Lecture - ~~16~~ 17 (a) 10-08-20

Cramer's Rule

As we have learnt previously,

$$A^{-1} = \frac{1}{\det(A)} (\text{cof } A)^T$$

For the equation,

$Ax = b$, we solve for x.

$$\Rightarrow x = A^{-1} b$$

$$\Rightarrow x = \frac{1}{\det(A)} (\text{cof } A)^T b$$

To under Cramer's rule, we need to take a look at $\vec{x}$ which has multiple components x_1, x_2 and so on. Let's say, it's 2 dimensional.

So, $x_1 = \dfrac{\det B_1}{\det A}$ [B_1 is some matrix which is the first row after multiplying $(\text{cof } A)^T$ and b]

and $x_2 = \dfrac{\det B_2}{\det A}$ [Same as B_1 but 2nd row]

There can be n-such components if x is n-dimensional.

The value of B_1 will have b in the first column and rest of the columns will be the same as A.

So, it's basically A matrix but column-1 will be replaced by the right-hand side b.

$$B_1 = \left[\begin{array}{c|c} b & \begin{matrix}(n-1)\\ \text{columns}\\ \text{of}\\ A\end{matrix}\end{array}\right]$$

This perfectly fits our formula. The entries from the given scenario is $\frac{1}{\det A} C^T b$. The first component of $C^T b$ will have $c_{11}b_1 + c_{21}b_2$ ---- and so on.

So, in Cramer's rule to find a particular component, we need to replace that column in the original matrix A with b and find the determinant. The determinant is then divided by determinant of A.

Mathematically, for equation $Ax=b$,

$$x_i = \frac{\det B_i}{\det A}$$ (where $i \leq n$ denoting the column number)

and $B_i = A$ with column i replaced by b.

Here's another way to look at the problem. We can re-write $Ax=b$ as a system of linear equations, say n, linear equations.

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 \cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 \cdots\cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 \cdots\cdots + a_{nn}x_n = b_n \end{array}$$

Using Cramer's rule, we find each individual component x_1, x_2 and so on.

Now, $\det(A) = C_1 a_{1j} + \cdots\cdots + C_n a_{nj}$

It can be derived by ~~combining~~ combining the system of equations by taking C_1 times the first equation, C_2 times the second and so on where $C_1, C_2 \dots C_n$ are co-effecients which depend on the columns of A. This is also known as Laplace expansion or co-factor expansion.

With this established the right hand side will face a similiar situation with $c_1 b_1 + c_2 b_2 - \dots + c_n b_n$ multiplied to the column vector b.

Hence, we can conclude that,

$$\cancel{\det(A) \cdot x_\theta}$$

$$(c_1 a_{1j} + \dots c_n a_{nj}) \cdot x_j = (c_1 b_1 + \dots c_n b_n) \cdot b$$

$$\Rightarrow \det(A) \cdot x_j = \det(B_j) \text{ where}$$

[B_j is matrix A with j^{th} column replaced by b]

$$\therefore x_j = \frac{\det(B_j)}{\det(A)} \text{ (proved).}$$

Example: Solve $Ax = b$ where,

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$ using

Cramer's rule.

Ans: We need to find $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{\det\left(\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}\right)}$$

$$= \frac{\begin{vmatrix} 4 & -3 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix}}$$

$$\therefore x_1 = \frac{-2}{17}$$

$$x_2 = \frac{\det(B_2)}{\det A} = \frac{\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix}}$$

$$\therefore x_2 = \frac{-24}{17}$$

$$\text{So, } x = \begin{bmatrix} -\frac{2}{17} \\ -\frac{24}{17} \end{bmatrix} \text{ (Ans.)}$$

Similarly, any $(n \times n)$ dimensional matrices or n linear equations can be solved using Cramer's rule.

Ex-2: Use Cramer's Rule to solve

$$x+y-z=6$$
$$3x-2y+z=-5$$
$$x+3y-2z=14$$

Ans: The corresponding matrices are:

$$A=\begin{bmatrix}1&1&-1\\3&-2&1\\1&3&-2\end{bmatrix},\quad B_1=\begin{bmatrix}6&1&-1\\-5&-2&1\\14&3&-2\end{bmatrix},\quad B_2=\begin{bmatrix}1&6&-1\\3&-5&1\\1&14&-2\end{bmatrix}$$

and $B_3=\begin{bmatrix}1&1&6\\3&-2&-5\\1&3&14\end{bmatrix}$.

$$\therefore x=\frac{\det(B_1)}{\det(A)}=\frac{\begin{vmatrix}6&1&-1\\-5&-2&1\\14&3&2\end{vmatrix}}{\begin{vmatrix}1&1&-1\\3&-2&1\\1&3&-2\end{vmatrix}}=\frac{-3}{-3}=1$$

$$\therefore y=\frac{\det(B_2)}{\det(A)}=\frac{\begin{vmatrix}1&6&-1\\3&-5&1\\1&14&-2\end{vmatrix}}{\begin{vmatrix}1&1&-1\\3&-2&1\\1&3&-2\end{vmatrix}}=\frac{-9}{-3}=3$$

$$\therefore z=\frac{\det(B_3)}{\det(A)}=\frac{\begin{vmatrix}1&1&6\\3&-2&-5\\1&3&14\end{vmatrix}}{\begin{vmatrix}1&1&-1\\3&-2&1\\1&3&-2\end{vmatrix}}=\frac{6}{-3}=-2$$

Ans: $(x,y,z)\equiv(1,3,-2)$