Lecture - 18(b)

12-08-20

Fibonacci Numbers and the Golden Ratio

0, 1, 1, 2, 3, 5, 8 ....

The next number will be the sum of the previous two numbers. The Fibonacci number are perhaps the common numbers as it is seen everywhere - in nature, music, geometry and so on. Patterns in nature, for example arrangement of seeds in plants, follow this sequence. But perhaps, what's more interesting is the ratio that crafts these numbers.

What if we wanted to find the 100^{th} number of the sequence? We can just find the next number and repeat the process. But it's ineffecient and will take a lot of time. Luckily, linear algebra provides us a better solution. Often dubbed as the most irrational number, the golden ratio can be used to compute Fibonacci numbers.

Let's find this number, by looking at the sequence again. We can generalize the sequence as the first and second row of a matrix.

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_k + F_{k-1} \\ F_k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$$

$$= A \begin{bmatrix} F_k \\ F_k - 1 \end{bmatrix}$$

$$= A \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix} \right)$$

$$= A^2 \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix}$$

We can generalize it as,

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^k \begin{bmatrix} F_{k-(k-1)} \\ F_{k-k} \end{bmatrix}$$

$$= A^k \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

The first term, $F_0 = 0$

" 2nd " , $F_1 = 1$

$$\therefore \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So, we can see at every step, the matrix is multiplied by A, to get the next value.

After 100 steps,

$$u_{100} = A^{100} u_0 \quad \text{where} \quad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

Every step ~~it~~ u_0 is multiplied by the matrix A,

The pattern will be like

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \; \cdots \cdots \; u_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$$\left(u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}\right), \left(u_1 = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}\right), \left(u_2 = \begin{bmatrix} F_3 \\ F_2 \end{bmatrix}\right)$$

As we can see that for a high value of A we need to use diagonalization to resolve the problem.

$$A = S \Lambda S^{-1}$$

$$\Rightarrow A^{100} = S \Lambda^{100} S^{-1}$$

Now, $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(1-\lambda) - 1 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

$$\therefore \lambda = \frac{1 \pm \sqrt{5}}{2} \quad \left[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right]$$

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$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$$

Now, taking $\lambda_1 = 1.618$,

$$\begin{bmatrix} 1-1.618 & 1 \\ 1 & -1.618 \end{bmatrix} \vec{x_1} = \vec{0}$$

If we look at the second row then it is $(1, -\lambda)$. So if $x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ then the multiplication will result in zero.

$$\therefore x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

And, for λ_2, we get,

$$\begin{bmatrix} 1+0.618 & 1 \\ 1 & 0.618 \end{bmatrix} \vec{x_2} = \vec{0}$$

Similarly, $\vec{x_2} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$$\therefore S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \text{ (matrix of eigenvectors)}$$

and $S^{-1} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$

And, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (diagonal matrix of eigenvalues)

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^k \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= S\Lambda^k S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k \\ -\lambda_2^k \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 \lambda_1^k - \lambda_2 \lambda_2^k \\ \lambda_1^k - \lambda_2^k \end{bmatrix}$$

$$\therefore \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2} \\ \dfrac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} \end{bmatrix}$$

$$F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$$

$$F_{100} = \frac{(1+\sqrt{5})^{100} - (1-\sqrt{5})^{100}}{2^{100}\sqrt{5}}$$

(Ans)

$$\lambda_1 - \lambda_2 = \frac{1+\sqrt{5}-1+\sqrt{5}}{2} = \sqrt{5}$$

Home-task: Find F_{1000} where

$$F_{k+2} = F_{k+1} + 3F_k$$

Ans: For the given series, the sequence is

$F_0 = 0$; $F_1 = 1$; $F_2 = F_1 + 3F_0 = 1$; $F_3 = F_2 + 3F_1 = 4$

and so on.

$\therefore u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $u_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ where $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

Now, $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \cancel{\begin{bmatrix} F_k + 3F_{k-1} \\ F_{k-1} + 3F_{k-2} \end{bmatrix}}$

$$\begin{bmatrix} F_k + 3F_{k-1} \\ F_k \end{bmatrix}$$

$$\therefore \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^k \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

Since, $F_0 = 0$ and $F_1 = 0$.

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $k = 1000$, $u_{1000} = A^{1000} u_0$

Now, $A = S \Lambda S^{-1}$

$$\Rightarrow A^{1000} = S \Lambda^{1000} S^{-1}$$

And, $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda + \lambda^2 - 3 = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{13}}{2}$$

$$\therefore \lambda_1 = \frac{1+\sqrt{13}}{2} \approx 2.3$$

$$\lambda_2 = \frac{1-\sqrt{13}}{2} \approx -1.3$$

Now, taking $\lambda_1 = 2\cdot3$

$$\begin{bmatrix} 1-2\cdot3 & 3 \\ 1 & -2\cdot3 \end{bmatrix} \vec{x_1} = \vec{0}$$

$$\therefore \vec{x_1} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

Similarly, $\vec{x_2} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

So, $S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$

$$S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

And, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Now, $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = S \Lambda^k S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k \\ \lambda_2^k \end{bmatrix}$$

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 \lambda_1^k - \lambda_2 \lambda_2^k \\ \lambda_1^k - \lambda_2^k \end{bmatrix}$$

$$\therefore F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$$

$$\Rightarrow F_{1000} = \frac{(1+\sqrt{13})^{1000} - (1-\sqrt{13})^{1000}}{2^{1000}\sqrt{13}} \quad \text{(Ans.)}$$