Limit and continuity of complex functions

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Limits

Let $f(z)$ be defined and single-valued in a neighborhood of $z = z_0$ with the possible exception of $z = z_0$ itself (i.e., in a deleted δ neighborhood of z_0). We say that the number l is the *limit* of $f(z)$ as z approaches z_0 and write $\lim_{z \to z_0} f(z) = l$ if for any positive number ϵ (however small), we can find some positive number δ (usually depending on ϵ) such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

In such a case, we also say that $f(z)$ approaches l as z approaches z_0 and write $f(z) \to l$ as $z \to z_0$. The limit must be independent of the manner in which z approaches z_0.

Geometrically, if z_0 is a point in the complex plane, then $\lim_{z \to z_0} f(z) = l$ if the difference in absolute value between $f(z)$ and l can be made as small as we wish by choosing points z sufficiently close to z_0 (excluding $z = z_0$ itself).

EXAMPLE Let

$$f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$$

Then, as z gets closer to i (i.e., z approaches i), $f(z)$ gets closer to $i^2 = -1$. We thus *suspect* that $\lim_{z \to i} f(z) = -1$. To *prove* this, we must see whether the above definition of limit is satisfied. For this proof, see Problem 2.23.

Note that $\lim_{z \to i} f(z) \neq f(i)$, i.e., the limit of $f(z)$ as $z \to i$ is not the same as the value of $f(z)$ at $z = i$, since $f(i) = 0$ by definition. The limit would, in fact, be -1 even if $f(z)$ were not defined at $z = i$.

When the limit of a function exists, it is unique, i.e., it is the only one (see Problem 2.26). If $f(z)$ is multiple-valued, the limit as $z \to z_0$ may depend on the particular branch.

Theorems on Limits

THEOREM 2.1. Suppose $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then

1. $\lim_{z\to z_0}\{f(z) + g(z)\} = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z) = A + B$
2. $\lim_{z\to z_0}\{f(z) - g(z)\} = \lim_{z\to z_0} f(z) - \lim_{z\to z_0} g(z) = A - B$
3. $\lim_{z\to z_0}\{f(z)g(z)\} = \left\{\lim_{z\to z_0} f(z)\right\}\left\{\lim_{z\to z_0} g(z)\right\} = AB$
4. $\displaystyle\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} = \frac{A}{B}$ if $B \neq 0$

Continuity

Let $f(z)$ be defined and single-valued in a neighborhood of $z = z_0$ as well as at $z = z_0$ (i.e., in a δ neighborhood of z_0). The function $f(z)$ is said to be *continuous* at $z = z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$. Note that this implies three conditions that must be met in order that $f(z)$ be continuous at $z = z_0$:

1. $\lim_{z\to z_0} f(z) = l$ must exist
2. $f(z_0)$ must exist, i.e., $f(z)$ is defined at z_0
3. $l = f(z_0)$

Equivalently, if $f(z)$ is continuous at z_0, we can write this in the suggestive form

$$\lim_{z\to z_0} f(z) = f\left(\lim_{z\to z_0} z\right).$$

EXAMPLE

(a) Suppose

$$f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$$

Then, $\lim_{z \to i} f(z) = -1$. But $f(i) = 0$. Hence, $\lim_{z \to i} f(z) \neq f(i)$ and the function is not continuous at $z = i$.

(b) Suppose $f(z) = z^2$ for all z. Then $\lim_{z \to i} f(z) = f(i) = -1$ and $f(z)$ is continuous at $z = i$.

Points in the z plane where $f(z)$ fails to be continuous are called *discontinuities* of $f(z)$, and $f(z)$ is said to be *discontinuous* at these points. If $\lim_{z \to z_0} f(z)$ exists but is not equal to $f(z_0)$, we call z_0 a *removable discontinuity* since by redefining $f(z_0)$ to be the same as $\lim_{z \to z_0} f(z)$, the function becomes continuous.

Alternative to the above definition of continuity, we can define $f(z)$ as continuous at $z = z_0$ if for any $\epsilon > 0$, we can find $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. Note that this is simply the definition of limit with $l = f(z_0)$ and removal of the restriction that $z \neq z_0$.

To examine the continuity of $f(z)$ at $z = \infty$, we let $z = 1/w$ and examine the continuity of $f(1/w)$ at $w = 0$.

Continuity in a Region

A function $f(z)$ is said to be *continuous in a region* if it is continuous at all points of the region.

Theorems on Continuity

THEOREM 2.2. Given $f(z)$ and $g(z)$ are continuous at $z = z_0$. Then so are the functions $f(z) + g(z)$, $f(z) - g(z)$, $f(z)g(z)$ and $f(z)/g(z)$, the last if $g(z_0) \neq 0$. Similar results hold for continuity in a region.

THEOREM 2.3. Among the functions continuous in every finite region are (a) all polynomials, (b) e^z, (c) $\sin z$ and $\cos z$.

THEOREM 2.4. Suppose $w = f(z)$ is continuous at $z = z_0$ and $z = g(\zeta)$ is continuous at $\zeta = \zeta_0$. If $z_0 = g(\zeta_0)$, then the function $w = f[g(\zeta)]$, called a *function of a function* or *composite function*, is continuous at $\zeta = \zeta_0$. This is sometimes briefly stated as: A continuous function of a continuous function is continuous.

THEOREM 2.5. Suppose $f(z)$ is continuous in a closed and bounded region. Then it is bounded in the region; i.e., there exists a constant M such that $|f(z)| < M$ for all points z of the region.

THEOREM 2.6. If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$ are also continuous in the region.

EX-1: Let $w = f(z) = z^2$. Find the values of w that correspond to (a) $z = -2 + i$ and (b) $z = 1 - 3i$, and show how the correspondence can be represented graphically.

Solution

(a) $w = f(-2 + i) = (-2 + i)^2 = 4 - 4i + i^2 = 3 - 4i$

(b) $w = f(1 - 3i) = (1 - 3i)^2 = 1 - 6i + 9i^2 = -8 - 6i$

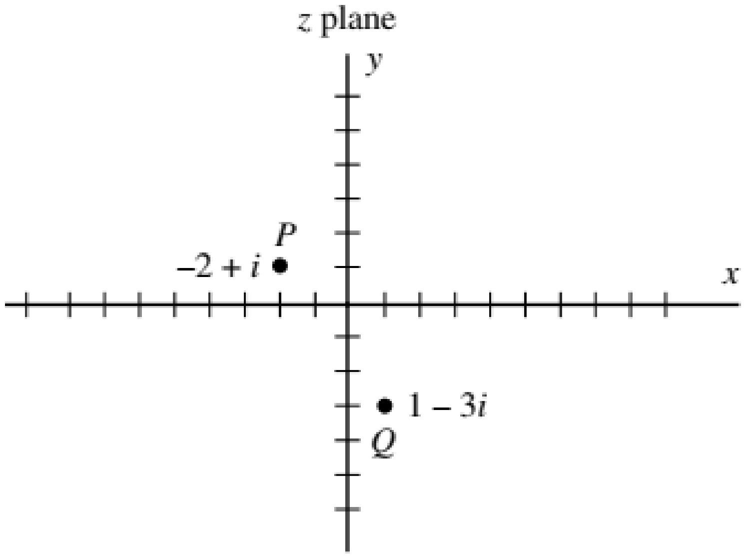

Fig. 2-6

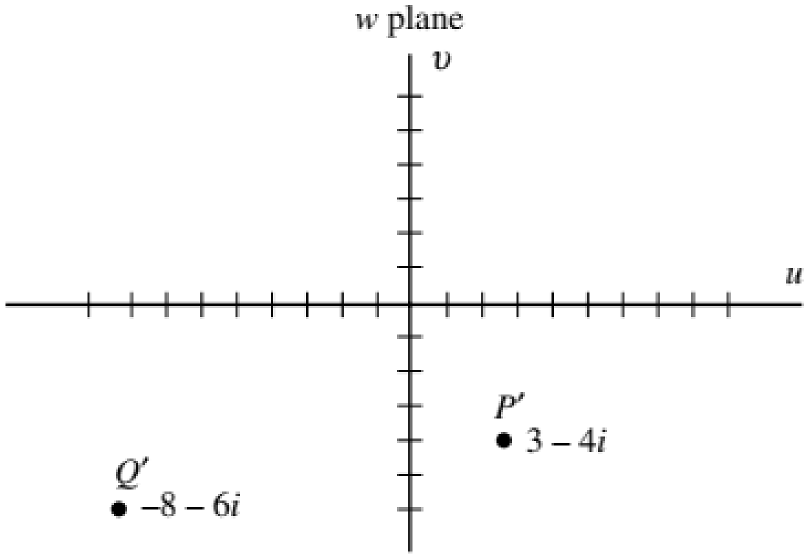

Fig. 2-7

The point $z = -2 + i$, represented by point P in the z plane of Fig. 2-6, has the *image point* $w = 3 - 4i$ represented by P' in the w plane of Fig. 2-7. We say that P is *mapped* into P' by means of the *mapping function* or *transformation* $w = z^2$. Similarly, $z = 1 - 3i$ [point Q of Fig. 2-6] is mapped into $w = -8 - 6i$ [point Q' of Fig. 2-7]. To each point in the z plane, there corresponds one and only one point (image) in the w plane, so that w is a single-valued function of z.

EX-2:

A point P moves in a counterclockwise direction around a circle in the z plane having center at the origin and radius 1. If the mapping function is $w = z^3$, show that when P makes one complete revolution, the image P' of P in the w plane makes three complete revolutions in a counterclockwise direction on a circle having center at the origin and radius 1.

Solution

Let $z = re^{i\theta}$. Then, on the circle $|z| = 1$ [Fig. 2-8], $r = 1$ and $z = e^{i\theta}$. Hence, $w = z^3 = (e^{i\theta})^3 = e^{3i\theta}$. Letting (ρ, ϕ) denote polar coordinates in the w plane, we have $w = \rho e^{i\phi} = e^{3i\theta}$ so that $\rho = 1$, $\phi = 3\theta$.

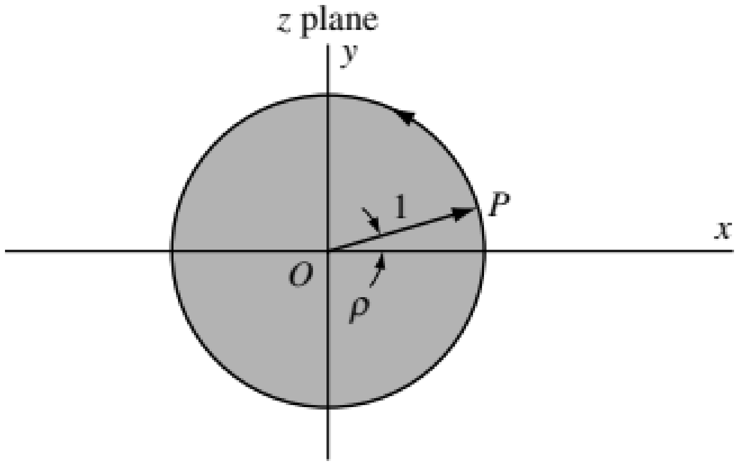

Fig. 2-8

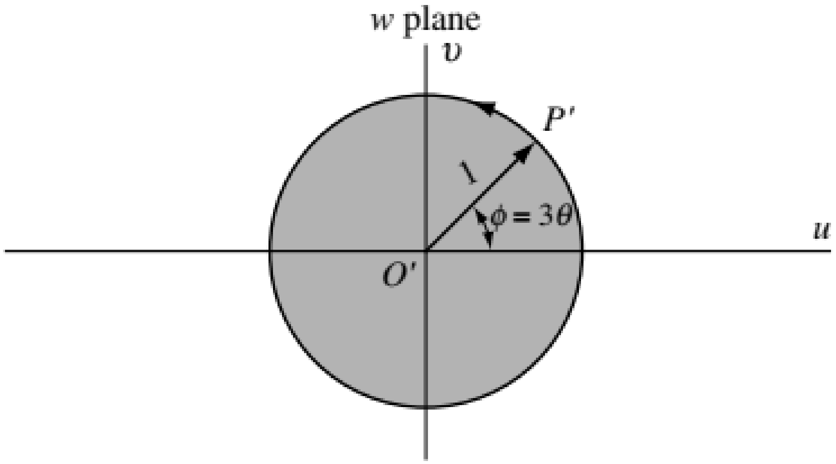

Fig. 2-9

Since $\rho = 1$, it follows that the image point P' moves on a circle in the w plane of radius 1 and center at the origin [Fig. 2-9]. Also, when P moves counterclockwise through an angle θ, P' moves counterclockwise through an angle 3θ. Thus, when P makes one complete revolution, P' makes three complete revolutions. In terms of vectors, it means that vector $O'P'$ is rotating three times as fast as vector OP.

EX-3: Suppose c_1 and c_2 are any real constants. Determine the set of all points in the z plane that map into the lines (a) $u = c_1$, (b) $v = c_2$ in the w plane by means of the mapping function $w = z^2$. Illustrate by considering the cases $c_1 = 2, 4, -2, -4$ and $c_2 = 2, 4, -2, -4$.

Solution

We have $w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$ so that $u = x^2 - y^2$, $v = 2xy$. Then lines $u = c_1$ and $v = c_2$ in the w plane correspond, respectively, to hyperbolas $x^2 - y^2 = c_1$ and $2xy = c_2$ in the z plane as indicated in Figs. 2-10 and 2-11.

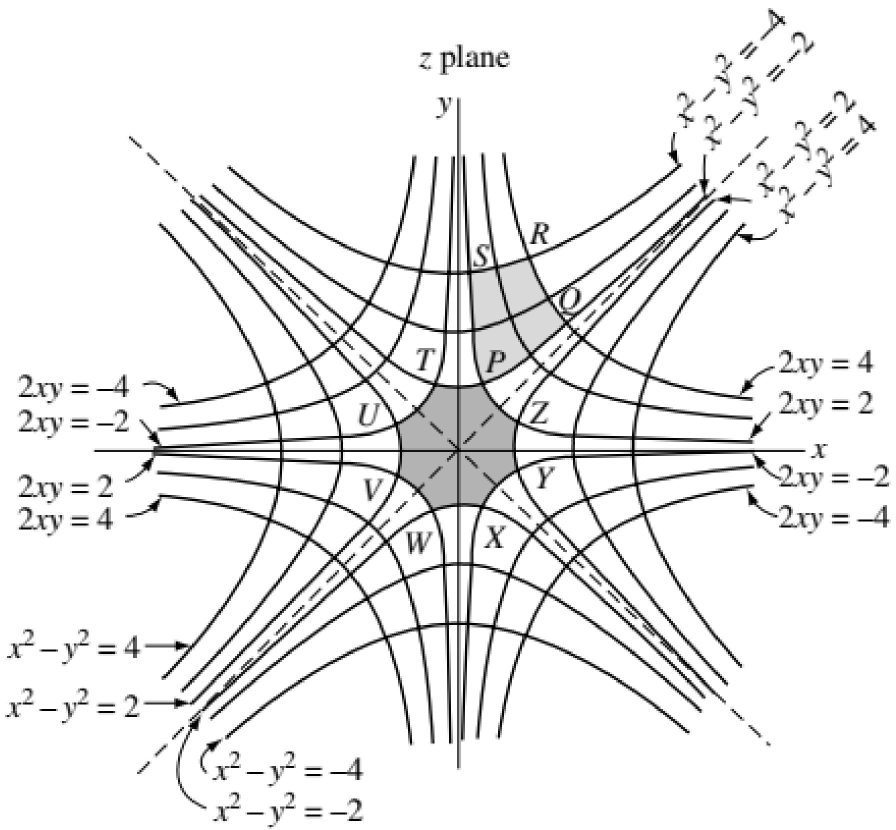

Fig. 2-10

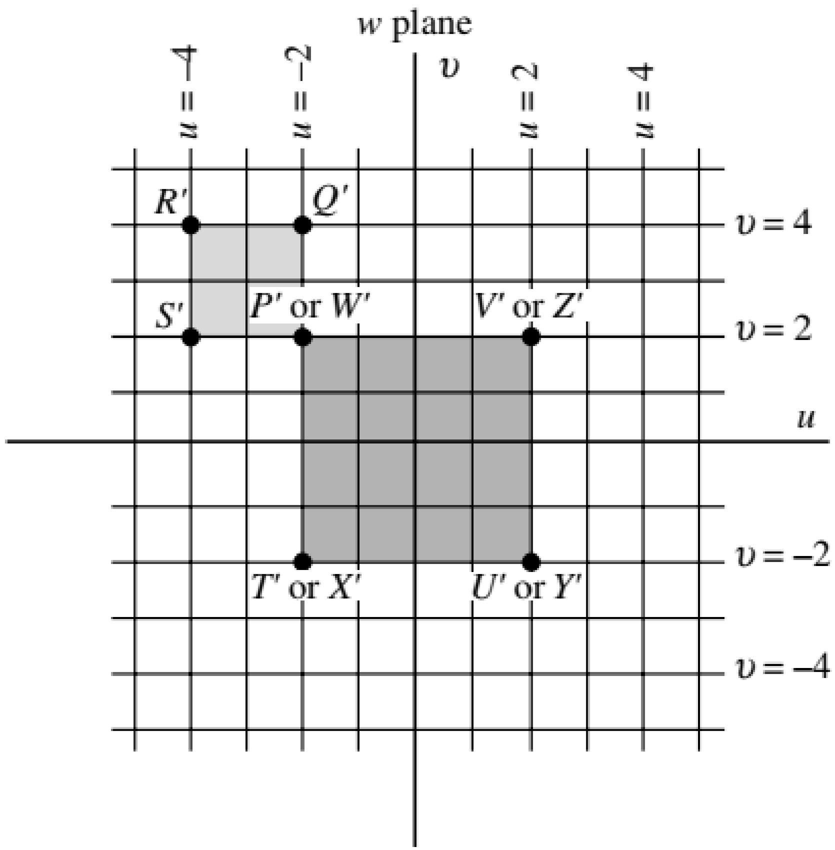

Fig. 2-11

EX-4: Prove that (a) $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, (b) $|e^z| = e^x$, (c) $e^{z+2k\pi i} = e^z$, $k = 0, \pm 1, \pm 2, \ldots$.

Solution

(a) By definition $e^z = e^x(\cos y + i \sin y)$ where $z = x + iy$. Then, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\begin{aligned} e^{z_1} \cdot e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) \cdot e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1} \cdot e^{x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}\{\cos(y_1 + y_2) + i \sin(y_1 + y_2)\} = e^{z_1+z_2} \end{aligned}$$

(b) $|e^z| = |e^x(\cos y + i \sin y)| = |e^x||\cos y + i \sin y| = e^x \cdot 1 = e^x$

(c) By part (a),

$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z(\cos 2k\pi + i \sin 2k\pi) = e^z$$

This shows that the function e^z has *period* $2k\pi i$. In particular, it has period $2\pi i$.

EX-5: Prove:

(a) $\sin^2 z + \cos^2 z = 1$ (c) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(b) $e^{iz} = \cos z + i \sin z,\ e^{-iz} = \cos z - i \sin z$ (d) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

Solution

By definition, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$. Then

(a)

$$\sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2$$

$$= -\left(\frac{e^{2iz} - 2 + e^{-2iz}}{4}\right) + \left(\frac{e^{2iz} + 2 + e^{-2iz}}{4}\right) = 1$$

(b)

$$e^{iz} - e^{-iz} = 2i \sin z \tag{1}$$

$$e^{iz} + e^{-iz} = 2 \cos z \tag{2}$$

Adding (1) and (2): $2e^{iz} = 2\cos z + 2i \sin z$ and $e^{iz} = \cos z + i \sin z$

Subtracting (1) from (2):

$$2e^{-iz} = 2\cos z - 2i\sin z \quad \text{and} \quad e^{-iz} = \cos z - i\sin z$$

(c) $$\sin(z_1 + z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \frac{e^{iz_1} \cdot e^{iz_2} - e^{-iz_1} \cdot e^{-iz_2}}{2i}$$

$$= \frac{(\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) - (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2)}{2i}$$

$$= \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

(d) $$\cos(z_1 + z_2) = \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2} = \frac{e^{iz_1} \cdot e^{iz_2} + e^{-iz_1} \cdot e^{-iz_2}}{2}$$

$$= \frac{(\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) + (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2)}{2}$$

$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

EX-6: Prove that the zeros of (a) $\sin z$ and (b) $\cos z$ are all real and find them.

Solution

(a) If $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = 0$, then $e^{iz} = e^{-iz}$ or $e^{2iz} = 1 = e^{2k\pi i}$, $k = 0, \pm 1, \pm 2, \ldots$.
Hence, $2iz = 2k\pi i$ and $z = k\pi$, i.e., $z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are the zeros.

(b) If $\cos z = \dfrac{e^{iz} + e^{-iz}}{2} = 0$, then $e^{iz} = -e^{-iz}$ or $e^{2iz} = -1 = e^{(2k+1)\pi i}$, $k = 0, \pm 1, \pm 2, \ldots$.
Hence, $2iz = (2k+1)\pi i$ and $z = (k + \frac{1}{2})\pi$, i.e., $z = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$ are the zeros.

EX-7: Prove that (a) $\sin(-z) = -\sin z$, (b) $\cos(-z) = \cos z$, (c) $\tan(-z) = -\tan z$.

Solution

(a) $$\sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i} = \frac{e^{-iz} - e^{iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\sin z$$

(b) $$\cos(-z) = \frac{e^{i(-z)} + e^{-i(-z)}}{2} = \frac{e^{-iz} + e^{iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

(c) $$\tan(-z) = \frac{\sin(-z)}{\cos(-z)} = \frac{-\sin z}{\cos z} = -\tan z,$$ using (a) and (b).

Functions of z having the property that $f(-z) = -f(z)$ are called *odd functions*, while those for which $f(-z) = f(z)$ are called *even functions*. Thus $\sin z$ and $\tan z$ are odd functions, while $\cos z$ is an even function.

EX-8: (a) Suppose $z = e^w$ where $z = r(\cos\theta + i\sin\theta)$ and $w = u + iv$. Show that $u = \ln r$ and $v = \theta + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ so that $w = \ln z = \ln r + i(\theta + 2k\pi)$. (b) Determine the values of $\ln(1 - i)$. What is the principal value?

Solution

(a) Since $z = r(\cos\theta + i\sin\theta) = e^w = e^{u+iv} = e^u(\cos v + i\sin v)$, we have on equating real and imaginary parts,

$$e^u \cos v = r\cos\theta \tag{1}$$

$$e^u \sin v = r\sin\theta \tag{2}$$

Squaring (1) and (2) and adding, we find $e^{2u} = r^2$ or $e^u = r$ and $u = \ln r$. Then, from (1) and (2), $r\cos v = r\cos\theta$, $r\sin v = r\sin\theta$ from which $v = \theta + 2k\pi$. Hence, $w = u + iv = \ln r + i(\theta + 2k\pi)$.

If $z = e^w$, we say that $w = \ln z$. We thus see that $\ln z = \ln r + i(\theta + 2k\pi)$. An equivalent way of saying the same thing is to write $\ln z = \ln r + i\theta$ where θ can assume infinitely many values which differ by 2π.

Note that *formally* $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$ using laws of real logarithms familiar from elementary mathematics.

(b) Since $1 - i = \sqrt{2}e^{7\pi i/4 + 2k\pi i}$, we have $\ln(1 - i) = \ln\sqrt{2} + \left(\frac{7\pi i}{4} + 2k\pi i\right) = \frac{1}{2}\ln 2 + \frac{7\pi i}{4} + 2k\pi i$.

The principal value is $\frac{1}{2}\ln 2 + \frac{7\pi i}{4}$ obtained by letting $k = 0$.

EX-9: Consider the transformation $w = \ln z$. Show that (a) circles with center at the origin in the z plane are mapped into lines parallel to the v axis in the w plane, (b) lines or *rays* emanating from the origin in the z plane are mapped into lines parallel to the u axis in the w plane, (c) the z plane is mapped into a strip of width 2π in the w plane. Illustrate the results graphically.

Solution

We have $w = u + iv = \ln z = \ln r + i\theta$ so that $u = \ln r$, $v = \theta$.
Choose the principal branch as $w = \ln r + i\theta$ where $0 \le \theta < 2\pi$.

(a) Circles with center at the origin and radius α have the equation $|z| = r = \alpha$. These are mapped into lines in the w plane whose equations are $u = \ln a$. In Figs. 2-17 and 2-18, the circles and lines corresponding to $\alpha = 1/2,\ 1,\ 3/2,\ 2$ are indicated.

(b) Lines or rays emanating from the origin in the z plane (dashed in Fig. 2-17) have the equation $\theta = \alpha$. These are mapped into lines in the w plane (dashed in Fig. 2-18) whose equations are $v = \alpha$. We have shown the corresponding lines for $\alpha = 0,\ \pi/6,\ \pi/3$, and $\pi/2$.

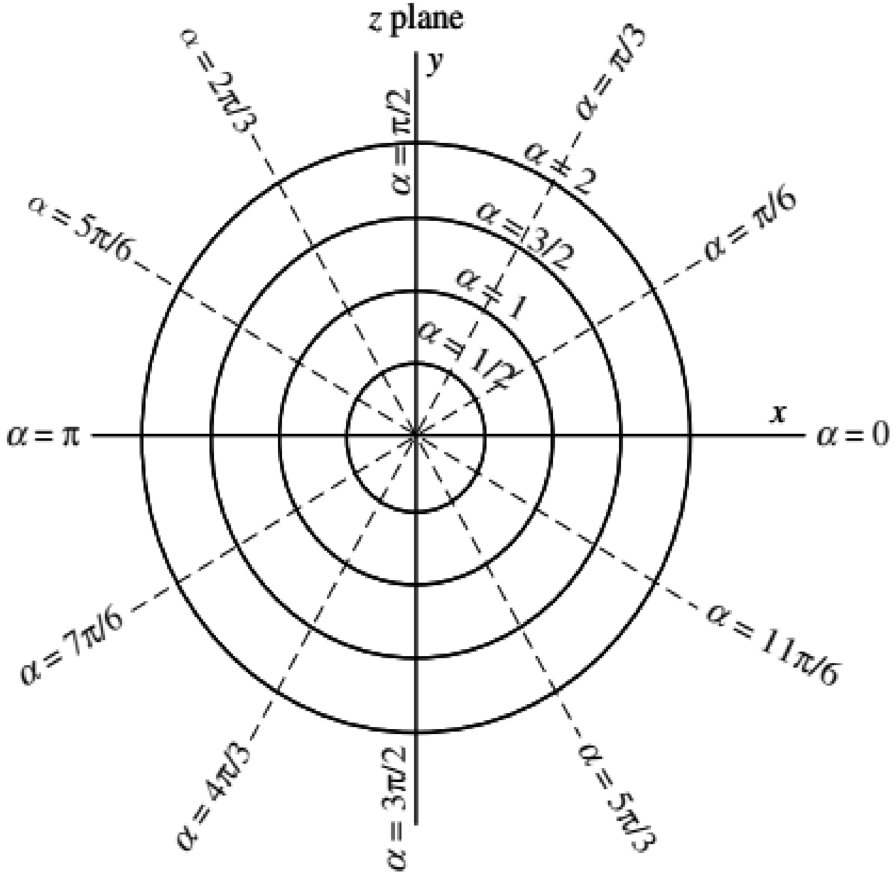

Fig. 2-17

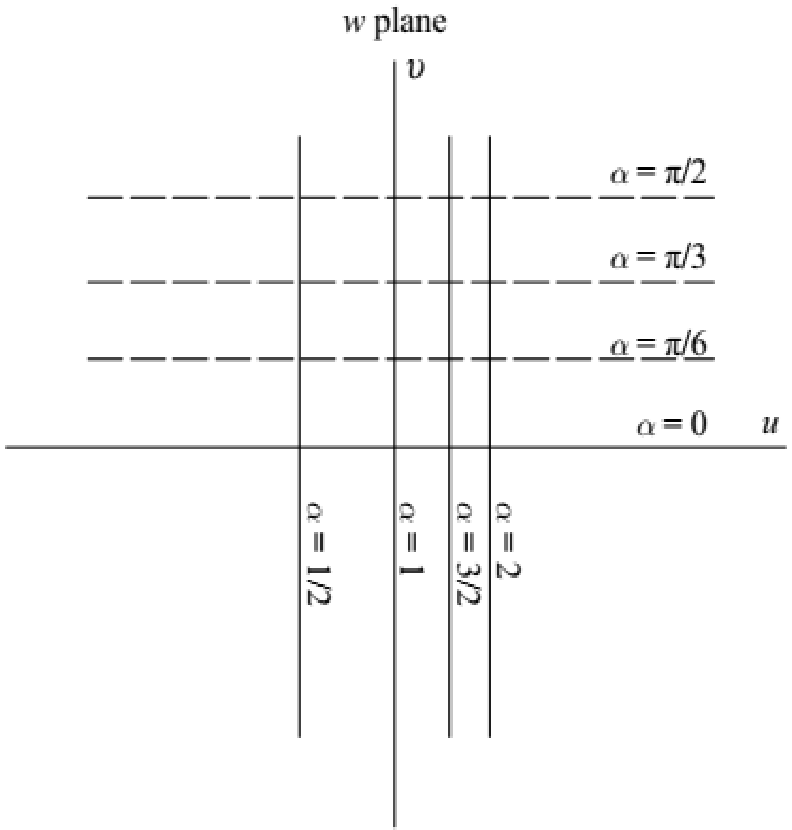

Fig. 2-18

(c) Corresponding to any given point P in the z plane defined by $z \neq 0$ and having polar coordinates (r, θ) where $0 \leq \theta < 2\pi$, $r > 0$ [as in Fig. 2-19], there is a point P' in the strip of width 2π shown shaded in Fig. 2-20. Thus, the z plane is mapped into this strip. The point $z = 0$ is mapped into a point of this strip sometimes called the *point at infinity*.

If θ is such that $2\pi \leq \theta < 4\pi$, the z plane is mapped into the strip $2\pi \leq v < 4\pi$ of Fig. 2-20. Similarly, we obtain the other strips shown in Fig. 2-20.

It follows that given any point $z \neq 0$ in the z plane, there are infinitely many image points in the w plane corresponding to it.

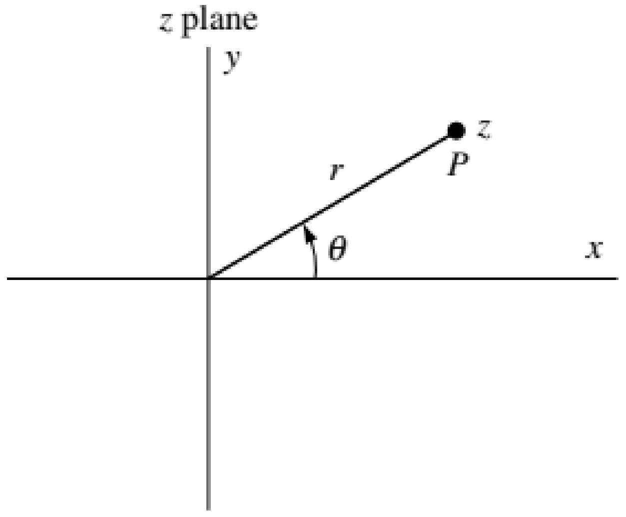

Fig. 2-19

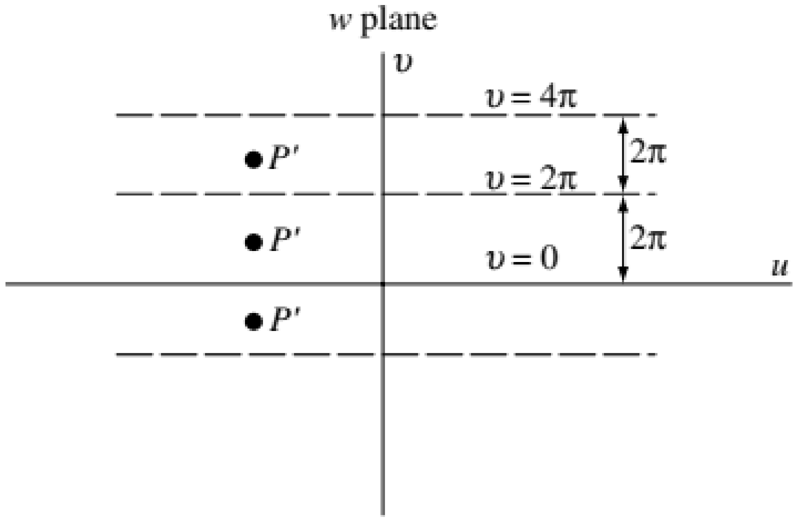

Fig. 2-20

It should be noted that if we had taken θ such that $-\pi \leq \theta < \pi$, $\pi \leq \theta < 3\pi$, etc., the strips of Fig. 2-20 would be shifted vertically a distance π.

EX-10: (a) Suppose $z = re^{i\theta}$. Prove that $z^i = e^{-(\theta+2k\pi)}\{\cos(\ln r) + i\sin(\ln r)\}$ where $k = 0, \pm1, \pm2, \ldots$.

(b) Suppose z is a point on the unit circle with center at the origin. Prove that z^i represents infinitely many real numbers and determine the principal value.

(c) Find the principal value of i^i.

Solution

(a) By definition,

$$z^i = e^{i\ln z} = e^{i\{\ln r + i(\theta+2k\pi)\}}$$

$$= e^{i\ln r-(\theta+2k\pi)} = e^{-(\theta+2k\pi)}\{\cos(\ln r) + i\sin(\ln r)\}$$

The principal branch of the many-valued function $f(z) = z^i$ is obtained by taking $k = 0$ and is given by $e^{-\theta}\{\cos(\ln r) + i\sin(\ln r)\}$ where we can choose θ such that $0 \le \theta < 2\pi$.

(b) If z is any point on the unit circle with center at the origin, then $|z| = r = 1$. Hence, by part (a), since $\ln r = 0$, we have $z^i = e^{-(\theta+2k\pi)}$ which represents infinitely many real numbers. The principal value is $e^{-\theta}$ where we choose θ such that $0 \le \theta < 2\pi$.

(c) By definition, $i^i = e^{i\ln i} = e^{i\{i(\pi/2+2k\pi)\}} = e^{-(\pi/2+2k\pi)}$ since $i = e^{i(\pi/2+2k\pi)}$ and $\ln i = i(\pi/2 + 2k\pi)$.

The principal value is given by $e^{-\pi/2}$.

EX-11: (a) Suppose $f(z) = z^2$. Prove that $\lim_{z\to z_0} f(z) = z_0^2$.

(b) Find $\lim_{z\to z_0} f(z)$ if $f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{cases}$.

Solution

(a) We must show that, given any $\epsilon > 0$, we can find δ (depending in general on ϵ) such that $|z^2 - z_0^2| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

If $\delta \le 1$, then $0 < |z - z_0| < \delta$ implies that

$$|z^2 - z_0^2| = |z - z_0||z + z_0| < \delta|z - z_0 + 2z_0| < \delta\{|z - z_0| + |2z_0|\} < \delta(1 + 2|z_0|)$$

Take δ as 1 or $\epsilon/(1 + 2|z_0|)$, whichever is smaller. Then, we have $|z^2 - z_0^2| < \epsilon$ whenever $|z - z_0| < \delta$, and the required result is proved.

(b) There is no difference between this problem and that in part (a), since in both cases we exclude $z = z_0$ from consideration. Hence, $\lim_{z\to z_0} f(z) = z_0^2$. Note that the limit of $f(z)$ as $z \to z_0$ has nothing whatsoever to do with the value of $f(z)$ at z_0.

EX-12: Interpret Problem **11** geometrically.

Solution

(a) The equation $w = f(z) = z^2$ defines a transformation or mapping of points of the z plane into points of the w plane. In particular, let us suppose that point z_0 is mapped into $w_0 = z_0^2$. [See Fig. 2-25 and 2-26.]

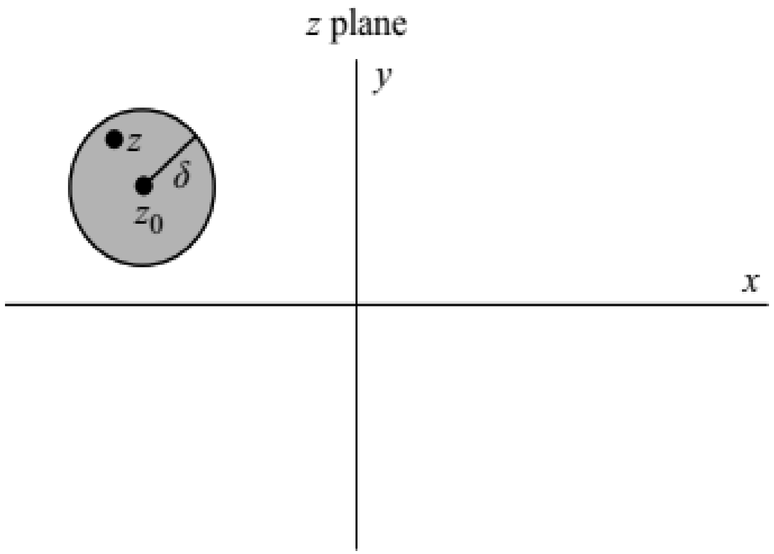

Fig. 2-25

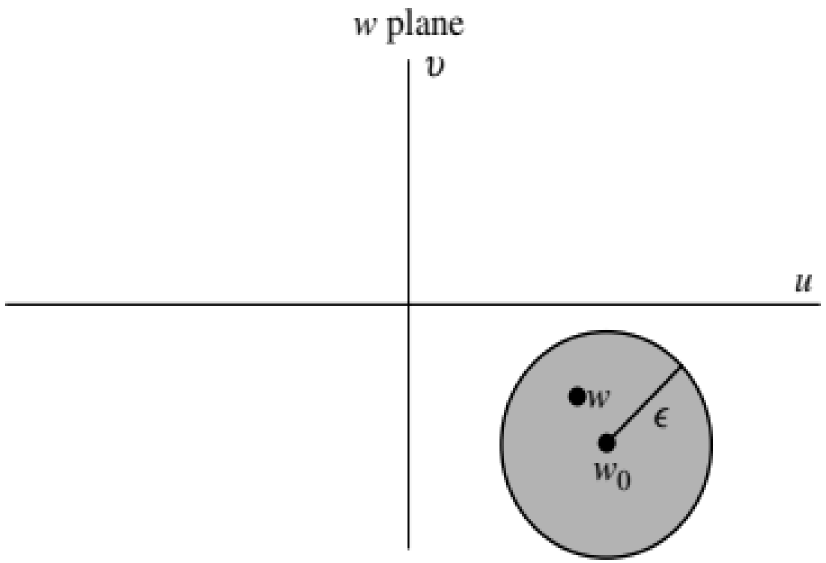

Fig. 2-26

In Problem **11** (a), we prove that given any $\epsilon > 0$ we can find $\delta > 0$ such that $|w - w_0| < \epsilon$ whenever $|z - z_0| < \delta$. Geometrically, this means that if we wish w to be inside a circle of radius ϵ [see Fig. 2-26] we must choose δ (depending on ϵ) so that z lies inside a circle of radius δ [see Fig. 2-25]. According to Problem **10** (a), this is certainly accomplished if δ is the smaller of 1 and $\epsilon/(1 + 2|z_0|)$.

(b) In Problem **11** (a), $w = w_0 = z_0^2$ is the image of $z = z_0$. However, in Problem **11** (b), $w = 0$ is the image of $z = z_0$. Except for this, the geometric interpretation is identical with that given in part (a).

EX-13: Evaluate each of the following using theorems on limits:

(a) $\lim_{z\to 1+i} (z^2 - 5z + 10)$ (b) $\lim\limits_{z\to -2i} \dfrac{(2z+3)(z-1)}{z^2 - 2z + 4}$ (c) $\lim\limits_{z\to 2e^{\pi i/3}} \dfrac{z^3 + 8}{z^4 + 4z^2 + 16}$

Solution

(a)
$$\begin{aligned}\lim_{z\to 1+i} (z^2 - 5z + 10) &= \lim_{z\to 1+i} z^2 + \lim_{z\to 1+i} (-5z) + \lim_{z\to 1+i} 10\\ &= (\lim_{z\to 1+i} z)(\lim_{z\to 1+i} z) + (\lim_{z\to 1+i} -5)(\lim_{z\to 1+i} z) + \lim_{z\to 1+i} 10\\ &= (1+i)(1+i) - 5(1+i) + 10 = 5 - 3i\end{aligned}$$

In practice, the intermediate steps are omitted.

(b) $\lim_{z\to -2i}\dfrac{(2z+3)(z-1)}{z^2-2z+4}=\dfrac{\lim_{z\to -2i}(2z+3)\lim_{z\to -2i}(z-1)}{\lim_{z\to -2i}(z^2-2z+4)}=\dfrac{(3-4i)(-2i-1)}{4i}=-\dfrac{1}{2}+\dfrac{11}{4}i$

(c) In this case, the limits of the numerator and denominator are each zero and the theorems on limits fail to apply. However, by obtaining the factors of the polynomials, we see that

$$\begin{aligned}\lim_{z\to 2e^{\pi i/3}}\frac{z^3+8}{z^4+4z^2+16}&=\lim_{z\to 2e^{\pi i/3}}\frac{(z+2)(z-2e^{\pi i/3})(z-2e^{5\pi i/3})}{(z-2e^{\pi i/3})(z-2e^{2\pi i/3})(z-2e^{4\pi i/3})(z-2e^{5\pi i/3})}\\&=\lim_{z\to 2e^{\pi i/3}}\frac{(z+2)}{(z-2e^{2\pi i/3})(z-2e^{4\pi i/3})}=\frac{e^{\pi i/3}+1}{2(e^{\pi i/3}-e^{2\pi i/3})(e^{\pi i/3}-e^{4\pi i/3})}\\&=\frac{3}{8}-\frac{\sqrt{3}}{8}i\end{aligned}$$

EX-14: Prove that $\lim_{z\to 0}(\bar{z}/z)$ does not exist.

Solution

If the limit is to exist, it must be independent of the manner in which z approaches the point 0.

Let $z \to 0$ along the x axis. Then $y = 0$, and $z = x + iy = x$ and $\bar{z} = x - iy = x$, so that the required limit is

$$\lim_{x\to 0} \frac{x}{x} = 1$$

Let $z \to 0$ along the y axis. Then $x = 0$, and $z = x + iy = iy$ and $\bar{z} = x - iy = -iy$, so that the required limit is

$$\lim_{y\to 0} \frac{-iy}{iy} = -1$$

Since the two approaches do not give the same answer, the limit does not exist.

EX-15: (a) Prove that $f(z) = z^2$ is continuous at $z = z_0$.

(b) Prove that $f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{cases}$, where $z_0 \neq 0$, is discontinuous at $z = z_0$.

Solution

(a) By Problem 2.23(a), $\lim_{z \to z_0} f(z) = f(z_0) = z_0^2$ and so $f(z)$ is continuous at $z = z_0$.

Another Method. We must show that given any $\epsilon > 0$, we can find $\delta > 0$ (depending on ϵ) such that $|f(z) - f(z_0)| = |z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$. The proof patterns that given in Problem 2.23(a).

(b) By Problem 2.23(b), $\lim_{z \to z_0} f(z) = z_0^2$, but $f(z_0) = 0$. Hence, $\lim_{z \to z_0} f(z) \neq f(z_0)$ and so $f(z)$ is discontinuous at $z = z_0$ if $z_0 \neq 0$.

If $z_0 = 0$, then $f(z) = 0$; and since $\lim_{z \to z_0} f(z) = 0 = f(0)$, we see that the function is continuous.

EX-16: Is the function $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$?

Solution

$f(i)$ does not exist, i.e., $f(x)$ is not defined at $z = i$. Thus $f(z)$ is not continuous at $z = i$.

By redefining $f(z)$ so that $f(i) = \lim_{z \to i} f(z) = 4 + 4i$ (see Problem 2.25), it becomes continuous at $z = i$. In such a case, we call $z = i$ a *removable discontinuity*.

EX-17: Prove that $f(z) = z^2$ is continuous in the region $|z| \leq 1$.

Solution

Let z_0 be any point in the region $|z| \leq 1$. By Problem **10** (a), $f(z)$ is continuous at z_0. Thus, $f(z)$ is continuous in the region since it is continuous at any point of the region.

EX-18: For what values of z are each of the following functions continuous?

Solution

(a) $f(z) = z/(z^2 + 1) = z/(z - i)(z + i)$. Since the denominator is zero when $z = \pm i$, the function is continuous everywhere except $z = \pm i$.

(b) $f(z) = \csc z = 1/\sin z$. By Problem 2.10(a), $\sin z = 0$ for $z = 0, \pm\pi, \pm 2\pi, \ldots$. Hence, $f(z)$ is continuous everywhere except at these points.

EX-19: Let $w = (z^2 + 1)^{1/2}$. (a) If $w = 1$ when $z = 0$, and z describes the curve C_1 shown in Fig. 2-27, find the value of w when $z = 1$. (b) If z describes the curve C_2 shown in Fig. 2-28, is the value of w, when $z = 1$, the same as that obtained in (a)?

Solution

(a) The branch points of $w = f(z) = (z^2 + 1)^{1/2} = \{(z - i)(z + i)\}^{1/2}$ are at $z = \pm i$

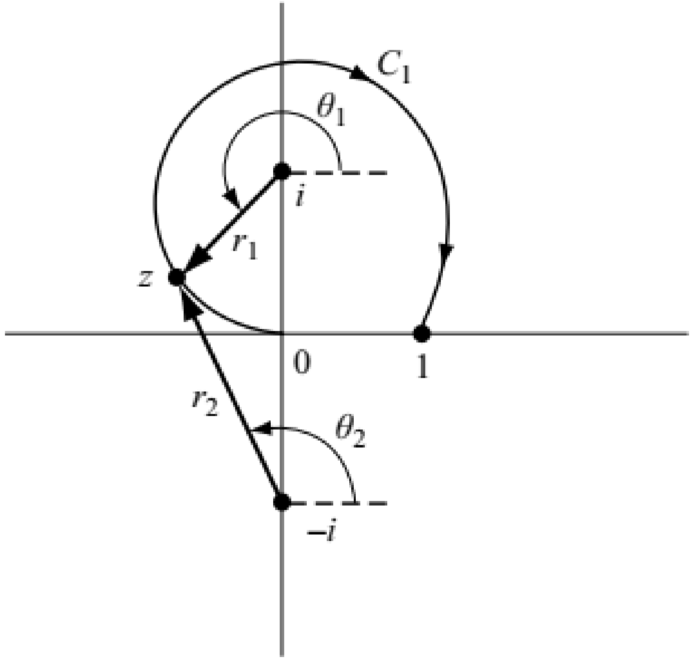

Fig. 2-27

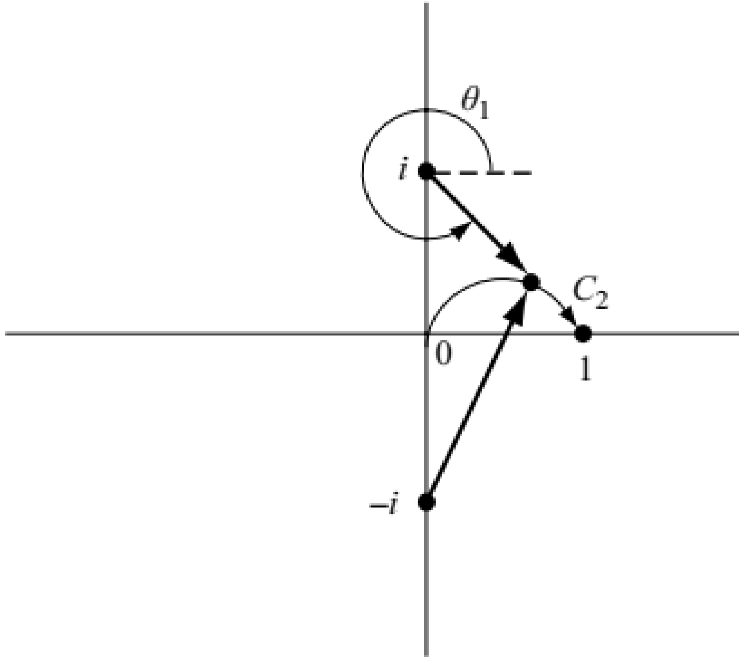

Fig. 2-28

Let (1) $z - i = r_1 e^{i\theta_1}$, (2) $z + i = r_2 e^{i\theta_2}$. Then, since θ_1 and θ_2 are determined only within integer multiples of $2\pi i$, we can write

$$w = \sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2} e^{2k\pi i/2} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2} e^{k\pi i} \tag{3}$$

Referring to Fig. 2-27 [or by using the equations (1) and (2)], we see that when z is at 0, $r_1 = 1$, $\theta_1 = 3\pi/2$, and $r_2 = 1$, $\theta_2 = \pi/2$. Since $w = 1$ at $z = 0$, we have from (3), $1 = e^{(k+1)\pi i}$ and we choose $k = -1$ [or 1, $-3, \ldots$]. Then

$$w = -\sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$$

As z traverses C_1 from 0 to 1, r_1 changes from 1 to $\sqrt{2}$, θ_1 changes from $3\pi/2$ to $-\pi/4$, r_2 changes from 1 to $\sqrt{2}$, θ_2 changes from $\pi/2$ to $\pi/4$. Then

$$w = -\sqrt{(\sqrt{2})(\sqrt{2})}\; e^{i(-\pi/4+\pi/4)/2} = -\sqrt{2}$$

(b) As in part (a), $w = -\sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2}$. Referring to Fig. 2-28, we see that as z traverses C_2, r_1 changes from 1 to $\sqrt{2}$, θ_1 changes from $3\pi/2$ to $7\pi/4$, r_2 changes from 1 to $\sqrt{2}$ and θ_2 changes from $\pi/2$ to $\pi/4$. Then

$$w = -\sqrt{(\sqrt{2})(\sqrt{2})}\, e^{i(7\pi/4+\pi/4)/2} = \sqrt{2}$$

which is not the same as the value obtained in (a).

EX-20: Let $\sqrt{1-z^2} = 1$ for $z = 0$. Show that as z varies from 0 to $p > 1$ along the real axis, $\sqrt{1-z^2}$ varies from 1 to $-i\sqrt{p^2-1}$.

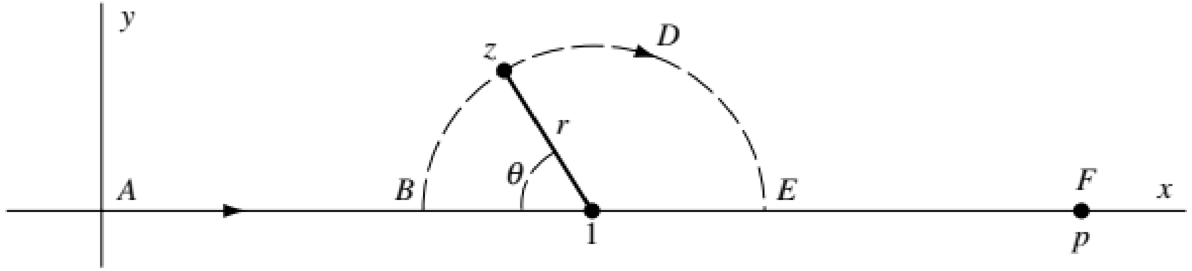

Fig. 2-29

Solution

Consider the case where z travels along path $ABDEF$, where BDE is a semi-circle as shown in Fig. 2-29. From this figure, we have

$$1 - z = 1 - x - iy = r\cos\theta - ir\sin\theta$$

so that $\sqrt{1-z^2} = \sqrt{(1-z)(1+z)} = \sqrt{r}(\cos\theta/2 - i\sin\theta/2)\sqrt{2 - r\cos\theta + ir\sin\theta}$

Along AB: $z = x$, $r = 1 - x$, $\theta = 0$ and $\sqrt{1-z^2} = \sqrt{1-x}\sqrt{1+x} = \sqrt{1-x^2}$.

Along EF: $z = x$, $r = x - 1$, $\theta = \pi$ and $\sqrt{1-z^2} = -i\sqrt{x-1}\sqrt{x+1} = -i\sqrt{x^2-1}$.

Hence, as z varies from 0 [where $x = 0$] to p [where $x = p$], $\sqrt{1-z^2}$ varies from 1 to $-i\sqrt{p^2-1}$.

EX-21: Prove that if $|a| < 1$,

(a) $$1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \cdots = \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}$$

(b) $$a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \cdots = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}$$

Solution

Let $z = ae^{i\theta}$ in Problem 2.41. We can do this since $|z| = |a| < 1$. Then

$$1 + ae^{i\theta} + a^2e^{2i\theta} + a^3e^{3i\theta} + \cdots = \frac{1}{1 - ae^{i\theta}}$$

or

$$(1 + a\cos\theta + a^2\cos 2\theta + \cdots) + i(a\sin\theta + a^2\sin 2\theta + \cdots) = \frac{1}{1 - ae^{i\theta}} \cdot \frac{1 - ae^{-i\theta}}{1 - ae^{-i\theta}}$$

$$= \frac{1 - a\cos\theta + ia\sin\theta}{1 - 2a\cos\theta + a^2}$$

The required results follow on equating real and imaginary parts.

Thanks a lot …