Solutions to Problems in Chapter 9 of

Simulation Modeling and Analysis, 5th ed., 2015, McGraw-Hill, New York

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9.1. If independent random numbers are used for each replication, then the input random variables for the different replications are independent. This is turn makes the output random variables for different replications independent.

9.2. $P[Y(t) = 1 \mid Y(0) = j] \to \omega/(\lambda + \omega)$ as $t \to \infty$ for $j = 0$ or 1. Thus, if Y is the steady-state random variable corresponding to $\{Y(t), t \geq 0\}$, then

$$Y = \begin{cases} 1 & \text{w.p. } \omega/(\lambda + \omega) \\ 0 & \text{w.p. } \lambda/(\lambda + \omega) \end{cases}$$

9.3. No. The productions in the first and last hours of each shift will always be different.

9.4. If a part is in the system at the end of a day (2 shifts) or at the end of a week (5 days), its time in system will not be computed correctly with this approach.

9.5. The number of tellers utilized in a bank can be varied with the time of the day, day of the week, etc. However, in a computer or communication system, the system resources are present at all times. Thus, the system designer must plan for the peak load.

9.6. The throughput for the fifth hour of each shift will be smaller regardless of how long the simulation is run.

9.7. (*a*) Terminating, $E = \{\text{end of delay for 100th call}\}$.
(*b*) Nonterminating, steady-state parameter.
(*c*) Terminating, $E = \{\text{thirteen days of production have been completed}\}$.
(*d*) Terminating, $E = \{\text{last plane departs}\}$.
(*e*) Nonterminating. Suppose that

$$\rho = \frac{\lambda}{6\omega(2/3) + 4\omega(1/3)} < 1$$

so that the system is well defined.
Case 1: Suppose that $\lambda > 4\omega$. Then we are interested in the steady-state cycle parameter mean throughput for a day (3 shifts). The cycle length is one day.
Case 2: Suppose that $\lambda < 4\omega$. The stochastic process $N_1, N_2, \ldots$ has a steady-state distribution and we are interested in the steady-state parameter $E(N)$.

9.8. Let X_j be the weekly throughput from the jth replication. We obtained $X_1 = 7221$, $X_2 = 7155$, $X_3 = 7250$, $X_4 = 7260$, and $X_5 = 7260$, from which we get $\overline{X}(5) = 7229.2$ and $S^2(5) = 1975.7$; the 95 percent confidence interval is thus 7229.2 ± 55.2. Furthermore,

$$n_a^*(50) = \min\left\{i \geq 5 : t_{i-1,0.975}\sqrt{1975.7/i} \leq 50\right\} = 6 \quad \text{and}$$

$$n_r^*(0.05) = \min\left\{i \geq 5 : \frac{t_{i-1,0.975}\sqrt{1975.7/i}}{7229.2} \leq 0.048\right\} = 5$$

9.9. Let

$$Y_j = \begin{cases} 1 & \text{if } X_j \in B \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, ..., n$. Then the Y_j's are IID random variables with $E(Y_j) = p$, and $\bar{Y}(n) = S/n = \hat{p}$. Let $S^2(n)$ be the sample variance of the Y_j's. An approximate $100(1 - \alpha)$ percent confidence interval for p is given by

$$\bar{Y}(n) \pm t_{n-1,1-\alpha/2}\sqrt{S^2(n)/n}$$

Furthermore,

$$\begin{aligned}
\frac{S^2(n)}{n} &= \frac{\sum_{j=1}^{n}\left[Y_j - \bar{Y}(n)\right]^2}{n(n-1)} \\
&= \frac{\sum_{j=1}^{n} Y_j^2 - 2\hat{p}\sum_{j=1}^{n} Y_j + n\hat{p}^2}{n(n-1)} \\
&= \frac{\sum_{j=1}^{n} Y_j^2 - n\hat{p}^2}{n(n-1)} \\
&= \frac{\sum_{j=1}^{n} Y_j - n\hat{p}^2}{n(n-1)} \\
&= \frac{n\hat{p} - n\hat{p}^2}{n(n-1)} \\
&= \frac{\hat{p}(1-\hat{p})}{n-1}
\end{aligned}$$

9.10. $\hat{x}(0.5) = X_{(5)} = 1.70$, which is less than $\overline{X}(10) = 2.03$.

9.11. Let S be the random variable corresponding to the number of customers present at time 0. Then

$$
\begin{aligned}
P(D_1 \le x) &= \sum_{s=0}^{\infty} P(D_1 \le x \mid S = s)P(S = s) \\
&= P(D_1 \le x \mid S = 0)(1-\rho) + \sum_{s=1}^{\infty} P(D_1 \le x \mid S = s)(1-\rho)\rho^s \\
&= (1)(1-\rho) + (1-\rho)\left\{\sum_{s=1}^{\infty}\left[1 - e^{-\omega x}\sum_{j=0}^{s-1}\frac{(\omega x)^j}{j!}\right]\right\}\rho^s \\
&= (1-\rho) + (1-\rho)\left[\frac{\rho}{1-\rho} - e^{-\omega x}\sum_{j=0}^{\infty}\frac{(\omega x)^j}{j!}\sum_{s=j+1}^{\infty}\rho^s\right] \\
&= (1-\rho) + (1-\rho)\left\{\frac{\rho}{1-\rho}\left[1 - e^{-\omega x}\sum_{j=0}^{\infty}\frac{(\rho\omega x)^j}{j!}\right]\right\} \\
&= (1-\rho) + \rho\left[1 - e^{-\omega x}e^{\rho\omega x}\right] \\
&= (1-\rho) + \rho\left[1 - e^{-(\omega-\lambda)x}\right]
\end{aligned}
$$

which is the distribution function of the steady-state delay-in-queue distribution. Thus,

$$E(D_1) = 0(1-\rho) + \rho/(\omega - \lambda) = \rho/(\omega - \lambda)$$

9.12. Since $\overline{Y}_i = \sum_{j=1}^{n} Y_{ji} \Big/ n$ and the Y_{ji}'s are IID, the desired result follows from Sec. 4.4.

9.13.

$$
\begin{aligned}
\mathrm{Var}\left[\overline{Y}_i(w)\right] &= \mathrm{Var}\left(\frac{1}{2w-1}\sum_{s=-w}^{w}\overline{Y}_{i+s}\right)\\
&= \frac{\mathrm{Var}\left(\overline{Y}_i\right)}{2w+1}\left[1+2\sum_{j=1}^{2w}\left(1-\frac{j}{2w+1}\right)\rho_j\right]\\
&< \frac{\mathrm{Var}\left(\overline{Y}_i\right)}{2w+1}\left[1+2\sum_{j=1}^{2w}\left(1-\frac{j}{2w+1}\right)\right]\\
&= \frac{\mathrm{Var}\left(\overline{Y}_i\right)}{2w+1}(1+2w)\\
&= \mathrm{Var}\left(\overline{Y}_i\right)
\end{aligned}
$$

9.14. Suppose for definiteness that $E(Y_i)$ converges monotonically to ν from below as $i \to \infty$. (Many queueing systems that are started empty and idle behave like this when, for example, Y_i is the delay in queue of the ith customer; see Fig. 9.2.) Let l be a time index such that $E(Y_i) \approx \nu$ for $i > l$, with l perhaps determined by Welch's graphical method. We would expect l' to be somewhat larger than l, since, in general, $E[\overline{Y}(m)] < E(Y_m)$ for $m \geq 2$. [Clearly, $E(Y_i) \approx \nu$ for $i > l'$.]

9.15.

$$
\begin{aligned}
E(X_j) &= E\left(\frac{1}{m'-l}\sum_{i=l+1}^{m'} Y_{ji}\right) \\
&= \frac{1}{m'-l} E\left(\sum_{i=l+1}^{m'} Y_{ji}\right) \\
&= \frac{1}{m'-l}\sum_{i=l+1}^{m'} E(Y_{ji}) \\
&\approx \frac{1}{m'-l}(m'-l)\nu \\
&= \nu
\end{aligned}
$$

The confidence interval is approximate in terms of coverage since the X_j's are not normally distributed and $E(X_j) \approx \nu$.

9.16. No. The X_j's depend on the value of l, which is determined from the Y_{ji}'s for all j. However, in a few test cases, the correlation between the X_j's did not seem significant.

9.17. The steady-state hourly throughput should be 60, since the arrival rate is 60 per hour.

9.18. Suppose that $Q(t)$ is available for $0 \le t \le m$. One approach is to let $Q_i = Q(i)$ for $i = 1, 2, \ldots, m$. Another approach is to let $Q_i = \int_{i-1}^{i} Q(t)\,dt \Big/ 1$ for $i = 1, 2, \ldots, m$.

9.20. The fact that

$$b(n,k) = \frac{\left\{ n \Big/ \left[1 + 2 \sum_{i=1}^{n-1} \left(1 - \frac{i}{n} \right) \rho_i(k) \right] \right\} - 1}{n-1}$$

follows from Eqs. (4.7) and (4.8) in Chapter 4. Since the expression for $b(n, k)$ has a finite number of correlation terms and since $\rho_i(k) \to 0$ as $k \to \infty$, it follows that $b(n, k) \to 1$ as $k \to \infty$. Thus,

$$E\{\widehat{\text{Var}}[\bar{\bar{Y}}(n, k)]\} \to \text{Var}[\bar{\bar{Y}}(n, k)] \text{ as } k \to \infty.$$

9.21.

$$\frac{\bar{Z}(n')}{\bar{N}(n')} = \frac{\sum_{j=1}^{n'} Z_j}{\sum_{j=1}^{n'} N_j} = \frac{\sum_{i=1}^{M(n')} Y_i}{M(n')}$$

The left quotient converges to $E(Z)/E(N)$ (w.p. 1) as $n' \to \infty$ by the strong law of large numbers (see Sec. 4.6). Since

$$\nu = \lim_{m\to\infty} \frac{\sum_{i=1}^{m} Y_i}{m} \quad \text{(w.p. 1)}$$

by definition (see Sec. 9.5.3) and since $M(n') \to \infty$ as $n' \to \infty$, it also follows from the strong law that

$$\frac{\sum_{i=1}^{M(n')} Y_i}{M(n')} \to \nu \quad \text{(w.p. 1) as } n' \to \infty$$

Therefore, $\nu = E(Z)/E(N)$. It is also clear that $\hat{\nu}(n') = \bar{Z}(n')/\bar{N}(n') \to \nu$ as $n' \to \infty$ (w.p. 1).

9.22. No. The distribution of the time to the next arrival will be different for successive customers who depart and leave exactly l customers behind. These indices will be regeneration points if the interarrival times are exponential random variables.

9.23. Regeneration points for the process $C_1, C_2, \ldots$ are the indices of those months for which the inventory level at the beginning of the month is exactly equal to l ($l \le S$ and fixed). If the interdemand times are continuous random variables that are not exponential, there is no sequence of regeneration points with a corresponding finite expected length.

9.24. Each regeneration cycle is just as good as any other. Therefore, not using l of the n' cycles will increase the bias in $\hat{v}(n')$.

9.25.

$$
\begin{aligned}
E(N) &= E(N \mid \text{Arrival before})\, P(\text{Arrival before}) + E(N \mid \text{Arrival after})\, P(\text{Arrival after}) \\
&= 2E(N)[\lambda/(\lambda + \omega)] + (1)[\omega/(\lambda + \omega)]
\end{aligned}
$$

Therefore, $E(N) = \omega/(\omega - \lambda) = 1/(1 - \rho)$. When an arrival occurs before the first departure, things probabilistically start over at the instant of the second arrival due to the memoryless property of the exponential service times (see Prob. 4.26). If we ignore temporarily the second customer, then the expected number of customers in the first customer's cycle is $E(N)$. At the end of this cycle, the second arrival (which was put aside) can be thought of as starting his own cycle with the expected number of customers also being $E(N)$. Thus, when the second arrival occurs first, the expected number of customers in the entire cycle is $2E(N)$.

9.26. Let K be the number of times a part is machined and inspected before actually leaving. Then $K = 1 + X$, where $X \sim \text{geom}(0.9)$. Thus, $E(K) = 1 + E(X) = 1 + 0.1/0.9 = 1.111$. Therefore, the effective arrival rate is $\lambda' = 1.111\lambda = 1.111$ and the utilization factor for the machine is

$$\rho_{\text{M}} = \frac{1.111}{(360/370)(1/0.675) + (10/370)(0)} = 0.771$$

Similarly, the utilization factor for the inspector is

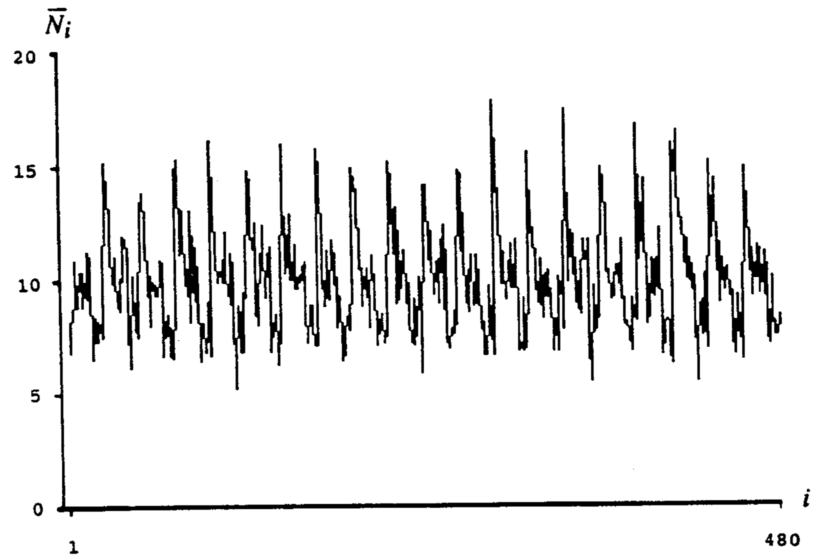
$$\rho_{\text{I}} = \frac{1.111}{(15/16)(1/0.775) + (1/16)(0)} = 0.918$$

9.27. We should have $\nu^C = 60$ per hour, since this is the arrival rate.

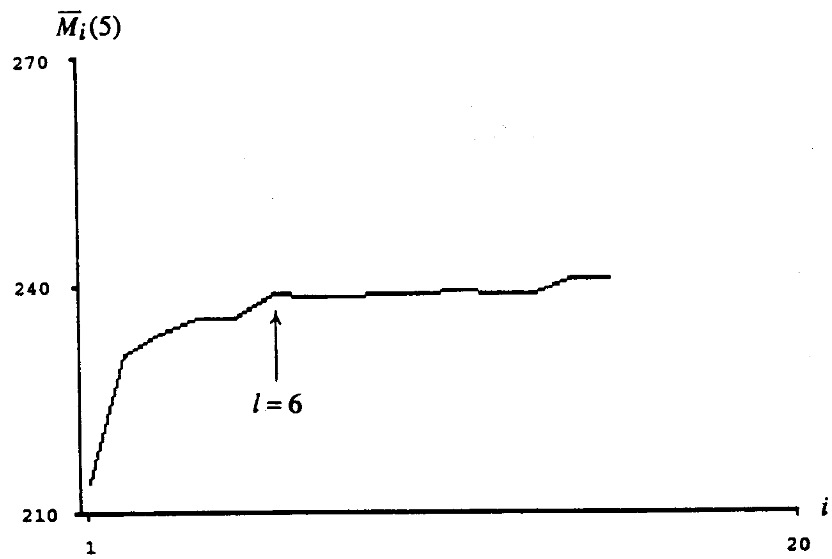
9.28. (*a*) $\rho = \lambda/\omega = 10/[16(2/3) + 8(1/3)] = 0.75$. Note that $\rho = 0.75$ is the proportion of machine-available time that the servers are busy.

(*b*) No. The hourly throughputs will be less during the third 8-hour shift; see (*c*).

(*c*) The averaged process $\overline{N}_i$ is plotted below; note the spikes of approximate height 16 at the beginning of days 2, 3, ..., 20. These correspond to "cleaning out" the queue that built up during the third shift of the previous day.



(*d*) We plot below the moving average $\overline{M}_i(w)$ for $w = 5$, from which we chose a warmup period of $l = 6$ days and obtained the 90 percent confidence interval 240.63 ± 2.28.



9.29. The point estimate for $\nu = E(M)$ is $\overline{M}(200) = 239.60$. For $n = 10$ batches, we obtained $\widehat{\text{Var}}[\overline{\overline{M}}(10, 20)] = 0.85$ and the 90 percent confidence interval 239.60 ± 1.69 for ν. For $n = 5$ batches, we obtained $\widehat{\text{Var}}[\overline{\overline{M}}(5,40)] = 1.06$ and the 90 percent confidence interval 239.60 ± 2.20 for ν. Note that both confidence intervals contain 240.

9.31. The anchor step ($k = 2$) for the induction is

$$\begin{aligned} P(E_1 \cap E_2) &= P(E_1) + P(E_2) - P(E_1 \cup E_2) \\ &\geq (1 - \alpha_1) + (1 - \alpha_2) - 1 \\ &= 1 - \alpha_1 - \alpha_2 \end{aligned}$$

The induction step (i.e., assume the result for $k - 1$ and prove it for k) is

$$\begin{aligned} P\left(\bigcap_{s=1}^{k} E_s\right) &= P\left[\left(\bigcap_{s=1}^{k-1} E_s\right) \cap E_k\right] \\ &\geq P\left(\bigcap_{s=1}^{k-1} E_s\right) + P(E_k) - 1 && \text{(see above)} \\ &\geq \left(1 - \sum_{s=1}^{k-1} \alpha_s\right) + (1 - \alpha_k) - 1 && \text{(by the induction hypothesis)} \\ &= 1 - \sum_{s=1}^{k} \alpha_s \end{aligned}$$

9.32. Assume that $\hat{p}$ will not change appreciably as we make more replications. Let n' be the required number of replications, which can be approximated by solving the following equation for n':

$$0.05 = 1.645\sqrt{0.77(0.23)/n'}$$

which gives $n' = 192$.

9.33. The traffic rate for the link from node SP-1 to node SP-2 is $\lambda_{\mathrm{A}} = 9600$ messages/minute or 0.16 message/ms. Solving the following equation for the required number of links, $s_{\text{1-2}}$:

$$0.4 = \rho_{1-2} = \frac{\lambda_{1-2}}{s_{\text{1-2}}\omega_{1-2}} = \frac{0.16 \text{ message/ms}}{s_{\text{1-2}}(7 \text{ bytes/ms})/(33 \text{ bytes/message})}$$

we get $s_{\text{1-2}} = 1.886$ links, which we round up to 2 links.

9.34. We assume that processing times for a message are a constant 2.5 milliseconds, rather than 3 milliseconds as stated in Example 9.26 (see the errata for the book).

The total traffic rate through nodes STP-A and STP-B is

(7200 + 4800) + (4800 + 7200) + (6400 + 4800) + (4800 + 5600) messages/minute

= 0.76 message/millesecond

Since the traffic through STP-A and STP-B is symmetric, the traffic rate through STP-A is $\lambda_{\mathrm{A}} = 0.38 \text{ message/millisecond}$. (We henceforth denote millisecond by ms.) Solving the following equation for the number of processors, s_{A}:

$$0.4 = \rho_{\mathrm{A}} = \frac{\lambda_{\mathrm{A}}}{s_{\mathrm{A}}\omega_{\mathrm{A}}} = \frac{0.38 \text{ message/ms}}{s_{\mathrm{A}}[1/(2.5 \text{ ms/message})]}$$

we get $s_{\mathrm{A}} = 2.375$ processors, which we round up to 3 processors.

Similarly,

$$1 = \rho_1 = \frac{\lambda_1}{s_1\omega_1} = \frac{0.547 \text{ message/ms}}{s_1[1/(2.5 \text{ ms/message})]}$$

which gives $s_1 = 1.367$ processors, which we round up to 2 processors.

9.35. The overall arrival rate is

$$
\begin{aligned}
& (9600 + 7200 + 4800) \\
& + (8000 + 4800 + 7200) \\
& + (6400 + 4800 + 6400) \\
& + (4800 + 5600 + 4800) \\
& = 74{,}400 \text{ messages/minute} \\
& = \quad 1240 \text{ messages/second}
\end{aligned}
$$

9.36. $\rho_{A} = \dfrac{\lambda_{A}}{3\omega_{A}} = \dfrac{0.38 \text{ message/ms}}{3[1/(2.5 \text{ ms/message})]} = 0.317$

$$\rho_{1-2} = \frac{\lambda_{1-2}}{2\omega_{1-2}} = \frac{0.16 \text{ message/ms}}{2[(7 \text{ bytes/ms})/(33 \text{ bytes/message})]} = 0.377$$

9.37. Let $I_i = \begin{cases} 1 & \text{if } i\text{th confidence interval does not contain its measure of performance} \\ 0 & \text{otherwise} \end{cases}$

$N =$ number of confidence intervals that do not contain their respective measures

Then $N = \sum_{i=1}^{k} I_i$ and $E(N) = E\sum_{i=1}^{k} I_i = \sum_{i=1}^{k} E(I_i) = \sum_{i=1}^{k} \alpha = k\alpha$

9.38. For approximately 90 out of the 100 banks, each of the three confidence intervals will contain its respective performance measure.