30-07

Lecture - 16

Methods to find out determinants:

(i) The Pivot formula → (most useful among 3)

(ii) The big formula

(iii) The co-factor formula

(i) The pivot formula -

→ We need to first perform elimination to make $A = LU$ form

→ then, $\det A = (\det L)(\det U)$

$= 1 \; (d_1 \cdot d_2 \cdots d_n)$

(elements in diagonal are one) property vii

$L = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix}$

$U = \begin{bmatrix} d_1 & x & x \\ 0 & d_2 & x \\ 0 & 0 & d_3 \end{bmatrix}$

forms of L and U

$= \pm (d_1 \cdots d_2 \cdots d_n)$

single row exchange can cause sign to be (+ve) or (-ve)

Example: Suppose $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ 1 & 4 & -6 & 3 \end{bmatrix}$

$= - \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 7 \end{vmatrix}$

[Upper triangular matrix after row exchange and elimination]

$$\therefore \det(A) = -(1 \times 3 \times 2 \times 7) = -42$$

Example-2:

Suppose $B = \begin{vmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{vmatrix}$

Performing, $R_1 = R_1 + \cdots R_5$

$$= \begin{vmatrix} x+4y & x+4y & x+4y & x+4y & x+4y \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$= (x+4y) \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{vmatrix}$$

Now, we transpose the matrix since

$$\det(A) = \det(A^T)$$

$$= (x+4y)\begin{vmatrix} 1 & y & y & y & y \\ 1 & x & y & y & y \\ 1 & y & x & y & y \\ 1 & y & y & x & y \\ 1 & y & y & y & x \end{vmatrix}$$

Performing $(R_2 = R_2 - R_1)$, $(R_3 = R_3 - R_1)$, $(R_4 = R_4 - R_1)$ and $(R_5 = R_5 - R_1)$, we get,

$$(x+4y)\begin{vmatrix} 1 & y & y & y & y \\ 0 & x-y & 0 & 0 & 0 \\ 0 & 0 & x-y & 0 & 0 \\ 0 & 0 & 0 & x-y & 0 \\ 0 & 0 & 0 & 0 & x-y \end{vmatrix}$$

(Upper triangular matrix is derived)

$$\therefore \det(B) = (x+4y)(1)(x-y)(x-y)(x-y)(x-y)$$
$$= (x+4y)(x-y)^4$$

In this way determinant of any matrix can be found using elimination and converting it to an upper triangular matrix.

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(ii) The big formula:

For a_{ij}

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

In this case we find determinant directly from a_{ij}.

Now, for a 2×2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= 0 + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + 0 + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

[∵ A single colum is zero]

$$= ad + (-)\begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix}$$

$$= ad - bc$$ [Diagonal property of determinants]

As we can see, based on the number of combinations, there will be $n!$ terms, So, for a 3×3 matrix there will be $3! = 6$ terms,

But if n is too high, say 11, the terms will be very high ($11! = 40$ million terms).

(iii) Determinant by co-factors:

→ In this element we find the sum of the elements of a row multiplied to their co-factors,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}}_{C_{11}} + \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}}_{C_{12}} + \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}}_{C_{13}}$$

$\therefore \det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$ (same for other rows)

$\approx a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{12}$ (for n ~~terms~~ columns.)

General formula for co-factors would be

$$(-1)^{i+j}\det(M_{ij}) \quad \left|\begin{matrix} i = \text{row no.}\\ j = \text{col no.}\end{matrix}\right|$$

For example
$$\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & 9 \end{vmatrix} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$
$$= (-1)^3 (36-42)$$
$$= 6$$

For a larger matrix, we need to find individual cofactors, of the smaller matrices.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 1 \begin{vmatrix} 6 & 7 & 8 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{vmatrix} - 2 \begin{vmatrix} 5 & 7 & 8 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{vmatrix} + 3 \begin{vmatrix} - & - & - \\ \cdot & & \\ - & - & - \end{vmatrix} - 4 \begin{vmatrix} - & - \\ - & - \\ - & - \end{vmatrix}$$

Then each particular determinant is found.

This is a lengthy process.

Example:
$$A = \begin{vmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ y & 0 & 0 & 0 & x \end{vmatrix}$$

Transposing A, we get,

$$\begin{vmatrix} x & 0 & 0 & 0 & y \\ y & x & 0 & 0 & 0 \\ 0 & y & x & 0 & 0 \\ 0 & 0 & y & x & 0 \\ 0 & 0 & 0 & y & x \end{vmatrix}$$

$$= x c_{11} + 0 \cdot c_{12} + 0 \cdot c_{13} + 0 \cdot c_{14} + y c_{15}.$$

$$\cancel{= x}$$

$$= x(-1)^{1+1} \begin{vmatrix} x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \end{vmatrix} + (-1)^{1+5} \cdot y \begin{vmatrix} y & x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \\ 0 & 0 & 0 & y \end{vmatrix}$$

$$= x(x \cdot x \cdot x \cdot x) + y(y \cdot y \cdot y \cdot y)$$

$$= x^5 + y^5$$

<u>False expansion theorem</u>–

If A is an $n \times n$ matrix, and $i \neq k$, then

$$a_{i1} c_{k1} + a_{i2} c_{k2} + \cdots a_{in} c_{kn} = 0.$$

For example, in case of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det A = a c_{11} + b c_{12} = ad + b(-c) = ad - bc$$

~~but, c_{11} =~~

Now, instead of taking the first row, we take the second row,

$$\Rightarrow -c \cdot d + d(c) \quad \begin{bmatrix} d = c_{11} \\ c = c_{12} \end{bmatrix}$$

We took a different row but used co-factors of first row.

Then, $= cd - cd$

$= 0$

$\therefore a_{21} c_{11} + a_{22} c_{12} = 0$

Proof: Previously,

$$a_{11} c_{11} + a_{12} c_{12} + \dots = \det(A)$$

Generalizing, $a_{i1} c_{k1} + \dots a_{in} c_{kn} = \det(A)$ [when $i = k$]

Now, another matrice B is produced by swapping i^{th} and k^{th} row.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{12} & \cdots & a_{2n} \\ \vdots & & \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \\ a_{k1} & \cdots & a_{kn} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ a_{i2} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

same rows

[i^{th} row = k^{th} row]

Since, two rows are same,

$$\det(B) = 0.$$

$$\therefore \det(B) = a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0 \text{ (proved).}$$

$$\det A = a_i^T c_i^T$$

$$0 = a_i^T c_k^T \quad [i \neq k]$$

From these two $\left(A\,(\text{~~coeff~~ fac } A)^T\right)_{ii} = a_i^T a_i^T$

$$\left(A\,(\text{coeff } A)^T\right)_{ik} = 0 \quad [i \neq k]$$

Example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\text{co-factor}\ \cancel{\text{co-eff}}\ A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$A\,(\text{coeff } A)^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$ (diagonal elements only remain)

$$\therefore A\,(\text{coeff } A)^T = \begin{bmatrix} \det A & 0 & 0 & 0 & \cdots \\ 0 & \det A & 0 & 0 & \cdots \\ 0 & 0 & \det A & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$= \det A \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

identity matrix

$$\therefore A\,(\text{co-eff } A)^T = (\det A)\, I$$

So, if A is an invertible matrix i.e. $\det A \neq 0$

$$A \cdot \frac{1}{\det A}\left(\text{coff } A^T\right) = I = AA^{-1}$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{\det A}\,\cancel{\text{co-eff}}\,(\text{cof } A)^T}$$

Formula to find inverse.

Q: $B = \begin{bmatrix} 7 & 1 & 2 \\ 4 & -2 & -5 \\ 9 & 8 & -3 \end{bmatrix}$. What is B^{-1}?

$$\text{co-f } B = \begin{bmatrix} 46 & -33 & 50 \\ 27 & -48 & -47 \\ 1 & 47 & -18 \end{bmatrix}$$

and $\det B = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$

$$\therefore \det B = 46 \times 7 + (-33) \times 1 + 2 \times 50$$

$$= 389$$

$$\therefore B^{-1} = \frac{1}{389}(\text{cof } B)^T = \frac{1}{389}\begin{bmatrix} 46 & -33 & 50 \\ 27 & -48 & -47 \\ 1 & 47 & -18 \end{bmatrix}^T$$

$$= \frac{1}{389}\begin{bmatrix} 46 & 27 & 1 \\ -33 & -48 & 41 \\ 50 & -47 & -18 \end{bmatrix} \text{ (Ans.)}$$