PECOS

Predictive Engineering and Computational Sciences

Locally Conservative Discontinuous Petrov-Galerkin for Convection-Diffusion

Truman E. Ellis

Advisors: Leszek Demkowicz, Robert Moser
Collaborators: Jesse Chan, Nate Roberts

The University of Texas at Austin

March 8-9, 2013

A Summary of DPG

Overview of Features

- Robust for singularly perturbed problems
- Stable in the preasymptotic regime
- Designed for adaptive mesh refinement

DPG is a minimum residual method:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} ||Bw_h - l||^2_{V'}$$

$$\Updownarrow$$

$$b(u_h, R_V^{-1} B\delta u_h) = l(R_V^{-1} B\delta u_h) \quad \forall \delta u_h \in U_h$$

where $v_{\delta u_h} := R_V^{-1} B\delta u_h$ are the optimal test functions.

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\begin{aligned} \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= g \\ \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= \mathbf{0} \end{aligned}$$

Multiply by test functions and integrate by parts over each element, K.

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + ((\boldsymbol{\beta} u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v)_{\partial K} &= (g, v)_K \\ \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - (u, \tau_n)_{\partial K} &= 0 \end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + (\hat{f}, v)_{\partial K} + \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K & \\ + (u, \nabla \cdot \boldsymbol{\tau})_K - (\hat{u}, \tau_n)_{\partial K} &= (g, v)_K \end{aligned}$$

Local Conservation

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{f} = \int_K g \,,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[1], we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2}\left|\left|R_V^{-1}(Bu_h - l)\right|\right|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K\rangle}_{\langle \hat{f}, 1_K\rangle_{\partial K} - \langle g, 1_K\rangle_K} \,,$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$.

[1] D. Moro, N.C. Nguyen, and J. Peraire. A hybridized discontinuous Petrov-Galerkin scheme for scalar conservation laws. *Int.J. Num. Meth. Eng.*, 2011. in print

Local Conservation

Finding the critical points of $L(u, \boldsymbol{\lambda})$, we get the following equations.

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B \delta u_h) - l(R_V^{-1} B \delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddlepoint problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{f}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable. Chan et al[2] developed the following robust norm for convection-diffusion.

$$\begin{aligned} \|(v,\boldsymbol{\tau})\|^2_{\mathbf{V},\Omega_h} &= \|\nabla\cdot\boldsymbol{\tau}\|^2 + \left\|\min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\}\boldsymbol{\tau}\right\|^2 \\ &+ \epsilon\|\nabla v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|^2 \quad \underbrace{+\left\|\min\left\{\sqrt{\frac{\epsilon}{|K|}}, 1\right\} v\right\|^2}_{\text{no longer necessary to make this a norm}} \end{aligned}$$

[2] J. Chan, N. Heuer, T Bui-Thanh, and L. Demkowicz. Robust DPG method for convection-dominated diffusion problems ii: a natural inflow condition. Technical Report 21, ICES, 2012

Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{f}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable. Chan et al[2] developed the following robust norm for convection-diffusion.

$$\begin{aligned} ||(v,\boldsymbol{\tau})||^2_{\mathbf{V},\Omega_h} &= ||\nabla\cdot\boldsymbol{\tau}||^2 + \left|\left|\min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\}\boldsymbol{\tau}\right|\right|^2 \\ &\quad + \epsilon\,||\nabla v||^2 + ||\boldsymbol{\beta}\cdot\nabla v||^2 \underbrace{+\left(\frac{1}{|K|}\int_K v\right)^2}_{\text{zero mean term}} \end{aligned}$$

[2] J. Chan, N. Heuer, T Bui-Thanh, and L. Demkowicz. Robust DPG method for convection-dominated diffusion problems ii: a natural inflow condition. Technical Report 21, ICES, 2012

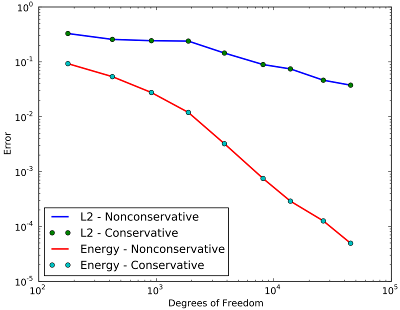
Erickson-Johnson Problem

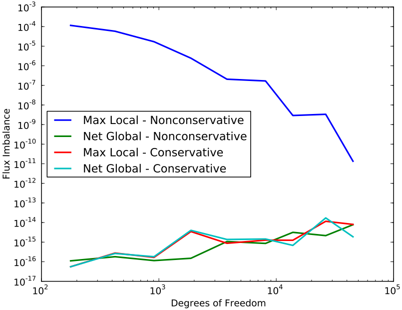
On domain $\Omega = [0,1]^2$, with $\beta = (1,0)^T$, $f = 0$ and boundary conditions

$$\hat{f} = u_0, \quad \beta_n \leq 0\,, \qquad \hat{u} = 0, \quad \beta_n > 0$$

Separation of variabes gives an analytic solution

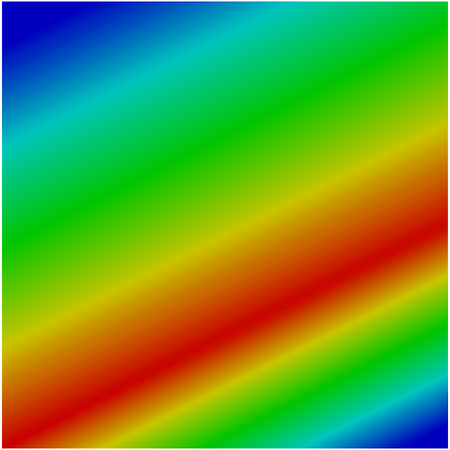
$$u(x,y) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\exp(r_2(x-1)) - \exp(r_1(x-1))}{r_1 \exp(-r_2) - r_2 \exp(-r_1)} \cos(n\pi y)$$

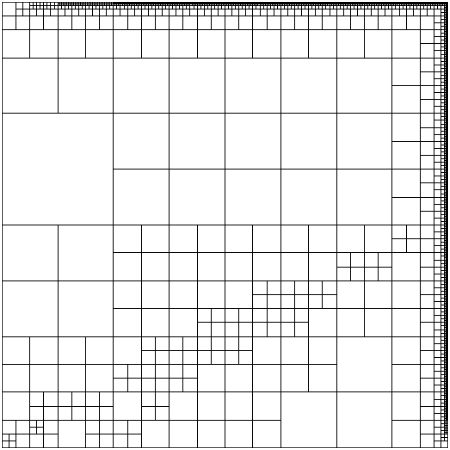




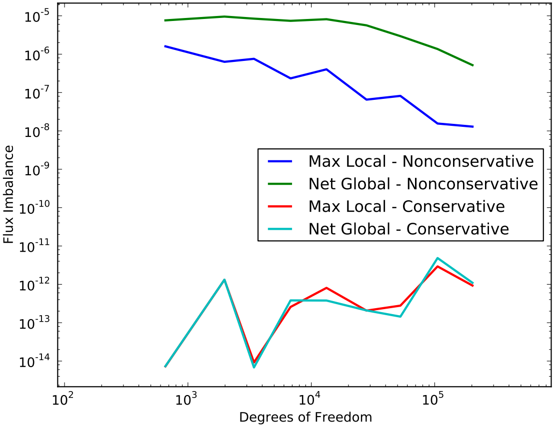
Skewed Convection-Diffusion Problem

After 8 refinement steps, $\epsilon = 10^{-4}$, $\boldsymbol{\beta} = (1, 0)^T$



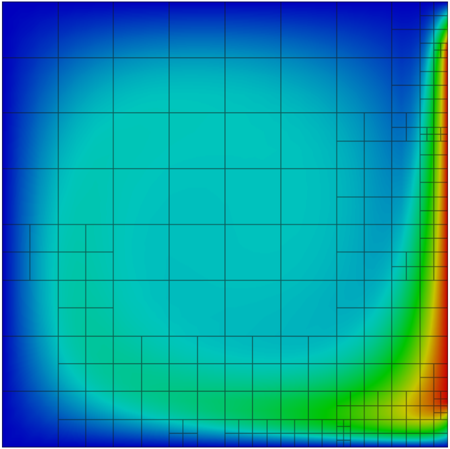


Skewed Convection-Diffusion Problem

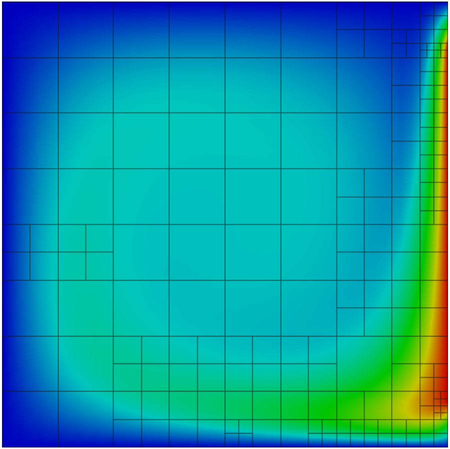


Double Glazing Problem

After 6 refinement steps, $\epsilon = 10^{-2}$, $\boldsymbol{\beta} = \begin{pmatrix} 2(2y-1)(1-(2x-1)^2) \\ -2(2x-1)(1-(2y-1)^2) \end{pmatrix}$

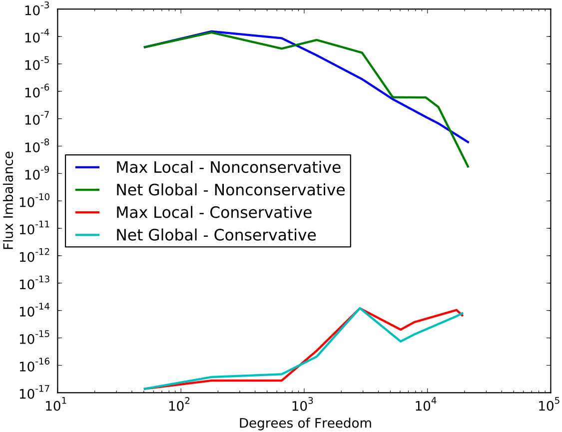


Without local conservation



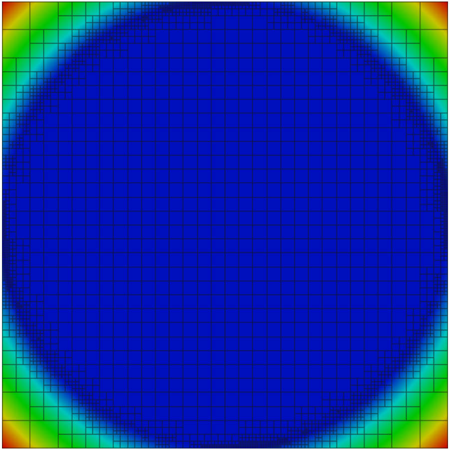
With local conservation

Double Glazing Problem

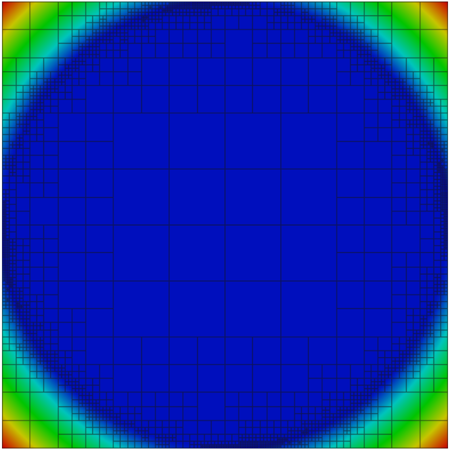


Vortex Problem

After 6 refinement steps, $\epsilon = 10^{-4}$, $\boldsymbol{\beta} = (-y, x)^T$

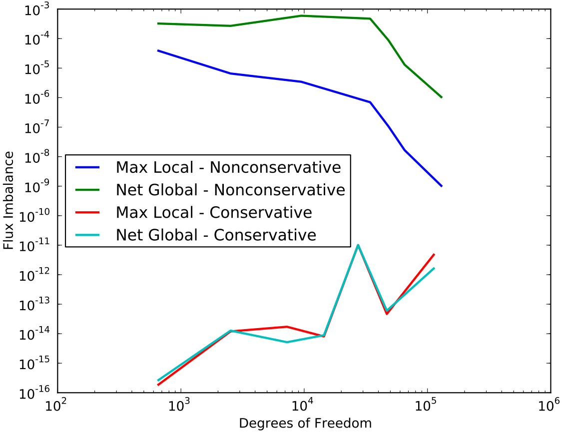


Without local conservation



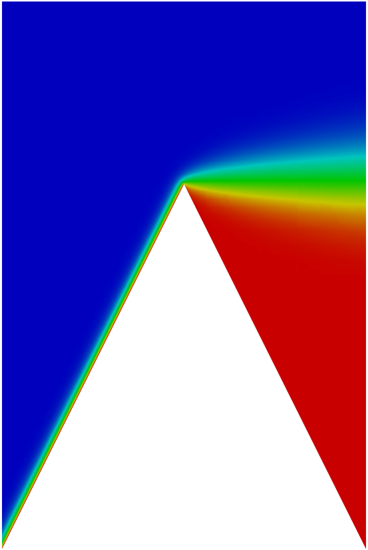
With local conservation

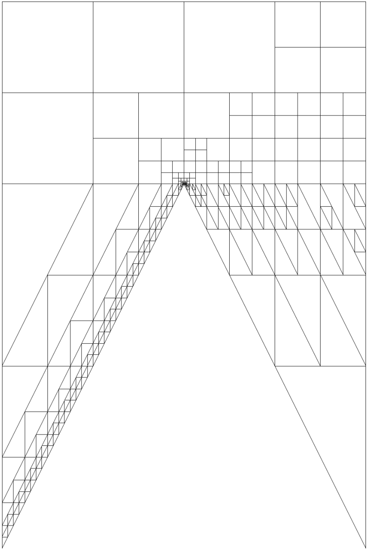
Vortex Problem



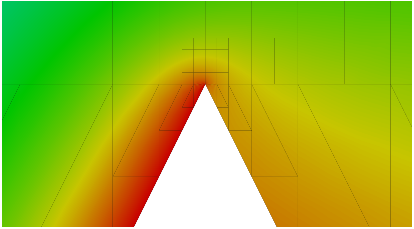
Wedge Problem

After 16 refinement steps, $\epsilon = 10^{-2}$, $\boldsymbol{\beta} = (1, 0)^T$

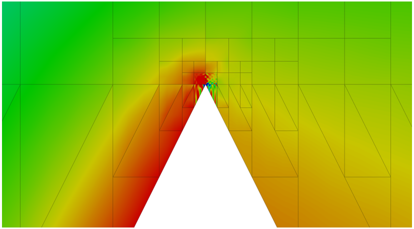




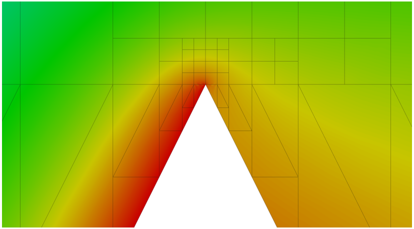
Wedge Problem



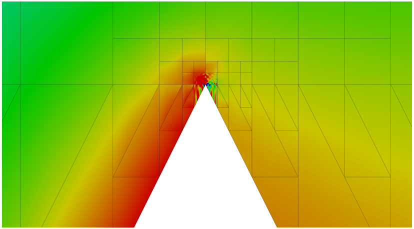
Nonconservative graph norm



Nonconservative robust norm

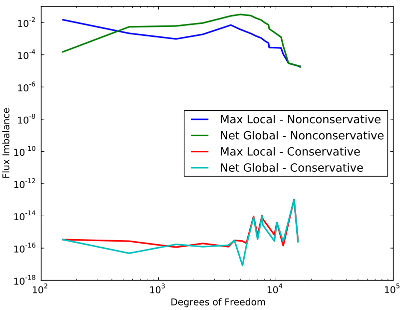


Conservative graph norm

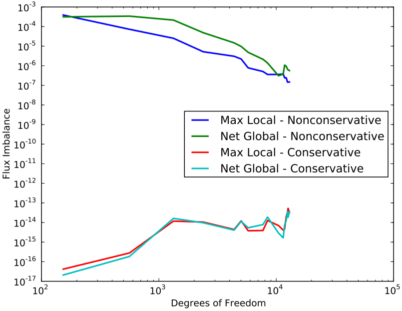


Conservative robust norm

Wedge Problem



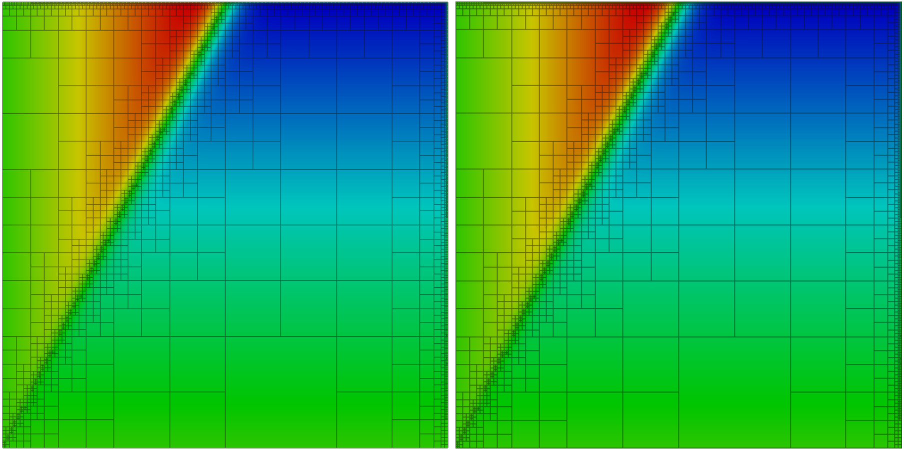
Graph norm



Robust norm

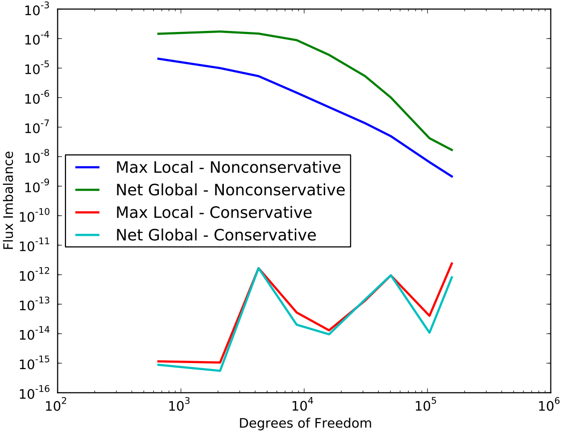
Discontinuous Source Problem

After 8 refinement steps, $\epsilon = 10^{-3}$, $\boldsymbol{\beta} = (0.5/\sqrt{1.25}, 1/\sqrt{1.25})^T$, $f = \pm 1$



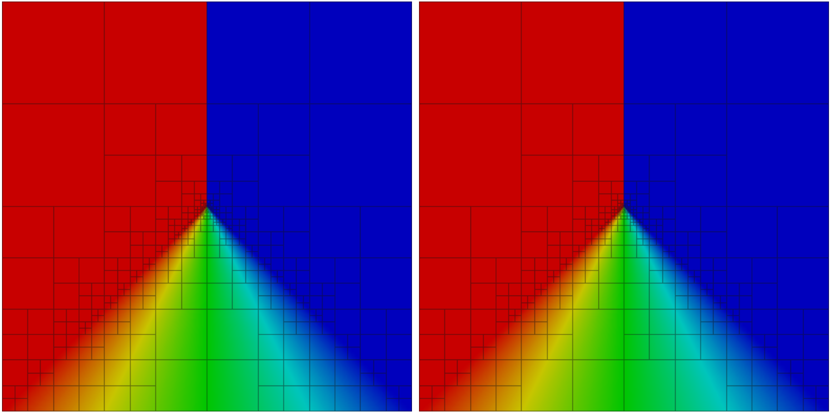
Without local conservation　　　　With local conservation

Discontinuous Source Problem



Inviscid Burgers' Equation

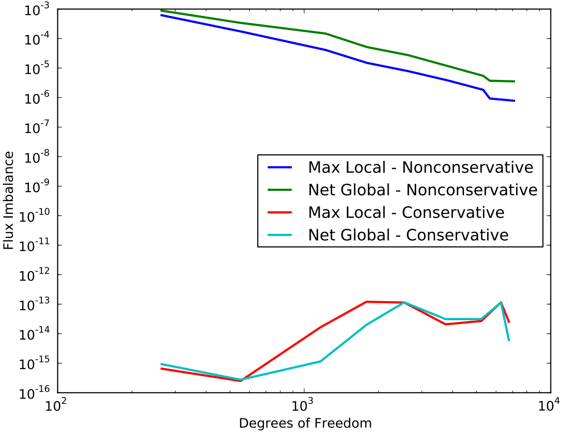
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \Leftrightarrow \quad \nabla_{x,t} \cdot \begin{pmatrix} \frac{u^2}{2} \\ u \end{pmatrix} = 0$$



Without local conservation

With local conservation

Inviscid Burgers' Equation



Summary

What have we done?

- We've turned our minimization problem into a saddlepoint problem.
- The change is computationally feasible.
- Mathematically, it gets rid of troublesome term.

Does it make a difference?

- Enforcement changes refinement strategy.
- Standard DPG is nearly conservative in practice.
- Usually we get the same results with better conservation.
- Some improvement on condition number for local solves.

Summary

What have we done?

- We've turned our minimization problem into a saddlepoint problem.
- The change is computationally feasible.
- Mathematically, it gets rid of troublesome term.

Does it make a difference?

- Enforcement changes refinement strategy.
- Standard DPG is nearly conservative in practice.
- Usually we get the same results with better conservation.
- Some improvement on condition number for local solves.

We need to study the effect on real fluid dynamics.

J. Chan, N. Heuer, T Bui-Thanh, and L. Demkowicz.
Robust DPG method for convection-dominated diffusion problems ii: a natural inflow condition.
Technical Report 21, ICES, 2012.

D. Moro, N.C. Nguyen, and J. Peraire.
A hybridized discontinuous Petrov-Galerkin scheme for scalar conservation laws.
Int.J. Num. Meth. Eng., 2011.
in print.