23-07-20

Lecture - 15:

Determinants:

Linear Transformation: The function that changes a vector space or a system and preserves its linear properties (which are addition and multiplication) is called linear transformation.

Basically it determines how a vector space is stretched or squeezed and hence the word "transformation" is used.

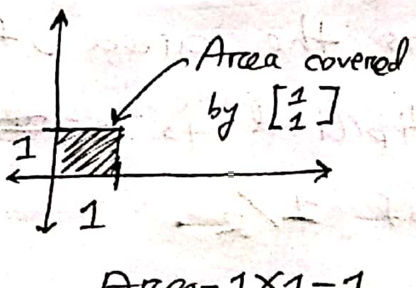
Determinants determine the measure of how we are stretching or squeezing the system, in this case, the area covered by vectors.

Example: $A = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}$ be a matrix. We can already see linear transformation in this matrix.

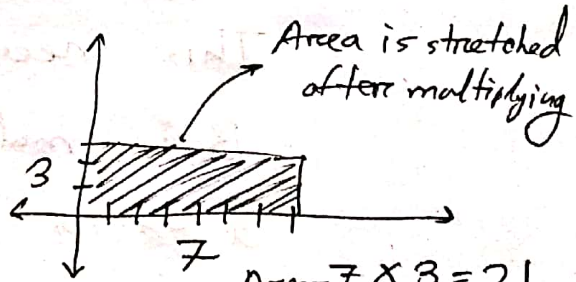
$\begin{bmatrix} 7 \\ 0 \end{bmatrix}$ scales $\hat{i}$ by a factor of 7 ~~aite~~ while

$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ scales $\hat{j}$ by a factor of 3.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} 7 \\ 3 \end{bmatrix}$



Area = $1 \times 1 = 1$



Area = $7 \times 3 = 21$

After multiplying by the matrix of A the area was streatched out by a factor of 21. This factor that stretches/squeezes the area is called determinant.

Here, 21 is the determinant.

Q: What is the significance of determinant?

→ Let's say we have $A=\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ which we will multiply to any matrix.

The determinant is $\det\left(\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\right) = 2-(-1)$
$= 3$

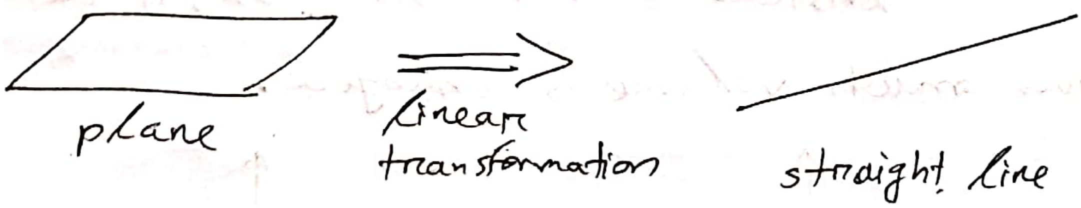
This means that the area of the matrix ~~which~~ to which A is multipled to is STRETCHED BY 3.

Say, for matrix, $B = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.05 \end{bmatrix}$, determinant is 0.5 or 1/2.

This means the area of the matrix to which B is multiplied is ~~squeez~~ SQUEEZED BY a factor of 1/2.

Q: What if a determinant is zero?

→ For matrix, say, $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ the determinant is zero. Further observing this matrix, we can see the columns are Linearly dependent. It looks like a 2D matrix covering 2D space but infact it is squeezed to a single straight line. Hence, area is zero. So if ~~o~~ we multiply this, area will be reduced ~~reduced~~ to zero



Zero determinant case

Q: What do we mean by determinant being negative?

~~A~~ → For $\det\left(\begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}\right) = -5.$

Because of the ~~order ori~~ orientation, determinant sometimes gets a negative value. By multiplying ~~with~~ it to a matrix, the orientation ~~axis~~ will get INVERTED or FLIPPED, i.e. the space will get flipped but the magnitude of stretching or squeezing will be same.

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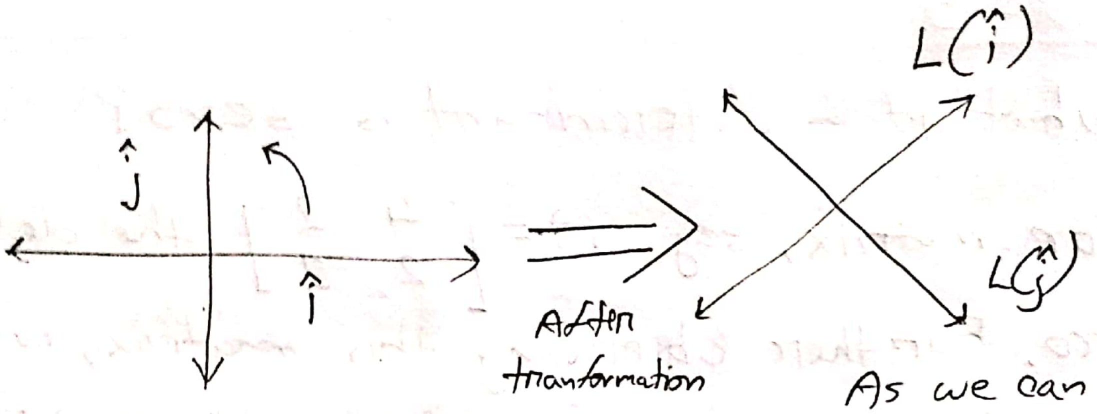
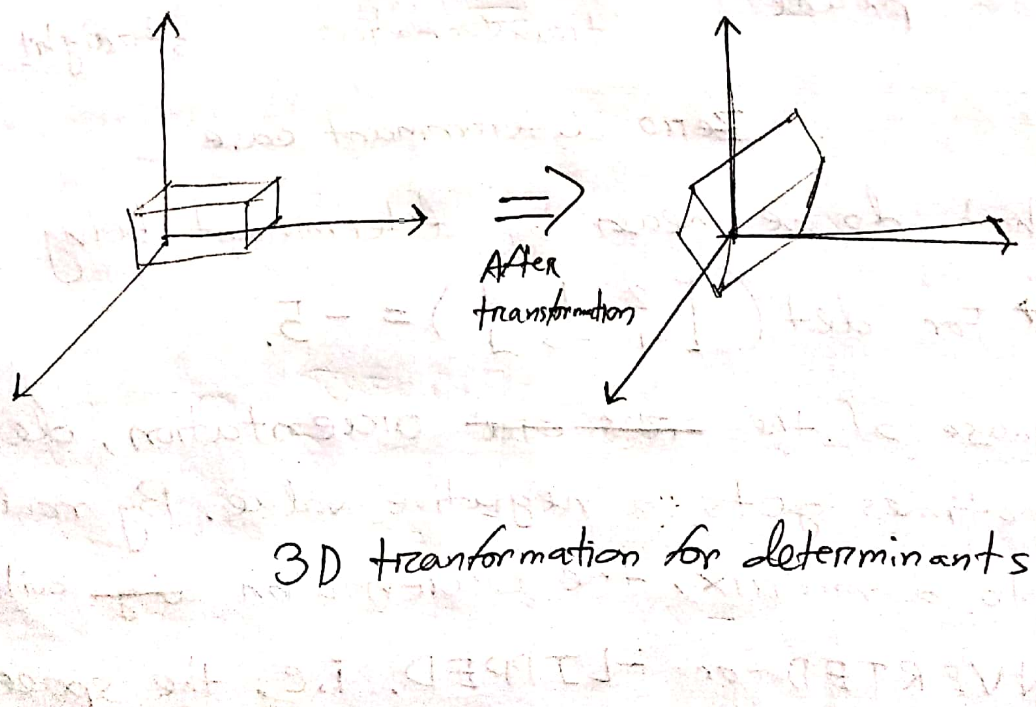


As we can see transformed co-odd axes are flipped.

Negative determinant case

Determinants in 3D matrices:

Instead of Area, in 3D, it determines how much volume is changed.



3D tranformation for determinants

Properties of Determinants:

(i) $\det(I) = 1$ — Determinant of identity matries is 1.

For example $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. For both, $\det(I) = 1$

(2D) (3 D)

(ii) Exchanging 2 rows reverses the sign of determinant.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\text{Det}(A) = 1$ $\text{Det}(A) = -1$.

We see these things in permutation matrices where we change rows once or twice.

That's why for permutation matrices, determinant is either 1 or -1.

(iii) About linear combinations

- a) Multiplying by scalar factor.
- b) Adding a quantity.

(iii) (a) $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[\text{multiplying}]{t} \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

If the original matrix has $\det(A) = ad - bc$
then, transformed matrix has $\det(A) = t(ad-bc)$

(iii) (b) ~~Adding someth~~

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

This property is linearity but it doesn't work like matrices. So, the following statement occurs

$$\longrightarrow \det(A+B) \neq \det(A) + \det(B)$$

(iv) If 2 rows are equal then det is 0.

To prove this, we us property (ii). Let,

Proof: $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. If two rows are equal, the ~~ro~~we can write it as –

[Suppose, row 1 = row 2]

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$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix} \longrightarrow \det(A) = x \text{ (suppose)}$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix} \longrightarrow \det(A) = -x$$ (since we ~~other~~ exchanged rows) [Property - 2]

But, both matrices are same!

So, det (A) has to be positive and negative at the same time. And the only number that solves this is zero.

$\therefore \det(A) = 0$.

(v) While elimination, if $l \times$ row i is subtracted from row k, the determinant doesn't change.

Proof: For,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\text{tranform}]{} \begin{bmatrix} a & b \\ c-la & d-lb \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

~~[But from (iv)~~

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \times 0 \text{ [Property (iv)]}$$

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(vi) All the elements of a row being zero leads to $\det(A) = 0$

Proof: $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0 - lc & 0 - ld \\ c & d \end{vmatrix}$ [from (v)]

$= \begin{vmatrix} -lc & -ld \\ c & d \end{vmatrix}$

$= -l \begin{vmatrix} c & d \\ c & d \end{vmatrix}$

$= -l \times 0$ [∵ from (iv)]

$= 0$

(vii) For any upper triangular matrix, determinant is product of diagonals.

i.e. $\det(U) = \begin{vmatrix} d_1 & x & x \\ 0 & d_2 & x \\ 0 & 0 & d_3 \end{vmatrix} = (d_1) \times (d_2) \times (d_3)$

Proof: $U = \begin{bmatrix} d_1 & x & x \\ 0 & d_2 & x \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ [After some elimination]

$= d_1 d_2 d_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= d_1 \times d_2 \times d_3$ (proved)

(viii) From (vi) and (vii), we can say,

$\det(A)=0$ when A is singular

$\det(A)\neq 0$ when A is invertible

Let, say, $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\xrightarrow{\text{elimination}} \begin{bmatrix} a & b \\ c-\frac{c}{a}a & d-\frac{b}{d}c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{bmatrix}$$

$$= a\times\left(d-\frac{c}{a}b\right)$$ [∵ Upper triangular matrix] [vii]

$\therefore \det(A)=ad-bc$

(ix) $\det(AB)=(\det A)(\det B)$

$\det(A+B)\neq\det(A)+\det(B)$

But what is determinant of any A^{-1}?

We know, $A\cdot A^{-1}=I$

$\Rightarrow \det(A)(\det A^{-1})=1$ [For simplification I is written as 1]

$\Rightarrow \det A^{-1}=\dfrac{1}{\det(A)}$ [A has to be invertible]

(x) $\det(A^T) = \det(A)$

can be transformed into LU

[We know, $A = LU$

$\Rightarrow A^T = U^T L^T$]

$\det(LU) = \det(L) \cdot \det(U)$

$= 1 \cdot x$ [Let, $\det(U)$ be x]

$= x$

$\Rightarrow \det(A) = x$

Again, $\det(A^T) = \det(U^T L^T)$

$= \det(U^T) \cdot \det(L^T)$ [Diagonal elements of transpose of lower triangular matrix will have 1]

$= \det(U^T) \cdot 1$

$= x \cdot 1$

$\Rightarrow \det(A^T) = x$

$\therefore \det(A) = \det(A^T)$

Homework-1: What is det (A^2)?

From (ix), we know,

$$\det(AB) = (\det A)(\det B)$$

If $A = B$, then,

$$\det(A \cdot A) = (\det A) \cdot (\det A)$$
$$\Rightarrow \det(A^2) = (\det A)^2 \text{ (Ans.)}$$

Homework-2: What is det $(2A)$?

If it is a $(n \times n)$ matrix then it will be

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \cdots & \cdots & \cdots \\ a_{13} & \cdots & \cdots & \cdots \\ \vdots & & & \\ a_{1n} & \cdots & \cdots & \cdots \end{bmatrix}$$

From (iii)(a), we get, $\det(A) = t(ad - bc)$ when multiplied by a factor of 2.

So, to get $\det(2A)$, we end up multiplying n number of 2 for the row.

$$\det(2A) = \{2 \times 2 \times 2 \ldots \text{(n times)}\} \times \det(A)$$
$$= 2^n \det(A) \text{ (Ans.)}$$