^2$



(b) $D_{\mathbf{u}} f(1, 2) = -2\left(\dfrac{\sqrt{2}}{2} - \sqrt{2}\right) = \sqrt{2}$

(c) $D_{\mathbf{u}} f(1, 2) = -2\left(\dfrac{1}{2} + \sqrt{3}\right) = -\left(1 + 2\sqrt{3}\right)$

(d) $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$\|\nabla f(1, 2)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

(e) $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$$\frac{\nabla f(1, 2)}{\|\nabla f(1, 2)\|} = \frac{1}{\sqrt{5}}(-\mathbf{i} - 2\mathbf{j})$$

Therefore, $\mathbf{u} = \left(1/\sqrt{5}\right)(-2\mathbf{i} + \mathbf{j})$ and

$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = 0.$

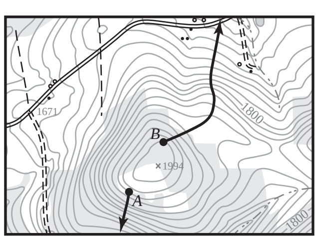
61. $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$

$\nabla h = -0.002x\mathbf{i} - 0.008y\mathbf{j}$

$\nabla h(500, 300) = -\mathbf{i} - 2.4\mathbf{j}$ or

$5\nabla h = -(5\mathbf{i} + 12\mathbf{j})$

62.

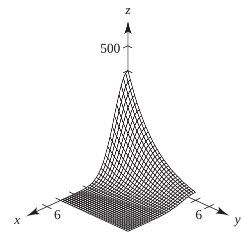


63. $T = \dfrac{x}{x^2 + y^2}$

$$\nabla T = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\mathbf{i} - \frac{2xy}{\left(x^2 + y^2\right)^2}\mathbf{j}$$

$$\nabla T(3, 4) = \frac{7}{625}\mathbf{i} - \frac{24}{625}\mathbf{j} = \frac{1}{625}(7\mathbf{i} - 24\mathbf{j})$$

64. (a)



(b) $\nabla T(x, y) = 400e^{-\left(x^2 + y\right)/2}\left[(-x)\mathbf{i} - \tfrac{1}{2}\mathbf{j}\right]$

$\nabla T(3, 5) = 400e^{-7}\left[-3\mathbf{i} - \tfrac{1}{2}\mathbf{j}\right]$

There will be no change in directions perpendicular to the gradient: $\pm(\mathbf{i} - 6\mathbf{j})$

(c) The greatest increase is in the direction of the gradient: $-3\mathbf{i} - \tfrac{1}{2}\mathbf{j}$

65. $T(x, y) = 80 - 3x^2 - y^2$, $P(-1, 5)$

$\nabla T(x, y) = -6x\mathbf{i} - 2y\mathbf{j}$

Maximum increase in direction:

$\nabla T(-1, 5) = (-6)(-1)\mathbf{i} - 2(5)\mathbf{j} = 6\mathbf{i} - 10\mathbf{j}$

Maximum rate:

$\|\nabla T(-1, 5)\| = \sqrt{6^2 + (-10)^2} = 2\sqrt{34}$

$\approx 11.66°$ per centimeter

66. $T(x, y) = 50 - x^2 - 4y^2$, $P(2, -1)$

$\nabla T(x, y) = -2x\mathbf{i} - 8y\mathbf{j}$

Maximum increase in direction:

$\nabla T(2, -1) = -2(2)\mathbf{i} - 8(-1)\mathbf{j} = -4\mathbf{i} + 8\mathbf{j}$

Maximum rate:

$\|\nabla T(2, -1)\| = \sqrt{16 + 64} = 4\sqrt{5}$

$\approx 8.94°$ per centimeter

67. $T(x, y) = 400 - 2x^2 - y^2$, $P = (10, 10)$

$\dfrac{dx}{dt} = -4x$ $\qquad$ $\dfrac{dy}{dt} = -2y$

$x(t) = C_1e^{-4t}$ $\qquad$ $y(t) = C_2e^{-2t}$

$10 = x(0) = C_1$ $\qquad$ $10 = y(0) = C_2$

$x(t) = 10e^{-4t}$ $\qquad$ $y(t) = 10e^{-2t}$

$x = \dfrac{y^2}{10}$ $\qquad$ $y^2(t) = 100e^{-4t}$

$y^2 = 10x$

68. $T(x, y) = 100 - x^2 - 2y^2,\ P = (4, 3)$

$\dfrac{dx}{dt} = -2x \qquad \dfrac{dy}{dt} = -4y$

$x(t) = C_1e^{-2t} \qquad y(t) = C_2e^{-4t}$

$4 = x(0) = C_1 \qquad 3 = y(0) = C_2$

$x(t) = 4e^{-2t} \qquad y(t) = 3e^{-4t}$

$\dfrac{3x^2}{16} = e^{-4t} = y \Rightarrow u = \dfrac{3}{16}x^2$

69. True

70. False

$D_{\mathbf{u}} f(x, y) = \sqrt{2} > 1$ when $\mathbf{u} = \left(\cos\dfrac{\pi}{4}\right)\mathbf{i} + \left(\sin\dfrac{\pi}{4}\right)\mathbf{j}.$

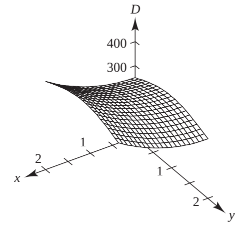
71. True

72. True

73. Let $f(x, y, z) = e^x \cos y + \dfrac{z^2}{2} + C.$ Then

$\nabla f(x, y, z) = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j} + z\mathbf{k}.$

74. (a)



(b) The graph of $-D = -250 - 30x^2 - 50\sin(\pi y/2)$ would model the ocean floor.

(c) $D(1, 0.5) = 250 + 30(1) + 50\sin\dfrac{\pi}{4} \approx 315.4$ ft

(d) $\dfrac{\partial D}{\partial x} = 60x$ and $\dfrac{\partial D}{\partial x}(1, 0.5) = 60$

(e) $\dfrac{\partial D}{\partial y} = 25\pi\cos\dfrac{\pi y}{2}$ and

$\dfrac{\partial D}{\partial y}(1, 0.5) = 25\pi\cos\dfrac{\pi}{4} \approx 55.5$

(f) $\nabla D = 60x\mathbf{i} + 25\pi\cos\left(\dfrac{\pi y}{2}\right)\mathbf{j}$

$\nabla D(1, 0.5) = 60\mathbf{i} + 55.5\mathbf{j}$

75. (a) $f(x, y) = \sqrt[3]{xy}$ is the composition of two continuous functions, $h(x, y) = xy$ and $g(z) = z^{1/3}$, and therefore continuous by Theorem 13.2.

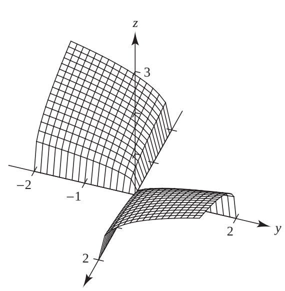
(b) $f_x(0, 0) = \displaystyle\lim_{\Delta x\to 0}\frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(0\cdot\Delta x)^{1/3} - 0}{\Delta x} = 0$

$f_y(0, 0) = \displaystyle\lim_{\Delta y\to 0}\frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{(0\cdot\Delta y)^{1/3} - 0}{\Delta y} = 0$

Let $\mathbf{u} = \cos\theta i + \sin\theta\mathbf{j},\ \theta \ne 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}.$ Then

$D_{\mathbf{u}} f(0, 0) = \displaystyle\lim_{t\to 0}\frac{f(0 + t\cos\theta, 0 + t\sin\theta) - f(0, 0)}{t} = \lim_{t\to 0}\frac{\sqrt[3]{t^2\cos\theta\sin\theta}}{t} = \lim_{t\to 0}\frac{\sqrt[3]{\cos\theta\sin\theta}}{t^{1/3}}$, does not exist.

(c)



76. We cannot use Theorem 13.9 because f is not a differentiable function of x and y. So, we use the definition of directional derivatives.

$$D_{\mathbf{u}}\, f(x, y) = \lim_{t\to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

$$D_{\mathbf{u}}\, f(0, 0) = \lim_{t\to 0} \frac{f\left[0 + \left(\frac{t}{\sqrt{2}}\right), 0 + \left(\frac{t}{\sqrt{2}}\right)\right] - f(0, 0)}{t} = \lim_{t\to 0} \frac{1}{t}\left[\frac{4\left(\frac{t}{\sqrt{2}}\right)\left(\frac{t}{\sqrt{2}}\right)}{\left(\frac{t^2}{2}\right) + \left(\frac{t^2}{2}\right)}\right] = \lim_{t\to 0} \frac{1}{t}\left[\frac{2t^2}{t^2}\right] = \lim_{t\to 0} \frac{2}{t} \text{ which does not exist.}$$

If $f(0, 0) = 2$, then $D_{\mathbf{u}}\, f(0, 0) = \lim_{t\to 0} \dfrac{f\left(0 + \frac{t}{\sqrt{2}}, 0 + \frac{t}{\sqrt{2}}\right) - 2}{t} = \lim_{t\to 0} \dfrac{1}{t}\left[\dfrac{2t^2}{t^2} - 2\right] = 0$

which implies that the directional derivative exists.

Section 13.7 Tangent Planes and Normal Lines

1. $F(x, y, z) = 3x - 5y + 3z - 15 = 0$

$3x - 5y + 3z = 15$ Plane

2. $F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$

$x^2 + y^2 + z^2 = 25$

Sphere, radius 5, centered at origin.

3. $F(x, y, z) = 4x^2 + 9y^2 - 4z^2 = 0$

$4x^2 + 9y^2 = 4z^2$ Elliptic cone

4. $F(x, y, z) = 16x^2 - 9y^2 + 36z = 0$

$16x^2 - 9y^2 + 36z = 0$ Hyperbolic paraboloid

5. $F(x, y, z) = 3x + 4y + 12z = 0$

$\nabla F = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}, \ \|\nabla F\| = \sqrt{9 + 16 + 144} = 13$

$\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{3}{13}\mathbf{i} + \dfrac{4}{13}\mathbf{j} + \dfrac{12}{13}\mathbf{k}$

6. $F(x, y, z) = x^2 + y^2 + z^2 - 6$

$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(1, 1, 2) = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

$\|\nabla F(1, 1, 2)\| = \sqrt{4 + 4 + 16} = 2\sqrt{6}$

$\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{1}{\sqrt{6}}\mathbf{i} + \dfrac{1}{\sqrt{6}}\mathbf{j} + \dfrac{2}{\sqrt{6}}\mathbf{k}$

7. $F(x, y, z) = x^2 + 3y + z^3 - 9$

$\nabla F(x, y, z) = 2x\mathbf{i} + 3\mathbf{j} + 3z^2\mathbf{k}$

$\nabla F(2, -1, 2) = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$

$\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$

8. $F(x, y, z) = x^2y^3 - y^2z + 2xz^3 - 4$

$\nabla F = (2xy^3 + 2z^3)\mathbf{i} + (3x^2y^2 - 2yz)\mathbf{j} + (6xz^2 - y^2)\mathbf{k}$

$\nabla F(-1, 1, -1) = -4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$

$\|\nabla F(-1, 1, -1)\| = 3\sqrt{10}$

$\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{1}{3\sqrt{10}}(-4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k})$

9. $z = x^2 + y^2 + 3, (2, 1, 8)$

$F(x, y, z) = x^2 + y^2 + 3 - z$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = -1$

$F_x(2, 1, 8) = 4 \quad F_y(2, 1, 8) = 2 \quad F_z(2, 1, 8) = -1$

$4(x - 2) + 2(y - 1) - 1(z - 8) = 0$

$4x + 2y - z = 2$

10. $f(x, y) = \dfrac{y}{x}, (1, 2, 2)$

$F(x, y, z) = \dfrac{y}{x} - z$

$F_x(x, y, z) = -\dfrac{y}{x^2} \quad F_y(x, y, z) = \dfrac{1}{x} \quad F_z(x, y, z) = -1$

$F_x(1, 2, 2) = -2 \quad F_y(1, 2, 2) = 1 \quad F_z(1, 2, 2) = -1$

$-2(x - 1) + (y - 2) - (z - 2) = 0$

$-2x + y - z + 2 = 0$

$2x - y + z = 2$

11. $z = \sqrt{x^2 + y^2}, (3, 4, 5)$

$F(x, y, z) = \sqrt{x^2 + y^2} - z$

$F_x(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}} \quad F_y(x, y, z) = \dfrac{y}{\sqrt{x^2 + y^2}} \quad F_z(x, y, z) = -1$

$F_x(3, 4, 5) = \dfrac{3}{5} \quad F_y(3, 4, 5) = \dfrac{4}{5} \quad F_z(3, 4, 5) = -1$

$\dfrac{3}{5}(x - 3) + \dfrac{4}{5}(y - 4) - (z - 5) = 0$

$3(x - 3) + 4(y - 4) - 5(z - 5) = 0$

$3x + 4y - 5z = 0$

12. $g(x, y) = \arctan\dfrac{y}{x}, (1, 0, 0)$

$G(x, y, z) = \arctan\dfrac{y}{x} - z$

$G_x(x, y, z) = \dfrac{-(y/x^2)}{1 + (y^2/x^2)} = \dfrac{-y}{x^2 + y^2} \quad G_y(x, y, z) = \dfrac{1/x}{1 + (y^2/x^2)} = \dfrac{x}{x^2 + y^2} \quad G_z(x, y, z) = -1$

$G_x(1, 0, 0) = 0 \quad G_y(1, 0, 0) = 1 \quad G_z(1, 0, 0) = -1$

$y - z = 0$

13. $g(x, y) = x^2 + y^2, (1, -1, 2)$

$G(x, y, z) = x^2 + y^2 - z$

$G_x(x, y, z) = 2x \quad G_y(x, y, z) = 2y \quad G_z(x, y, z) = -1$

$G_x(1, -1, 2) = 2 \quad G_y(1, -1, 2) = -2 \quad G_z(1, -1, 2) = -1$

$2(x - 1) - 2(y + 1) - 1(z - 2) = 0$

$2x - 2y - z = 2$

14. $f(x, y) = x^2 - 2xy + y^2, (1, 2, 1)$

$F(x, y, z) = x^2 - 2xy + y^2 - z$

$F_x(x, y, z) = 2x - 2y \quad F_y(x, y, z) = -2x + 2y \quad F_z(x, y, z) = -1$

$F_x(1, 2, 1) = -2 \quad F_y(1, 2, 1) = 2 \quad F_z(1, 2, 1) = -1$

$-2(x - 1) + 2(y - 2) - (z - 1) = 0$

$-2x + 2y - z - 1 = 0$

$2x - 2y + z = -1$

15. $h(x, y) = \ln\sqrt{x^2 + y^2}, (3, 4, \ln 5)$

$$H(x, y, z) = \ln\sqrt{x^2 + y^2} - z = \frac{1}{2}\ln(x^2 + y^2) - z$$

$$H_x(x, y, z) = \frac{x}{x^2 + y^2} \qquad H_y(x, y, z) = \frac{y}{x^2 + y^2} \qquad H_z(x, y, z) = -1$$

$$H_x(3, 4, \ln 5) = \frac{3}{25} \qquad H_y(3, 4, \ln 5) = \frac{4}{25} \qquad H_z(3, 4, \ln 5) = -1$$

$$\frac{3}{25}(x - 3) + \frac{4}{25}(y - 4) - (z - \ln 5) = 0$$

$$3(x - 3) + 4(y - 4) - 25(z - \ln 5) = 0$$

$$3x + 4y - 25z = 25(1 - \ln 5)$$

16. $h(x, y) = \cos y, \left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$$H(x, y, z) = \cos y - z$$

$$H_x(x, y, z) = 0 \qquad H_y(x, y, z) = -\sin y \qquad H_z(x, y, z) = -1$$

$$H_x\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = 0 \qquad H_y\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \qquad H_z\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -1$$

$$-\frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) - \left(z - \frac{\sqrt{2}}{2}\right) = 0$$

$$-\frac{\sqrt{2}}{2}y - z + \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} = 0$$

$$4\sqrt{2}y + 8z = \sqrt{2}(\pi + 4)$$

17. $x^2 + 4y^2 + z^2 = 36, (2, -2, 4)$

$$F(x, y, z) = x^2 + 4y^2 + z^2 - 36$$

$$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 8y \qquad F_z(x, y, z) = 2z$$

$$F_x(2, -2, 4) = 4 \qquad F_y(2, -2, 4) = -16 \qquad F_z(2, -2, 4) = 8$$

$$4(x - 2) - 16(y + 2) + 8(z - 4) = 0$$

$$(x - 2) - 4(y + 2) + 2(z - 4) = 0$$

$$x - 4y + 2z = 18$$

18. $x^2 + 2z^2 = y^2, (1, 3, -2)$

$$F(x, y, z) = x^2 - y^2 + 2z^2$$

$$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = -2y \qquad F_z(x, y, z) = 4z$$

$$F_x(1, 3, -2) = 2 \qquad F_y(1, 3, -2) = -6 \qquad F_z(1, 3, -2) = -8$$

$$2(x - 1) - 6(y - 3) - 8(z + 2) = 0$$

$$(x - 1) - 3(y - 3) - 4(z + 2) = 0$$

$$x - 3y - 4z = 0$$

19. $xy^2 + 3x - z^2 = 8, (1, -3, 2)$

$F(x, y, z) = xy^2 + 3x - z^2 - 8$

$F_x(x, y, z) = y^2 + 3 \quad F_y(x, y, z) = 2xy \quad F_z(x, y, z) = -2z$

$F_x(1, -3, 2) = 12 \quad F_y(1, -3, 2) = -6 \quad F_z(1, -3, 2) = -4$

$$12(x - 1) - 6(y + 3) - 4(z - 2) = 0$$
$$12x - 6y - 4z = 22$$
$$6x - 3y - 2z = 11$$

20. $z = e^x(\sin y + 1), \left(0, \frac{\pi}{2}, 2\right)$

$F(x, y, z) = e^x(\sin y + 1) - z$

$F_x(x, y, z) = e^x(\sin y + 1) \quad F_y(x, y, z) = e^x \cos y \quad F_z(x, y, z) = -1$

$F_x\left(0, \frac{\pi}{2}, 2\right) = 2 \quad F_y\left(0, \frac{\pi}{2}, 2\right) = 0 \quad F_z\left(0, \frac{\pi}{2}, 2\right) = -1$

$2x - z = -2$

21. $x + y + z = 9, (3, 3, 3)$

$F(x, y, z) = x + y + z - 9$

$F_x(x, y, z) = 1 \quad F_y(x, y, z) = 1 \quad F_z(x, y, z) = 1$

$F_x(3, 3, 3) = 1 \quad F_y(3, 3, 3) = 1 \quad F_z(3, 3, 3) = 1$

$$(x - 3) + (y - 3) + (z - 3) = 0$$
$$x + y + z = 9 \text{ (same plane!)}$$

Direction numbers: 1, 1, 1

Line: $x - 3 = y - 3 = z - 3$

22. $x^2 + y^2 + z^2 = 9, (1, 2, 2)$

$F(x, y, z) = x^2 + y^2 + z^2 - 9$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 2z$

$F_x(1, 2, 2) = 2 \quad F_y(1, 2, 2) = 4 \quad F_z(1, 2, 2) = 4$

Direction numbers: 1, 2, 2

Plane: $(x - 1) + 2(y - 2) + 2(z - 2) = 0, x + 2y + 2z = 9$

Line: $\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}$

23. $x^2 + y^2 + z = 9, (1, 2, 4)$

$F(x, y, z) = x^2 + y^2 + z - 9$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 1$

$F_x(1, 2, 4) = 2 \quad F_y(1, 2, 4) = 4 \quad F_z(1, 2, 4) = 1$

Direction numbers: 2, 4, 1

Plane: $2(x - 1) + 4(y - 2) + (z - 4) = 0, \ 2x + 4y + z = 14$

Line: $\frac{x - 1}{2} = \frac{y - 2}{4} = \frac{z - 4}{1}$

24. $z = 16 - x^2 - y^2, (2, 2, 8)$

$F(x, y, z) = 16 - x^2 - y^2 - z$

$F_x(x, y, z) = -2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = -1$

$F_x(2, 2, 8) = -4 \quad F_y(2, 2, 8) = -4 \quad F_z(2, 2, 8) = -1$

$$-4(x - 2) - 4(y - 2) - (z - 8) = 0$$
$$-4x - 4y - z = -24$$
$$4x + 4y + z = 24$$

Direction numbers: 4, 4, 1

Line: $\dfrac{x - 2}{4} = \dfrac{y - 2}{4} = z - 8$

25. $z = x^2 - y^2, (3, 2, 5)$

$F(x, y, z) = x^2 - y^2 - z$

$F_x(x, y, z) = 2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = -1$

$F_x(3, 2, 5) = 6 \quad F_y(3, 2, 5) = -4 \quad F_z(3, 2, 5) = -1$

$$6(x - 3) - 4(y - 2) - (z - 5) = 0$$
$$6x - 4y - z = 5$$

Direction numbers: 6, −4, −1

Line: $\dfrac{x - 3}{6} = \dfrac{y - 2}{-4} = \dfrac{z - 5}{-1}$

26. $xy - z = 0, (-2, -3, 6)$

$F(x, y, z) = xy - z$

$F_x(x, y, z) = y \quad F_y(x, y, z) = x \quad F_z(x, y, z) = -1$

$F_x(-2, -3, 6) = -3 \quad F_y(-2, -3, 6) = -2 \quad F_z(-2, -3, 6) = -1$

Direction numbers: 3, 2, 1

Plane: $3(x + 2) + 2(y + 3) + (z - 6) = 0, 3x + 2y + z = -6$

Line: $\dfrac{x + 2}{3} = \dfrac{y + 3}{2} = \dfrac{z - 6}{1}$

27. $xyz = 10, (1, 2, 5)$

$F(x, y, z) = xyz - 10$

$F_x(x, y, z) = yz \quad F_y(x, y, z) = xz \quad F_z(x, y, z) = xy$

$F_x(1, 2, 5) = 10 \quad F_y(1, 2, 5) = 5 \quad F_z(1, 2, 5) = 2$

Direction numbers: 10, 5, 2

Plane: $10(x - 1) + 5(y - 2) + 2(z - 5) = 0, 10x + 5y + 2z = 30$

Line: $\dfrac{x - 1}{10} = \dfrac{y - 2}{5} = \dfrac{z - 5}{2}$

28. $z = ye^{2xy}, (0, 2, 2)$

$F(x, y, z) = ye^{2xy} - z$

$F_x(x, y, z) = 2y^2e^{2xy} \quad F_y(x, y, z) = (1 + 2xy)e^{2xy} \quad F_z(x, y, z) = -1$

$F_x(0, 2, 2) = 8 \quad F_y(0, 2, 2) = 1 \quad F_z(0, 2, 2) = -1$

$$8(x - 0) + (y - 2) - (z - 2) = 0$$
$$8x + y - z = 0$$

Direction number: $8, 1, -1$

Line: $\dfrac{x}{8} = \dfrac{y - 2}{1} = \dfrac{z - 2}{-1}$

29. $z = \arctan \dfrac{y}{x}, \left(1, 1, \dfrac{\pi}{4}\right)$

$F(x, y, z) = \arctan \dfrac{y}{x} - z$

$F_x(x, y, z) = \dfrac{-y}{x^2 + y^2} \quad F_y(x, y, z) = \dfrac{x}{x^2 + y^2} \quad F_z(x, y, z) = -1$

$F_x\left(1, 1, \dfrac{\pi}{4}\right) = -\dfrac{1}{2} \quad F_y\left(1, 1, \dfrac{\pi}{4}\right) = \dfrac{1}{2} \quad F_z\left(1, 1, \dfrac{\pi}{4}\right) = -1$

Direction numbers: $1, -1, 2$

Plane: $(x - 1) - (y - 1) + 2\left(z - \dfrac{\pi}{4}\right) = 0, x - y + 2z = \dfrac{\pi}{2}$

Line: $\dfrac{x - 1}{1} = \dfrac{y - 1}{-1} = \dfrac{z - (\pi/4)}{2}$

30. $y \ln(xz^2) = 2, (e, 2, 1)$

$F(x, y, z) = y[\ln x + 2 \ln z] - 2$

$F_x(x, y, z) = \dfrac{y}{x} \quad F_y(x, y, z) = \ln x + 2 \ln z \quad F_z(x, y, z) = \dfrac{2y}{z}$

$F_x(e, 2, 1) = \dfrac{2}{e} \quad F_y(e, 2, 1) = 1 \quad F_z(e, 2, 1) = 4$

$$\frac{2}{e}(x - e) + (y - 2) + 4(z - 1) = 0$$
$$\frac{2}{e}x + y + 4z = 8$$

Direction numbers: $\dfrac{2}{e}, 1, 4$

$\dfrac{x - e}{(2/e)} = \dfrac{y - 2}{1} = \dfrac{z - 1}{4}$

31. $F(x, y, z) = x^2 + y^2 - 2 \quad G(x, y, z) = x - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} \quad \nabla G(x, y, z) = \mathbf{i} - \mathbf{k}$

$\nabla F(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} \qquad \nabla G(1, 1, 1) = \mathbf{i} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - \mathbf{j} + \mathbf{k})$

Direction numbers: $1, -1, 1$

Line: $x - 1 = \dfrac{y - 1}{-1} = z - 1$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{2}{(2\sqrt{2})\sqrt{2}} = \dfrac{1}{2}$

Not orthogonal

32. $F(x, y, z) = x^2 + y^2 - z \qquad G(x, y, z) = 4 - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \quad \nabla G(x, y, z) = -\mathbf{j} - \mathbf{k}$

$\nabla F(2, -1, 5) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \quad \nabla G(2, -1, 5) = -\mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

Direction numbers: $1, 4, -4.\ \dfrac{x - 2}{1} = \dfrac{y + 1}{4} = \dfrac{z - 5}{-4}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{3}{\sqrt{21}\sqrt{2}} = \dfrac{3}{\sqrt{42}} = \dfrac{\sqrt{42}}{14}$; not orthogonal

33. $F(x, y, z) = x^2 + z^2 - 25 \quad G(x, y, z) = y^2 + z^2 - 25$

$\nabla F = 2x\mathbf{i} + 2z\mathbf{k} \qquad \nabla G = 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(3, 3, 4) = 6\mathbf{i} + 8\mathbf{k} \qquad \nabla G(3, 3, 4) = 6\mathbf{j} + 8\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & 6 & 8 \end{vmatrix} = -48\mathbf{i} - 48\mathbf{j} + 36\mathbf{k} = -12(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

Direction numbers: $4, 4, -3.\ \dfrac{x - 3}{4} = \dfrac{y - 3}{4} = \dfrac{z - 4}{-3}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{64}{(10)(10)} = \dfrac{16}{25}$; not orthogonal

34. $F(x, y, z) = \sqrt{x^2 + y^2} - z \qquad G(x, y, z) = 5x - 2y + 3z - 22$

$\nabla F(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k} \qquad \nabla G(x, y, z) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\nabla F(3, 4, 5) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j} - \mathbf{k} \qquad \nabla G(3, 4, 5) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix} = \dfrac{2}{5}\mathbf{i} - \dfrac{34}{5}\mathbf{j} - \dfrac{26}{5}\mathbf{k}$

Direction numbers: $1, -17, -13$

$\dfrac{x - 3}{1} = \dfrac{y - 4}{-17} = \dfrac{z - 5}{-13}$; tangent line

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = \dfrac{-(8/5)}{\sqrt{2}\sqrt{38}} = \dfrac{-8}{5\sqrt{76}}$; not orthogonal

35. $F(x, y, z) = x^2 + y^2 + z^2 - 14 \qquad G(x, y, z) = x - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \qquad \nabla G(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

$\nabla F(3, 1, 2) = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \qquad \nabla G(3, 1, 2) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 4 \\ 1 & -1 & -1 \end{vmatrix} = 2\mathbf{i} + 10\mathbf{j} - 8\mathbf{k} = 2[\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}]$

Direction numbers: $1, 5, -4$

Line: $\dfrac{x - 3}{1} = \dfrac{y - 1}{5} = \dfrac{z - 2}{-4}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = 0 \Rightarrow$ orthogonal

36. $F(x, y, z) = x^2 + y^2 - z \qquad G(x, y, z) = x + y + 6z - 33$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \qquad \nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

$\nabla F(1, 2, 5) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} \qquad \nabla G(1, 2, 5) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 1 & 1 & 6 \end{vmatrix} = 25\mathbf{i} - 13\mathbf{j} - 2\mathbf{k}$

Direction numbers: $25, -13, -2$. $\dfrac{x - 1}{25} = \dfrac{y - 2}{-13} = \dfrac{z - 5}{-2}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\|\nabla G\|} = 0$; orthogonal

37. $F(x, y, z) = 3x^2 + 2y^2 - z - 15, (2, 2, 5)$

$\nabla F(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$

$\nabla F(2, 2, 5) = 12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

$$\cos\theta = \frac{|\nabla F(2, 2, 5) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 5)\|} = \frac{1}{\sqrt{209}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{209}}\right) = 86.03°$$

38. $F(x, y, z) = 2xy - z^3, (2, 2, 2)$

$\nabla F = 2y\mathbf{i} + 2x\mathbf{j} - 3z^2\mathbf{k}$

$\nabla F(2, 2, 2) = 4\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$

$$\cos\theta = \frac{|\nabla F(2, 2, 2) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 2)\|} = \frac{|-12|}{\sqrt{176}} = \frac{3\sqrt{11}}{11}$$

$$\theta = \arccos\left(\frac{3\sqrt{11}}{11}\right) \approx 25.24°$$

39. $F(x, y, z) = x^2 - y^2 + z, (1, 2, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 3) = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

$$\cos\theta = \frac{|\nabla F(1, 2, 3) \cdot \mathbf{k}|}{\|\nabla F(1, 2, 3)\|} = \frac{1}{\sqrt{21}}$$

$$\theta = \arccos\frac{1}{\sqrt{21}} \approx 77.40°$$

40. $F(x, y, z) = x^2 + y^2 - 5, (2, 1, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$

$\nabla F(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j}$

$$\cos\theta = \frac{|\nabla F(2, 1, 3) \cdot \mathbf{k}|}{\|\nabla F(2, 1, 3)\|} = 0$$

$$\theta = \arccos 0 = 90°$$

41. $F(x, y, z) = 3 - x^2 - y^2 + 6y - z$

$\nabla F(x, y, z) = -2x\mathbf{i} + (-2y + 6)\mathbf{j} - \mathbf{k}$

$-2x = 0, x = 0$

$-2y + 6 = 0, y = 3$

$z = 3 - 0^2 - 3^2 + 6(3) = 12$

$(0, 3, 12)$ (vertex of paraboloid)

42. $F(x, y, z) = 3x^2 + 2y^2 - 3x + 4y - z - 5$

$\nabla F(x, y, z) = (6x - 3)\mathbf{i} + (4y + 4)\mathbf{j} - \mathbf{k}$

$6x - 3 = 0, x = \frac{1}{2}$

$4y + 4 = 0, y = -1$

$z = 3\left(\frac{1}{2}\right)^2 + 2(-1)^2 - 3\left(\frac{1}{2}\right) + 4(-1) - 5 = -\frac{31}{4}$

$\left(\frac{1}{2}, -1, -\frac{31}{4}\right)$

43. $F(x, y, z) = x^2 - xy + y^2 - 2x - 2y - z$

$\nabla F(x, y, z) = (2x - y - 2)\mathbf{i} + (-x + 2y - 2)\mathbf{j} - \mathbf{k}$

$$\begin{aligned} 2x - y - 2 &= 0 \\ -x + 2y - 2 &= 0 \end{aligned}$$

$$\begin{aligned} y &= 2x - 2 \Rightarrow -x + 2(2x - 2) - 2 \\ &= 3x - 6 = 0 \Rightarrow x = 2 \end{aligned}$$

$y = 2, z = -4$

Point: $(2, 2, -4)$

44. $F(x, y, z) = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4 - z$

$\nabla F(x, y, z) = (8x + 4y + 8)\mathbf{i} + (4x - 4y - 5)\mathbf{j} - \mathbf{k}$

$$\begin{aligned} 8x + 4y + 8 &= 0 \\ 4x - 4y - 5 &= 0 \end{aligned}$$

Adding, $12x + 3 = 0 \Rightarrow x = -\frac{1}{4} \Rightarrow y = -\frac{3}{2}$, and $z = -\frac{5}{4}$

Point: $\left(-\frac{1}{4}, -\frac{3}{2}, -\frac{5}{4}\right)$

45. $F(x, y, z) = 5xy - z$

$\nabla F(x, y, z) = 5y\mathbf{i} + 5x\mathbf{j} - \mathbf{k}$

$5y = 0$

$5x = 0$

$x = y = z = 0$

Point: $(0, 0, 0)$

46. $F(x, y, z) = xy + \dfrac{1}{x} + \dfrac{1}{y} - z$

$$\nabla F(x, y, z) = \left(y - \frac{1}{x^2}\right)\mathbf{i} + \left(x - \frac{1}{y^2}\right)\mathbf{j} - \mathbf{k}$$

$$y = \frac{1}{x^2}$$

$$x = \frac{1}{y^2} = x^4 \Rightarrow x = 1, y = 1, z = 3$$

Point: $(1, 1, 3)$

47. $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 3, (-1, 1, 0)$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 4y \quad F_z(x, y, z) = 6z$$
$$F_x(-1, 1, 0) = -2 \quad F_y(-1, 1, 0) = 4 \quad F_z(-1, 1, 0) = 0$$

$$\begin{aligned} -2(x + 1) + 4(y - 1) + 0(z - 0) &= 0 \\ -2x + 4y &= 6 \\ -x + 2y &= 3 \end{aligned}$$

$G(x, y, z) = x^2 + y^2 + z^2 + 6x - 10y + 14, (-1, 1, 0)$

$$G_x(x, y, z) = 2x + 6 \quad G_y(x, y, z) = 2y - 10 \quad G_z(x, y, z) = 2z$$
$$G_x(-1, 1, 0) = 4 \quad G_y(-1, 1, 0) = -8 \quad G_z(-1, 1, 0) = 0$$

$$\begin{aligned} 4(x + 1) - 8(y - 1) + 0(z - 0) &= 0 \\ 4x - 8y + 12 &= 0 \\ -x + 2y &= 3 \end{aligned}$$

The tangent planes are the same.

48. $F(x, y, z) = x^2 + y^2 + z^2 - 8x - 12y + 4z + 42, (2, 3, -3)$

$$F_x(x, y, z) = 2x - 8 \quad F_y(x, y, z) = 2y - 12 \quad F_z(x, y, z) = 2z + 4$$
$$F_x(2, 3, -3) = -4 \quad F_y(2, 3, -3) = -6 \quad F_z(2, 3, -3) = -2$$

$$\begin{aligned} -4(x - 2) - 6(y - 3) - 2(z + 3) &= 0 \\ -4x - 6y - 2z + 20 &= 0 \\ 2x + 3y + z &= 10 \end{aligned}$$

$G(x, y, z) = x^2 + y^2 + 2z - 7, (2, 3, -3)$

$$G_x(x, y, z) = 2x \quad G_y(x, y, z) = 2y \quad G_z(x, y, z) = 2$$
$$G_x(2, 3, -3) = 4 \quad G_y(2, 3, -3) = 6 \quad G_z(x, y, z) = 2$$

$$\begin{aligned} 4(x - 2) + 6(y - 3) + 2(z + 3) &= 0 \\ 4x + 6y + 2z - 20 &= 0 \\ 2x + 3y + z &= 10 \end{aligned}$$

The tangent planes are the same.

49. (a) $F(x, y, z) = 2xy^2 - z,\ F(1, 1, 2) = 2 - 2 = 0$

$G(x, y, z) = 8x^2 - 5y^2 - 8z + 13,\ G(1, 1, 2) = 8 - 5 - 16 + 13 = 0$

So, $(1, 1, 2)$ lies on both surfaces.

(b) $\nabla F = 2y^2\mathbf{i} + 4xy\mathbf{j} - \mathbf{k},\ \nabla F(1, 1, 2) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

$\nabla G = 16x\mathbf{i} - 10y\mathbf{j} - 8\mathbf{k},\ \nabla G(1, 1, 2) = 16\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$

$\nabla F \cdot \nabla G = 2(16) + 4(-10) + (-1)(-8) = 0$

The tangent planes are perpendicular at $(1, 1, 2)$.

50. (a) $F(x, y, z) = x^2 + y^2 + z^2 + 2x - 4y - 4z - 12$

$F(1, -2, 1) = 0$

$G(x, y, z) = 4x^2 + y^2 + 16z^2 - 24$

$G(1, -2, 1) = 0$

So, $(1, -2, 1)$ lies on both surfaces.

(b) $\nabla F = (2x + 2)\mathbf{i} + (2y - 4)\mathbf{j} + (2z - 4)\mathbf{k}$

$\nabla F(1, -2, 1) = 4\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

$\nabla G = 8x\mathbf{i} + 2y\mathbf{j} + 32z\mathbf{k}$

$\nabla G(1, -2, 1) = 8\mathbf{i} - 4\mathbf{j} + 32\mathbf{k}$

$\nabla F \cdot \nabla G = 32 + 32 - 64 = 0$

The planes are perpendicular at $(1, -2, 1)$.

51. $F(x, y, z) = x^2 + 4y^2 + z^2 - 9$

$\nabla F = 2x\mathbf{i} + 8y\mathbf{j} + 2z\mathbf{k}$

This normal vector is parallel to the line with direction number $-4, 8, -2$.

So, $2x = -4t \Rightarrow x = -2t$

$8y = 8t \Rightarrow y = t$

$2z = -2t \Rightarrow z = -t$

$x^2 + 4y^2 + z^2 - 9 = 4t^2 + 4t^2 + t^2 - 9 = 0 \Rightarrow t = \pm 1$

There are two points on the ellipse where the tangent plane is perpendicular to the line:

$(-2, 1, -1)$ $(t = 1)$

$(2, -1, 1)$ $(t = -1)$

52. $F(x, y, z) = x^2 + 4y^2 - z^2 - 1$

$\nabla F = 2x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$

The normal to the plane, $\mathbf{n} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$

must be parallel to ∇F.

So, $2x = t \Rightarrow x = \dfrac{t}{2}$

$8y = 4t \Rightarrow y = \dfrac{t}{2}$

$-2z = -t \Rightarrow z = \dfrac{t}{2}$

$x^2 + 4y^2 - z^2 = \dfrac{t^2}{4} + t^2 - \dfrac{t^2}{4} = t^2 = 1 \Rightarrow t = \pm 1.$

Two points: $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$ $(t = 1)$ and $\left(-\dfrac{1}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}\right)$ $(t = -1)$

53. $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

(Theorem 13.13)

54. For a sphere, the common object is the center of the sphere. For a right circular cylinder, the common object is the axis of the cylinder.

55. Answers will vary.

56. (a) At $(4, 0, 0)$, the tangent plane is parallel to the *yz*-plane.

Equation: $x = 4$

(b) At $(0, -2, 0)$, the tangent plane is parallel to the *xz*-plane.

Equation: $y = -2$

(c) At $(0, 0, -4)$, the tangent plane is parallel to the *xy*-plane.

57. $z = f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}, -2 \le x \le 2, 0 \le y \le 3$

(a) Let $F(x, y, z) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)} - z$

$$\nabla F(x, y, z) = \frac{4y}{y^2 + 1}\left(\frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}\right)\mathbf{i} + \frac{4x}{x^2 + 1}\left(\frac{y^2 + 1 - 2y^2}{(y^2 + 1)^2}\right)\mathbf{j} - \mathbf{k} = \frac{4y(1 - x^2)}{(y^2 + 1)(x^2 + 1)^2}\mathbf{i} + \frac{4x(1 - y^2)}{(x^2 + 1)(y^2 + 1)^2}\mathbf{j} - \mathbf{k}$$

$\nabla F(1, 1, 1) = -\mathbf{k}$

Direction numbers: 0, 0, –1

Line: $x = 1, y = 1, z = 1 - t$

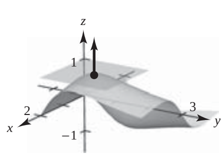
Tangent plane: $0(x - 1) + 0(y - 1) - 1(z - 1) = 0 \Rightarrow z = 1$

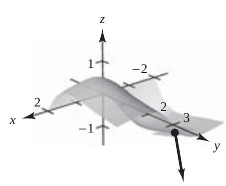
(b) $\nabla F\left(-1, 2, -\dfrac{4}{5}\right) = 0\mathbf{i} + \dfrac{-4(-3)}{(2)(5)^2}\mathbf{j} - \mathbf{k} = \dfrac{6}{25}\mathbf{j} - \mathbf{k}$

Line: $x = -1, y = 2 + \dfrac{6}{25}t, z = -\dfrac{4}{5} - t$

Plane:
$$\begin{aligned} 0(x + 1) + \frac{6}{25}(y - 2) - 1\left(z + \frac{4}{5}\right) &= 0 \\ 6y - 12 - 25z - 20 &= 0 \\ 6y - 25z - 32 &= 0 \end{aligned}$$

(c)





58. (a) $f(x, y) = \dfrac{\sin y}{x}, -3 \le x \le 3, 0 \le y \le 2\pi$

Let $F(x, y, z) = \dfrac{\sin y}{x} - z$

$\nabla F(x, y, z) = \dfrac{-\sin y}{x^2}\mathbf{i} + \dfrac{\cos y}{x}\mathbf{j} - \mathbf{k}$

$\nabla F\left(2, \dfrac{\pi}{2}, \dfrac{1}{2}\right) = -\dfrac{1}{4}\mathbf{i} - \mathbf{k}$

Direction numbers: $-\dfrac{1}{4}, 0, -1$ or $1, 0, 4$

Line: $x = 2 + t, y = \dfrac{\pi}{2}, z = \dfrac{1}{2} + 4t$

Tangent plane: $1(x - 2) + 0\left(y - \dfrac{\pi}{2}\right) + 4\left(z - \dfrac{1}{2}\right) = 0 \Rightarrow x + 4z - 4 = 0$

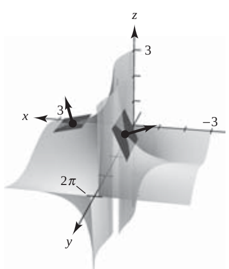
(b) $\nabla F\left(-\dfrac{2}{3}, \dfrac{3\pi}{2}, \dfrac{3}{2}\right) = \dfrac{9}{4}\mathbf{i} - \mathbf{k}$

Direction numbers: $\dfrac{9}{4}, 0, -1$ or $9, 0, -4$

Line: $x = -\dfrac{2}{3} + 9t, y = \dfrac{3\pi}{2}, z = \dfrac{3}{2} - 4t$

Tangent plane: $9\left(x + \dfrac{2}{3}\right) + 0\left(y - \dfrac{3\pi}{2}\right) - 4\left(z - \dfrac{3}{2}\right) = 0 \Rightarrow 9x - 4z + 12 = 0$

(c)



59. $f(x, y) = 6 - x^2 - \dfrac{y^2}{4}, g(x, y) = 2x + y$

(a) $F(x, y, z) = z + x^2 + \dfrac{y^2}{4} - 6 \quad G(x, y, z) = z - 2x - y$

$\nabla F(x, y, z) = 2x\mathbf{i} + \dfrac{1}{2}y\mathbf{j} + \mathbf{k} \quad \nabla G(x, y, z) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 4) = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \quad \nabla G(1, 2, 4) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

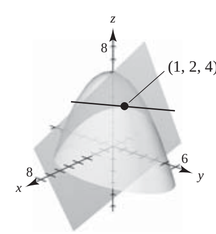
The cross product of these gradients is parallel to the curve of intersection.

$$\nabla F(1, 2, 4) \times \nabla G(1, 2, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j}$$

Using direction numbers $1, -2, 0$, you get $x = 1 + t, y = 2 - 2t, z = 4$.

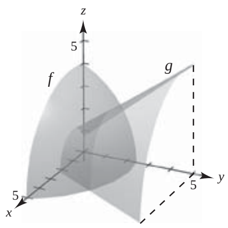
$\cos\theta = \dfrac{\nabla F \cdot \nabla G}{\|\nabla F\|\|\nabla G\|} = \dfrac{-4 - 1 + 1}{\sqrt{6}\sqrt{6}} = \dfrac{-4}{6} \Rightarrow \theta \approx 48.2°$

(b)



60. (a) $f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$

$g(x, y) = \frac{\sqrt{2}}{2}\sqrt{1 - 3x^2 + y^2 + 6x + 4y}$



(b)
$$f(x, y) = g(x, y)$$
$$16 - x^2 - y^2 + 2x - 4y = \frac{1}{2}(1 - 3x^2 + y^2 + 6x + 4y)$$
$$32 - 2x^2 - 2y^2 + 4x - 8y = 1 - 3x^2 + y^2 + 6x + 4y$$
$$x^2 - 2x + 31 = 3y^2 + 12y$$
$$(x^2 - 2x + 1) + 42 = 3(y^2 + 4y + 4)$$
$$(x - 1)^2 + 42 = 3(y + 2)^2$$

To find points of intersection, let $x = 1$. Then

$$3(y + 2)^2 = 42$$
$$(y + 2)^2 = 14$$
$$y = -2 \pm \sqrt{14}$$

$\nabla f\left(1, -2 + \sqrt{14}\right) = -\sqrt{2}\mathbf{j}$, $\nabla g\left(1, -2 + \sqrt{14}\right) = \left(1/\sqrt{2}\right)\mathbf{j}$. The normals to f and g at this point are $-\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(-1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are orthogonal.

Similarly, $\nabla f\left(1, -2 - \sqrt{14}\right) = \sqrt{2}\mathbf{j}$ and $\nabla g\left(1, -2 - \sqrt{14}\right) = \left(-1/\sqrt{2}\right)\mathbf{j}$ and the normals are $\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(-1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are also orthogonal.

(c) No, showing that the surfaces are orthogonal at 2 points does not imply that they are orthogonal at every point of intersection.

61. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$F_x(x, y, z) = \frac{2x}{a^2}$

$F_y(x, y, z) = \frac{2y}{b^2}$

$F_z(x, y, z) = \frac{2z}{c^2}$

Plane: $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

62. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1$

$F_x(x, y, z) = \frac{2x}{a^2}$

$F_y(x, y, z) = \frac{2y}{b^2}$

$F_z(x, y, z) = \frac{-2z}{c^2}$

Plane: $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c_2}(z - z_0) = 0$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 1$$

63. $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$

$F_x(x, y, z) = 2a^2x$

$F_y(x, y, z) = 2b^2y$

$F_z(x, y, z) = -2z$

Plane: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$

$$a^2x_0x + b^2y_0y - z_0z = a^2x_0^2 + b^2y_0^2 - z_0^2 = 0$$

So, the plane passes through the origin.

64. $z = xf\left(\frac{y}{x}\right)$

$F(x, y, z) = xf\left(\frac{y}{x}\right) - z$

$F_x(x, y, z) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$

$F_y(x, y, z) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$

$F_x(x, y, z) = -1$

Tangent plane at (x_0, y_0, z_0):

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) - (z - z_0) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x - x_0f\left(\frac{y_0}{x_0}\right) + y_0f'\left(\frac{y_0}{x_0}\right) + yf'\left(\frac{y_0}{x_0}\right) - y_0f'\left(\frac{y_0}{x_0}\right) - z + x_0f\left(\frac{y_0}{x_0}\right) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x + f'\left(\frac{y_0}{x_0}\right)y - z = 0$$

So, the plane passes through the origin $(x, y, z) = (0, 0, 0)$.

65. $f(x, y) = e^{x-y}$

$f_x(x, y) = e^{x-y}, \; f_y(x, y) = -e^{x-y}$

$f_{xx}(x, y) = e^{x-y}, \; f_{yy}(x, y) = e^{x-y}, \; f_{xy}(x, y) = -e^{x-y}$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1 + x - y$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2 = 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$

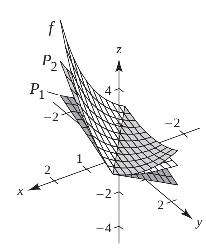
(c) If $x = 0$, $P_2(0, y) = 1 - y + \frac{1}{2}y^2$. This is the second-degree Taylor polynomial for e^{-y}.

If $y = 0$, $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$. This is the second-degree Taylor polynomial for e^x.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250

(e)



66. $f(x, y) = \cos(x + y)$

$f_x(x, y) = -\sin(x + y),\ f_y(x, y) = -\sin(x + y)$

$f_{xx}(x, y) = -\cos(x + y),\ f_{yy}(x, y) = -\cos(x + y),\ f_{xy}(x, y) = -\cos(x + y)$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$

$= 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$

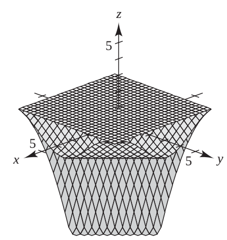
(c) If $x = 0$, $P_2(0, y) = 1 - \frac{1}{2}y^2$. This is the second-degree Taylor polynomial for $\cos y$.

If $y = 0$, $P_2(x, 0) = 1 - \frac{1}{2}x^2$. This is the second-degree Taylor polynomial for $\cos x$.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9950	1	0.9950
0.2	0.1	0.9553	1	0.9950
0.2	0.5	0.7648	1	0.7550
1	0.5	0.0707	1	−0.1250

(e)



67. Given $z = f(x, y)$, then:

$$F(x, y, z) = f(x, y) - z = 0$$

$$\nabla F(x_0, y_0, z_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$$

$$\cos\theta = \frac{|\nabla F(x_0, y_0, z_0) \cdot \mathbf{k}|}{\|\nabla F(x_0, y_0, z_0)\|\|\mathbf{k}\|}$$

$$= \frac{|-1|}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}$$

68. Given $w = F(x, y, z)$ where F is differentiable at

(x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$,

the level surface of F at (x_0, y_0, z_0) is of the form $F(x, y, z) = C$ for some constant C. Let

$G(x, y, z) = F(x, y, z) - C = 0.$

Then $\nabla G(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$ where $\nabla G(x_0, y_0, z_0)$ is normal to $F(x, y, z) - C = 0$ at (x_0, y_0, z_0). So, $\nabla F(x_0, y_0 z_0)$ is normal to the level surface through (x_0, y_0, z_0).

Section 13.8 Extrema of Functions of Two Variables

1. $g(x, y) = (x - 1)^2 + (y - 3)^2 \geq 0$

Relative minimum: $(1, 3, 0)$

Check: $g_x = 2(x - 1) = 0 \Rightarrow x = 1$

$g_y = 2(y - 3) = 0 \Rightarrow y = 3$

$g_{xx} = 2, g_{yy} = 2, g_{xy} = 0, d = (2)(2) - 0 = 4 > 0$

At critical point $(1, 3)$, $d > 0$ and $g_{xx} > 0 \Rightarrow$ relative minimum at $(1, 3, 0)$.

2. $g(x, y) = 5 - (x - 3)^2 - (y + 2)^2 \leq 5$

Relative maximum: $(3, -2, 5)$

Check: $g_x = -2(x - 3) = 0 \Rightarrow x = 3$

$g_y = -2(y + 2) = 0 \Rightarrow y = -2$

$g_{xx} = -2, g_{yy} = -2, g_{xy} = 0$

$d = (-2)(-2) - 0 = 4 > 0$

At critical point $(3, -2)$, $d > 0$ and $g_{xx} < 0 \Rightarrow$ relative maximum at $(3, -2, 5)$.

3. $f(x, y) = \sqrt{x^2 + y^2 + 1} \geq 1$

Relative minimum: $(0, 0, 1)$

Check: $f_x = \dfrac{x}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow x = 0$

$$f_y = \frac{y}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow y = 0$$

$$f_{xx} = \frac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}}$$

$$f_{yy} = \frac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}}$$

$$f_{xy} = \frac{-xy}{(x^2 + y^2 + 1)^{3/2}}$$

At the critical point $(0, 0)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(0, 0, 1)$ is a relative minimum.

4. $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \leq 5$

Relative maximum: $(2, 0, 5)$

Check: $f_x = -\dfrac{x - 2}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \Rightarrow x = 2$

$$f_y = -\frac{y}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \Rightarrow y = 0$$

$$f_{xx} = -\frac{25 - y^2}{\left[25 - (x - 2)^2 - y^2\right]^{3/2}}$$

$$f_{yy} = -\frac{25 - (x - 2)^2}{\left[25 - (x - 2)^2 - y^2\right]^{3/2}}$$

$$f_{xy} = -\frac{y(x - 2)}{\left[25 - (x - 2)^2 - y^2\right]^{3/2}}$$

At the critical point $(2, 0)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(2, 0, 5)$ is a relative maximum.

5. $f(x, y) = x^2 + y^2 + 2x - 6y + 6 = (x + 1)^2 + (y - 3)^2 - 4 \geq -4$

Relative minimum: $(-1, 3, -4)$

Check: $f_x = 2x + 2 = 0 \Rightarrow x = -1$

$f_y = 2y - 6 = 0 \Rightarrow y = 3$

$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$

At the critical point $(-1, 3)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. So, $(-1, 3, -4)$ is a relative minimum.

6. $f(x, y) = -x^2 - y^2 + 10x + 12y - 64$

$$= -(x^2 - 10x + 25) - (y^2 - 12y + 36) + 25 + 36 - 64 = -(x - 5)^2 - (y - 6)^2 - 3 \le -3$$

Relative maximum: $(5, 6, -3)$

Check: $f_x = -2x + 10 = 0 \Rightarrow x = 5$

$f_y = -2y + 12 = 0 \Rightarrow y = 6$

$f_{xx} = -2, f_{yy} = -2, f_{xy} = 0, d = (-2)(-2) - 0 = 4 > 0$

At critical point $(5, 6)$, $d > 0$ and $f_{xx} < 0 \Rightarrow$ relative maximum at $(5, 6, -3)$.

7. $h(x, y) = 80x + 80y - x^2 - y^2$

$$\left.\begin{aligned} h_x &= 80 - 2x = 0 \\ h_y &= 80 - 2y = 0 \end{aligned}\right\} x = y = 40$$

$h_{xx} = -2, h_{yy} = -2, h_{xy} = 0,$

$d = (-2)(-2) - 0 = 4 > 0$

At the critical point $(40, 40)$, $d > 0$ and

$h_{xx} < 0 \Rightarrow (40, 40, 3200)$ is a relative maximum.

8. $g(x, y) = x^2 - y^2 - x - y$

$$\left.\begin{aligned} g_x &= 2x - 1 = 0 \\ g_y &= -2y - 1 = 0 \end{aligned}\right\} \begin{aligned} x &= 1/2 \\ y &= -1/2 \end{aligned}$$

$g_{xx} = 2, g_{yy} = -2, g_{xy} = 0, d = 2(-2) - 0 = -4 < 0$

At the critical point $(1/2, -1/2)$, $d < 0$

$\Rightarrow (1/2, -1/2, 0)$ is a saddle point.

9. $g(x, y) = xy$

$$\left.\begin{aligned} g_x &= y \\ g_y &= x \end{aligned}\right\} x = 0 \text{ and } y = 0$$

$g_{xx} = 0, g_{yy} = 0, g_{xy} = 1$

At the critical point $(0, 0)$, $g_{xx}g_{yy} - (g_{xy})^2 < 0$.

So, $(0, 0, 0)$ is a saddle point.

10. $h(x, y) = x^2 - 3xy - y^2$

$$\left.\begin{aligned} h_x &= 2x - 3y = 0 \\ h_y &= -3x - 2y = 0 \end{aligned}\right\} \begin{array}{l} \text{Solving simultaneously} \\ \text{yields } x = 0 \text{ and } y = 0. \end{array}$$

$h_{xx} = 2, h_{yy} = -2, h_{xy} = -3$

At the critical point $(0, 0)$, $h_{xx}h_{yy} - (h_{xy})^2 < 0$.

So, $(0, 0, 0)$ is a saddle point.

11. $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$

$f_x = -6x + 3 = 0$ when $x = \frac{1}{2}$.

$f_y = -4y - 4 = 0$ when $y = -1$.

$f_{xx} = -6, f_{yy} = -4, f_{xy} = 0$

At the critical point $\left(\frac{1}{2}, -1\right)$, $f_{xx} < 0$

and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $\left(\frac{1}{2}, -1, \frac{31}{4}\right)$ is a relative maximum.

12. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

$$\left.\begin{aligned} f_x &= 4x + 2y + 2 = 0 \\ f_y &= 2x + 2y = 0 \end{aligned}\right\} \begin{array}{l} \text{Solving simultaneously} \\ \text{yields } x = -1 \text{ and } y = 1. \end{array}$$

$f_{xx} = 4, f_{yy} = 2, f_{xy} = 2$

At the critical point $(-1, 1)$, $f_{xx} > 0$

and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(-1, 1, -4)$ is a relative minimum.

13. $f(x, y) = z = x^2 + xy + \frac{1}{2}y^2 - 2x + y$

$$\left.\begin{aligned} f_x &= 2x + y - 2 = 0 \\ f_y &= x + y + 1 = 0 \end{aligned}\right\} \begin{array}{l} \text{Solving simultaneously} \\ \text{yields } x = 3, y = -4 \end{array}$$

$f_{xx} = 2, f_{yy} = 1, f_{xy} = 1, d = 2(1) - 1 = 1 > 0.$

At the critical point $(3, -4)$, $d > 0$

and $f_{xx} > 0 \Rightarrow (3, -4, -5)$ is a relative minimum.

14. $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$$\left.\begin{aligned} f_x &= -10x + 4y + 16 = 0 \\ f_y &= 4x - 2y = 0 \end{aligned}\right\} \begin{array}{l} \text{Solving simultaneously} \\ \text{yields } x = 8 \text{ and } y = 16. \end{array}$$

$f_{xx} = -10, f_{yy} = -2, f_{xy} = 4$

At the critical point $(8, 16)$, $f_{xx} < 0$

and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

So, $(8, 16, 74)$ is a relative maximum.

15. $f(x, y) = \sqrt{x^2 + y^2}$

$$\left.\begin{aligned} f_x &= \frac{x}{\sqrt{x^2 + y^2}} = 0 \\ f_y &= \frac{y}{\sqrt{x^2 + y^2}} = 0 \end{aligned}\right\} x = y = 0$$

Because $f(x, y) \geq 0$ for all (x, y) and $f(0, 0) = 0$, $(0, 0, 0)$ is a relative minimum.

16. $h(x, y) = (x^2 + y^2)^{1/3} + 2$

$$\left.\begin{aligned} h_x &= \frac{2x}{3(x^2 + y^2)^{2/3}} = 0 \\ h_y &= \frac{2y}{3(x^2 + y^2)^{2/3}} = 0 \end{aligned}\right\} x = 0, y = 0$$

Because $h(x, y) \geq 2$ for all (x, y), $(0, 0, 2)$ is a relative minimum.

17. $f(x, y) = x^2 - xy - y^2 - 3x - y$

$f_x = 2x - y - 3 = 0$

$f_y = -x - 2y - 1 = 0$

Solving simultaneously yields $x = 1, y = -1$.

$f_{xx} = 2, f_{yy} = -2, f_{xy} = -1$

$d = (2)(-2) - (-1)^2 = -5 < 0$

At the critical point $(1, -1)$, $d < 0 \Rightarrow (1, -1, -1)$ is a saddle point.

18. $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^2) + 1$

$$\left.\begin{aligned} f_x &= 2y - 2x^3 \\ f_y &= 2x - 2y^3 \end{aligned}\right\} \begin{array}{l}\text{Solving by substitution yields 3 critical points:} \\ (0, 0), (1, 1), (-1, -1)\end{array}$$

$f_{xx} = -6x^2, f_{yy} = -6y^2, f_{xy} = 2$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0 \Rightarrow (0, 0, 1)$ saddle point.

At $(1, 1)$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \Rightarrow (1, 1, 2)$ relative maximum.

At $(-1, -1)$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \Rightarrow (-1, -1, 2)$ relative maximum.

19. $f(x, y) = e^{-x} \sin y$

$$\left.\begin{aligned} f_x &= -e^{-x} \sin y = 0 \\ f_y &= e^{-x} \cos y = 0 \end{aligned}\right\} \begin{array}{l}\text{Because } e^{-x} > 0 \text{ for all } x \text{ and } \sin y \text{ and } \cos y \text{ are never} \\ \text{both zero for a given value of } y\text{, there are no critical points.}\end{array}$$

20. $f(x, y) = \left(\frac{1}{2} - x^2 + y^2\right)e^{1-x^2-y^2}$

$$\left.\begin{aligned} f_x &= (2x^3 - 2xy^2 - 3x)e^{1-x^2-y^2} = 0 \\ f_y &= (2x^2y - 2y^3 + y)e^{1-x^2-y^2} = 0 \end{aligned}\right\} \text{Solving yields the critical points } (0, 0), \left(0, \pm\frac{\sqrt{2}}{2}\right), \left(\pm\frac{\sqrt{6}}{2}, 0\right).$$

$f_{xx} = (-4x^4 + 4x^2y^2 + 12x^2 - 2y^2 - 3)e^{1-x^2-y^2}$

$f_{yy} = (4y^4 - 4x^2y^2 + 2x^2 - 8y^2 + 1)e^{1-x^2-y^2}$

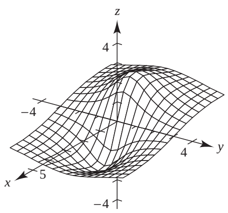
$f_{xy} = (-4x^3y + 4xy^3 + 2xy)e^{1-x^2-y^2}$

At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. So, $(0, 0, e/2)$ is a saddle point. At the critical points $(0, \pm\sqrt{2}/2)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. So, $(0, \pm\sqrt{2}/2, \sqrt{e})$ are relative maxima. At the critical points $(\pm\sqrt{6}/2, 0)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. So, $(\pm\sqrt{6}/2, 0, -\sqrt{e}/e)$ are relative minima.

21. $z = \dfrac{-4x}{x^2 + y^2 + 1}$

Relative minimum: $(1, 0, -2)$

Relative maximum: $(-1, 0, 2)$

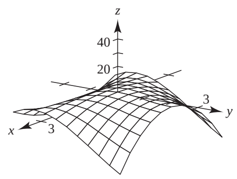


22. $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$

Relative maximum: $(0, 0, 1)$

Saddle points:

$(0, 2, -3), (\pm\sqrt{3}, -1, -3)$

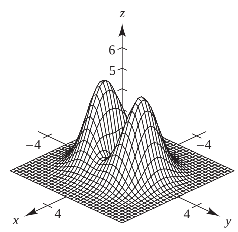


23. $z = (x^2 + 4y^2)e^{1-x^2-y^2}$

Relative minimum: $(0, 0, 0)$

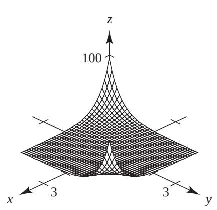
Relative maxima: $(0, \pm 1, 4)$

Saddle points: $(\pm 1, 0, 1)$



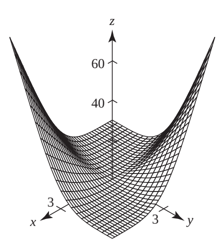
24. $z = e^{xy}$

Saddle point: $(0, 0, 1)$



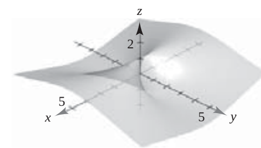
25. $z = \dfrac{(x - y)^4}{x^2 + y^2} \geq 0.\ z = 0$ if $x = y \neq 0.$

Relative minimum at all points $(x, x), x \neq 0.$



26. $z = \dfrac{(x^2 - y^2)^2}{x^2 + y^2} \geq 0.\ z = 0$ if $x^2 = y^2 \neq 0.$

Relative minima at all points (x, x) and $(x, -x), x \neq 0.$



27. $f_{xx}f_{yy} - (f_{xy})^2 = (9)(4) - 6^2 = 0$

Insufficient information.

28. $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = (-3)(-8) - 2^2 > 0$

f has a relative maximum at (x_0, y_0)

29. $f_{xx}f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2 < 0$

f has a saddle point at (x_0, y_0).

30. $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = (25)(8) - 10^2 > 0$

f has a relative minimum at (x_0, y_0)

31. $d = f_{xx}f_{yy} - f_{xy}{}^2 = (2)(8) - f_{xy}{}^2 = 16 - f_{xy}{}^2 > 0$

$\Rightarrow f_{xy}{}^2 < 16 \Rightarrow -4 < f_{xy} < 4$

32. $d = f_{xx}f_{yy} - f_{xy}{}^2 < 0$ if f_{xx} and f_{yy} have opposite signs. So, $(a, b, f(a, b))$ is a saddle point. For example, consider $f(x, y) = x^2 - y^2$ and $(a, b) = (0, 0)$.

33. $f(x, y) = x^3 + y^3$

(a) $\left.\begin{aligned} f_x &= 3x^2 = 0 \\ f_y &= 3y^2 = 0 \end{aligned}\right\} x = y = 0$

Critical point: $(0, 0)$

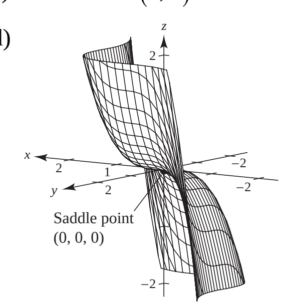
(b) $f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = 0$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0.$

$(0, 0, 0)$ is a saddle point.

(c) Test fails at $(0, 0)$.

(d)



34. $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

(a) $\left.\begin{aligned} f_x &= 3x^2 - 12x + 12 = 0 \\ f_y &= 3y^2 + 18y + 27 = 0 \end{aligned}\right\}$ Solving yields $x = 2$ and $y = -3$.

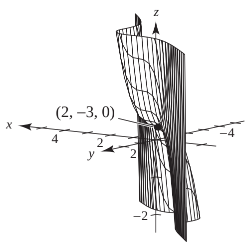
(b) $f_{xx} = 6x - 12,\ f_{yy} = 6y + 18,\ f_{xy} = 0$

At $(2, -3)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$.

$(2, -3, 0)$ is a saddle point.

(c) Test fails at $(2, -3)$.

(d)



35. $f(x, y) = (x - 1)^2(y + 4)^2 \geq 0$

(a) $\left.\begin{aligned} f_x &= 2(x - 1)(y + 4)^2 = 0 \\ f_y &= 2(x - 1)^2(y + 4) = 0 \end{aligned}\right\}$ critical points: $(1, a)$ and $(b, -4)$

(b) $f_{xx} = 2(y + 4)^2$

$f_{yy} = 2(x - 1)^2$

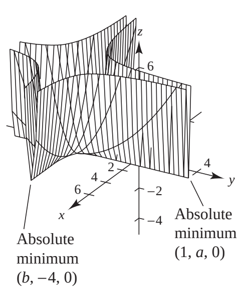
$f_{xy} = 4(x - 1)(y + 4)$

At both $(1, a)$ and $(b, -4)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$.

Because $f(x, y) \geq 0$, there are absolute minima at $(1, a, 0)$ and $(b, -4, 0)$.

(c) Test fails at $(1, a)$ and $(b, -4)$.

(d)



36. $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2} \geq 0$

(a) $\left.\begin{aligned} f_x &= \frac{x - 1}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \\ f_y &= \frac{y + 2}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \end{aligned}\right\}$ Solving yields $x = 1$ and $y = -2$.

(b) $f_{xx} = \dfrac{(y + 2)^2}{\left[(x - 1)^2 + (y + 2)^2\right]^{3/2}}$

$f_{yy} = \dfrac{(x - 1)^2}{\left[(x - 1)^2 + (y + 2)^2\right]^{3/2}}$

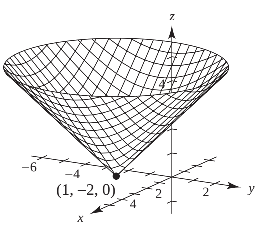
$f_{xy} = \dfrac{(x - 1)(y + 2)}{\left[(x - 1)^2 + (y + 2)^2\right]^{3/2}}$

At $(1, -2)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined.

$(1, -2, 0)$ is an absolute minimum.

(c) Test fails at $(1, -2)$.

(d)



37. $f(x, y) = x^{2/3} + y^{2/3} \geq 0$

(a) $\left.\begin{aligned} f_x &= \frac{2}{3x^{1/3}} \\ f_y &= \frac{2}{3y^{1/3}} \end{aligned}\right\}$ f_x and f_y are undefined at $x = 0$ and $y = 0$. Critical point: $(0, 0)$

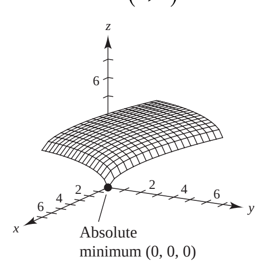
(b) $f_{xx} = \dfrac{-2}{9x^{4/3}},\ f_{yy} = \dfrac{-2}{9y^{4/3}},\ f_{xy} = 0$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined.

$(0, 0, 0)$ is an absolute minimum.

(c) Test fails at $(0, 0)$.

(d)



38. $f(x, y) = (x^2 + y^2)^{2/3} \geq 0$

(a) $\left.\begin{aligned} f_x &= \dfrac{4x}{3(x^2 + y^2)^{1/3}} \\ f_y &= \dfrac{4y}{3(x^2 + y^2)^{1/3}} \end{aligned}\right\}$ f_x and f_y are undefined at $x = 0$, $y = 0$. Critical Point: $(0, 0)$

(b) $f_{xx} = \dfrac{4(x^2 + 3y^2)}{9(x^2 + y^2)^{4/3}}$

$f_{yy} = \dfrac{4(3x^2 + y^2)}{9(x^2 + y^2)^{4/3}}$

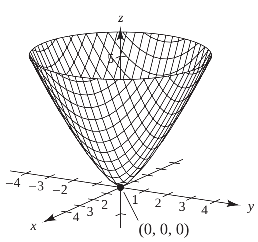
$f_{xy} = \dfrac{-8xy}{9(x^2 + y^2)^{4/3}}$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined.

$(0, 0, 0)$ is an absolute minimum.

(c) Test fails at $(0, 0)$.

(d)



39. $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2 \geq 0$

$\left.\begin{aligned} f_x &= 2x = 0 \\ f_y &= 2(y - 3) = 0 \\ f_z &= 2(z + 1) = 0 \end{aligned}\right\}$ Solving yields the critical point $(0, 3, -1)$.

Absolute minimum: 0 at $(0, 3, -1)$

40. $f(x, y, z) = 9 - [x(y - 1)(z + 2)]^2 \leq 9$

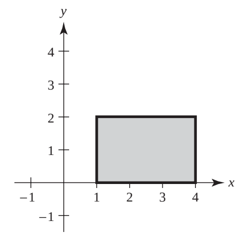
The absolute maximum value of f is 9, and realized at all points where $x(y - 1)(z + 2) = 0$.

So, the critical points are of the form $(0, a, b)$, $(c, 1, d)$, $(e, f, -z)$

where a, b, c, d, e, f are real numbers.

41. $f(x, y) = x^2 - 4xy + 5, R = \{(x, y): 1 \le x \le 4, 0 \le y \le 2\}$

$$\left.\begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= -4x = 0 \end{aligned}\right\} x = y = 0 \quad \text{(not in region R)}$$



Along $y = 0, 1 \le x \le 4$: $f = x^2 + 5$, $f(1, 0) = 6$, $f(4, 0) = 21$.

Along $y = 2, 1 \le x \le 4$: $f = x^2 - 8x + 5$, $f' = 2x - 8 = 0$
$f(1, 2) = -2$, $f(4, 2) = -11$.

Along $x = 1, 0 \le y \le 2$: $f = -4y + 6$, $f(1, 0) = 6$, $f(1, 2) = -2$.

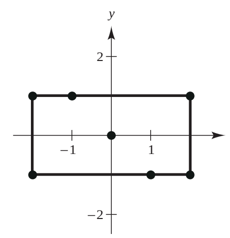
Along $x = 4, 0 \le y \le 2$: $f = 21 - 16y$, $f(4, 0) = 21$, $f(4, 2) = -11$.

So, the maximum is $(4, 0, 21)$ and the minimum is $(4, 2, -11)$.

42. $f(x, y) = x^2 + xy, R = \{(x, y): |x| \le 2, |y| \le 1\}$

$$\left.\begin{aligned} f_x &= 2x + y = 0 \\ f_y &= x = 0 \end{aligned}\right\} x = y = 0$$

$f(0, 0) = 0$



Along $y = 1, -2 \le x \le 2$, $f = x^2 + x$, $f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2$, $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f(2, 1) = 6$.

Along $y = -1, -2 \le x \le 2$, $f = x^2 - x$, $f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

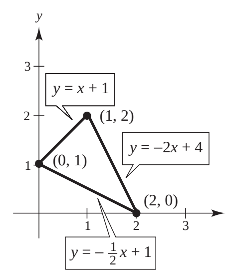
Thus, $f(-2, -1) = 6$, $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$, $f(2, -1) = 2$.

Along $x = 2, -1 \le y \le 1$, $f = 4 + 2y \Rightarrow f' = 2 \ne 0$.

Along $x = -2, -1 \le y \le 1$, $f = 4 - 2y \Rightarrow f' = -2 \ne 0$.

So, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$.

43. $f(x, y) = 12 - 3x - 2y$ has no critical points. On the line $y = x + 1, 0 \le x \le 1$,



$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$

and the maximum is 10, the minimum is 5. On the line $y = -2x + 4, 1 \le x \le 2$,

$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$

and the maximum is 6, the minimum is 5. On the line $y = -\frac{1}{2}x + 1, 0 \le x \le 2$,

$f(x, y) = f(x) = 12 - 3x - 2\left(-\frac{1}{2}x + 1\right) = -2x + 10$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at $(0, 1)$

Absolute minimum: 5 at $(1, 2)$

44. $f(x, y) = (2x - y)^2$

$f_x = 4(2x - y) = 0 \Rightarrow 2x = y$

$f_y = -2(2x - y) = 0 \Rightarrow 2x = y$

On the line $y = x + 1, 0 \le x \le 1$,

$f(x, y) = f(x) = (2x - (x + 1))^2 = (x - 1)^2$

and the maximum is 1, the minimum is 0. On the line $y = -\frac{1}{2}x + 1, 0 \le x \le 2$,

$f(x, y) = f(x) = \left(2x - \left(-\frac{1}{2}x + 1\right)\right)^2 = \left(\frac{5}{2}x - 1\right)^2$

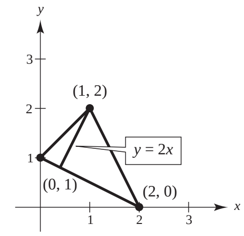
and the maximum is 16, the minimum is 0. On the line $y = -2x + 4, 1 \le x \le 2$,

$f(x, y) = f(x) = (2x - (-2x + 4))^2 = (4x - 4)^2$

and the maximum is 16, the minimum is 0.

Absolute maximum: 16 at $(2, 0)$

Absolute Minimum: 0 at $(1, 2)$ and along the line $y = 2x$.



45. $f(x, y) = 3x^2 + 2y^2 - 4y$

$$\left.\begin{aligned} f_x &= 6x = 0 \Rightarrow x = 0 \\ f_y &= 4y - 4 = 0 \Rightarrow y = 1 \end{aligned}\right\} f(0, 1) = -2$$

On the line $y = 4, -2 \le x \le 2$,

$f(x, y) = f(x) = 3x^2 + 32 - 16 = 3x^2 + 16$

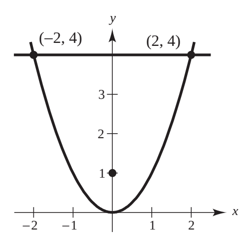
and the maximum is 28, the minimum is 16. On the curve $y = x^2, -2 \le x \le 2$,

$f(x, y) = f(x) = 3x^2 + 2(x^2)^2 - 4x^2 = 2x^4 - x^2 = x^2(2x^2 - 1)$

and the maximum is 28, the minimum is $-\frac{1}{8}$.

Absolute maximum: 28 at $(\pm 2, 4)$

Absolute minimum: -2 at $(0, 1)$



46. $f(x, y) = 2x - 2xy + y^2$

$$\left.\begin{aligned} f_x &= 2 - 2y = 0 \Rightarrow y = 1 \\ f_y &= 2y - 2x = 0 \Rightarrow y = x \Rightarrow x = 1 \end{aligned}\right\} f(1, 1) = 1$$

On the line $y = 1, -1 \le x \le 1$,

$f(x, y) = f(x) = 2x - 2x + 1 = 1.$

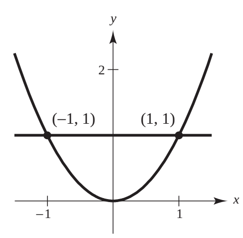
On the curve $y = x^2, -1 \le x \le 1$

$f(x, y) = f(x) = 2x - 2x(x^2) + (x^2)^2 = x^4 - 2x^3 + 2x$

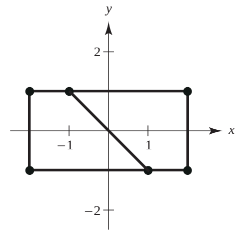
and the maximum is 1, the minimum is $-\frac{11}{16}$.

Absolute maximum: 1 at $(1, 1)$ and on $y = 1$

Absolute minimum: $-\frac{11}{16} = -0.6875$ at $\left(-\frac{1}{2}, \frac{1}{4}\right)$



47. $f(x, y) = x^2 + 2xy + y^2, R = \{(x, y): |x| \le 2, |y| \le 1\}$



$$\left.\begin{aligned} f_x &= 2x + 2y = 0 \\ f_y &= 2x + 2y = 0 \end{aligned}\right\} y = -x$$

$$f(x, -x) = x^2 - 2x^2 + x^2 = 0$$

Along $y = 1, -2 \le x \le 2$,

$$f = x^2 + 2x + 1,\ f' = 2x + 2 = 0 \Rightarrow x = -1,\ f(-2, 1) = 1,\ f(-1, 1) = 0,\ f(2, 1) = 9.$$

Along $y = -1, -2 \le x \le 2$,

$$f = x^2 - 2x + 1,\ f' = 2x - 2 = 0 \Rightarrow x = 1,\ f(-2, -1) = 9,\ f(1, -1) = 0,\ f(2, -1) = 1.$$

Along $x = 2, -1 \le y \le 1,\ f = 4 + 4y + y^2,\ f' = 2y + 4 \ne 0.$

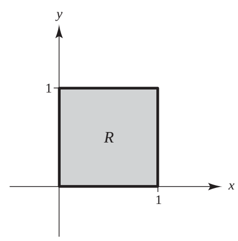
Along $x = -2, -1 \le y \le 1,\ f = 4 - 4y + y^2,\ f' = 2y - 4 \ne 0.$

So, the maxima are $f(-2, -1) = 9$ and $f(2, 1) = 9$, and the minima are $f(x, -x) = 0, -1 \le x \le 1$.

48. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} \Rightarrow x = 0 \text{ or } y = 1$$



For $x = 0$, $y = 0$, also, and $f(0, 0) = 0$.

For $x = 1$, $y = 1$, $f(1, 1) = 1$.

The absolute maximum is $1 = f(1, 1)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$.)

49. (a) The function f has a relative minimum at (x_0, y_0) if $f(x, y) \ge f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0).

(b) The function f has a relative maximum at (x_0, y_0) if $f(x, y) \le f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0).

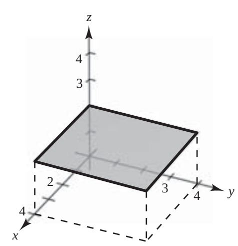
(c) The point (x_0, y_0) is a critical point if either

(1) $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or

(2) $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

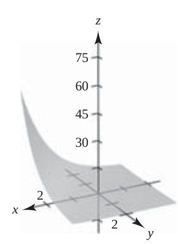
(d) A critical point is a saddle point if it is neither a relative minimum nor a relative maximum.

50.



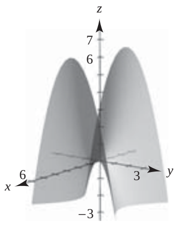
Extrema at all (x, y)

51.



No extrema

52.



Saddle point

53. $f(x, y) = x^2 - y^2, g(x, y) = x^2 + y^2$

(a) $f_x = 2x = 0, f_y = -2y = 0 \Rightarrow (0, 0)$ is a critical point.

$g_x = 2x = 0, g_y = 2y = 0 \Rightarrow (0, 0)$ is a critical point.

(b) $f_{xx} = 2, f_{yy} = -2, f_{xy} = 0$

$d = 2(-2) - 0 < 0 \Rightarrow (0, 0)$ is a saddle point.

$g_{xx} = 2, g_{yy} = 2, g_{xy} = 0$

$d = 2(2) - 0 > 0 \Rightarrow (0, 0)$ is a relative minimum.

54. A and B are relative extrema.

C and D are saddle points.

55. False.

Let $f(x, y) = 1 - |x| - |y|$.

$(0, 0, 1)$ is a relative maximum, but $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

56. False. Consider $f(x, y) = x^2 - y^2$.

Then $f_x(0, 0) = f_y(0, 0) = 0$, but $(0, 0, 0)$ is a saddle point.

57. False. Let $f(x, y) = x^2y^2$ (See Example 4 on page 940).

58. False.

Let $f(x, y) = x^4 - 2x^2 + y^2$.

Relative minima: $(\pm 1, 0, -1)$

Saddle point: $(0, 0, 0)$

Section 13.9 Applications of Extrema of Functions of Two Variables

1. A point on the plane is given by $(x, y, z) = (x, y, 3 - x + y)$. The square of the distance from $(0, 0, 0)$ to this point is

$$S = x^2 + y^2 + (3 - x + y)^2.$$
$$S_x = 2x - 2(3 - x + y)$$
$$S_y = 2y + 2(3 - x + y)$$

From the equations $S_x = 0$ and $S_y = 0$ we obtain

$$4x - 2y = 6$$
$$-2x + 4y = -6.$$

Solving simultaneously, we have $x = 1, y = -1, z = 1$.

So, the distance is $\sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$.

2. A point on the plane is given by $(x, y, z) = (x, y, 3 - x + y)$. The square of the distance from $(1, 2, 3)$ to this point is

$$S = (x - 1)^2 + (y - 2)^2 + (3 - x + y - 3)^2$$
$$= (x - 1)^2 + (y - 2)^2 + (y - x)^2.$$

$$S_x = 2(x - 1) - 2(y - x)$$
$$S_y = 2(y - 2) + 2(y - x)$$

From the equation $S_x = 0$ and $S_y = 0$ we obtain

$$4x - 2y = 2$$
$$-2x + 4y = 4.$$

Solving simultaneously, we have $x = 4/3, y = 5/3, z = 10/3$.

So, the distance is

$$\sqrt{\left(\frac{4}{3} - 1\right)^2 + \left(\frac{5}{3} - 2\right)^2 + \left(\frac{5}{3} - \frac{4}{3}\right)^2} = \frac{\sqrt{13}}{3}.$$

3. A point on the surface is given by $(x, y, z) = \left(x, y, \sqrt{1 - 2x - 2y}\right)$. The square of the distance from $(-2, -2, 0)$ to a point on the surface is given by

$$S = (x + 2)^2 + (y + 2)^2 + \left(\sqrt{1 - 2x - 2y} - 0\right)^2 = (x + 2)^2 + (y + 2)^2 + 1 - 2x - 2y.$$

$$S_x = 2(x + 2) - 2$$
$$S_y = 2(y + 2) - 2$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain $\left.\begin{aligned} 2x + 2 &= 0 \\ 2y + 2 &= 0 \end{aligned}\right\} \Rightarrow x = y = -1, z = \sqrt{5}.$

So, the distance is $\sqrt{(-1+2)^2 + (-1+2)^2 + \left(\sqrt{5}\right)^2} = \sqrt{7}$.

4. A point on the surface is given by $(x, y, z) = \left(x, y, \sqrt{1 - 2x - 2y}\right)$. The square of the distance from $(-4, 1, 0)$ to a point on the surface is given by

$$S = (x + 4)^2 + (y - 1)^2 + (1 - 2x - 2y).$$

$$S_x = 2(x + 4) - 2 = 2x + 6$$

$$S_y = 2(y - 1) - 2 = 2y - 4$$

From the equations $S_x = S_y = 0$, we obtain $x = -3, y = 2$. Hence, $z = \sqrt{3}$.

So the distance is

$$\sqrt{(-3 + 4)^2 + (2 - 1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{5}.$$

5. Let $x, y,$ and z be the numbers. Because $xyz = 27$,

$$z = \frac{27}{xy}.$$

$$S = x + y + z = x + y + \frac{27}{xy}.$$

$$S_x = 1 - \frac{27}{x^2 y} = 0, S_y = 1 - \frac{27}{xy^2} = 0.$$

$$\left.\begin{aligned} x^2 y &= 27 \\ xy^2 &= 27 \end{aligned}\right\} x = y = 3$$

So, $x = y = z = 3$.

6. Because $x + y + z = 32, z = 32 - x - y$. So,

$$P = xy^2 z = 32xy^2 - x^2y^2 - xy^3$$
$$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0$$
$$P_y = 64xy - 2x^2y - 3xy^2 = y\left(64x - 2x^2 - 3xy\right) = 0.$$

Ignoring the solution $y = 0$ and substituting $y = 32 - 2x$ into $P_y = 0$, we have

$$64x - 2x^2 - 3x(32 - 2x) = 0$$
$$4x(x - 8) = 0.$$

So, $x = 8, y = 16,$ and $z = 8$.

7. Let $x, y,$ and z be the numbers and let $S = x^2 + y^2 + z^2$. Because $x + y + z = 30$, we have

$$S = x^2 + y^2 + (30 - x - y)^2$$

$$\left.\begin{aligned} S_x &= 2x + 2(30 - x - y)(-1) = 0 \\ S_y &= 2y + 2(30 - x - y)(-1) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 30 \\ x + 2y &= 30. \end{aligned}$$

Solving simultaneously yields $x = 10$, $y = 10$, and $z = 10$.

8. Let $x, y,$ and z be the numbers. Because $xyz = 1, z = 1/xy$.

$$S = x^2 + y^2 + z^2 = x^2 + y^2 + \frac{1}{x^2y^2}$$

$$S_x = 2x - \frac{2}{x^3y^2} = 0, S_y = 2y - \frac{2}{x^2y^3} = 0$$

$$\left.\begin{aligned} x\left(x^3y^2\right) &= 1 \\ y\left(x^2y^3\right) &= 1 \end{aligned}\right\} x^4y^2 = x^2y^4 \Rightarrow x = y$$

So, $x = y = z = 1$.

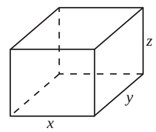
9. The volume is $668.25 = xyz \Rightarrow z = \dfrac{668.25}{xy}$.

$$C = 0.06(2yz + 2xz) + 0.11(xy) = 0.12\left(\frac{668.25}{x} + \frac{668.25}{y}\right) + 0.11(xy)$$

$$C = \frac{80.19}{x} + \frac{80.19}{y} + 0.11(xy)$$

$$C_x = \frac{-80.19}{x^2} + 0.11y = 0$$

$$C_y = \frac{-8.19}{y^2} + 0.11x = 0$$



Solving simultaneously, $x = y = 9$ and $z = 8.25$.

Minimum cost: $\dfrac{80.19}{9} + \dfrac{80.19}{9} + 0.11(xy) = \$26.73$

10. Let x, y, and z be the length, width, and height, respectively. Then $C_0 = 1.5xy + 2yz + 2xz$ and $z = \dfrac{C_0 - 1.5xy}{2(x + y)}$.

The volume is given by

$$V = xyz = \frac{C_0xy - 1.5x^2y^2}{2(x + y)}$$

$$V_x = \frac{y^2(2C_0 - 3x^2 - 6xy)}{4(x + y)^2}$$

$$V_y = \frac{x^2(2C_0 - 3y^2 - 6xy)}{4(x + y)^2}.$$

In solving the system $V_x = 0$ and $V_y = 0$, we note by the symmetry of the equations that $y = x$.

Substituting $y = x$ into $V_x = 0$ yields

$$\frac{x^2(2C_0 - 9x^2)}{16x^2} = 0,\ 2C_0 = 9x^2,\ x = \frac{1}{3}\sqrt{2C_0},\ y = \frac{1}{3}\sqrt{2C_0},\text{ and } z = \frac{1}{4}\sqrt{2C_0}.$$

11. Let x, y, and z be the length, width, and height, respectively and let V_0 be the given volume.

Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right)$$

$$\left.\begin{aligned} S_x &= 2\left(y - \frac{V_0}{x^2}\right) = 0 \\ S_y &= 2\left(x - \frac{V_0}{y^2}\right) = 0 \end{aligned}\right\} \begin{aligned} x^2y - V_0 &= 0 \\ xy^2 - V_0 &= 0. \end{aligned}$$

Solving simultaneously yields $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

12. Consider the sphere given by $x^2 + y^2 + z^2 = r^2$ and let a vertex of the rectangular box be $\left(x, y, \sqrt{r^2 - x^2 - y^2}\right)$.

Then the volume is given by

$$V = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$$

$$V_x = 8\left(xy\frac{-x}{\sqrt{r^2 - x^2 - y^2}} + y\sqrt{r^2 - x^2 - y^2}\right) = \frac{8y}{\sqrt{r^2 - x^2 - y^2}}\left(r^2 - 2x^2 - y^2\right) = 0$$

$$V_y = 8\left(xy\frac{-y}{\sqrt{r^2 - x^2 - y^2}} + x\sqrt{r^2 - x^2 - y^2}\right) = \frac{8x}{\sqrt{r^2 - x^2 - y^2}}\left(r^2 - x^2 - 2y^2\right) = 0.$$

Solving the system

$$2x^2 + y^2 = r^2$$

$$x^2 + 2y^2 = r^2$$

yields the solution $x = y = z = r/\sqrt{3}$.

13. $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$$R_{x_1} = -10x_1 - 2x_2 + 42 = 0, 5x_1 + x_2 = 21$$

$$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, x_1 + 8x_2 = 51$$

Solving this system yields $x_1 = 3$ and $x_2 = 6$.

$$R_{x_1x_1} = -10$$

$$R_{x_1x_2} = -2$$

$$R_{x_2x_2} = -16$$

$R_{x_1x_1} < 0$ and $R_{x_1x_1}R_{x_2x_2} - \left(R_{x_1x_2}\right)^2 > 0$

So, revenue is maximized when $x_1 = 3$ and $x_2 = 6$.

14. $P(x_1, x_2) = 15(x_1 + x_2) - C_1 - C_2$

$$= 15x_1 + 15x_2 - \left(0.02x_1^2 + 4x_1 + 500\right) - \left(0.05x_2^2 + 4x_2 + 275\right) = -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 - 775$$

$$P_{x_1} = -0.04x_1 + 11 = 0, x_1 = 275$$

$$P_{x_2} = -0.10x_2 + 11 = 0, x_2 = 110$$

$$P_{x_1x_1} = -0.04$$

$$P_{x_1x_2} = 0$$

$$P_{x_2x_2} = -0.10$$

$P_{x_1x_1} < 0$ and $P_{x_1x_1}P_{x_2x_2} - \left(P_{x_1x_2}\right)^2 > 0$

So, profit is maximized when $x_1 = 275$ and $x_2 = 110$.

15. $P(p, q, r) = 2pq + 2pr + 2qr.$

$p + q + r = 1$ implies that $r = 1 - p - q.$

$$P(p, q) = 2pq + 2p(1 - p - q) + 2q(1 - p - q)$$
$$= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2 = -2pq + 2p + 2q - 2p^2 - 2q^2$$

$$\frac{\partial P}{\partial p} = -2q + 2 - 4p; \frac{\partial P}{\partial q} = -2p + 2 - 4q$$

Solving $\frac{\partial P}{\partial p} = \frac{\partial P}{\partial q} = 0$ gives $\begin{aligned} q + 2p &= 1 \\ p + 2q &= 1 \end{aligned}$

and so $p = q = \frac{1}{3}$ and $P\left(\frac{1}{3}, \frac{1}{3}\right) = -2\left(\frac{1}{9}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{9}\right) - 2\left(\frac{1}{9}\right) = \frac{6}{9} = \frac{2}{3}.$

16. $H = -x \ln x - y \ln y - z \ln z,\ x + y + z = 1 = -x \ln x - y \ln y - (1 - x - y)\ln(1 - x - y)$

$$H_x = -1 - \ln x + 1 + \ln(1 - x - y) = 0$$
$$H_y = -1 - \ln y + 1 + \ln(1 - x - y) = 0$$

$\ln(1 - x - y) = \ln x = \ln y \Rightarrow x = y.$

So, $\ln(1 - 2x) = \ln x \Rightarrow 1 - 2x = x \Rightarrow x = y = z = \frac{1}{3}.$

$$H = -\tfrac{1}{3}\ln\left(\tfrac{1}{3}\right) - \tfrac{1}{3}\ln\left(\tfrac{1}{3}\right) - \tfrac{1}{3}\ln\left(\tfrac{1}{3}\right) = -\ln\left(\tfrac{1}{3}\right) = \ln 3$$

17. The distance from P to Q is $\sqrt{x^2 + 4}$. The distance from Q to R is $\sqrt{(y - x)^2 + 1}$. The distance from R to S is $10 - y$.

$$C = 3k\sqrt{x^2 + 4} + 2k\sqrt{(y - x)^2 + 1} + k(10 - y)$$

$$C_x = 3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(\frac{-(y - x)}{\sqrt{(y - x)^2 + 1}}\right) = 0$$

$$C_y = 2k\left(\frac{y - x}{\sqrt{(y - x)^2 + 1}}\right) - k = 0 \Rightarrow \frac{y - x}{\sqrt{(y - x)^2 + 1}} = \frac{1}{2}$$

$$3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{3}$$

$$3x = \sqrt{x^2 + 4}$$

$$9x^2 = x^2 + 4$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

$$2(y - x) = \sqrt{(y - x)^2 + 1}$$

$$4(y - x)^2 = (y - x)^2 + 1$$

$$(y - x)^2 = \frac{1}{3}$$

$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3} + 3\sqrt{2}}{6}$$

So, $x = \frac{\sqrt{2}}{2} \approx 0.707$ km and $y = \frac{2\sqrt{3} + 3\sqrt{2}}{6} \approx 1.284$ km.

18. $A = \frac{1}{2}\left[(30 - 2x) + (30 - 2x) + 2x\cos\theta\right]x\sin\theta = 30x\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta$

$$\frac{\partial A}{\partial x} = 30\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta = 0$$

$$\frac{\partial A}{\partial \theta} = 30x\cos\theta - 2x^2\cos\theta + x^2\left(2\cos^2\theta - 1\right) = 0$$

From $\frac{\partial A}{\partial x} = 0$ we have $15 - 2x + x\cos\theta = 0 \Rightarrow \cos\theta = \frac{2x - 15}{x}$.

From $\frac{\partial A}{\partial \theta} = 0$ we obtain $30x\left(\frac{2x - 15}{x}\right) - 2x^2\left(\frac{2x - 15}{x}\right) + x^2\left(2\left(\frac{2x - 15}{x}\right)^2 - 1\right) = 0$

$$30(2x - 15) - 2x(2x - 15) + 2(2x - 15)^2 - x^2 = 0$$

$$3x^2 - 30x = 0$$

$$x = 10.$$

Then $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$.

19. Write the equation to be maximized or minimized as a function of two variables. Set the partial derivatives equal to zero (or undefined) to obtain the critical points. Use the Second Partials Test to test for relative extrema using the critical points. Check the boundary points, too.

20. See pages 946 and 947.

21. (a)

x	y	xy	x^2
-2	0	0	4
0	1	0	0
2	3	6	4
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 8$

$$a = \frac{3(6) - 0(4)}{3(8) - 0^2} = \frac{3}{4}, b = \frac{1}{3}\left[4 - \frac{3}{4}(0)\right] = \frac{4}{3}, y = \frac{3}{4}x + \frac{4}{3}$$

(b) $S = \left(-\frac{3}{2} + \frac{4}{3} - 0\right)^2 + \left(\frac{4}{3} - 1\right)^2 + \left(\frac{3}{2} + \frac{4}{3} - 3\right)^2 = \frac{1}{6}$

22. (a)

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
1	1	1	1
3	2	6	9
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 20$

$$a = \frac{4(6) - 0(4)}{4(20) - (0)^2} = \frac{3}{10}, b = \frac{1}{4}\left[4 - \frac{3}{10}(0)\right] = 1, y = \frac{3}{10}x + 1$$

(b) $S = \left(\frac{1}{10} - 0\right)^2 + \left(\frac{7}{10} - 1\right)^2 + \left(\frac{13}{10} - 1\right)^2 + \left(\frac{19}{10} - 2\right)^2 = \frac{1}{5}$

23. (a)

x	y	xy	x^2
0	4	0	0
1	3	3	1
1	1	1	1
2	0	0	4
$\sum x_i = 4$	$\sum y_i = 8$	$\sum x_i y_i = 4$	$\sum x_i^2 = 6$

$$a = \frac{4(4) - 4(8)}{4(6) - 4^2} = -2, b = \frac{1}{4}\left[8 + 2(4)\right] = 4, y = -2x + 4$$

(b) $S = (4 - 4)^2 + (2 - 3)^2 + (2 - 1)^2 + (0 - 0)^2 = 2$

24. (a)

x	y	xy	x^2
3	0	0	9
1	0	0	1
2	0	0	4
3	1	3	9
4	1	4	16
4	2	8	16
5	2	10	25
6	2	12	36
$\sum x_i = 28$	$\sum y_i = 8$	$\sum x_i y_i = 37$	$\sum x_i^2 = 116$

$$a = \frac{8(37) - (28)(8)}{8(116) - (28)^2} = \frac{72}{144} = \frac{1}{2}, b = \frac{1}{8}\left[8 - \frac{1}{2}(28)\right] = -\frac{3}{4}, y = \frac{1}{2}x - \frac{3}{4}$$

(b) $S = \left(\frac{3}{4} - 0\right)^2 + \left(-\frac{1}{4} - 0\right)^2 + \left(\frac{1}{4} - 0\right)^2 + \left(\frac{3}{4} - 1\right)^2 + \left(\frac{5}{4} - 1\right)^2 + \left(\frac{5}{4} - 2\right)^2 + \left(\frac{7}{4} - 2\right)^2 + \left(\frac{9}{4} - 2\right)^2 = \frac{3}{2}$

25. $(0, 0), (1, 1), (3, 4), (4, 2), (5, 5)$

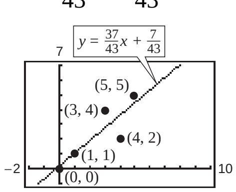
$\sum x_i = 13, \quad \sum y_i = 12,$

$\sum x_i y_i = 46, \quad \sum x_i^2 = 51$

$$a = \frac{5(46) - 13(12)}{5(51) - (13)^2} = \frac{74}{86} = \frac{37}{43}$$

$$b = \frac{1}{5}\left[12 - \frac{37}{43}(13)\right] = \frac{7}{43}$$

$$y = \frac{37}{43}x + \frac{7}{43}$$



26. $(1, 0), (3, 3), (5, 6)$

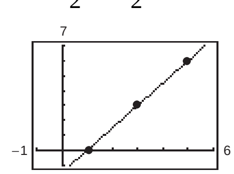
$\sum x_i = 9, \quad \sum y_i = 9,$

$\sum x_i y_i = 39, \quad \sum x_i^2 = 35$

$$a = \frac{3(39) - 9(9)}{3(35) - (9)^2} = \frac{36}{24} = \frac{3}{2}$$

$$b = \frac{1}{3}\left[9 - \frac{3}{2}(9)\right] = -\frac{9}{6} = -\frac{3}{2}$$

$$y = \frac{3}{2}x - \frac{3}{2}$$



27. $(0, 6), (4, 3), (5, 0), (8, -4), (10, -5)$

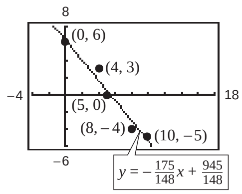
$\sum x_i = 27, \quad \sum y_i = 0,$

$\sum x_i y_i = -70, \quad \sum x_i^2 = 205$

$$a = \frac{5(-70) - (27)(0)}{5(205) - (27)^2} = \frac{-350}{296} = -\frac{175}{148}$$

$$b = \frac{1}{5}\left[0 - \left(-\frac{175}{148}\right)(27)\right] = \frac{945}{148}$$

$$y = -\frac{175}{148}x + \frac{945}{148}$$



28. $(6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8); n = 6$

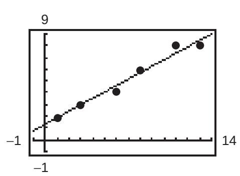
$\sum x_i = 42 \quad \sum y_i = 31$

$\sum x_i y_i = 275 \quad \sum x_i^2 = 400$

$$a = \frac{6(275) - (42)(31)}{6(400) - (42)^2} = \frac{29}{53} \approx 0.5472$$

$$b = \frac{1}{6}\left(31 - \frac{29}{53}42\right) = \frac{425}{318} \approx 1.3365$$

$$y = \frac{29}{53}x + \frac{425}{318}$$



29. (a) Using a graphing utility, $y = 1.6x + 84$.

(b) For each one-year increase in age, the pressure changes by approximately 1.6, the slope of the line.

30. (a) Using a graphing utility, $y = 0.2x - 3$.

(b) When $x = 1300$, $y \approx \$257$ billion.

Answers will vary.

31. $S(a, b, c) = \sum_{i=1}^{n}(y_i - ax_i^2 - bx_i - c)^2$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} -2x_i^2(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} -2x_i(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial c} = -2\sum_{i=1}^{n}(y_i - ax_i^2 - bx_i - c) = 0$$

$$a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$$

$$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$$

32. (a) Matches (iv) because the slope in (iv) is approximately 0.22.

(b) Matches (i) because the slope in (i) is approximately −0.35.

(c) Matches (iii) because the slope in (iii) is approximately 0.09.

(d) Matches (ii) because the slope in (ii) is approximately −1.29.

33. $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

$\sum x_i = 0$

$\sum y_i = 8$

$\sum x_i^2 = 10$

$\sum x_i^3 = 0$

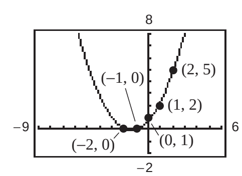
$\sum x_i^4 = 34$

$\sum x_i y_i = 12$

$\sum x_i^2 y_i = 22$

$34a + 10c = 22, 10b = 12, 10a + 5c = 8$

$a = \frac{3}{7}, b = \frac{6}{5}, c = \frac{26}{35}, y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$



34. $(-4, 5), (-2, 6), (2, 6)\,(4, 2)$

$\sum x_i = 0$

$\sum y_i = 19$

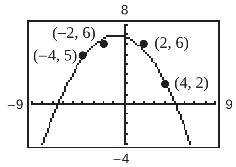
$\sum x_i^2 = 40$

$\sum x_i^3 = 0$

$\sum x_i^4 = 544$

$\sum x_i y_i = -12$

$\sum x_i^2 y_i = 160$



$544a + 40c = 160,\ 40b = -12,\ 40a + 4c = 19$

$a = -\frac{5}{24}, b = -\frac{3}{10}, c = \frac{41}{6}, y = -\frac{5}{24}x^2 - \frac{3}{10}x + \frac{41}{6}$

35. $(0, 0), (2, 2), (3, 6), (4, 12)$

$\sum x_i = 9$

$\sum y_i = 20$

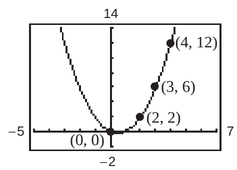
$\sum x_i^2 = 29$

$\sum x_i^3 = 99$

$\sum x_i^4 = 353$

$\sum x_i y_i = 70$

$\sum x_i^2 y_i = 254$



$$353a + 99b + 29c = 254$$
$$99a + 29b + 9c = 70$$
$$29a + 9b + 4c = 20$$

$a = 1, b = -1, c = 0, y = x^2 - x$

36. $(0, 10), (1, 9), (2, 6), (3, 0)$

$\sum x_i = 6$

$\sum y_i = 25$

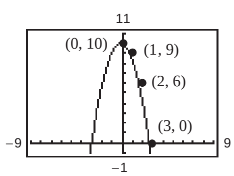
$\sum x_i^2 = 14$

$\sum x_i^3 = 36$

$\sum x_i^4 = 98$

$\sum x_i y_i = 21$

$\sum x_i^2 y_i = 33$



$$98a + 36b + 14c = 33$$
$$36a + 14b + 6c = 21$$
$$14a + 6b + 4c = 25$$

$a = -\frac{5}{4}, b = \frac{9}{20}, c = \frac{199}{20}, y = -\frac{5}{4}x^2 + \frac{9}{20}x + \frac{199}{20}$

37. (a) $(0, 0), (2, 15), (4, 30),$
$(6, 50), (8, 65), (10, 70)$

$\sum x_i = 30$

$\sum y_i = 230$

$\sum x_i^2 = 220$

$\sum x_i^3 = 1800$

$\sum x_i^4 = 15{,}664$

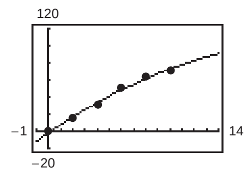
$\sum x_i y_i = 1670$

$\sum x_i^2 y_i = 13{,}500$

$$15{,}664a + 1800b + 220c = 13{,}500$$
$$1800a + 220b + 30c = 1670$$
$$220a + 30b + 6c = 230$$

$y = -\frac{25}{112}x^2 + \frac{541}{56}x - \frac{25}{14} \approx -0.22x^2 + 9.66x - 1.79$

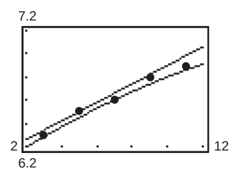
(b)



38. (a) Using a graphing utility, $y = 0.08x + 6.1$.

(b) Using a graphing utility,
$y = -0.002x^2 + 0.10x + 6.0$.

(c)



(d) For 2020, $x = 20$,

Linear model:
$y = 0.075(20) + 6.095 \approx 7.6$ billion

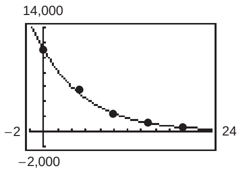
Quadratic model:
$y = -0.0018(20)^2 + 0.10(20) + 6.02 \approx 7.3$ billion

The quadratic model is less accurate because of the negative x^2 coefficient

39. (a) $\ln P = -0.1499h + 9.3018$

(b) $\ln P = -0.1499h + 9.3018$

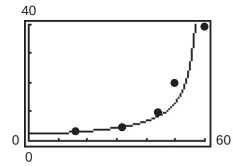
$P = e^{-0.1499h+9.3018} = 10{,}957.7e^{-0.1499h}$

(c) 

(d) Same answers

40. (a) $\dfrac{1}{y} = ax + b = -0.0074x + 0.445$

$y = \dfrac{1}{-0.0074x + 0.445}$

(b) 

(c) No. For $x = 70$, $y \approx -14$, which is nonsense.

41. $S(a, b) = \sum_{i=1}^{n}(ax_i + b - y_i)^2$

$S_a(a, b) = 2a\sum_{i=1}^{n}x_i^2 + 2b\sum_{i=1}^{n}x_i - 2\sum_{i=1}^{n}x_iy_i$

$S_b(a, b) = 2a\sum_{i=1}^{n}x_i + 2nb - 2\sum_{i=1}^{n}y_i$

$S_{aa}(a, b) = 2\sum_{i=1}^{n}x_i^2$

$S_{bb}(a, b) = 2n$

$S_{ab}(a, b) = 2\sum_{i=1}^{n}x_i$

$S_{aa}(a, b) > 0$ as long as $x_i \neq 0$ for all i. (**Note:** If $x_i = 0$ for all i, then $x = 0$ is the least squares regression line.)

$$d = S_{aa}S_{bb} - S_{ab}^2 = 4n\sum_{i=1}^{n}x_i^2 - 4\left(\sum_{i=1}^{n}x_i\right)^2 = 4\left[n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2\right] \geq 0 \text{ since } n\sum_{i=1}^{n}x_i^2 \geq \left(\sum_{i=1}^{n}x_i\right)^2.$$

As long as $d \neq 0$, the given values for a and b yield a minimum.

Section 13.10 Lagrange Multipliers

1. Maximize $f(x, y) = xy$

Constraint: $x + y = 10$

$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$$\left.\begin{aligned} y &= \lambda \\ x &= \lambda \\ x + y &= 10 \end{aligned}\right\} \quad x = y = 5$$

$f(5, 5) = 25$

2. Minimize $f(x, y) = 2x + y$

Constraint: $xy = 32$

$\nabla f = \lambda \nabla g$

$2\mathbf{i} + \mathbf{j} = \lambda y\mathbf{i} + \lambda x\mathbf{j}$

$2 = \lambda y \Rightarrow y = 2/\lambda$

$1 = \lambda x \Rightarrow x = 1/\lambda$

$xy = (1/\lambda)(2/\lambda) = 2/\lambda^2 = 32$

$\lambda^2 = 1/16$

$\lambda = 1/4, x = 4, y = 8$

$f(4, 8) = 16$

3. Minimize $f(x, y) = x^2 + y^2$.

Constraint: $x + 2y - 5 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = \lambda(\mathbf{i} + 2\mathbf{j})$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= 2\lambda \end{aligned}\right\} \begin{aligned} x &= \lambda/2 \\ y &= \lambda \end{aligned}$$

$x + 2y - 5 = 0$

$\dfrac{\lambda}{2} + 2\lambda = 5 \Rightarrow \lambda = 2, x = 1, y = 2$

$f(1, 2) = 5$

4. Maximize $f(x, y) = x^2 - y^2$.

Constraint: $2y - x^2 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = -2x\lambda\mathbf{i} + 2\lambda\mathbf{j}$

$2x = -2x\lambda \Rightarrow x = 0$ or $\lambda = -1$

If $x = 0$, then $y = 0$ and $f(0, 0) = 0$.

If $\lambda = -1$,

$-2y = 2\lambda = -2 \Rightarrow y = 1 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}$.

$f(\sqrt{2}, 1) = 2 - 1 = 1$, Maximum

5. Maximize $f(x, y) = 2x + 2xy + y$.

Constraint: $2x + y = 100$

$\nabla f = \lambda \nabla g$

$(2 + 2y)\mathbf{i} + (2x + 1)\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$

$$\left.\begin{aligned} 2 + 2y &= 2\lambda \Rightarrow y = \lambda - 1 \\ 2x + 1 &= \lambda \Rightarrow x = \frac{\lambda - 1}{2} \end{aligned}\right\} y = 2x$$

$2x + y = 100 \Rightarrow 4x = 100$

$x = 25, y = 50$

$f(25, 50) = 2600$

6. Minimize $f(x, y) = 3x + y + 10$.

Constraint: $x^2y = 6$

$\nabla f = \lambda \nabla g$

$3\mathbf{i} + \mathbf{j} = 2xy\lambda\mathbf{i} + x^2\lambda\mathbf{j}$

$$\left.\begin{aligned} 3 &= 2xy\lambda \Rightarrow \lambda = \frac{3}{2xy} \\ 1 &= x^2\lambda \Rightarrow \lambda = \frac{1}{x^2} \end{aligned}\right\} 3x^2 = 2xy \Rightarrow y = \frac{3x}{2} \quad (x \neq 0)$$

$x^2y = 6 \Rightarrow x^2\left(\dfrac{3x}{2}\right) = 6$

$x^3 = 4$

$x = \sqrt[3]{4}, y = \dfrac{3\sqrt[3]{4}}{2}$

$f\left(\sqrt[3]{4}, \dfrac{3\sqrt[3]{4}}{2}\right) = \dfrac{9\sqrt[3]{4} + 20}{2}$

7. Note: $f(x, y) = \sqrt{6 - x^2 - y^2}$ is maximum when $g(x, y)$ is maximum.

Maximize $g(x, y) = 6 - x^2 - y^2$.

Constraint: $x + y - 2 = 0$

$$\left.\begin{aligned} -2x &= \lambda \\ -2y &= \lambda \end{aligned}\right\} x = y$$

$x + y = 2 \Rightarrow x = y = 1$

$f(1, 1) = \sqrt{g(1, 1)} = 2$

8. Note: $f(x, y) = \sqrt{x^2 + y^2}$ is minimum when $g(x, y)$ is minimum.

Minimize $g(x, y) = x^2 + y^2$.

Constraint: $2x + 4y - 15 = 0$

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 4\lambda \end{aligned}\right\} y = 2x$$

$2x + 4y = 15 \Rightarrow 10x = 15$

$x = \dfrac{3}{2}, y = 3$

$f\left(\dfrac{3}{2}, 3\right) = \sqrt{g\left(\dfrac{3}{2}, 3\right)} = \dfrac{3\sqrt{5}}{2}$

9. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z - 9 = 0$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = 9 \Rightarrow x = y = z = 3$$

$$f(3, 3, 3) = 27$$

10. Maximize $f(x, y, z) = xyz$.

Constraint: $x + y + z - 3 = 0$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} yz = xz = xy \Rightarrow x = y = z$$

$$x + y + z = 3 \Rightarrow x = y = z = 1$$

$$f(1, 1, 1) = 1$$

11. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 1$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = 1 \Rightarrow x = y = z = \tfrac{1}{3}$$

$$f\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = \tfrac{1}{3}$$

12. Maximize $f(x, y, z) = x + y + z$

Constraint: $x^2 + y^2 + z^2 = 1$

$$\left.\begin{aligned} 1 &= \lambda 2x \\ 1 &= \lambda 2y \\ 1 &= \lambda 2z \end{aligned}\right\} \quad x = y = z = \frac{1}{2\lambda}$$

$$x^2 + y^2 + z^2 = \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = \frac{3}{4\lambda^2} = 1$$

$$\lambda^2 = 3/4 \Rightarrow \lambda = \sqrt{3}/2 \Rightarrow x = y = z = \frac{1}{\sqrt{3}}$$

$$f(x, y, z) = 3/\sqrt{3} = \sqrt{3}$$

13. Maximize or minimize $f(x, y) = x^2 + 3xy + y^2$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} 2x + 3y &= 2x\lambda \\ 3x + 2y &= 2y\lambda \end{aligned}\right\} x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}, y = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{5}{2}$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= 2x + 3y = 0 \\ f_y &= 3x + 2y = 0 \end{aligned}\right\} x = y = 0$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, f_{xx}f_{yy} - (f_{xy})^2 \le 0$$

Saddle point: $f(0, 0) = 0$

By combining these two cases, we have a maximum of $\frac{5}{2}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ and a minimum of $-\frac{1}{2}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$.

14. Maximize or minimize $f(x, y) = e^{-xy/4}$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned}-(y/4)e^{-xy/4} &= 2x\lambda\\ -(x/4)e^{-xy/4} &= 2y\lambda\end{aligned}\right\} \Rightarrow x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = e^{1/8} \approx 1.1331$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = e^{-1/8} \approx 0.8825$

Case 2: Inside the circle

$$\left.\begin{aligned}f_x &= -(y/4)e^{-xy/4} = 0\\ f_y &= -(x/4)e^{-xy/4} = 0\end{aligned}\right\} \Rightarrow x = y = 0$$

$$f_{xx} = \frac{y^2}{16}e^{-xy/4},\ f_{yy} = \frac{x^2}{16}e^{-xy/4},\ f_{xy} = e^{-xy}\left(\frac{1}{16}xy - \frac{1}{4}\right)$$

At $(0, 0)$, $f_{xx}f_{yy} - \left(f_{xy}\right)^2 < 0$.

Saddle point: $f(0, 0) = 1$

Combining the two cases, we have a maximum of $e^{1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$ and a minimum of $e^{-1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

15. Maximize $f(x, y, z) = xyz$.

Constraints: $x + y + z = 32$

$x - y + z = 0$

$$\nabla f = \lambda\nabla g + \mu\nabla h$$

$$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\left.\begin{aligned}yz &= \lambda + \mu\\ xz &= \lambda - \mu\\ xy &= \lambda + \mu\end{aligned}\right\} yz = xy \Rightarrow x = z$$

$$\left.\begin{aligned}x + y + z &= 32\\ x - y + z &= 0\end{aligned}\right\} 2x + 2z = 32 \Rightarrow x = z = 8$$

$$y = 16$$

$$f(8, 16, 8) = 1024$$

16. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraints: $x + 2z = 6$

$x + y = 12$

$$\nabla f = \lambda\nabla g + \mu\nabla h$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j})$$

$$\left.\begin{aligned}2x &= \lambda + \mu\\ 2y &= \mu\\ 2z &= 2\lambda\end{aligned}\right\} 2x = 2y + z$$

$$x + 2z = 6 \Rightarrow z = \frac{6 - x}{2} = 3 - \frac{x}{2}$$

$$x + y = 12 \Rightarrow y = 12 - x$$

$$2x = 2(12 - x) + \left(3 - \frac{x}{2}\right) \Rightarrow \frac{9}{2}x = 27 \Rightarrow x = 6$$

$$x = 6, z = 0$$

$$f(6, 6, 0) = 72$$

17. Minimize the square of the distance $f(x, y) = (x - 0)^2 + (y - 0)^2 = x^2 + y^2$ subject to the constraint $x + y = 1$.

$$\left.\begin{aligned}2x &= \lambda\\ 2y &= \lambda\end{aligned}\right\} \quad \left.\begin{aligned}x &= \lambda/2\\ y &= \lambda/2\end{aligned}\right\} \Rightarrow x = y$$

$$x + y = 1$$

$$x = y = \frac{1}{2}$$

The minimum distance is $d = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$.

18. Minimize the square of the distance $f(x, y) = x^2 + y^2$ subject to the constraint $2x + 3y = -1$.

$$\left.\begin{aligned}2x &= 2\lambda\\ 2y &= 3\lambda\end{aligned}\right\} y = \frac{3x}{2}$$

$$2x + 3y = -1 \Rightarrow x = -\frac{2}{13}, y = -\frac{3}{13}$$

The minimum distance is $d = \sqrt{\left(-\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{\sqrt{13}}{13}$.

19. Minimize the square of the distance

$f(x, y) = x^2 + (y - 2)^2$

subject to the constraint $x - y = 4$.

$$\left.\begin{aligned} 2x &= \lambda \\ 2(y - 2) &= -\lambda \end{aligned}\right\} \begin{aligned} x &= \lambda/2 \\ y &= \frac{4 - \lambda}{2} \end{aligned}$$

$x - y = 4$

$\dfrac{\lambda}{2} - \left(\dfrac{4 - \lambda}{2}\right) = 4$

$\lambda = 6$

$x = 3, y = -1$

The minimum distance is $d = \sqrt{3^2 + (-1 - 2)^2} = 3\sqrt{2}$.

20. Minimize the square of the distance

$f(x, y) = (x - 1)^2 + y^2$ subject to the constraint $x + 4y = 3$.

$$\left.\begin{aligned} 2(x - 1) &= \lambda \\ 2y &= 4\lambda \end{aligned}\right\} \begin{aligned} x &= \frac{\lambda + 2}{2} \\ y &= 2\lambda \end{aligned}$$

$x + 4y = 3$

$\dfrac{\lambda + 2}{2} + 4(2\lambda) = 3$

$\lambda + 2 + 16\lambda = 6$

$17\lambda = 4$

$\lambda = \dfrac{4}{17}$

$x = \dfrac{19}{17}, y = \dfrac{8}{17}$

The minimum distance is $d = \sqrt{\left(\dfrac{19}{17}\right)^2 + \left(\dfrac{8}{17}\right)^2} = \dfrac{5\sqrt{17}}{17}$.

21. Minimize the square of the distance

$f(x, y) = x^2 + (y - 3)^2$ subject to the constraint $y - x^2 = 0$.

$2x = -2x\lambda$

$2(y - 3) = \lambda$

$y = x^2$

If $x = 0, y = 0,$ and $f(0, 0) = 9 \Rightarrow$ distance $= 3$.

If $x \neq 0, \lambda = -1, y = 5/2, x = \pm\sqrt{5/2}$

$f\left(\pm\sqrt{5/2}, 5/2\right) = 5/2 + \left(\dfrac{1}{2}\right)^2 = \dfrac{11}{4} < 3$

The minimum distance is $d = \dfrac{\sqrt{11}}{2}$.

22. Minimize the square of the distance

$f(x, y) = (x + 3)^2 + y^2$ subject to the constraint $y - x^2 = 0$.

$2(x + 3) = -2\lambda x$

$2y = \lambda$

$y = x^2$

$\lambda = 2y = 2x^2$

$2(x + 3) = -2(2x^3)$

$4x^3 + 2x + 6 = 0$

$2(x + 1)(2x^2 - 2x + 3) = 0 \Rightarrow x = -1, y = 1,$

The minimum distance is $d = \sqrt{(-1)^2 + (1^2)} = \sqrt{2}$.

23. Minimize the square of the distance $f(x, y) = (x - 4)^2 + (y - 4)^2$ subject to the constraint $x^2 + (y - 1)^2 = 9$.

$2(x - 4) = 2x\lambda$

$2(y - 4) = 2(y - 1)\lambda$

$x^2 + (y - 1)^2 = 9$

Solving these equations, you obtain

$x = 12/5, y = 14/5$ and $\lambda = -2/3$.

The minimum distance is $d = \sqrt{\left(\dfrac{12}{5} - 4\right)^2 + \left(\dfrac{14}{5} - 4\right)^2} = \sqrt{\dfrac{64}{25} + \dfrac{36}{25}} = 2.$

24. Minimize the square of the distance $f(x, y) = x^2 + (y - 10)^2$ subject to the constraint $(x - 4)^2 + y^2 = 4$.

$$\left.\begin{aligned} 2x &= 2(x-4)\lambda \\ 2(y-10) &= 2y\lambda \end{aligned}\right\} \frac{x}{x-4} = \frac{y-10}{y} \Rightarrow y = -\frac{5}{2}x + 10$$

$$(x-4)^2 + y^2 = 4 \Rightarrow (x^2 - 8x + 16) + \left(\frac{25}{4}x^2 - 50x + 100\right) = 4$$

$$\frac{29}{4}x^2 - 58x + 112 = 0$$

Using a graphing utility, we obtain $x \approx 3.2572$ and $x \approx 4.7428$ or by the Quadratic Formula,

$$x = \frac{58 \pm \sqrt{58^2 - 4(29/4)(112)}}{2(29/4)} = \frac{58 \pm 2\sqrt{29}}{29/2} = 4 \pm \frac{4\sqrt{29}}{29}.$$

Using the smaller value, we have $x = 4\left(1 - \frac{\sqrt{29}}{29}\right)$ and $y = \frac{10\sqrt{29}}{29} \approx 1.8570$.

The minimum distance is $d = \sqrt{16\left(1 - \frac{\sqrt{29}}{29}\right)^2 + \left(\frac{10\sqrt{29}}{29} - 10\right)^2} \approx 8.77$.

The larger x-value does not yield a minimum.

25. Minimize the square of the distance

$$f(x, y, z) = (x-2)^2 + (y-1)^2 + (z-1)^2$$

subject to the constraint $x + y + z = 1$.

$$\left.\begin{aligned} 2(x-2) &= \lambda \\ 2(y-1) &= \lambda \\ 2(z-1) &= \lambda \end{aligned}\right\} y = z \text{ and } y = x - 1$$

$$x + y + z = 1 \Rightarrow x + 2(x-1) = 1$$

$$x = 1, y = z = 0$$

The minimum distance is

$$d = \sqrt{(1-2)^2 + (0-1)^2 + (0-1)^2} = \sqrt{3}.$$

26. Minimize the square of the distance

$$f(x, y, z) = (x-4)^2 + y^2 + z^2$$

subject to the constraint $\sqrt{x^2 + y^2} - z = 0$.

$$\left.\begin{aligned} 2(x-4) &= \frac{x}{\sqrt{x^2+y^2}}\lambda = \frac{x}{z}\lambda \\ 2y &= \frac{y}{\sqrt{x^2+y^2}}\lambda = \frac{y}{z}\lambda \\ 2z &= -\lambda \end{aligned}\right\} \begin{aligned} 2(x-4) &= -2x \\ 2y &= -2y \end{aligned}$$

$$\sqrt{x^2+y^2} - z = 0, x = 2, y = 0, z = 2$$

The minimum distance is

$$d = \sqrt{(2-4)^2 + 0^2 + 2^2} = 2\sqrt{2}.$$

27. Maximize $f(x, y, z) = z$ subject to the constraints

$x^2 + y^2 - z^2 = 0$ and $x + 2z = 4$.

$$\begin{aligned} 0 &= 2x\lambda + \mu \\ 0 &= 2y\lambda \Rightarrow y = 0 \\ 1 &= -2z\lambda + 2\mu \end{aligned}$$

$$x^2 + y^2 - z^2 = 0$$

$$x + 2z = 4 \Rightarrow x = 4 - 2z$$

$$\begin{aligned} (4-2z)^2 + 0^2 - z^2 &= 0 \\ 3z^2 - 16z + 16 &= 0 \\ (3z-4)(z-4) &= 0 \\ z = \tfrac{4}{3} \text{ or } z &= 4 \end{aligned}$$

The maximum value of f occurs when $z = 4$ at the point of $(-4, 0, 4)$.

28. Maximize $f(x, y, z) = z$ subject to the constraints

$x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

$$\left.\begin{aligned} 0 &= 2x\lambda + 2\mu \\ 0 &= 2y\lambda + \mu \\ 1 &= 2z\lambda - \mu \end{aligned}\right\} x = 2y$$

$$x^2 + y^2 + z^2 = 36$$

$$2x + y - z = 2 \Rightarrow z = 2x + y - 2 = 5y - 2$$

$$(2y)^2 + y^2 + (5y - 2)^2 = 36$$

$$30y^2 - 20y - 32 = 0$$

$$15y^2 - 10y - 16 = 0$$

$$y = \frac{5 \pm \sqrt{265}}{15}$$

Choosing the positive value for y we have the point

$$\left(\frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3}\right).$$

29. Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solution are called contrained optimization problems.

30. See explanation at the bottom of page 953.

31. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $g(x, y, z) = x - y + z = 3$

$$2x = \lambda \Rightarrow x = \lambda/2$$

$$2y = -\lambda \Rightarrow y = -\lambda/2$$

$$2z = \lambda \Rightarrow z = \lambda/2$$

$$x - y + z = 3$$

$$\frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) + \frac{\lambda}{2} = 3$$

$$\frac{3\lambda}{2} = 3$$

$$\lambda = 2$$

$$x = 1, y = -1, z = 1$$

Minimum distance $= \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

32. Minimize $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$.

Constraint: $g(x, y, z) = x - y + z = 3$

$$2(x - 1) = \lambda \Rightarrow x = \frac{2 + \lambda}{2}$$

$$2(y - 2) = -\lambda \Rightarrow y = \frac{4 - \lambda}{2}$$

$$2(z - 3) = \lambda \Rightarrow z = \frac{6 + \lambda}{2}$$

$$x - y + z = 3$$

$$\frac{2 + \lambda}{2} - \frac{4 - \lambda}{2} + \frac{6 + \lambda}{2} = 3$$

$$3\lambda + 4 = 6$$

$$\lambda = \frac{2}{3}$$

$$x = \frac{4}{3}, y = \frac{5}{3}, z = \frac{10}{3}$$

Minimum distance $= \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{3}$

33. Minimize $f(x, y, z) = x + y + z$.

Constraint: $g(x, y, z) = xyz = 27$

$$\left.\begin{aligned} 1 = \lambda yz &\Rightarrow x = \lambda xyz \\ 1 = \lambda xz &\Rightarrow y = \lambda xyz \\ 1 = \lambda xy &\Rightarrow z = \lambda xyz \end{aligned}\right\} \Rightarrow x = y = z$$

$$xyz = 27$$

$$x^3 = 27 \Rightarrow x = y = z = 3$$

34. Maximize $P(x, y, z) = xy^2z$.

Constraint: $g(x, y, z) = x + y + z = 32$

$$y^2z = \lambda$$

$$2xyz = \lambda$$

$$xy^2 = \lambda$$

$$x + y + z = 32$$

$$xy^2 = y^2z \Rightarrow x = z \qquad (y \neq 0)$$

$$2xyz = xy^2 \Rightarrow 2x^2y = xy^2 \Rightarrow 2x = y$$

$$x + 2x + x = 32$$

$$x = 8$$

$$y = 16$$

$$z = 8$$

35. Minimize $f(x, y, z) = 0.06(2yz + 2xz) + 0.11(xy)$.

Constraint: $g(x, y, z) = xyz = 668.25$

$$\begin{aligned} 0.12z + 0.11y &= yz\lambda \\ 0.12z + 0.11x &= xz\lambda \\ 0.12(y + x) &= xy\lambda \\ xyz &= 668.25 \end{aligned}$$

$0.12xz + 0.11yx = xyz\lambda = 0.12yz + 0.11xy \Rightarrow x = y$

$0.12(2x) = x^2\lambda \Rightarrow \lambda = \dfrac{0.24}{x}$

$0.12z + 0.11x = xz\left(\dfrac{0.24}{x}\right) = 0.24z \Rightarrow z = \dfrac{0.11x}{0.12} = \dfrac{11x}{12}$

$xyz = x^2\left(\dfrac{11}{12}x\right) = 668.25 \Rightarrow x = y = 9, z = \dfrac{33}{4}$

$f\left(9, 9, \dfrac{33}{4}\right) = \$26.73$

36. Maximize $f(x, y, z) = xyz$ (volume).

Constraint: $g(x, y, z) = 1.5xy + 2xz + 2yz = C$

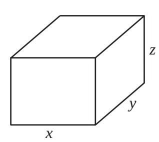
$$\begin{aligned} yz &= 1.5y\lambda + 2z\lambda \\ xz &= 1.5x\lambda + 2z\lambda \\ xy &= 2x\lambda + 2y\lambda \\ 1.5xy + 2xz + 2yz &= C \end{aligned}$$

$$\begin{aligned} xyz = x[1.5y\lambda + 2z\lambda] &= y[1.5x\lambda + 2z\lambda] \\ 2xz\lambda &= 2yz\lambda \\ x &= y \qquad \text{(also by symmetry)} \end{aligned}$$

$x^2 = 2x\lambda + 2x\lambda \Rightarrow \lambda = x/4.$

$xz = 1.5x\left(\dfrac{x}{4}\right) + 2z\left(\dfrac{x}{4}\right) \Rightarrow z = \dfrac{3}{4}x$

$1.5x^2 + 2x\left(\dfrac{3}{4}x\right) + 2x\left(\dfrac{3}{4}x\right) = C \Rightarrow x^2 = \dfrac{2}{9}C \Rightarrow x = \dfrac{\sqrt{2C}}{3},$



$y = \dfrac{\sqrt{2C}}{3}, z = \dfrac{\sqrt{2C}}{4}$

37. Maximize $P(p, q, r) = 2pq + 2pr + 2qr$.

Constraint: $g(p, q, r) = p + q + r = 1$

$$\left.\begin{aligned} 2q + 2r &= \lambda \\ 2p + 2r &= \lambda \\ 2p + 2q &= \lambda \end{aligned}\right\} p = q = r$$

$p + q + r = 3p = 1 \Rightarrow p = \frac{1}{3}$ and

$P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3\left(\frac{2}{9}\right) = \frac{2}{3}.$

38. Maximize $H(x, y, z) = -x \ln x - y \ln y - y \ln z$.

Constraint: $g(x, y, z) = x + y + z = 1$

(a) $$\left.\begin{aligned} -\ln x - 1 &= \lambda \\ -\ln y - 1 &= \lambda \\ -\ln z - 1 &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 3x = 1 \Rightarrow x = y = z = \frac{1}{3}$

(b) $H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3\left[-\frac{1}{3}\ln\left(\frac{1}{3}\right)\right] = \ln 3$

39. Maximize $V(x, y, z) = (2x)(2y)(2z) = 8xyz$ subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\left.\begin{aligned} 8yz &= \frac{2x}{a^2}\lambda \\ 8xz &= \frac{2y}{b^2}\lambda \\ 8xy &= \frac{2z}{c^2}\lambda \end{aligned}\right\} \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1, \frac{3y^2}{b^2} = 1, \frac{3z^2}{c^2} = 1$$

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

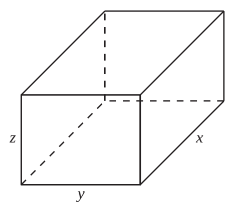
So, the dimensions of the box are $\frac{2\sqrt{3}a}{3} \times \frac{2\sqrt{3}b}{3} \times \frac{2\sqrt{3}c}{3}$.

40. (a) $f(1, 2) = 2$

(b) $f(2, 2) = 8$

41. Minimize $C(x, y, z) = 5xy + 3(2xz + 2yz + xy)$ subject to the constraint $xyz = 480$.

$$\left.\begin{aligned} 8y + 6z &= yz\lambda \\ 8x + 6z &= xz\lambda \\ 6x + 6y &= xy\lambda \end{aligned}\right\} x = y, 4y = 3z$$



$$xyz = 480 \Rightarrow \tfrac{4}{3}y^3 = 480$$

$$x = y = \sqrt[3]{360}, z = \tfrac{4}{3}\sqrt[3]{360}$$

Dimensions: $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ feet.

42. (a) Maximize $P(x, y, z) = xyz$ subject to the constraint $x + y + z = S$.

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = S \Rightarrow x = y = z = \frac{S}{3}$$

So, $xyz \le \left(\frac{S}{3}\right)\left(\frac{S}{3}\right)\left(\frac{S}{3}\right), x, y, z > 0$

$$xyz \le \frac{S^3}{27}$$

$$\sqrt[3]{xyz} \le \frac{S}{3}$$

$$\sqrt[3]{xyz} \le \frac{x + y + z}{3}.$$

(b) Maximize $P = x_1x_2x_3\cdots x_n$ subject to the constraint $\sum_{i=1}^{n} x_i = S$.

$$\left.\begin{aligned} x_2x_3\cdots x_n &= \lambda \\ x_1x_3\cdots x_n &= \lambda \\ x_1x_2\cdots x_n &= \lambda \\ &\vdots \\ x_1x_2x_3\cdots x_{n-1} &= \lambda \end{aligned}\right\} x_1 = x_2 = x_3 = \cdots = x_n$$

$$\sum_{i=1}^{n} x_i = S \Rightarrow x_1 = x_2 = x_3 = \cdots = x_n = \frac{S}{n}$$

So,

$$x_1x_2x_3\cdots x_n \le \left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\cdots\left(\frac{S}{n}\right), x_i \ge 0$$

$$x_1x_2x_3\cdots x_n \le \left(\frac{S}{n}\right)^n$$

$$\sqrt[n]{x_1x_2x_3\cdots x_n} \le \frac{S}{n}$$

$$\sqrt[n]{x_1x_2x_3\cdots x_n} \le \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

43. Minimize $A(\pi, r) = 2\pi rh + 2\pi r^2$ subject to the constraint $\pi r^2 h = V_0$.

$$\left.\begin{aligned} 2\pi h + 4\pi r &= 2\pi rh\lambda \\ 2\pi r &= \pi r^2\lambda \end{aligned}\right\} h = 2r$$

$$\pi r^2 h = V_0 \Rightarrow 2\pi r^3 = V_0$$

Dimensions: $r = \sqrt[3]{\dfrac{V_0}{2\pi}}$ and $h = 2\sqrt[3]{\dfrac{V_0}{2\pi}}$

44. Maximize $T(x, y, z) = 100 + x^2 + y^2$ subject to the constraints $x^2 + y^2 + z^2 = 50$ and $x - z = 0$.

$$\left.\begin{aligned} 2x &= 2x\lambda + \mu \\ 2y &= 2y\lambda \\ 0 &= 2z\lambda - \mu \end{aligned}\right\}$$

If $y \neq 0$, then $\lambda = 1$ and $\mu = 0, z = 0$.

So, $x = z = 0$ and $y = \sqrt{50}$.

$$T\left(0, \sqrt{50}, 0\right) = 100 + 50 = 150$$

If $y = 0$ then $x^2 + z^2 = 2x^2 = 50$ and $x = z = \sqrt{50}/2$.

$$T\left(\frac{\sqrt{50}}{2}, 0, \frac{\sqrt{50}}{2}\right) = 100 + \frac{50}{4} = 112.5$$

So, the maximum temperature is 150.

45. Using the formula $\text{Time} = \dfrac{\text{Distance}}{\text{Rate}}$, minimize $T(x, y) = \dfrac{\sqrt{d_1^2 + x^2}}{v_1} + \dfrac{\sqrt{d_2^2 + y^2}}{v_2}$ subject to the constraint $x + y = a$.

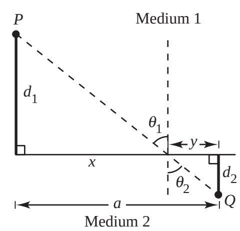
$$\left.\begin{aligned} \frac{x}{v_1\sqrt{d_1^2 + x^2}} &= \lambda \\ \frac{y}{v_2\sqrt{d_2^2 + y^2}} &= \lambda \end{aligned}\right\} \frac{x}{v_1\sqrt{d_1^2 + x^2}} = \frac{y}{v_2\sqrt{d_2^2 + y^2}}$$

$$x + y = a$$

Because $\sin\theta_1 = \dfrac{x}{\sqrt{d_1^2 + x^2}}$

and $\sin\theta_2 = \dfrac{y}{\sqrt{d_2^2 + y^2}}$,

we have $\dfrac{x/\sqrt{d_1^2 + x^2}}{v_1} = \dfrac{y/\sqrt{d_2^2 + y^2}}{v_2}$ or $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.



46. Case 1: Minimize $P(l, h) = 2h + l + \left(\dfrac{\pi l}{2}\right)$ subject to the constraint $lh + \left(\dfrac{\pi l^2}{8}\right) = A$.

$$1 + \frac{\pi}{2} = \left(h + \frac{\pi l}{4}\right)\lambda$$

$$2 = l\lambda \Rightarrow \lambda = \frac{2}{l}, 1 + \frac{\pi}{2} = \frac{2h}{l} + \frac{\pi}{2}$$

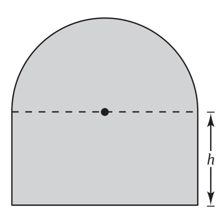
$$l = 2h$$

Case 2: Minimize $A(l, h) = lh + \left(\dfrac{\pi l^2}{8}\right)$ subject to the constraint $2h + l + \left(\dfrac{\pi l}{2}\right) = P$.

$$h + \frac{\pi l}{4} = \left(\perp + \frac{\pi}{2}\right)\lambda$$

$$l = 2\lambda \Rightarrow \lambda = \frac{l}{2}, h + \frac{\pi l}{4} = \frac{l}{2} + \frac{\pi l}{4}$$

$$h = \frac{l}{2} \text{ or } l = 2h$$



47. Maximize $P(x, y) = 100x^{0.25}y^{0.75}$ subject to the constraint $72x + 60y = 250{,}000$.

$$25x^{-0.75}y^{0.75} = 72\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{72\lambda}{25}$$

$$75x^{0.25}y^{-0.25} = 60\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{60\lambda}{75}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{72\lambda}{25}\right)\left(\frac{75}{60\lambda}\right)$$

$$\frac{y}{x} = \frac{18}{5}$$

$$y = \frac{18}{5}x$$

$$72x + 60\left(\frac{18}{5}x\right) = 288x = 250{,}000 \Rightarrow x = \frac{15{,}625}{18}$$

$$y = 3125$$

$$P\left(\frac{15625}{18}, 3125\right) \approx 226{,}869$$

48. Maximize $P(x, y) = 100x^{0.4}y^{0.6}$ subject to the constraint $72x + 60y = 250{,}000$.

$$40x^{-0.6}y^{0.6} = 72\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.6} = \frac{72\lambda}{40}$$

$$60x^{0.4}y^{-0.4} = 60\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.4} = \frac{60\lambda}{60} = \lambda$$

$$\left(\frac{y}{x}\right)^{0.6}\left(\frac{y}{x}\right)^{0.4} = \frac{72\lambda}{40}\cdot\frac{1}{\lambda}$$

$$\frac{y}{x} = \frac{9}{5} \Rightarrow y = \frac{9}{5}x$$

$$72x + 60\left(\frac{9}{5}x\right) = 180x = 250{,}000 \Rightarrow x = \frac{125{,}000}{9}$$

$$y = 2500$$

$$P\left(\frac{125{,}000}{9}, 2500\right) \approx 496{,}399$$

49. Minimize $C(x, y) = 72x + 60y$ subject to the constraint $100x^{0.25}y^{0.75} = 50{,}000$.

$$72 = 25x^{-0.75}y^{0.75}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{72}{25\lambda}$$

$$60 = 75x^{0.25}y^{-0.25}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{60}{75\lambda}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \frac{72}{25\lambda}\cdot\frac{75\lambda}{60}$$

$$\frac{y}{x} = \frac{18}{5} \Rightarrow y = \frac{18}{5}x = 3.6x$$

$$100x^{0.25}(3.6x)^{0.75} = 50{,}000$$

$$x = \frac{500}{3.6^{0.75}} \approx 191.3124$$

$$y = 3.6x \approx 688.7247$$

$$C(191.3124, 688.7247) \approx 55{,}097.97$$

50. Minimize $C(x, y) = 72x + 60y$ subject to the constraint $100x^{0.6}y^{0.4} = 50{,}000$.

$$72 = 60x^{-0.4}y^{0.4}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.4} = \frac{72}{60\lambda}$$

$$60 = 40x^{0.6}y^{-0.6}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.6} = \frac{60}{40\lambda} = \frac{3}{2\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4}\left(\frac{y}{x}\right)^{0.6} = \frac{72}{60\lambda}\cdot\frac{2\lambda}{3}$$

$$\frac{y}{x} = \frac{4}{5} \Rightarrow y = \frac{4}{5}x$$

$$100x^{0.6}\left(\frac{4}{5}x\right)^{0.4} = 50{,}000$$

$$x = \frac{500}{(4/5)^{0.4}}$$

$$y = \frac{400}{(4/5)^{0.4}}$$

$$C\left(\frac{500}{(4/5)^{0.4}}, \frac{400}{(4/5)^{0.4}}\right) \approx \$65{,}601.72$$

51. Let r = radius of cylinder, and h = height of cylinder = height of cone.

$S = 2\pi rh + 2\pi r\sqrt{h^2 + r^2}$ = constant surface area

$V = \pi r^2 h + \dfrac{2\pi r^2 h}{3} = \dfrac{5\pi r^2 h}{3}$ volume

We maximize $f(r, h) = r^2 h$ subject to $g(r, h) = rh + r\sqrt{h^2 + r^2} = C$.

$$(C - rh)^2 = r^2(h^2 + r^2)$$

$$C^2 - 2Crh = r^4$$

$$h = \frac{C^2 - r^4}{2Cr}$$

$$f(r, h) = F(r) = r^2\left[\frac{C^2 - r^4}{2Cr}\right] = \frac{Cr}{2} - \frac{r^5}{2C}$$

$$F'(r) = \frac{C}{2} - \frac{5r^4}{2C} = 0$$

$$C^2 = 5r^4$$

$$r^2 = \frac{C}{\sqrt{5}}$$

$$F''(r) = \frac{-10r^3}{C}$$

$$h = \frac{C^2 - r^4}{2Cr} = \frac{C^2 - C^2/5}{2C(C^2/5)^{1/4}}$$

$$= \frac{(4/5)C}{2(C^2/5)^{1/4}}$$

$$= \frac{2}{5}\frac{C}{r}$$

$$= \frac{2}{5r}\left(\sqrt{5}r^2\right)$$

$$= \frac{2\sqrt{5}}{5}r$$

So, $\dfrac{h}{r} = \dfrac{2\sqrt{5}}{5}$.

By the Second Derivative Test, this is a maximum.

Review Exercises for Chapter 13

1. $f(x, y) = 3x^2 y$

(a) $f(1, 3) = 3(1)^2(3) = 9$

(b) $f(-1, 1) = 3(-1)^2(1) = 3$

(c) $f(-4, 0) = 3(-4)^2(0) = 0$

(d) $f(x, z) = 3x^2(2) = 6x^2$

2. $f(x, y) = 6 - 4x - 2y^2$

(a) $f(0, 2) = 6 - 4(0) - 2(2)^2 = -2$

(b) $f(5, 0) = 6 - 4(5) - 2(0)^2 = -14$

(c) $f(-1, -2) = 6 - 4(-1) - 2(-2)^2 = 2$

(d) $f(-3, y) = 6 - 4(-3) - 2y^2 = 18 - 2y^2$

3. $f(x, y) = \dfrac{\sqrt{x}}{y}$

The domain is $\{(x, y) : x \geq 0, y \neq 0\}$.

The range is all real numbers.

4. $f(x, y) = \sqrt{36 - x^2 - y^2}$

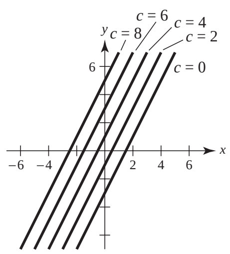
Domain: $\{(x, y) : x^2 + y^2 \leq 36\}$

Range: $0 \leq z \leq 6$

(The surface is a hemisphere.)

5. $z = 3 - 2x + y$

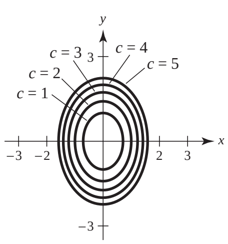
The level curves are parallel lines of the form $y = 2x - 3 + c$.



6. $z = 2x^2 + y^2$

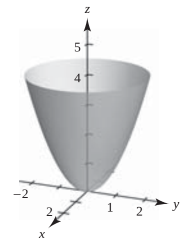
The level curves are ellipses of the form $2x^2 + y^2 = c$

(except $2x^2 + y^2 = 0$ is the point (0, 0)).



7. $f(x, y) = x^2 + y^2$

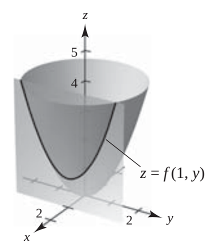
(a)



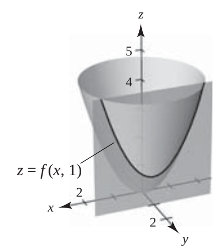
(b) $g(x, y) = f(x, y) + 2$ is a vertical translation of f two units upward.

(c) $g(x, y) = f(x, y - z)$ is a horizontal translation of f two units to the right. The vertex moves from $(0, 0, 0)$ to $(0, 2, 0)$.

(d)



$z = f(1, y)$



$z = f(x, 1)$

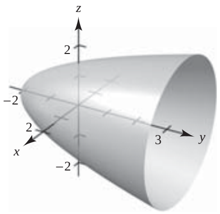
8. $A(r, t) = 2000e^{rt}$

	Number of years			
Rate	5	10	15	20
0.02	2210.34	2442.81	2699.72	2983.65
0.04	2442.81	2983.65	3644.24	4451.08
0.06	2699.72	3644.24	4919.21	6640.23
0.07	2838.14	4027.51	5715.30	8110.40

9. $f(x, y, z) = x^2 - y + z^2 = 2$

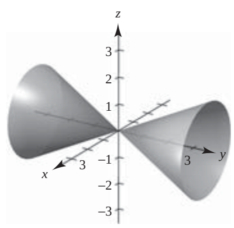
$$y = x^2 + z^2 - 2$$

Elliptic paraboloid



10. $f(x, y, z) = 4x^2 - y^2 + 4z^2 = 0$

Elliptic cone



11. $\lim_{(x, y)\to(1,1)} \dfrac{xy}{x^2 + y^2} = \dfrac{1}{2}$

Continuous except at $(0, 0)$.

12. $\lim_{(x, y)\to(1,1)} \dfrac{xy}{x^2 - y^2}$

Does not exist.

Continuous except when $y = \pm x$.

13. $\lim_{(x, y)\to(0,0)} \dfrac{y + xe^{-y^2}}{1 + x^2} = \dfrac{0 + 0}{1 + 0} = 0$

Continuous everywhere.

14. $\lim_{(x, y)\to(0,0)} \dfrac{x^2y}{x^4 + y^2}$

For $y = x^2$, $\dfrac{x^2y}{x^4 + y^2} = \dfrac{x^4}{x^4 + x^4} \to \dfrac{1}{2}$.

For $y = 0$, $\dfrac{x^2y}{x^4 + y^2} = 0$ for $x \neq 0$.

The limit does not exist.

Continuous to all $(x, y) \neq (0, 0)$

15. $f(x, y) = 5x^3 + 7y - 3$

$\dfrac{\partial f}{\partial x} = 15x^2 \qquad \dfrac{\partial f}{\partial y} = 7$

16. $f(x, y) = 4x^2 - 2xy + y^2$

$$\frac{\partial f}{\partial x} = 8x - 2y$$

$$\frac{\partial f}{\partial y} = -2x + 2y$$

17. $f(x, y) = e^x \cos y$

$$f_x = e^x \cos y$$

$$f_y = -e^x \sin y$$

18. $f(x, y) = \dfrac{xy}{x + y}$

$$f_x = \frac{y(x + y) - xy}{(x + y)^2} = \frac{y^2}{(x + y)^2}$$

$$f_y = \frac{x^2}{(x + y)^2}$$

19. $f(x, y) = y^3e^{4x}$

$$\frac{\partial f}{\partial x} = 4y^3e^{4x}$$

$$\frac{\partial f}{\partial y} = 3y^2e^{4x}$$

20. $z = \ln(x^2 + y^2 + 1)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

21. $f(x, y, z) = 2xz^2 + 6xyz - 5xy^3$

$$\frac{\partial f}{\partial x} = 2z^2 + 6yz - 5y^3$$

$$\frac{\partial f}{\partial y} = 6xz - 15xy^2$$

$$\frac{\partial f}{\partial z} = 4xz + 6xy$$

22. $w = \sqrt{x^2 - y^2 - z^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 - y^2 - z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - y^2 - z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2 - z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{-z}{\sqrt{x^2 - y^2 - z^2}}$$

23. $f(x, y) = 3x^2 - xy + 2y^3$

$f_x = 6x - y$

$f_y = -x + 6y^2$

$f_{xx} = 6$

$f_{yy} = 12y$

$f_{xy} = -1$

$f_{yx} = -1$

24. $h(x, y) = \dfrac{x}{x + y}$

$h_x = \dfrac{y}{(x + y)^2}$

$h_y = \dfrac{-x}{(x + y)^2}$

$h_{xx} = \dfrac{-2y}{(x + y)^3}$

$h_{yy} = \dfrac{2x}{(x + y)^3}$

$h_{xy} = \dfrac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \dfrac{x - y}{(x + y)^3}$

$h_{yx} = \dfrac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \dfrac{x - y}{(x + y)^3}$

25. $h(x, y) = x \sin y + y \cos x$

$h_x = \sin y - y \sin x$

$h_y = x \cos y + \cos x$

$h_{xx} = -y \cos x$

$h_{yy} = -x \sin y$

$h_{xy} = \cos y - \sin x$

$h_{yx} = \cos y - \sin x$

26. $g(x, y) = \cos(x - 2y)$

$g_x = -\sin(x - 2y)$

$g_y = 2 \sin(x - 2y)$

$g_{xx} = -\cos(x - 2y)$

$g_{yy} = -4 \cos(x - 2y)$

$g_{xy} = 2 \cos(x - 2y)$

$g_{yx} = 2 \cos(x - 2y)$

27. $z = x^2 \ln(y + 1)$

$\dfrac{\partial z}{\partial x} = 2x \ln(y + 1)$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial x} = 0$.

Slope in x-direction.

$\dfrac{\partial z}{\partial y} == \dfrac{x^2}{1 + y}$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial y} = 4$.

Slope in y-direction.

28. $R = 300x_1 + 300x_2 - 5x_1^2 - 10x_1x_2 - 5x_2^2$

(a) $\dfrac{\partial R}{\partial x_1} = 300 - 10x_1 - 10x_2$

At $(x_1, x_2) = (5, 8)$,

$\dfrac{\partial R}{\partial x_1} = 300 - 10(5) - 10(8) = 170.$

(b) $\dfrac{\partial R}{\partial x_2} = 300 - 10x_1 - 10x_2$

At $(x_1, x_2) = (5, 8)$,

$\dfrac{\partial R}{\partial x_2} = 300 - 10(5) - 10(8) = 170.$

29. $z = x \sin xy$

$dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = (xy \cos xy + \sin xy)dx + (x^2 \cos xy)dy$

30. $z = 5x^4y^3$

$dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy = 20x^3y^3\,dx + 15x^4y^2\,dy$

31. $w = 3xy^2 - 2x^3yz^2$

$dw = \dfrac{\partial w}{\partial x}\,dx + \dfrac{\partial w}{\partial y}\,dy + \dfrac{\partial w}{\partial z}\,dz$

$= (3y^2 - 6x^2yz^2)dx + (6xy - 2x^3z^2)dy - 4x^3yz\,dz$

32. $w = \dfrac{3x + 4y}{y + 3z}$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

$$= \frac{3}{y + 3z}dx + \frac{3(4z - x)}{(y + 3z)^2}dy + \frac{-3(3x + 4y)}{(y + 3z)^2}dz$$

33. $f(x, y) = 4x + 2y$

(a) $f(2, 1) = 4(2) + 2(1) = 10$

$f(2.1, 1.05) = 4(2.1) + 2(1.05) = 10.5$

$\Delta z = 10.5 - 10 = 0.5$

(b) $dz = 4dx + 2dy$

$= 4(0.1) + 2(0.05) = 0.5$

34. $f(x, y) = 36 - x^2 - y^2$

(a) $f(2, 1) = 36 - 2^2 - 1^2 = 31$

$f(2.1, 1.05) = 36 - (2.1)^2 - (1.05)^2 = 30.4875$

$\Delta z = 30.4875 - 31 = -0.5125$

(b) $dz = -2x\,dx - 2y\,dy$

$= -2(2)(0.1) - 2(1)(0.05) = -0.5$

35. $V = \dfrac{1}{3}\pi r^2 h$

$$dV = \frac{2}{3}\pi rh\,dr + \frac{1}{3}\pi r^2\,dh$$

$$= \frac{2}{3}\pi(2)(5)\left(\pm\frac{1}{8}\right) + \frac{1}{3}\pi(2)^2\left(\pm\frac{1}{8}\right)$$

$$= \pm\frac{5}{6}\pi + \frac{1}{6}\pi = \pm\pi \text{ in.}^3 \qquad \text{Propogated error}$$

$$V = \frac{1}{3}\pi(2)^2 5 = \frac{20}{3}\pi \text{ in.}^3$$

$$\text{Relative error} = \frac{dV}{V} = \frac{\pm\pi}{\left(\frac{20}{3}\pi\right)} = \frac{3}{20} = 15\%$$

36. $A = \pi r\sqrt{r^2 + h^2}$

$$dA = \left(\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}\right)dr + \frac{\pi rh}{\sqrt{r^2 + h^2}}dh$$

$$= \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}}dr + \frac{\pi rh}{\sqrt{r^2 + h^2}}dh = \frac{\pi(8 + 25)}{\sqrt{29}}\left(\pm\frac{1}{8}\right) + \frac{10\pi}{\sqrt{29}}\left(\pm\frac{1}{8}\right) = \pm\frac{43\pi}{8\sqrt{29}} \qquad \text{Propogated error}$$

$$A = 2\pi\sqrt{2^2 + 5^2}$$

$$= 2\pi\sqrt{29}$$

$$\text{Relative error} = \frac{dA}{A} = \frac{\pm\frac{43\pi}{8\sqrt{29}}}{2\pi\sqrt{29}} \approx 0.0927 = 9.27\%$$

37. $w = \ln(x^2 + y),\ x = 2t,\ y = 4 - t$

(a) Chain Rule: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

$$= \frac{2x}{x^2 + y}(2) + \frac{1}{x^2 + y}(-1)$$

$$= \frac{8t - 1}{4t^2 + 4 - t}$$

(b) Substitution: $w = \ln(x^2 + y) = \ln(4t^2 + 4 - t)$

$$\frac{dw}{dt} = \frac{1}{4t^2 + 4 - t}(8t - 1)$$

38. $w = y^2 - x,\ x = \cos t,\ y = \sin t$

(a) Chain Rule: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$

$$= -1(-\sin t) + 2y(\cos t)$$

$$= \sin t + 2(\sin t)\cos t$$

$$= \sin t(1 + 2\cos t)$$

(b) Substitution: $w = \sin^2 t - \cos t$

$$\frac{dw}{dt} = 2\sin t\cos t + \sin t$$

$$= \sin t(1 + 2\cos t)$$

39. $w = \dfrac{xy}{z},\ x = 2r + t,\ y = rt,\ z = 2r - t$

(a) Chain Rule: $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$
$$= \frac{y}{z}(2) + \frac{x}{z}(t) - \frac{xy}{z^2}(2)$$
$$= \frac{2rt}{2r - t} + \frac{(2r + t)t}{2r - t} - \frac{2(2r + t)(rt)}{(2r - t)^2}$$
$$= \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$= \frac{y}{z}(1) + \frac{x}{z}(r) = \frac{xy}{z^2}(-1)$$
$$= \frac{4r^2t - rt^2 + 4r^3}{(2r - t)^2}$$

(b) Substitution: $$w = \frac{xy}{z} = \frac{(2r + t)(rt)}{2r - t} = \frac{2r^2t + rt^2}{2r - t}$$
$$\frac{\partial w}{\partial r} = \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$
$$\frac{\partial w}{\partial t} = \frac{4r^2t - rt^2 + 4r^3}{(2r - t)^2}$$

40. $w = x^2 + y^2 + z^2,\ x = r\cos t,\ y = r\sin t,\ z = t$

(a) Chain Rule: $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$
$$= 2x\cos t + 2y\sin t + 2z(0)$$
$$= 2\left(r\cos^2 t + r\sin^2 t\right) = 2r$$
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$= 2x(-r\sin t) + 2y(r\cos t) + 2z = 2\left(-r^2\sin t\cos t + r^2\sin t\cos t\right) + 2t = 2t$$

(b) Substitution: $$w(r, t) = r^2\cos^2 t + r^2\sin^2 t + t^2 = r^2 + t^2$$
$$\frac{\partial w}{\partial r} = 2r$$
$$\frac{\partial w}{\partial t} = 2t$$

41. $$x^2 + xy + y^2 + yz + z^2 = 0$$
$$2x + y + y\frac{\partial z}{\partial x} + 2z\frac{\partial z}{\partial x} = 0$$
$$\frac{\partial z}{\partial x} = \frac{-2x - y}{y + 2z}$$
$$x + 2y + y\frac{\partial z}{\partial y} + z + 2z\frac{\partial z}{\partial y} = 0$$
$$\frac{\partial z}{\partial y} = \frac{-x - 2y - z}{y + 2z}$$

42. $$xz^2 - y\sin z = 0$$
$$2xz\frac{\partial z}{\partial x} + z^2 - y\cos z\frac{\partial z}{\partial x} = 0$$
$$\frac{\partial z}{\partial x} = \frac{z^2}{y\cos z - 2xz}$$
$$2xz\frac{\partial z}{\partial y} - y\cos z\frac{\partial z}{\partial y} - \sin z = 0$$
$$\frac{\partial z}{\partial y} = \frac{\sin z}{2xz - y\cos z}$$

43. $f(x, y) = x^2y,\ P(-5, 5),\ \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = 2xy\cos\theta + x^2\sin\theta$$

$$D_{\mathbf{u}}f(-5, 5) = 2(-5)(5)\left(\frac{3}{5}\right) + (-5)^2\left(-\frac{4}{5}\right) = -30 - 20 = -50$$

44. $f(x, y) = \frac{1}{4}y^2 - x^2,\ P(1, 4),\ \mathbf{v} = 2\mathbf{i} + \mathbf{j}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}}f(x, y) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = -2x\cos\theta + \frac{1}{2}y\sin\theta$$

$$D_{\mathbf{u}}f(1, 4) = -2\left(\frac{2}{\sqrt{5}}\right) + 2\left(\frac{1}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

45.

$$w = y^2 + xz$$
$$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$$
$$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$
$$\mathbf{u} = \tfrac{1}{3}\mathbf{v} = \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}$$
$$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2)\cdot\mathbf{u} = \tfrac{4}{3} - \tfrac{4}{3} + \tfrac{2}{3} = \tfrac{2}{3}$$

46. $w = 5x^2 + 2xy - 3y^2z$

$$\nabla w = (10x + 2y)\mathbf{i} + (2x - 6yz)\mathbf{j} - 3y^2\mathbf{k}$$
$$\nabla w(1, 0, 1) = 10\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$$
$$D_{\mathbf{u}}w(1, 0, 1) = \nabla w(1, 0, 1)\cdot\mathbf{u} = \frac{10}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$$

47.

$$z = x^2y$$
$$\nabla z = 2xy\mathbf{i} + x^2\mathbf{j}$$
$$\nabla_z(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$
$$\|\nabla z(2, 1)\| = 4\sqrt{2}$$

48.

$$z = e^{-x}\cos y$$
$$\nabla z = -e^{-x}\cos y\,\mathbf{i} - e^{-x}\sin y\,\mathbf{j}$$
$$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$$
$$\left\|\nabla z\left(0, \frac{\pi}{4}\right)\right\| = 1$$

49.

$$z = \frac{y}{x^2 + y^2}$$
$$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$$
$$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0\right\rangle$$
$$\|\nabla z(1, 1)\| = \frac{1}{2}$$

50.

$$z = \frac{x^2}{x - y}$$
$$\nabla z = \frac{x^2 - 2xy}{(x - y)^2}\mathbf{i} + \frac{x^2}{(x - y)^2}\mathbf{j}$$
$$\nabla z(2, 1) = 4\mathbf{j}$$
$$\|\nabla z(2, 1)\| = 4$$

51. $f(x, y) = 9x^2 - 4y^2,\ c = 65,\ P(3, 2)$

(a) $\nabla f(x, y) = 18x\mathbf{i} - 8y\mathbf{j}$

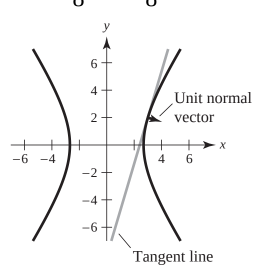
$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$

(b) Unit normal: $\dfrac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \dfrac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$

(c) Slope $= \dfrac{27}{8}$.

$$y - z = \frac{27}{8}(x - 3)$$
$$y = \frac{27}{8}x - \frac{65}{8} \quad \text{Tangent line}$$

(d)



52. $f(x, y) = 4y \sin x - y, c = 3, P\left(\dfrac{\pi}{2}, 1\right)$

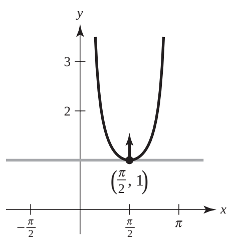
(a) $\nabla f(x, y) = 4y \cos x\mathbf{i} + (4 \sin x - 1)\mathbf{j}$

$\nabla f\left(\dfrac{\pi}{2}, 1\right) = 3\mathbf{j}$

(b) Unit normal vector: $\mathbf{j}$

(c) Tangent line horizontal: $y = 1$

(d)



53. $F(x, y, z) = x^2 + y^2 + 2 - z = 0, (1, 3, 12)$

$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$

$\nabla F(1, 3, 12) = 2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$

Tangent Plane:

$2(x - 1) + 6(y - 3) - (z - 12) = 0$

$2x + 6y - z = 8$

54. $F(x, y, z) = 9x^2 + y^2 + 4z^2 - 25 = 0, (0, -3, 2)$

$\nabla F = 18x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$

$\nabla F(0, -3, 2) = -6\mathbf{j} + 16\mathbf{k}$

Tangent Plane:

$0(x - 0) - 6(y + 3) + 16(z - 2) = 0$

$-6y + 16z = 50$

$-3y + 8z = 25$

55. $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$

$\nabla F(2, -3, 4) = \mathbf{k}$

So, the equation of the tangent plane is

$z - 4 = 0$ or $z = 4$.

56. $F(x, y, z) = y^2 + z^2 - 25 = 0$

$\nabla F = 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(2, 3, 4) = 6\mathbf{j} + 8\mathbf{k} = 2(3\mathbf{j} + 4\mathbf{k})$

So, the equation of the tangent plane is

$3(y - 3) + 4(z - 4) = 0$ or $3y + 4z = 25$.

57. $F(x, y, z) = x^2y - z = 0$

$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$

$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

So, the equation of the tangent plane is

$4(x - 2) + 4(y - 1) - (z - 4) = 0$ or

$4x + 4y - z = 8$,

and the equation of the normal line is

$x = 4t + 2, y = 4t + 1, z = -t + 4$.

Symmetric equations:

$$\frac{x - 2}{4} = \frac{y - 1}{4} = -\frac{z - 4}{1}$$

58. $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

So, the equation of the tangent plane is

$(x - 1) + 2(y - 2) + 2(z - 2) = 0$ or

$x + 2y + 2z = 9$,

and the equation of the normal line is

$$\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}.$$

59. $f(x, y, z) = x^2 + y^2 + z^2 - 14$

$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ Normal vector to plane.

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$$

$\theta = 36.7°$

60. (a) $f(x, y) = \cos x + \sin y,\ f(0, 0) = 1$

$f_x = -\sin x,\ f_x(0, 0) = 0$

$f_y = \cos y,\ f_y(0, 0) = 1$

$P_1(x, y) = 1 + y$

(b) $f_{xx} = -\cos x,\ f_{xx}(0, 0) = -1$

$f_{yy} = -\sin y,\ f_{yy}(0, 0) = 0$

$f_{xy} = 0,\ f_{xy}(0, 0) = 0$

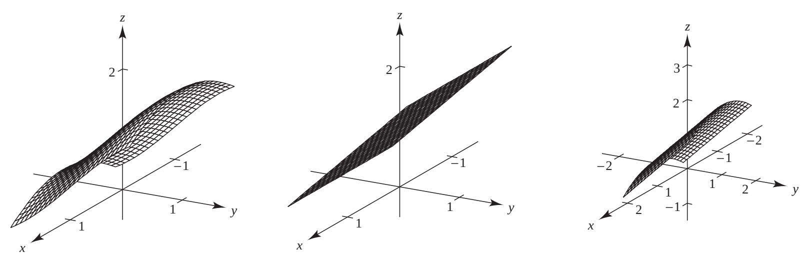
$P_2(x, y) = 1 + y - \frac{1}{2}x^2$

(c) If $y = 0$, you obtain the 2nd degree Taylor polynomial for $\cos x$.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1.0	1.0	1.0
0	0.1	1.0998	1.1	1.1
0.2	0.1	1.0799	1.1	1.095
0.5	0.3	1.1731	1.3	1.175
1	0.5	1.0197	1.5	1.0

(e)



The accuracy lessens as the distance from $(0, 0)$ increases.

61. $f(x, y) = -x^2 - 4y^2 + 8x - 8y - 11$

$f_x = -2x + 8 = 0 \Rightarrow x = 4$

$f_y = -8y - 8 = 0 \Rightarrow y = -1$

$f_{xx} = -2,\ f_{yy} = -8,\ f_{xy} = 0$

$f_{xx}\, f_{yy} - \left(f_{xy}\right)^2 = (-2)(-8) - 0 = 16 > 0$

So, $(4, -1, 9)$ is a relative minimum.

62. $f(x, y) = x^2 - y^2 - 16x - 16y$

$f_x = 2x - 16 = 0 \Rightarrow x = 8$

$f_y = -2y - 16 = 0 \Rightarrow y = -8$

$f_{xx} = 2,\ f_{yy} = -2,\ f_{xy} = 0$

$f_{yy}\, f_{yy} - \left(f_{xy}\right)^2 = 2(-2) - 0 = -4 < 0$

So, $(8, -8, 0)$ is a saddle point.

63. $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

$f_x = 4x + 6y + 8 = 0$

$f_y = 6x + 18y = 0, x = -3y$

$4(-3y) + 6y = -8 \Rightarrow y = \frac{4}{3}, x = -4$

$f_{xx} = 4$

$f_{yy} = 18$

$f_{xy} = 6$

$f_{xx} f_{yy} - \left(f_{xy}\right)^2 = 4(18) - (6)^2 = 36 > 0.$

So, $\left(-4, \frac{4}{3}, -2\right)$ is a relative minimum.

64. $f(x, y) = x^2 + 3xy + y^2 - 5x$

$f_x = 2x + 3y - 5 = 0$

$f_y = 3x + 2y = 0 \quad \Rightarrow y = -\frac{3}{2}x$

$2x + 3\left(-\frac{3}{2}x\right) = 5$

$4x - 9x = 10$

$x = -2, y = 3$

$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, d = 4 - 9 < 0$

$\Rightarrow (-2, 3)$ is a saddle point.

65. $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

$f_x = y - \frac{1}{x^2} = 0, x^2y = 1$

$f_y = x - \frac{1}{y^2} = 0, xy^2 = 1$

So, $x^2y = xy^2$ or $x = y$ and substitution yields the critical point $(1, 1)$.

$f_{xx} = \frac{2}{x^3}$

$f_{xy} = 1$

$f_{yy} = \frac{2}{y^3}$

At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and

$f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0.$

So, $(1, 1, 3)$ is a relative minimum.

66. $f(x, y) = -8x^2 + 4xy - y^2 + 12x + 7$

$f_x = -16x + 4y + 12 = 0 \Rightarrow y - 4x = -3$

$f_y = 4x - 2y = 0 \Rightarrow y = 2x$

So, $x = 3/2, y = 3.$

$f_{xx} = -16, f_{yy} = -2, f_{xy} = 4$

$f_{xx}\,f_{yy} - (f_{xy})^2 = (-16)(-2) - 4^2 = 16 > 0$

So, $(3/2, 3, 16)$ is a relative maximum.

67. A point on the plane is given by $(x, y, 4 - x - y)$

The square of the distance from $(2, 1, 4)$ to a point on the plane is

$$S = (x - 2)^2 + (y - 1)^2 + (4 - x - y - 4)^2$$
$$= (x - 2)^2 + (y - 1)^2 + (-x - y)^2.$$

$S_x = 2(x - 2) - 2(-x - y) = 4x + 2y - 4$

$S_y = 2(y - 1) - 2(-x - y) = 2x + 4y - 2$

$S_x = S_y = 0 \Rightarrow \begin{Bmatrix} 4x + 2y = 4 \\ 2x + 4y = 2 \end{Bmatrix} \Rightarrow x = 1, y = 0, z = 3$

The distance is $\sqrt{(1 - 2)^2 + (0 - 1)^2 + (-1)^2} = \sqrt{3}$.

68. $xyz = 64 \Rightarrow z = \frac{64}{xy}$

$S = x + y + z = x + y + \frac{64}{xy}$

$Sx = 1 - \frac{64}{x^2y} = 0$

$Sy = 1 - \frac{64}{xy^2} = 0$

$$\left.\begin{aligned} \frac{64}{x^2y} &= 1 \Rightarrow 64 = x^2y \\ \frac{64}{xy^2} &= 1 \Rightarrow 64 = xy^2 \end{aligned}\right\} \quad x = y = 4$$

So, $x = y = z = 4$.

69. $R = -6x_1^2 - 10x_2^2 - 2x_1x_2 + 32x_1 + 84x_2$

$Rx_1 = -12x_1 - 2x_2 + 32 = 0 \Rightarrow 6x_1 + x_2 = 16$

$Rx_2 = -20x_2 - 2x_1 + 84 = 0 \Rightarrow x_1 + 10x_2 = 42$

Solving this system yields $x_1 = 2$ and $x_2 = 4$.

70. $P = 180(x_1 + x_2) - C_1 - C_2$

$= 180x_1 + 180x_2 - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$

$= -0.05x_1^2 - 0.03x_2^2 + 165x_1 + 165x_2 - 11{,}500$

$Px_1 = -0.1x_1 + 165 = 0$

$Px_2 = -0.06x_2 + 165 = 0$

Solving this system yields

$x_1 = 1650$ and

$x_2 = 2750.$

By the Second Derivative Test, this is a maximum.

71. $(0, 4), (1, 5), (3, 6), (6, 8), (8, 10)$

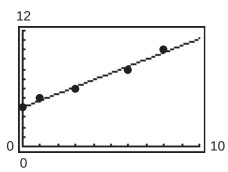
$\sum x_i = 18 \qquad \sum y_i = 33$

$\sum x_i y_i = 151 \qquad \sum x_i^2 = 110$

$$a = \frac{5(151) - 18(33)}{5(110) - (18)^2} = \frac{161}{226} \approx 0.7124$$

$$b = \frac{1}{5}\left(33 - \frac{161}{226}(18)\right) = \frac{456}{113} \approx 4.0354$$

$$y = \frac{161}{226}x + \frac{456}{113}$$



72. $(0, 10), (2, 8), (4, 7), (7, 5), (9, 3), (12, 0)$

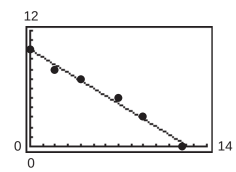
$\sum x_i = 34 \qquad \sum y_i = 33$

$\sum x_i y_i = 106 \qquad \sum x_i^2 = 294$

$$a = \frac{6(106) - 34(33)}{6(294) - (34)^2} = -\frac{243}{304} \approx -0.7993$$

$$b = \frac{1}{6}\left(33 - \left(\frac{-243}{304}\right)(34)\right) = \frac{3049}{304} \approx 10.0296$$

$$y = -\frac{243}{304}x + \frac{3049}{304}$$



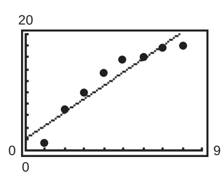
73. $(100, 35), (150, 44), (200, 50), (250, 56)$

(a) Using a graphing utility, you obtain

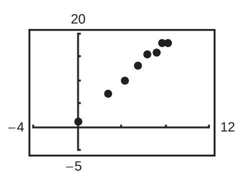
$y = 0.138x + 22.1.$

(b) If $x = 175$, $y = 0.138(175) + 22.1 = 46.25$ bushels per acre.

74. (a) $y = 2.29t + 2.0$



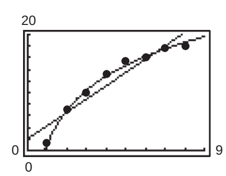
(b)



Yes, the data appear linear.

(c) $y = 1.24 + 8.37 \ln t$

(d)



75. Minimize $f(x, y) = x^2 + y^2$

Constraint: $x + y - 8 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \end{aligned}\right\} \quad x = y$

$x + y - 8 = 2x - 8 = 0 \Rightarrow x = y = 4$

$f(4, 4) = 32$

76. Maximize $f(x, y) = xy$

Constraint: $x + 3y - 6 = 0$

$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + 3\mathbf{j})$

$\left.\begin{aligned} y &= \lambda \\ x &= 3\lambda \end{aligned}\right\} \quad x = 3y$

$x + 3y - 6 = 6y - 6 = 0 \Rightarrow y = 1, x = 3$

$f(3, 1) = 3$

77. Maximize $f(x, y) = 2x + 3xy + y$

Constraint: $x + 2y = 29$

$\nabla f = \lambda \nabla g$

$\left.\begin{aligned} 2 + 3y &= \lambda \\ 3x + 1 &= 2\lambda \end{aligned}\right\} \quad 4 + 6y = 3x + 1 \Rightarrow x - 2y = 1$

$\left.\begin{aligned} x - 2y &= 1 \\ x + 2y &= 29 \end{aligned}\right\} \quad x = 15, y = 7$

$f(15, 7) = 2(15) + 3(15)(7) + 7 = 352$

78. Minimize $f(x, y) = x^2 - y^2$

Constraint: $x - 2y + 6 = 0$

$\nabla f = \lambda \nabla g$

$\left.\begin{aligned} 2x &= \lambda \\ -2y &= -2\lambda \end{aligned}\right\} \quad -4x = -2y \Rightarrow y = 2x$

$x - 2y + 6 = x - 4x + 6 = 0 \Rightarrow x = 2, y = 4$

$f(2, 4) = 4 - 16 = -12$

79. Maximize $f(x, y) = 2xy$

Constraint: $2x + y = 12$

$\nabla f = \lambda \nabla g$

$\left.\begin{aligned} 2y &= 2\lambda \\ 2x &= \lambda \end{aligned}\right\} \quad 4x = 2y \Rightarrow y = 2x$

$2x + y = 2x + 2x = 12 \Rightarrow x = 3, y = 6$

$f(3, 6) = 2(3)(6) = 36$

80. Minimize $f(x, y) = 3x^2 - y^2$

Constraint: $2x - 2y + 5 = 0$

$\nabla f = \lambda \nabla g$

$\left.\begin{aligned} 6x &= 2\lambda \\ -2y &= -2\lambda \end{aligned}\right\} \quad 6x = 2y \Rightarrow y = 3x$

$2x - 2y + 5 = 2x - 2(3x) + 5 = 0 \Rightarrow -4x + 5 = 0$

$\Rightarrow x = \frac{5}{4}, y = \frac{15}{4}$

$f\left(\frac{5}{4}, \frac{15}{4}\right) = -\frac{75}{8}$

81. $PQ = \sqrt{x^2 + 4}$,

$QR = \sqrt{y^2 + 1}$,

$RS = z; x + y + z = 10$

$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + z$

Constraint: $x + y + z = 10$

$\nabla C = \lambda \nabla g$

$$\frac{3x}{\sqrt{x^2 + 4}}\mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}}\mathbf{j} + \mathbf{k} = \lambda[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$3x = \lambda\sqrt{x^2 + 4}$

$2y = \lambda\sqrt{y^2 + 1}$

$1 = \lambda$

$9x^2 = x^2 + 4 \Rightarrow x^2 = \dfrac{1}{2}$

$4y^2 = y^2 + 1 \Rightarrow y^2 = \dfrac{1}{3}$

So, $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km,

$y = \dfrac{\sqrt{3}}{3} \approx 0.577$ km,

$z = 10 - \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{3} \approx 8.716$ km.

Problem Solving for Chapter 13

1. (a) The three sides have lengths 5, 6, and 5.

Thus, $s = \frac{16}{2} = 8$ and $A = \sqrt{8(3)(2)(3)} = 12$.

(b) Let $f(a, b, c) = (\text{area})^2 = s(s-a)(s-b)(s-c)$, subject to the constraint

$a + b + c =$ constant (perimeter).

Using Lagrange multipliers,

$-s(s-b)(s-c) = \lambda$

$-s(s-a)(s-c) = \lambda$

$-s(s-a)(s-b) = \lambda$.

From the first 2 equations

$s - b = s - a \Rightarrow a = b$.

Similarly, $b = c$ and hence $a = b = c$ which is an equilateral triangle.

(c) Let $f(a, b, c) = a + b + c$, subject

to $(\text{Area})^2 = s(s-a)(s-b)(s-c)$ constant.

Using Lagrange multipliers,

$1 = -\lambda s(s-b)(s-c)$

$1 = -\lambda s(s-a)(s-c)$

$1 = -\lambda s(s-a)(s-b)$

So, $s - a = s - b \Rightarrow a = b$ and $a = b = c$.

2. $V = \frac{4}{3}\pi r^3 + \pi r^2 h$

Material $= M = 4\pi r^2 + 2\pi rh$

$V = 1000 \Rightarrow h = \dfrac{1000 - (4/3)\pi r^3}{\pi r^2}$

So,
$$M = 4\pi r^2 + 2\pi r\left(\frac{1000 - (4/3)\pi r^3}{\pi r^2}\right)$$
$$= 4\pi r^2 + \frac{2000}{r} - \frac{8}{3}\pi r^2$$
$$\frac{dM}{dr} = 8\pi r - \frac{2000}{r^2} - \frac{16}{3}\pi r = 0$$
$$8\pi r - \frac{16}{3}\pi r = \frac{2000}{r^2}$$
$$r^3\left(\frac{8}{3}\pi\right) = 2000$$
$$r^3 = \frac{750}{\pi} \Rightarrow r = 5\left(\frac{6}{\pi}\right)^{1/3}.$$

Then, $h = \dfrac{1000 - (4/3)\pi(750/\pi)}{\pi r^2} = 0$.

The tank is a sphere of radius $r = 5\left(\dfrac{6}{\pi}\right)^{1/3}$.

3. (a) $F(x, y, z) = xyz - 1 = 0$

$F_x = yz, F_y = xz, F_z = xy$

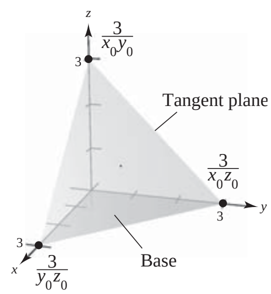
Tangent plane:

$y_0 z_0(x - x_0) + x_0 z_0(y - y_0) + x_0 y_0(z - z_0) = 0$

$y_0 z_0 x + x_0 z_0 y + x_0 y_0 z = 3x_0 y_0 z_0 = 3$

(b) $V = \frac{1}{3}(\text{base})(\text{height})$

$= \frac{1}{3}\left(\frac{1}{2}\frac{3}{y_0 z_0}\frac{3}{x_0 z_0}\right)\left(\frac{3}{x_0 y_0}\right) = \frac{9}{2}$



4. (a) As $x \to \pm\infty$, $f(x) = (x^3 - 1)^{1/3} \to x$ and

hence $\lim_{x\to\infty}[f(x) - g(x)] = \lim_{x\to\infty}[f(x) - g(x)] = 0$.

(b) Let $\left(x_0, (x_0^3 - 1)^{1/3}\right)$ be a point on the graph of f.

The line through this point perpendicular to g is $y = -x + x_0 + \sqrt[3]{x_0^3 - 1}$.

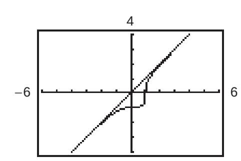
This line intersects g at the point

$\left(\frac{1}{2}\left[x_0 + \sqrt[3]{x_0^3 - 1}\right], \frac{1}{2}\left[x_0 + \sqrt[3]{x_0^3 - 1}\right]\right)$.

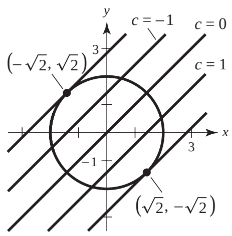
The square of the distance between these two points is $h(x_0) = \frac{1}{2}\left(x_0 - \sqrt[3]{x_0^3 - 1}\right)^2$.

h is a maximum for $x_0 = \dfrac{1}{\sqrt[3]{2}}$. So, the point

on f farthest from g is $\left(\dfrac{1}{\sqrt[3]{2}}, -\dfrac{1}{\sqrt[3]{2}}\right)$.



5. (a)



Maximum value of f is $f\left(\sqrt{2}, -\sqrt{2}\right) = 2\sqrt{2}$.

Maximize $f(x, y) = x - y$.

Constraint: $g(x, y) = x^2 + y^2 = 4$

$$\nabla f = \lambda \nabla g: \quad 1 = 2\lambda x$$
$$-1 = 2\lambda y$$
$$x^2 + y^2 = 4$$

$2\lambda x = -2\lambda y \Rightarrow x = -y$

$2x^2 = 4 \Rightarrow x = \pm\sqrt{2}, y = \mp\sqrt{2}$

$f\left(\sqrt{2}, -\sqrt{2}\right) = 2\sqrt{2},\ f\left(-\sqrt{2}, \sqrt{2}\right) = -2\sqrt{2}$

(b) $f(x, y) = x - y$

Constraint: $x^2 + y^2 = 0 \Rightarrow (x, y) = (0, 0)$

Maximum and minimum values are 0.

Lagrange multipliers does not work:

$$\left.\begin{aligned} 1 &= 2\lambda x \\ -1 &= 2\lambda y \end{aligned}\right\} x = -y = 0, \text{ a contradiction.}$$

Note that $\nabla g(0, 0) = \mathbf{0}$.

6. Heat Loss $= H = k(5xy + xy + 3xz + 3xz + 3yz + 3yz)$

$$= k(6xy + 6xz + 6yz)$$

$V = xyz = 1000 \Rightarrow z = \dfrac{1000}{xy}$.

Then $H = 6k\left(xy + \dfrac{1000}{y} + \dfrac{1000}{x}\right)$.

Setting $H_x = H_y = 0$, you obtain $x = y = z = 10$.

7. $H = k(5xy + 6xz + 6yz)$

$z = \dfrac{1000}{xy} \Rightarrow H = k\left(5xy + \dfrac{6000}{y} + \dfrac{6000}{x}\right)$.

$H_x = 5y - \dfrac{6000}{x^2} = 0 \Rightarrow 5yx^2 = 6000$

By symmetry, $x = y \Rightarrow x^3 = y^3 = 1200$.

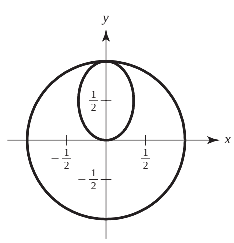
So, $x = y = 2\sqrt[3]{150}$ and $z = \dfrac{5}{3}\sqrt[3]{150}$.

8. (a) $T(x, y) = 2x^2 + y^2 - y + 10 = 10$

$$2x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$

$$2x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{x^2}{1/8} + \frac{(y - (1/2))^2}{1/4} = 1 \qquad \text{ellipse}$$



(b) On $x^2 + y^2 = 1$, $T(x, y) = T(y) = 2(1 - y^2) + y^2 - y + 10 = 12 - y^2 - y$

$T'(y) = -2y - 1 = 0 \Rightarrow y = -\dfrac{1}{2}, x = \pm\dfrac{\sqrt{3}}{2}$.

Inside: $T_x = 4x - 0, T_y = 2y - 1 = 0 \Rightarrow \left(0, \dfrac{1}{2}\right)$

$T\left(0, \dfrac{1}{2}\right) = \dfrac{39}{4}$ minimum

$T\left(\pm\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right) = \dfrac{49}{4}$ maximum

9. (a) $\dfrac{\partial f}{\partial x} = Cax^{a-1}y^{1-a}, \dfrac{\partial f}{\partial y} = C(1-a)x^a y^{-a}$

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= Cax^a y^{1-a} + C(1-a)x^a y^{1-a} \\ &= [Ca + C(1-a)]x^a y^{1-a} \\ &= Cx^a y^{1-a} = f \end{aligned}$$

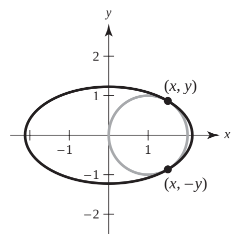
(b) $f(tx, ty) = C(tx)^a (ty)^{1-a} = Ct^a x^a t^{1-a} y^{1-a} = Cx^a y^{1-a}(t) = tf(x, y)$

10. $x^2 + y^2 = 2x$

$(x-1)^2 + y^2 = 1$ Circle

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Ellipse

The circle and ellipse intersect at (x, y) and $(x, -y)$ for a unique value of x.



$y^2 = \dfrac{b^2}{a^2}(a^2 - x^2)$ Ellipse

$x^2 + \dfrac{b^2}{a^2}(a^2 - x^2) = 2x$ Circle

$\left(1 - \dfrac{b^2}{a^2}\right)x^2 - 2x + b^2 = 0$ Quadratic

For these to be a unique x-value, the discriminant must be 0.

$$4 - 4\left(1 - \frac{b^2}{a^2}\right)b^2 = 0$$

$$a^2 - a^2b^2 + b^4 = 0$$

We use lagrange multipliers to minimize the area $f(a, b) = \pi ab$ of the ellipse subject to the constraint

$g(a, b) = a^2 - a^2b^2 + b^4 = 0.$

$\nabla f = \lambda \nabla g$

$\langle \pi b, \pi a \rangle = \lambda \langle 2a - 2ab^2, -2a^2b + 4b^3 \rangle$

$\pi b = \lambda(2a - 2ab^2)$

$\pi a = \lambda(-2a^2b + 4b^3)$

$$\lambda = \frac{\pi b}{2a - 2ab^2} = \frac{\pi a}{4b^3 - 2a^2b} \Rightarrow 4b^4 - 2a^2b^2 = 2a^2 - 2a^2b^2 \Rightarrow 2b^4 = a^2 \Rightarrow b^2 = \frac{a}{\sqrt{2}}$$

Using the constraint, $a^2 - a^2b^2 + b^4 = 0,\ a^2 - a^2\dfrac{a}{\sqrt{2}} + \dfrac{a^2}{2} = 0$

$$\frac{3}{2} = \frac{a}{\sqrt{2}}$$

$$a = \frac{3}{2}\sqrt{2}, b = \frac{\sqrt{6}}{2}.$$

Ellipse: $\dfrac{x^2}{(9/2)} + \dfrac{y^2}{(3/2)} = 1$

11. (a) $x = 64(\cos 45°)t = 32\sqrt{2}t$

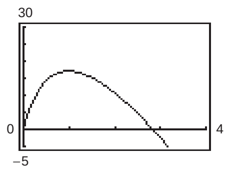
$y = 64(\sin 45°)t - 16t^2 = 32\sqrt{2}t - 16t^2$

(b) $\tan\alpha = \dfrac{y}{x + 50}$

$\alpha = \arctan\left(\dfrac{y}{x + 50}\right) = \arctan\left(\dfrac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$

(c) $$\frac{d\alpha}{dt} = \frac{1}{1 + \left(\dfrac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)^2} \cdot \frac{-64\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{\left(32\sqrt{2}t + 50\right)^2} = \frac{-16\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$$

(d)



No. The rate of change of α is greatest when the projectile is closest to the camera.

(e) $\dfrac{d\alpha}{dt} = 0$ when

$8\sqrt{2}t^2 + 25t - 25\sqrt{2} = 0$

$$t = \frac{-25 + \sqrt{25^2 - 4\left(8\sqrt{2}\right)\left(-25\sqrt{2}\right)}}{2\left(8\sqrt{2}\right)} \approx 0.98 \text{ second.}$$

No, the projectile is at its maximum height when $dy/dt = 32\sqrt{2} - 32t = 0$ or $t = \sqrt{2} \approx 1.41$ seconds.

12. (a) $d = \sqrt{x^2 + y^2} = \sqrt{\left(32\sqrt{2}t\right)^2 + \left(32\sqrt{2}t - 16t^2\right)^2} = \sqrt{4096t^2 - 1024\sqrt{2}t^3 + 256t^4} = 16t\sqrt{t^2 - 4\sqrt{2}t + 16}$

(b) $\dfrac{dd}{dt} = \dfrac{32\left(t^2 - 3\sqrt{2}t + 8\right)}{\sqrt{t^2 - 4\sqrt{2}t + 16}}$

(c) When $t = 2$:

$\dfrac{dd}{dt} = \dfrac{32\left(12 - 6\sqrt{2}\right)}{\sqrt{20 - 8\sqrt{2}}} \approx 38.16$ ft/sec

(d) $\dfrac{d^2d}{dt^2} = \dfrac{32\left(t^3 - 6\sqrt{2}t^2 + 36t - 32\sqrt{12}\right)}{\left(t^2 - 4\sqrt{2}t + 16\right)^{3/2}} = 0$ when $t \approx 1.943$ seconds. No. The projectile is at its maximum height when $t = \sqrt{2}$.

13. (a) There is a minimum at $(0, 0, 0)$, maxima at $(0, \pm 1, 2/e)$ and saddle point at $(\pm 1, 0, 1/e)$:

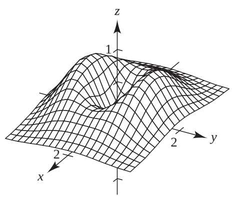
$$f_x = \left(x^2 + 2y^2\right)e^{-\left(x^2+y^2\right)}(-2x) + (2x)e^{-\left(x^2+y^2\right)}$$

$$= e^{-\left(x^2+y^2\right)}\left[\left(x^2 + 2y^2\right)(-2x) + 2x\right] = e^{-\left(x^2+y^2\right)}\left[-2x^3 + 4xy^2 + 2x\right] = 0 \Rightarrow x^3 + 2xy^2 - x = 0$$

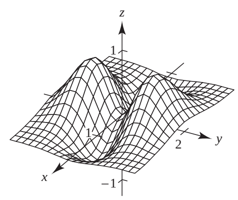
$$f_y = \left(x^2 + 2y^2\right)e^{-\left(x^2+y^2\right)}(-2y) + (4y)e^{-\left(x^2+y^2\right)}$$

$$= e^{-\left(x^2+y^2\right)}\left[\left(x^2 + 2y^2\right)(-2y) + 4y\right] = e^{-\left(x^2+y^2\right)}\left[-4y^3 - 2x^2y + 4y\right] = 0 \Rightarrow 2y^3 + x^2y - 2y = 0$$

Solving the two equations $x^3 + 2xy^2 - x = 0$ and $2y^3 + x^2y - 2y = 0$, you obtain the following critical points: $(0, \pm 1), (\pm 1, 0), (0, 0)$. Using the second derivative test, you obtain the results above.



(b) As in part (a), you obtain



$$f_x = e^{-\left(x^2+y^2\right)}\left[2x\left(x^2 - 1 - 2y^2\right)\right]$$

$$f_y = e^{-\left(x^2+y^2\right)}\left[2y\left(2 + x^2 - 2y^2\right)\right]$$

The critical numbers are $(0, 0), (0, \pm 1), (\pm 1, 0)$.

These yield

$(\pm 1, 0, -1/e)$ minima

$(0, \pm 1, 2/e)$ maxima

$(0, 0, 0)$ saddle

(c) In general, for $\alpha > 0$ you obtain

$(0, 0, 0)$ minimum

$(0, \pm 1, \beta/e)$ maxima

$(\pm 1, 0, \alpha/e)$ saddle

For $\alpha < 0$, you obtain

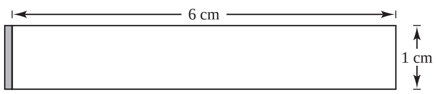
$(\pm 1, 0, \alpha/e)$ minima

$(0, \pm 1, \beta/e)$ maxima

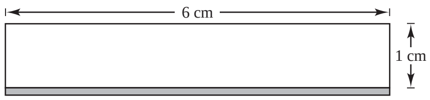
$(0, 0, 0)$ saddle

14. Given that f is a differentiable function such that $\nabla f(x_0, y_0) = \mathbf{0}$, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Therefore, the tangent plane is $-(z - z_0) = 0$ or $z = z_0 = f(x_0, y_0)$ which is horizontal.

15. (a)



(b)



(c) The height has more effect since the shaded region in (b) is larger than the shaded region in (a).

(d) $A = hl \Rightarrow dA = l\,dh + h\,dl$

If $dl = 0.01$ and $dh = 0$, then $dA = 1(0.01) = 0.01$.

If $dh = 0.01$ and $dl = 0$, then $dA = 6(0.01) = 0.06$.

16. Let $g(x, y) = yf\left(\frac{x}{y}\right)$.

$$g_y(x, y) = f\left(\frac{x}{y}\right) + yf'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) = f\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right)$$

$$g_x(x, y) = yf'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) = f'\left(\frac{x}{y}\right)$$

Tangent plane at (x_0, y_0, z_0) is $f'\left(\frac{x_0}{y_0}\right)(x - x_0) + \left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right)\right](y - y_0) - 1\left(z - y_0 f\left(\frac{x_0}{y_0}\right)\right) = 0$

$$\Rightarrow f'\left(\frac{x_0}{y_0}\right)x + \left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right)\right]y - z = 0.$$

This plane passes through the origin, the common point of intersection.

17. $\frac{\partial u}{\partial t} = \frac{1}{2}\left[-\cos(x - t) + \cos(x + t)\right]$

$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\left[-\sin(x - t) - \sin(x + t)\right]$

$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\cos(x - t) + \cos(x + t)\right]$

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left[-\sin(x - t) - \sin(x + t)\right]$

Then, $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$.

18. $u(x, t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right]$

Let $r = x - ct$ and $s = x + ct$.

Then $u(r, s) = \frac{1}{2}\left[f(r) + f(s)\right]$.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \frac{1}{2}\frac{df}{dr}(-c) + \frac{1}{2}\frac{df}{ds}(c)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\frac{d^2 f}{dr^2}(-c)^2 + \frac{1}{2}\frac{d^2 f}{ds^2}(c)^2 = \frac{c^2}{2}\left[\frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2}\right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{1}{2}\frac{df}{dr}(1) + \frac{1}{2}\frac{df}{ds}(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{d^2 f}{2dr^2}(1)^2 + \frac{1}{2}\frac{d^2 f}{ds^2}(1)^2 = \frac{1}{2}\left[\frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2}\right]$$

So, $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.

19. $w = f(x, y),\ x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

(a)

$$\begin{aligned}
r\cos\theta\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}r\cos^2\theta + \frac{\partial w}{\partial y}r\sin\theta\cos\theta\\
-\sin\theta\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}(r\sin^2\theta) - \frac{\partial w}{\partial y}r\sin\theta\cos\theta\\
r\cos\theta\frac{\partial w}{\partial r} - \sin\theta\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}(r\cos^2\theta + r\sin^2\theta)\\
r\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial r}(r\cos\theta) - \frac{\partial w}{\partial \theta}\sin\theta\\
\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r} \quad \text{(First Formula)}\\
r\sin\theta\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}r\sin\theta\cos\theta + \frac{\partial w}{\partial y}r\sin^2\theta\\
\cos\theta\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}(-r\sin\theta\cos\theta) + \frac{\partial w}{\partial y}(r\cos^2\theta)\\
r\sin\theta\frac{\partial w}{\partial r} + \cos\theta\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial y}(r\sin^2\theta + r\cos^2\theta)\\
r\frac{\partial w}{\partial y} &= \frac{\partial w}{\partial r}r\sin\theta + \frac{\partial w}{\partial \theta}\cos\theta\\
\frac{\partial w}{\partial y} &= \frac{\partial w}{\partial r}\sin\theta + \frac{\partial w}{\partial \theta}\frac{\cos\theta}{r} \quad \text{(Second Formula)}
\end{aligned}$$

(b)

$$\begin{aligned}
\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 &= \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta\\
&\quad - 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2
\end{aligned}$$

20. $w = \arctan\dfrac{y}{x},\ x = r\cos\theta,\ y = r\sin\theta$

$$= \arctan\left(\frac{r\sin\theta}{r\cos\theta}\right) = \arctan(\tan\theta) = \theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{\partial w}{\partial x} = \frac{-y}{x^2 + y^2},\ \frac{\partial w}{\partial y} = \frac{x}{x^2 + y^2},\ \frac{\partial w}{\partial r} = 0,\ \frac{\partial w}{\partial \theta} = 1$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \frac{y^2}{(x^2 + y^2)^2} + \frac{x^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{r^2}$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2 = 0 + \frac{1}{r^2}(1) = \frac{1}{r^2}$$

So, $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = \left(\dfrac{\partial w}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial w}{\partial \theta}\right)^2.$

21. $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\frac{\partial u}{\partial\theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial\theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta \text{ Similarly,}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

$$\frac{\partial^2 u}{\partial\theta^2} = (-r\sin\theta)\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial\theta} + \frac{\partial^2 u}{\partial x\partial z}\frac{\partial z}{\partial\theta}\right] - r\frac{\partial u}{\partial x}\cos\theta + (r\cos\theta)\left[\frac{\partial^2 u}{\partial y\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\theta} + \frac{\partial^2 u}{\partial y\partial z}\frac{\partial z}{\partial\theta}\right] - r\frac{\partial u}{\partial y}\sin\theta$$

$$= \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - 2\frac{\partial^2 u}{\partial x\partial y}r^2\sin\theta\cos\theta - \frac{\partial u}{\partial x}r\cos\theta - \frac{\partial u}{\partial y}r\sin\theta$$

Similarly, $\dfrac{\partial^2 u}{\partial r^2} = \dfrac{\partial^2 u}{\partial x^2}\cos^2\theta + \dfrac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\dfrac{\partial^2 u}{\partial x\partial y}\cos\theta\sin\theta.$

Now observe that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} = \left[\frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\cos\theta\sin\theta\right] + \frac{1}{r}\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]$$

$$+\left[\frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta - \frac{1}{r}\frac{\partial u}{\partial x}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial y}\sin\theta\right] + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

So, Laplace's equation in cylindrical coordinates, is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$