Analytical Solution of Homework-1

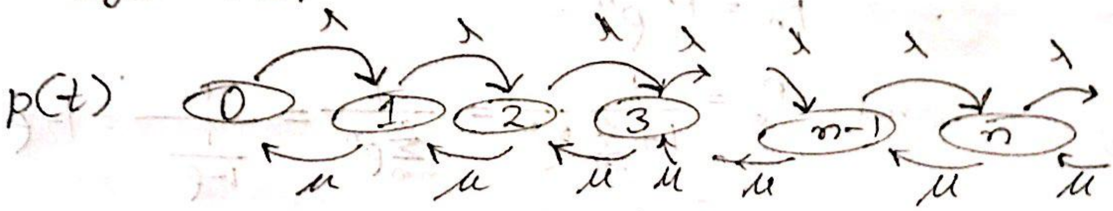
The system follows M/M/1 model as it has a single server.

Let, packet arrive at a ~~...~~ rate λ_i

and packets depart at a rate μ_i

Packet departion occurs due to token generation.

The state transition diagram for number of packets in the system is.



state Transition Diagram

For a state, $P_{i-1}\lambda_{i-1} + P_{i+1}\mu_{i+1} = P_i\mu_i + P_i\lambda_i$

For every state,

state-0, $\lambda_0 P_0 = \mu_1 P_1$ —— (i)

$$(\mu_1+\lambda_1)P_1 = \mu_2 P_2 + \lambda_0 P_0 \quad \text{—— (ii)}$$

$$(\mu_2+\lambda_2)P_2 = \mu_2 P_3 + \lambda_1 P_1 \quad \text{—— (iii)}$$

Adding, the equations with their previous ones,

$$\lambda_1 P_1 = \mu_2 P_2 \qquad \Rightarrow P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$\lambda_2 P_2 = \mu_3 P_3 \qquad \Rightarrow P_2 = \frac{\lambda_1}{\mu_2} P_1$$

.....

$$\therefore P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} \times P_0$$

If all the rates are equal i.e.

$$\lambda_i = \lambda$$

$$\mu_i = \mu \text{ then}$$

$$P_n = P_0 \left(\frac{\lambda}{\mu}\right)^n \text{ and } P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}$$

Taking $\frac{\lambda}{\mu} = \rho$, we get,

$$P_n = P_0 \rho^n$$

$$\text{and, } P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \rho^n} = \frac{1}{\sum_{n=0}^{\infty} \rho^n} = \frac{1}{\frac{1}{1-\rho}} = 1-\rho$$

$$\therefore P_n = (1-\rho) \cdot \rho^n$$

Average no. of packets in the queue,

$$\overline{N} = \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{\infty} (1-\rho) n \rho^n$$

$$= (1-\rho) \frac{\partial}{\partial \rho} \sum_{n=0}^{\infty} \rho^n$$

$$= (1-\rho) \rho \frac{\partial}{\partial \rho} \frac{1}{1-\rho}$$

$$= \frac{\rho}{1-\rho}$$

Analytically, the average queue length is $\frac{\rho}{1-\rho}$ (Ans.)

Using Little's formula, we get,

→ average time in system as

$$T_{sw} = \frac{\bar{N}}{\lambda}$$

$$= \frac{\rho}{1-\rho} \cdot \frac{1}{\lambda} = \frac{1/\mu}{1-\rho}$$

→ average waiting time in queue is found since we know, total time in system = average time in queue + average service time

$$T_W = T_d + T_s$$

$$\Rightarrow T_d = T_w - T_s$$

Average time in queue

$$= \frac{1/\mu}{1-\rho} - T_s$$, where T_s is average service time