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Ans. to Q. no. 1(a)

We know, the nth root a complex number can be found using,

$$z^{1/n} = (x+iy)^{1/n} = r^{1/n}\left\{\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\}$$

Given, the number $z = 1+i$. We need to find 4^{th} root

Now, $r = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

So, $$(1+i)^{1/4} = (\sqrt{2})^{1/4}\left\{\cos\left(\frac{\pi/4+2k\pi}{4}\right) + i\sin\left(\frac{\pi/4+2k\pi}{4}\right)\right\}$$

$$= 2^{1/8}\cdot\left\{\cos\left(\frac{\pi/4+2k\pi}{4}\right) + i\sin\left(\frac{\pi/4+2k\pi}{4}\right)\right\}$$

$k = 0, 1, 2, 3,$

Putting $k=0$, $z_1 = 2^{1/8}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)$

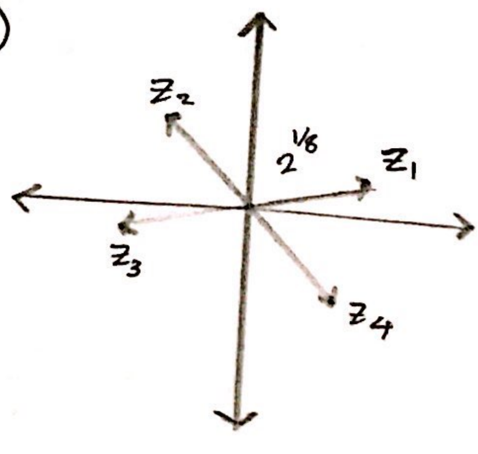
Putting $k=1$, $z_2 = 2^{1/8}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$

Putting, $k=2$, $z_3 = 2^{1/8}\left(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right)$

~~Putting, k=3, z4 =~~

Putting $k=3$, $z_4 = 2^{1/8}\left(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right)$

So, z_1, z_2, z_3 and z_4 are the four roots of $z = 1+i$ (Ans.)



Ans. to Q. no. 1 (b)

~~We know, the n^{th} root of a number is~~

$$\cancel{z^{1/n} = (x+iy)^{1/n} = r^{1/n}\left\{\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\}}$$

~~Here, $z = 1+\sqrt{3}i$ and $n =$~~

Given, $z = 1+\sqrt{3}i$

Find z^9.

Now, $z^2 = (1+\sqrt{3}i) \times (1+\sqrt{3}i)$

$= 1 + 2\sqrt{3}i - 3$

$= -2 + 2\sqrt{3}i$

Squaring, we get,

$z^4 = (-2+2\sqrt{3}i)^2$

$= 4 - 8\sqrt{3}i - 12$

$= -8 - 8\sqrt{3}i$

Squaring, we get,

$z^8 = (-8-8\sqrt{3}i)^2$

$= 64 + 128\sqrt{3}i - 192$

$= -128 + 128\sqrt{3}i$

$\therefore z^9 = z^8 \times z = (-128+128\sqrt{3}i)(1+\sqrt{3}i)$

$= -128 + 128\sqrt{3}i - 128\sqrt{3}i - 384$

$= -512$ (Ans.)

So, $z^9 = -512$

Ans. to Q no. 1(c)

Given, $z^2(1 - z^2) = 16.$

$$\Rightarrow z^2 - z^4 = 16$$

$$\Rightarrow z^4 - z^2 + 16 = 0$$

$$\Rightarrow z^4 + 8z^2 + 16 - 9z^2 = 0$$

$$\Rightarrow (z^2+4)^2 - 9z^2 = 0$$

$$\Rightarrow (z^2+4)^2 - (3z)^2 = 0$$

$$\Rightarrow (z^2 + 4 + 3z)(z^2 + 4 - 3z) = 0$$

So, the salutions are

$$z^2 + 3z + 4 = 0$$
$$z^2 - 3z + 4 = 0$$

Now, solving $z^2 + 3z + 4 = 0$,

$$\cancel{\Rightarrow z^2 + 4z + 4 - z = 0}$$

$$\cancel{\Rightarrow (z+2)^2 - z}$$

$$\Rightarrow z^2 + 2 \cdot \frac{3}{2} \cdot z + \left(\frac{3}{2}\right)^2 + 4 - \left(\frac{3}{2}\right)^2 = 0$$

$$\Rightarrow \left(z + \frac{3}{2}\right)^2 + \frac{7}{4} = 0$$

$$\Rightarrow \left(z + \frac{3}{2}\right)^2 = -\frac{7}{4}$$

$$\Rightarrow z + \frac{3}{2} = \pm \frac{\sqrt{7}}{2} i$$

$$\therefore z = -\frac{3}{2} \pm \frac{\sqrt{7}}{2} i$$

And, solving $z^2-3z+4=0$,

$$\Rightarrow z^2 - 2\cdot\frac{3}{2}\cdot z + \left(\frac{3}{2}\right)^2 + 4 - \left(\frac{3}{2}\right)^2 = 0$$

$$\Rightarrow \left(z - 3/2\right)^2 + 7/4 = 0$$

$$\Rightarrow \left(z - 3/2\right)^2 = -7/4$$

$$\Rightarrow z - 3/2 = \pm\frac{\sqrt{7}}{2}i$$

$$\therefore z = 3/2 \pm \frac{\sqrt{7}}{2}i$$

So, the values of z are

$$3/2 \pm \frac{\sqrt{7}i}{2} \text{ and } -3/2 \pm \frac{\sqrt{7}}{2}i \text{ (Ans.)}$$

Ans. to Q. no. 2(a)

$$\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$$

Let, $f(z) = (3+i)z^4 - z^2 + 2z$

$g(z) = z+1$

$f(i) = (3+i)i^4 - i^2 + 2i \neq 0$

$g(i) = i+1 \neq 0.$

$f(i) \neq 0;\ g(i) \neq 0$

So, $\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$

$$= \frac{(3+i)i^4 - i^2 + 2i}{i+1}$$

$$= \frac{3+i+1+2i}{i+1}$$

$$= \frac{4+3i}{i+1} \text{ (Ans.)}$$

Ans. to Q. no. 2(b)

$$f(z) = 2x^2 + y + i(y^2 - x)$$

~~Now,~~ $\cancel{\frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}}$

~~Given,~~ $\cancel{\frac{df(z)}{dz} = 2x^2 + y + iy^2 - ix}$

$$\cancel{= (\sqrt{2}x)^2 +}$$

At every point, the functional value is equal to the limiting value. There is no point where limiting value $\neq$ functional value.

So, this function is analytical everywhere.

Ans. to Q. no. ~~BR~~ 2(e)

Given, $u(x,y) = x^3 - 3xy^2 - 5y$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

Again, $\frac{\partial u}{\partial y} = -6xy - 5$

$$\frac{\partial^2 u}{\partial y^2} = -6x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

So, u is a harmonic function anywhere in the complex plane. (proved).

We need to find the harmonic conjugate function.

By total differezentiation,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

Using, C.R. equations, $dv = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$

$$dv = (6xy + 5) dx + (3x^2 - 3y^2) dy$$

$$\Rightarrow v = \int 6xy + 5 \, dx + \int 3x^2 - 3y^2 \, dy$$

$$\Rightarrow v = \int \frac{6x^2y}{2} + 5x + 3x^2y - y^3 + c$$

$$\therefore v = 3x^2y + 5x + 3x^2y - y^3 + c$$

$$= 5x + 6x^2y - y^3 + c$$

$$\therefore f(z) = u + iv$$

$$= x^3 - 3xy^2 - 5y + i5x + i6x^2y - iy^3 + ic$$

$$= x^3 + 3x(iy)^2 + 3x^2iy + (iy)^3 + 3x^2iy - 5y + i5x$$

$$= (x+iy)^3 + 5i(x+iy) + 3x^2iy$$

$$\therefore f(z) = z^3 + 5iz + 3x^2iy$$

Ans. to Q. no 3(a)

Given, $\sin z = 5$

$$\frac{e^{iz} - e^{-iz}}{2i} = 5$$

$$\Rightarrow e^{iz} - e^{-iz} = 10i$$

$$\Rightarrow e^{iz} - \frac{1}{e^{iz}} = 10i$$

$$\Rightarrow e^{2iz} - 1 = 10i\, e^{iz}$$

$$\Rightarrow e^{2iz} - 10i e^{iz} - 1 = 0$$

$\Rightarrow$ Let $e^{iz} = \omega$,

Now, $\omega^2 - 10i\omega - 1 = 0$

$$\Rightarrow \omega^2 - 2\cdot\omega\cdot 5i + (5i)^2 - (5i)^2 - 1 = 0$$

$$\Rightarrow (\omega - 5i)^2 = (5i)^2 + 1$$

$$\Rightarrow (\omega - 5i)^2 = -24$$

$$\Rightarrow \omega - 5i = 2\sqrt{6}\, i$$

$$\Rightarrow \omega = (2\sqrt{6} + 5)i$$

So, $e^{iz} = (2\sqrt{6} + 5)i$

$$\Rightarrow iz = \ln\{(2\sqrt{6} + 5)i\} + i\theta$$ (taking principal branch)

$$\Rightarrow z = \frac{\ln\{(2\sqrt{6} + 5)i\}}{i} + (\theta + 2k\pi)$$

where $k = 0, 1, 2, 3 \ldots$

(Ans)

Ans. to Q.no. 3(b)

(b) Given, $\int_c \bar{z}\, dz$ by $x = 3t$

$y = t^2$

$-1 \le t \le 4.$

~~Now, $\int_c \bar{z}$~~ $dx = 3 \cdot dt$

$dy = 2t \cdot dt$

Now, $\int_c \bar{z}\, dz$

$= \int_c (x - iy)\, d(x + iy)$

$= \int_{t=-1}^{4} (3t - it^2)(dx + i\,dy)$

$= \int_{-1}^{4} (3t - it^2)(3dt + i2t\,dt)$

$= \int_{-1}^{4} (3t - it^2)(3 + i2t)\, dt$

$= \int_{-1}^{4} 9t - i3t^2 + i6t^2 + 2t^3\, dt$

$= \int_{-1}^{4} 2t^3 + 3it^2 + 9t\, dt$

$= \left[\frac{2t^4}{4} + \frac{3it^3}{3} + \frac{9t^2}{2}\right]_{-1}^{4}$

$= \left[\frac{t^4}{2} + it^3 + \frac{9t^2}{2}\right]_{-1}^{4}$

$$= \left(\frac{4^4}{2} + i4^3 + \frac{9 \cdot 4^2}{2}\right) - \left(\frac{1}{2} - i + \frac{9}{2}\right)$$

$$= (128 + 64i + 72) - (0.5 - i + 4.5)$$

$$= 205 + 63i \text{ (Ans.)}$$

Ans. to Q. no. 3(c)

Given, $\oint_c \frac{5z+7}{z^2+2z-3} dz$; $|z-2| = 2$

Now, $z^2 + 2z - 3 = z^2 + 2z + 1 - 4$

$$= (z+1)^2 - 2^2$$

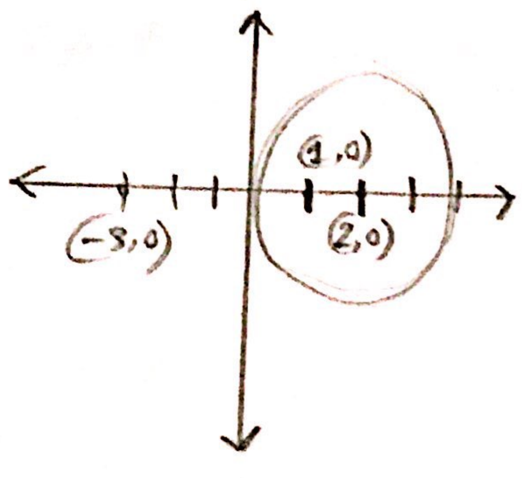
$$= (z+1+2)(z+1-2)$$

$$= (z+3)(z-1)$$

So, the values of $z = 1, -3$.

The given, circle contains the point $z = 1$.

but the other ~~point~~ pole $z = -3$ lies outside the circle.



Center is $(2,0)$ and radius is 2.

So, $\oint_c \frac{5z+7}{z^2+2z-3}\,dz$

$$= \oint_c \frac{5z+7}{(z+3)(z-1)}\,dz$$

$$= \oint_c \frac{\frac{5z+7}{z+3}}{z-1}\,dz$$

$$= 2\pi i \left[\frac{5z+7}{z+3}\right]_{z=1}$$

$$= 2\pi i \left(\frac{5+7}{1+3}\right)$$

$$= 2\pi i \left(\frac{12}{4}\right)$$

$$= 6\pi i \text{ (Ans.)}$$

$$\therefore \oint_c \frac{5z+7}{z^2+2z-3}\,dz = 6\pi i \text{ (Ans.)}$$