Complex function

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Variables and Functions

A symbol, such as z, which can stand for any one of a set of complex numbers is called a *complex variable*.

Suppose, to each value that a complex variable z can assume, there corresponds one or more values of a complex variable w. We then say that w is a *function* of z and write $w = f(z)$ or $w = G(z)$, etc. The variable z is sometimes called an *independent variable*, while w is called a *dependent variable*. The *value of a function* at $z = a$ is often written $f(a)$. Thus, if $f(z) = z^2$, then $f(2i) = (2i)^2 = -4$.

Single and Multiple-Valued Functions

If only one value of w corresponds to each value of z, we say that w is a *single-valued* function of z or that $f(z)$ is single-valued. If more than one value of w corresponds to each value of z, we say that w is a *multiple-valued* or *many-valued* function of z.

A multiple-valued function can be considered as a collection of single-valued functions, each member of which is called a *branch* of the function. It is customary to consider one particular member as a *principal branch* of the multiple-valued function and the value of the function corresponding to this branch as the *principal value*.

EXAMPLE

(a) If $w = z^2$, then to each value of z there is only one value of w. Hence, $w = f(z) = z^2$ is a single-valued function of z.

(b) If $w^2 = z$, then to each value of z there are two values of w. Hence, $w^2 = z$ defines a multiple-valued (in this case two-valued) function of z.

Whenever we speak of *function*, we shall, unless otherwise stated, assume *single-valued function*.

Inverse Functions

If $w = f(z)$, then we can also consider z as a function, possibly multiple-valued, of w, written $z = g(w) = f^{-1}(w)$. The function f^{-1} is often called the *inverse* function corresponding to f. Thus, $w = f(z)$ and $w = f^{-1}(z)$ are *inverse functions* of each other.

Transformations

If $w = u + iv$ (where u and v are real) is a single-valued function of $z = x + iy$ (where x and y are real), we can write $u + iv = f(x + iy)$. By equating real and imaginary parts, this is seen to be equivalent to

$$u = u(x, y), \quad v = v(x, y) \tag{2.1}$$

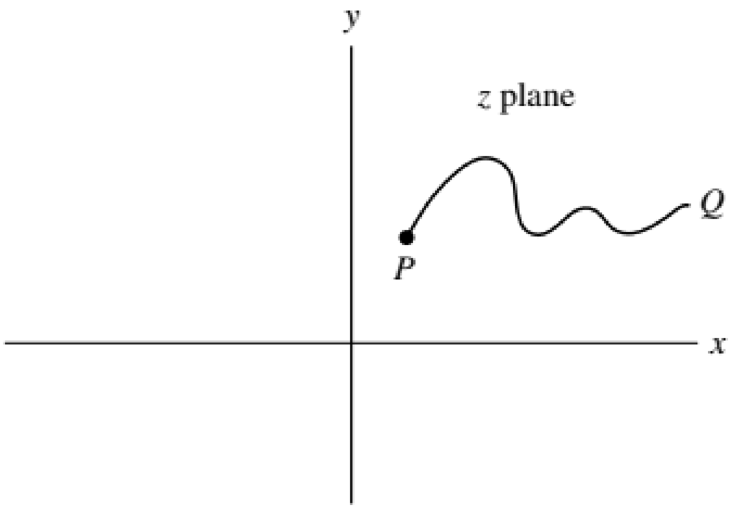

Fig. 2-1

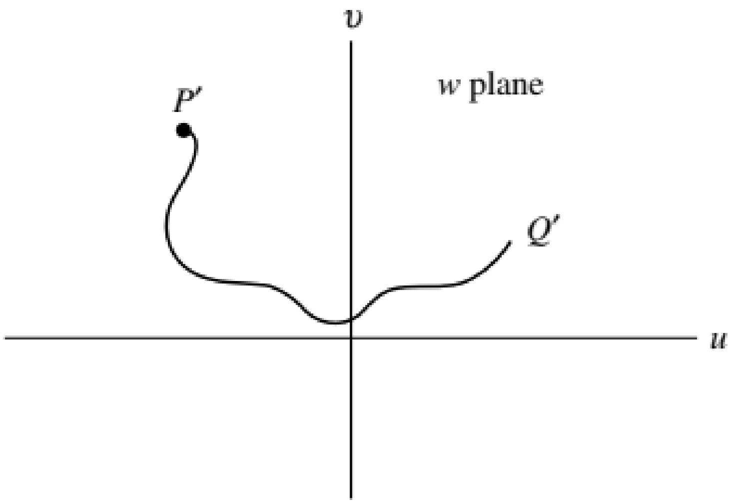

Fig. 2-2

Thus given a point (x, y) in the z plane, such as P in Fig. 2-1, there corresponds a point (u, v) in the w plane, say P' in Fig. 2-2. The set of equations (2.1) [or the equivalent, $w = f(z)$] is called a *transformation*. We say that point P is *mapped* or *transformed* into point P' by means of the transformation and call P' the *image* of P.

EXAMPLE If $w = z^2$, then $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$ and the transformation is $u = x^2 - y^2$, $v = 2xy$. The image of a point $(1, 2)$ in the z plane is the point $(-3, 4)$ in the w plane.

In general, under a transformation, a set of points such as those on curve PQ of Fig. 2-1 is mapped into a corresponding set of points, called the *image*, such as those on curve $P'Q'$ in Fig. 2-2. The particular characteristics of the image depend of course on the type of function $f(z)$, which is sometimes called a *mapping function*. If $f(z)$ is multiple-valued, a point (or curve) in the z plane is mapped in general into more than one point (or curve) in the w plane.

Curvilinear Coordinates

Given the transformation $w = f(z)$ or, equivalently, $u = u(x, y)$, $v = v(x, y)$, we call (x, y) the rectangular coordinates corresponding to a point P in the z plane and (u, v) the *curvilinear coordinates* of P.

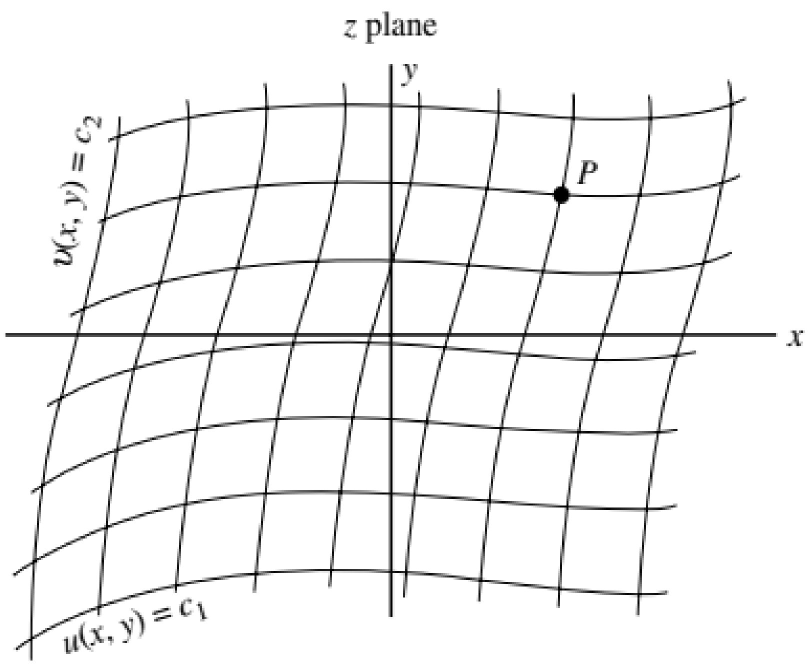

Fig. 2-3

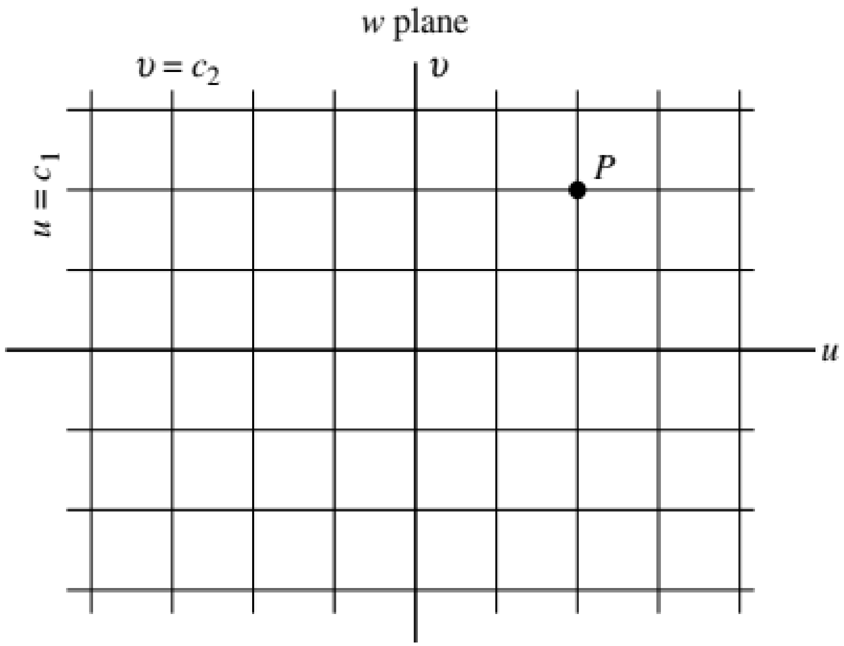

Fig. 2-4

The curves $u(x, y) = c_1$, $v(x, y) = c_2$, where c_1 and c_2 are constants, are called *coordinate curves* [see Fig. 2-3] and each pair of these curves intersects in a point. These curves map into mutually orthogonal lines in the w plane [see Fig. 2-4].

The Elementary Functions

1. **Polynomial Functions** are defined by

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z) \tag{2.2}$$

where $a_0 \neq 0$, $a_1, \ldots, a_n$ are complex constants and n is a positive integer called the *degree* of the polynomial $P(z)$.

The transformation $w = az + b$ is called a *linear transformation*.

2. **Rational Algebraic Functions** are defined by

$$w = \frac{P(z)}{Q(z)} \tag{2.3}$$

where $P(z)$ and $Q(z)$ are polynomials. We sometimes call (2.3) a *rational transformation*. The special case $w = (az + b)/(cz + d)$ where $ad - bc \neq 0$ is often called a *bilinear* or *fractional linear transformation*.

3. **Exponential Functions** are defined by

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) \tag{2.4}$$

where e is the *natural base of logarithms*. If a is real and positive, we define

$$a^z = e^{z \ln a} \tag{2.5}$$

where $\ln a$ is the *natural logarithm of a*. This reduces to (4) if $a = e$.

Complex exponential functions have properties similar to those of real exponential functions. For example, $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, $e^{z_1}/e^{z_2} = e^{z_1-z_2}$.

4. Trigonometric Functions. We define the trigonometric or circular functions $\sin z, \cos z$, etc., in terms of exponential functions as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \qquad \csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}, \qquad \cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

Many of the properties familiar in the case of real trigonometric functions also hold for the complex trigonometric functions. For example, we have:

$$\sin^2 z + \cos^2 z = 1, \qquad 1 + \tan^2 z = \sec^2 z, \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(-z) = -\sin z, \qquad \cos(-z) = \cos z, \qquad \tan(-z) = -\tan z$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

5. Hyperbolic Functions are defined as follows:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}, \qquad \operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \qquad \coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

The following properties hold:

$$\cosh^2 z - \sinh^2 z = 1, \qquad 1 - \tanh^2 z = \operatorname{sech}^2 z, \qquad \coth^2 z - 1 = \operatorname{csch}^2 z$$

$$\sinh(-z) = -\sinh z, \qquad \cosh(-z) = \cosh z, \qquad \tanh(-z) = -\tanh z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$$

The following relations exist between the trigonometric or circular functions and the hyperbolic functions:

$$\sin iz = i \sinh z, \qquad \cos iz = \cosh z, \qquad \tan iz = i \tanh z$$
$$\sinh iz = i \sin z, \qquad \cosh iz = \cos z, \qquad \tanh iz = i \tan z$$

6. **Logarithmic Functions.** If $z = e^w$, then we write $w = \ln z$, called the *natural logarithm* of z. Thus the natural logarithmic function is the inverse of the exponential function and can be defined by

$$w = \ln z = \ln r + i(\theta + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots$$

where $z = re^{i\theta} = re^{i(\theta+2k\pi)}$. Note that $\ln z$ is a multiple-valued (in this case, infinitely-many-valued) function. The *principal-value* or *principal branch* of $\ln z$ is sometimes defined as $\ln r + i\theta$ where $0 \le \theta < 2\pi$. However, any other interval of length 2π can be used, e.g., $-\pi < \theta \le \pi$, etc.

The logarithmic function can be defined for real bases other than e. Thus, if $z = a^w$, then $w = \log_a z$ where $a > 0$ and $a \ne 0, 1$. In this case, $z = e^{w \ln a}$ and so, $w = (\ln z)/(\ln a)$.

7. Inverse Trigonometric Functions. If $z = \sin w$, then $w = \sin^{-1} z$ is called the *inverse sine of z* or *arc sine of z*. Similarly, we define other inverse trigonometric or circular functions $\cos^{-1} z$, $\tan^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases, we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm:

$$\sin^{-1} z = \frac{1}{i}\ln\left(iz + \sqrt{1 - z^2}\right), \qquad \csc^{-1} z = \frac{1}{i}\ln\left(\frac{i + \sqrt{z^2 - 1}}{z}\right)$$

$$\cos^{-1} z = \frac{1}{i}\ln\left(z + \sqrt{z^2 - 1}\right), \qquad \sec^{-1} z = \frac{1}{i}\ln\left(\frac{1 + \sqrt{1 - z^2}}{z}\right)$$

$$\tan^{-1} z = \frac{1}{2i}\ln\left(\frac{1 + iz}{1 - iz}\right), \qquad \cot^{-1} z = \frac{1}{2i}\ln\left(\frac{z + i}{z - i}\right)$$

8. Inverse Hyperbolic Functions. If $z = \sinh w$, then $w = \sinh^{-1} z$ is called the *inverse hyperbolic sine of* z. Similarly, we define other inverse hyperbolic functions $\cosh^{-1} z$, $\tanh^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases, we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm:

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right), \qquad \operatorname{csch}^{-1} z = \ln\left(\frac{1+\sqrt{z^2+1}}{z}\right)$$

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right), \qquad \operatorname{sech}^{-1} z = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \qquad \coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

Thanks a lot …