AN OVERVIEW OF THE BLACK-SCHOLES PARTIAL DIFFERENIAL EQUATION

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INTRODUCTION

Partial differential equations serve humans as a compass for interaction and behavior, in physical, metaphysical, heuristic, and natural applications. No function better documents the natural progression of often erratic consumer behaviour than the Black-Scholes model and consequent function when applied to global derivatives markets. By analyzing the trend of several parameters, including stock price, exercise price, timeframe of option, risk-free interest rate, and volatility of stock option, and the Black-Scholes model can analyze and accurately model the value of an option over its lifetime, given certain restrictions and assumptions. In the Black-Scholes model, the stock is the risky asset, and the money market, (or the cash in circulation) is the riskless asset. This assumption implies that the model is very resistant to intermittent or erratic shocks of inflation. Furthermore, the model assumes that the risk-free interest rate and volatility rates are constant over the lifetime of the option, so as to simplify operations, and retain the assumption of *ceteris paribus*, or the idea of examining on variable while keeping others constant. However, the greater laws of economics dictate that this is a rare occurrence and later interpretations of this model address this issue. If the assumption were broken in this context, a large error term would appear, altering the interpretation of the values received as outputs. It also assumes the call and put options discussed are European call and put options, so the exercise date is not a random variable subject to termination at any time. The model furthermore assumes that no dividends are paid on the option, so as not to over or undervalue the option. Finally, the option's value is a stochastic process that follows a geometric Brownian pedesis, known as an infinitesimal random walk with drift. The Black-Scholes model

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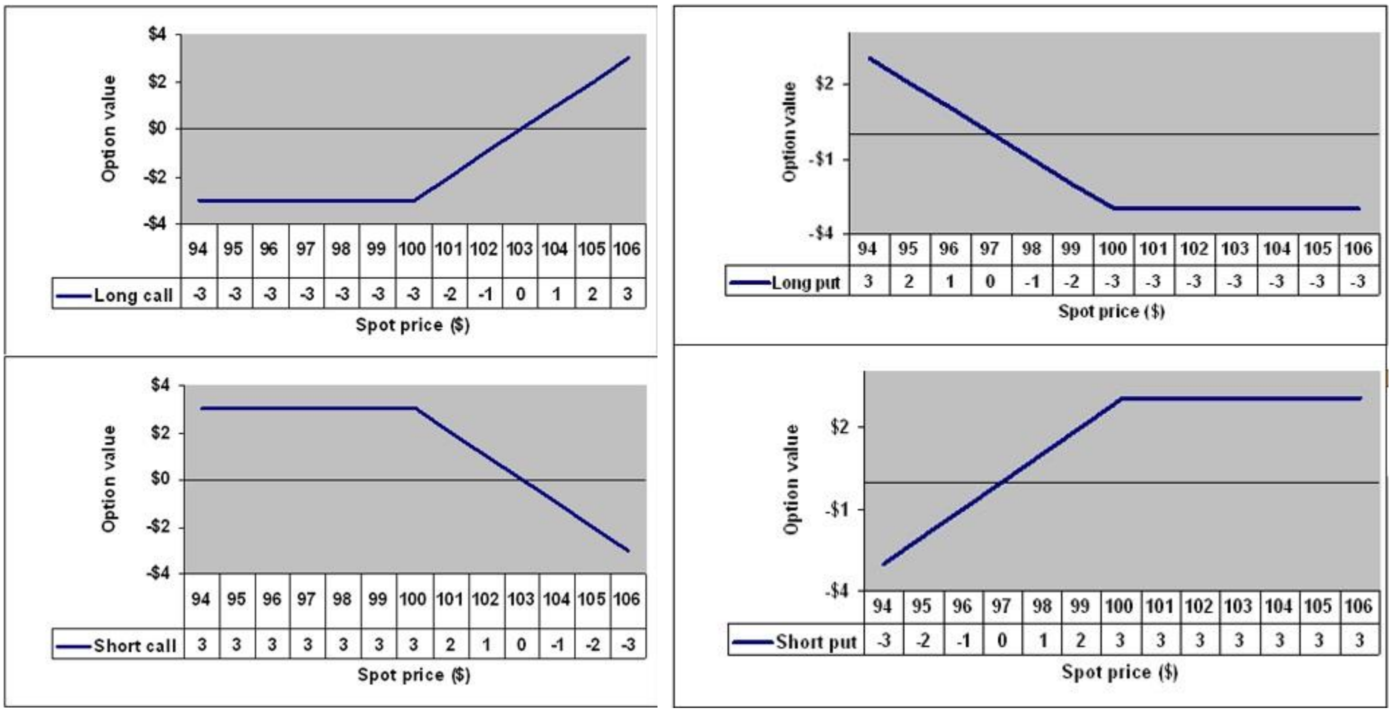
market assumptions are very conservative: there is no arbitrage opportunity (so as to avoid over or undervaluing the value of the option), no transactional fees or taxes paid (this issue is resolved in later models and adaptations of the model), derivative security, and the opportunity to borrow or lend an incremental level of money and stock at the riskless interest rate. Vanilla options – vanilla indicates the level of complexity to the option is low – are a common tool of the financial and derivatives market and their value has been a quandary. “Options are valuable when there is uncertainty. For example, one option contract traded on the financial exchanges gives the buyer the opportunity to buy a stock [or collection of stocks] at a specified price on a specified date and will be exercised (used) only if the price of the stock on that date exceeds the specified price” (Amram and Kulatilaka 5). The two types of options used are call and put options: “A call is a financial instrument that gives its owner the right to purchase an underlying [asset]... [and a] put option is a financial instrument that gives its owner the right to sell the underlying [asset]” (Kolb 309). When purchasing an option, either call or put, the buyer assumes either a long or short position and respectively the seller assumes either a short or long position. To assume a long position on a call option is to desire an increase in stock price as it will increase option value and to assume a short position on a call option is to desire a decrease in stock price to avoid a high payout. In the event of a provisional price incline, the long buyer will profit since the value of the option is greater than the original cost. Conversely, the short seller will experience a profit in the event that the price of the instrument should decrease prior to purchase. The potential loss on a short sale is theoretically unlimited; however, in practice, the long buyer will exercise the option prior to its expiration date if potential losses exceed the strike price, or if the option is “out-of-the-money”. To assume a long position on a put option is to desire a decrease in stock price as it will decrease payout and to assume a short position on a put option is to desire an increase in stock price to increase option value. In the event of a provisional price decline, the long buyer of a put option will experience a loss since the equipment is valued less and therefore receives less than the original value. In the following figures, an example is shown to indicate the change of value over time and value of price for both call and put options.

An options investor goes long on the underlying asset by buying call options or selling put options, exercising the option at the maximum of the market value or the strike price, thereby maximizing profits



and minimizing losses as well. Note that maximizing profits and minimizing losses are intrinsically not the same concept, indicating that they are not mutually exclusive, a topic discussed further when analyzing the purpose and roles of The Greeks. Option analysis aims to address two issues: intrinsic value – the difference between the market value and the strike price of the underlying asset of the given option, and time value – a multi-variable, non-linear interrelationship reflecting the discounted expected value of that difference at expiration. There are multitudes of hedging strategies that stem from targeting either of these values, once more a topic furthered when analyzing The Greeks. In 1973, Fischer Black and Myron Scholes presented a paper entitled *The Pricing of Options and Corporate Liabilities* to the Journal of Political Economy and was published by the University of Chicago Press. Addendums to the same concepts presented in this paper were published by the same press in 1976 by Robert C. Merton and Johnathan Ingersoll, delving further into issues that the Black-Scholes model originally failed to anticipate. These functional addendums incorporate dynamic interest rates, transaction costs and taxes, and dividend payouts. Additional addendum models incorporate the inclusion of other factors such as the jump from European vanilla options to more-complex American and Asian options, inclusion of foreign currencies and their respective risk rates (Capińksi and Kopp 79), path-dependency (Capińksi and Kopp 107), and various others that have greater relevance in the current derivatives market.

ORIGIN

The theoretical origin of the Black-Scholes partial differential equation as per Mr. Fischer Black himself (Chriss 119):

> "Suppose there is a formula that tells how the value of a [European] call option depends on the price of the underlying stock, the volatility of the stock, the exercise price and maturity, and the interest rate. Such a formula will tell us…how much the option value will change when the stock price changes by a small amount within a short time. Then you can create a hedged position by going short two option contracts and long one round lot of stock. Such a position will be close to riskless for small moves in the stock in the short run, your losses on one side will mostly offset by gains on the other side… As the hedged position will close to riskless, it should return an amount equal to the short-term interest rate on close-to-riskless securities. This one principle gives us the formula."

The mathematical origin of the Black-Scholes partial differential equation, however, is innately complex, delving into material far outside the scope of this paper such as use of Itô and stochastic calculus and martingales. The equation originates from the following Itô process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where μ is the drift of the Brownian pedesis and σ is the volatility of the risky asset – the stock. Following Lebesgue and Itô calculus and constringent to the laws of Itô processes and the market and stock assumptions previously made, the following formula is procured:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

All of the aforementioned and proceeding values will be defined in the coming sections when the solution is derived.

THE GREEKS

With the derivation of the Black-Scholes partial differential equation, investors have a multitude of methods to maximize their profits while minimizing losses. These are all represented as solutions to the first, second, and third order derivatives of the solution.

The following five are the first order derivatives.

1. **Delta** $\left(\Delta = \frac{\partial V}{\partial S}\right)$ measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. Given a vanilla option, usually for a long call/short put $0.0 < delta < 1.0$ and usually for a long put/short call $-1.0 < delta < 0.0$. Delta hedging is the term for when an investor takes the portfolio and minimizes the sum of all deltas for all underlying assets. Delta is also

close to – but not the exact value of – the approximate probability that the option will expire and create profit, or be "in-the-money". The following equation holds to link call and put deltas:

$$\Delta_{call} = 1 + \Delta_{put}$$
$$\Delta_{call} = \Phi(d_1)$$
$$\Delta_{put} = \Phi(-d_1)$$

2. **Vega** $\left(\nu = \frac{\partial V}{\partial \sigma}\right)$ measures the sensitivity to volatility (i.e., how likely the stock price is going to change given a variable volatility of the market). Low vegas are more desirable than higher vegas, for a low vega indicates a high resilience to volatility of stock price.

$$\nu = S\Phi(d_1)\sqrt{\tau} = K\Phi(d_2)\sqrt{\tau}$$

3. **Theta** $\left(\theta = -\frac{\partial V}{\partial \tau}\right)$ measures the sensitivity of the value of the derivative to the passage of time. It is interpreted as the amount of money per share lost in a given period of time. Knowing this value can lead to maximizing the minimum loss in selling or purchasing of the underlying asset.

$$\theta_{call} = \frac{-S\Phi(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2)$$
$$\theta_{put} = \frac{-S\Phi(d_1)\sigma}{2\sqrt{\tau}} + rKe^{-r\tau}\Phi(d_2)$$

4. **Rho** $\left(\rho = \frac{\partial V}{\partial r}\right)$ measures the sensitivity of the value of the underlying asset to change in the risk-free interest rate. This is one of the more difficult trends to follow since the interest rate is one of the few parameters completely out of control of the investor.

$$\rho_{call} = K\tau e^{-r\tau}\Phi(d_2)$$
$$\rho_{put} = -K\tau e^{-r\tau}\Phi(-d_2)$$

5. **Lambda** $\left(\lambda = \frac{\partial V}{\partial S} \cdot \frac{S}{V}\right)$ measures the economics elasticity factor of the underlying asset, or the percent change in underlying value per percentage change in underlying price. This is used as a measure of leverage, or a method of multiplying gains or losses.

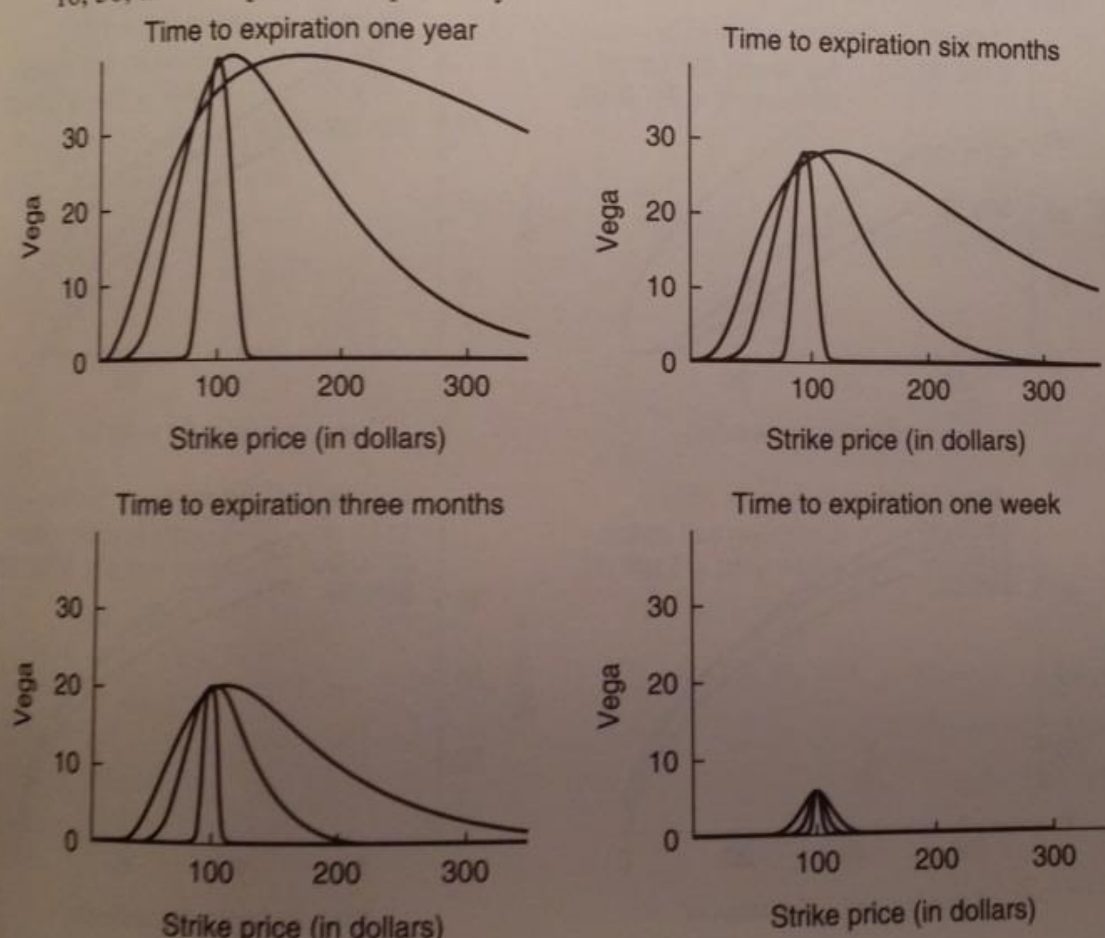
$$\lambda_{call} = \frac{S\Delta_{call}}{V} = \frac{S\Phi(d_1)}{V}$$

$$\lambda_{put} = \frac{S\Delta_{put}}{V} = \frac{S\Phi(-d_1)}{V}$$

Finding the value $V$of the stock option alone is not enough in true application; the Greeks are incredibly important in determining the actual worth of the option at hand. Based on an investment group's standards or a project's definition, these values can define the success of one and the downfall of many. In some cases, the evaluation of the Greeks hold more worth than just the value of the stock option, for there is much more information on the historical financial context.

FIGURE 4.10.2

European option vega versus strike price for an option on an underlying worth \$100. In each main plot, the three plots represent underlyings with volatilities of 10, 50, and 100 percent respectively.

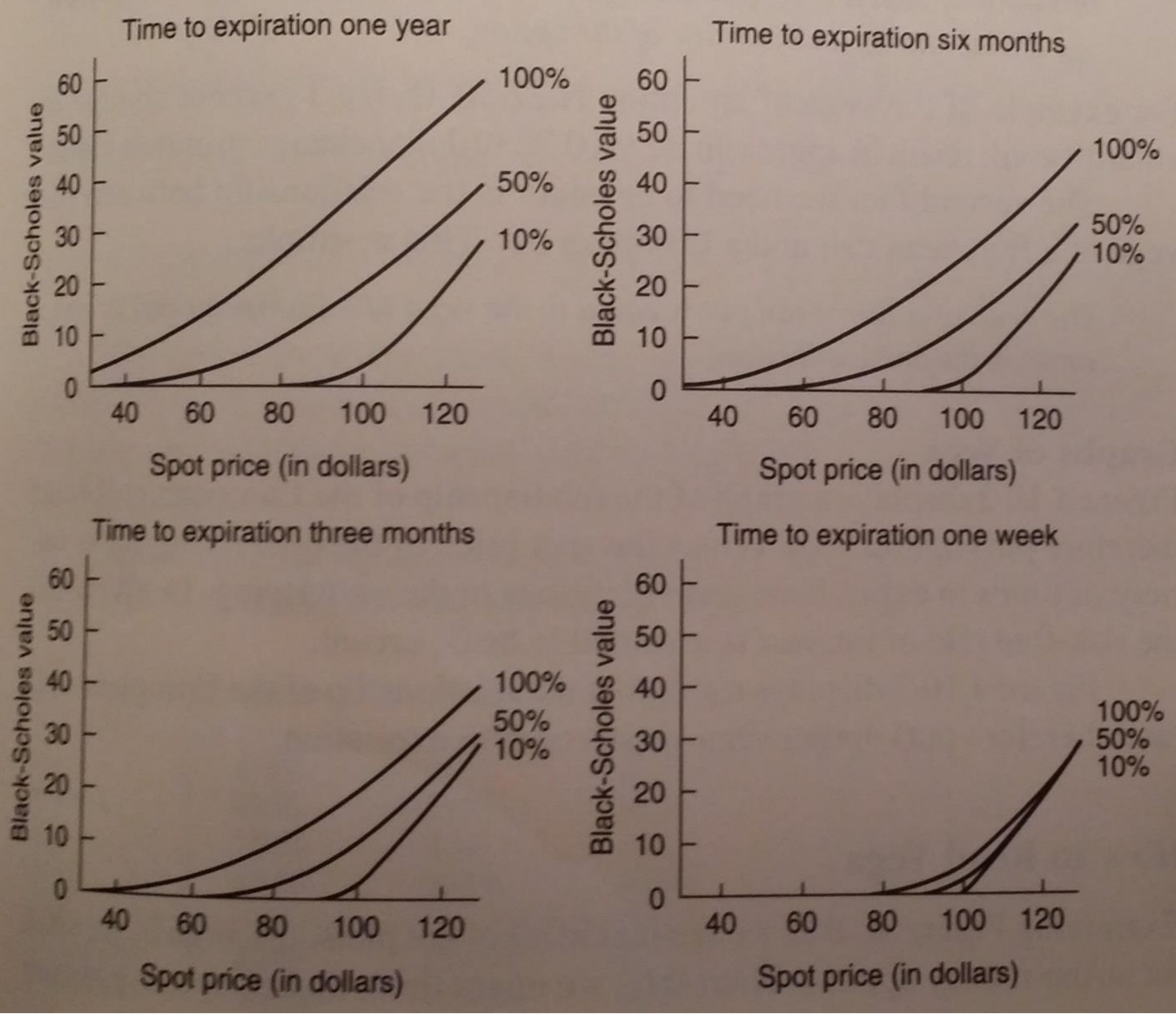


Notice in Figure 3 that as τ decreases, the range of ν diminishes incrementally. This visual representation of vega is incredibly important in an investing strategy known as vega-hedging, or hedging a portfolio to minimize the effects of vega to maximize profits and minimize losses. Minimizing vega has the effect of minimizing the influence changes in market volatility have on the overall portfolio. Market volatility is one of the more complicated variables in the Black-Scholes, but is also one of the more fascinating. Black and Scholes created a backwards and forward solving function with regards to volatility. In application, market prices are already established at a set value for τ and therefore the stock price can be used in reverse order to calculate the market volatility, or what is known as the implied volatility. This calculation can be useful in calculating or predicting general trends for the market, emerging, technological, or otherwise.

The following figure denotes the impact σ has on the stock value as τ decreases with passage of time. This indicates the drastic impact volatility has upon the value of the stock with the passage of time, a concept highly recognized in delta, vega, and gamma hedging strategies.

FIGURE 4.10.1

Black-Scholes call value versus spot price for different volatilities and times to expiration. Each graph has three plots representing options on stock with volatility of 100%, 50% and 10%.



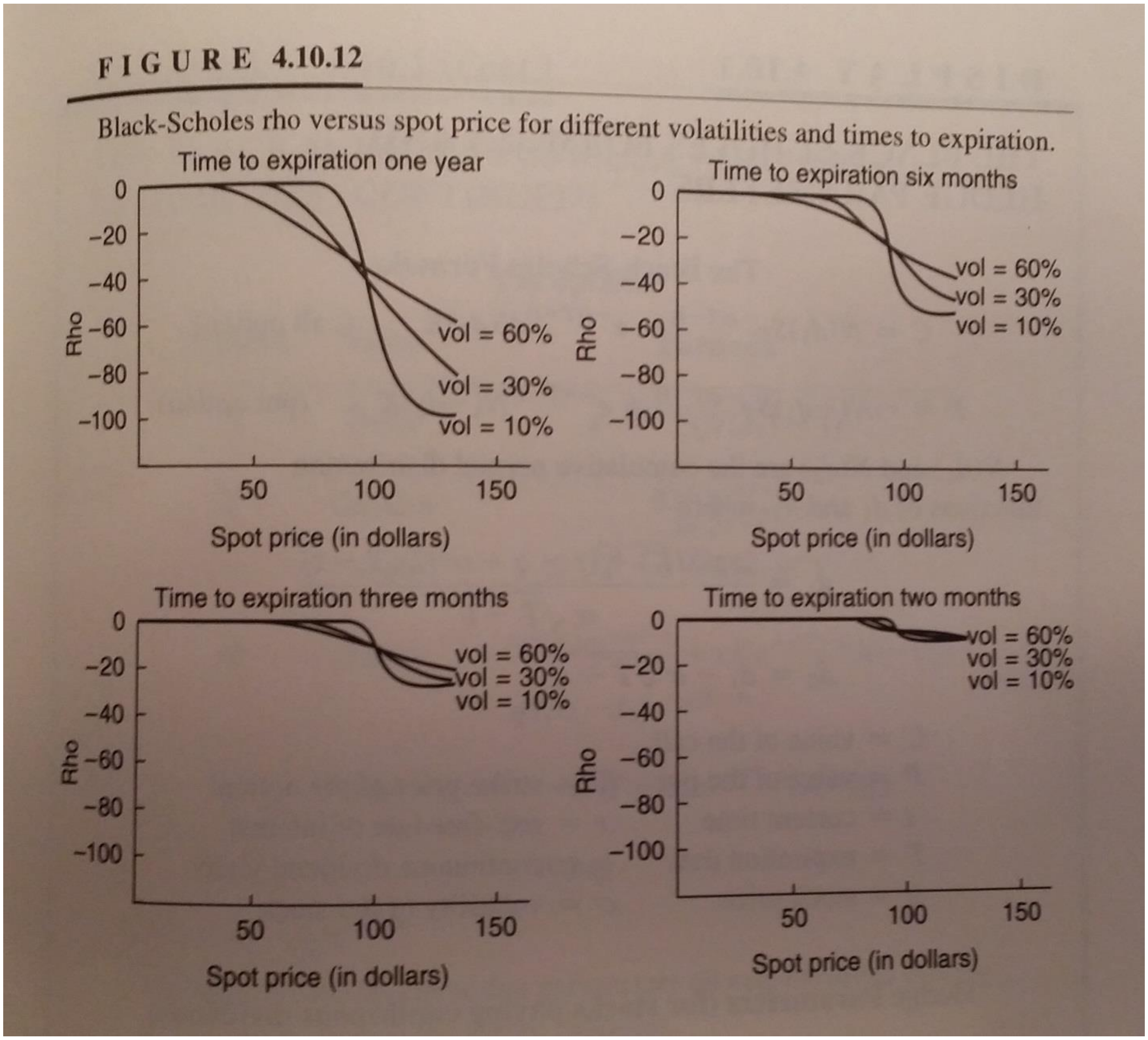
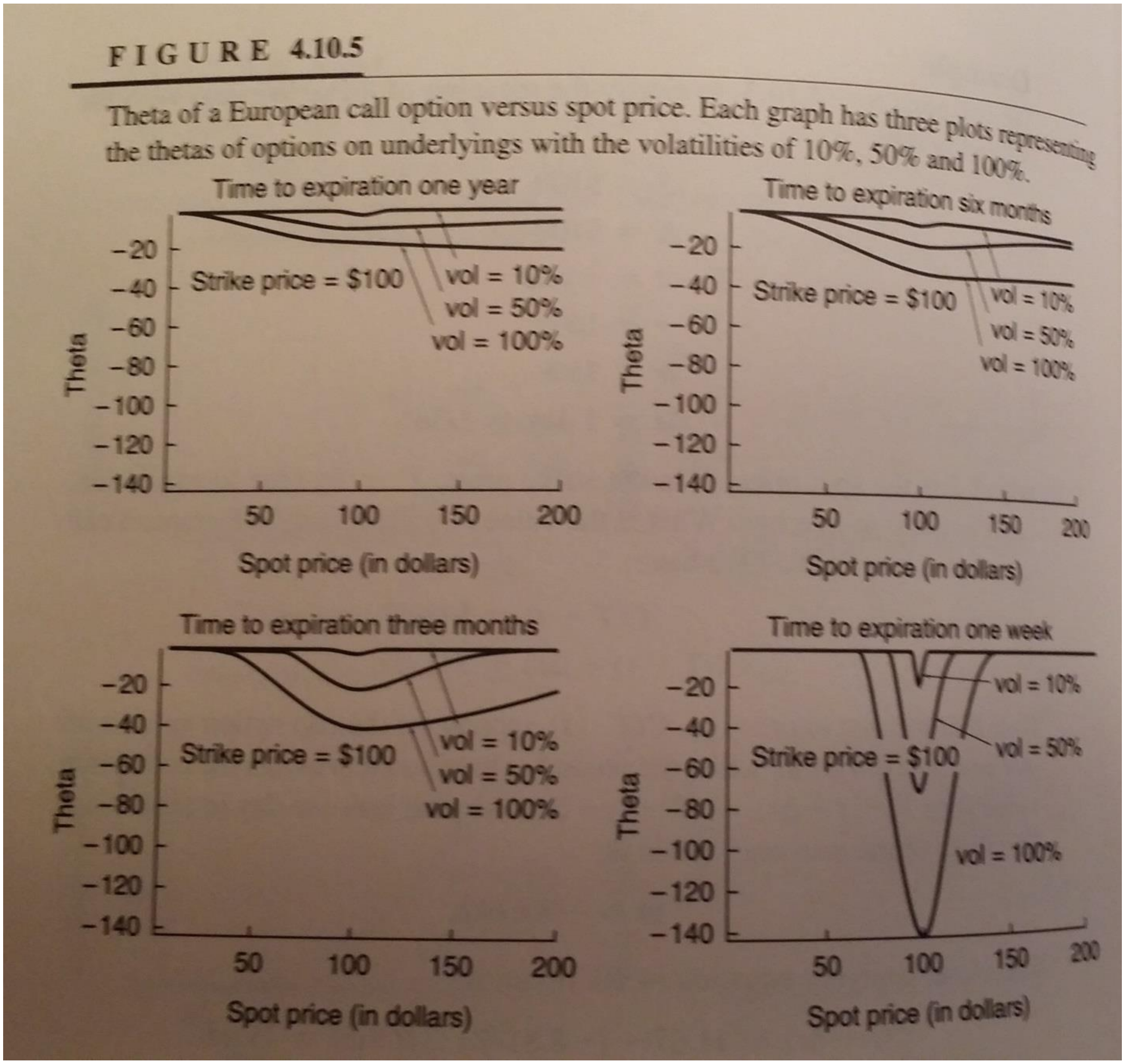


FIGURE 4.10.12

Black-Scholes rho versus spot price for different volatilities and times to expiration.

Notice in Figure 3 that as τ decreases, the range of ρ diminishes incredibly. This visual representation of rho is important in an investing strategy known as rho-hedging, or hedging a portfolio to minimize the effects of rho to maximize profits and minimize losses. Minimizing rho has the effect of minimizing the influence changes in the fluid, non-consant, risk-free interest rate has on the overall portfolio. This strategy is especially useful in periods of economic downturn or growth, as the market rate is changing the least during these times. As a part of expansionary monetary policy, the interest rate is set low to increase market consumption and is therefore predictable. Similary, as a part of contractionary monetary policy, the interest rate is set high to decrease market consumption, thereby making it predictable and in some cases more profitable of a venture. The reason for using the risk-free interest rate is to minimize as much exterior influence upon calculation as possible. This rate is assumed constant during the lifetime of the option, but it is known from economic predicitions that rarely does this occur, producing a downfall to the concept of rho-hedging.



Notice in Figure 4, that as τ decreases, the range of θ increases exponentially. Theta-hedging is not particularly useful in the investment field, as it is the factor with the most variability hidden and the least impact. Minimizing theta has the effect of minimizing the influence changes in time has on the overall portfolio. The graphs above imply that as the time to expiration decreases, or approaches zero, the value of the portfolio can become sporadic and highly precarious, since mathematically the value of the portfolio is aproaching a limit.

The second order derivatives, or second order Greeks include the following values

1. **Gamma** $\left(\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}\right)$ measures the rate of change in delta with respect to changes in the underlying price. More intuitively, this is the acceleration of value with respect to price.

$$\Gamma = \frac{\Phi(d_1)}{S\sigma\sqrt{\tau}}$$

2. **Vanna** $\left(\frac{\partial \Delta}{\partial \sigma} \sim \frac{\partial \nu}{\partial S} \sim \frac{\partial^2 V}{\partial S \partial \sigma}\right)$ measures the sensitivity of the option delta with respect to change in volatility. Vanna is useful in monitoring a delta-hedged or vega-hedged portfolio as vanna can help the trader anticipate changes to the effectiveness of their method of hedging.

$$Vanna = \frac{\Phi(d_1)d_2}{\sigma} = \frac{\nu}{S}\left[1 - \frac{d_1}{\sigma\sqrt{\tau}}\right]$$

3. **Vomma** $\left(\frac{\partial \nu}{\partial \sigma} = \frac{\partial^2 V}{\partial \sigma^2}\right)$ measures the rate of change of vega as volatility changes. This Greek develops the purpose of vega, as it is the acceleration of the value with respect to volatility.

$$Vomma = S\Phi(d_1)\sqrt{\tau}\frac{d_1 d_2}{\sigma} = \nu\frac{d_1 d_2}{\sigma}$$

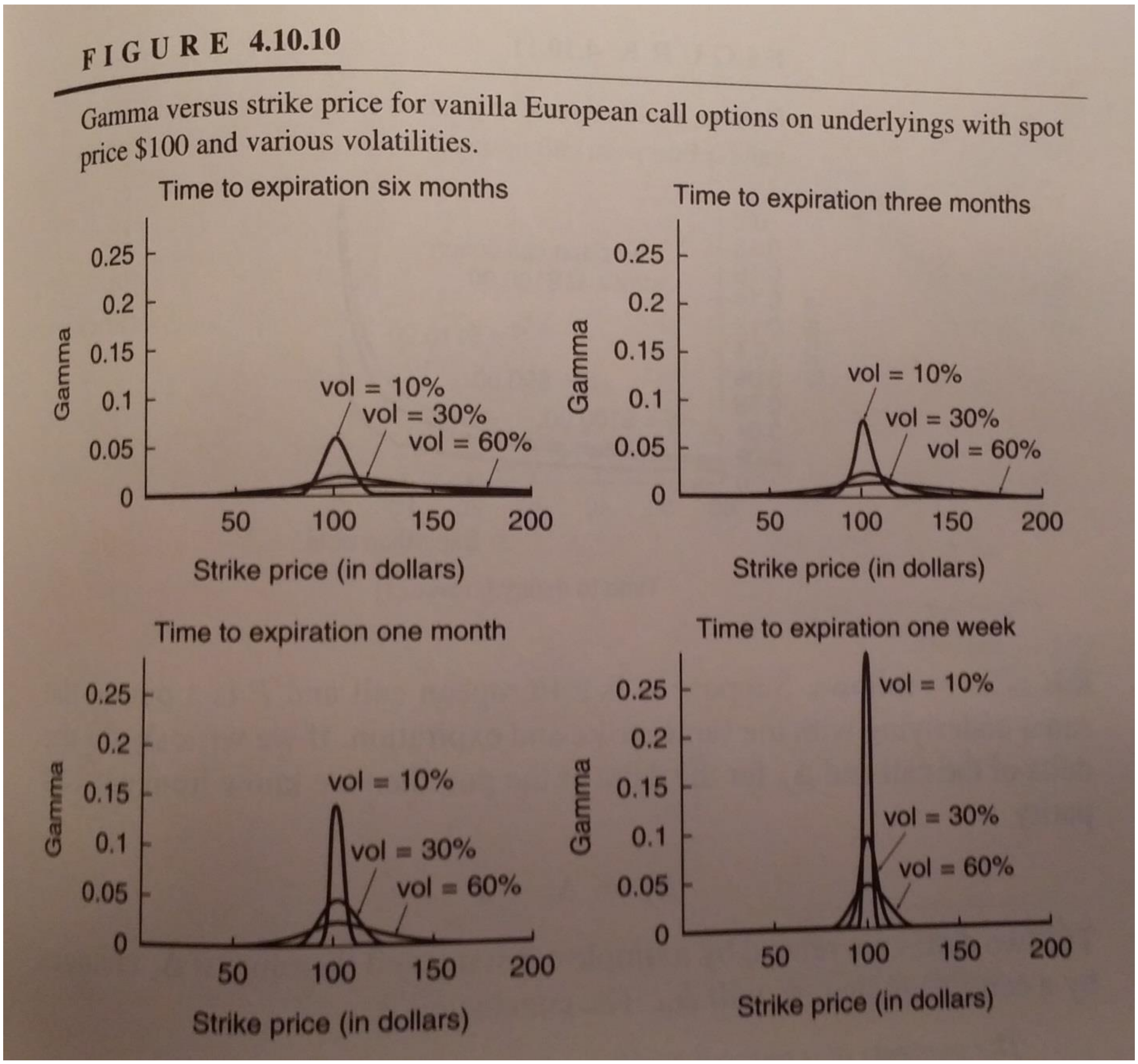
4. **Charm** $\left(-\frac{\partial \Delta}{\partial \tau} = \frac{\partial \Theta}{\partial S} = \frac{\partial^2 V}{\partial S \partial \tau}\right)$ measures the instantaneous rate of change of delta over time. Its purpose is to see the delta growth or decay per (business) day.

$$Charm = -\Phi(d_1)\frac{2r\tau - d_2\sigma\sqrt{\tau}}{2\tau\sigma\sqrt{\tau}}$$

5. **Veta** $\left(\frac{\partial \nu}{\partial \tau} = \frac{\partial^2 V}{\partial \sigma \partial \tau}\right)$ measures the rate of change in vega with respect to time, usually interpreted as the percentage change in vega per business day.

$$Veta = S\Phi(d_1)\sqrt{\tau}\left[\frac{rd_1}{\sigma\sqrt{\tau}} - \frac{1 + d_1 d_2}{2\tau}\right]$$

6. **Vera** $\left(\frac{\partial\rho}{\partial\sigma} = \frac{\partial^2 V}{\partial\sigma\partial r}\right)$ measures the rate of change in rho with respect to volatility, often used to assess the impact of volatility change on rho-hedging.



Notice in Figure 5, that Γ decreases in spread as the time to expiration aproaches zero. This implies that the acceleration, as it were, of the value of the portfolio with respect to stock price is minimal in the youth of the stock option and grows more volatile as τ approaches zero. Mathematically, this is a reasonable assumption since as τ approaches zero, the pure Black-Scholes model function value for Γ approaches an incalculable and infinite asymptote.

Third order Greeks include

1. **Color** $\left(\frac{\partial \Gamma}{\partial \tau} = \frac{\partial^3 V}{\partial S^2 \partial \tau}\right)$ measures the rate of change of gamma over time. This parameter is especially useful when gamma-hedging, observed in the change of gamma per day.

$$Color = -\frac{\Phi(d_1)}{2S\tau\sigma\sqrt{\tau}}\left[1 + \frac{2r\tau - d_2\sigma\sqrt{\tau}}{\sigma\sqrt{\tau}} d_1\right]$$

2. **Speed** $\left(\frac{\partial \Gamma}{\partial S} = \frac{\partial^3 V}{\partial S^3}\right)$ measures the rate of change of gamma with respect to price. This parameter is especially useful when gamma or delta-hedging a portfolio.

$$Speed = -\frac{\Phi(d_1)}{S^2\sigma\sqrt{\tau}}\left[1 + \frac{d_1}{\sigma\sqrt{\tau}}\right] = -\frac{\Gamma}{S}\left[1 + \frac{d_1}{\sigma\sqrt{\tau}}\right]$$

3. **Ultima** $\left(\frac{\partial (Vomma)}{\partial \sigma} = \frac{\partial^3 V}{\partial \sigma^3}\right)$ measures the rate of change of vomma with respect to change in volatility. Useful in delta-hedging

$$Ultima = -\frac{\nu}{\sigma^2}[d_1 d_2(1 - d_1 d_2) + d_1^2 + d_2^2]$$

4. **Zomma** $\left(\frac{\partial \Gamma}{\partial \sigma} = \frac{\partial (Vanna)}{\partial S} = \frac{\partial^3 V}{\partial S^2 \partial \sigma}\right)$ measures the rate of change of gamma with respect to changes in volatility. Zomma helps a trader anticipate the changes to the effectiveness of trading methods as volatility changes given a gamma-hedged portfolio.

$$Zomma = \frac{\Phi(d_1)(d_1 d_2 - 1)}{S\sigma^2\sqrt{\tau}} = \Gamma \cdot \left[\frac{d_1 d_2 - 1}{\sigma}\right]$$

FURTHER NOTES ON THE FUNCTION

It is also incredibly worthwhile to note the path-independency trait of the function, for this structure alters the definition and intrinstic thought process behind the construction of this model. To know that the random variable of stock price at any given time does not strictly depend on the previous price of any other previous time gives great insight into the future of the stock. Knowing that a downward trend in price behaviour does not indicate a succeeding downard trend in price behaviour, and similary that a upward trend in price behaviour does not indicate a succeeding upward trend in price behaviour, drastically affects investor decisions and hedging options. Though in the truest of maret replications, it may seem that stock option prices are in fact path-dependent, there are many behavioural, personal, and subjective parameters than cannot be mathematically added to a function to describe its true nature such as a seller's influence on the option, the buyer's influence on the option, the knowledgability of either aformentioned parties, the caliber of risk of market given current economic state, the finanicial, accounting, and economic history of the firm, corporation, or entity selling or buying the stock option, etc. It is also integral to note the discrepancies in endpoints from start to end business day transactions. The following figures are examples of the phenomenon that often occurs from the end of one business day to the start of the next. (Source: http://www.google.com/finance)

Figure 6: Google, Incorporated (GOOGL) index on the National Association of Securities Dealers Automated Quotations (NASDAQ) traded from the start of December 2, 2014 to the end of December 4, 2014; notice the jump discountinuties between December 2 and Deember 3 as well as December 3 and December 4. (Source: http://www.google.com/finance)

Figure 7: Berkshire Hathatway, Incorporated (BRK.A) index on the New York Stock Exchange (NYSE) traded from the start of December 2, 2014 to the end of December 4, 2014; notice the jump discountinuties and the more plateaued behaviour of function, which in itself would cause more issues in creating a Fourier transform given a simple understanding of the topic. (Source: http://www.google.com/finance)

Figure 8: The London Stock Exchange Group, Public Limited Company (LON) traded on the London Stock Exchange (LSE) traded from the start of December 2, 2014 to the end of December 4, 2014; a European example of how the price of stock can flucatuate unbeknownst to the public eye. (Source: http://www.google.com/finance)

This discrepancy would usually cause errors in creating a Fourier transform for the two dimensional equation evaluating the price of the stock, but this issue is resolved with the Black-Scholes assumptions on the market, creating a fluid and piecewise smooth function that accurately represents the value of the stock option from birth to death. These fluctuations are infinitessminally insignificant with respect to the overall life of the stock option that they can be ignored. The function as seen in the market place is not formally continuous since

$$\lim_{x \to x_0^+} u(x,t) \neq \lim_{x \to x_0^-} u(x,t) \ \forall\, x \in [0,\infty]\ and\ constant\ t > 0$$

However, an infinite amount of jump discontinuities are allowed when given an infinite interval and a Fourier series and thereby transform is still allowed to be created.

DERIVATION

The mathematical origin of the Black-Scholes partial differential equation, as previously mentioned is Itô calculus, giving us the following start point

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Initial boundary conditions are then instrinsicly defined as

1) $V(0,t) = 0$
2) $\frac{V(S,t)}{S} \xrightarrow{S\to\infty} 1$
3) $V(S,T) = \max\{(S-K),0\}$

The first boundary condition indicates that the value of a stock option at a price of 0 USD is itself worthless, an intuitive but necessary condition. The third boundary condition indicates that the final value of the stock option is the maximum of the current price subtracted by the strike price (indicating a roft if found) or 0, if the strike price is higher than the current value of the stock option. In order to minimze losses, as is the structure of the derivatives market, the option will be exercised at the strike price since losses otherwise have the potential to be infinite. The second boundary condition is less intuitive; the interpretation indicates that the ratio of the value of the stock option to the price is one as the price reaches its infinite boundary. This simply means that the value of the stock option is directly related to the price of the call option, with one increasing when the other increases and coversely decreasing when the other decreases. This lemma is important to the exact description of the formula: if this relationship did not hold, the price could grow exponentially without the value of the call option receiving any of the benefits, thereby rendering the call option intrinsically useless. Similarly, if the call option were to simply grow regardless of how quickly the price is increasing, this would render the Security Exchange Commission (SEC) useless, as its purpose is to ensure all market trades are fair. The solution of this partial differential equation requires multiple complex variable transformations, but the end result creates "an application of the heat equation" or more formally known as the diffusion equation (Pinsky 315). The following transformations take place:

$$x = \log\left(\frac{S}{K}\right) \qquad -\infty < x < +\infty \qquad \tau = T - t \qquad 0 \le \tau < T$$

Where S is the price of the option, K is the market determined strike price, and T is the time to expiration.

$$v(x,\tau) = \frac{V(S,t)}{K}$$

$$V_t = \frac{\partial V}{\partial t} = -K\frac{\partial v}{\partial \tau} = -Kv_\tau$$

$$V_S = \frac{\partial V}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x} = \frac{K}{S}v_x$$

$$V_{SS} = \frac{\partial^2 V}{\partial S^2} = \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2} - \frac{K}{S^2}\frac{\partial v}{\partial x} = \frac{K}{S^2}(v_{xx} - v_x)$$

Taking all of this information and replacing the variables in the original partial differential equation, the following equation is procured,

$$v_\tau = \frac{\sigma^2}{2}v_{xx} + \left(r - \frac{\sigma^2}{2}\right)v_x - rv$$

where σ is the standard deviation of the stock's return – this is the square root of the quadratic variation of the stock's log price process – also referred to as the volatility, and r is the annualized risk-free interest rate, continuously compounded. This function looks slightly similar to an ordinary diffusion equation, but further alterations to the initial boundary conditions and terms are required;

$$v(x,0) = \max(e^x - 1,0)$$

To determine weights for each portion, a new function is prescribed:

$$v(x,\tau) = u(x,\tau)e^{\alpha x + \beta\tau}$$

From this, the obvious derivatives are constructed:

$$v_\tau = (\beta u + u_\tau)e^{\alpha x + \beta\tau}$$

$$v_x = (\alpha u + u_x)e^{\alpha x + \beta\tau}$$

$$v_{xx} = (\alpha^2 u + 2\alpha u_x + u_{xx})e^{\alpha x + \beta\tau}$$

With previous equations, a final form is observed:

$$u_\tau = \frac{\sigma^2}{2}u_{xx} + \left(\alpha\sigma^2 + r - \frac{\sigma^2}{2}\right)u_x + \left(\frac{\alpha(\alpha-1)\sigma^2}{2} + r(\alpha - 1) - \beta\right)u$$

Knowing that the coefficients for u_x and u should be zero to provide a pure diffusion equation, setting the coefficients to zero and thereby solving for α and β – by setting $k = \frac{r}{\sigma^2}$ – produces the following equations, notably for initial establishment and used heavily in later procedures:

$$\alpha = -\frac{r - \frac{\sigma^2}{2}}{\sigma^2} = \frac{1}{2}(1-k);\ \ \beta = \frac{\alpha(\alpha-1)\sigma^2}{2} + r(\alpha - 1) = -\frac{\sigma^2(1+k)^2}{8}$$

Finally, a pure diffusion equation is produced:

$$u_\tau = \frac{\sigma^2}{2} u_{xx}$$

Or, more formally:

$$\frac{\partial u(x,\tau)}{\partial \tau} = \frac{\sigma^2}{2} \cdot \frac{\partial\,^2 u(x,\tau)}{\partial x^2}$$

Initial conditions are further adjusted, as so:

$$\nu(x,\tau) = u(x,\tau)e^{\alpha x + \beta\tau}$$

$$\nu(x,0) = u(x,0)e^{\alpha x} = \max(e^x - 1{,}0)$$

$$u(x,0) = \max(e^{x(1-\alpha)} - e^{-\alpha x}, 0)$$

$$= \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)$$

Since this has been reduced to a more simplified diffusion equation, the Gauß – Weierstraß kernel can provide the truest form of the function $u(x,t)$. The Gauß – Weierstraß kernel mentioned is as follows:

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)\frac{e^{\frac{-(x-\xi)^2}{4K\tau}}}{\sqrt{4\pi K\tau}}\,d\xi$$

Note that in this instance $K = \frac{\sigma^2}{2}$ for further simplicity in integration. The adapted Gauß – Weierstraß kernel looks like:

$$u(x,\tau) = \int_{0}^{\infty} u(\xi,0)\frac{e^{\frac{-(x-\xi)^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}\,d\xi = \int_{0}^{\infty} \left[e^{\frac{1}{2}(k+1)\xi} - e^{\frac{1}{2}(k-1)\xi}\right]\frac{e^{\frac{-(x-\xi)^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}\,d\xi$$

This can be reduced to a probabilistic normal distribution integral with the following variable transformation

$$z = \frac{(\xi - x)}{\sigma\sqrt{\tau}}$$

Note, that when $= 0,\ z = \frac{-x}{\sigma\sqrt{\tau}}$, so this changes the lower bounds of integration

Furthermore

$$dz = \frac{d\xi}{\sigma\sqrt{\tau}}$$

So, the transcribed equation becomes

$$u(x,\tau) = e^{\frac{1}{2}(k+1)x} e^{\frac{1}{8}(\sigma\sqrt{\tau}(k+1))^2} \int_{\frac{-x}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{z^2}{2}+\frac{1}{2}(k+1)z\sigma\sqrt{\tau}-\frac{1}{8}(\sigma\sqrt{\tau}(k+1))^2} \frac{dz}{\sqrt{2\pi}}$$

$$+\ e^{\frac{1}{2}(k-1)x} e^{\frac{1}{8}(\sigma\sqrt{\tau}(k-1))^2} \int_{\frac{-x}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{z^2}{2}+\frac{1}{2}(k-1)z\sigma\sqrt{\tau}-\frac{1}{8}(\sigma\sqrt{\tau}(k-1))^2} \frac{dz}{\sqrt{2\pi}}$$

After some further substitution and progress, the final result becomes

$$u(x,\tau) = \ e^{\frac{1}{2}\left(\frac{r}{\sigma^2}+1\right)x} e^{\frac{1}{8}\left(\sigma\sqrt{\tau}\left(\frac{r}{\sigma^2}+1\right)\right)^2} \Phi\left(\frac{x+\frac{\sigma^2\tau\left(\frac{r}{\sigma^2}+1\right)}{2}}{\sigma\sqrt{\tau}}\right)$$

$$-\ e^{\frac{1}{2}\left(\frac{r}{\sigma^2}-1\right)x} e^{\frac{1}{8}\left(\sigma\sqrt{\tau}\left(\frac{r}{\sigma^2}-1\right)\right)^2} \Phi\left(\frac{x+\frac{\sigma^2\tau\left(\frac{r}{\sigma^2}-1\right)}{2}}{\sigma\sqrt{\tau}}\right)$$

A more common description of this final equation seperates it into context, since calculating x and τ is complicated, requiring greater effort and less intuition. Equations for values on call and put options are as follows:

$$V_{call}(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}$$

$$V_{put}(S,t) = N(-d_2)Ke^{-r(T-t)} - \ N(-d_1)S$$

Where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + (r+\frac{\sigma^2}{2})(T-t)\right]$$

$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + (r-\frac{\sigma^2}{2})(T-t)\right] = \ d_1 - \ \sigma\sqrt{T-t}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\,dz$$

$N(d_1)S$ is the value of the stock weighted by the probability the stock achieves this value and $N(d_2)Ke^{-r(T-t)}$ is the value of the discounted strike price (evaluated with a present value calculation) weighted by the probability of exercising the option at the strike price value.

ANALYSIS OF MODEL

The Black-Scholes model is a straightforward, pragmatic way of evaluating a complex topic, and yields quite a useful approximation. The preceding Black-Scholes formula is a robust basis for further developed models - addendums have now been created to formally and unequivocally represent the modern day derivatives market. An interesting and exceptionally useful corollary is the reversibility as the model's original variables – risk-free interest rate, strike price, market established price, and time of expiration – can be used as an input and the volatility can be solved for. The implied volatility calculated in this way is often used to quote option prices. Rather than assuming volatility a priori and computing prices from a market assumption of volatility, the model actually enables users to back solve for volatility, creating an implied volatility surface domain. The limitations of the pure Black-Scholes model are greater, but not unsolvable, having already paved the way for thirty years of improvement. The pure Black-Scholes model underestimates the extreme moves of underlying asset prices – within the range of $\pm 3\sigma$ from mean – yielding tail risk. In application, this can be hedged with out-of-the-money options. The pure Black-Scholes model has an initial premise of instant, cost-less trading implying a frictionless market. This in turn yields liquidity risk, which is difficult to hedge against. The pure Black-Scholes method assumes a stationary process, yielding volatility risk. This, however, can be hedged with specified volatility or sigma-hedging. The pure Black-Scholes method assumes continuous time intervals and continuous trading along these intervals, yielding gap risk as previously mentioned creates problems for which can be hedged with Gamma-hedging. In reality, security prices do not follow a strictly stationary logarithmic-normal process, nor is the risk-free interest rate actually known as it is market driven, and it certainly isn't constant over time. The pure Black-Scholes model tends to undervalue the portfolio in extreme price changes, indicating a conservative nature which in itself is not a heinous characteristic of an equation, but it certainly is not a desired trait in this field. Furthermore, the pure Black-Scholes model does not apply to municipal, federal, or corporate bonds because of pull-to-par and complications that arise, since the comparison of a risky asset to a riskless asset is deemed inadequate for this type of option. The equation does not reflect the lack of volatility within these options, but there are addendums to the equation to help support this, such as the Black-Scholes-Myron Model. When creating actual Fourier Transformations for the final Black-Scholes formula, the following figures describe the accuracy of the transformation with respect to the number of nodes used given certain trite or market usual conditions.

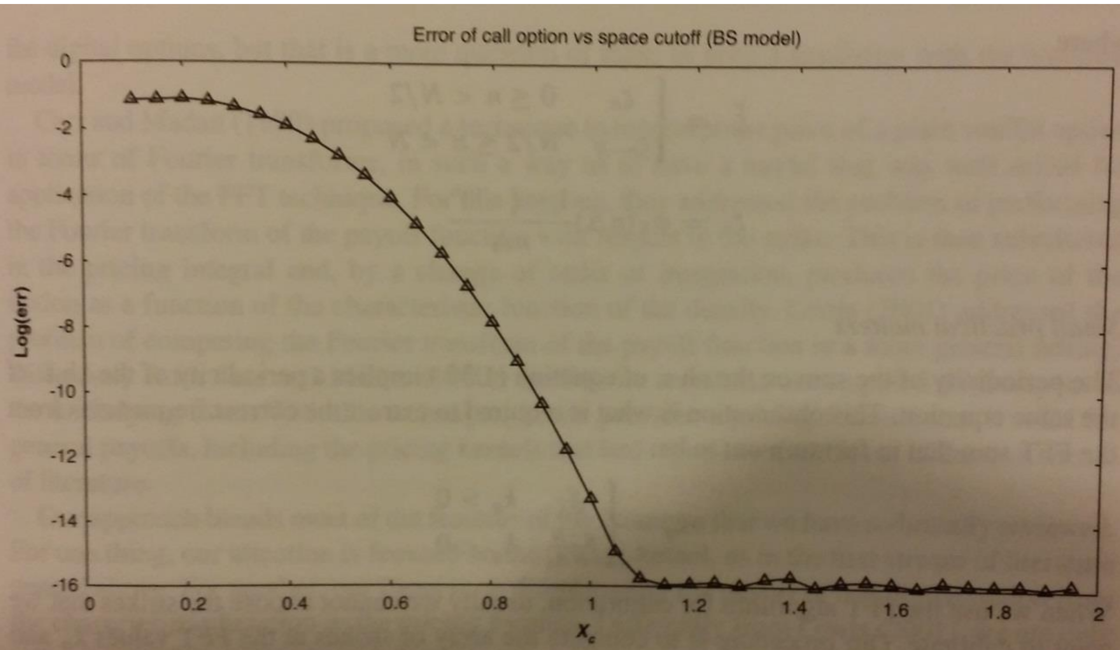


Figure 1.10 The dependency of the error on the spatial cutoff used to compute a call option. Parameters are $T = 1$, $K = 1.0$, $\sigma = 0.1$, $r = 0.05$. The number of Fourier modes is 128

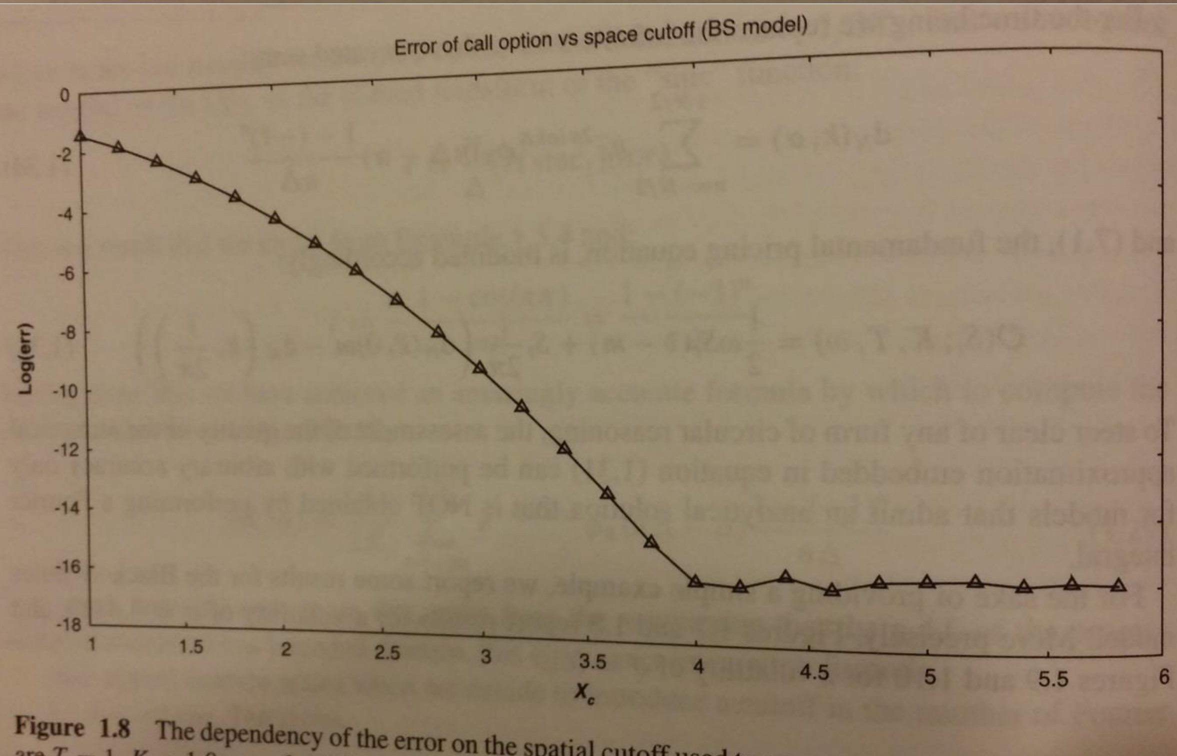


Figure 1.8 The dependency of the error on the spatial cutoff used to compute a call option. Parameters are $T = 1$, $K = 1.0$, $\sigma = 0.4423$, $r = 0.05$. The number of Fourier modes is 128

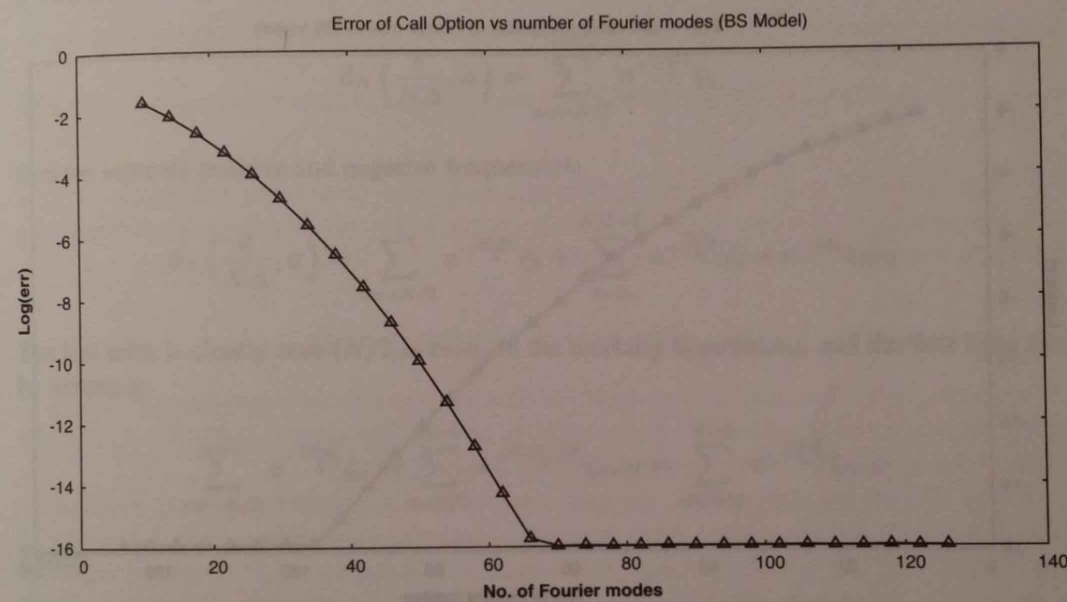


Figure 1.7 The dependency of the error on the number of Fourier modes used to compute a call option. Parameters are $T = 1$, $K = 1.0$, $\sigma = 0.4423$, $r = 0.05$. The spatial cutoff is $X_c = 6.0$

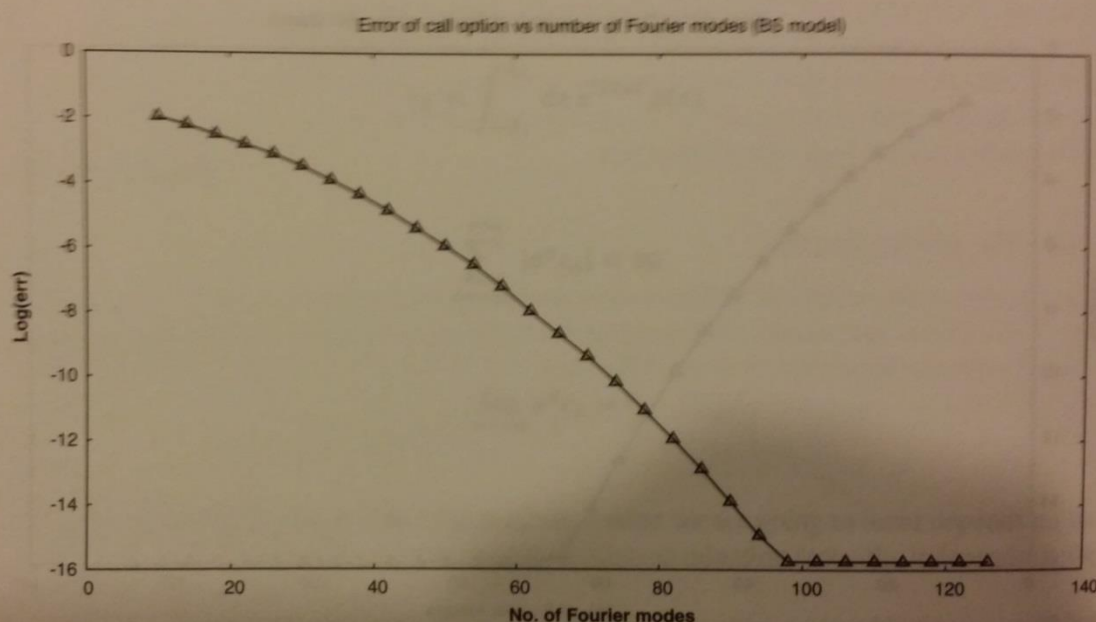


Figure 1.9 The dependency of the error on the number of Fourier modes used to compute a call option. Parameters are $T = 1$, $K = 1.0$, $\sigma = 0.1$, $r = 0.05$. The spatial cutoff is $X_c = 2.0$

EXPERIMENTATION AND APPLICATION

In practice, it is rather difficult to find a company of any market magnitude that still strictly follows the Black-Scholes assumptions previously mentioned. Analysis of the stock market at this stage is not limited, however, since with some financial intuition, relatively conceivable stock options can be created and hence analyzed. See Appendices 1 and 2 for formulas used henceforth. The first option, option 1, is an expensive and lengthy option in a very volatile field, perhaps low grade junk debt or highly experimental options in technological development. The interest rate was set to 0.12% as per data received from the Federal Reserve website (http://www.federalreserve.gov/releases/h15/update/). As previously assumed, the delta for the call option lies between 0 and 1 and the delta for the put option lies between -1 and 0, so this assumption is verified, as well as for the other 4 options examined. There lies slight error in the delta calculations since the first equation mentioned ($\Delta_{call} = 1 + \Delta_{put}$) is not accurate. This error can be traced to Microsoft Excel's rounding error and nothing more. This delta value, however, indicates that with one unit of increase in [spot] price, the value of the call will increase by 0.9675 units and the value of the put will decrease by 0.033 units. The vega for this option, as indicated by the market volatility, is very high and implies that the impact of small changes in volatility in the market have enormous repercussions on the value of the option. Thetas for call options are negative, since intrinsically value is slowly detracted from the option due to the time value of money among other factors. Similarly, value is detracted from put options as well, but this indicates a positive theta. A negative theta indicates losses for the long position and gains for the short, and identically a positive theta indicates losses for the short position and gains for the long. Gamma is the same for both call and put options and in this case gamma is rather miniscule, indicating delta is very resilient to change. This is incredibly useful in delta-hedging, knowing that delta is rarely a greater variant than the value calculated in determining the effectivity of delta-hedging strategies. The implied volatility is identical to the volatility used in market calculations, save for rounding error. This is rather intuitive since all of the volatility is controlled within the system and no outside fluctuations – such as time value of money, dividend payouts, exchange rate of currency, or inflation – can affect the system in this pure model. Time value of money, exchange rate of currency, and inflation all violate assumptions in the pure model that the riskless asset – the currency in exchange – is purely riskless and perfect, however inaccurate this assumption may be. Dividend payouts project an extra layer of volatility with the introduction of common versus preferred stock options. In times of turmoil, buyers of common stock options may not even receive payouts until the company or firm is in-the-money. In this same example, if the option were to have a dividend payout of 3.1415%, the implied volatility would decrease to 0.5981, since the assumption is that if a company can provide dividend payouts, the volatility is not as high as previously expected. Nonetheless, its call value is 367.91 at this exact time and price. From this, whether or not to sell the call option is strictly the discretion of the investor and the strategies set forth as goals. The fourth option is a less expensive, less volatile, and shorter option, lasting only six months. The delta values once more do not strictly follow the relationship, but imply a slightly less strict statement similar to option one. Delta for option four is much less resilient – by about a factor of three – and vega is much less than for option one, as expected given the time frame and volatility level. Theta for the fourth call is lower than the theta value for the first call, indicating that the call is subject to greater change as time proceeds. Theta for the fourth put is smaller, however, than the theta for the first option, but only since the actual put value is a fraction of the first value. The put option in this case is essentially useless, but once more the true interpretation stringently depends on the goals of the investor or investment group at hand.

ANALYSIS OF RESULTS AND ERROR

As previously stated, a majority of the error experienced in this experiment originates with Microsoft Excel's odd rounding techniques, for the formulae are exact and precise given such malleable numbers as inputs. However, the grander context of error in this field of study goes beyond even calculation error.

> "Real world stock price uncertainty is more complicated than in the Black-Scholes world because the dynamics of stock price movements are more complicated than geometric Brownian motion. For example, in reality the volatility of the underlying [asset] can change unexpectedly, and changes in volatility will change the value of the option even when the spot/strike price does not change.
>
> There is also jump risk embedded in the price of an option. If there is a probability the underlying stock price will unexpectedly jump downward, then this risk is implicit in the price of the option. However, since this risk is no included in the stock price model (i.e., the geometric Brownian motion model does not account for possible stock price jumps), jump risk is not included in the Black-Scholes price of an option. This relates to the delta of an option, because if there are many sources of uncertainty in the value of an option, then changes in its value cannot be explained by changes in the underlying [asset's] price alone… Unfortunately, there are many reasons why this may not give an accurate estimate of the delta of the option. Underlying all of these is a single reason: The price of the option is not controlled by the price of the stock alone. For example, if the volatility suddenly 'spikes up' between time t_0 and time t_1, then there will be a larger change

> in the value of the option than we would predict on the basis of stock price uncertainty alone". (Chriss 134)

Error occurs when there is uncertainty. In the derivatives market where millions of units of currency are placed on uncertainties, error in inescapable and an inevitable factor in calculations and interpretations. In this case, as previously mentioned, the only surface level error comes from volatility judgment, as all other cases of error have been eliminated in the pure Black-Scholes model assumptions. These error eliminating assumptions carry their lack of error through the Gauß – Weierstraß kernel into the origins of the Fourier series of the function of both time and value and the recalculation into something valuable, the call and put option value functions. Error presents itself more clearly when solving for a similar Fourier series given dividend payouts or when the exercising date is itself a random variable recursively dependent on previous data.

CONCLUSION

The Black-Scholes model of 1973 was an immense contribution to the field of derivatives evaluation and has been the foundation of all work done today and for years to come. The Black-Scholes model is accurate and precise for options matching its assumptions and criteria, however difficult it may be to find companies that exist such that these criteria coincide with company structure. Nonetheless extensions have been made that have proved equally viable as functions of analysis. Creating Fourier series with these functions requires knowledge of the Gauß – Weierstraß theorem as well as stochastic, Lebesgue, and Itô calculus. The inclusion of these methods also surpasses the issue of business-day-to-business-day discontinuities previously seen. These Fourier series are unique to the stock option and provide a detailed analysis of the pattern and behaviour of the option, barring the financial wherewithal to interpret this easily acquired information. This complicated function is usually transcribed to display two separate functions, one for calculating the value of a call option and the other for a put option. These functions themselves also contain more meaning and intuitive interpretation and set the foundation for the calculation of the Greeks, intrinsic values based in elucidation of exact behaviours and patterns of these call and put options. Given these easily calculable values an investor or investment group can examine the option from day to day and already have the evidence to support a gain or loss. This information and these functions are also applicable to most other forms of derivatives and exchanges on financial markets, government and highly rated corporate bonds withstanding. These values can be deceptive and misleading, as seen in the Financial Collapse of 2008, and require immense mental fortitude to accurately construe information from. In the experiment performed above, financially sound numbers were chosen to represent stock price, strike price, volatility, and time period, whereas the risk-free interest rate was information provided by the Federal Reserve’s most relevant and latest posting. No call options values were preposterous and most put option values followed the same trend. The calculated implied volatility was almost identical, barring slight rounding error, as all exterior volatility was removed. This experiment was highly successful in attaining truthful and accurate results with minimal error. Further questions worth exploring include the use of Fourier Transforms to perhaps even more accurately track the derivative value and the inclusion of the put-call-parity, a financial function used to describe the exact relationship between call and put values, in order to create one cohesive intuitive function rather than one for call options and another for put options.

APPENDIX

Appendix 1

	Option 1	Option 2	Option 3	Option 4	Option 5
Spot Price	*400*	*150*	*350*	*75*	*200*
Strike/Exercise Price	*400*	*140*	*370*	*75*	*195*
Risk-free Rate	*0.12*	*0.12*	*0.12*	*0.12*	*0.12*
Volatility	*0.9*	*0.6*	*0.2*	*0.1*	*0.3*
Expiration Time	*10*	*1*	*5*	*0.5*	*20*
Call Value	*367.91*	*46.7809*	*153.2852*	*4.9395*	*183.2514*
Call Delta	*0.9675*	*0.7307*	*0.9252*	*0.8116*	*0.9934*
Call Gamma	*6.40E-05*	*0.0036*	*0.0009*	*0.0509*	*6.89E-05*
Call Vega	*92.059*	*49.5306*	*110.5522*	*14.3156*	*16.5378*
Call Theta	*-6.4315*	*-22.3984*	*-22.6755*	*-8.144*	*-1.9756*
Put Value	*88.386*	*20.9498*	*6.3455*	*0.5673*	*0.9414*
Put Delta	*-0.033*	*-0.2693*	*-0.0747*	*-0.1884*	*-0.0065*
Put Gamma	*6.39E-05*	*0.0036*	*0.0009*	*0.0509*	*6.89E-05*
Put Vega	*92.059*	*49.5306*	*110.5522*	*14.3156*	*16.5378*
Put Theta	*8.0258*	*-7.4982*	*1.6918*	*0.3319*	*0.1472*
Target Call Value for Implied Volatility	*367.9*	*46.781*	*153.2852*	*4.9349*	*183.2514*
Implied Volatility	*0.8999*	*0.5999*	*0.1999*	*0.1000*	*0.3000*

Appendix 2

(Source: http://investexcel.net/calculate-implied-volatility-with-the-bisection-method/)

```
Function BlackScholesCall( _
    ByVal S As Double, _
    ByVal X As Double, _
    ByVal T As Double, _
    ByVal r As Double, _
    ByVal d As Double, _
    ByVal v As Double) As Double
    Dim d1 As Double
    Dim d2 As Double
    d1 = (Log(S / X) + (r - d + v ^ 2 / 2) * T) / v / Sqr(T)
    d2 = d1 - v * Sqr(T)
    BlackScholesCall = Exp(-d * T) * S * Application.NormSDist(d1) - X * Exp(-r * T) * Application.NormSDist(d2)
  End Function
```

```
  Function ImpliedVolatility( _
    ByVal S As Double, _
    ByVal X As Double, _
    ByVal T As Double, _
    ByVal r As Double, _
    ByVal d As Double, _
    ByVal Price As Double) As Double

    Dim epsilonABS As Double
    Dim epsilonSTEP As Double
    Dim volMid As Double
    Dim niter As Integer
    Dim volLower As Double
    Dim volUpper As Double

    epsilonABS = 0.0000001
    epsilonSTEP = 0.0000001
    niter = 0
    volLower = 0.001
    volUpper = 1

    Do While volUpper - volLower >= epsilonSTEP Or Abs(BlackScholesCall(S, X, T, r, d, volLower) - Price) >= epsilonABS And epsilonABS <= Abs(BlackScholesCall(S, X, T, r, d, volUpper) - Price) >= epsilonABS
      volMid = (volLower + volUpper) / 2
      If Abs(BlackScholesCall(S, X, T, r, d, volMid) - Price) <= epsilonABS Then
        Exit Do
      ElseIf ((BlackScholesCall(S, X, T, r, d, volLower) - Price) * (BlackScholesCall(S, X, T, r, d, volMid) - Price) < 0) Then
        volUpper = volMid
      Else
        volLower = volMid
      End If
      niter = niter + 1
    Loop
```

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