Global and local DPG test functions for convection-diffusion

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Ultra-weak formulation

Given a first order system $Au = f$, multiply by test function v and integrate

$$(Au, v) = \langle \gamma\left(Au\right), v\rangle + (u, A_h^* v) = (f, v)$$

We identify boundary terms $\langle \gamma\left(Au\right), v\rangle_{\Gamma_h} = \langle \widehat{u}, v\rangle_{\Gamma_h}$ as unknowns $\widehat{u}$ on Γ_h. This gives us the bilinear form.

$$\begin{aligned} b(\left(u, \widehat{u}\right), v) &:= \langle \widehat{u}, v\rangle + (u, A_h^* v) \\ l(v) &:= (f, v) \end{aligned}$$

DPG approximates optimal test functions $v_{\delta u}$ for all $\delta u \in U_h$ by solving on a local level

$$(v_{\delta u}, \delta v) = b(\delta u, \delta v), \quad \delta v \in V_h(K)$$

L^2 best approximations under the ultra-weak formulation

If our optimal test functions satisfy for all $\delta u_h \in U_h$

$$A^* v = \delta u_h, \quad \text{on } \Omega$$

with boundary conditions on v such that the boundary terms disappear, we get back the best L^2 approximation by virtue of

$$\begin{aligned} b((u_h, \widehat{u}_h), v) &= \langle \widehat{u}_h, v \rangle + (u_h, A^* v) = (u_h, \delta u_h) \\ (f, v) = b((u, \widehat{u}), v) &= \langle \widehat{u}, v \rangle + (u, A^* v) = (u, \delta u_h) \end{aligned}$$

Corresponds to a graph norm choice of test norm: under assumptions of boundedness below of B, for $\delta > 0$, we can define as a DPG test norm

$$\|v\|_V := \|A^* v\|_{L^2} + \delta \|v\|_{L^2}$$

which, as $\delta \to 0$, gives an equivalent result.

Globally optimal test functions

Recall that DPG optimal test functions are from a local inversion of the Riesz operator. We can choose a conforming test space and invert the Riesz operator over the entire domain:

$$V_{\mathrm{global}} := \{v \in \oplus V_K : \langle \widehat{u}_h, [\![v]\!] \rangle_{\Gamma_h^0}, \forall \widehat{u}_h \in \widehat{U}_h\}$$
$$(v, \delta v)_\Omega = b(u_h, \delta v), \quad v \in V_{\mathrm{global}}$$

We refer to these as *globally* optimal test functions.[1] The test space resulting from the local inversion of the Riesz operator is related to the *globally* optimal test space through the following lemma:

Lemma

The globally optimal test space is contained in the locally optimal test space.

[1] Selecting these test functions removes internal traces from the big picture!

Optimal test functions for convection-diffusion

For convection-diffusion,

$$b\left(\left(u,\sigma,\widehat{u},\widehat{f}_n\right),(v,\tau)\right) = (u,\nabla_h\cdot\tau-\beta\cdot\nabla_h v)_{\Omega_h} + \left(\sigma,\epsilon^{-1}\tau+\nabla_h v\right)_{\Omega_h} \\ - \langle[\![\tau\cdot n]\!],\widehat{u}\rangle_{\Gamma_h} + \left\langle\widehat{f}_n,[\![v]\!]\right\rangle_{\Gamma_h},$$

where

$$\widehat{f}_n := \beta_n u - \sigma_n \in H^{-1/2}(\Gamma_h), \quad \widehat{u}\in H^{1/2}(\Gamma_h)$$

The adjoint problem for L^2 best optimality under convection diffusion is

$$\begin{aligned} \nabla\cdot\tau-\beta\cdot\nabla v &= u \\ \frac{1}{\epsilon}\tau+\nabla v &= 0 \end{aligned},$$

with boundary condition $v=0$ on Γ. A boundary layer forms at the inflow.

The graph norm for convection-diffusion

The graph norm for convection diffusion forms boundary layers on each element, which are only resolvable using special subgrid meshes. However,

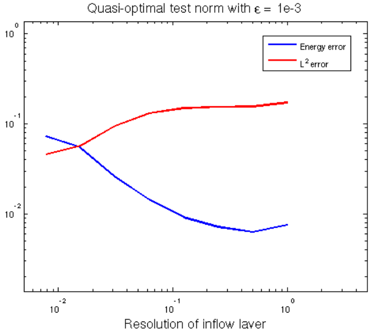
1. The globally optimal test functions only have boundary layers at the boundary
2. The test functions for L^2 optimality only have boundary layers at the inflow boundary

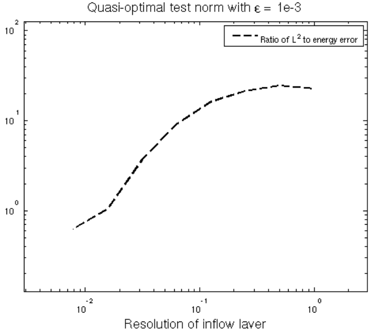
Question: do we need to resolve the boundary layers in optimal test functions everywhere?

Numerical experiment

- Given the Erikkson-Johnson problem, we use the graph test norm and compute both energy and L^2 errors.
- We then refine near the inflow and compare the energy and L^2 errors.

Best approximation error is small near the inflow, so changes in energy/L^2 error are due to resolution of test functions, not the solution.





Distribution of error

We expect that the term $\|\nabla \cdot \tau - \beta \cdot \nabla v\|_{L^2}$ is bounded uniformly in ϵ. However, the term $\left\|\frac{1}{\epsilon}\tau - \nabla v\right\|_{L^2}$ is not.[2]
As the boundedness of this term determines the robustness of σ, our energy norm is

$$
\begin{aligned}
\|\mathbf{U}\|_E &:= \sup_{(v,\tau)} \left(\frac{(u, \nabla_h \cdot \tau - \beta \cdot \nabla_h v)_\Omega}{\|(v,\tau)\|_V} + \frac{\left(\sigma, \frac{1}{\epsilon}\tau + \nabla_h v\right)_\Omega}{\|(v,\tau)\|_V} \right) + \text{etc} \\
&\approx \|u\|_{L^2} + C_{\text{Pe}} \|\sigma\|_{L^2} + \left\| \left(\widehat{u}, \widehat{f}_n\right) \right\|
\end{aligned}
$$

where Pe is the element Peclet number near the inflow.

[2] Analytical calculations in 1D show that this term grows with the Peclet number in the element at the inflow, where the adjoint solution develops a boundary layer.

Three levels of test functions

- L^2 optimal test functions resulting from a global adjoint problem,
- Global test functions resulting from a global Riesz inversion,
- Local test functions resulting from local Riesz inversions.

Thoughts:

- The complete local resolution of test functions may not be necessary
- You can have optimal test functions with boundary layers even when there are none in the global adjoint: resolution of these on a global level may still be necessary to maintain robustness.