Discontinuous Petrov-Galerkin methods for transport (and convection-diffusion problems)

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Talk outline

1. The Discontinuous Petrov-Galerkin (DPG) method: discrete stability.
   - An issue for higher order methods and singular perturbation problems.
2. Convection-diffusion and Navier-Stokes with small diffusion.
   - Stable in pre-asymptotic regions, automatic adaptivity.
   - Avoids artificial diffusion and stabilization parameters.
3. Mixed success and lessons with pure convection/Euler.

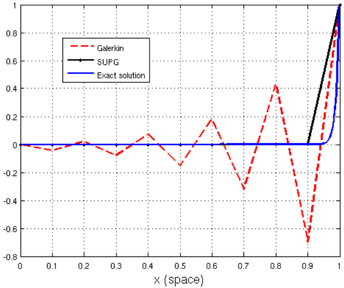


Figure: Discrete stability issues in convection-diffusion.

DPG: a minimum residual method via optimal testing

- Given a trial space U and Hilbert test space V,

$$b(u,v) = \ell(v) \Longleftrightarrow Bu = \ell, \qquad \begin{aligned} \langle Bu, v\rangle_V &:= b(u,v) \\ \langle \ell, v\rangle_V &:= \ell(v). \end{aligned}$$

- We seek to minimize the dual residual over $U_h \subset U$

$$J(u_h) = \frac{1}{2}\|Bu_h - \ell\|_{V'}^2 \Longleftrightarrow b(u_h, v_{\delta u}) = \ell(v_{\delta u}), \quad \forall \delta u \in U_h$$

- Computation of $v_{\delta u} := R_V^{-1} B\delta u$ involves solving

$$(v_{\delta u}, \delta v)_V = b(\delta u, \delta v), \quad \delta u \in U_h, \quad \forall \delta v \in V.$$

This is global and infinite-dimensional. Solution: localize using discontinuous test functions, and approximate using an enriched space $V_h \subset V$, where $\dim(V_h) > \dim(U_h)$ elementwise.

Properties of DPG

- Stiffness matrices are symmetric positive-definite. For trial/test bases $\{\phi_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$, with $B_{ji} = b(\phi_j, v_i)$ and $l_i = \ell(v_i)$. DPG solves

$$\left(B^T R_V^{-1} B\right) u = \left(B^T R_V^{-1}\right) l,$$

  For localizable norms and discontinuous testing, R_V is block diagonal.

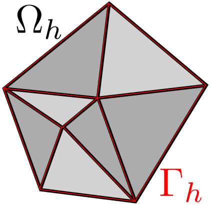
- DPG provides the best approximation in the energy norm

$$\|u\|_E = \|Bu\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{|b(u,v)|}{\|v\|_V}.$$

- The energy error is computable through the error representation function e defined through $(e, \delta v)_V = \ell(v) - b(u_h, \delta v)$ for all $\delta v \in V$.

$$\|u - u_h\|_E = \|B(u - u_h)\|_{V'} = \left\|R_V^{-1}(l - Bu_h)\right\|_V = \|e\|_V$$

Ultra-weak formulation for convection-diffusion



The first order convection-diffusion system:

$$A(u,\sigma) := \begin{bmatrix} \nabla\cdot(\beta u - \sigma) \\ \frac{1}{\epsilon}\sigma - \nabla u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

The variational formulation is

$$b\left(\left(u,\sigma,\widehat{u},\widehat{f}_n\right),(v,\tau)\right) = \left(u, \nabla_h\cdot\tau - \beta\cdot\nabla_h v\right)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla_h v\right)_{\Omega_h} \\ - \left\langle \llbracket \tau\cdot n \rrbracket, \widehat{u}\right\rangle_{\Gamma_h} + \left\langle \widehat{f}_n, \llbracket v \rrbracket \right\rangle_{\Gamma_h},$$

where $\widehat{f}_n := \beta_n u - \sigma_n$ and $\left\langle \widehat{f}_n, \llbracket v \rrbracket \right\rangle_{\Gamma_h}$ is defined

$$\left\langle \widehat{f}_n, \llbracket v \rrbracket \right\rangle_{\Gamma_h} := \sum_K \int_{\partial K} \operatorname{sgn}(\vec{n})\, \widehat{f}_n v.$$

Construction of a test norm: adjoints and energy estimates

$$b(\mathbf{U},\mathbf{V}) = (u, \nabla\cdot\tau - \beta\cdot\nabla v)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla v\right)_{\Omega_h} + \text{boundary terms}$$

Recover $\|u,\sigma\|^2_{L^2(\Omega)}$ with conforming (v,τ) satisfying the *adjoint equations*

$$\begin{aligned} \nabla\cdot\tau - \beta\cdot\nabla v &= u \\ \frac{1}{\epsilon}\tau + \nabla v &= \sigma \end{aligned}, \quad \text{boundary terms} = 0$$

"Necessary" conditions for robustness (independence from ϵ) —

$$\|u,\sigma\|^2_{L^2(\Omega)} = b\left(\mathbf{U},(v,\tau)\right) = \frac{b\left(\mathbf{U},(v,\tau)\right)}{\|(v,\tau)\|_V}\|(v,\tau)\|_V \leq \|\mathbf{U}\|_E\,\|(v,\tau)\|_V$$

Let $\lesssim$ denote a robust bound - if $\|(v,\tau)\|_V \lesssim \|u,\sigma\|_{L^2(\Omega)}$, then we have

$$\|u,\sigma\|_{L^2(\Omega)} \lesssim \|\mathbf{U}\|_E$$

Main idea: the test norm should measure adjoint solutions robustly.

Results for convection-diffusion

By constructing $\|v\|_V$ carefully, we prove an ϵ-independent bound[1]

$$\|u\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} + \epsilon \left\|\widehat{u}\right\| + \sqrt{\epsilon}\left\|\widehat{f}_n\right\| \lesssim \left\|\left(u,\sigma,\widehat{u},\widehat{f}_n\right)\right\|_E .$$

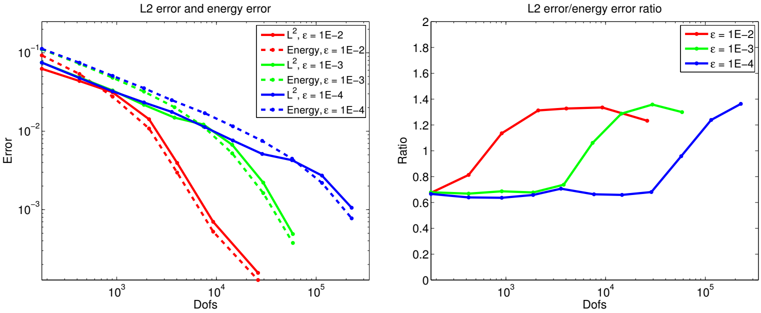


Figure: L^2/energy errors for $\epsilon = .01, .001, .0001$ and a boundary layer solution.

[1] J. Chan, N. Heuer, T. Bui Thanh, and L. Demkowicz. Robust DPG method for convection-diffusion problems II: natural inflow conditions. Technical Report 12-21, ICES, June 2012. Submitted

2D test case: Burgers equation

$$\frac{\partial\left(u^2/2\right)}{\partial x}+\frac{\partial u}{\partial y}+\epsilon\Delta u=f$$

Burgers equation can be written with $\beta(u)=(u/2,1)$

$$\nabla\cdot(\beta(u)u-\sigma)=f$$
$$\frac{1}{\epsilon}\sigma-\nabla u=0.$$

i.e. nonlinear convection-diffusion on domain $[0,1]^2$.

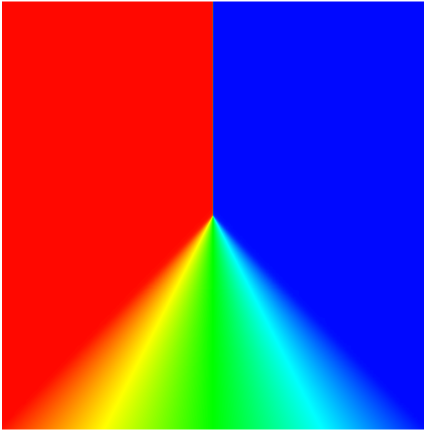


Figure: Shock solution for Burgers' equation, $\epsilon=1e-4$, using Newton-Raphson.

Adaptivity begins with a cubic 4×4 mesh.

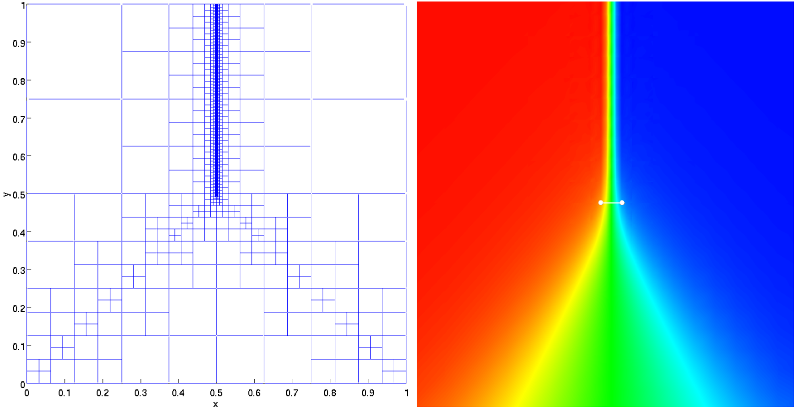


Figure: Adaptive mesh after 9 refinements, and zoom view at point (.5,.5) with shock formation and $1e-3$ width line for reference.

2D Compressible Navier-Stokes - Carter's flat plate

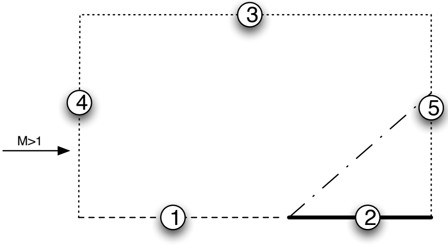
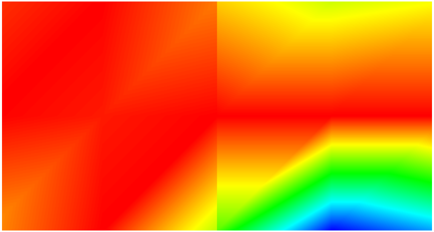


Figure: Carter flat plate problem on domain $[0,2] \times [0,1]$. Plate begins at $x = 1$, $\mathrm{Re} = 1000$.

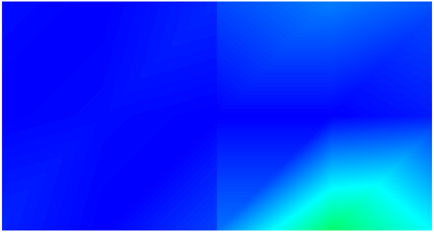
1. Symmetry boundary conditions.
2. Prescribed temperature and wall stagnation conditions.
3. Symmetry boundary conditions.
4. Inflow: conserved quantities specified using far-field values.
5. No outflow condition set.

Stress/heat flux boundary conditions are set in terms of the momentum and energy fluxes.

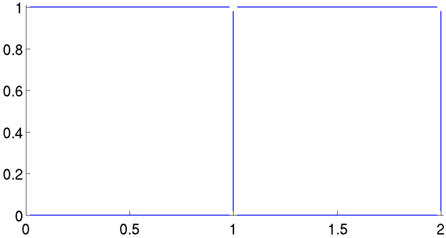
Refinement level 0



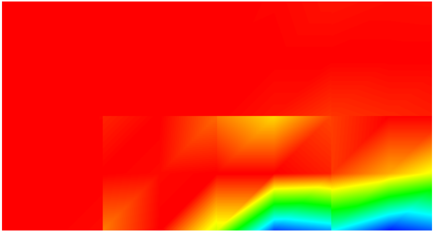
(a) u_1



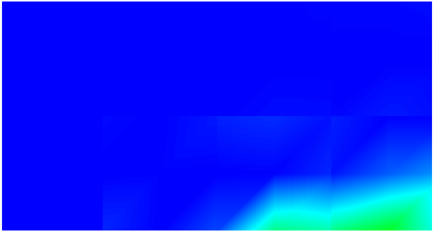
(b) T



Refinement level 1



(a) u_1



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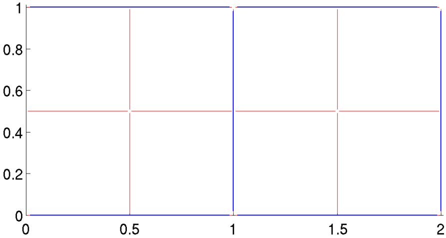
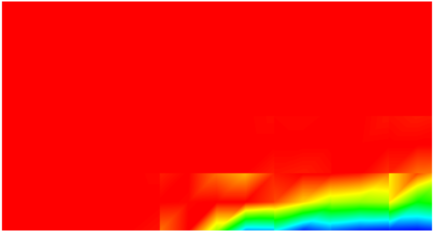
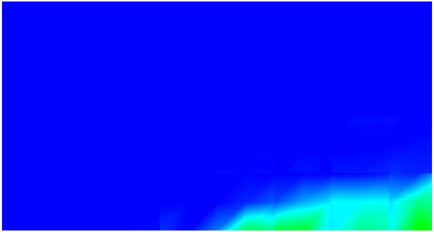


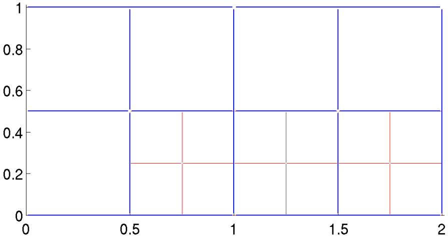
Refinement level 2



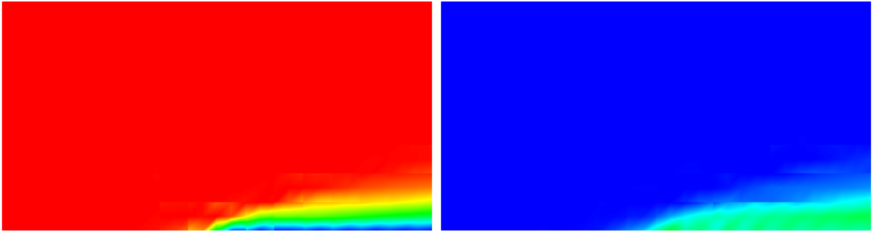
(a) u_1



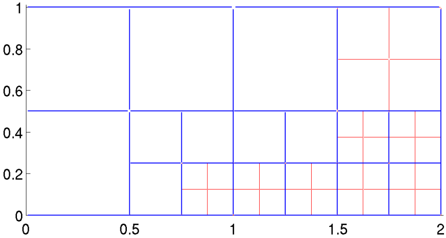
(b) T



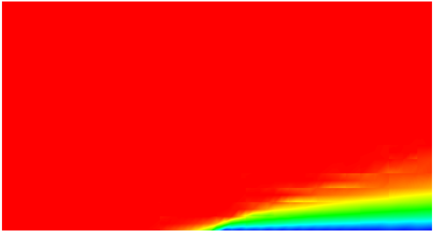
Refinement level 3



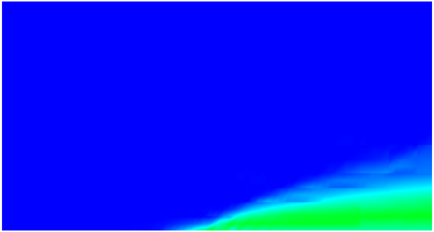
(a) u_1 (b) T



Refinement level 4



(a) u_1



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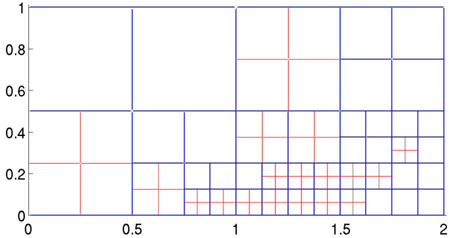
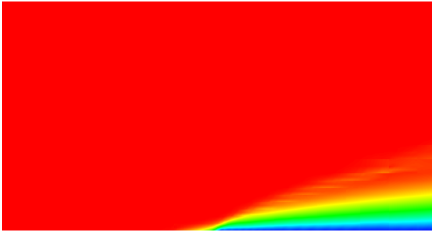
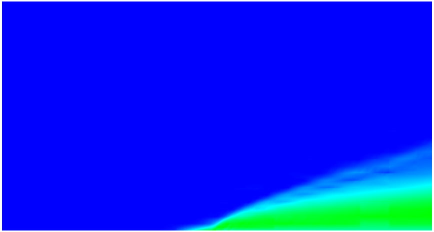


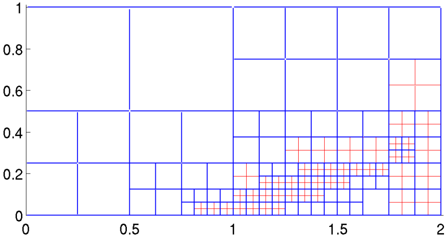
Refinement level 5



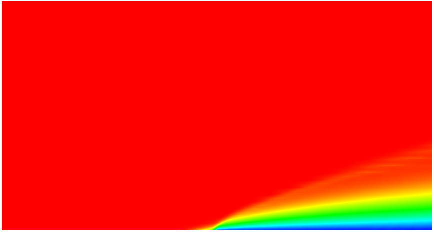
(a) u_1



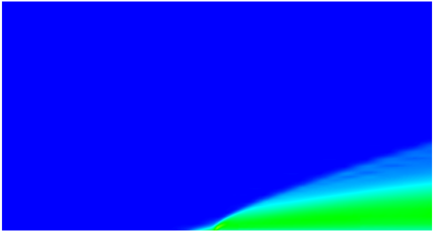
(b) T



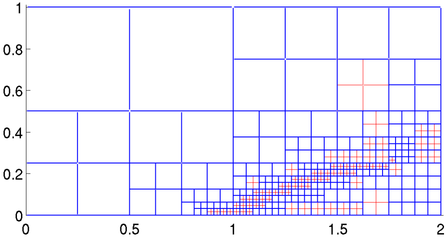
Refinement level 6



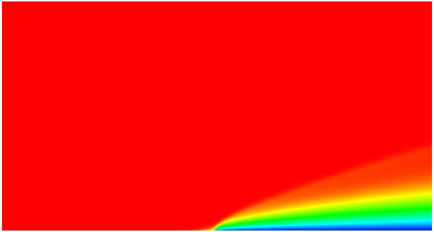
(a) u_1



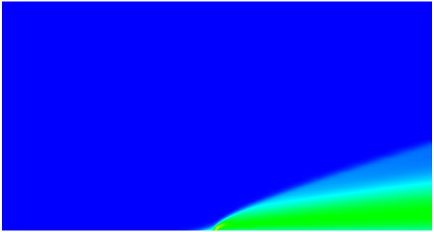
(b) T



Refinement level 7



(a) u_1



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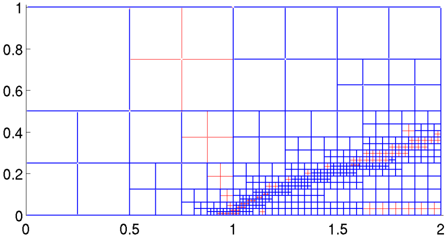
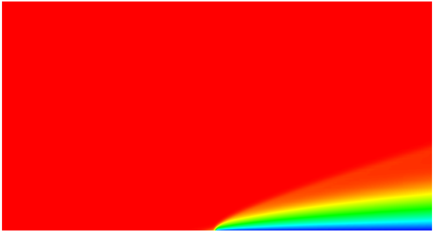
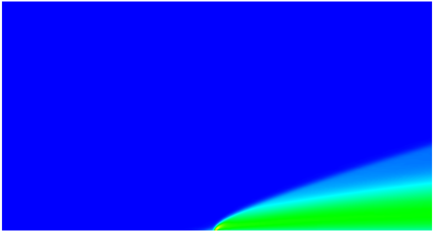


Refinement level 8



(a) u_1



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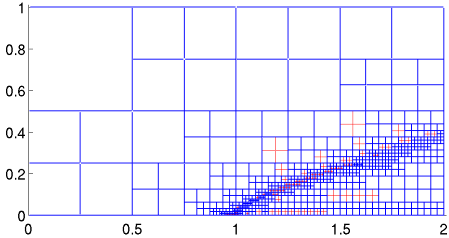
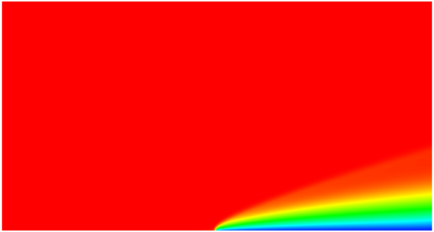
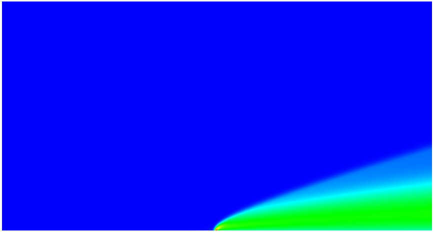


Refinement level 9



(a) u_1



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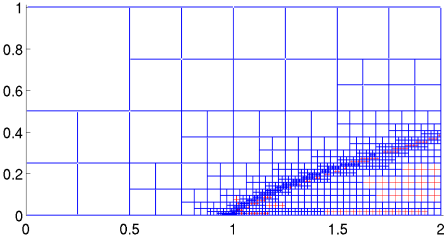
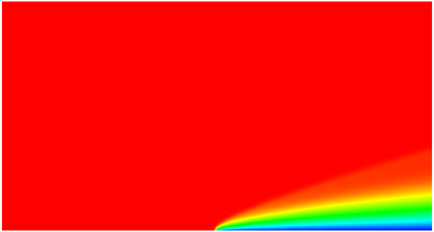
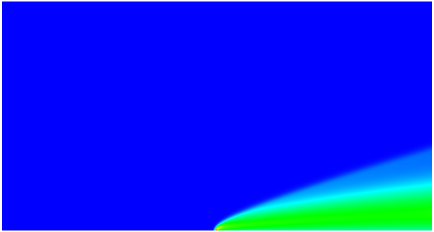


Refinement level 10



(a) u_1



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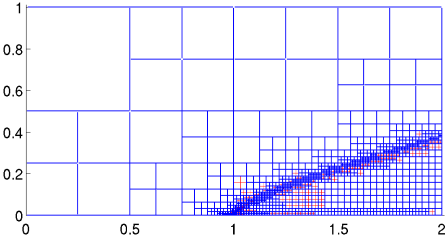
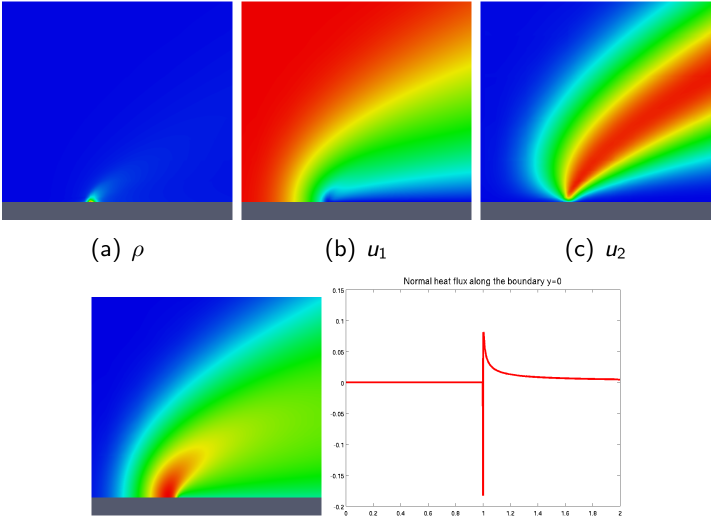


Zoomed solutions at plate/stagnation point



(a) ρ (b) u_1 (c) u_2

(d) T (e) q_n

Automatic extension to anisotropic/*hp* meshes

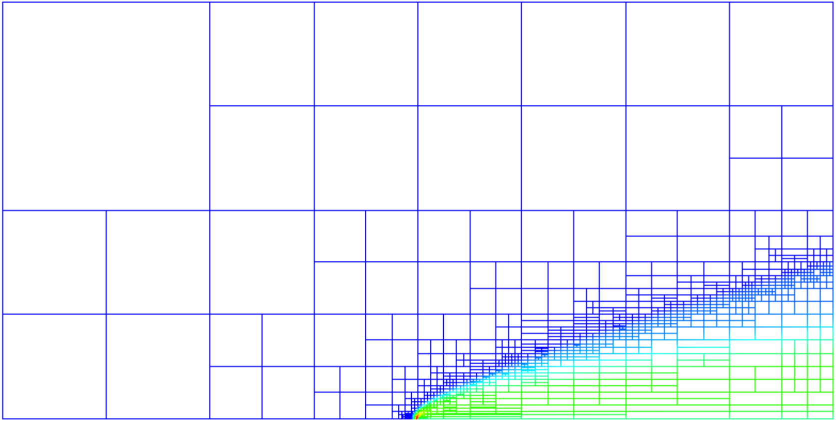


Figure: Trace $\widehat{T}$ for $\mathrm{Re} = 1000$ using an anisotropic refinement scheme[2].

[2] N. Roberts, D. Ridzal, P. Bochev, and L. Demkowicz. A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos. Technical Report SAND2011-6678, Sandia National Laboratories, 2011

The pure convection equation

- Our primary focus has been on the resolution of boundary layers and viscous effects with ϵ as small as possible ($O(10^{-7})$) without artificial diffusion or stabilization.[3]
- What about when $\epsilon = 0$ - the pure convection equation?

$$\nabla \cdot (\beta u) = f$$
$$u = u_0, \quad \text{on } \Gamma_{\text{in}}$$

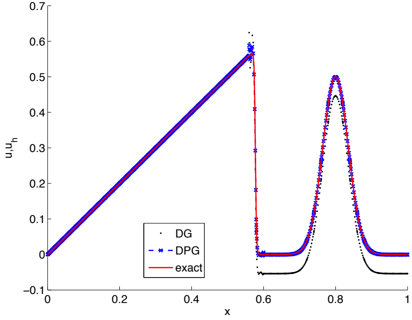
- How does DPG compare to standard methods?

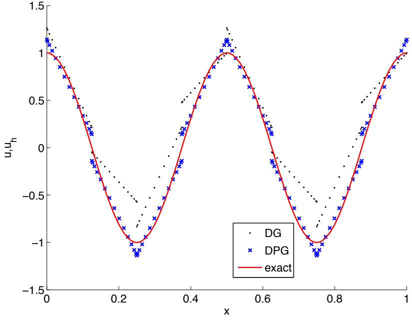
[3] L. Demkowicz, J. Gopalakrishnan, and A. Niemi. A class of discontinuous Petrov-Galerkin methods. Part III: Adaptivity. *Appl. Numer. Math.*, 62(4):396–427, April 2012

A connection between DPG and upwind DG for convection

Upwind DG can be derived from the DPG method[4]:

- Derive strong equations that test functions must satisfy, use downwind DG to solve for test functions.
- Without enriching V_h, DPG corresponds to the upwind DG method.

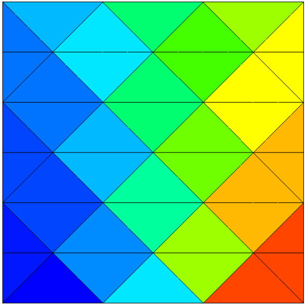




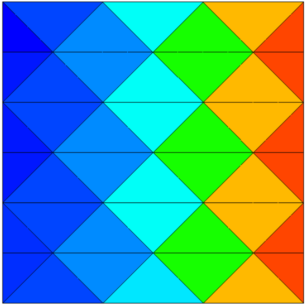
[4] T. Bui-Thanh, O. Ghattas, and L. Demkowicz. A relation between the Discontinuous Petrov-Galerkin method and the Discontinuous Galerkin method. Technical report, ICES, 2011

Peterson problem: suboptimality of DG

Example of $h^{p+1/2}$ suboptimal convegence of upwind DG.



(c) Upwind DG

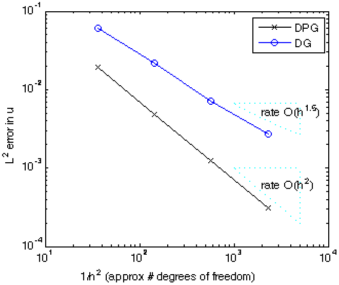


(d) DPG

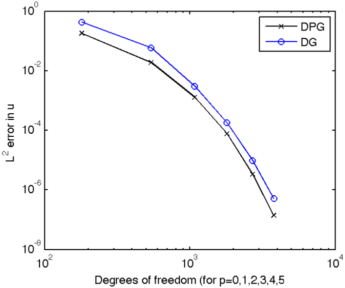
Figure: Comparison of DG and DPG for the Peterson mesh example for $p = 0$.[5]

[5] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. Part I: The transport equation. *Comput. Methods Appl. Mech. Engrg.*, 2009. accepted, see also ICES Report 2009-12

Peterson problem: convergence rates



(a) h



(b) p

- DPG error in u converges at the optimal rate in h, and is lower than DG error for p-convergence.
- Optimal rate observed for convergence of flux $\widehat{f}_n$, proven rate is suboptimal by one order. Why?

Convection equation: degeneration of fluxes

For pure convection, the ultra-weak variational formulation is

$$\left\langle \widehat{f}_n, v \right\rangle - (u, \beta \cdot \nabla v) = (f, v),$$

where $\widehat{f}_n := \beta_n u$. The proper spaces for $\widehat{f}_n$, u, and v are

$$\begin{aligned} u &\in L^2(\Omega) \\ v \in H_\beta(\Omega_h) &:= \left\{ v \in L^2(\Omega) : \beta \cdot \nabla v = 0, \text{ on } K \in \Omega_h \right\} \\ \widehat{f}_n \in L^2(\Gamma_h) &:= \left\{ f \in L^2(\Gamma_h) : \int_{\partial K} |\beta \cdot n| \, |f|^2 < \infty, \text{ on } K \in \Omega_h \right\} \end{aligned}$$

When $\beta_n = 0$, v has only a streamline derivative, and $\widehat{f}_n$ becomes an ill-defined trace in the cross-stream direction. This is not observed numerically for convection. However...

Linearized Euler equations: degeneration of fluxes

For hyperbolic *systems* of equations such as the linearized Euler equations $(A_i U)_{,i} = 0$, $\beta_n = 0$ corresponds to $\lambda(A_n) = \{u_n, u_n - c, u_n + c\} = 0$, and we observe this along *sonic lines*.

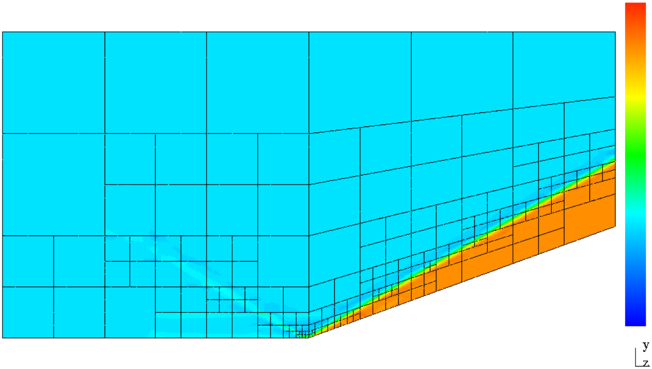


Figure: Sonic lines appear in the y-velocity for linearized Euler. Without a second-order viscosity term, traces are not always well defined.

Viscous regularization: the vortex example

For $\beta = (-y, x)^T$ on $\Omega = [-1,1]^2$. Ill posed in the convection setting.
Similar tests have been done with discontinuous data.

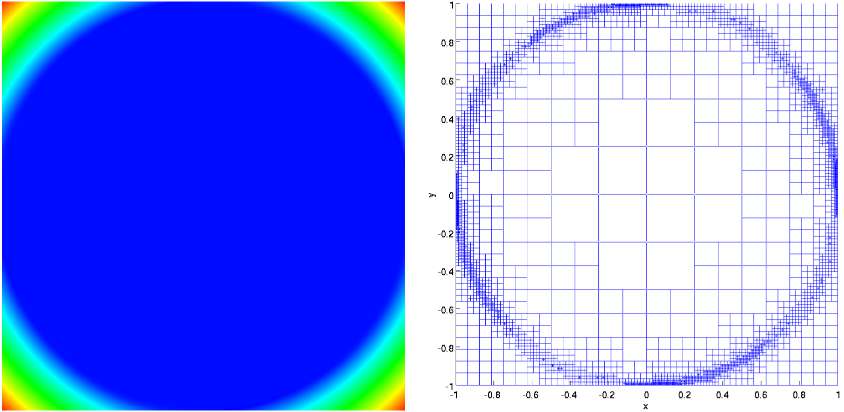


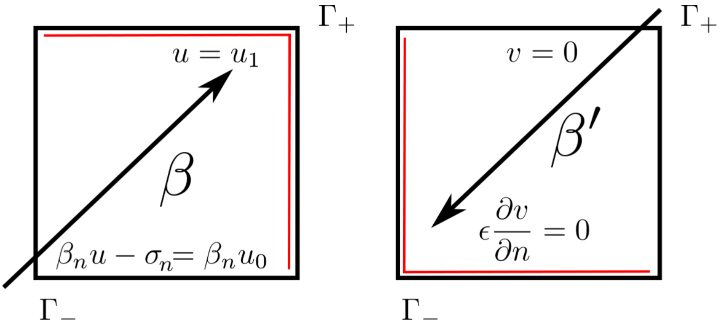
Figure: Steady vortex problem with $\epsilon = 1e - 4$.

Thank you!

Questions?

A new inflow boundary condition for a better adjoint

Non-standard choice of boundary condition: $\widehat{f}_n = \beta_n u - \sigma_n \approx \beta_n u_0$ on Γ_{in}, induces smoother adjoint problems and stronger energy estimates.



(a) Primal problem (b) Adjoint problem

Figure: Under the new inflow condition, the wall-stop boundary condition is relaxed to a zero-stress condition at the outflow boundary of the adjoint problem.

Convection-diffusion test norm

For solutions (v, τ) of the adjoint equations, we derive quantities that are robustly bounded from above by $\|u\|_{L^2(\Omega)}$. Our test norm, as defined over a single element K, is now

$$\|(v,\tau)\|_{V,K}^2 = \min\left\{\frac{\epsilon}{|K|}, 1\right\}\|v\|^2 + \epsilon\|\nabla v\|^2 + \|\beta\cdot\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \min\left\{\frac{1}{\epsilon}, \frac{1}{|K|}\right\}\|\tau\|^2.$$

which induces the proven *robust* bound[6]

$$\|u\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} + \epsilon\|\widehat{u}\| + \sqrt{\epsilon}\left\|\widehat{f}_n\right\| \lesssim \left\|\left(u, \sigma, \widehat{u}, \widehat{f}_n\right)\right\|_E.$$

[6] J. Chan, N. Heuer, T. Bui Thanh, and L. Demkowicz. Robust DPG method for convection-diffusion problems II: natural inflow conditions. Technical Report 12-21, ICES, June 2012. Submitted

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