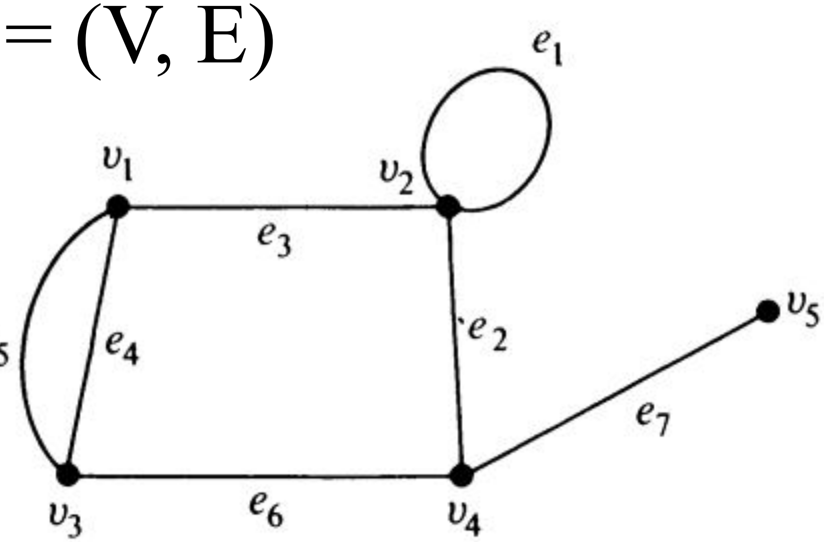
Graph Theory

Basic Definitions

What is Graph?

- A linear* graph G = (V, E)
- Set of vertices V
- Set of Edges E
- Self Loop
- Parallel Edge
- Edges don’t have to be a straight line

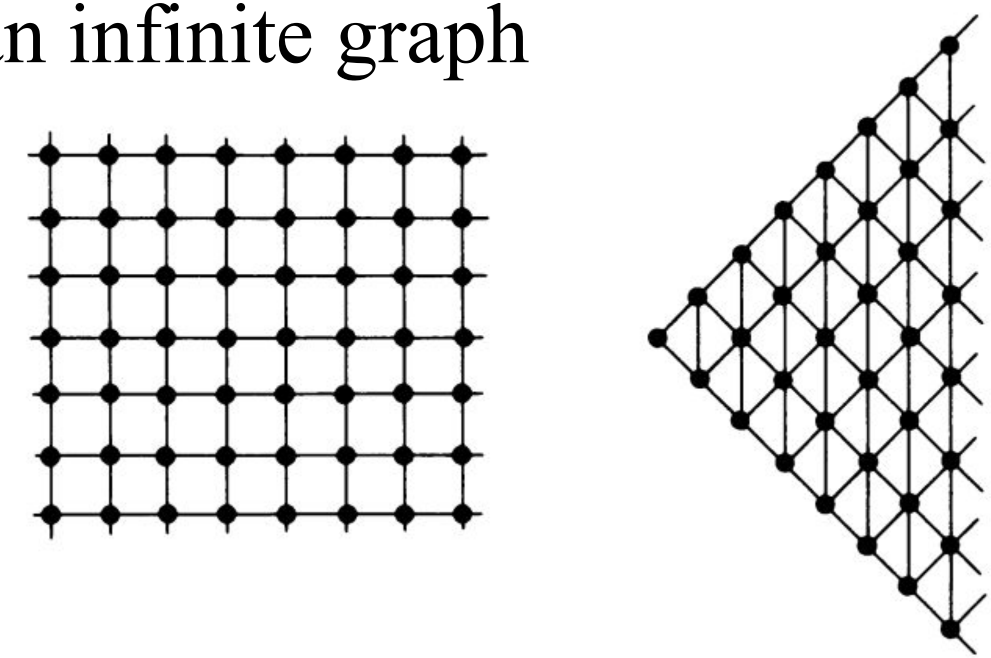


Graph continued

- A graph without any selfloop or parallel edge is called simple graph
- Crossover of edges not necessarily means a connection
- A graph is also called a liner complex, a 1-complex or a one dimensional complex
- A vertex is referred to as a node, a junction, a point, 0-cell or an 0-simplex
- An edges is called a branch, a line, an element, a 1-cell, an arc, or a 1-simplex

Finite & Infinite Graphs

- A graph with finite number of vertices as well as edges is called a finite graph, otherwise, is called an infinite graph



Incidence & Degree

- When a vertex v is an end vertex of some edge e, v & e are said to incident with each other
- Two non parallel edges are said to be adjacent if they are incident on a common vertex. Similar reasoning can be applied for vertex

Let us now consider a graph G with e edges and n vertices $v_1, v_2, \ldots, v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

$$\sum_{i=1}^{n} d(v_i) = 2e. \tag{1-1}$$

Theorem 1.1

The number of vertices of odd degree in a graph is always even.

Proof: If we consider the vertices with odd and even degrees separately, the quantity in the left side of Eq. (1-1) can be expressed as the sum of two sums, each taken over vertices of even and odd degrees, respectively, as follows:

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k). \tag{1-2}$$

Since the left-hand side in Eq. (1-2) is even, and the first expression on the right-hand side is even (being a sum of even numbers), the second expression must also be even:

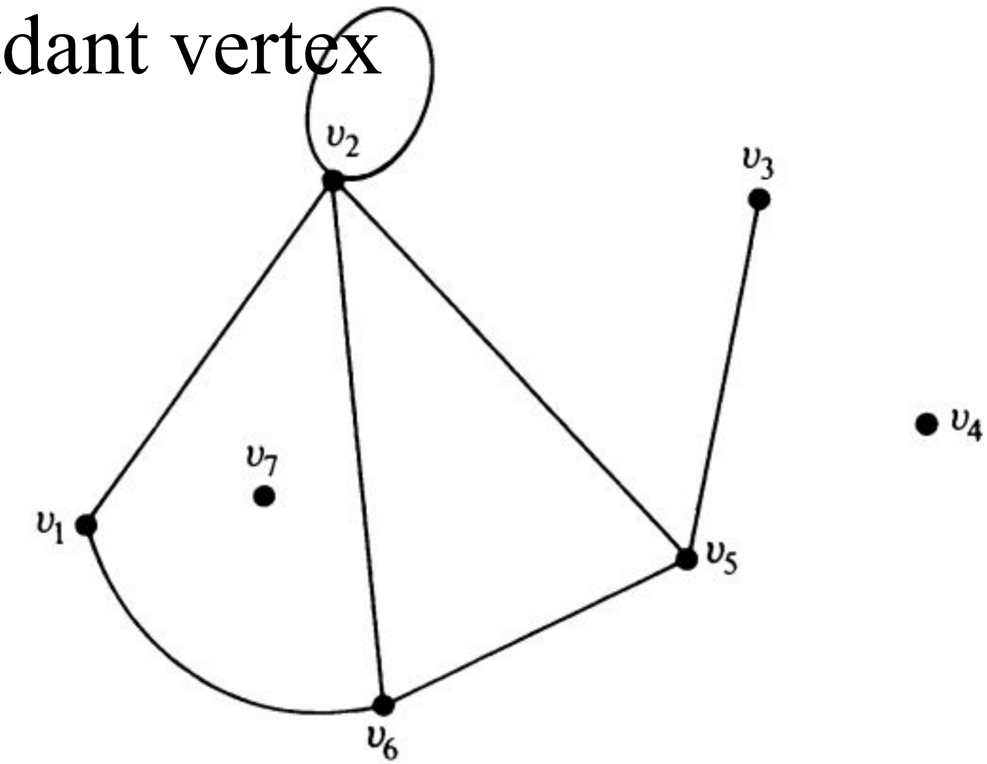
$$\sum_{\text{odd}} d(v_k) = \text{an even number.} \tag{1-3}$$

Because in Eq. (1-3) each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem. ■

- Regular Graph

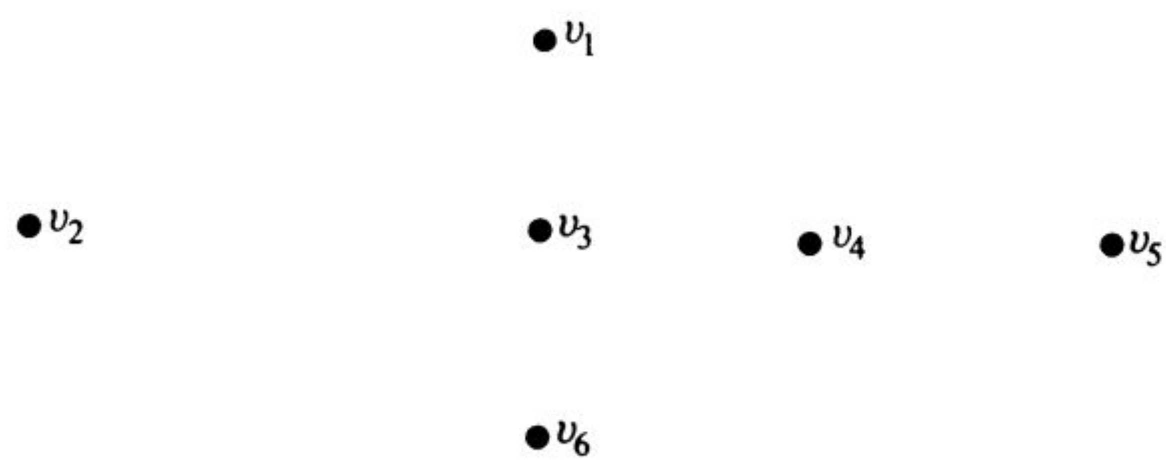
Isolated & Pendant Vertex

- If deg(v) = 0, isolated vertex
- If deg(v) = 1, pendant vertex



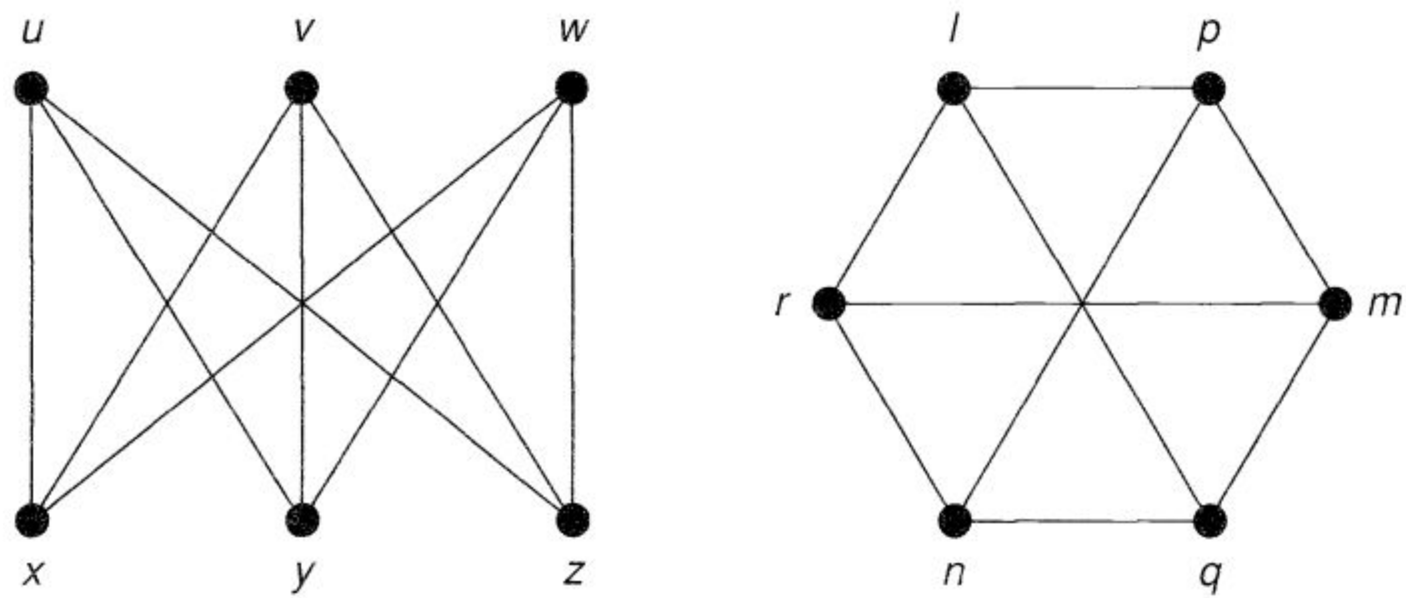
Null Graph

- Set of E can be empty(this leads to null graph)
- But if set of V is empty, no graph is formed



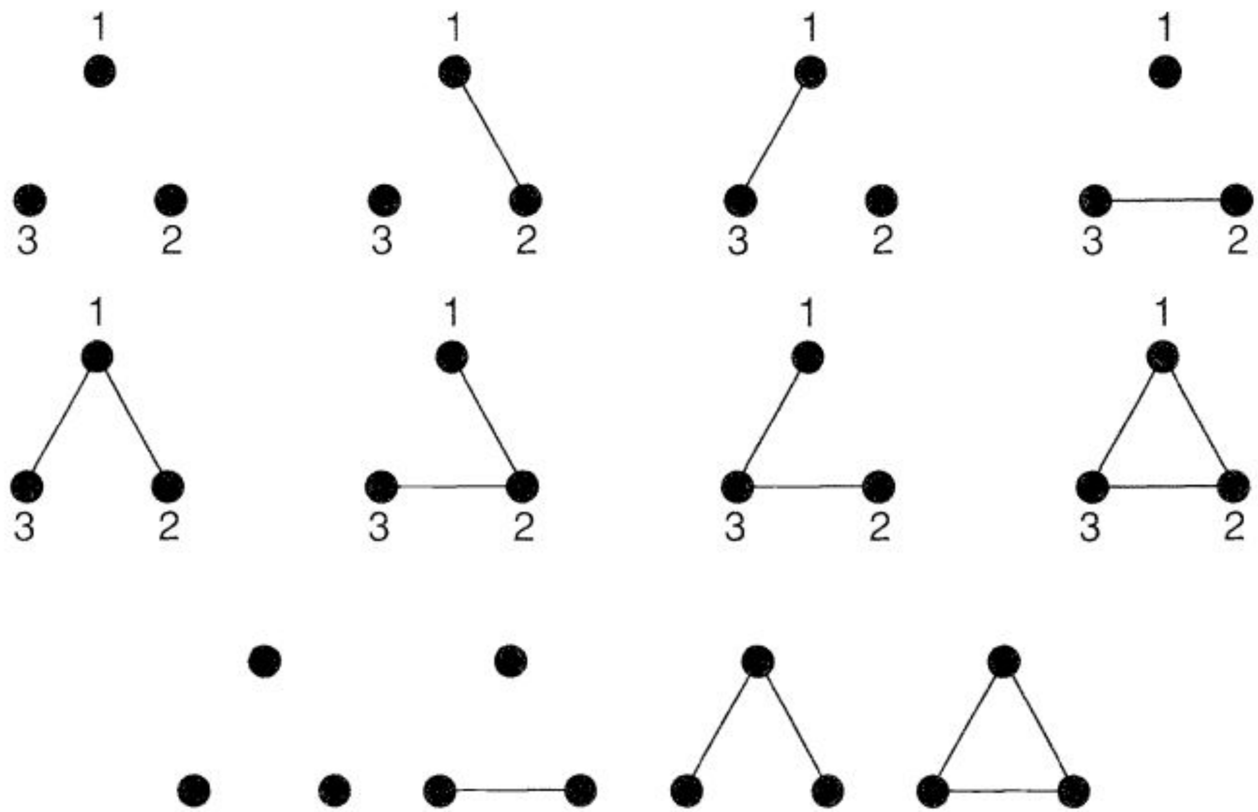
Isomorphism

- Two graphs G_1 and G_2 are isomorphic if there is a one-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2



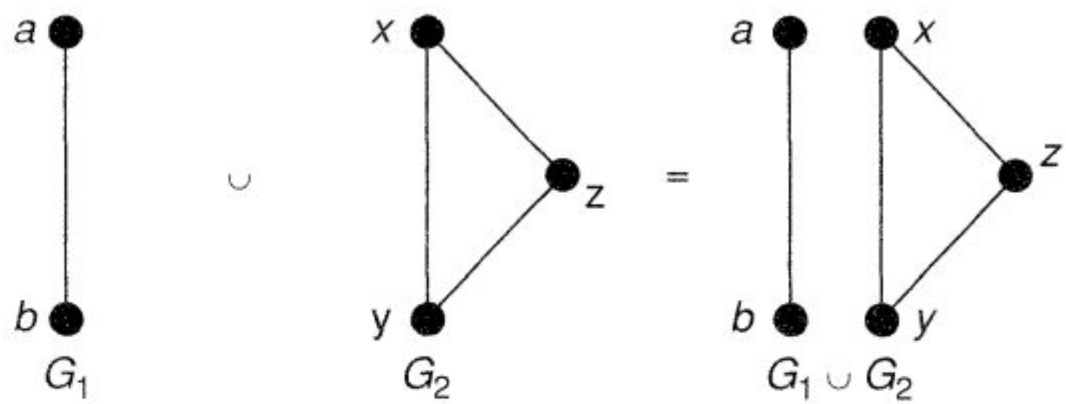
Labelled & Unlabelled

- The difference between labelled and unlabelled graphs becomes more apparent when we try to count them.



Connectedness

- We can combine two graphs to make a larger graph. If the two graphs are $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, where $V(G_1)$ and $V(G_2)$ are disjoint, then their union $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge family $E(G_1) \cup E(G_2)$

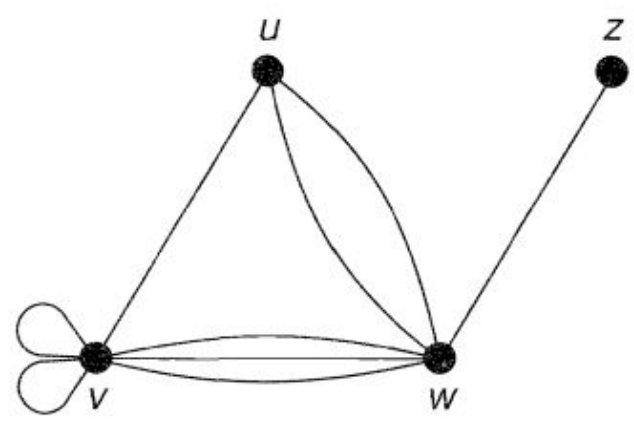


Degree Sequence

- The degree sequence of a graph consists of the degrees written in increasing order, with repeats where necessary.

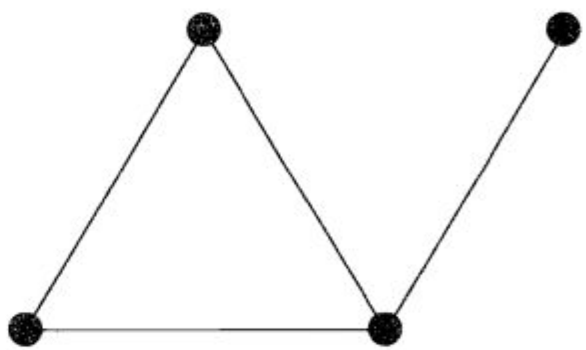
- (1,1,2,2,2)

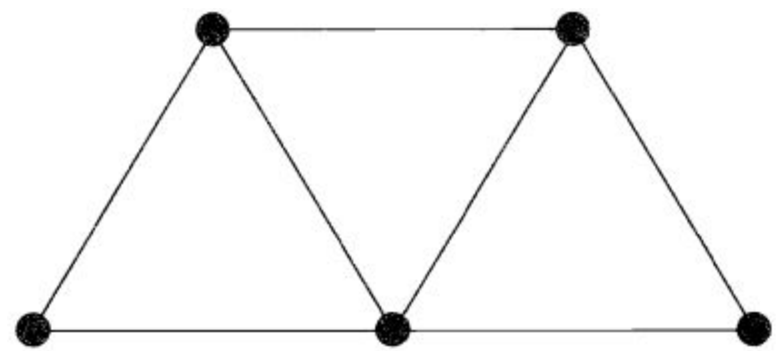
- (1,3,6,8)

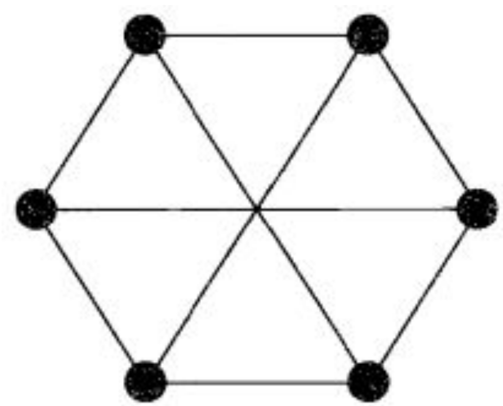


Subgraphs

- A subgraph of a graph *G* is a graph, each of whose vertices belongs to *V(G)* and each of whose edges belongs to *E(G)*.

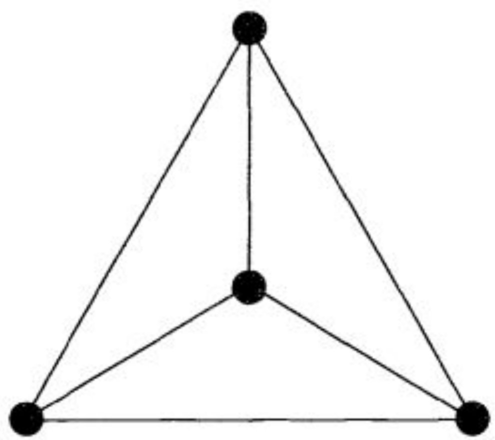


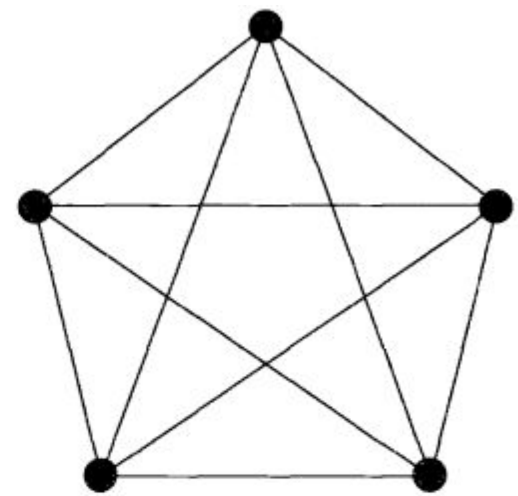




Complete Graphs

- A simple graph in which each pair of distinct vertices are adjacent is a complete graph. We denote the complete graph on n vertices by K_n; You should check that Kn has n(n-1)/2 edges

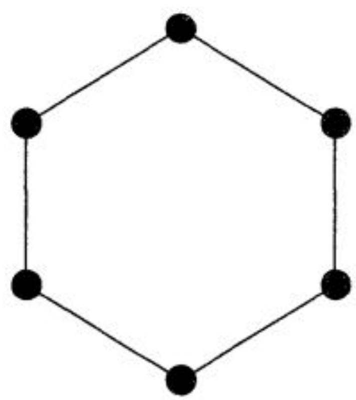




K4 K5

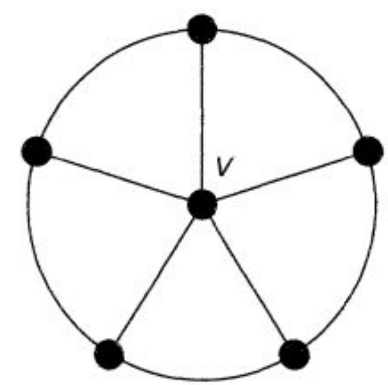
Cycle, Path & Wheel

- A connected graph that is regular of degree 2 is a **cycle graph.** We denote the cycle graph on n vertices by C_n.
- The graph obtained from C_n by removing an edge is the **path graph** on n vertices, denoted by P_n.
- The graph obtained from C_{n-1} by joining each vertex to a new vertex v is the **wheel** on n vertices, denoted by W_n.



C_6

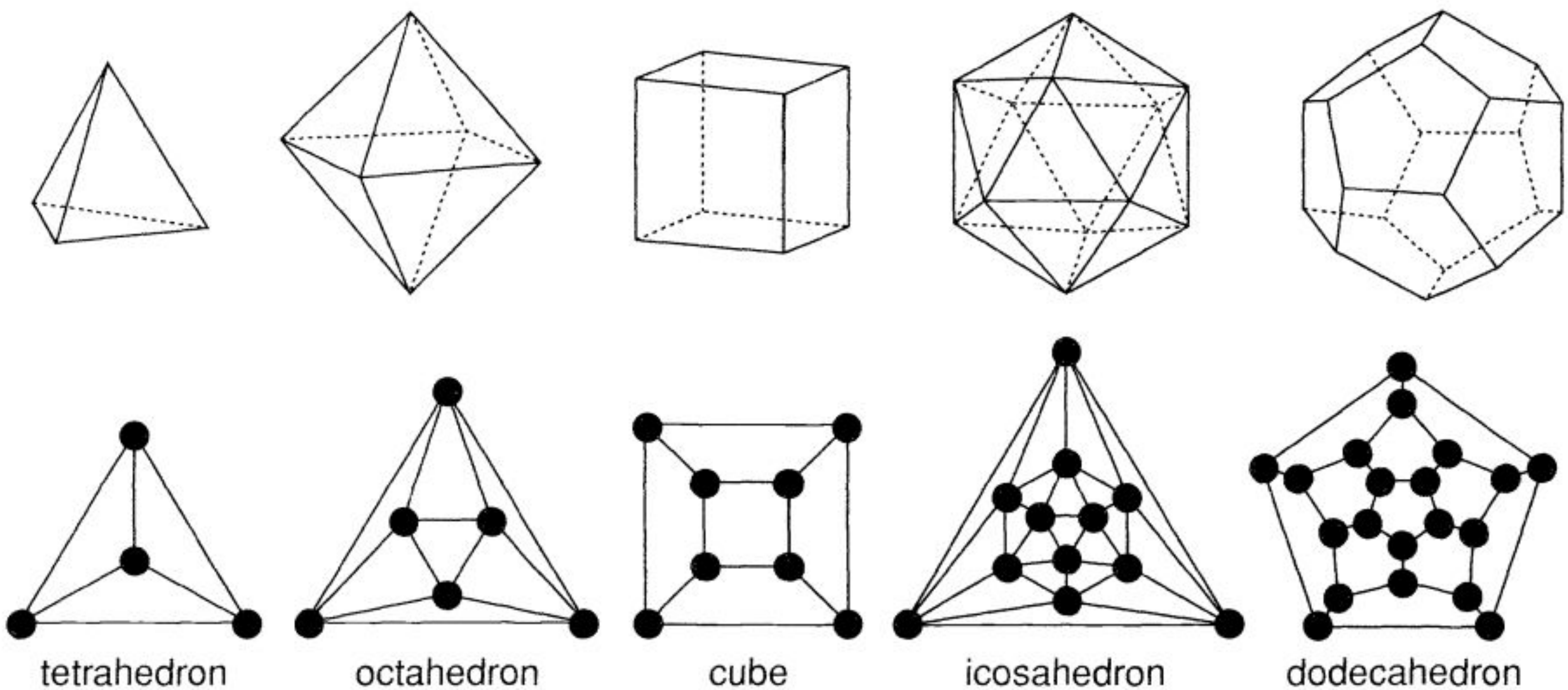
P_6



W_6

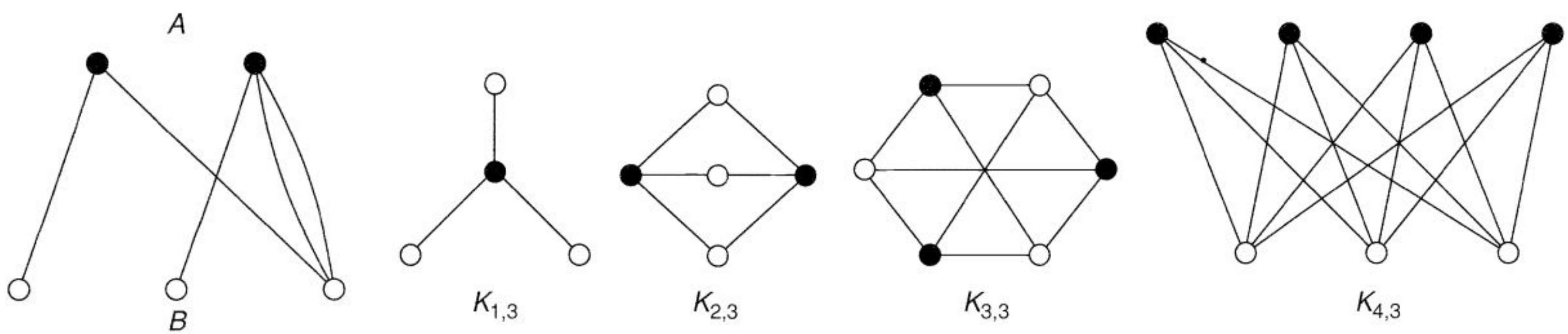
Platonic Graphs

- Of interest among the regular graphs are the **Platonic graphs,** formed from the vertices and edges of the five regular (Platonic) solids - the tetrahedron, octahedron, cube, icosahedron and dodecahedron



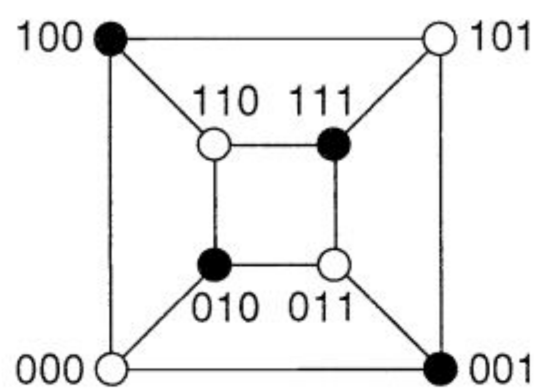
Bipartite Graphs

- If the vertex set of a graph *G* can be split into two disjoint sets *A* and *B* so that each edge of *G* joins a vertex of A and a vertex of *B,* then *G* is a **bipartite graph**. Alternatively, a bipartite graph is one whose vertices can be coloured black and white in such a way that each edge joins a black vertex (in *A)* and a white vertex (in ***B)***
- A **complete bipartite graph** is a bipartite graph in which each vertex in *A* is joined to each vertex in *B* by just one edge. We denote the bipartite graph with r black vertices and *s* white vertices by $K_{r,s}$



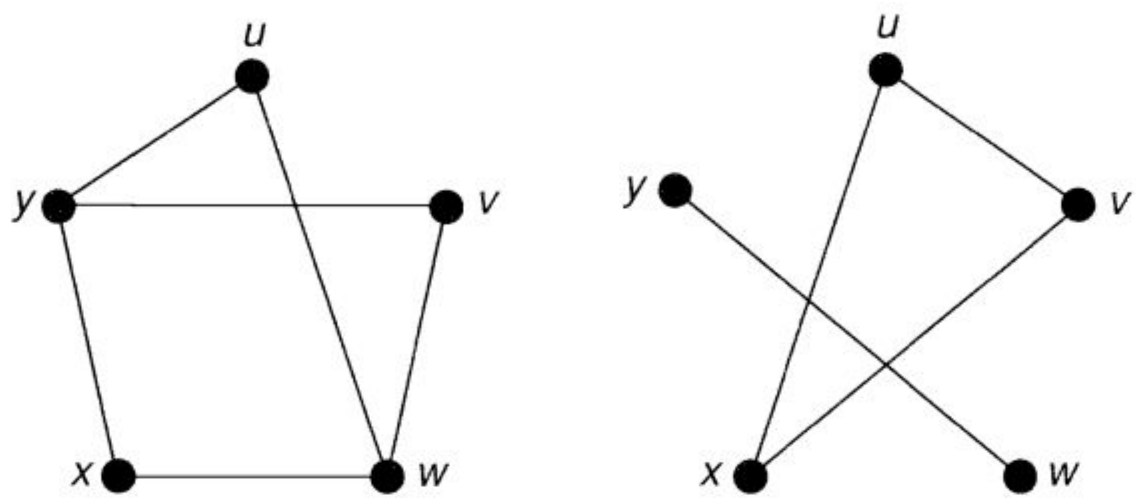
Cubes

- Of special interest among the regular bipartite graphs are the cubes. The **k-cube** Q_k is the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$, where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place. Note that the graph of the cube is the graph Q_3. You should check that Q_k has 2^k vertices and $k2^{k-1}$ edges, and is regular of degree k.



The complement of a simple graph

- If G is a simple graph with vertex set V(G), its complement G is the simple graph with vertex set V(G) in which two vertices are adjacent if and only if they are not adjacent in G. For example, figure below shows a graph and its complement. Note that the complement of a complete graph is a null graph, and that the complement of a complete bipartite graph is the union of two complete graphs.



Directed & Undirected graph