#1 $R\sim U[0,1)$, $X=f(R) \Rightarrow F(x)=P(X\le x)$, $f(x)=F'(x)$, $E(X)=\int_{-\infty}^{+\infty} x f(R)\,dR$

#2 $\int_a^b f(x)dx = \int_a^b (b-a) f(x)\frac{1}{b-a}dx = E((b-a)f(x))$, $x\sim U[a,b)$

$\int_a^b\int_{f(x)}^{g(x)} F(x,y)\,dy\,dx = E[(b-a)(g(x)-f(x))F(x,y)]$, $X\sim U[a,b)$, $Y\sim U[f(x),g(x))$

#3

Independent: $E(\alpha X+\beta)=\alpha E(X)+\beta$ $E(\alpha X+\beta Y)=\alpha E(X)+\beta E(Y)$ $\quad E(g(X)h(Y))=E(g(X))E(h(Y))$ $Var(X)=E(X^2)-(E(X))^2$ $\quad Var(\sum X_i)=\sum Var(X_i)$ $\quad Var(\alpha X)=\alpha^2 Var(X)$
Dependent: $Cov(X,Y)=E(XY)-E(X)E(Y)$ $\quad Cov(\alpha X,Y)=\alpha Cov(X,Y)$ $Cov(\sum_{i=1}^m X_i,\sum_{j=1}^n Y_j)=\sum_{i=1}^m\sum_{j=1}^n Cov(X_i,Y_j)$ $Var(\sum_{i=1}^n X_i)=\sum_{i=1}^n Var(X_i)+2\sum_{i=1}^n\sum_{j>i}^n Cov(X_i,X_j)$ $\underset{[-1,1]}{\rho(X,Y)}=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$ $\quad \rho(\alpha X,Y)=\rho(X,\alpha Y)=\frac{\alpha}{\lvert\alpha\rvert}\rho(X,Y)$

$f_{X\mid Y}(x\mid y)=\frac{f(x,y)}{f_Y(y)}=\frac{f(x,y)}{\int_{-\infty}^{+\infty} f(x,y)\,dx}$

$E(X\mid Y=y)=\int_{-\infty}^{+\infty} x f_{X\mid Y}(x\mid y)\,dx$

$\Rightarrow E(X)=E(E(X\mid Y))$

$Var(X\mid Y)=E(X^2\mid Y)-(E(X\mid Y))^2$

$Var(X)=E(Var(X\mid Y))+Var(E(X\mid Y))$

$M(t)=E(e^{tX}) \Rightarrow M^{k}(0)=E(X^k)$

$M_{X+Y}(t)=M_X(t)M_Y(t)$

$Y=\min\{X_1,\cdots,X_n\} \Rightarrow F_Y(y)=1-\prod_{k=1}^n(1-F_{X_k}(y))$

$Y=\max\{X_1,\cdots,X_n\} \Rightarrow F_Y(y)=\prod_{k=1}^n F_{X_k}(y)$

Discrete + Discrete: $P_Z(z)=\sum_{x\in R_X} P_X(x)P_Y(z-x)$ Discrete X + Continuous Y: $f_Z(z)=\sum_{x\in R_X} P_X(x) f_Y(z-x)$ Continuous + Continuous: $f_Z(z)=\int_{x\in R_X} f_X(x) f_Y(z-x)\,dx$

#3 Modes: $X=\arg\max f(x)$

Bernoulli Distribution: $Pr(X=x)=\begin{cases}1-p, & x=0\\ p, & x=1\end{cases}$ $\quad E(X)=p\quad Var(X)=p(1-p)$

$X\sim Binomial(n,p)$, $Y\sim Pascal(r,p) \Rightarrow Pr(Y>n)=Pr(X<r)$

$X_1\sim Poisson(\lambda_1)$, $X_2\sim Poisson(\lambda_2) \Rightarrow X_1+X_2\sim Poisson(\lambda_1+\lambda_2)$

Gamma Distribution: $\Gamma(\beta)=(\beta-1)\Gamma(\beta-1)$

$X\sim Gamma(\beta,\theta)$, $E(X)=\frac{1}{\theta}$, $Var(X)=\frac{1}{\beta\theta^2}$, mode $=\frac{\beta-1}{\beta\theta}$

Hypergeometric Distribution: $X\sim Hypergeometric(g,b,n)$

$Pr(X=x)=\frac{C_g^x C_b^{n-x}}{C_{g+b}^n}$ $\quad E(X)=\frac{ng}{g+b}$

λ: mean arrival rate $\quad \Delta t\sim Exp(\lambda)$

Poisson: $\underset{\lambda(t)}{\underset{\sim\sim}{\text{Stationary}}}$: $Pr(N(t)=n)=P_n(t)=\frac{e^{-\lambda t}(\lambda t)^n}{n!}\approx N(\lambda t,\lambda t)$

$N(t)=N(t_1)+N(t_2)+\cdots+N(t_n) \Rightarrow \Lambda=\lambda_1+\cdots+\lambda_n$

Nonstationary: $\Lambda(t)=\int_0^t\lambda(s)\,ds$, $\bar\lambda=\frac{1}{t}\Lambda(t)$

$Pr(N(t)=n\mid\Lambda(t))=\frac{e^{-\bar\lambda t}(\bar\lambda t)^n}{n!}$

$p_1+\cdots p_k=1$ $N(t)=N_1(t)+\cdots+N_k(t)$ $N_j(t)\sim Poisson(\lambda p_j)$

#2021

$$\boxed{\#1\ f(x)=\begin{cases}Ax, & 0<x<a\\ Aae^{-\lambda(x-a)}, & a<x\end{cases}}$$

a) $\int_0^{+\infty} f(x)dx = \frac{1}{2}Ax^2\Big|_0^a + (-\frac{1}{\lambda})Aae^{-\lambda x}\Big|_0^{+\infty}$

$= \frac{1}{2}Aa^2 + \frac{1}{\lambda}Aa = 1$

$\Rightarrow A(\frac{1}{2}a^2+\frac{a}{\lambda})=1 \Rightarrow A=\frac{2\lambda}{a(\lambda a+2)}$

b) $0<x<a$:

$F(x)=\int_0^x At\,dt = \frac{1}{2}Ax^2 = \frac{\lambda x^2}{a(\lambda a+2)}$

$x\geq a$:

$F(x)=\frac{1}{2}Aa^2 + (\frac{1}{-\lambda})Aae^{-\lambda(t-a)}\Big|_a^x$

$= 1-\frac{2}{a\lambda+2}e^{-\lambda(x-a)}$

c) $M_X(t)=\int_{-\infty}^{+\infty} e^{tx} f(x)dx$

$=\int_0^a e^{tx}\frac{2\lambda}{a(\lambda a+2)}x\,dx$

$+\int_a^{+\infty} e^{tx}\frac{2\lambda}{a(a\lambda+2)}ae^{-\lambda(x-a)}dx$

$=\frac{2\lambda}{a\lambda+2}\left(\frac{1+(at-1)e^{at}}{at^2}+\frac{e^{at}}{\lambda-t}\right)$

d) $M'_X(0) = E(X) = \frac{2\lambda}{a\lambda+2}\left(\frac{a^2\lambda^2+3a\lambda+3}{3\lambda^2}\right)$

$\boxed{\text{\#2 p: win, plan to play n, if } n_{th} \text{ win, continue, until lost}}$

a) $X \sim n-1+\text{Geometric}(1-p)$

$E(X)=n-1+\frac{1}{1-p}$

b) $X_{lose}=(1-p)(n-1)+(1-p)\cdot 1+p\cdot 1$

$= 1+(1-p)(n-1)$

$X_{win}=p(n-1)+\frac{1}{1-p}-1$

$\boxed{\#3\ g(a)=E(|X-a|)}$

a) $g'(a)=0$

$g(a)=\int_{-\infty}^{+\infty}|x-a|f(x)dx$

$=\int_{-\infty}^a (a-x)f(x)dx+\int_a^{+\infty}(x-a)f(x)dx$

$g'(a)=\int_{-\infty}^a f(x)dx+\int_a^{+\infty}-f(x)dx$

$=F(a)-(1-F(a))=2F(a)-1$

$\Rightarrow F(a)=\frac{1}{2} \Rightarrow \int_{-\infty}^a f(x)dx=\int_a^{+\infty}f(x)dx=\frac{1}{2}$

$g''(a)=2f(a)\geq 0 \Rightarrow g'(a)$ increasing

$\Rightarrow g_{min}(a)=\cancel{a\int_{-\infty}^a f(x)dx}-\int_{-\infty}^a x f(x)dx$

$+\int_a^{+\infty} x f(x)dx - \cancel{a\int_a^{+\infty} f(x)dx}$

$= E(X)-2\int_{-\infty}^a x f(x)dx$

b) $F(x)=1-e^{-\lambda x}$

$\Rightarrow F(a)=1-e^{-\lambda a}=\frac{1}{2}$

$a=\frac{\ln 2}{\lambda}$

$\boxed{\#4\ \text{item} \sim Exp(\lambda),\ \begin{cases}A & \text{sell all}\\ B & \text{sell one by one}\end{cases}}$

a) $Pr=(1-e^{-\lambda T})^3$

b) $X=X_1+X_2+X_3 \sim Erlang(3,\frac{\lambda}{3})$

$Pr=1-\sum_{i=0}^{2}\frac{e^{-\lambda T}(\lambda T)^i}{i!}$

$=1-e^{-\lambda T}(1+\lambda T+\frac{\lambda T^2}{2})$

$\boxed{\#5\ \text{car } A\sim U[0,20)\ \text{car } B\sim U[0,30)}$

a) $Pr(|\frac{1200}{a}-\frac{1200}{b}|\leq 28)$

$=Pr(\frac{300a}{300+7a}<b<\frac{300a}{300-7a})$

$=\int_0^{\frac{300}{17}}\int_{\frac{300a}{300+7a}}^{\frac{300a}{300-7a}}\frac{1}{20}\cdot\frac{1}{30}\,db\,da$

$+\int_{\frac{300}{17}}^{30}\int_{\frac{300a}{300+7a}}^{30}\frac{1}{20}\cdot\frac{1}{30}\,db\,da$

$=\frac{150}{49}\ln(\frac{187}{75})-\frac{18}{7}$

#2020

#1 $f(x) = A \times \begin{cases} \frac{1}{x^2}, & -\infty < x \le -1 \\ x^2, & -1 \le x \le 1 \\ \frac{1}{x^2}, & 1 \le x < +\infty \end{cases}$

a) $\int_{-\infty}^{+\infty} f(x)dx = A(1+\frac{2}{3}+1) = 1$

$\Rightarrow A = \frac{3}{8}$

b) $F(x) = \begin{cases} -\frac{3}{8x}, & -\infty < x \le -1 \\ \frac{1}{8}x^3 + \frac{1}{2}, & -1 \le x \le 1 \\ -\frac{3}{8x}+1, & 1 \le x < +\infty \end{cases}$

#2 $x_1, x_2, x_3 \sim f(x) = \frac{1}{2}x^2e^{-x}$

$p = Pr(X>4) = (-\frac{1}{2}x^2 - x - 1)e^{-x}\Big|_4^{+\infty} = 13e^{-4}$

a) $Pr = 1-(1-p)^3$

b) $Pr = C_3^2 p^2(1-p) + C_3^3 p^3$

c) $Pr = p^3$

#3 $\lambda(t) = \frac{3t^2(8-t)}{8}$ $\frac{3}{4}$ are men

$\Lambda(t) = \int_0^t \frac{3s^2(8-s)}{8} ds = t^3 - \frac{3}{32}t^4$

$\bar\lambda(t) = t^2 - \frac{3}{32}t^3 \Rightarrow \bar\lambda(8) = 16$

$P_n(t) = \frac{e^{-\bar\lambda t}(\bar\lambda t)^n}{n!}$

$\bar\lambda_{men} = 16 \times \frac{3}{4} = 12 \Rightarrow P_n(t) = \frac{e^{-96}96^n}{n!}$

$Pr = 1 - \sum_{i=0}^{100} \frac{e^{-96}96^n}{n!}$

#4 $X, Y \sim f(x) = \begin{cases} 0, & x \le 0 \\ xe^{-x}, & x > 0 \end{cases}$

$Z = X+Y > 6$

a) $f_Z(z) = \int_0^z ye^{-y} \cdot (z-y)e^{y-z} dy$

$= e^{-z}\int_0^z (z-y)y\,dy = \frac{1}{6}z^3e^{-z}$

$p = 61e^{-6}$

b) $P_n = C_{12}^3 p^3(1-p)^9$

c) $P_n = C_4^1 p^2(1-p)^3$

#5 coin head $1-p$, tossed until head twice, $Pr(n_2 - n_1 > n_1)$

$X, Y \sim \text{Geometric}(1-p)$

$Pr(Y>X) = \sum_{j=1}^{\infty}\sum_{k>j}^{\infty}(1-p)p^{k-1}(1-p)p^{j-1}$

$= (1-p)^2\sum_{j=1}^{\infty}p^{j-1}\sum_{k=j+1}^{\infty}p^{k-1}$

$= (1-p)^2\sum_{j=1}^{\infty}p^{j-1}\cdot p^j\sum_{k=1}^{\infty}p^{k-1}$

$= (1-p)^2\sum_{j=1}^{\infty}p^{2j-1}\cdot\frac{1}{1-p}$

$= (1-p)\cdot\frac{1}{p}\sum_{j=1}^{\infty}(p^2)^j = (1-p)\cdot\frac{1}{p}\cdot\frac{p^2}{1-p^2}$

$= \frac{p}{1+p}$

#6 $p(x) = (1-p)^2 x p^{x-1}$ $M_t(x)$

$M_t(x) = \sum_{k=1}^{\infty} e^{kx}(1-p)^2 k p^{k-1}$

$= \frac{(1-p)^2}{p}\sum_{k=1}^{\infty} k(pe^x)^k$

$= \frac{(1-p)^2}{p}\frac{pe^x}{(1-pe^x)^2} = \frac{(1-p)^2e^x}{(1-pe^x)^2}$

$M_t'(x) = \frac{e^x(1+pe^x)(1-p)^2}{(1-pe^x)^3}$

$E(X) = M_t'(0) = \frac{(1+p)(1-p)^2}{(1-p)^3} = \frac{1+p}{1-p}$

#7 nth customer arrives after t

$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$

$Pr = \sum_{i=0}^{n-1}\frac{e^{-\lambda t}(\lambda t)^i}{i!}$

#2019

#1 $P_{x|a}(x)=\frac{e^{-a}a^x}{x!}$ $f_{a|\lambda}=\lambda e^{-\lambda a}$

Show $P_{x|\lambda}(x)=(1-q)^x q$

$$P_{x|\lambda}(x)=\int_{-\infty}^{+\infty}P_{x|a}(x)f_{a|\lambda}(a)\,da$$
$$=\frac{\lambda}{(\lambda+1)^{x+1}}=\left(1-\frac{\lambda}{\lambda+1}\right)^x\left(\frac{\lambda}{\lambda+1}\right)$$

#2 $X_1,X_2,X_3\sim Exp(\frac{1}{50})$

a) $p=Pr(X\geq 25)=1-F(25)=e^{-\frac{1}{2}}$

$Pr=1-(1-e^{-\frac{1}{2}})^3$

b) $Pr=e^{-\frac{3}{2}}$

c) $X=X_1+X_2+X_3\sim Erlang(3,\frac{1}{150})$

$$Pr=\sum_{t=0}^{2}\frac{e^{-\frac{3}{2}}(\frac{3}{2})^t}{t!}$$

#3 $f(x)=A|x|e^{-\lambda|x|}$, $A>0,\lambda>0$

a) $\int_{-\infty}^{+\infty}f(x)dx=2\int_0^{+\infty}Axe^{-\lambda x}$
$$=2A\cdot\int_0^{+\infty}xe^{-\lambda x}$$
$$=2A\cdot\frac{(-\lambda x-1)e^{-\lambda x}}{\lambda^2}\Big|_0^{+\infty}$$
$$=\frac{2A}{\lambda^2}=1$$
$\Rightarrow A=\frac{1}{2}\lambda^2$

b) $f(x)=\frac{1}{2}\lambda^2|x|e^{-\lambda|x|}$

For $x\leq 0$, $f(x)=-\frac{1}{2}\lambda^2xe^{\lambda x}$

$F(x)=-\frac{1}{2}(\lambda x-1)e^{\lambda x}$

For $x>0$, $f(x)=\frac{1}{2}\lambda^2xe^{-\lambda x}$

$F(x)=1-\frac{1}{2}(\lambda x+1)e^{-\lambda x}$

#4 $L_1,L_2,L_3\sim Exp(0.02)$

$L=L_1+L_2+L_3\sim Erlang(3,\frac{0.02}{3})$

a) $p=1-F(155)=\sum_{k=0}^{2}e^{-3.1}\frac{(3.1)^k}{k!}$

b) $Pr=C_{10}^3p^3(1-p)^7$

c) $Pr=C_6^2p^3(1-p)^4$

#5 the number of components the need replacement $\sim$ Poisson(4), how long at least N=30 components will need replacement with P=0.985

$P_n(t)=\frac{e^{-\lambda t}(\lambda t)!}{n!}$ $\Rightarrow n\sim N(\lambda t,\lambda t)$

$\lambda=4$

$Pr(n>N)=0.985$

$Pr(n\leq N)=1-P$

$\Rightarrow \phi\left(\frac{n-\lambda t}{\sqrt{\lambda t}}\leq\frac{N-\lambda t}{\sqrt{\lambda t}}\right)=1-P$

$\Rightarrow \frac{N-\lambda t}{\sqrt{\lambda t}}=\phi^{-1}(1-P)$

$\Rightarrow \lambda t+\phi^{-1}(1-P)\sqrt{\lambda t}-N=0$

$\Rightarrow t=\left(\frac{-\phi^{-1}(1-P)+\sqrt{(\phi^{-1}(1-P))^2+4N}}{2\sqrt{\lambda}}\right)^2$

#2018

$\boxed{\#1\ A\sim U[-1,1),\ C\sim U[-1,1),\ Pr(Ax^2-x+C=0 \text{ positive roots})}$

$$\begin{cases} 1-4AC \geq 0 \\ \dfrac{1\pm\sqrt{1-4AC}}{2A} \geq 0 \end{cases}$$

$$\Rightarrow \begin{cases} AC \leq \frac{1}{4} \\ A>0,\ AC\geq 0 \end{cases} \Rightarrow \begin{cases} A>0,\ C\geq 0 \\ 0\leq AC\leq \frac{1}{4} \end{cases}$$

$$Pr = \int_0^{\frac{1}{4}}\int_0^1 \frac{1}{2}\cdot\frac{1}{2}\,da\,dc + \int_{\frac{1}{4}}^1\int_0^{\frac{1}{4c}} \frac{1}{2}\cdot\frac{1}{2}\,da\,dc$$
$$= \frac{1}{16}+\frac{1}{8}\ln 2$$

$\boxed{\#2\ X_1\sim Exp(\frac{1}{600}),\ X_2\sim Exp(\frac{1}{400})}$

a) $Pr = 1-(1-e^{-\frac{1}{600}\times 1200}) = e^{-2}$

b) $Pr = Pr(X_1<1200)\,Pr(X_2>1200-X_1)$
$$=\int_0^{1200}\int_{1200-x_1}^{+\infty} \frac{1}{600}e^{-\frac{1}{600}x_1}\,\frac{1}{400}e^{-\frac{1}{400}x_2}\,dx_2dx_1$$
$$=\int_0^{1200}\frac{1}{600}e^{-\frac{1}{600}x_1}\,e^{-\frac{1}{400}(1200-x_1)}dx_1$$
$$=2e^{-2}-2e^{-3}$$

c) $Pr = Pr(X_1<1200)\,Pr(X_2<1200-X_1)$
$$=\int_0^{1200}\int_0^{1200-x_1}\frac{1}{600}e^{-\frac{1}{600}x_1}\cdot\frac{1}{400}e^{-\frac{1}{400}x_2}\,dx_2dx_1$$
$$=\int_0^{1200}\frac{1}{600}e^{-\frac{1}{600}x_1}\left(1-e^{-\frac{1}{400}(1200-x_1)}\right)dx_1$$
$$=1-(3e^{-2}-2e^{-3})$$

#3 customers ~ Poisson
first 2 hours, $\lambda_B=60$
next 4 hours, $\lambda_M=40$
last 2 hours, $\lambda_E=30$
$Pr(>350)$

$$\lambda(t)=\begin{cases}60, & 0\leq t<2\\ 40, & 2<t\leq 6\\ 30, & 6<t\leq 8\end{cases}$$

$$\Lambda(8)=\int_0^2 60\,dt+\int_2^6 40\,dt+\int_6^8 30\,dt$$
$$=340 \Rightarrow \lambda=42.5$$

$$P_n(t)=\frac{e^{-\lambda t}(\lambda t)^n}{n!}\sim N(340,340)$$

$$Pr(n>350)=1-Pr(n\leq 350)$$
$$=1-\phi\left(\frac{350-340}{\sqrt{340}}\right)$$
$$=1-\phi\left(\frac{\sqrt{85}}{17}\right)=0.294$$

$\boxed{\#4\ \text{men}\sim Poisson(6),\ \text{women}\sim Poisson(4)}$

$$Pr(X_1=n)=\frac{e^{-6}6^n}{n!}$$
$$Pr(X_2=n)=\frac{e^{-4}4^n}{n!}$$

a) $Pr=\dfrac{e^{-6}\cdot 6^5}{5!}\cdot\dfrac{e^{-4}\cdot 4^3}{3!}=0.0314$

b) $Pr=\dfrac{e^{-6}6^3}{3!}\cdot\dfrac{e^{-4}4^5}{5!}=0.0139$

c) $Pr=\displaystyle\sum_{k=0}^{8}\frac{e^{-6}6^k}{k!}\cdot\frac{e^{-4}4^{8-k}}{(8-k)!}$

Or $X=X_1+X_2\sim Poisson(10)$
$$Pr=\frac{e^{-10}10^8}{8!}=0.1126$$

$\boxed{\#5\ \Delta t\sim N(\mu,\sigma^2),\ \mu=3,\ \sigma=0.5}$

a) $Pr(\Delta t<0)=\phi\left(\frac{0-3}{0.5}\right)=\phi(-6)$

b) $t=\Delta t_1+\cdots+\Delta t_{225}\sim N(675,56.25)$

$$Pr(t<T)=0.985$$
$$\Rightarrow \phi\left(\frac{t-675}{7.5}\right)=0.985$$
$$\frac{t-675}{7.5}=\phi^{-1}(0.985)$$
$$\Rightarrow t=675+7.5\,\phi^{-1}(0.985)=691.275 \text{ min}$$