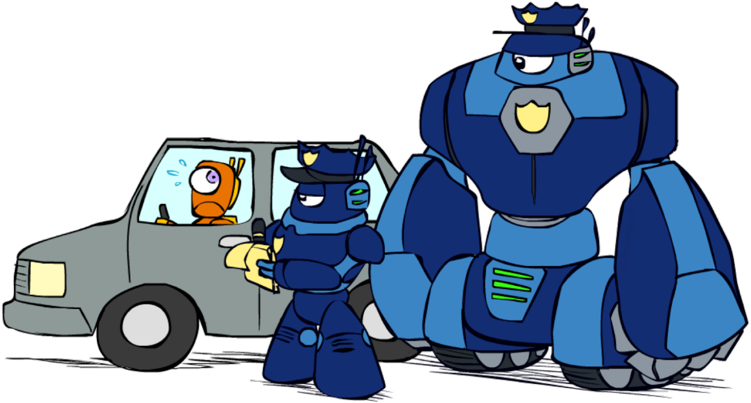
Constraint Satisfaction Problem II

CSE 4711: Artificial Intelligence

Md. Bakhtiar Hasan

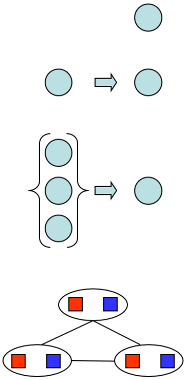
Assistant Professor
Department of Computer Science and Engineering
Islamic University of Technology

k-Consistency



k-Consistency

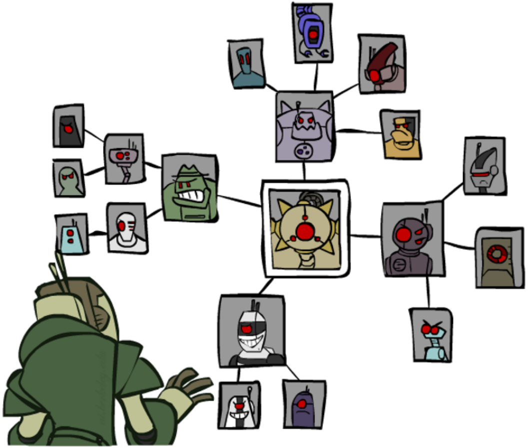
- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - k-Consistency: For each k nodes, any consistent assignment to $k-1$ can be extended to the k^{th} node.
- The higher the k, the more expensive to compute



Strong k-Consistency

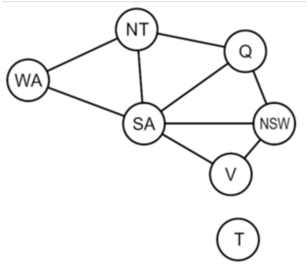
- Also $k-1$, $k-2$, $\ldots$, 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - $\ldots$
- Lots of middle ground between arc consistency and n-consistency! (e.g. $k = 3$t, called path consistency)

Problem Structure



Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact



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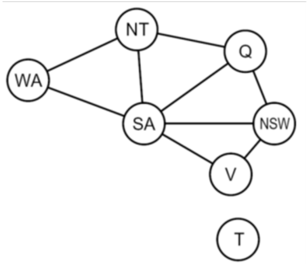
- Independent subproblems are identifiable as connected components of constraint graph
  - Use DFS!



Problem Structure

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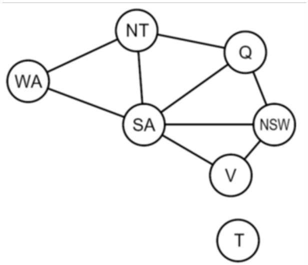
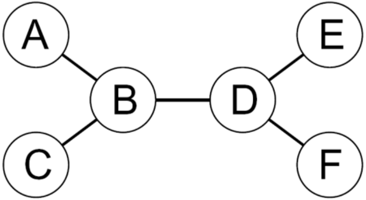
- Independent subproblems are identifiable as connected components of constraint graph
  - Use DFS!
- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O\left((n/c)(d^c)\right)$, linear in n
  - Compared to $O\left(d^n\right)$ for naïve backtracking
    - e.g., for $n = 80, d = 2, c = 20$
    - $2^{80} = 4$ billion years at 10 million nodes/sec
    - $(4)(2^{20}) = 0.4$ seconds at 10 million nodes/sec



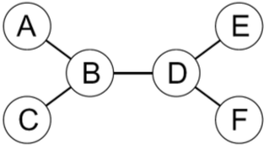
Tree-Structured CSPs



- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(nd^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$
- Only one incoming arc per node
- Also applies to probabilistic reasoning

Tree-Structured CSPs

- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children



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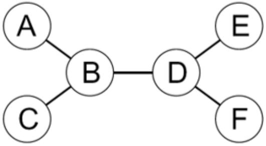

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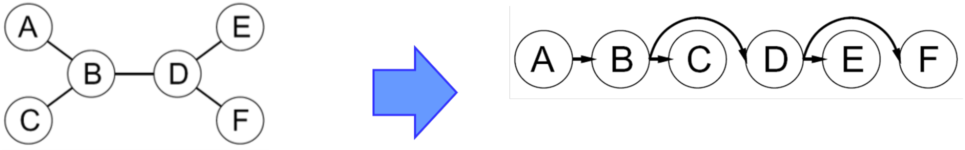

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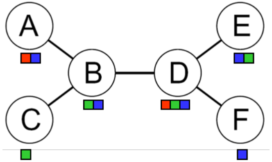

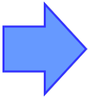

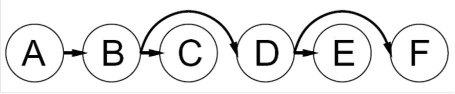

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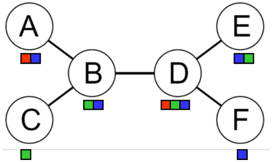

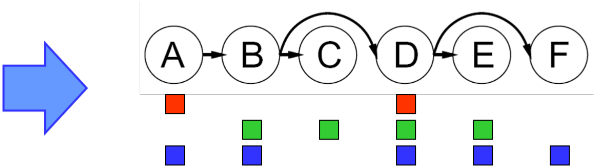

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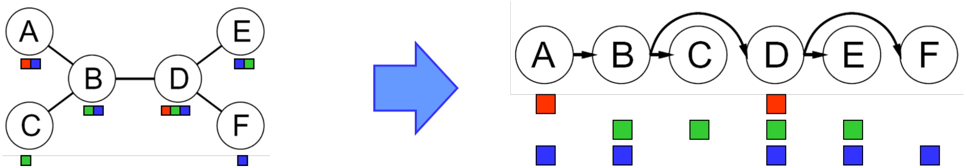

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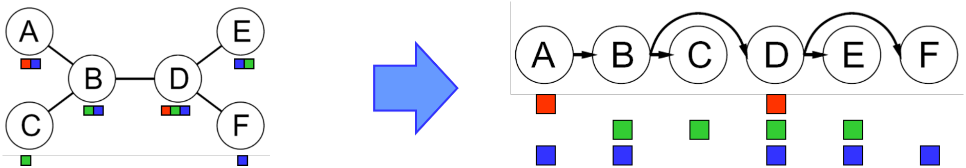

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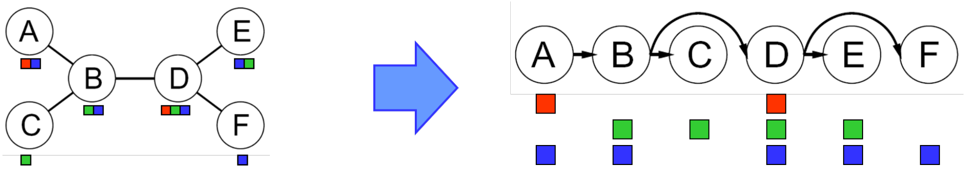

  - Remove backward: For $i = n : 2$, apply RemoveInconsistent(Parent(X_i), X_i)
  - Assign forward: For $i = 1 : n$, assign X_i consistently with Parent(X_i)
- Runtime: $O(nd^2)$
  - Go from tail to head, and then head to tail $\rightarrow O(n)$
  - Check pairs of values for consistency/assignment $\rightarrow O(d^2)$

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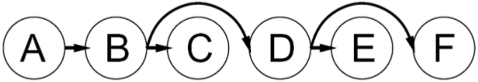

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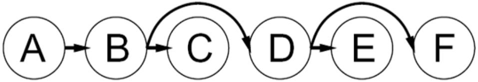

- Claim #1: After backward pass, all root-to-leaf arcs are consistent
  - Proof: Each $X \rightarrow Y$ was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)

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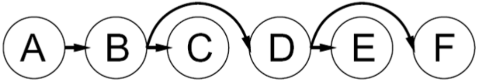

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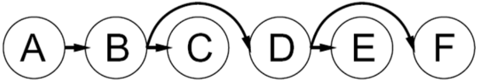

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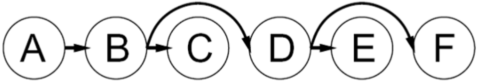

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- Why doesn't this algorithm work with cycles in the constraint graph?

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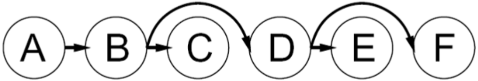

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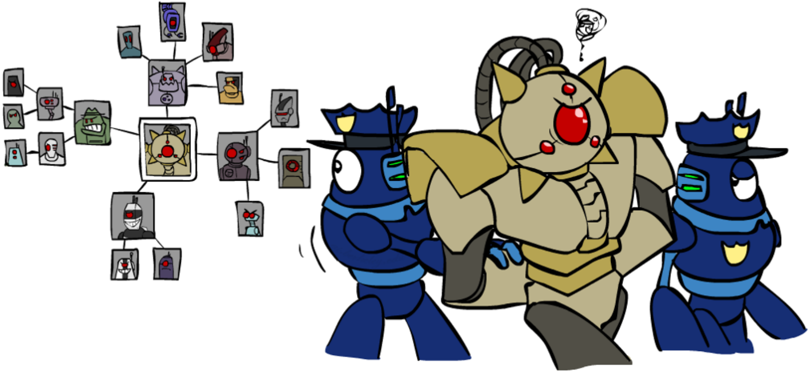
- Why doesn't this algorithm work with cycles in the constraint graph?
- Note: we'll see this basic idea again with Bayes' nets

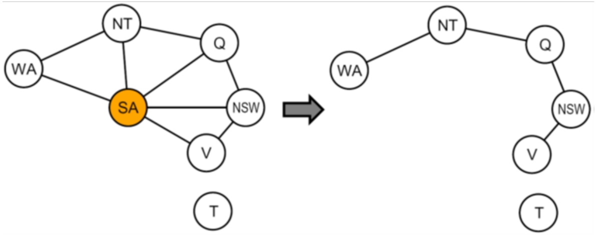
Tree-Structured CSPs

function TREE-CSP-SOLVER(*csp*) **returs** a solution, or failure
  local variables: X, set of variables
    n, number of variables in X
    $root$, any variable in X
    D, set of values
  $X \leftarrow$ TOPOLOGICALSORT(X, $root$)
  for $i \rightarrow$ n **down to** 2 **do**
    MAKEARCCONSISTENT(PARENT(X_i), X_i)
    if no consistency **then return** failure
  for $i \leftarrow$ 1 **to** n
    $X_i \leftarrow$ Any consistent value from D_i
    if no consistency **then return** failure
  return X

Improving Structure

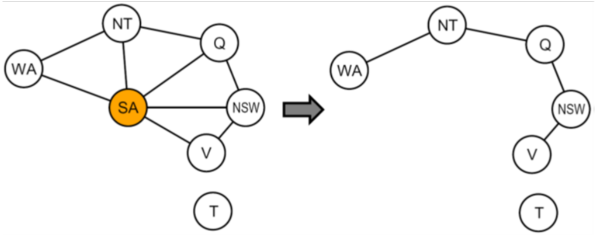


Nearly Tree-Structured CSPs



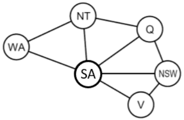
- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

Nearly Tree-Structured CSPs



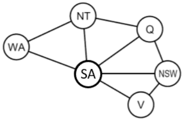
- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O((d^c)(n-c)d^2)$, very fast for small c
  - Total number of instantiation: $O(d^c)$
  - Total number of remaining subproblems: $(n-c)$

Cutset Conditioning



Cutset Conditioning

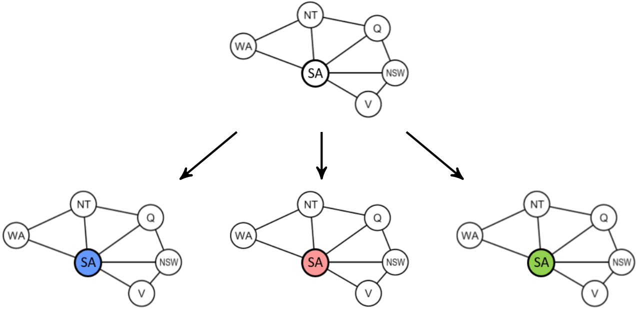
Choose a cutset



Cutset Conditioning

Choose a cutset

Instantiate the cutset (all possible ways)



Cutset Conditioning

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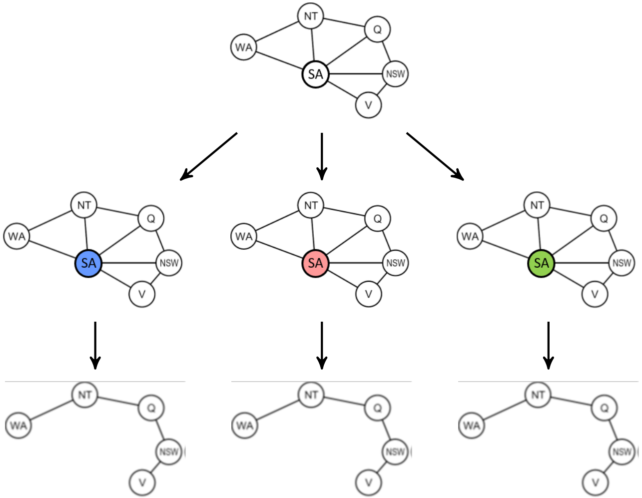
Compute residual CSP for each assignment



Cutset Conditioning

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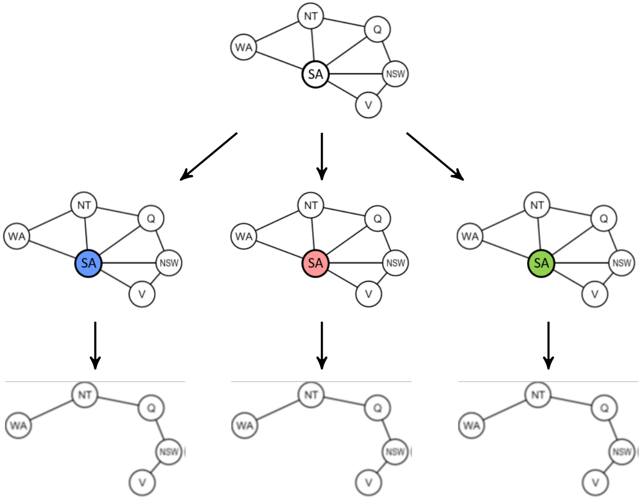
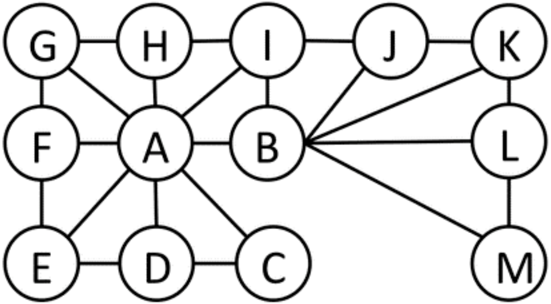
Compute residual CSP for each assignment

Solve the residual CSPs (tree structured)

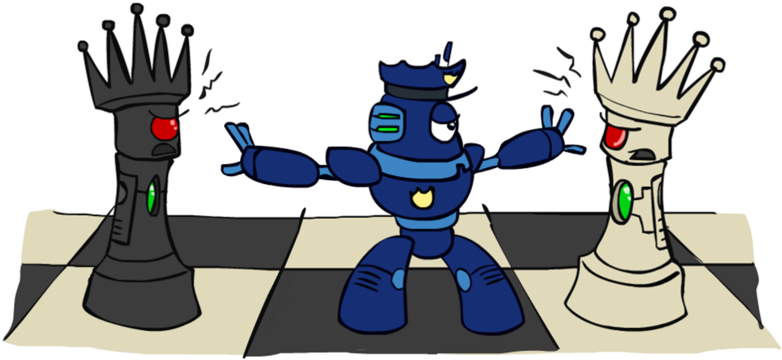


Cutset Quiz

- Find the smallest cutset for the graph below:



Iterative Improvement



Iterative Algorithms for CSPs

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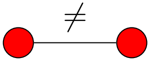

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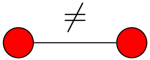
- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe!



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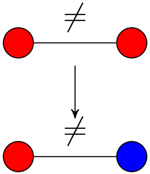

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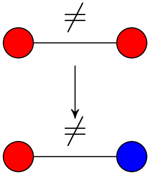
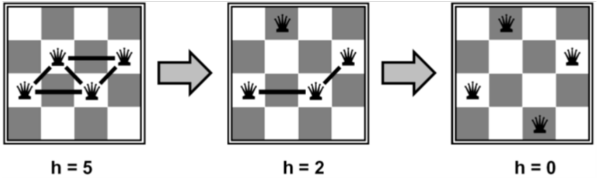


- Algorithm: While not solved
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - i.e., hill climb with h(n) = total number of violated constraints

Example: 4-Queens



- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: $c(n)$ = number of attacks

Video: 5-queens-iterative-improvement
Website: complex - iterating improvement

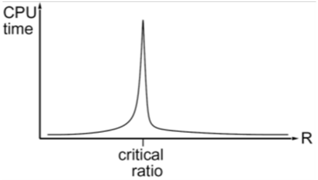
Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!

Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!
- The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio

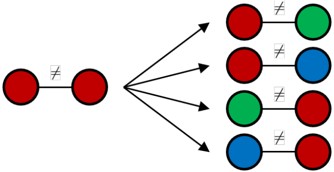
$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Local Search

Local Search

- Tree search keeps unexplored alternatives on the fringe (ensures completeness)
- Local search: improve a single option until you can't make it better (no fringe!)
- New successor function: local changes

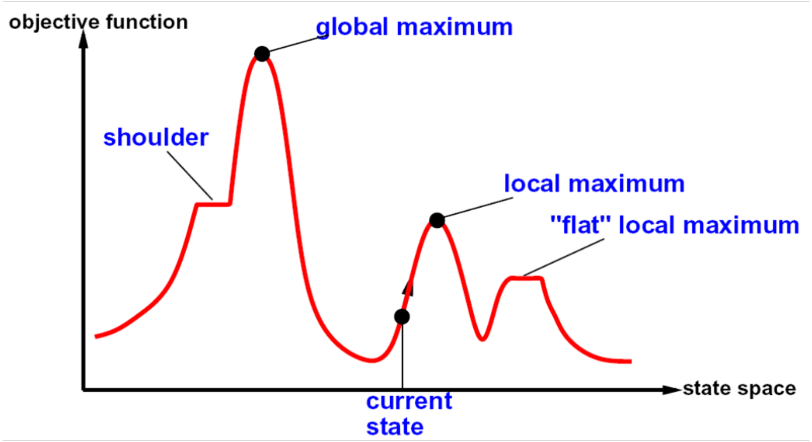


- Generally much faster and more memory efficient (but incomplete and suboptimal)

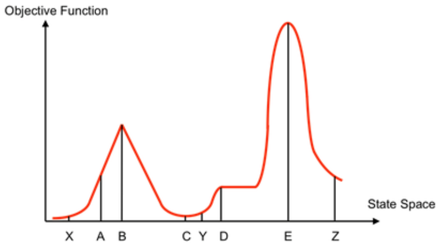
Hill Climbing

- Simple, general idea
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit
- What's bad about this approach?
  - Complete?
  - Optimal?
- What's good about it?

Hill Climbing Diagram



Hill Climbing Quiz



Starting from X, where do you end up?

Starting from Y, where do you end up?

Starting from Z, where do you end up?

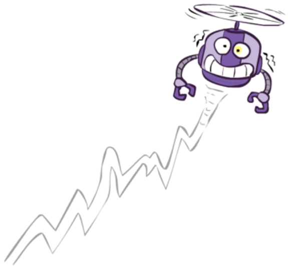
Simulated Annealing

- Escape local maxima by allowing downhill moves $\rightarrow$ Make them rarer as time goes on

```
function SIMULATED-ANNELING(problem, schedule) returns a solution state
   inputs: problem, a problem
           schedule, a mapping from time to "temperature"
   local variables: current, a node
                    next, a node
                    T, a "temperature" controlling prob. of downward steps
   current ← MAKE-NODE(INITIAL-STATE[problem])
   for t ← 1 to ∞ do
      T ← schedule[t]
      if T = 0 then return current
      next ← a randomly selected successor of current
      ΔE ← VALUE[next] - VALUE[current]
      if ΔE > 0 then current ← next
      else current ← next only with probability e^(ΔE/T)
```

Simulated Annealing

- Theoretical guarantee
  - Stationary distribution: $p(x) \propto e^{\frac{E(x)}{kT}}$
  - If T decreased slowly enough, will converge to optimal state!



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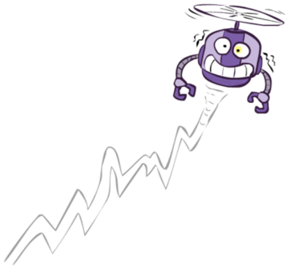

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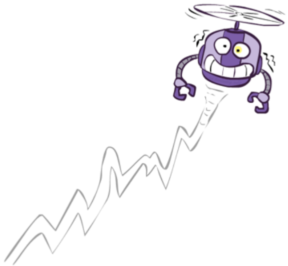
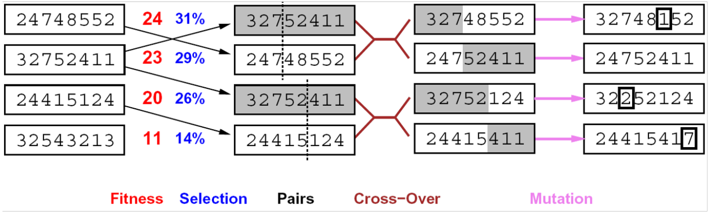
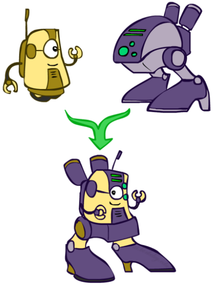
  - If T decreased slowly enough, will converge to optimal state!
- Is this an interesting guarantee
- Sounds like magic, but reality is reality
  - The more downhill steps you need to escape a local optimum, the less likely you are to ever make them all in a row
  - People think hard about *ridge operators* which let you jump around the space in better ways



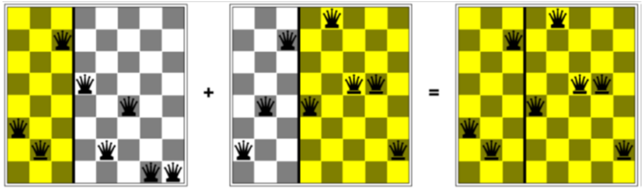
Genetic Algorithms





- Genetic algorithms use a natural selection metaphor
  - Keep best N hypotheses at each step (selection) based on a fitness function
  - Also have pairwise crossover operators, with optional mutation to give variety
- Possibly the most misunderstood, misapplied (and even maligned) technique around

Example: N-Queens



- Why does crossover make sense here?
- When wouldn't it make sense?
- What would mutation be?
- What would a good fitness function be?

Suggested Reading

- Russell & Norvig: Chapter 6.2-6.5