Singularities and Cauchy's Integral Formula

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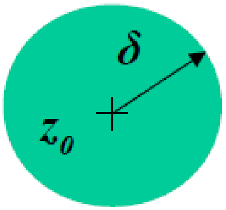
Singular Points

A point at which f (z) is not analytic is called a singular point. There are various types of singular points:

1. Isolated Singularity

The point z_0 at which f (z) is not analytic is called an isolated singular point, if we can a neighborhood of z_0 in which there are not singular points.



If no such a neighborhood of z_0 can be found then we call z_0 a non-isolated singular point.

2. Poles

If we can find a positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$ *and* $\varphi(z) = (z - z_0)^n f(z)$ *is analytic at $z=z_0$*

then $z = z_0$ is called a pole of order n. If n = 1, z is called a simple pole.

Example $$f(z)=\frac{(4z+3)(8z+1)}{(z-3)^2(z-5)^3(z+3)(z+5)}$$

has a pole of order 2 at $z = 3$, a pole of order 3 at $z = 5$, and two simple poles at $z = -3$ and $z = -5$.

3. Branch Points

If $f(z)$ is a multiple valued function at z_0, then this is a branch point.

Examples:

$f(z)=(z-z_0)^{1/n}$ *has a branch point at $z=z_0$*

$f(z)=\ln[(z-z_{01})(z-z_{02})]$ *has a branch points at $z=z_{01}$ and $z=z_{02}$*

4. Removable Singularities

The singular point z_0 is a removable singularity of $f(z)$ if $\lim_{z\to z_0} f(z)$ *exists.*

Examples: The singular point $z = 0$ of $\frac{\sin z}{z}$ *is a removable singularity*

$$\lim_{z\to 0}\frac{\sin z}{z}=1$$

5. Essential Singularities

A singularity which is not a pole, branch point or a removable singularity is called an essential singularity.

Example: $f(z) = e^{1/(z-z_0)}$ *has an essential singularity at* $z = z_0$.

6. Singularities at Infinity

If $\lim_{z\to\infty} f(z) = 0$ *we say that* $f(z)$ *has singularities at* $z \to \infty$. *The type of the singularity is the same as that of* $f(1/w)$ *at* $w = 0$.

Example: The function $f(z) = z^5$ *has a pole of order 5 at* $z = \infty$, *since* $f(1/w) = 1/w^5$ *has a pole of order 5 at* $w = 0$.

Ex-1: For each of the following functions, locate and name the singularities in the finite z plane and determine whether they are isolated singularities or not.

(a) $f(z) = \dfrac{z}{(z^2+4)^2}$, (b) $f(z) = \sec(1/z)$, (c) $f(z) = \dfrac{\ln(z-2)}{(z^2+2z+2)^4}$, (d) $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$

Solution

(a) $$f(z) = \frac{z}{(z^2+4)^2} = \frac{z}{\{(z+2i)(z-2i)\}^2} = \frac{z}{(z+2i)^2(z-2i)^2}.$$

Since

$$\lim_{z\to 2i}(z-2i)^2 f(z) = \lim_{z\to 2i}\frac{z}{(z+2i)^2} = \frac{1}{8i} \neq 0$$

$z = 2i$ is a pole of order 2. Similarly, $z = -2i$ is a pole of order 2.

Since we can find δ such that no singularity other than $z = 2i$ lies inside the circle $|z - 2i| = \delta$ (e.g., choose $\delta = 1$), it follows that $z = 2i$ is an isolated singularity. Similarly, $z = -2i$ is an isolated singularity.

(b) Since $\sec(1/z) = 1/\cos(1/z)$, the singularities occur where $\cos(1/z) = 0$, i.e., $1/z = (2n+1)\pi/2$ or $z = 2/(2n+1)\pi$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Also, since $f(z)$ is not defined at $z = 0$, it follows that $z = 0$ is also a singularity.

Now, by L'Hospital's rule,

$$\lim_{z \to 2/(2n+1)\pi} \left\{ z - \frac{2}{(2n+1)\pi} \right\} f(z) = \lim_{z \to 2/(2n+1)\pi} \frac{z - 2/(2n+1)\pi}{\cos(1/z)}$$

$$= \lim_{z \to 2/(2n+1)\pi} \frac{1}{-\sin(1/z)\{-1/z^2\}}$$

$$= \frac{\{2/(2n+1)\pi\}^2}{\sin(2n+1)\pi/2} = \frac{4(-1)^n}{(2n+1)^2\pi^2} \neq 0$$

Thus the singularities $z = 2/(2n+1)/\pi$, $n = 0, \pm 1, \pm 2, \ldots$ are *poles of order one*, i.e., *simple poles*. Note that these poles are located on the real axis at $z = \pm 2/\pi$, $\pm 2/3\pi$, $\pm 2/5\pi, \ldots$ and that there are infinitely many in a finite interval which includes 0 (see Fig. 3-9).

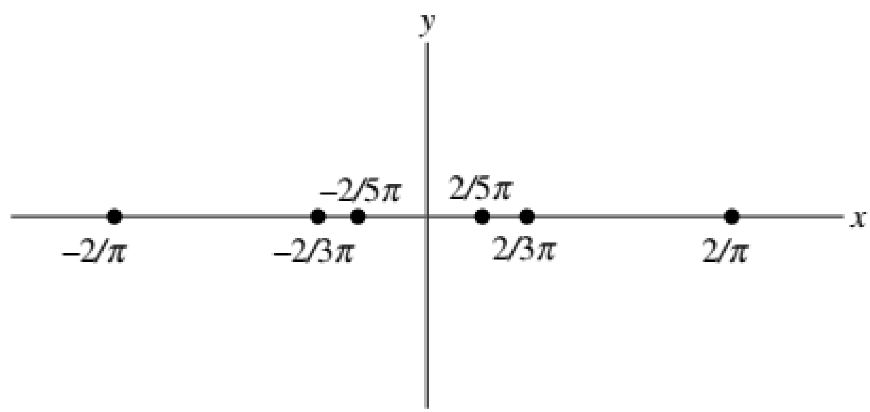


Fig. 3-9

Since we can surround each of these by a circle of radius δ, which contains no other singularity, it follows that they are isolated singularities. It should be noted that the δ required is smaller the closer the singularity is to the origin.

Since we cannot find any positive integer n such that $\lim_{z\to 0}(z-0)^n f(z) = A \neq 0$, it follows that $z = 0$ is an *essential singularity*. Also, since every circle of radius δ with center at $z = 0$ contains singular points other than $z = 0$, no matter how small we take δ, we see that $z = 0$ is a *non-isolated singularity*.

(c) The point $z = 2$ is a *branch point* and is a *non-isolated singularity*. Also, since $z^2 + 2z + 2 = 0$ where $z = -1 \pm i$, it follows that $z^2 + 2z + 2 = (z + 1 + i)(z + 1 - i)$ and that $z = -1 \pm i$ are *poles of order 4* which are *isolated singularities*.

(d) At first sight, it appears as if $z = 0$ is a branch point. To test this, let $z = re^{i\theta} = re^{i(\theta+2\pi)}$ where $0 \leq \theta < 2\pi$.

If $z = re^{i\theta}$, we have

$$f(z) = \frac{\sin(\sqrt{r}e^{i\theta/2})}{\sqrt{r}e^{i\theta/2}}$$

If $z = re^{i(\theta+2\pi)}$, we have

$$f(z) = \frac{\sin(\sqrt{r}e^{i\theta/2}e^{\pi i})}{\sqrt{r}e^{i\theta/2}e^{\pi i}} = \frac{\sin(-\sqrt{r}e^{i\theta/2})}{-\sqrt{r}e^{i\theta/2}} = \frac{\sin(\sqrt{r}e^{i\theta/2})}{\sqrt{r}e^{i\theta/2}}$$

Thus, there is actually only one branch to the function, and so $z = 0$ cannot be a branch point. Since $\lim_{z\to 0} \sin\sqrt{z}/\sqrt{z} = 1$, it follows in fact that $z = 0$ is a *removable singularity*.

Ex-2: (a) Locate and name all the singularities of $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$.

(b) Determine where $f(z)$ is analytic.

Solution

(a) The singularities in the finite z plane are located at $z = 1$ and $z = -2/3$; $z = 1$ is a *pole of order 3* and $z = -2/3$ is a *pole of order 2*.

To determine whether there is a singularity at $z = \infty$ (the point at infinity), let $z = 1/w$. Then

$$f(1/w) = \frac{(1/w)^8 + (1/w)^4 + 2}{(1/w - 1)^3(3/w + 2)^2} = \frac{1 + w^4 + 2w^8}{w^3(1-w)^3(3+2w)^2}$$

Thus, since $w = 0$ is a pole of order 3 for the function $f(1/w)$, it follows that $z = \infty$ is a pole of order 3 for the function $f(z)$.

Then the given function has three singularities: a pole of order 3 at $z = 1$, a pole of order 2 at $z = -2/3$, and a pole of order 3 at $z = \infty$.

(b) From (a) it follows that $f(z)$ is analytic everywhere in the finite z plane except at the points $z = 1$ and $-2/3$.

Cauchy's Integral Formulas

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C [Fig. 5-1]. Then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \tag{5.1}$$

where C is traversed in the positive (counterclockwise) sense.

Also, the nth derivative of $f(z)$ at $z = a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad n = 1, 2, 3, \ldots \tag{5.2}$$

The result (5.1) can be considered a special case of (5.2) with $n = 0$ if we define $0! = 1$.

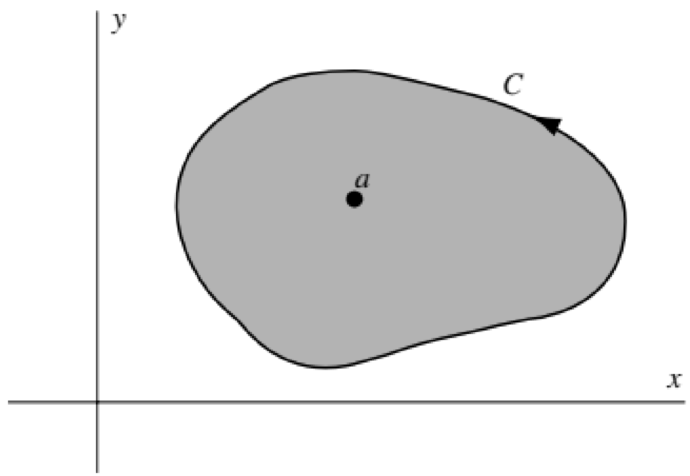


Fig. 5-1

The results (5.1) and (5.2) are called *Cauchy's integral formulas* and are quite remarkable because they show that if a function $f(z)$ is known *on* the simple closed curve C, then the values of the function and all its derivatives can be found at all points *inside* C. Thus, if a function of a complex variable has a first derivative, i.e., is analytic, in a simply-connected region $\mathcal{R}$, all its higher derivatives exist in $\mathcal{R}$. This is not necessarily true for functions of real variables.

Ex-1: Let $f(z)$ be analytic inside and on the boundary C of a simply-connected region $\mathcal{R}$. Prove *Cauchy's integral formula*

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

Solution

The function $f(z)/(z-a)$ is analytic inside and on C except at the point $z = a$ (see Fig. 5-2).

we have

$$\oint_C \frac{f(z)}{z-a} dz = \oint_\Gamma \frac{f(z)}{z-a} dz \qquad (1)$$

where we can choose Γ as a circle of radius ϵ with center at a. Then an equation for Γ is $|z-a| = \epsilon$ or $z - a = \epsilon e^{i\theta}$ where $0 \le \theta < 2\pi$. Substituting $z = a + \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta}$, the integral on the right of (1) becomes

$$\oint_\Gamma \frac{f(z)}{z-a} dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta}) i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} d\theta = i \int_0^{2\pi} f(a+\epsilon e^{i\theta})\, d\theta$$

Thus we have from (1),

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta \tag{2}$$

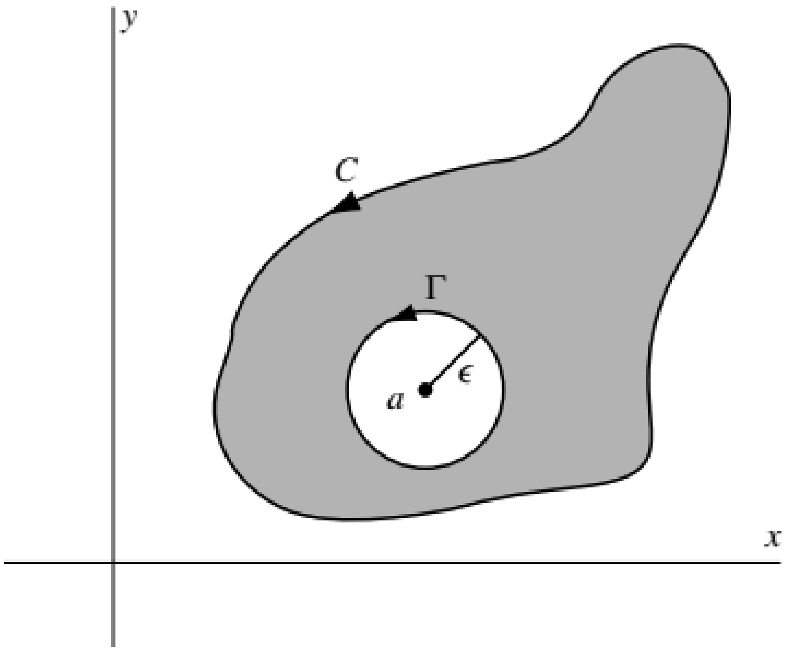


Fig. 5-2

Taking the limit of both sides of (2) and making use of the continuity of $f(z)$, we have

$$\oint_C \frac{f(z)}{z-a} dz = \lim_{\epsilon \to 0} i \int_0^{2\pi} f(a + \epsilon e^{i\theta})\, d\theta$$

$$= i \int_0^{2\pi} \lim_{\epsilon \to 0} f(a + \epsilon e^{i\theta})\, d\theta = i \int_0^{2\pi} f(a)\, d\theta = 2\pi i\, f(a) \tag{3}$$

so that we have, as required,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

Ex-2: Evaluate:

(a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, (b) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z| = 3$.

Solution

(a) Since $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

By Cauchy's integral formula with $a = 2$ and $a = 1$, respectively, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz = 2\pi i\{\sin \pi(2)^2 + \cos \pi(2)^2\} = 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz = 2\pi i\{\sin \pi(1)^2 + \cos \pi(1)^2\} = -2\pi i$$

since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then, the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

(b) Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \tag{1}$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$. Hence (1) becomes

$$8e^{-2} = \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4} dz$$

from which we see that the required integral has the value $8\pi i e^{-2}/3$.

Ex-1: Evaluate $\dfrac{1}{2\pi i}\oint\limits_C \dfrac{e^z}{z-2}dz$ if C is: (a) the circle $|z|=3$, (b) the circle $|z|=1$.

Ex-2: Evaluate $\oint\limits_C \dfrac{\sin\ 3z}{z+\pi/2}dz$ if C is the circle $|z|=5$.

Ex-3: Evaluate $\oint\limits_C \dfrac{e^{3z}}{z-\pi i}dz$ if C is: (a) the circle $|z-1|=4$, (b) the ellipse $|z-2|+|z+2|=6$.

Ex-4: Evaluate $\dfrac{1}{2\pi i}\oint\limits_C \dfrac{\cos\ \pi 2}{z^2-1}dz$ around a rectangle with vertices at: (a) $2\pm i,\ -2\pm i$; (b) $-i,\ 2-i,\ 2+i,\ i$.

Ex-5: Show that $\dfrac{1}{2\pi i}\oint\limits_C \dfrac{e^{zt}}{z^2+1}dz=\sin t$ if $t>0$ and C is the circle $|z|=3$.

Ex-6: Evaluate $\oint\limits_C \dfrac{e^{iz}}{z^3}dz$ where C is the circle $|z|=2$.

Ex-7: Suppose C is a simple closed curve enclosing $z=a$ and $f(z)$ is analytic inside and on C. Prove that

$$f'''(a)=\frac{3!}{2\pi i}\oint\limits_C \frac{f(z)\,dz}{(z-a)^4}.$$

Ex-8: Prove Cauchy's integral formulas for all positive integral values of n. [*Hint*: Use mathematical induction.]

Ex-9: Given C is the circle $|z| = 1$. Find the value of (a) $\oint_C \frac{\sin^6 z}{z - \pi/6} dz$, (b) $\oint_C \frac{\sin^6 z}{(z - \pi/6)^3} dz$.

Ex-10: Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2 + 1)^2} dz$ when $t > 0$ and C is the circle $|z| = 3$.

Thanks a lot …