Trees

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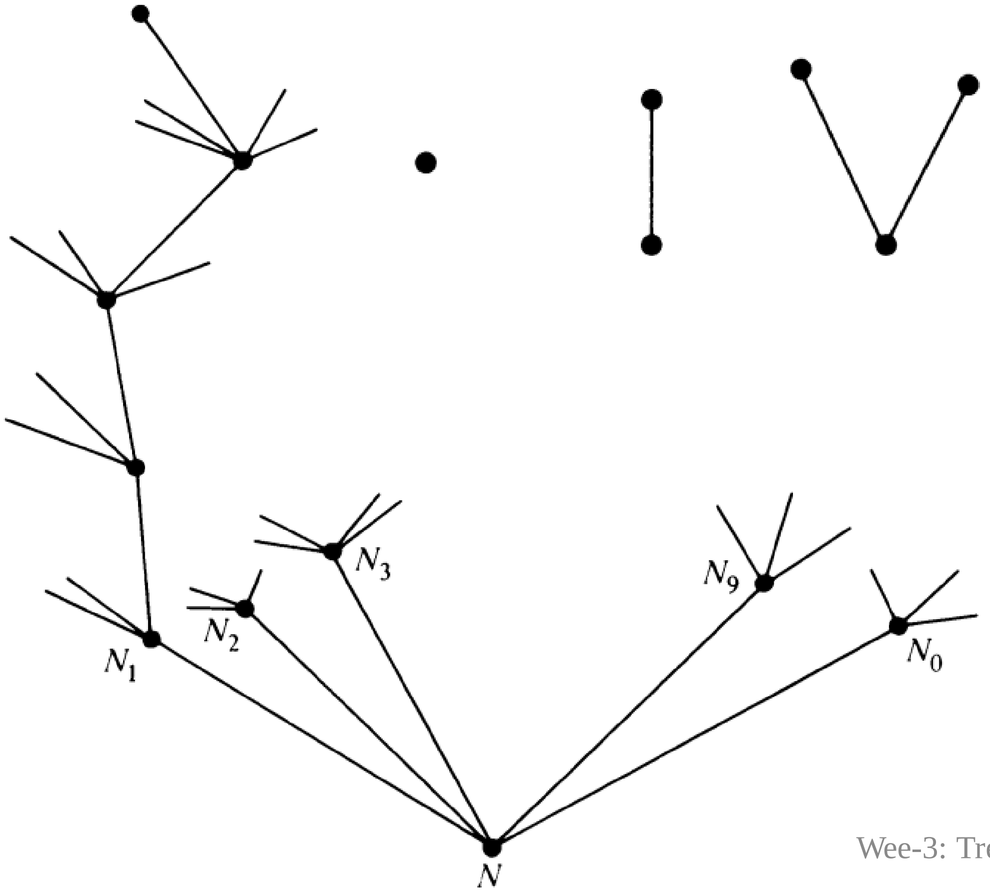
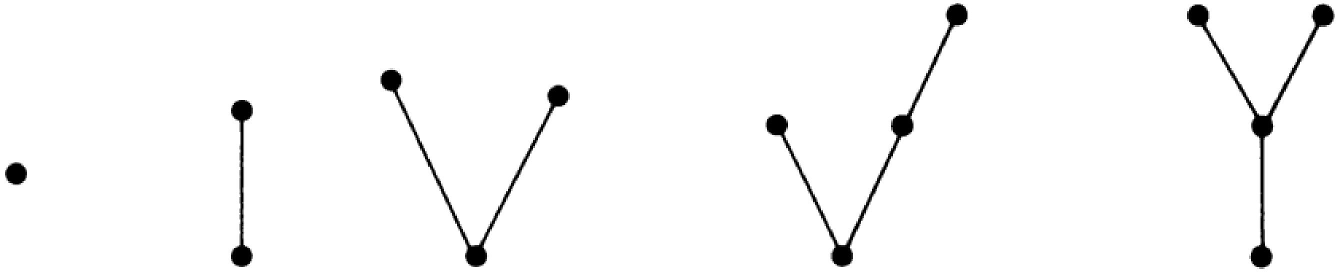
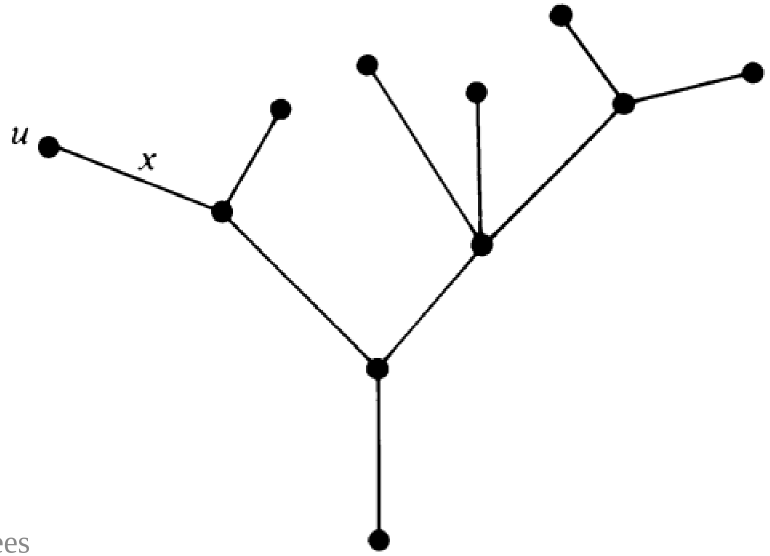
Definition

- A connected acyclic graph







Properties of Tree

- Theorem 3.1 – There is one and only one path between every pair of vertices in a tree, *T*.
- Theorem 3.2 – If in a graph *G* there is one and only one path between every pair of vertices, *G* is a tree.
- Theorem 3.3 – A tree with *n* vertices has *n-1* edges.

Properties of Tree

- Theorem 3.4 – Any connected graph with *n* vertices and *n-1* edges is a tree.

- Theorem 3.5 – A graph is a tree if and only if it is minimally connected.

- Theorem 3.6 – A graph *G* with n vertices, *n-1* edges and no circuits is connected.

Summary of the Properties

- A graph *G* with *n* vertices is called a tree if –
  - *G* is *connected* & is *circuitless*, or
  - *G* is *connected* & has *n-1* edges, or
  - *G* is *circuitless* and has *n-1* edges, or
  - There is *exactly one path* between every pair of vertices in *G*, or
  - *G* is a *minimally connected* graph.

Properties of tree

Question: Cycle in tree. Possible?

Answer:

In graph theory: Nope.

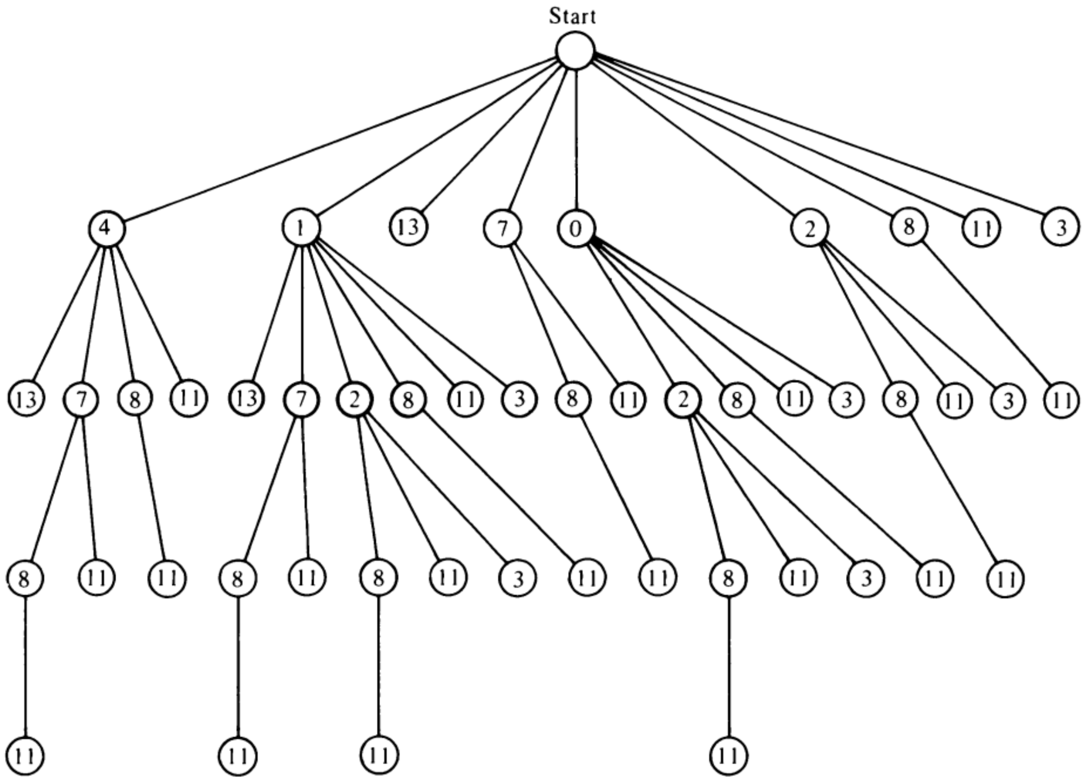
In jungle: May be.

Pendant Vertices in a Tree

- Theorem 3.7 – In any tree (with two or more vertices), there are at least two pendant vertices.
- Finding monotonically increasing

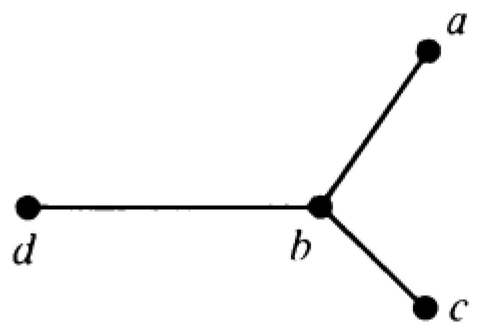
  subsequence of

  4, 1, 13, 7, 0, 2, 8, 11, 3



Distance & Centers in a Tree

- Distance
  - *distance* $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path
- Metric – is a function f that satisfies the conditions below
  - Nonnegativity: $f(x,y) \geq 0$, and $f(x,y) = 0$ if and only if $x = y$.
  - Symmetry: $f(x,y) = f(y,x)$.
  - Triangle inequality: $f(x,y) \leq f(x,z) + f(z,y)$ for any z.



Distance & Centers in a Tree

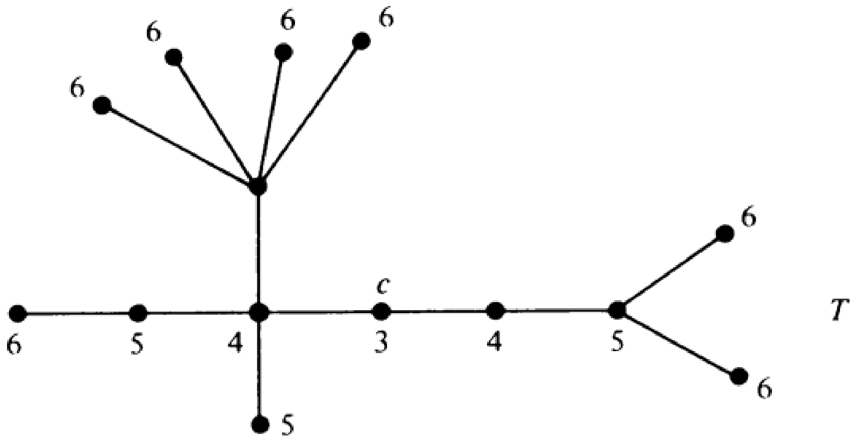
- Theorem 3.8 – The distance between vertices of a connected graph is a metric.
- Eccentricity(also referred to as assosiated number or separation) *E(v)* of a vertex *v* in a graph *G* is distance from *v* to the vertex farthest from *v* in *G*; that is

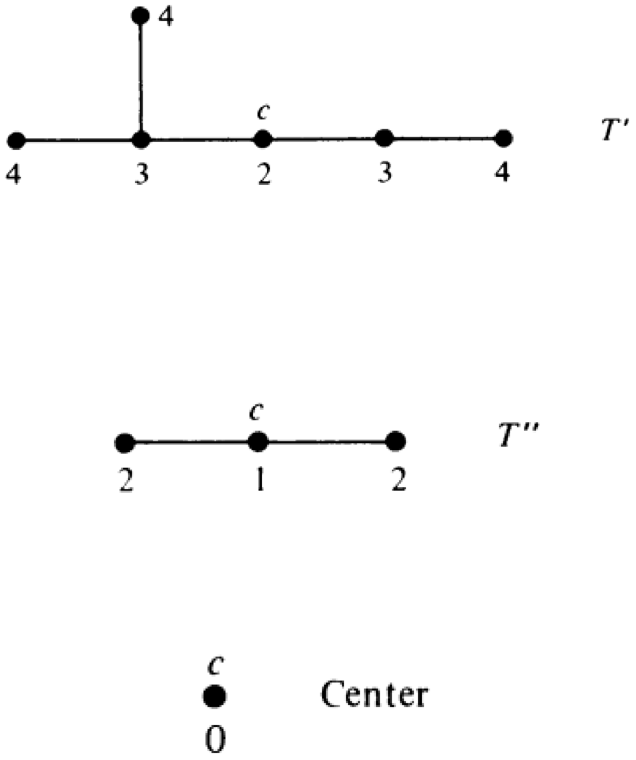
$$E(v) = \max_{v_i \in G} d(v, v_i).$$

- Center : A vertex with minimum eccentricity in graph G is called a center of G.

Distance & Centers in a Tree

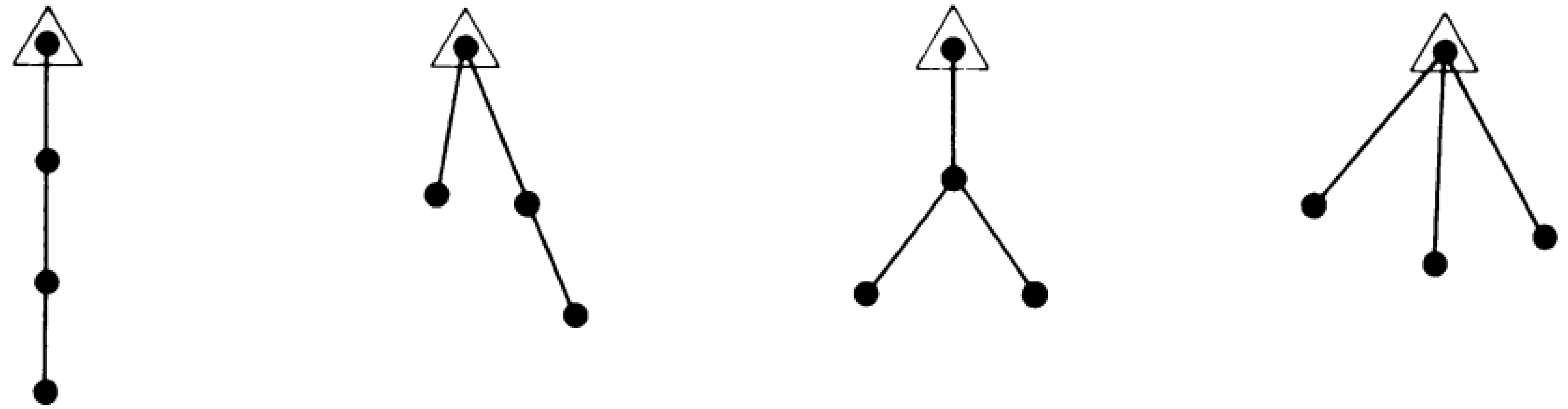
- Theorem 3.9 – Every tree has either one or two centers.
- Radius & Diameter





Rooted & Binary Trees

- Rooted trees with 4 vertices



- Binary Trees
- Internal Vertex

Properties Binary Trees

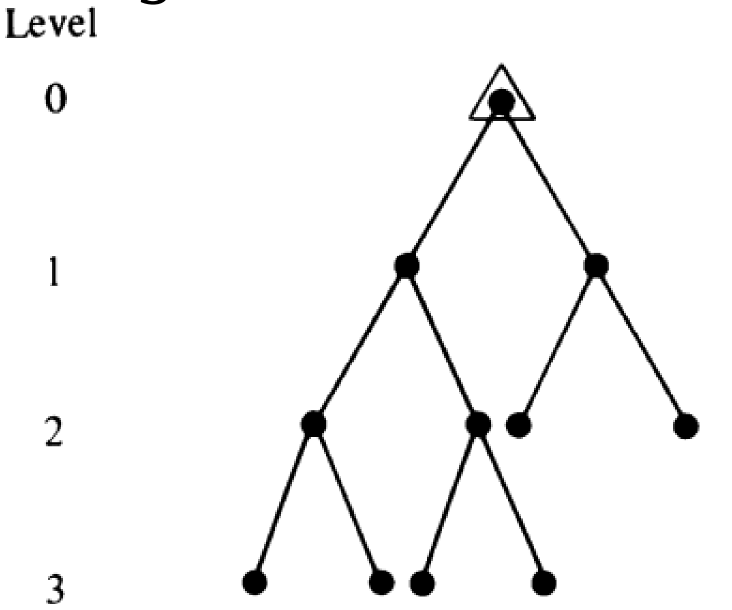
- The number of vertices *n* in a binary tree is always odd. This because there is exactly one vertex of even degree and the remaining *n-1* vertices are of odd degrees. Since the number of vertices of odd degree is even, *n-1* is even. Hence *n* is odd.
- Let *p* be the number of pendant vertices in a binary tree *T*. Then *n-p-1* is the number of vertices of degree three. Therefore the number of edges in *T* equals

$$\frac{1}{2}[p + 3(n - p - 1) + 2] = n - 1$$

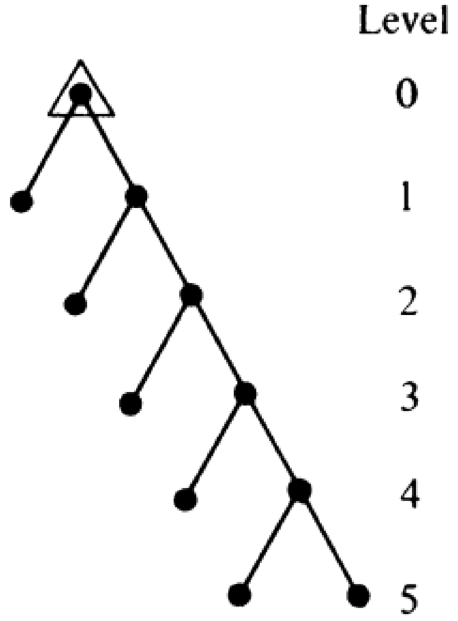
Hence $$p = \frac{n+1}{2}$$

Properties Binary Trees

- Height : The maximum level, l_{max}, of any vertex in a binary tree is called the *height* of the tree.



$\min l_{max} = \lceil (\log_2 12) - 1 \rceil$



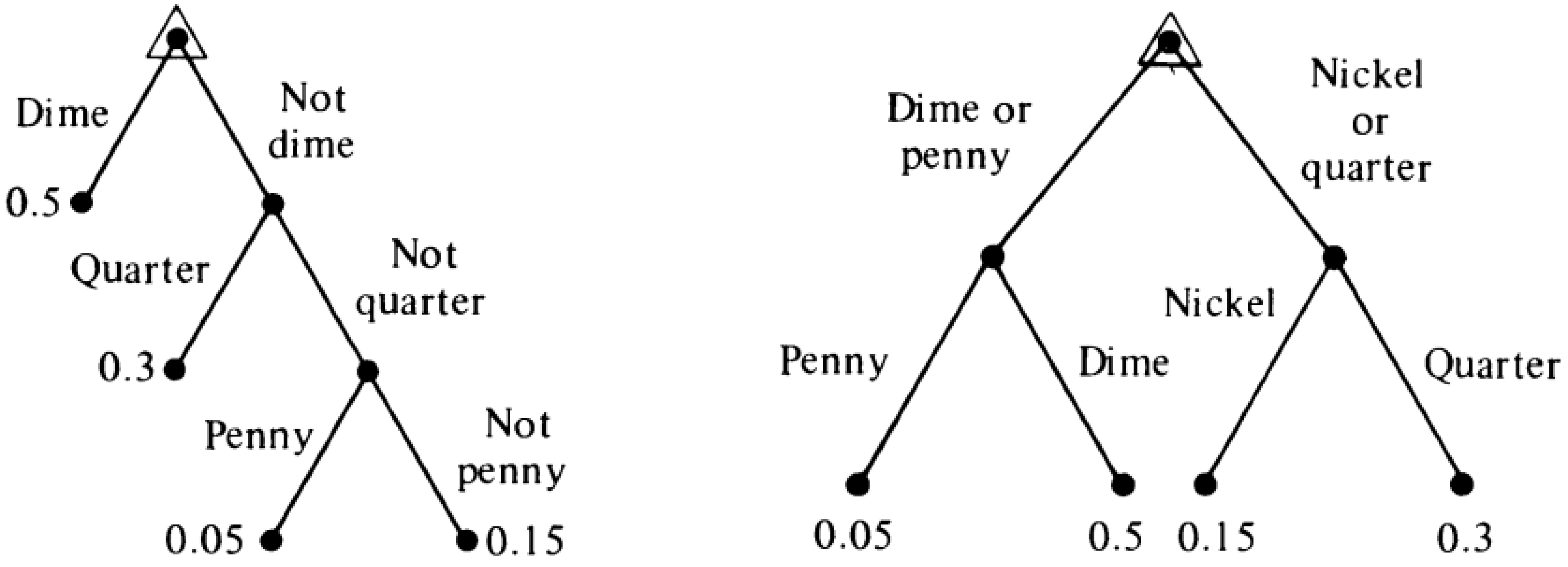
$\max l_{max} = \frac{11 - 1}{2} = 5$

Weighted Path Length

- Minimize: $\sum w_j l_j$
- A Coke machine is to identify, by a sequence of tests, the coin that is put into the machine. Only pennies, nickels, dimes, and quarters can go through the slot. Let us assume that the probabilities of a coin being a penny, a nickel, a dime, and a quarter are .05, .15, .5, and .30, respectively. Each test has the effect of partitioning the four types of coins into two complementary sets and asserting the unknown coin to be in one of the two sets. Thus for four coins we have $2^3 - 1$ such tests. If the time taken for each test is the same, what sequence of tests will minimize the expected time taken by the Coke machine to identify the coin?

Weighted Path Length

- Minimize: $\sum w_j l_j$



$\Sigma w_i \cdot l_i = 1.7$ $\quad$ $\Sigma w_i \cdot l_i = 2$

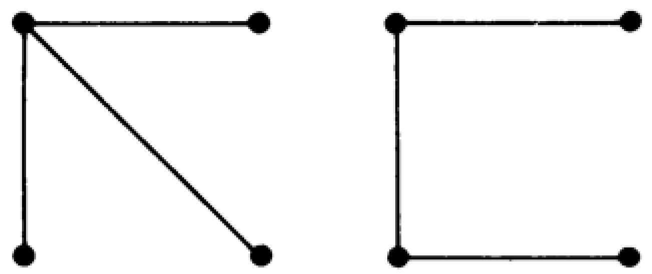
Counting Trees

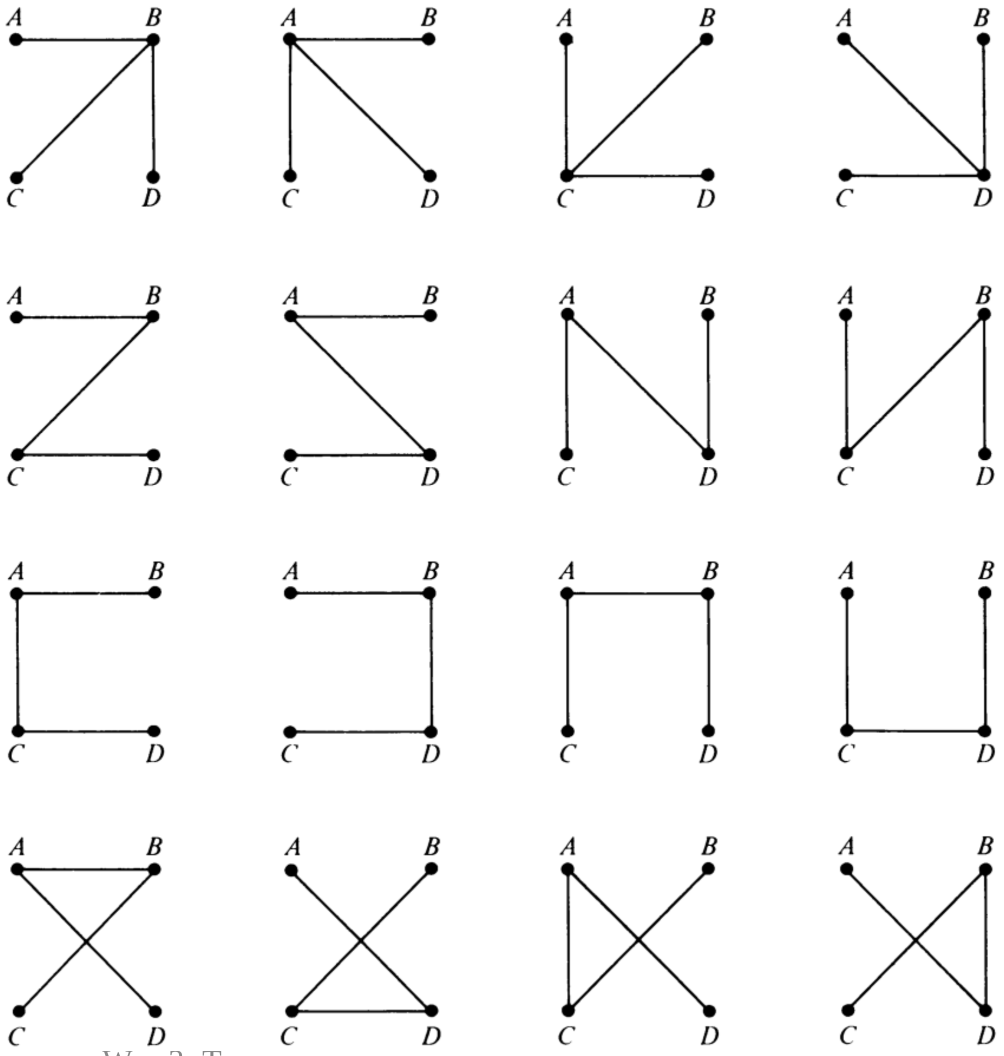
- On 1857, Arthur Cayley discovered tree while studying structural isomers of saturated hydrocarbons C_kH_{2k+2} molecule.
- The total number of vertices in such a graph is n = 3k + 2
- And total number of edge
  - e = .5 x (sum of degree)
  - e = .5 x (4k + 2k +2)
  - e = 3k + 1

This structure is tree.

Counting Trees

- When
  - n = 4
  - 16 labeled trees
  - 2 unlabeled trees



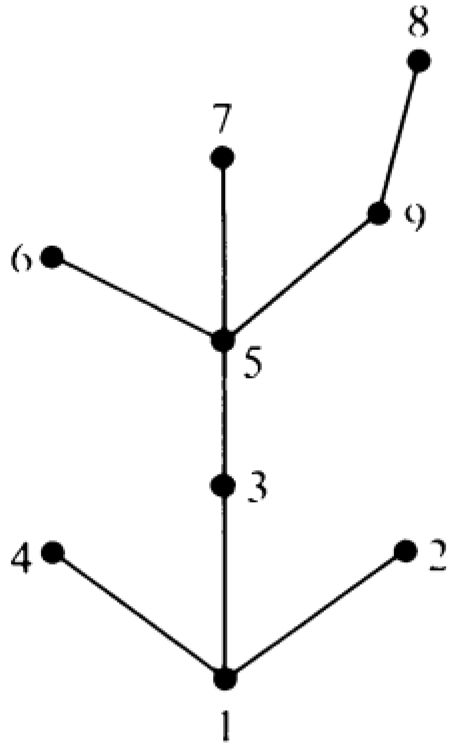


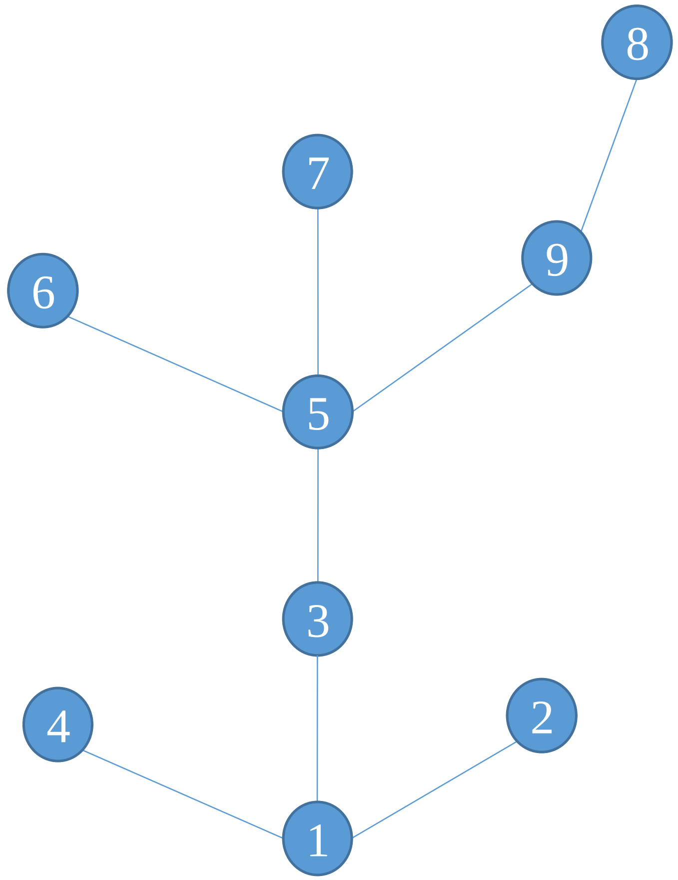
Cayley's Theorem

- Theorem 3.10 – The number of labeled trees with $n(\geq 2)$ vertices is n^{n-2}.
- Proof:
  - Let the n vertices of the tree T be labeled $1, 2, 3, .., n$.
  - Remove the pendant vertex(and the edge incident on it) having the smallest label.
  - Repeat until n-2 steps or only 2 vertices are left.
  - The tree T defines the sequence : $(b_1, b_2, \ldots, b_{n-2})$ uniquely.

Cayley's Theorem

- Remove $a_1 = 2$, get $b_1 = 1$
- Remove $a_2 = 4$, get $b_2 = 1$
- Remove $a_3 = 1$, get $b_3 = 3$
- Remove $a_4 = 3$, get $b_4 = 5$
- Remove $a_5 = 6$, get $b_5 = 5$
- Remove $a_6 = 7$, get $b_6 = 5$
- Remove $a_7 = 5$, get $b_7 = 9$
- Only vertex 8 and 9 are left
- So our sequence of $b = (1,1,3,5,5,5,9)$; is only unique to this labeled graph





Cayley’s Theorem

- Conversely given a sequence *b* of *n-2* labels, an *n*-vertex tree can be constructed uniquely
  - First construct the sequence of $a = (1, 2, 3, \ldots, n)$.
  - Pick the first number in sequence *a* which is not in *b* and pair them.
  - Remove both the vertices from the list.
  - Continue pairing until sequence of *b* has no element left.
  - Finally join the last two remaining vertices of sequence *a*.

Cayley's Theorem

Reconstruction:

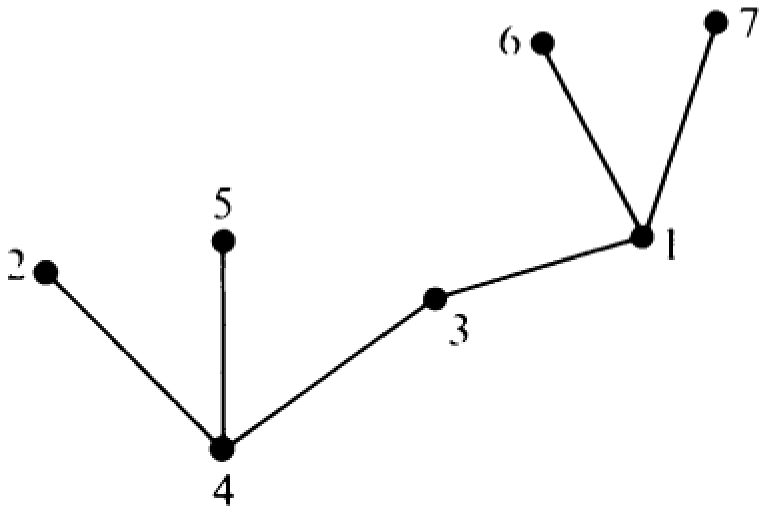
$b = (1,1,3,5,5,5,9)$

$a = (1,2,3,4,5,6,7,8,9)$

**Follow the class lecture for reconstruction process*

Cayley's Theorem & Proof

- For example, given a sequence (4, 4, 3, 1, 1)
  - Given sequence is the sequence of *b*
  - So, we construct $a = (1, 2, 3, 4, 5, 6, 7)$
  - Pair (2, 4) as $a_1 = 2$, $b_1 = 4$
  - Pair (5, 4) as $a_2 = 5$, $b_2 = 4$
  - Pair (4, 3) as $a_3 = 4$, $b_3 = 3$
  - Pair (3, 1) as $a_4 = 3$, $b_4 = 1$
  - Pair (6, 1) as $a_5 = 6$, $b_5 = 1$
  - Finally pair (1, 7)

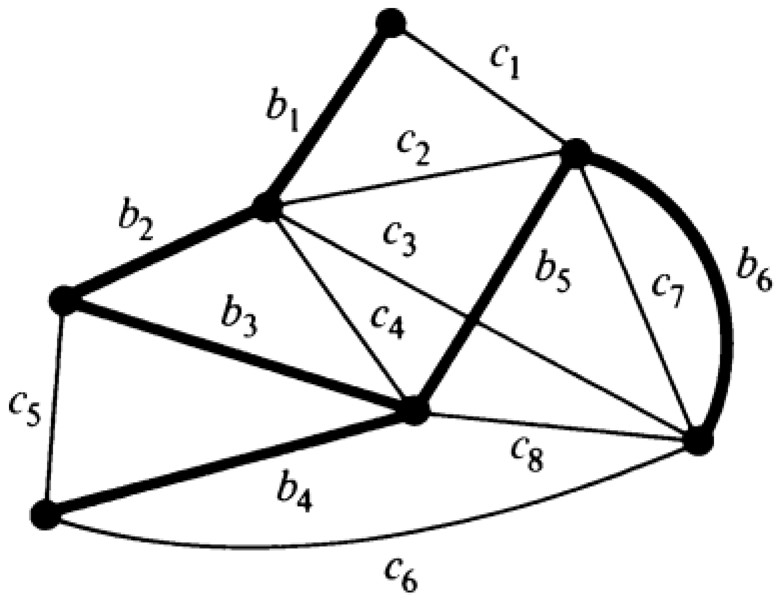


Cayley's Theorem & Proof

- For each of the *n-2* element in sequence *b* we can choose any one of the *n* numbers, thus froming n^{n-2}; (*n*-2) tuples, each defining a distinct labeled tree of *n* vertices.
- Since each tree defines one of the sequence uniquely, there is a 1-to-1 correspondence between the trees and the n^{n-2} sequences.
- Thus the theorem is proved.

Spanning Trees

- A tree T is said to be a spanning tree of connected graph G if T is a subgraph of G and T contains all vertices of G.
- Maxiaml Tree/Subgraph, skeleton
- Sppaning forest of disconnected graph
- Finding Spanning Tree
  - Find Circuits
  - delete an edge from the circuit
  - continue

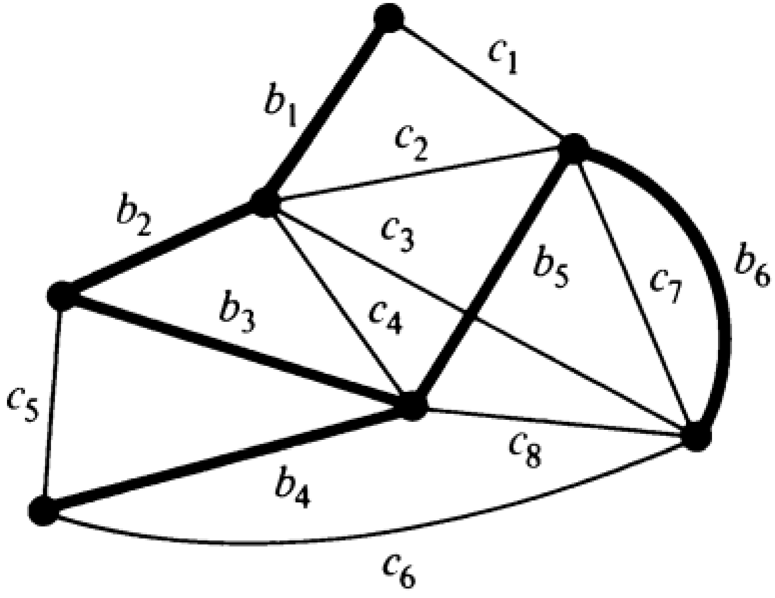


Spanning Trees

- Theroem 3.11 – Every connected graph has at least one spanning tree.

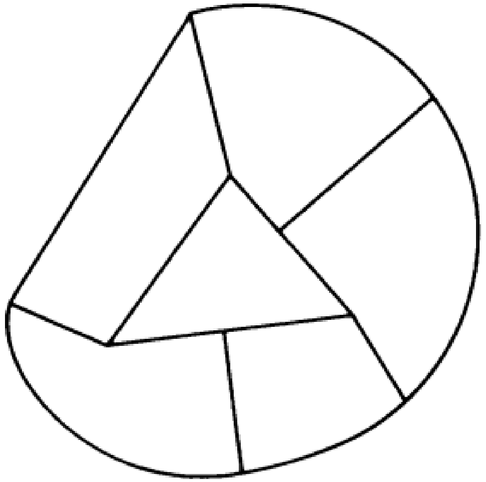
  - Branch (an edge which is part of spanning tree T and graph G),

  - Chord (an edge which is not part of spanning tree T but of graph G)



Spanning Trees

- Theroem 3.12 – With respect to any of its spanning trees, a connected graph of *n* vertices and e edges has *n-1* tree branches and *e-n+1* chords.
  - Electrical network with e edges and n nodes, how to be sure that we don't have a circuit?
  - How many walls have to be broken to drain the lands?



Spanning Trees: Rank & Nullity

- n, e, k are independent of each other except for two constrains
  - n-$k \geq 0$
  - e-n+$k \geq 0$
- Rank, $r = n$-k; number of branches
- Nullity, $\mu = e$-n+k; number of chords
- Rank + Nullity = total number of edges

Fundamental Circuits

- Theroem 3.13 – A connected graph G is a tree if and only if adding an edge between any two vertices in G creates exactly one circuit.
- Consider a spanning tree T in a connected graph G. Adding any chord to T will create exactly one circuit. Such a circuit, formed by adding a chord to a spanning tree, is called a *fundamental circuit*.

- How many fundamental circuits does a graph have?

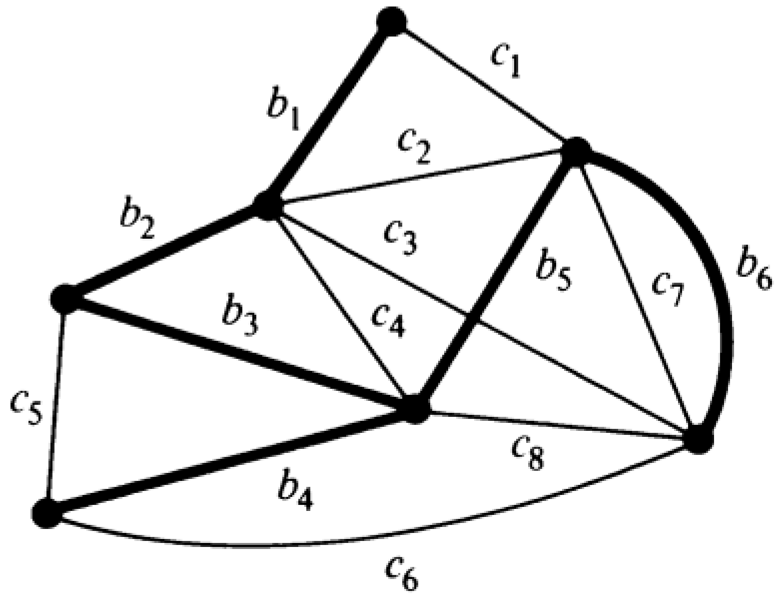
Fundamental Circuits

- Has 8 fundamental circuits
- A circuit may be fundamental with respect to a spanning tree but not other spanning tree of the same graph
- Has 75 total circuits
- A set of edges comprising

  a circuit will not be

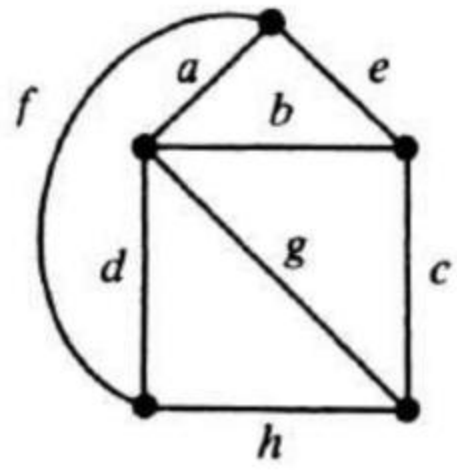
  fundamental if it has

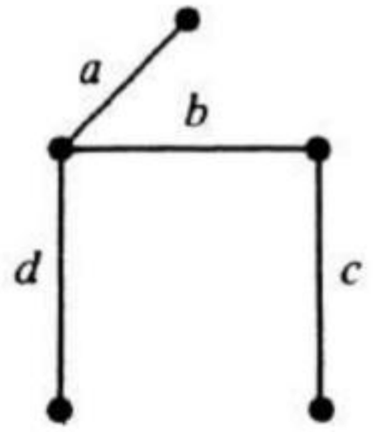
  more than one chord



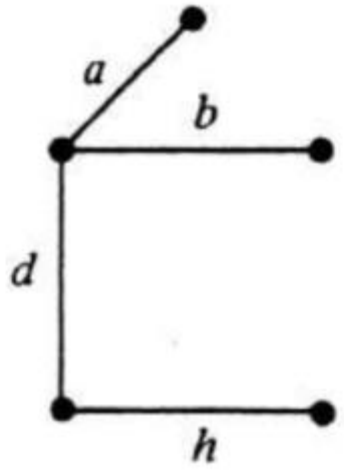
Finding All Spanning Tree

- *Cyclic interchange:*
  - start with a given spanning tree
  - add a chord forming a fundamental circuit
  - delete an appropriate branch

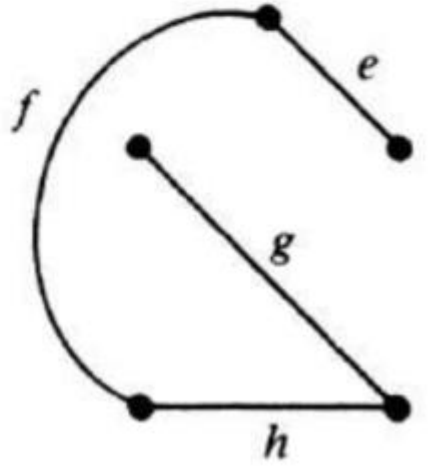




T_1



T_2



T_3

Cyclic interchange

- Can we start from any spanning tree and get a desired spanning tree by a number of *cyclic exchanges*?
- Can we get all spanning trees of a given graph in this fashion?
- How long will we have to continue exchanging edges?
- Is there a preferred spanning tree for starting?

Cyclic interchange

- *The distance between two spanning trees* of a graph G is defined as the number of edges of G present in one tree but not in the other
- Written as $d(T_i, T_j)$

$$d(T_i, T_j) = \frac{1}{2} N(T_i \oplus T_j)$$

*$N(g)$ denote the number of edges in a graph g

Metric

The distance between the spanning trees of a graph is a *metric*. That is, it satisfies-

1. $d(T_i, T_j) \geq 0$ and $d(T_i, T_j) = 0$ if and only if $T_i = T_j$,
2. $d(T_i, T_j) = d(T_j, T_i)$,
3. $d(T_i, T_j) \leq d(T_i, T_k) + d(T_k, T_j)$.

Spanning Trees In A Weighted Graph

- Edges have weights
- Minimal spanning tree
- Kruskal’s or Prim’s algorithm