

Locally Conservative Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

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Abstract

1 Introduction

The discontinuous Petrov-Galerkin (DPG) method with optimal test functions has been under active development for convection-diffusion type systems [4, 5, 6, 1, 7, 2, 8]. In this paper, we develop a locally conservative formulation of DPG for convection-diffusion type equations and supplement this with numerical results.

1.1 Importance of Local Conservation

Locally conservative methods hold a special place for numerical analysts in the field of fluid dynamics. Perot[9] argues

Accuracy, stability, and consistency are the mathematical concepts that are typically used to analyze numerical methods for partial differential equations (PDEs). These important tools quantify how well the mathematics of a PDE is represented, but they fail to say anything about how well the physics of the system is represented by a particular numerical method. In practice, physical fidelity of a numerical solution can be just as important (perhaps even more important to a physicist) as these more traditional mathematical concepts. A numerical solution that violates the underlying physics (destroying mass or entropy, for example) is in many respects just as flawed as an unstable solution.

The discontinuous Petrov-Galerkin finite element method has been described as least squares finite elements with a twist. The key difference is that least square methods seek to minimize the residual of the solution in some Hilbert space norm, while DPG seeks the minimization in a dual norm through the inverse Riesz map. Exact mass conservation has been an issue that has plagued least squares finite elements for a long time. Several approaches have been used to try to address this. Chang and Nelson[3] developed the 'restricted LSFEM'[3] by augmenting the least squares equations with a Lagrange multiplier explicitly enforcing mass conservation element-wise. Our conservative formulation of DPG takes a similar approach and both methods share similar negative of transforming a minimization method to a saddle-point problem.

1.2 DPG is a Minimum Residual Method

Roberts *et al.* presents a brief history and derivation of DPG with optimal test functions in [10]. We follow his derivation of the standard DPG method as a minimum residual method. Let U be the trial Hilbert

space and V the test Hilbert space for a well-posed variational problem $b(u, v) = l(v)$. In operator form this is $Bu = l$, where $B : U \rightarrow V'$. We seek to minimize the residual for the discrete space $U_h \subset U$:

$$u_h = \arg \min_{u_h \in U_h} \frac{1}{2} \|Bu_h - l\|_{V'}^2. \quad (1)$$

Recalling that the Riesz operator $R_V : V \rightarrow V'$ is an isometry defined by

$$\langle R_V v, \delta v \rangle = (v, \delta v)_V, \quad \forall \delta v \in V,$$

we can use the Riesz inverse to minimize on V rather than its dual:

$$\frac{1}{2} \|Bu_h - l\|_{V'}^2 = \frac{1}{2} \|R_V^{-1}(Bu_h - l)\|_V^2 = \frac{1}{2} (R_V^{-1}(Bu_h - l), R_V^{-1}(Bu_h - l))_V. \quad (2)$$

The first order optimality condition for (2) requires the Gâteaux derivative to be zero in all directions $\delta u \in U_h$, i.e.,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u)_V = 0, \quad \forall \delta u \in U.$$

By definition of the Riesz operator, this is equivalent to

$$\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle = 0 \quad \forall \delta u_h \in U_h. \quad (3)$$

Now, we can identify $v_{\delta u_h} := R_V^{-1}B\delta u_h$ as the optimal test function for trial function u_h and we can rewrite (3) as

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}). \quad (4)$$

The DPG method then is to solve (4) with optimal test functions $v_{\delta u_h} \in V$ that solve the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v), \quad \forall \delta v \in V. \quad (5)$$

Using a continuous test basis would result in a global solve for every optimal test function. Therefore DPG uses a discontinuous test basis which makes each solve element-local and much more computationally tractable. Of course, (5) still requires the inversion of the infinite-dimensional Riesz map, but approximating V by finite dimensional V_h which is of a higher polynomial degree than U_h (hence “enriched space”) works well in practice.

No assumptions have been made so far on the definition of the inner product on V . In fact, proper choice of $(\cdot, \cdot)_V$ can make the difference between a solid DPG method and one that suffers from robustness issues.

2 Analysis

We now proceed to develop a locally conservative formulation of DPG for convection-diffusion type problems, but there are a few terms that we need to define first. If Ω is our problem domain, then we can partition it into finite elements K such that

$$\bar{\Omega} = \bigcup_K \bar{K}, \quad K \text{ open},$$

with corresponding *skeleton* Γ_h and *interior skeleton* Γ_h^0 ,

$$\Gamma_h := \bigcup_K \partial K \quad \Gamma_h^0 := \Gamma_h - \Gamma.$$

We define broken Sobolev spaces element-wise:

$$\begin{aligned} H^1(\Omega_h) &:= \prod_K H^1(K), \\ \mathbf{H}(\text{div}, \Omega_h) &:= \prod_K \mathbf{H}(\text{div}, K). \end{aligned}$$

We also need the trace spaces:

$$\begin{aligned} H^{\frac{1}{2}}(\Gamma_h) &:= \{\hat{v} = \{\hat{v}_K\} \in \prod_K H^{1/2}(\partial K) : \exists v \in H^1(\Omega) : v|_{\partial K} = \hat{v}_K\}, \\ H^{-\frac{1}{2}}(\Gamma_h) &:= \{\hat{\sigma}_n = \{\hat{\sigma}_{Kn}\} \in \prod_K H^{-1/2}(\partial K) : \exists \boldsymbol{\sigma} \in \mathbf{H}(\text{div}, \Omega) : \hat{\sigma}_{Kn} = (\boldsymbol{\sigma} \cdot \mathbf{n})|_{\partial K}\}, \end{aligned}$$

which are developed more precisely in [10].

2.1 Element Conservative Convection-Diffusion

Now that we have briefly outlined the abstract DPG method, let us apply it to the convection-diffusion equation. The strong form of the steady convection-diffusion problem with homogeneous Dirichlet boundary conditions reads

$$\begin{cases} \nabla \cdot (\beta u) - \epsilon \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where u is the property of interest, β is the convection vector, and f is the source term. Nonhomogeneous Dirichlet and Neumann boundary conditions are straightforward but would add technicality to the following discussion. Let us write this as an equivalent system of first order equations:

$$\begin{aligned} \nabla \cdot (\beta u - \sigma) &= f \\ \frac{1}{\epsilon} \sigma - \nabla u &= \mathbf{0}. \end{aligned}$$

If we then multiply the top equation by some scalar test function v and the bottom equation by some vector-valued test function τ , we can integrate by parts over each element K :

$$\begin{aligned} -(\beta u - \sigma, \nabla v)_K + ((\beta u - \sigma) \cdot \mathbf{n}, v)_{\partial K} &= (f, v)_K \\ \frac{1}{\epsilon} (\sigma, \tau)_K + (u, \nabla \cdot \tau)_K - (u, \tau_n)_{\partial K} &= 0. \end{aligned} \tag{6}$$

The discontinuous Petrov-Galerkin method refers to the fact that we are using discontinuous optimal test functions that come from a space differing from the trial space. It does not specify our choice of trial space. Nevertheless, many considerations of DPG in the literature [] associate DPG with the so-called “ultra-weak formulation.” We will follow the same derivation for the convection-diffusion equation, but we emphasize that other formulations are available. Thus, we seek field variables $u \in L^2(K)$ and $\sigma \in \mathbf{L}^2(K)$. Mathematically, this leaves their traces on element boundaries undefined, and in a manner similar to the hybridized discontinuous Galerkin method, we define new unknowns for trace \hat{u} and flux \hat{t} . Applying these definitions to (6) and adding the two equations together, we arrive at our desired variational problem.

Find $\mathbf{u} := (u, \sigma, \hat{u}, \hat{t}) \in \mathbf{U} := L^2(\Omega_h) \times \mathbf{L}^2(\Omega_h) \times H^{1/2}(\Gamma_h) \times H^{-1/2}(\Gamma_h)$ such that

$$-(\beta u - \sigma, \nabla v)_K + (\hat{t}, v)_{\partial K} + \frac{1}{\epsilon} (\sigma, \tau)_K + (u, \nabla \cdot \tau)_K - (\hat{u}, \tau_n)_{\partial K} = (f, v)_K \tag{7}$$

for all $\mathbf{v} := (v, \tau) \in \mathbf{V} := H^1(\Omega_h) \times \mathbf{H}(\text{div}, \Omega_h)$. Thus, our bilinear form is $b(\mathbf{u}, \mathbf{v}) = l(\mathbf{v})$.

Let $\mathbf{U}_h := U_h \times \mathbf{S}_h \times \hat{U} \times \hat{F} \subset L^2(\Omega_h) \times \mathbf{L}^2(\Omega_h) \times H^{1/2}(\Gamma_h) \times H^{-1/2}(\Gamma_h)$ be a finite-dimensional subspace, and let $\mathbf{u}_h := (u_h, \sigma_h, \hat{u}_h, \hat{t}_h) \in \mathbf{U}_h$ be a group variable. The element conservative DPG scheme is derived from the Lagrangian:

$$L(\mathbf{u}_h, \lambda_k) = \frac{1}{2} \|R_V^{-1}(b(\mathbf{u}_h, \cdot) - (f, \cdot))\|_{\mathbf{V}}^2 - \sum_K \lambda_K (b(\mathbf{u}_h, 1_K) - l(1_K)). \tag{8}$$

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