B.Sc. Engg. / HD CSE 4th Semester 13 November 2020

ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

MID EXAMINATION **SUMMER SEMESTER, 2019-2020**

DURATION: 1 Hour 30 Minutes **FULL MARKS: 75**

Math 4441: Probability and Statistics

Programmable calculators are not allowed. Answer all the questions

Figures in the right margin indicate full marks.

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1. There are 12 red balls and 10 white balls in an urn. Two balls are drawn, at random, from the urn and the balls are discarded without looking at their colors. Finally, a third ball is drawn, at random, from the urn.
   a) If it is given that the colors of both the discarded balls are red, find the probability that the color of the third ball is red. 5
   b) Find the probability that the color of the third ball is red. 10
   c) If the color of the third ball is red, find the probability that both the discarded balls were white. 10

2. a) There are 10 red balls and 15 white balls in an urn. Suppose you pick balls randomly from the urn – one at a time and without replacement – until you get 5 red balls. Let the random variable X represent the number of balls you picked to get 5 red balls. Find the PMF of X. 8
   Note: Balls are picked without replacement, probability of picking a ball of particular color changes after each pick up. Also, the experiment ends with the pick up of a red ball.
   b) Suppose you need to know whether a group of 100 peoples in your area have a certain disease or not. Say there is a special blood test available and you need to test the blood of 100 peoples. Assuming that the test is expensive, you decide to divide the peoples into 10 groups with 10 peoples in each group. Now blood samples of all the 10 peoples of a group are pooled and analyzed together, instead of doing this for each individual separately. If the test result of a group is negative, a single test is enough for the group. However, if the test result is positive for a group, each of the 10 people in the group is individually tested again. 10
   Suppose the probability that a person is infected by the disease is 0.10 for all peoples independent from each other. Let X represent the number of groups with at least one people infected. Find E[X].
   c) Suppose you have two coins – one of which is a fair coin and the other one is a biased coin with probability of getting head is p, where p = 1/[mod (your_student_id, 3) + 3], mod returns the remainder of the division of the first number by the second. 7
   You pick one coin randomly and give the other coin to your friend. Then you and your friend continue to toss the respective coins repeatedly until both of you get the same outcome (an outcome of a toss is either a head or a tail). Let X be the number of tosses required. Find the PMF of X.

3. a) Suppose you invite your friend for a dinner at a five-star hotel. You and your friend decide to meet in the lobby of the hotel between 7:30 pm and 8:00 pm. If you arrive at the lobby at a random time uniformly distributed during this period, find the probability that the first to arrive has to wait at least 12 minutes. 7
   b) A fair die is rolled twice. Let the random variable X represent the difference of the number of dots on the top face of the two rolls and the random variable Y represent the maximum of the number of dots on the top face of the two rolls. Find the joint PMF of X and Y. 10

c) Random variables X and Y are uniformly distributed in a triangular area defined by (0, 0), (0, 1) and (1, 0). Find the following two probabilities:

i. $P[0 < X < ½, 0 < Y < ½, X>Y]$. 4

ii. $P[½ < X < 1, 0 < Y < ½]$ 4

Appendix A: PMF/PDF, expected values and Variance of some Random Variables

Distribution	PMF/PDF	Expected value	Variance
Bernoulli	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & otherwise \end{cases}$	$E[X] = p$	$Var[X] = p(1-p)$
Geometric	$P_X(x) = \begin{cases} p(1-p)^{x-1} & x \geq 1 \\ 0 & otherwise \end{cases}$	$E[X] = 1/p$	$Var[X] = (1-p)/p^2$
Binomial	$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 1, \ldots, n \\ 0 & otherwise \end{cases}$	$E[X] = np$	$Var[X] = np(1-p)$
Pascal	$P_X(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x = k, k+1, \ldots \\ 0 & otherwise \end{cases}$	$E[X] = k/p$	$Var[X] = k(1-p)/p^2$
Poisson	$P_X(x) = \begin{cases} \frac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = \alpha$ $\alpha = \lambda T$	$Var[X] = \alpha$
Hypergeometric	$P_X(x) = \begin{cases} \frac{\binom{m}{x}\binom{n}{r-x}}{\binom{m+n}{r}} & x = 0,1, \ldots, r \\ 0 & otherwise \end{cases}$; m, n and r are integer $r \leq \min(m, n)$	$E[X] = \frac{rm}{m+n}$	$Var[X] = \frac{rmn}{(m+n)^2}\left(1 - \frac{r-1}{m+n-1}\right)$
Uniform (discrete)	$P_X(x) = \begin{cases} \frac{1}{b-a+1}, & x = a, a+1, a+2, \ldots, b \\ 0, & \text{otherwise} \end{cases}$	$E[X] = \frac{a+b}{2}$	$Var[X] = \frac{(b-a)(b-a+2)}{12}$
Exponential	$f_X(x) = \begin{cases} ae^{-ax} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = 1/a$	$Var[X] = 1/a^2$
Gaussian	$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} & \sigma > 0 \\ 0 & otherwise \end{cases}$	$E[X] = \mu$	$Var[X] = \sigma^2$
Uniform (Continuous)	$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & otherwise \end{cases}$	$E[X] = \frac{a+b}{2}$	$Var[X] = \frac{(b-a)^2}{12}$

Appendix B: Basic Rules in Probability

$P[ABC] = P[A]P[B\|A]P[C\|AB]$	$P[A] = \sum_i^n P[A\|B_i]P[B_i]$	$P[B_i\|A] = \frac{P[A\|B_i]P[B_i]}{\sum_i^n P[A\|B_i]P[B_i]}$
$P_X(x) = P[X = x] = \sum_y P_{XY}(x, y).$	$E[Y] = \sum_y \sum_x y P_{XY}(x, y).$	$Var[Y] = \sum_y \sum_x (y - \mu_y)^2 P_{XY}(x, y).$
$F_{X,Y}(x_1 \leq X < x_2, y_1 \leq Y < y_2) = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1)$		
$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x, y)\, dy\, dx$		
$f_X(x) = \int_{-\infty}^{\infty} f_{x,y}(x, y)\, dy$	$\text{Cov}[X, Y] = E\Big[(X - E[X])(Y - E[Y])\Big]$	$Cov[X, Y] = E[XY] - \mu_X \mu_Y$
$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}} = \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y}.$	$r_{XY} = E[XY]$	