ColumbiaX: Machine Learning

Lecture 23

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Association analysis

Setup

Many businesses have massive amounts of customer purchasing data.

- Amazon has your order history
- A grocery store knows objects purchased in each transaction
- Other retailers have data on purchases in their stores

Using this data, we may want to find sub-groups of products that tend to co-occur in purchasing or viewing behavior.

- Retailers can use this to cross-promote products through "deals"
- Grocery stores can use this to strategically place items
- Online retailers can use this to recommend content
- This is more general than finding purchasing patterns

MARKET BASKET ANALYSIS

Association analysis is the task of understanding these patterns.

For example consider the following "market baskets" of five customers.

TID	Items
1	{Bread, Milk}
2	{Bread, Diapers, Beer, Eggs}
3	{Milk, Diapers, Beer, Cola}
4	{Bread, Milk, Diapers, Beer}
5	{Bread, Milk, Diapers, Cola}

Using such data, we want to analyze patterns of co-occurance within it. We can use these patterns to define *association rules*. For example,

$$\{\text{diapers}\} \;\Rightarrow\; \{\text{beer}\}$$

Association analysis and rules

Imagine we have:

- p different objects indexed by $\{1, \ldots, p\}$
- A collection of subsets of these objects $X_n \subset \{1, \ldots, p\}$. Think of X_n as the index of things purchased by customer $n = 1, \ldots, N$.

Association analysis: Find subsets of objects that often appear together. For example, if $\mathcal{K} \subset \{1, \ldots, p\}$ indexes objects that frequently co-occur, then

$$P(\mathcal{K}) = \frac{\#\{n \text{ such that } \mathcal{K} \subseteq X_n\}}{N} \quad \text{is large relatively speaking}$$

Example: $\mathcal{K} = \{$`peanut_butter, jelly, bread`$\}$

Association rules: Learn correlations. Let A and B be disjoint sets. Then $A \Rightarrow B$ means purchasing A increases likelihood of also purchasing B.

Example: $\{$`peanut_butter, jelly`$\} \Rightarrow \{$`bread`$\}$

PROCESSING THE BASKET

TID	Items
1	{Bread, Milk}
2	{Bread, Diapers, Beer, Eggs}
3	{Milk, Diapers, Beer, Cola}
4	{Bread, Milk, Diapers, Beer}
5	{Bread, Milk, Diapers, Cola}

Figure: An example of 5 baskets.

TID	Bread	Milk	Diapers	Beer	Eggs	Cola
1	1	1	0	0	0	0
2	1	0	1	1	1	0
3	0	1	1	1	0	1
4	1	1	1	1	0	0
5	1	1	1	0	0	1

Figure: A binary representation of these 5 baskets for analysis.

PROCESSING THE BASKET

TID	Bread	Milk	Diapers	Beer	Eggs	Cola
1	1	1	0	0	0	0
2	1	0	1	1	1	0
3	0	1	1	1	0	1
4	1	1	1	1	0	0
5	1	1	1	0	0	1

Want to find subsets that occur with probability above some threshold.

For example, does {`bread, milk`} occur relatively frequently?

- Go to each of the 5 baskets and count the number that contain both.
- Divide this number by 5 to get the frequency.
- Aside: Notice that the basket might have more items in it.

When $N = 5$ and $p = 6$ as in this case, we can easily check every possible combination. However, real problems might have $N \approx 10^8$ and $p \approx 10^4$.

Some combinatorics

Some combinatorial analysis will show that brute-force search isn't possible.

Q: How many different subsets $\mathcal{K} \subseteq \{1, \ldots, p\}$ are there?

A: Each subset can be represented by a binary indicator vector of length p. The total number of possible vectors is 2^p.

Q: Nobody will have a basket with every item in it, so we shouldn't check every combination. How about if we only check up to k items?

A: The number of sets of size k picked from p items is $\binom{p}{k} = \frac{p!}{k!(p-k)!}$. For example, if $p = 10^4$ and $k = 5$, then $\binom{p}{k} \approx 10^{18}$.

Takeaway: Though the problem only requires counting, we need an algorithm that can tell us which $\mathcal{K}$ we should count and which we can ignore.

QUANTITIES OF INTEREST

Before we find an efficient counting algorithm, what do we want to count?

- Again, let $\mathcal{K} \subset \{1, \ldots, p\}$ and $A, B \subset \mathcal{K}$, where $A \cup B = \mathcal{K}$, $A \cap B = \emptyset$.

We're interested in the following empirically-calculated probabilities:

1. $P(\mathcal{K}) = P(A, B)$: The *prevalence* (or support) of items in set $\mathcal{K}$. We want to find which combinations co-occur often.
2. $P(B|A) = \frac{P(\mathcal{K})}{P(A)}$: The *confidence* that B appears in the basket given A is in the basket. We use this to define a *rule* $A \Rightarrow B$.
3. $L(A, B) = \frac{P(A,B)}{P(A)P(B)} = \frac{P(B|A)}{P(B)}$: The *lift* of the rule $A \Rightarrow B$. This is a measure of how much *more* confident we are in B given that we see A.

EXAMPLE

For example, let

$$\mathcal{K} = \{\texttt{peanut\_butter, jelly, bread}\},$$

$$A = \{\texttt{peanut\_butter, jelly}\}, B = \{\texttt{bread}\}$$

- A *prevalence* of 0.03 means that `peanut_butter`, `jelly` and `bread` appeared together in 3% of baskets.
- A *confidence* of 0.82 means that when both `peanut_butter` and `jelly` were purchased, 82% of the time `bread` was also purchased.
- A *lift* of 1.95 means that it's 1.95 more probable that `bread` will be purchased given that `peanut_butter` and `jelly` were purchased.

The goal of the **Apriori algorithm** is to quickly find all of the subsets $\mathcal{K} \subset \{1, \ldots, p\}$ that have probability greater than a predefined threshold t.

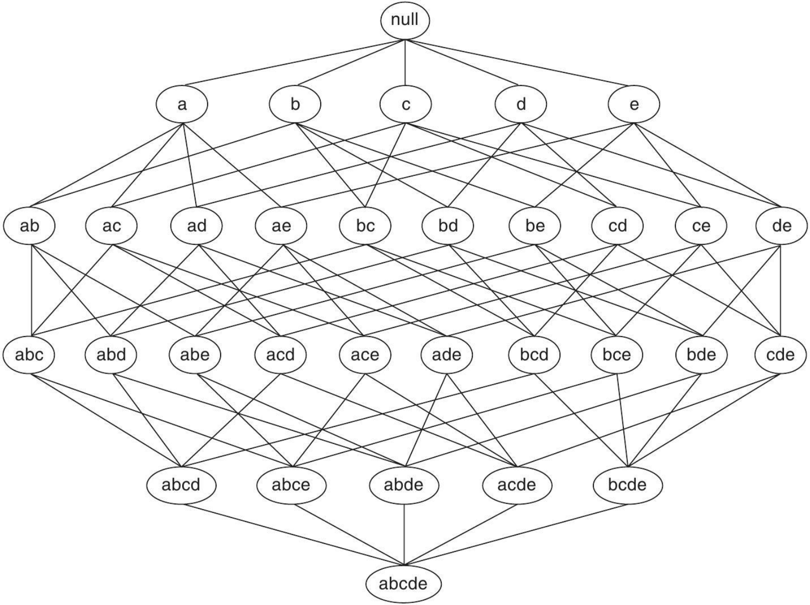
- Such a $\mathcal{K}$ will contain items that appear in at least $N \cdot t$ of the N baskets.
- A small fraction of such $\mathcal{K}$ should exist out of the 2^p possibilities.

Apriori uses properties about $P(\mathcal{K})$ to reduce the number of subsets that need to be checked to a small fraction of all 2^p sets.

- It starts with $\mathcal{K}$ containing 1 item. It then moves to 2 items, etc.
- Sets of size $k - 1$ that "survive" help determine sets of size k to check.
- Important: Apriori finds *every* set $\mathcal{K}$ such that $P(\mathcal{K}) > t$.

Next slide: The structure of the problem can be organized in a lattice.

LATTICE REPRESENTATION



FREQUENCY DEPENDENCE

We can use two properties to develop an algorithm for efficiently counting.

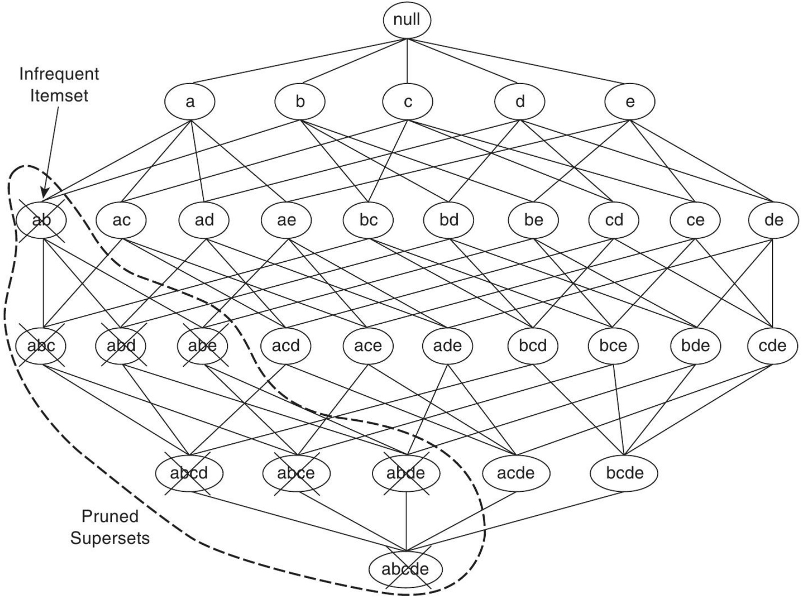
1. If the set $\mathcal{K}$ is not big enough, then $\mathcal{K}' = \mathcal{K} \cup A$ with $A \subset \{1, \ldots, p\}$ is not big enough. In other words: $P(\mathcal{K}) < t$ implies $P(\mathcal{K}') < t$

   e.g., Let $\mathcal{K} = \{a, b\}$. If these items appear together in x baskets, then the set of items $\mathcal{K}' = \{a, b, c\}$ appears in $\leq x$ baskets since $\mathcal{K} \subset \mathcal{K}'$.

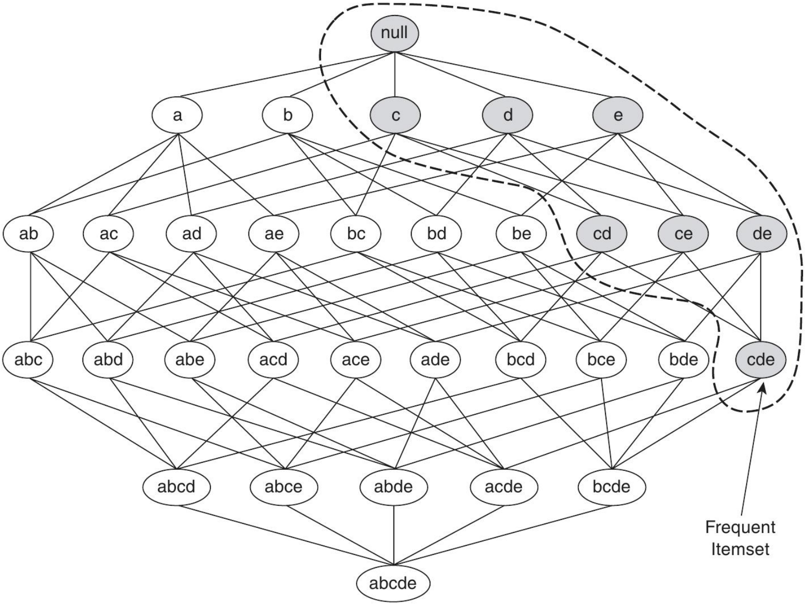
   Mathematically: $P(\mathcal{K}') = P(\mathcal{K}, A) = P(A|\mathcal{K})P(\mathcal{K}) \leq P(\mathcal{K}) < t$

2. By the converse, if $P(\mathcal{K}) > t$ and $A \subset \mathcal{K}$, then $P(A) > P(\mathcal{K}) > t$.

Frequency dependence: Property 1



Frequency dependence: Property 2



Apriori algorithm (one version)

Here is a basic version of the algorithm. It can be improved in clever ways.

Apriori algorithm

Set a threshold $N \cdot t$, where $0 < t < 1$ (but relatively small).

1. $|\mathcal{K}| = 1$: Check each object and keep those that appear in $\geq N \cdot t$ baskets.
2. $|\mathcal{K}| = 2$: Check all pairs of objects that survived Step 1 and keep the sets that appear in $\geq N \cdot t$ baskets.

   ⋮

k. $|\mathcal{K}| = k$: Using all sets of size $k - 1$ that appear in $\geq N \cdot t$ baskets,
   - Increment each set with an object surviving Step 1 not already in the set.
   - Keep all sets that appear in $\geq N \cdot t$ baskets

It should be clear that as k increases, we can hope that the number of sets that survive decrease. At a certain $k < p$, no sets will survive and we're done.

MORE CONSIDERATIONS

1. We can show that this algorithm returns *every* set $\mathcal{K}$ for which $P(\mathcal{K}) > t$.

- Imagine we know every set of size $k-1$ for which $P(\mathcal{K}) > t$. Then every potential set of size k that could have $P(\mathcal{K}) > t$ will be checked.

  e.g. Let $k = 3$: The set $\{a, b, c\}$ appears in $> N \cdot t$ baskets. Will we check it?
  Known: $\{a, b\}$ and $\{c\}$ must appear in $> N \cdot t$ baskets.
  Assumption: We've found $\mathcal{K} = \{a, b\}$ as a set satisfying $P(\mathcal{K}) > t$.
  Apriori algorithm: We know $P(\{c\}) > t$ and so will check $\{a, b\} \cup \{c\}$.
  Induction: We have all $|\mathcal{K}| = 1$ by brute-force search (start induction).

2. As written, this can lead to duplicate sets for checking, e.g., $\{a, b\} \cup \{c\}$ and $\{a, c\} \cup \{b\}$. Indexing methods can ensure we create $\{a, b, c\}$ once.

3. For each proposed $\mathcal{K}$, should we iterate through each basket for checking? There are tricks to make this faster that takes structure into account.

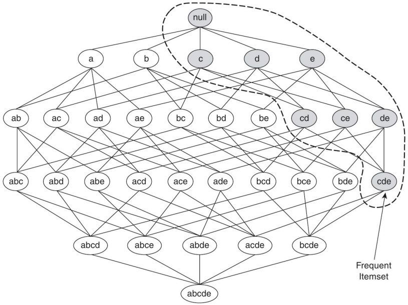
FINDING ASSOCIATION RULES

We've found all $\mathcal{K}$ such that

$$P(\mathcal{K}) > t.$$

Now we want to find association rules.

These are of the form $P(A|B) > t_2$ where we split $\mathcal{K}$ into subsets A and B.



Notice:

1. $P(A|B) = \frac{P(\mathcal{K})}{P(B)}$.
2. If $P(\mathcal{K}) > t$ and A and B partition $\mathcal{K}$, then $P(A) > t$ and $P(B) > t$.
3. Since Apriori found all $\mathcal{K}$ such that $P(\mathcal{K}) > t$, it found $P(A)$ and $P(B)$, so we can calculate $P(A|B)$ without counting again.

Example

Feature	Demographic	# Values	Type
1	Sex	2	Categorical
2	Marital status	5	Categorical
3	Age	7	Ordinal
4	Education	6	Ordinal
5	Occupation	9	Categorical
6	Income	9	Ordinal
7	Years in Bay Area	5	Ordinal
8	Dual incomes	3	Categorical
9	Number in household	9	Ordinal
10	Number of children	9	Ordinal
11	Householder status	3	Categorical
12	Type of home	5	Categorical
13	Ethnic classification	8	Categorical
14	Language in home	3	Categorical

Data

$N = 6876$ questionnaires

14 questions coded into $p = 50$ items

For example:

- ordinal (2 items): Pick the item based on value being $\lesseqgtr$ median
- categorical: item = category

  x categories $\rightarrow$ x items

- Based on the item encoding, it's clear that no "basket" can have every item.
- We see that association analysis extends to more than consumer analysis.

Example

Association rule 1: Support 13.4%, confidence 80.8%, and lift 2.13.

$$\begin{bmatrix} \text{language in home} & = & \textit{English} \\ \text{householder status} & = & \textit{own} \\ \text{occupation} & = & \{\textit{professional/managerial}\} \end{bmatrix}$$

$$\Downarrow$$

$$\text{income} \geq \textit{\$40,000}$$

Association rule 2: Support 26.5%, confidence 82.8% and lift 2.15.

$$\begin{bmatrix} \text{language in home} & = & \textit{English} \\ \text{income} & < & \textit{\$40,000} \\ \text{marital status} & = & \textit{not married} \\ \text{number of children} & = & \textit{0} \end{bmatrix}$$

$$\Downarrow$$

$$\text{education} \notin \{\textit{college graduate, graduate study}\}$$