

## Chapter 07.03

### Simpson's 1/3 Rule of Integration

*After reading this chapter, you should be able to*

1. *derive the formula for Simpson's 1/3 rule of integration,*
2. *use Simpson's 1/3 rule to solve integrals,*
3. *develop the formula for multiple-segment Simpson's 1/3 rule of integration,*
4. *use multiple-segment Simpson's 1/3 rule of integration to solve integrals, and*
5. *derive the true error formula for multiple-segment Simpson's 1/3 rule.*

#### What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson's 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson's 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$  is called the integrand,

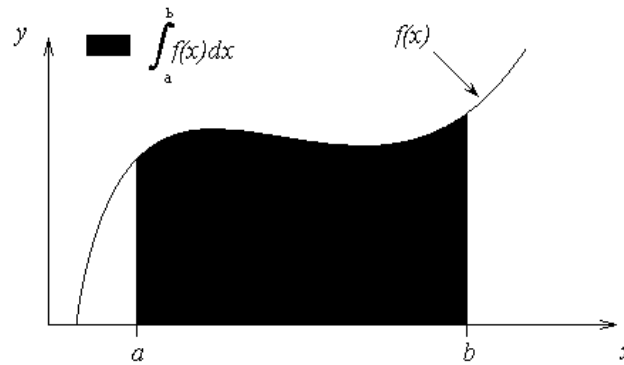
$a$  = lower limit of integration

$b$  = upper limit of integration

#### Simpson's 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an

extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.



**Figure 1** Integration of a function

Method 1:

Hence

$$I = \int_a^b f(x) dx \approx \int_a^b f_2(x) dx$$

where  $f_2(x)$  is a second order polynomial given by

$$f_2(x) = a_0 + a_1x + a_2x^2.$$

Choose

$$(a, f(a)), \left( \frac{a+b}{2}, f\left(\frac{a+b}{2}\right) \right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate  $a_0$ ,  $a_1$  and  $a_2$ .

$$f(a) = f_2(a) = a_0 + a_1a + a_2a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1b + a_2b^2$$

Solving the above three equations for unknowns,  $a_0$ ,  $a_1$  and  $a_2$  give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$\begin{aligned} I &\approx \int_a^b f_2(x) dx \\ &= \int_a^b (a_0 + a_1x + a_2x^2) dx \\ &= \left[ a_0x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \right]_a^b \\ &= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3} \end{aligned}$$

Substituting values of  $a_0$ ,  $a_1$  and  $a_2$  give

$$\int_a^b f_2(x) dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since for Simpson 1/3 rule, the interval  $[a, b]$  is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x) dx \approx \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

#### Method 2:

Simpson's 1/3 rule can also be derived by approximating  $f(x)$  by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

$$b_2 = \frac{\frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} - \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}}{b - a}$$

Integrating Newton's divided difference polynomial gives us

$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b f_2(x) dx \\ &= \int_a^b \left[ b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right) \right] dx \\ &= \left[ b_0 x + b_1 \left( \frac{x^2}{2} - ax \right) + b_2 \left( \frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2} \right) \right]_a^b \\ &= b_0(b-a) + b_1 \left( \frac{b^2 - a^2}{2} - a(b-a) \right) \\ &\quad + b_2 \left( \frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2} \right) \end{aligned}$$

Substituting values of  $b_0$ ,  $b_1$ , and  $b_2$  into this equation yields the same result as before

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ &= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \end{aligned}$$

### Method 3:

One could even use the Lagrange polynomial to derive Simpson's formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x) = \frac{\left(x - \frac{a+b}{2}\right)(x-b)}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{(x-a)(x-b)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)\left(x - \frac{a+b}{2}\right)}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)$$

Integrating this function gets

$$\begin{aligned}
\int_a^b f_2(x) dx &= \left[ \frac{\frac{x^3}{3} - \frac{(a+3b)x^2}{4} + \frac{b(a+b)x}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{\frac{x^3}{3} - \frac{(a+b)x^2}{2} + abx}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) \right]_a^b \\
&\quad + \left[ \frac{\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b) \right]_a^b \\
&= \frac{\frac{b^3 - a^3}{3} - \frac{(a+3b)(b^2 - a^2)}{4} + \frac{b(a+b)(b-a)}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) \\
&\quad + \frac{\frac{b^3 - a^3}{3} - \frac{(a+b)(b^2 - a^2)}{2} + ab(b-a)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{\frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)
\end{aligned}$$

Believe it or not, simplifying and factoring this large expression yields you the same result as before

$$\begin{aligned}
\int_a^b f(x) dx &\approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\
&= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].
\end{aligned}$$

#### Method 4:

Simpson's 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x) dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals  $\int_a^b 1 dx$ ,  $\int_a^b x dx$ , and  $\int_a^b x^2 dx$ . This

implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\int_a^b 1 dx = b - a = c_1 + c_2 + c_3$$

$$\int_a^b x dx = \frac{b^2 - a^2}{2} = c_1 a + c_2 \frac{a+b}{2} + c_3 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 \left( \frac{a+b}{2} \right)^2 + c_3 b^2$$

Solving the above three equations for  $c_0$ ,  $c_1$  and  $c_2$  give

$$c_1 = \frac{b-a}{6}$$

$$c_2 = \frac{2(b-a)}{3}$$

$$c_3 = \frac{b-a}{6}$$

This gives

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b) \\ &= \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ &= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \end{aligned}$$

The integral from the first method

$$\int_a^b f(x) dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2) dx$$

can be viewed as the area under the second order polynomial, while the equation from Method 4

$$\int_a^b f(x) dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

can be viewed as the sum of the areas of three rectangles.

### Example 1

The distance covered by a rocket in meters from  $t = 8s$  to  $t = 30s$  is given by

$$x = \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

- Use Simpson's 1/3 rule to find the approximate value of  $x$ .
- Find the true error,  $E_t$ .
- Find the absolute relative true error,  $|\epsilon_t|$ .

**Solution**

$$a) \quad x \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$a = 8$$

$$b = 30$$

$$\frac{a+b}{2} = 19$$

$$f(t) = 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t$$

$$f(8) = 2000 \ln \left[ \frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27 \text{ m/s}$$

$$f(30) = 2000 \ln \left[ \frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 \text{ m/s}$$

$$f(19) = 2000 \ln \left( \frac{140000}{140000 - 2100(19)} \right) - 9.8(19) = 484.75 \text{ m/s}$$

$$x \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$= \left( \frac{30-8}{6} \right) [f(8) + 4f(19) + f(30)]$$

$$= \frac{22}{6} [177.27 + 4 \times 484.75 + 901.67]$$

$$= 11065.72 \text{ m}$$

b) The exact value of the above integral is

$$x = \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

$$= 11061.34 \text{ m}$$

So the true error is

$$E_t = \text{True Value} - \text{Approximate Value}$$

$$= 11061.34 - 11065.72$$

$$= -4.38 \text{ m}$$

c) The absolute relative true error is

$$|\epsilon_t| = \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100$$

$$= \left| \frac{-4.38}{11061.34} \right| \times 100$$

$$= 0.0396\%$$

### Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval  $[a, b]$  into  $n$  segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that  $n$  needs to be even. Divide interval  $[a, b]$  into  $n$  equal segments, so that the segment width is given by

$$h = \frac{b - a}{n}.$$

Now

$$\int_a^b f(x) dx = \int_{x_0}^{x_n} f(x) dx$$

where

$$x_0 = a$$

$$x_n = b$$

$$\int_a^b f(x) dx = \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots + \int_{x_{n-4}}^{x_{n-2}} f(x) dx + \int_{x_{n-2}}^{x_n} f(x) dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\begin{aligned} \int_a^b f(x) dx &\cong (x_2 - x_0) \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + (x_4 - x_2) \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \dots \\ &+ (x_{n-2} - x_{n-4}) \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + (x_n - x_{n-2}) \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right] \end{aligned}$$

Since

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, \dots, n$$

then

$$\begin{aligned} \int_a^b f(x) dx &\cong 2h \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + 2h \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \dots \\ &+ 2h \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + 2h \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right] \\ &= \frac{h}{3} [f(x_0) + 4\{f(x_1) + f(x_3) + \dots + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + \dots + f(x_{n-2})\} + f(x_n)] \end{aligned}$$



$$= \frac{h}{3} \left[ f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\int_a^b f(x) dx \cong \frac{b-a}{3n} \left[ f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right]$$

**Example 2**

Use 4-segment Simpson's 1/3 rule to approximate the distance covered by a rocket in meters from  $t = 8$  s to  $t = 30$  s as given by

$$x = \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

- Use four segment Simpson's 1/3rd Rule to estimate  $x$ .
- Find the true error,  $E_t$  for part (a).
- Find the absolute relative true error,  $|\epsilon_t|$  for part (a).

**Solution:**

- a) Using  $n$  segment Simpson's 1/3 rule,

$$x \approx \frac{b-a}{3n} \left[ f(t_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(t_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(t_i) + f(t_n) \right]$$

$$n = 4$$

$$a = 8$$

$$b = 30$$

$$h = \frac{b-a}{n}$$

$$= \frac{30-8}{4}$$

$$= 5.5$$

$$f(t) = 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t$$

So

$$f(t_0) = f(8)$$

$$f(8) = 2000 \ln \left[ \frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27 \text{ m/s}$$

$$f(t_1) = f(8 + 5.5) = f(13.5)$$

$$f(13.5) = 2000 \ln \left[ \frac{140000}{140000 - 2100(13.5)} \right] - 9.8(13.5) = 320.25 \text{ m/s}$$

$$f(t_2) = f(13.5 + 5.5) = f(19)$$

$$f(19) = 2000 \ln \left( \frac{140000}{140000 - 2100(19)} \right) - 9.8(19) = 484.75 \text{ m/s}$$

$$f(t_3) = f(19 + 5.5) = f(24.5)$$

$$f(24.5) = 2000 \ln \left[ \frac{140000}{140000 - 2100(24.5)} \right] - 9.8(24.5) = 676.05 \text{ m/s}$$

$$f(t_4) = f(t_n) = f(30)$$

$$f(30) = 2000 \ln \left[ \frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 \text{ m/s}$$

$$\begin{aligned} x &= \frac{b-a}{3n} \left[ f(t_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(t_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(t_i) + f(t_n) \right] \\ &= \frac{30-8}{3(4)} \left[ f(8) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^3 f(t_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^2 f(t_i) + f(30) \right] \\ &= \frac{22}{12} [f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30)] \\ &= \frac{11}{6} [f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30)] \\ &= \frac{11}{6} [177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67] \\ &= 11061.64 \text{ m} \end{aligned}$$

b) The exact value of the above integral is

$$\begin{aligned} x &= \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \\ &= 11061.34 \text{ m} \end{aligned}$$

So the true error is

$$E_t = \text{True Value} - \text{Approximate Value}$$

$$E_t = 11061.34 - 11061.64$$

$$= -0.30 \text{ m}$$

c) The absolute relative true error is

$$\begin{aligned}
 |\epsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \\
 &= \left| \frac{-0.3}{11061.34} \right| \times 100 \\
 &= 0.0027\%
 \end{aligned}$$

**Table 1** Values of Simpson's 1/3 rule for Example 2 with multiple-segments

$n$	Approximate Value	$E_t$	$ \epsilon_t $
2	11065.72	-4.38	0.0396%
4	11061.64	-0.30	0.0027%
6	11061.40	-0.06	0.0005%
8	11061.35	-0.02	0.0002%
10	11061.34	-0.01	0.0001%

### Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3rd Rule is given<sup>1</sup> by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the  $n$  segments Simpson's 1/3rd Rule is given by

$$\begin{aligned}
 E_1 &= -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2 \\
 &= -\frac{h^5}{90} f^{(4)}(\zeta_1) \\
 E_2 &= -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4 \\
 &= -\frac{h^5}{90} f^{(4)}(\zeta_2) \\
 &\vdots \\
 E_i &= -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i} \\
 &= -\frac{h^5}{90} f^{(4)}(\zeta_i) \\
 &\vdots
 \end{aligned}$$

---

<sup>1</sup> The  $f^{(4)}$  in the true error expression stands for the fourth derivative of the function  $f(x)$ .

$$\begin{aligned}
 E_{\frac{n-1}{2}} &= -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n-1}{2}}\right), \quad x_{n-4} < \zeta_{\frac{n-1}{2}} < x_{n-2} \\
 &= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n-1}{2}}\right) \\
 E_{\frac{n}{2}} &= -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), \quad x_{n-2} < \zeta_{\frac{n}{2}} < x_n
 \end{aligned}$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$\begin{aligned}
 &= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right) \\
 E_t &= \sum_{i=1}^{\frac{n}{2}} E_i \\
 &= -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \\
 &= -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \\
 &= -\frac{(b-a)^5}{180n^4} \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}},
 \end{aligned}$$

The term  $\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}}$  is an approximate average value of  $f^{(4)}(x)$ ,  $a < x < b$ . Hence

$$E_t = -\frac{(b-a)^5}{180n^4} \bar{f}^{(4)}$$

where

$$\bar{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}}$$

---

#### INTEGRATION

---

Topic	Simpson's 1/3 rule
Summary	Textbook notes of Simpson's 1/3 rule
Major	General Engineering
Authors	Autar Kaw, Michael Keteltas

---

---

Date	December 3, 2017
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

---