Notes on the solution of the Navier-Stokes equations using DPG

1 Linearized Navier-Stokes equations

Our approach in applying DPG to the Navier-Stokes equations is to linearize, and then extrapolate DPG to the linearized equations. The Navier-Stokes equations can be written in terms of the Euler variables U and viscous stresses Σ as

$$\begin{aligned} \nabla \cdot (F(U) - G(U,\Sigma)) &= 0 \\ E\Sigma - \nabla U &= 0. \end{aligned}$$

The linearized Navier-Stokes equations can be written

$$\begin{aligned} \nabla \cdot \left(\left(F_{,U}(U) - G_{,U}(U,\Sigma) \right) dU - G_{,\Sigma}(U,\Sigma) d\Sigma \right) &= \text{conservation law residual} \\ E\Sigma - \nabla U &= \text{stress law residual}. \end{aligned}$$

We test the conservation laws with test functions V and stress laws with test functions V. The test norm is extrapolated by identifying like terms with convection-diffusion and extrapolating test norms for groups of variables

$$\begin{aligned} \|(V,W)\|_V^2 = \|V\|^2 + \left\| \left(F_{,U}(U) - G_{,U}(U,\Sigma) \right)^T V \right\|^2 + \frac{1}{\mathrm{Re}} \left\| G_{,\Sigma}(U,\Sigma)^T \nabla V \right\|^2 \\ + \min\left\{ \mathrm{Re}, \frac{1}{|K|} \right\} \|W\|^2 + \|\nabla \cdot W\|^2 \end{aligned}$$

2 Conditioning of local problems

Notice that, under the above test norm, V and W are decoupled. Likewise, our local problem decouples into three local problems with submatrices A_1, A_2, A_3, where A_1 is τ_1 and τ_2, coupled together by the stress equation. A_2 has to do with τ_3, which is decoupled from τ_1, τ_2 by merit that temperature T and heat flux q do not show up in the stress equations. A_3 is related to $v_1, \ldots, v_4$, and is the main cause of conditioning problems.

We use $\mathrm{Re} = 100$ to eliminate robustness issues. After 7 refinements, we have $h_{\min} \approx .002$. The magnitude of the $L^2(\Omega)$ term on v is proportional to dt^{-1}, and with $dt = .1$, we have a condition number of $\kappa(A_3) = O(1e8)$.

Our condition number is now of the order of single precision arithmetic, and we begin to see either divergence or non-monotonic convergence of the transient residual. Since condition numbers can be misleading, we computed a discrete residual for 100 random loads and averaged them to confirm that the error in our approximation is indeed $O(\kappa(A_3))$.

2.1 Cholesky stability

We found that Cholesky decompositions of the matrices led to factorizations with more diagonally dominant terms when compared to LU factorizations. Additionally, the failure of Cholesky due to numerical indefiniteness signals a clear stopping point for the method due to roundoff effects.

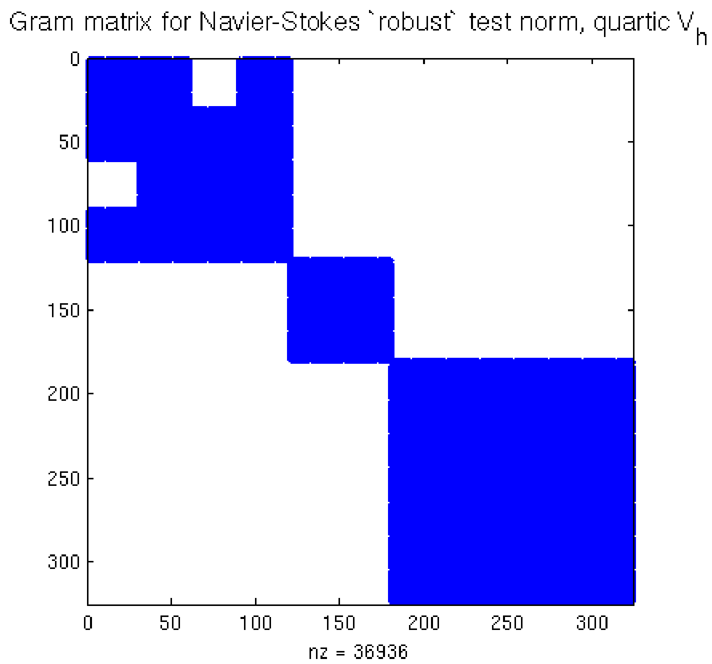

Figure 1: Spy plot of the Riesz operator under the robust test norm for Navier-Stokes.

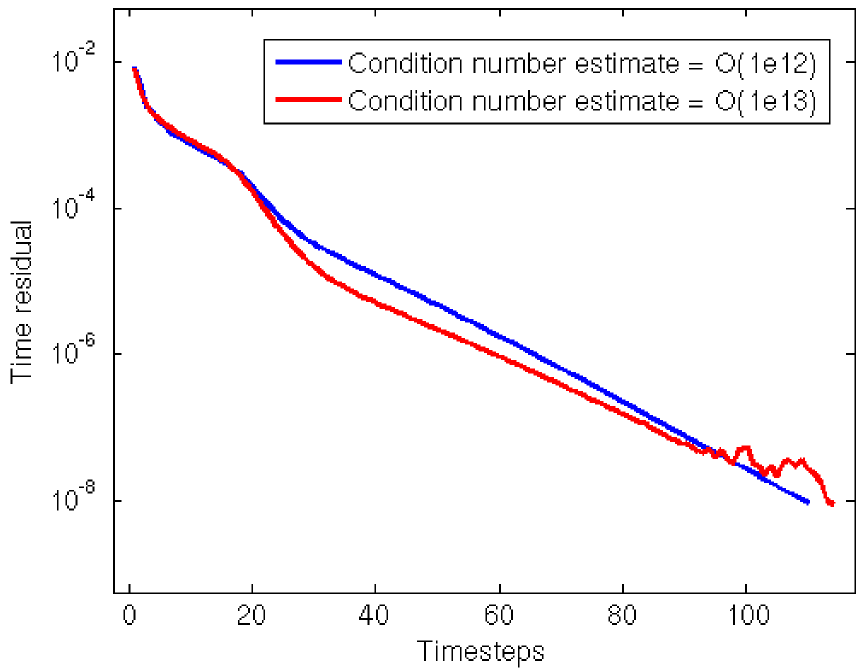

Figure 2: Transient residual behavior under poor conditioning. The condition number estimate is for the entire system; the individual decoupled blocks have significantly lower condition number estimates.

3 Singularity in ρ

The singularity in ρ at the plate edge is independent of the Reynolds number and appears to be unbounded. For $Re = 100$, the singularity continues to grow in magnitude despite the mesh having resolved the diffusive length scale.

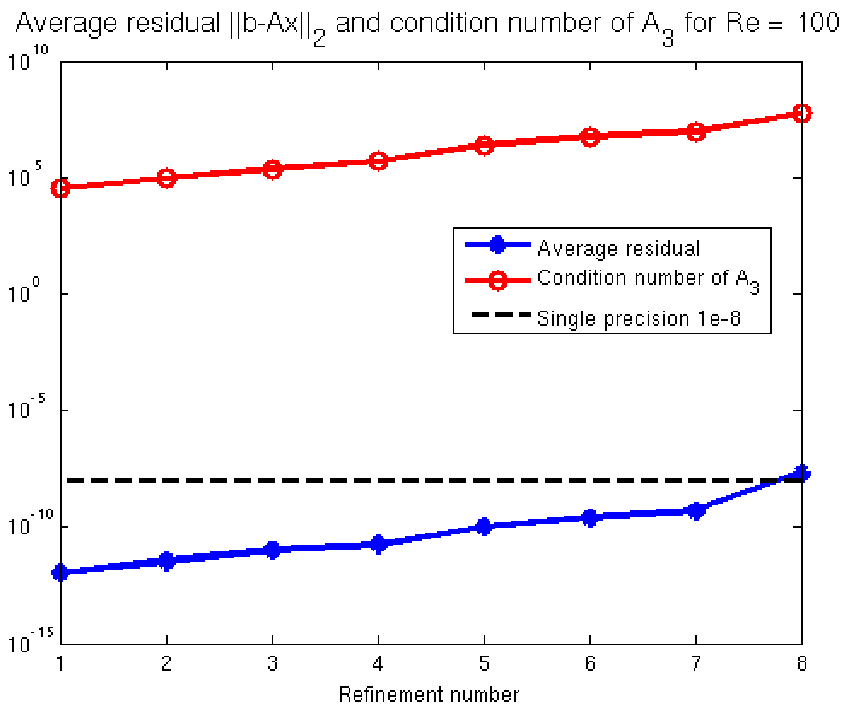

Figure 3: Average discrete residual $\|b - Ax\|_2$. Convergence of the pseudo-time iteration stalls after the 8th refinement step due to conditioning issues.

4 Solution strategy

Specify parameters and motivation, pseudo-time step and inner Newton iteration.

4.1 Nested solves

For the compressible Navier-Stokes equations, density ρ and temperature T must both be guaranteed positive to have a physically realistic solution. Numerically, non-positivity of ρ and T can cause non-convergence of our nonlinear iteration, and in some cases, solution blowup. In order to ensure positivity of ρ and T for these simulations, during each timestep, we apply a Newton-Raphson with line search to ensure that

$$\rho + \triangle\rho > 0$$
$$T + \triangle T > 0,$$

where $\triangle\rho$ and $\triangle T$ are the Newton-Raphson updates to the solution at each step. In similar applications, multiple Newton steps per timestep have been shown to accelerate convergence of the pseudo-time algorithm [1].

4.2 Adaptive time tolerance

A common technique in the solution of the steady-state Navier-Stokes equations under implicit time discretizations is the use of variable time-stepping. Typically, these schemes can be expressed as such: at a timestep i, the next time step can be expressed as a scaling of the timestep k steps ago

$$dt_{i+1} = dt_{i-k}\left(\frac{R_{i-k}}{R_i}\right)^r,$$

where R_{i-k} and R_i are the transient residuals at the ith and $(i-k)$th timesteps. The intuition behind this algorithm is that as transient behavior dies out, we can take larger and larger timesteps, thus accelerating our convergence to steady state.

Talk about DPG and "changing targets" with variable timestep.

References

[1] Benjamin Shelton Kirk. *Adaptive finite element simulation of flow and transport applications on parallel computers.* PhD thesis, University of Texas at Austin, Austin, TX, USA, 2007. AAI3285930.