Book.

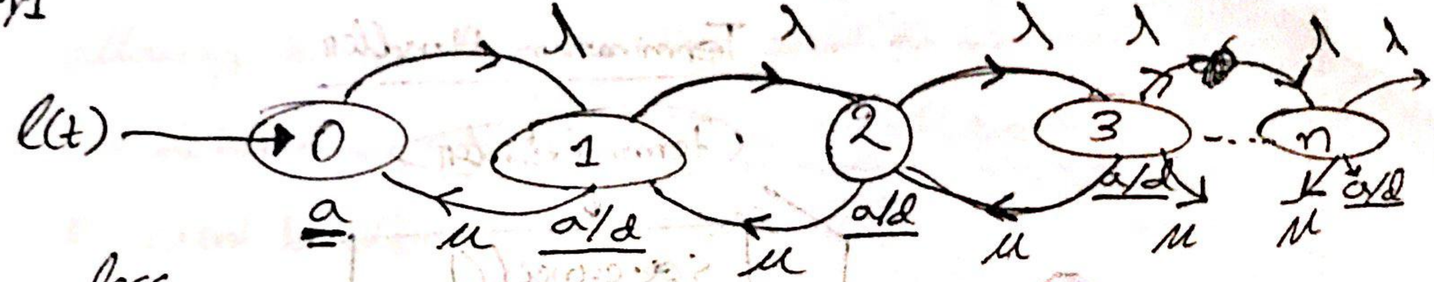
Ross: Introduction to Probability Models

Topic → Continuous time Markov Chain

$\lambda \rightarrow$ arrival

$\mu \rightarrow$ departure

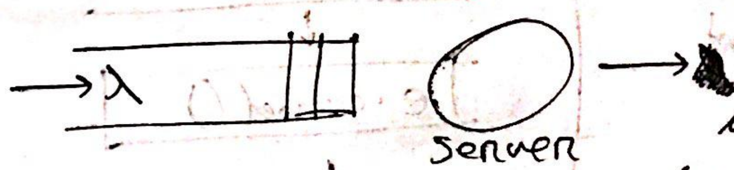
M/M/1



memoryless

State Transition Diagram

Arrival Rate



μ (Departure Rate)

(Poisson)

Interarrival time (exponential)

M/M/1 → Server

Arrival Service

→ Interdeparture time (exponential time)

The given state transition diagram is a form of birth-date process.

$\lambda_i \rightarrow$ arrival rate for i^{th} state

$\mu_i \rightarrow$ departure rate for i^{th} state

Steady State Scenario

$P_i \triangleq$ steady state prob. that system is in state i.

If we run system for 1 hour and it stays at 0 state for 6 minutes then.

$$P_0 = \frac{6\,min}{60\,min} = 10\%$$

(i) Average no. of customers

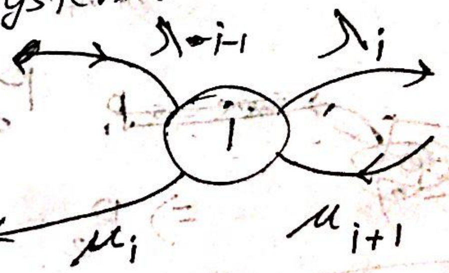
$$P_N(i) = \begin{cases} P_i & i = 0,1,2,3 \ldots \\ 0 & o/w \end{cases}$$

$$E[N] = \sum_{i=0}^{\infty} i \cdot P_i$$

Steady state * (thing in terms of that state only)

(i) Rate at which system enters state i. [Here, $\lambda_{i-1} + \mu_{i+1}$

(ii) Rate at which system Leaves state i. $\lambda_i + \mu_i$]



$i \neq 0$.

For each state,

the rate will be $P_{i-1}\lambda_{i-1} + P_{i+1}\mu_{i+1}$ for entering the state

and $P_i \mu_i + P_i \lambda_i$ for leaving the state.

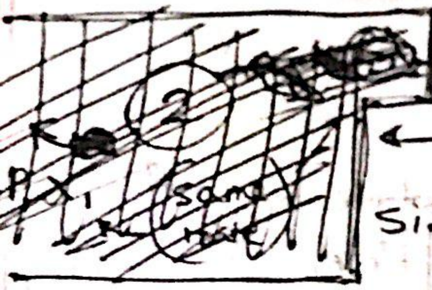
→ As it is a steady state system, this ~~states~~ rates will be equal.

Condition of Stable System

State	Rate of leaving = Rate of entering	
0	$\lambda_0 P_0 = \mu_1 P_1$	— (i)
1	$(\mu_1 + \lambda_1) P_1 = \mu_2 P_2 + \lambda_0 P_0$	— (ii)
2	$(\mu_2 + \lambda_2) P_2 = \mu_3 P_3 + \lambda_1 P_1$	— (iii)
...	...	
n	$(\mu_n + \lambda_n) P = \mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1}$	— (n)

Visualise

Adding, the equations with its previous ones i.e (i with ii, ii with iii, and so on), we get,

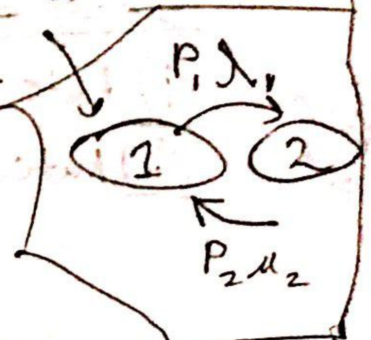


$$\lambda_0 P_0 + \cancel{\mu_1 P_1} + \lambda_1 P_1 = \mu_2 P_2 + \cancel{\lambda_0 P_0} + \cancel{\mu_1 P_1}$$

$$\Rightarrow \lambda_1 P_1 = \mu_2 P_2 \longrightarrow \text{Visualise}$$

Similarly, $\lambda_2 P_2 = \mu_3 P_3$ and so on.

$$\therefore \lambda_n P_n = \mu_{n+1} P_{n+1}$$

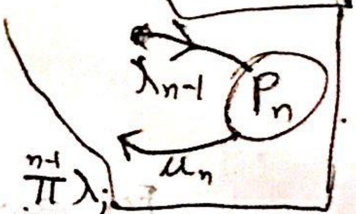


$$\Rightarrow \boxed{P_1 = \frac{\lambda_0 P_0}{\mu_1}}$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 \quad \text{~~same as~~ } P_1$$

$$\cancel{\Rightarrow P_2 = \frac{\lambda_1}{\mu_2} \times \frac{\lambda_0}{\mu_1} \times P_0}$$

$$\Rightarrow P_2 = \frac{\lambda_1}{\mu_2} \times \frac{\lambda_0}{\mu_1} \times P_0$$



$$\therefore \boxed{P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_{n-1}\lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} \times P} = \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} P$$

$$\sum_{n=0}^{\infty} P = 1$$

$$P_0 + P_0 \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} = 1$$

$$\boxed{P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1}} = 1}$$

This sum can't be ∞

If $\lambda_i = \lambda$, $\mu_i = \mu$, we get, $\boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot P_0}$ $\boxed{P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}}$

*

(continued)

#07 (continued)

Analytical Solution of SSQL

B) $\boxed{P_0 = \dfrac{1}{1+\sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n}}$

$\boxed{P_n = \left(\dfrac{\lambda}{\mu}\right)^n \times P_0}$

$$P_0 = \frac{1}{\left(\frac{\lambda}{\mu}\right)^0 + \left\{\left(\frac{\lambda}{\mu}\right)^1 + \left(\frac{\lambda}{\mu}\right)^2 \cdots \left(\frac{\lambda}{\mu}\right)^\infty\right\}}$$

So, the ~~suf~~ necessary condition is

$$\boxed{\frac{\lambda}{\mu} < 1}$$

$$\boxed{\Rightarrow \lambda < \mu}$$

$$P_0 = \frac{1}{\frac{1}{1-\frac{\lambda}{\mu}}} = 1 - \frac{\lambda}{\mu}$$

$\boxed{\therefore P_0 = 1 - \rho}$ ⊛

$\rho \triangleq \dfrac{\lambda}{\mu} \longrightarrow$ Traffic intensity

⊛ $\boxed{P_n = \rho^n (1-\rho)}$

⊛ Expected value, $\boxed{\overline{N} = \sum_{n=0}^{\infty} n \cdot P_n}$

$$= \sum_{n=0}^{\infty} n \rho^n (1-\rho)$$

$$= \frac{\rho}{1-\rho}$$

⊛ System Time, T : Using Little's formula

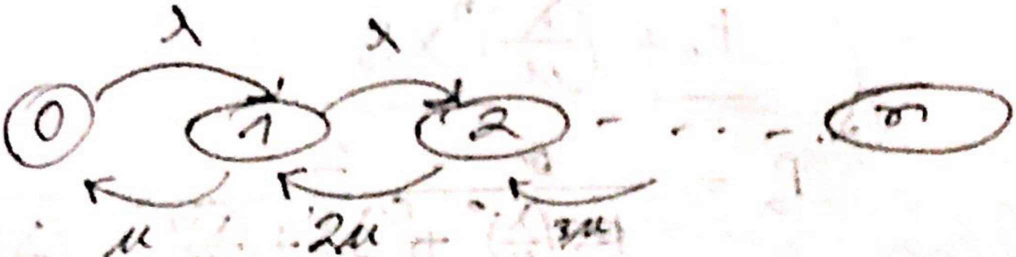
$$\lambda = \frac{\overline{N}}{T}$$

$$\boxed{T = \frac{1}{\lambda}\overline{N}} = \frac{1}{\lambda}\cdot\frac{\rho}{1-\rho} = \frac{1}{\mu(1-\rho)}$$

$\lambda \to$ Rate

$\rho \to \lambda/\mu$

M/M/∞



$$P_n = P_0 \frac{\lambda \cdot \lambda \cdots \lambda}{(n\mu)(n-1)\mu \cdots \mu}$$

$$= P_0 \prod_{i=0}^{n-1} \frac{\lambda}{(i+1)\mu}$$

$$\boxed{P_n = P_0 \left(\frac{\lambda}{\mu}\right)^n \underbrace{\left(\frac{1}{n!}\right)}_{\text{new term}}}$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu_n n!}}$$

$$\left(\frac{\lambda^0}{\mu_0 0!}\right) = \frac{1}{\sum_{n=0}^{\infty} \frac{\lambda^n}{\mu_n n!}} = \frac{1}{e^{\lambda/\mu}} = e^{-\lambda/\mu} = e^{-\rho}$$

$$\therefore \boxed{P_0 = e^{-\rho}} \leftarrow \text{rho}$$

$$\cancel{P_n = e^{-\rho} \rho^n \left(\frac{1}{n!}\right)}$$

$$\boxed{P_n = e^{-\rho} \cdot \rho^n \left(\frac{1}{n!}\right)}$$

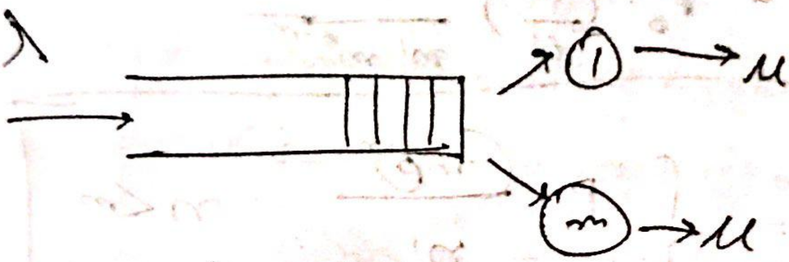
$$P_n = \left(\lambda/\mu\right)^n \cdot \frac{e^{-\lambda/\mu}}{n!}$$

Similar to Poisson distribution

$$\overline{N} = \frac{\lambda}{\mu} \text{ for Poisson.}$$

$$\therefore T = 1/\mu$$

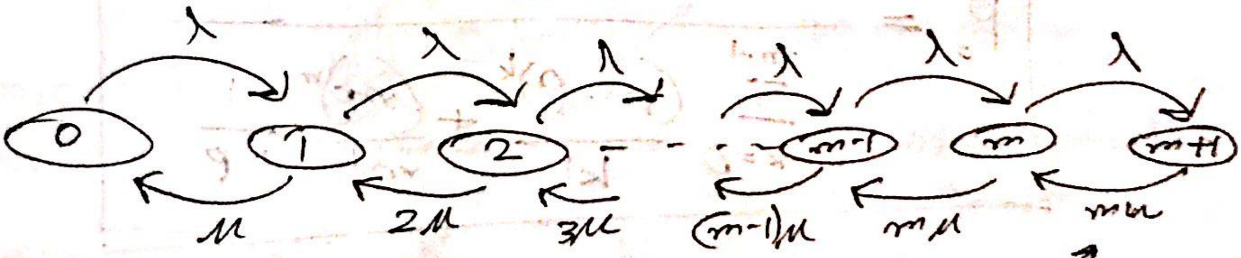
M/M/m → server



$$\lambda_n = \lambda \qquad n = 0, 1, 2, \ldots$$

$$\mu_n = \min(n\mu, m\mu) \longrightarrow$$ (Bottlenecked by number of people or servers)

$$= \begin{cases} n\mu, & n < m \\ m\mu & n \geq m \end{cases}$$



different

(i) For $n < m$, $P_n = P_0 \prod_{i=0}^{n-1} \frac{\lambda}{(i+1)\mu}$

(follows M/M/∞) $= P_0 \left(\frac{\lambda}{\mu}\right)^n \left(\frac{1}{n!}\right)$

(ii) For $n \geq m$, $P_n = P_0$

$$P_n = P_0 \underbrace{\boxed{\prod_{i=0}^{m-1} \left(\frac{\lambda}{(i+1)}\right) \frac{1}{\mu n!}}}_{\text{Infinite Server}} \cdot \underbrace{\boxed{\prod_{i=m}^{n-1} \frac{\lambda}{m\mu}}}_{\text{Finite Server}}$$

$$= P_0 \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{m!\, m^{n-m}}$$

$$= P_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{m!\, m^{n-m}}$$

$$\boxed{P_n = \begin{cases} P_0 \dfrac{(m\rho)^n}{n!} & n < m \\ P_0 \dfrac{m^m \rho^n}{m!} & n \geq m \end{cases}}$$

$$\left[\rho = \frac{\lambda}{m\mu} < 1\right]$$

$$\boxed{P_0 = \frac{1}{\displaystyle\sum_{k=0}^{m-1} \frac{(m-\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \cdot \frac{1}{1-\rho}}}$$