#13

Parametric Equations

$$z = x(t) + iy(t)$$

the parametric eq.

$$z = z_1(1-t) + t z_2 \quad [0 \le t \le 1]$$

z_1 ↳ first point z_2 ↳ last points

Example -1 $\frac{1}{2\pi i} \oint_c \frac{e^z}{z-2}$ $|z|=3$ and $|z|=1$

We know, the cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} d(z)$$

Putting $a=2$ and $f(z)=e^z$, we get,

$$e^2 = \frac{1}{2\pi i} \oint \frac{e^z}{z-2} dz$$

$$\Rightarrow 2\pi i \times e^2 = \oint \frac{e^z}{z-2} dz$$

$$\therefore \oint \frac{e^z}{z-2} dz = e^2 \times 2\pi i \text{ (Ans.)}$$

$z=2$ is inside $|z|=3$ and also anlytic inside.

So, $e^2 \times 2\pi i$ is the required integral.

But for $|z|=1$, the point $z=2$ is outside the circle.

So, the integral is 0 (Ans.)

Ex-2 $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ when $|z|=5$

Now, ~~a~~ the Cauchy Integral Formula is

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Here, putting $a=(-\pi/2)$ and $f(z)=\sin 3z$, we get,

$$\sin 3a=\frac{1}{2\pi i}\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$$

$$\Rightarrow \sin\left(-\frac{3\pi}{2}\right)=\frac{1}{2\pi i}\oint \frac{\sin 3z}{z+\pi/2}\,dz$$

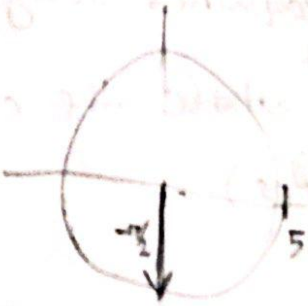
$$\Rightarrow \oint \frac{\sin 3z}{z+\pi/2}\,dz = 2\pi i \times \cancel{6}\left(-\sin\frac{3\pi}{2}\right)$$

$$= 2\pi i \quad \text{(Ans.)}$$

The value of $z=-\pi/2$ is inside circle C and $\sin 3z$ is analytic inside C. So, this is the required integral (Ans.)

Because a circle means z will be ~~within~~ from 0 to 2π. so $(-\pi/2)$ is ~~[illegible]~~ contained

$n = \cancel{\ldots} \ldots -1, \cancel{0}\, 0, 1, 2 \ldots$



Ex-3 Evaluate $\oint_c \frac{e^{3z}}{z-\pi i}$, $|z-1|=4$

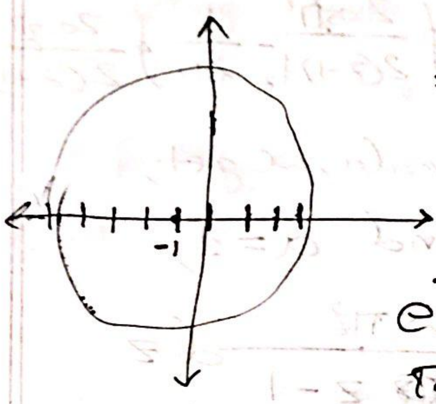
From Cauchy's integral formula,

$$\cancel{f(a)\cdot 2\pi i =}\ f(a)=\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$$

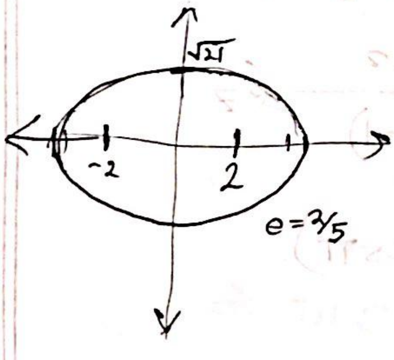
Putting $a=\pi i$, $f(z)=e^{3z}$

$$f(\pi i)=\frac{1}{2\pi i}\oint \frac{e^{3z}}{z-\pi i}\,dz$$

$$\Rightarrow e^{3\pi i}\cdot 2\pi i = \oint_c \frac{e^{3z}}{z-\pi i}\,dz$$



πi is within the circle C and, e^{3z} is analytic with C. So, this the required integral.



πi is ~~also~~ not within the ellipse $|z-2|+|z+2|=6$. ~~So, this is the required integral.~~

Putting $z=\pi i$, we get

$$|\pi i-2|+|\pi i+2| = 7.45 > 6$$

So, the point lies outside ellipse.

Ex-4 $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z^2}{z^2-1}$

~~Cauchy's integral with~~

$$\frac{1}{z^2-1} = \frac{2}{2(z+1)(z-1)} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$$

So, $\frac{1}{2\pi i}\oint_C \frac{2\cos\pi z^2}{z^2-1} = \frac{1}{2\pi i}\oint \frac{2\cos\pi z^2}{2(z-1)}dz - \frac{1}{2\pi i}\oint \frac{2\cos\pi z^2}{2(z+1)}dz$

Using cauch's integral formula, we get,

if $f(z) = \cos\pi z^2$ and $a = 1$,

$$f(1) = \frac{1}{2\pi i}\oint_C \frac{\cos\pi z^2}{z-1}dz$$

$$\frac{(\cos\pi)(2\pi i)}{2} = \oint_C \frac{\cos\pi z^2}{2(z-1)}dz$$

$$\therefore \oint_C \frac{\cos\pi z^2}{2(z-1)}dz = \pi i(\cos\pi).$$

And $a = -1$, we get,

$$f(-1) = \frac{1}{2\pi i}\oint_C \frac{\cos\pi z^2}{z+1}dz$$

$$\Rightarrow \frac{\cos(\pi)(2\pi i)}{2} = \oint \frac{\cos\pi z^2}{2(z+1)}dz$$

$$\Rightarrow \oint \frac{\cos \pi z^2}{2(z+1)} dz = \pi i \cos \pi$$

$$\therefore \frac{1}{2\pi i} \oint_C \frac{\cos \pi z^2}{z^2-1} = \pi i \cos \pi - \pi i \cos \pi = 0$$

(Ans)

<u>Ex-5</u> $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2+1} dz = \sin t$

Using, Cauchy's integral formula,

$$f(a) = 2\pi i \oint_C \frac{f(z)}{z-a} dz$$

~~Putting, a =~~

Now, $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2(z-i)} - \frac{1}{2(z+i)}$

So, $2\pi i \oint_C \frac{e^{zt}}{z^2+1} dz = 2\pi i \oint_C \frac{e^{zt}}{2(z-i)} dz - 2\pi i \oint_C \frac{e^{zt}}{2(z+i)} dz$

Now, putting $a=i$ and $f(z)=e^{zt}$,

$$2\pi i\, f(i) = \oint_C \frac{e^{zt}}{z-i} dz$$

$$\Rightarrow \pi i e^{it} = \oint_C \frac{e^{zt}}{2(z-i)} dz$$

$$\Rightarrow \oint_C \frac{e^z}{2(z-i)} dz = \pi i (\cos t + i \sin t)$$

$$\Rightarrow \frac{1}{2\pi i} \oint \frac{e^z}{2(z-i)} dz = \frac{1}{2} (\cos t + i \sin t) \longrightarrow (i)$$

Putting, $a=-i$ and $f(z)=e^{zt}$, we get,

$$f(-i)=\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z+i}\,dz$$

$$\Rightarrow \frac{e^{-it}}{2}=\frac{1}{2\pi i}\oint \frac{e^{zt}}{2(z+i)}\,dt$$

$$\Rightarrow \frac{1}{2\pi i}\oint_c \frac{e^{zt}}{2(z+i)}\,dt=\frac{1}{2}(\cos t-i\sin t) \quad \text{\_\_\_\_\_\_} \quad (ii)$$

~~So,~~ Adding (i) and (ii), we get,

$$\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2+1}=\frac{1}{2}(\cos t+i\sin t)-\frac{1}{2}(\cos t-i\sin t)$$

$$=\frac{2\cancel{\cos}\sin t}{2}$$

$$=\sin t$$

$$\therefore \frac{1}{2\pi i}\int_c \frac{e^{zt}}{z^2+1}=\sin t \ (\cancel{\text{shown}}) \ (\text{shown})$$

Both $+i$ and $-i$ are within the circle $|z|=3$ and e^{zt} is analytic in the circle when $t>0$.

So,

Ex-6 $\oint_c \frac{e^{iz}}{z^3} dz \quad |z|=2$

We know, $\boxed{f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz}$

~~Putting n=2~~ Now, if $n=2$,

$$f'(z) = ie^{iz}$$

$$f''(z) = -e^{iz}$$

Putting, $n=2$ and $a=0$, ~~we get~~ $f(z) = e^{iz}$, we get,

$$f^2(0) = \frac{2!}{2\pi i} \oint_c \frac{e^{iz}}{z^3} dz$$

$$\Rightarrow \oint_c \frac{e^{iz}}{z^3} dz = \pi i \times (-e^{i\times 0})$$

$$= -\pi i$$

The point $z=0$, is with in circle $|z|=3$ and the function e^{iz} is analytic within the circle, so, this is the required integral. (Ans)

Ex-7, Ex-8 Proof করি না, করবও না।

Ex-9 (a) $\oint_c \frac{\sin^6 z}{z-\pi/6}$

Cauchy's integral formula

$$f(a) = 2\pi i \oint_c \frac{f(z)}{z-a} dz$$

Putting $a = \pi/6$ and $f(z) = \sin^6 z$,

$$\sin^6(\pi/6) = \frac{1}{2\pi i} \oint_c \frac{\sin^6 z}{z-\pi/6} dz$$

$$\Rightarrow \oint_c \frac{\sin^6 z}{z-\pi/6} dz = 2\pi i \left(\frac{1}{2}\right)^6$$

$$= \frac{\pi i}{32} \quad \text{(Ans)}$$

$\pi/6$ is within $|z|=1$

(b) $f(z) = \sin^6 z$

$f'(z) = 6\sin^5 z \cdot \cos z$

$f''(z) = -6\sin^6 z + 30\sin^4 z\cos^2 z$

$a = \pi/6$, $f(z) = \sin^6 z$

$$\oint_c \frac{\sin^6 z}{(z-\pi/6)^3} = 2\pi i \left(-6\sin^6\frac{\pi}{6} + 30\sin^4\frac{\pi}{6}\cos^2\frac{\pi}{6}\right)$$

$$= 2\pi i \left(-\frac{3}{32} + \frac{30}{4} \times \frac{3}{4}\right.$$

$$= 22.3125\pi$$

10) $$\frac{1}{(z^2+1)^2} = \frac{1}{(z^2-i^2)^2} = \frac{1}{(z+i)^2(z-i)^2} = \frac{A}{z+i} + \frac{B}{(z+i)^2} + \frac{C}{z-i} + \frac{D}{(z-i)^2}$$

multiply with $\cancel{(z^2+1)^2}$ $(z+i)^2(z-i)^2$

$$\Rightarrow 1 = A(z+i)(z-i)^2 + B(z-i)^2 + C(z-i)(z+i)^2 + D(z+i)^2$$

$$\therefore 1 = A(z^2-i^2)(z-i) + B(z-i)^2 + C(z^2-i^2)(z+i) + D(z+i)^2$$

if $z=i$, $\quad 1 = D(i+i)^2 = -4D \quad \therefore D = -\frac{1}{4}$

if, $z=-i$, $\quad 1 = B(-2i)^2 = B(-4) \quad \therefore B = -\frac{1}{4}$

$$\Rightarrow 1 = A(z^3 - z^2 i + z - i) + B(z^2 - 2zi - 1) + C(z^3 + z^2 i + z + i) + D(z^2 + 2zi - 1)$$

$$= z^3(A+C) + z^2(-Ai + B + Ci + D) + z(A - 2Bi + C + 2Di) + (-Ai - B + Ci - D)$$

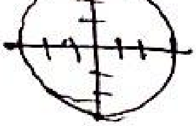
equate the coefficients $\longrightarrow$

$A + C = 0 \qquad -i(A-C) + B + D = 0$

$\therefore A = -C \qquad \Rightarrow -i(-2C) - \frac{1}{4} - \frac{1}{4} = 0$

$\Rightarrow 2iC = \frac{1}{2} \quad \therefore C = \frac{1}{4i} \quad \therefore A = -\frac{1}{4i}$

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz = \frac{1}{2\pi i}\left[\oint \frac{(-\frac{1}{4i})e^{zt}}{(z+i)}dz + \oint \frac{(-\frac{1}{4})e^{zt}}{(z+i)^2}dz + \oint \frac{(\frac{1}{4i})e^{zt}}{(z-i)}dz + \oint \frac{(-\frac{1}{4})e^{zt}}{(z-i)^2}dz\right] \quad \text{—(1)}$$

all of the points are inside the circle 

$$\frac{1}{2\pi i}\oint \frac{\left(-\frac{1}{4i}\right)e^{zt}}{(z+i)}dz = \left[-\frac{1}{4i}e^{zt}\right]_{z=-i} = \left[-\frac{1}{4i}e^{-it}\right]$$

$$\frac{1}{2\pi i}\oint \frac{\left(-\frac{1}{4}\right)e^{zt}}{(z+i)^2}dz = \cancel{f'(z)}\, f'(-i) = \frac{i}{4}e^{-it}$$

$$f(z) = -\frac{1}{4}e^{zt} \quad \therefore f'(z) = -\frac{1}{4}te^{zt}$$

$$\therefore f'(-i) = \cancel{\frac{i}{4}e^{-it}} = -\frac{1}{4}te^{-it}$$

$$\frac{1}{2\pi i}\oint \frac{\left(\frac{1}{4i}\right)e^{zt}}{(z-i)}dz = \left[\frac{1}{4i}e^{zt}\right]_{z=i} = \frac{e^{it}}{4i}$$

$$\frac{1}{2\pi i}\oint \frac{\left(-\frac{1}{4}\right)e^{zt}}{(z-i)^2}dz = f'(i)$$

$$f(z) = -\frac{1}{4}e^{zt} \quad f'(z) = -\frac{1}{4}e^{zt}t$$

$$\therefore f'(i) = -\frac{1}{4}e^{it}t$$

Putting in ①

$$\frac{1}{2\pi i}\oint \frac{e^{zt}}{(z^2+1)^2}dz = -\frac{1}{4i}e^{-it} - \frac{1}{4}te^{-it} + \frac{1}{4i}e^{it} - \frac{1}{4}te^{it}$$

$$= \frac{1}{4i}\left(-e^{-it} - ite^{-it} + e^{it} - ite^{it}\right)$$

$$= \frac{1}{4i}\left(-\cancel{\cos t} + i\sin t - it\cos t - \cancel{t\sin t}\right.$$
$$\left. + \cancel{\cos t} + i\sin t - ti\cos t + \cancel{t\sin t}\right)$$

$$= \frac{1}{4i}\left(2i\sin t - 2it\cos t\right) = \frac{1}{2}\left(\sin t - t\cos t\right)$$