Complex differentiation and Cauchy-Riemann equations

Md. Abul Kalam Azad
Assistant Professor, Mathematics
MPE, IUT

Derivatives

If $f(z)$ is single-valued in some region $\mathcal{R}$ of the z plane, the *derivative* of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{3.1}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such a case, we say that $f(z)$ is *differentiable* at z. In the definition (3.1), we sometimes use h instead of Δz. Although differentiability implies continuity, the reverse is not true

Analytic Functions

If the derivative $f'(z)$ exists at all points z of a region $\mathcal{R}$, then $f(z)$ is said to be *analytic in* $\mathcal{R}$ and is referred to as an *analytic function in* $\mathcal{R}$ or a function *analytic in* $\mathcal{R}$. The terms *regular* and *holomorphic* are sometimes used as synonyms for analytic.

A function $f(z)$ is said to be *analytic at a point* z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

Cauchy–Riemann Equations

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathcal{R}$ is that, in $\mathcal{R}$, u and v satisfy the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2}$$

If the partial derivatives in (3.2) are continuous in $\mathcal{R}$, then the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$.

The functions $u(x, y)$ and $v(x, y)$ are sometimes called *conjugate functions*. Given u having continuous first partials on a simply connected region $\mathcal{R}$, we can find v (within an arbitrary additive constant) so that $u + iv = f(z)$ is analytic

Harmonic Functions

If the second partial derivatives of u and v with respect to x and y exist and are continuous in a region $\mathcal{R}$, then we find from (3.2) that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{3.3}$$

It follows that under these conditions, the real and imaginary parts of an analytic function satisfy *Laplace's equation* denoted by

$$\frac{\partial^2 \Psi}{\partial^2 x} + \frac{\partial^2 \Psi}{\partial^2 y} = 0 \quad \text{or} \quad \nabla^2 \Psi = 0 \quad \text{where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3.4}$$

The operator ∇^2 is often called the *Laplacian*.

Functions such as $u(x, y)$ and $v(x, y)$ which satisfy Laplace's equation in a region $\mathcal{R}$ are called *harmonic functions* and are said to be *harmonic in* $\mathcal{R}$.

Rules for Differentiation

Suppose $f(z)$, $g(z)$, and $h(z)$ are analytic functions of z. Then the following differentiation rules (identical with those of elementary calculus) are valid.

1. $$\frac{d}{dz}\{f(z)+g(z)\}=\frac{d}{dz}f(z)+\frac{d}{dz}g(z)=f'(z)+g'(z)$$
2. $$\frac{d}{dz}\{f(z)-g(z)\}=\frac{d}{dz}f(z)-\frac{d}{dz}g(z)=f'(z)-g'(z)$$
3. $$\frac{d}{dz}\{cf(z)\}=c\frac{d}{dz}f(z)=cf'(z) \text{ where } c \text{ is any constant}$$
4. $$\frac{d}{dz}\{f(z)g(z)\}=f(z)\frac{d}{dz}g(z)+g(z)\frac{d}{dz}f(z)=f(z)g'(z)+g(z)f'(z)$$
5. $$\frac{d}{dz}\left\{\frac{f(z)}{g(z)}\right\}=\frac{g(z)(d/dz)f(z)-f(z)(d/dz)g(z)}{[g(z)]^2}=\frac{g(z)f'(z)-f(z)g'(z)}{[g(z)]^2} \quad \text{if } g(z)\neq 0$$

6. If $w = f(\zeta)$ where $\zeta = g(z)$ then

$$\frac{dw}{dz} = \frac{dw}{d\zeta} \cdot \frac{d\zeta}{dz} = f'(\zeta)\frac{d\zeta}{dz} = f'\{g(z)\}g'(z) \tag{3.10}$$

Similarly, if $w = f(\zeta)$ where $\zeta = g(\eta)$ and $\eta = h(z)$, then

$$\frac{dw}{dz} = \frac{dw}{d\zeta} \cdot \frac{d\zeta}{d\eta} \cdot \frac{d\eta}{dz} \tag{3.11}$$

The results (3.10) and (3.11) are often called *chain rules* for differentiation of composite functions.

7. If $w = f(z)$ has a single-valued inverse f^{-1}, then $z = f^{-1}(w)$, and dw/dz and dz/dw are related by

$$\frac{dw}{dz} = \frac{1}{dz/dw} \tag{3.12}$$

8. If $z = f(t)$ and $w = g(t)$ where t is a parameter, then

$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{g'(t)}{f'(t)} \tag{3.13}$$

Similar rules can be formulated for differentials. For example,

$$d\{f(z) + g(z)\} = df(z) + dg(z) = f'(z)\,dz + g'(z)\,dz = \{f'(z) + g'(z)\}\,dz$$
$$d\{f(z)g(z)\} = f(z)\,dg(z) + g(z)\,df(z) = \{f(z)g'(z) + g(z)f'(z)\}\,dz$$

L'Hospital's Rule

Let $f(z)$ and $g(z)$ be analytic in a region containing the point z_0 and suppose that $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$. Then, *L'Hospital's rule* states that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{3.14}$$

In the case of $f'(z_0) = g'(z_0) = 0$, the rule may be extended. See Problems 3.21–3.24.

We sometimes say that the left side of (3.14) has the "indeterminate form" $0/0$, although such terminology is somewhat misleading since there is usually nothing indeterminate involved. Limits represented by so-called indeterminate forms ∞/∞, $0 \cdot \infty$, ∞°, 0°, 1^{∞}, and $\infty - \infty$ can often be evaluated by appropriate modifications of L'Hospital's rule.

Ex-1: Using the definition, find the derivative of $w = f(z) = z^3 - 2z$ at the point where (a) $z = z_0$, (b) $z = -1$.

Solution

(a) By definition, the derivative at $z = z_0$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta_z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - \{z_0^3 - 2z_0\}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2 \Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 3z_0^2 + 3z_0 \Delta z + (\Delta z)^2 - 2 = 3z_0^2 - 2$$

In general, $f'(z) = 3z^2 - 2$ for all z.

(b) From (a), or directly, we find that if $z_0 = -1$, then $f'(-1) = 3(-1)^2 - 2 = 1$.

Ex-2: Show that $(d/dz)\bar{z}$ does not exist anywhere, i.e., $f(z) = \bar{z}$ is non-analytic anywhere.

Solution

By definition, $\dfrac{d}{dz}f(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

if this limit exists independent of the manner in which $\Delta z = \Delta x + i\Delta y$ approaches zero.

Then

$$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+iy+\Delta x + i\Delta y} - \overline{x+iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\Delta y - (x - iy)}{\Delta x + i\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is $\lim\limits_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1$

If $\Delta x = 0$, the required limit is $\lim\limits_{\Delta y \to 0} \dfrac{-i\Delta y}{i\Delta y} = -1$

Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e., $f(z) = \bar{z}$ is *non-analytic* anywhere.

Ex-3: Given $w = f(z) = (1 + z)/(1 - z)$, find (a) dw/dz and (b) determine where $f(z)$ is non-analytic.

Solution

(a) Using the definition

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\dfrac{1 + (z + \Delta z)}{1 - (z + \Delta z)} - \dfrac{1 + z}{1 - z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2}{(1 - z - \Delta z)(1 - z)} = \frac{2}{(1 - z)^2}$$

independent of the manner in which $\Delta z \to 0$, provided $z \neq 1$.

(b) The function $f(z)$ is analytic for all finite values of z except $z = 1$ where the derivative does not exist and the function is non-analytic. The point $z = 1$ is a *singular point* of $f(z)$.

Ex-4: Prove that a (a) necessary and (b) sufficient condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathcal{R}$ is that the Cauchy–Riemann equations $\partial u/\partial x = \partial v/\partial y,\ \partial u/\partial y = -(\partial v/\partial x)$ are satisfied in $\mathcal{R}$ where it is supposed that these partial derivatives are continuous in $\mathcal{R}$.

Solution

(a) *Necessity.* In order for $f(z)$ to be analytic, the limit

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y} \tag{1}$$

must exist independent of the manner in which Δz (or Δx and Δy) approaches zero. We consider two possible approaches.

Case 1. $\Delta y = 0,\ \Delta x \to 0$. In this case, (1) becomes

$$\lim_{\Delta x \to 0} \left\{ \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i\left[\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}\right]\right\} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

provided the partial derivatives exist.

Case 2. $\Delta x = 0$, $\Delta y \to 0$. In this case, (1) becomes

$$\lim_{\Delta y \to 0} \left\{ \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \right\} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now $f(z)$ cannot possibly be analytic unless these two limits are identical. Thus, a necessary condition that $f(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{or} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

(b) *Sufficiency*. Since $\partial u/\partial x$ and $\partial u/\partial y$ are supposed to be continuous, we have

$$\begin{aligned}
\Delta u &= u(x + \Delta x, y + \Delta y) - u(x, y) \\
&= \{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)\} + \{u(x, y + \Delta y) - u(x, y)\} \\
&= \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y
\end{aligned}$$

where $\epsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Similarly, since $\partial v/\partial x$ and $\partial v/\partial y$ are supposed to be continuous, we have

$$\Delta v = \left(\frac{\partial v}{\partial x} + \epsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_2 \Delta x + \eta_2 \Delta y$$

where $\epsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Then

$$\Delta w = \Delta u + i\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y \tag{2}$$

where $\epsilon = \epsilon_1 + i\epsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

By the Cauchy–Riemann equations, (2) can be written

$$\begin{aligned}\Delta w &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \epsilon\Delta x + \eta\Delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \epsilon\Delta x + \eta\Delta y\end{aligned}$$

Then, on dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0}\frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

so that the derivative exists and is unique, i.e., $f(z)$ is analytic in $\mathcal{R}$.

Harmonic Function: Any function which satisfies Laplace's equation $\nabla^2\Psi = 0$ is called a **Harmonic Function (HF). For example if** $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, then u is a HF.

Similarly if $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ then v is another HF.

 # Theorem: If f(z) = u + iv is an analytic function, then u and v are both HF.

Proof: For any analytic function, f(z) = u + iv, we know that u and v must satisfy C-R Equations. That is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} ------------------(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} -----------------(2)$$

Differentiation eqn (1) with respect to x, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} ---------------(3)$$

Similarly differentiation eqn (2) with respect to y, we have

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \text{---------------} (4)$$

Now adding eqn (3) and eqn (4), we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left[\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \text{ for any analytic function}\right]$$

Therefore, u satisfies Laplace's equation and thereby a HF. Similarly differentiating eqn (1) and (2) with respect to y and x respectively and carrying out logically similar steps one can show that

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

Thus, u and v both satisfy Laplace's equation and thereby HF[s].

Conjugate Functions: Any function which satisfy C-R Equation is called a Conjugate function (CF).

Harmonic Conjugate Functions: Functions satisfying both C-R Equation and Laplace's equation is called a Harmonic Conjugate Function (HCF).

Example1: Show that the function u(x, y) = 4xy – 3x +2 is HF. Construct the corresponding analytic function f(z) = u + iv

Solution: Given that

$$\boxed{\begin{aligned} & u(x, y) = 4xy - 3x + 2 \\ & \Rightarrow \frac{\partial u}{\partial x} = 4y - 3, \quad \frac{\partial u}{\partial y} = 4x, \\ & \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0 \\ & \therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \text{ u is a HF.} \end{aligned}}$$

From Total differentiation, we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy \quad \text{[By C-R Equations]}$$

$$\therefore dv = -4xdx + (4y-3)dy$$

On integration, we have,

$$v = -2x^2 + 2y^2 - 3y + c$$

$$\therefore f(z) = u(x,y) + iv(x,y)$$

$$= (4xy - 3x + 2) + i\left(-2x^2 + 2y^2 - 3y + c\right)$$

$$= -i2x^2 + 4xy + i2y^2 - 3x - i3y + 2 + ic$$

$$= -2i\left(x^2 + 2ixy - y^2\right) - 3(x+iy) + 2 + ic$$

$$= -2i(x+iy)^2 - 3z + 2 + ic$$

$$\therefore f(z) = -2iz^2 - 3z + 2 + ic \;\square$$

Ex-5: Given $f(z) = u + iv$ is analytic in a region $\mathcal{R}$. Prove that u and v are harmonic in $\mathcal{R}$ if they have continuous second partial derivatives in $\mathcal{R}$.

Solution

If $f(z)$ is analytic in $\mathcal{R}$, then the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2}$$

are satisfied in $\mathcal{R}$. Assuming u and v have continuous second partial derivatives, we can differentiate both sides of (1) with respect to x and (2) with respect to y to obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{3}$$

and

$$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \tag{4}$$

from which

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

i.e., u is harmonic.

Similarly, by differentiating both sides of (1) with respect to y and (2) with respect to x, we find

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

and v is harmonic.

Ex-6: (a) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

(b) Find v such that $f(z) = u + iv$ is analytic.

Solution

(a)
$$\frac{\partial u}{\partial x} = (e^{-x})(\sin y) + (-e^{-x})(x \sin y - y \cos y) = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y) = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \cos y \tag{1}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y) = xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y) = -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y \tag{2}$$

Adding (1) and (2) yields $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$ and u is harmonic.

(b) From the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y \tag{3}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x} \cos y - xe^{-x} \cos y - ye^{-z} \sin y \tag{4}$$

Integrate (3) with respect to y, keeping x constant. Then

$$\begin{aligned} v &= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x) \\ &= ye^{-x}\sin y + xe^{-x}\cos y + F(x) \end{aligned} \tag{5}$$

where $F(x)$ is an arbitrary real function of x.

Substitute (5) into (4) and obtain

$$-ye^{-x}\sin y - xe^{-x}\cos y + e^{-x}\cos y + F'(x) = -ye^{-x}\sin y - xe^{-x}\cos y - ye^{-x}\sin y$$

or $F'(x) = 0$ and $F(x) = c$, a constant. Then, from (5),

$$v = e^{-x}(y\sin y + x\cos y) + c$$

Ex-7: Find $f(z)$ in Problem **6**

Solution

Apart from an arbitrary additive constant, we have from the results of Problem **6**

$$
\begin{aligned}
f(z) = u + iv &= e^{-x}(x \sin y - y \cos y) + ie^{-x}(y \sin y + x \cos y) \\
&= e^{-x}\left\{x\left(\frac{e^{iy} - e^{-iy}}{2i}\right) - y\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} + ie^{-x}\left\{y\left(\frac{e^{iy} - e^{-iy}}{2i}\right) + x\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} \\
&= i(x + iy)e^{-(x+iy)} = ize^{-z}
\end{aligned}
$$

Ex-8: Prove that (a) $(d/dz)e^z = e^z$, (b) $(d/dz)e^{az} = ae^{az}$ where a is any constant.

Solution

(a) By definition, $w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = u + iv$ or $u = e^x \cos y$, $v = e^x \sin y$.

Since $\partial u/\partial x = e^x \cos y = \partial v/\partial y$ and $\partial v/\partial x = e^x \sin y = -(\partial u/\partial y)$, the Cauchy–Riemann equations are satisfied. Then, by Problem 3.5, the required derivative exists and is equal to

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^x \cos y + ie^x \sin y = e^z$$

(b) Let $w = e^{\zeta}$ where $\zeta = az$. Then, by part (a)

$$\frac{d}{dz}e^{az} = \frac{d}{dz}e^{\zeta} = \frac{d}{d\zeta}e^{\zeta} \cdot \frac{d\zeta}{dz} = e^{\zeta} \cdot a = ae^{az}$$

We can also proceed as in part (a).

Ex-9: Prove that $\dfrac{d}{dz} z^{1/2} = \dfrac{1}{2z^{1/2}}$, realizing that $z^{1/2}$ is a multiple-valued function.

Solution

A function must be single-valued in order to have a derivative. Thus, since $z^{1/2}$ is multiple-valued (in this case two-valued), we must restrict ourselves to one branch of this function at a time.

Case 1

Let us first consider that branch of $w = z^{1/2}$ for which $w = 1$ where $z = 1$. In this case, $w^2 = z$ so that

$$\frac{dz}{dw} = 2w \quad \text{and so} \quad \frac{dw}{dz} = \frac{1}{2w} \quad \text{or} \quad \frac{d}{dz} z^{1/2} = \frac{1}{2z^{1/2}}$$

Case 2

Next, we consider that branch of $w = z^{1/2}$ for which $w = -1$ where $z = 1$. In this case too, we have $w^2 = z$ so that

$$\frac{dz}{dw} = 2w \quad \text{and} \quad \frac{dw}{dz} = \frac{1}{2w} \quad \text{or} \quad \frac{d}{dz} z^{1/2} = \frac{1}{2z^{1/2}}$$

In both cases, we have $(d/dz)z^{1/2} = 1/(2z^{1/2})$. Note that the derivative does not exist at the branch point $z = 0$. In general, a function does not have a derivative, i.e., is not analytic, at a branch point. Thus branch points are singular points.

Ex-10: Using rules of differentiation, find the derivatives of each of the following:

(a) $\cos^2(2z+3i)$, (b) $z\tan^{-1}(\ln z)$, (c) $\{\tanh^{-1}(iz+2)\}^{-1}$, (d) $(z-3i)^{4z+2}$.

Solution

(a)
$$\frac{d}{dz}\{\cos(2z+3i)\}^2 = 2\{\cos(2z+3i)\}\left\{\frac{d}{dz}\cos(2z+3i)\right\}$$
$$= 2\{\cos(2z+3i)\}\{-\sin(2z+3i)\}\left\{\frac{d}{dz}(2z+3i)\right\}$$
$$= -4\cos(2z+3i)\sin(2z+3i)$$

(b)
$$\frac{d}{dz}\{(z)[\tan^{-1}(\ln z)]\} = z\frac{d}{dz}[\tan^{-1}(\ln z)] + [\tan^{-1}(\ln z)]\frac{d}{dz}(z)$$
$$= z\left\{\frac{1}{1+(\ln z)^2}\right\}\frac{d}{dz}(\ln z) + \tan^{-1}(\ln z) = \frac{1}{1+(\ln z)^2} + \tan^{-1}(\ln z)$$

(c)
$$\frac{d}{dz}\{\tanh^{-1}(iz+2)\}^{-1} = -1\{\tanh^{-1}(iz+2)\}^{-2}\frac{d}{dz}\{\tanh^{-1}(iz+2)\}$$
$$= -\{\tanh^{-1}(iz+2)\}^{-2}\left\{\frac{1}{1-(iz+2)^2}\right\}\frac{d}{dz}(iz+2)$$
$$= \frac{-i\{\tanh^{-1}(iz+2)\}^{-2}}{1-(iz+2)^2}$$

(d) $$\frac{d}{dz}\{(z-3i)^{4z+2}\} = \frac{d}{dz}\{e^{(4z+2)\ln(z-3i)}\} = e^{(4z+2)\ln(z-3i)}\frac{d}{dz}\{(4z+2)\ln(z-3i)\}$$
$$= e^{(4z+2)\ln(z-3i)}\left\{(4z+2)\frac{d}{dz}[\ln(z-3i)] + \ln(z-3i)\frac{d}{dz}(4z+2)\right\}$$
$$= e^{(4z+2)\ln(z-3i)}\left\{\frac{4z+2}{z-3i} + 4\ln(z-3i)\right\}$$
$$= (z-3i)^{4z+1}(4z+2) + 4(z-3i)^{4z+2}\ln(z-3i)$$

Ex-11: Suppose $w^3 - 3z^2 w + 4\ln z = 0$. Find dw/dz.

Solution

Differentiating with respect to z, considering w as an implicit function of z, we have

$$\frac{d}{dz}(w^3) - 3\frac{d}{dz}(z^2 w) + 4\frac{d}{dz}(\ln z) = 0 \quad \text{or} \quad 3w^2\frac{dw}{dz} - 3z^2\frac{dw}{dz} - 6zw + \frac{4}{z} = 0$$

Then, solving for dw/dz, we obtain $\dfrac{dw}{dz} = \dfrac{6zw - 4/z}{3w^2 - 3z^2}$.

Ex-12: Given $w = \sin^{-1}(t-3)$ and $z = \cos(\ln t)$. Find dw/dz.

Solution

$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{1/\sqrt{1-(t-3)^2}}{-\sin(\ln t)(1/t)} = -\frac{t}{\sin(\ln t)\sqrt{1-(t-3)^2}}$$

Ex-13: In Problem **11** , find d^2w/dz^2.

Solution

$$\frac{d^2w}{dz^2} = \frac{d}{dz}\left(\frac{dw}{dz}\right) = \frac{d}{dz}\left(\frac{6zw-4/z}{3w^2-3z^2}\right)$$

$$= \frac{(3w^2-3z^2)(6z\,dw/dz+6w+4/z^2)-(6zw-4/z)(6w\,dw/dz-6z)}{(3w^2-3z^2)^2}$$

The required result follows on substituting the value of dw/dz from Problem **11** and simplifying.

Ex-14: Evaluate (a) $\lim_{z\to i}\frac{z^{10}+1}{z^6+1}$, (b) $\lim_{z\to 0}\frac{1-\cos z}{z^2}$, (c) $\lim_{z\to 0}\frac{1-\cos z}{\sin z^2}$.

Solution

(a) Let $f(z)=z^{10}+1$ and $g(z)=z^6+1$. Then $f(i)=g(i)=0$. Also, $f(z)$ and $g(z)$ are analytic at $z=i$. Hence, by L'Hospital's rule,

$$\lim_{z\to i}\frac{z^{10}+1}{z^6+1}=\lim_{z\to i}\frac{10z^9}{6z^5}=\lim_{z\to i}\frac{5}{3}z^4=\frac{5}{3}$$

(b) Let $f(z)=1-\cos z$ and $g(z)=z^2$. Then $f(0)=g(0)=0$. Also, $f(z)$ and $g(z)$ are analytic at $z=0$. Hence, by L'Hospital's rule,

$$\lim_{z\to 0}\frac{1-\cos z}{z^2}=\lim_{z\to 0}\frac{\sin z}{2z}$$

Since $f_1(z)=\sin z$ and $g_1(z)=2z$ are analytic and equal to zero when $z=0$, we can apply L'Hospital's rule again to obtain the required limit,

$$\lim_{z\to 0}\frac{\sin z}{2z}=\lim_{z\to 0}\frac{\cos z}{2}=\frac{1}{2}$$

(c) By repeated application of L'Hospital's rule, we have

$$\lim_{z\to 0}\frac{1-\cos z}{\sin z^2}=\lim_{z\to 0}\frac{\sin z}{2z\cos z^2}=\lim_{z\to 0}\frac{\cos z}{2\cos z^2-4z^2\sin z^2}=\frac{1}{2}$$

Ex-15: Evaluate $\lim_{z\to 0}(\cos z)^{1/z^2}$.

Solution

Let $w = (\cos z)^{1/z^2}$. Then $\ln w = (\ln\cos z)/z^2$ where we consider the principal branch of the logarithm. By L'Hospital's rule,

$$\lim_{z\to 0}\ln w = \lim_{z\to 0}\frac{\ln\cos z}{z^2} = \lim_{z\to 0}\frac{(-\sin z)/\cos z}{2z}$$

$$= \lim_{z\to 0}\left(\frac{\sin z}{z}\right)\left(-\frac{1}{2\cos z}\right) = (1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

But since the logarithm is a continuous function, we have

$$\lim_{z\to 0}\ln w = \ln\left(\lim_{z\to 0} w\right) = -\frac{1}{2}$$

or $\lim_{z\to 0} w = e^{-1/2}$, which is the required value.

Note that since $\lim_{z\to 0}\cos z = 1$ and $\lim_{z\to 0} 1/z^2 = \infty$, the required limit has the "indeterminate form" 1^∞.

Ex-18: Let C be the curve in the xy plane defined by $3x^2y - 2y^3 = 5x^4y^2 - 6x^2$. Find a unit vector normal to C at $(1, -1)$.

Solution

Let $F(x, y) = 3x^2y - 2y^3 - 5x^4y^2 + 6x^2 = 0$. By Problem 3.33, a vector normal to C at $(1, -1)$ is

$$\nabla F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = (6xy - 20x^3y^2 + 12x) + i(3x^2 - 6y^2 - 10x^4y) = -14 + 7i$$

Then a unit vector normal to C at $(1, -1)$ is $\dfrac{-14 + 7i}{|-14 + 7i|} = \dfrac{-2 + i}{\sqrt{5}}$. Another such unit vector is $\dfrac{2 - i}{\sqrt{5}}$.

Ex-19: Suppose $A(x, y) = 2xy - ix^2y^3$. Find (a) grad A, (b) div A, (c) |curl A|, (d) Laplacian of A.

Solution

(a) $$\text{grad } A = \nabla A = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(2xy - ix^2y^3) = \frac{\partial}{\partial x}(2xy - ix^2y^3) + i\frac{\partial}{\partial y}(2xy - ix^2y^3)$$
$$= 2y - 2ixy^3 + i(2x - 3ix^2y^3) = 2y + 3x^2y^2 + i(2x - 2xy^3)$$

(b) $$\text{div } A = \nabla \cdot A = \text{Re}\{\overline{\nabla}A\} = \text{Re}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(2xy - ix^2y^3)\right\}$$
$$= \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y^3) = 2y - 3x^2y^2$$

(c) $$|\text{curl } A| = |\nabla \times A| = |\text{Im}\{\overline{\nabla}A\}| = \left|\text{Im}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(2xy - ix^2y^3)\right\}\right|$$
$$= \left|\frac{\partial}{\partial x}(-x^2y^3) - \frac{\partial}{\partial y}(2xy)\right| = \left|-2xy^3 - 2x\right|$$

(d) Laplacian $A = \nabla^2 A = \mathrm{Re}\{\overline{\nabla}\nabla A\} = \dfrac{\partial^2 A}{\partial x^2} + \dfrac{\partial^2 A}{\partial y^2} = \dfrac{\partial^2}{\partial x^2}(2xy - ix^2y^3) + \dfrac{\partial^2}{\partial y^2}(2xy - ix^2y^3)$

$$= \frac{\partial}{\partial x}(2y - 2ixy^3) + \frac{\partial}{\partial y}(2x - 3ix^2y^2) = -2iy^3 - 6ix^2y$$

Thanks a lot …