PECOS

Predictive Engineering and Computational Sciences

Locally Conservative Discontinuous Petrov-Galerkin for Convection-Diffusion

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A Summary of DPG

Overview of Features

- Robust for singularly perturbed problems
- Stable in the preasymptotic regime
- Designed for adaptive mesh refinement

DPG is a minimum residual method:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} ||Bw_h - l||^2_{V'}$$

$$\Updownarrow$$

$$b(u_h, R_V^{-1} B\delta u_h) = l(R_V^{-1} B\delta u_h) \quad \forall \delta u_h \in U_h$$

where $v_{\delta u_h} := R_V^{-1} B\delta u_h$ are the optimal test functions.

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\begin{aligned} \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= g \\ \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= \mathbf{0} \end{aligned}$$

Multiply by test functions and integrate by parts over each element, K.

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + ((\boldsymbol{\beta} u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v)_{\partial K} &= (g, v)_K \\ \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - (u, \tau_n)_{\partial K} &= 0 \end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + (\hat{f}, v)_{\partial K} &+ \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K \\ &+ (u, \nabla \cdot \boldsymbol{\tau})_K - (\hat{u}, \tau_n)_{\partial K} = (g, v)_K \end{aligned}$$

Local Conservation

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{f} = \int_K g \,,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[1], we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \left|\left| R_V^{-1}(Bu_h - l) \right|\right|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{f}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K} \,,$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$.

[1] D. Moro, N.C. Nguyen, and J. Peraire. A hybridized discontinuous Petrov-Galerkin scheme for scalar conservation laws. *Int.J. Num. Meth. Eng.*, 2011. in print

Local Conservation

Finding the critical points of $L(u, \boldsymbol{\lambda})$, we get the following equations.

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B \delta u_h) - l(R_V^{-1} B \delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddlepoint problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{f}\} \in \mathbf{U}_h$, find $\mathbf{v}_\mathbf{u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v}_\mathbf{u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable. Chan et al[2] developed the following robust norm for convection-diffusion.

$$||(v, \boldsymbol{\tau})||^2_{\mathbf{V},\Omega_h} = ||\nabla \cdot \boldsymbol{\tau}||^2 + \left|\left|\min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\}\boldsymbol{\tau}\right|\right|^2 + \epsilon\,||\nabla v||^2 + ||\boldsymbol{\beta} \cdot \nabla v||^2 \quad \underbrace{+\left|\left|\min\left\{\sqrt{\frac{\epsilon}{|K|}}, 1\right\} v\right|\right|^2}_{\text{no longer necessary to make this a norm}}$$

[2] J. Chan, N. Heuer, T Bui-Thanh, and L. Demkowicz. Robust DPG method for convection-dominated diffusion problems ii: a natural inflow condition. Technical Report 21, ICES, 2012

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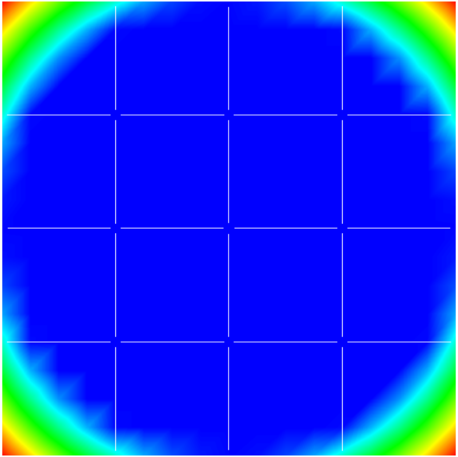
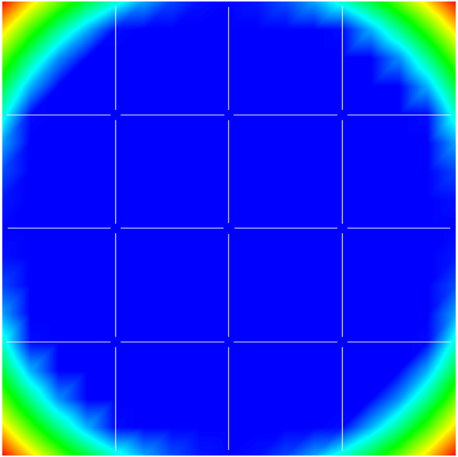
[2] J. Chan, N. Heuer, T Bui-Thanh, and L. Demkowicz. Robust DPG method for convection-dominated diffusion problems ii: a natural inflow condition. Technical Report 21, ICES, 2012

Vortex Problem

Does enforcing local conservation make a difference?



With Local Conservation



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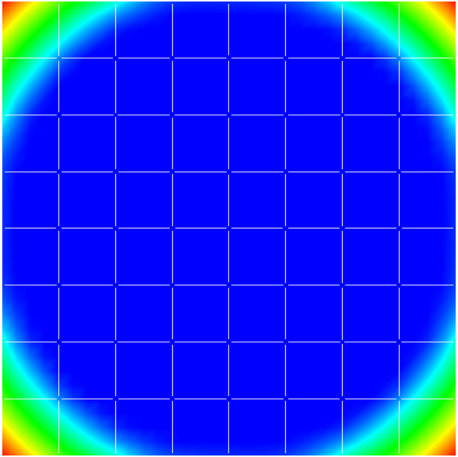

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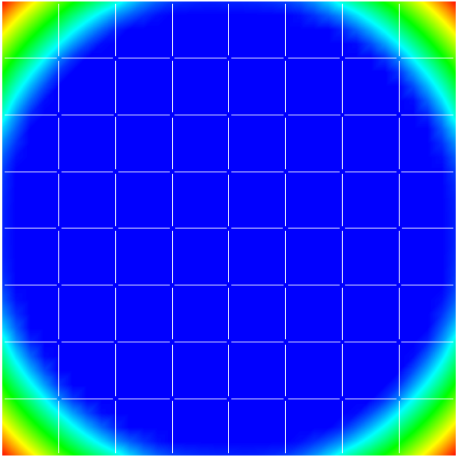

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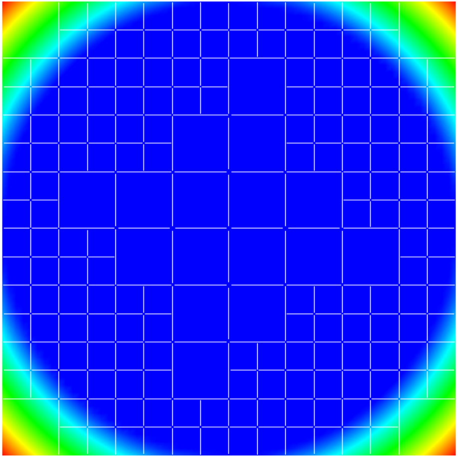

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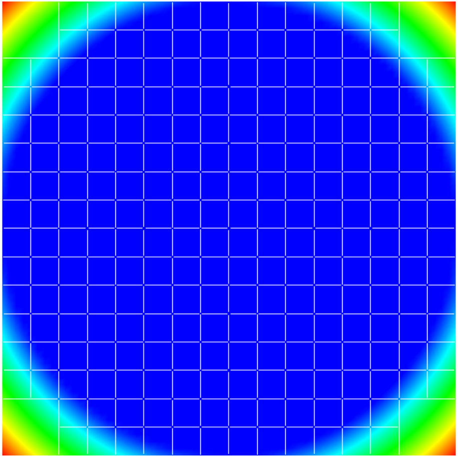

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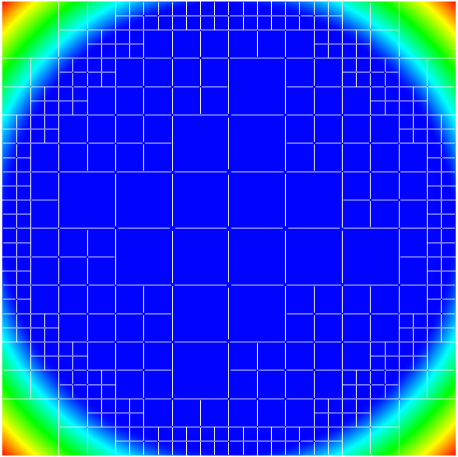

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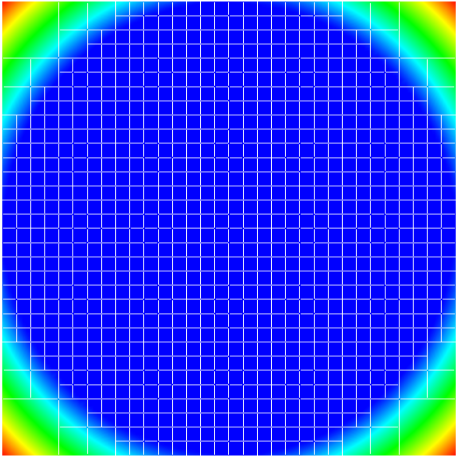

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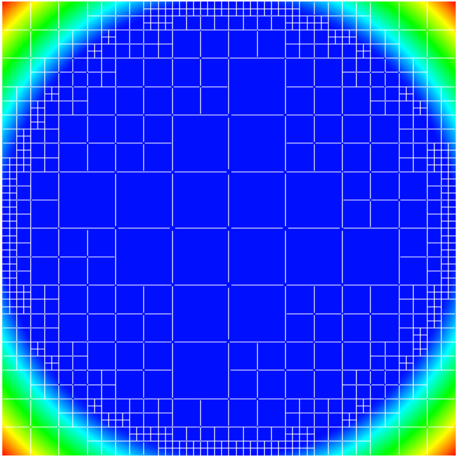

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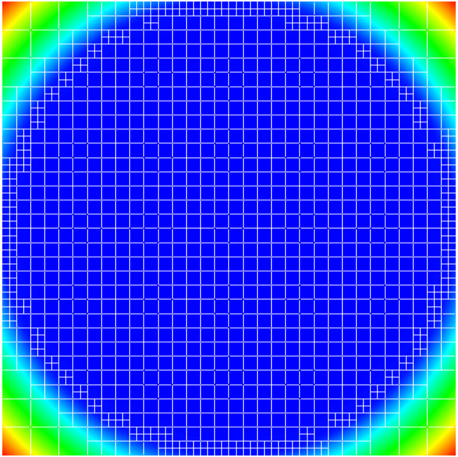
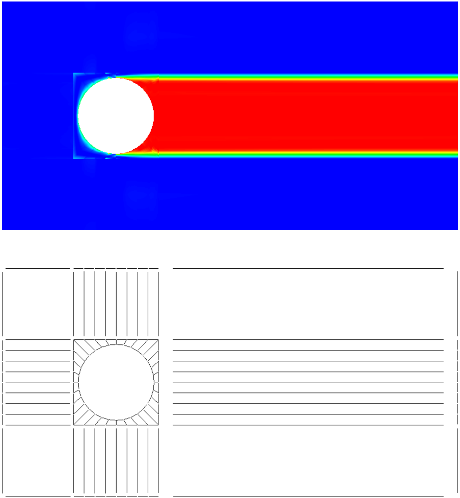


Without Local Conservation

Hemker Problem



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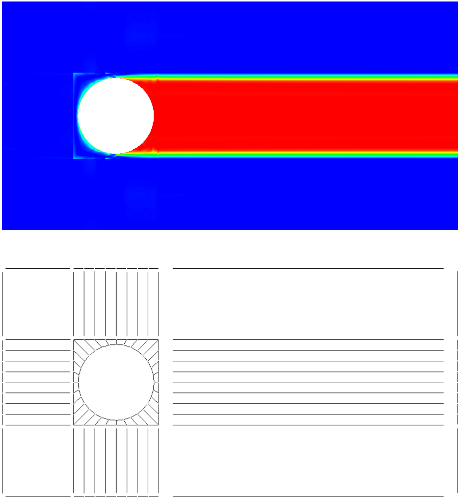

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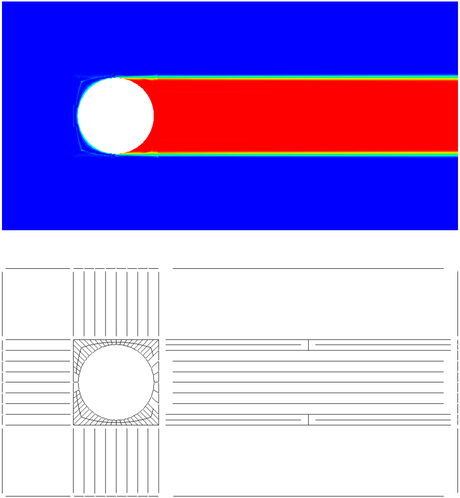

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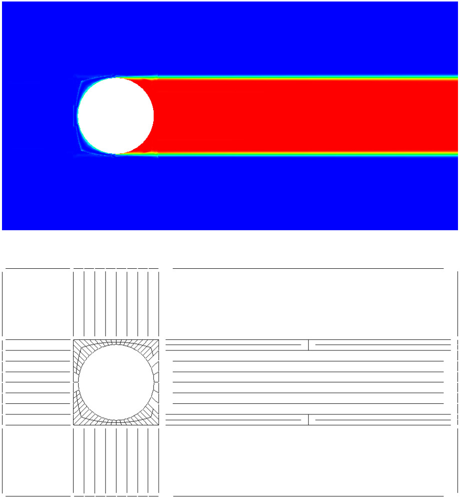

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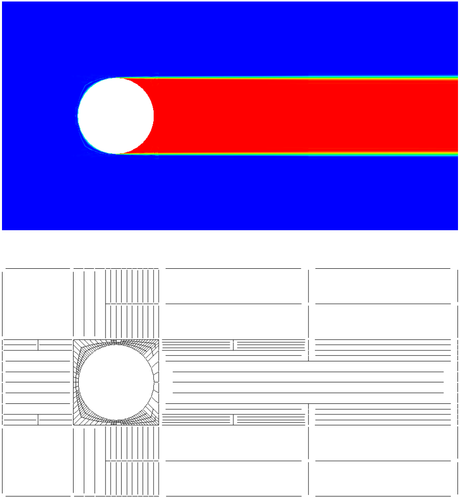

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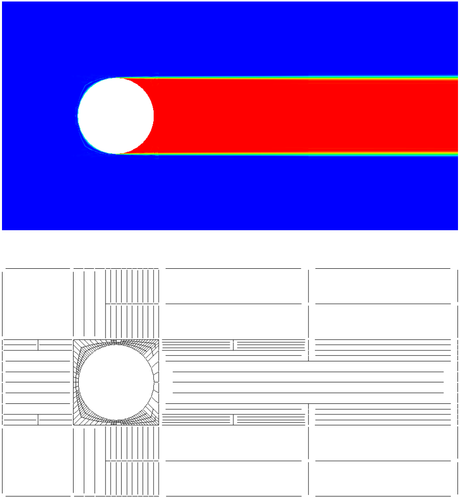

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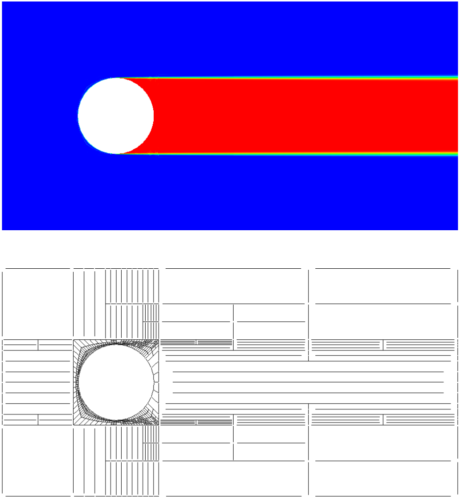

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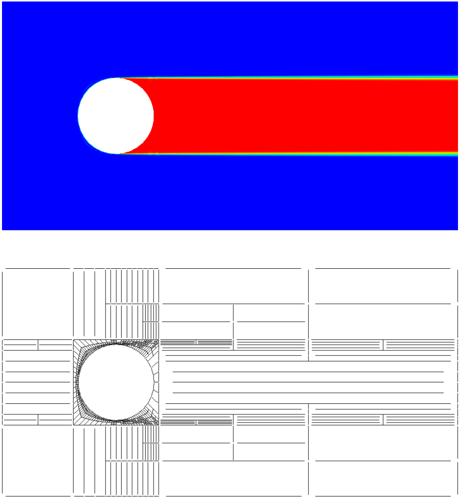

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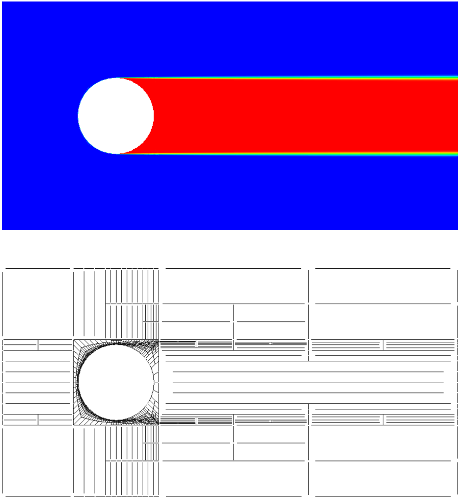

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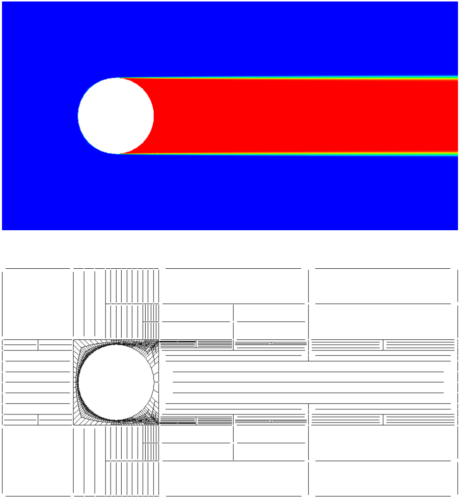
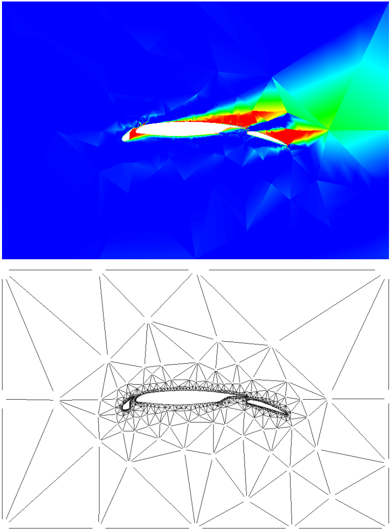


Without Local Conservation

High Lift Airfoil



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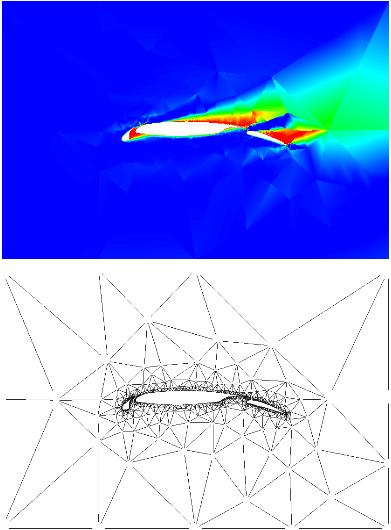

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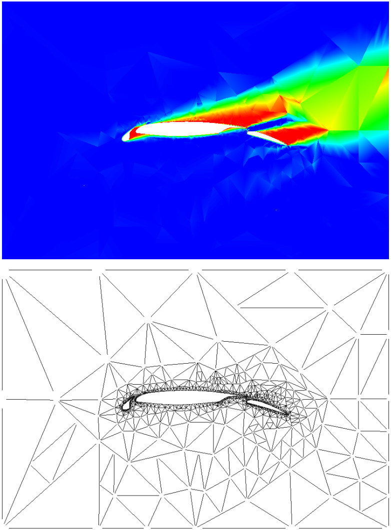

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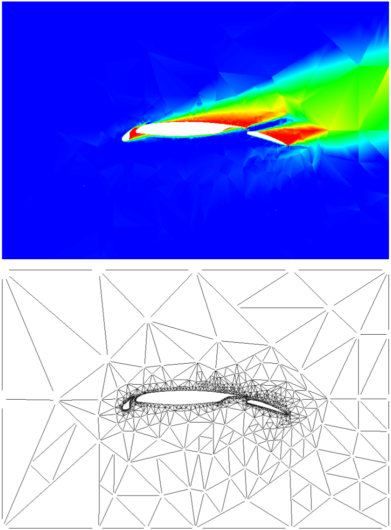

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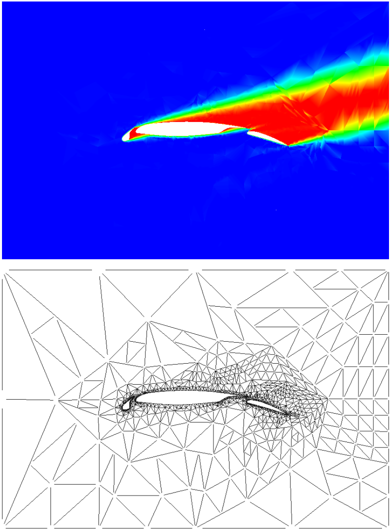

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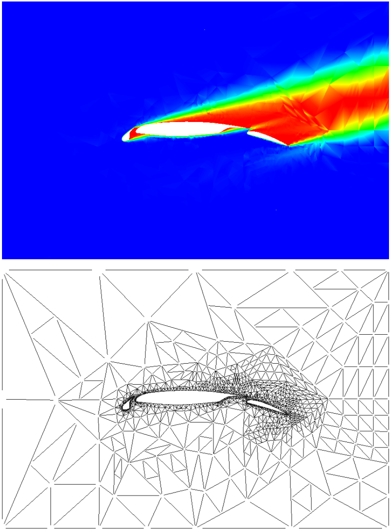

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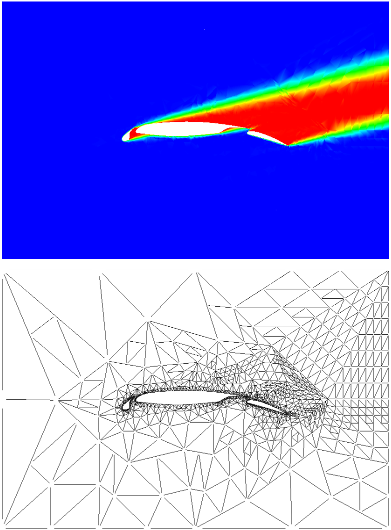

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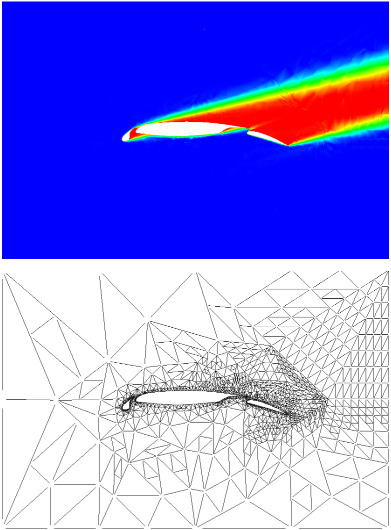

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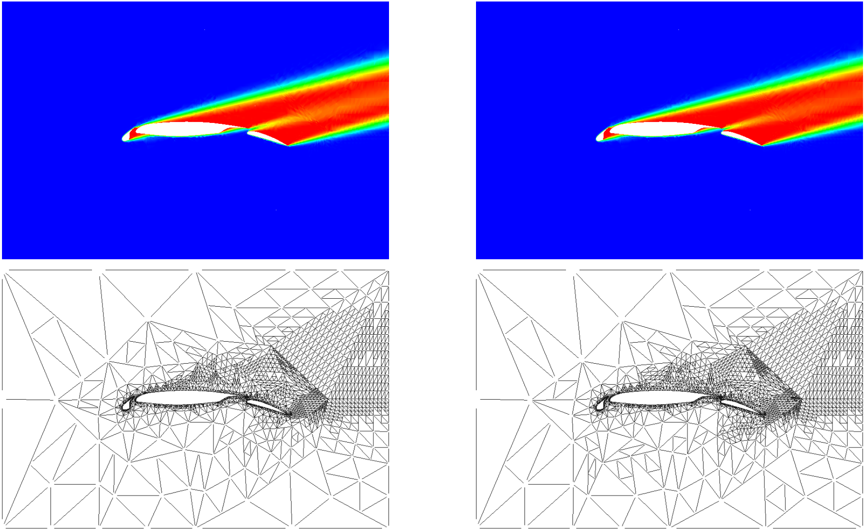

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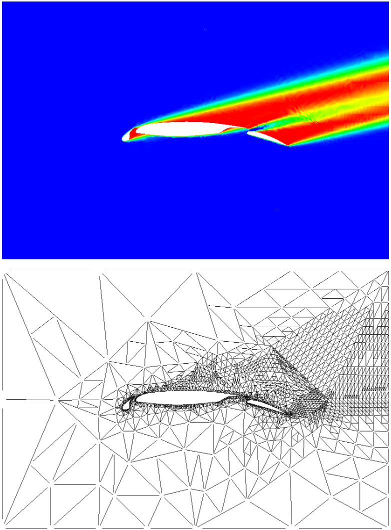

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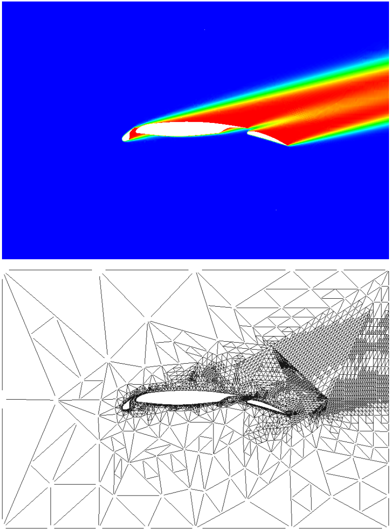

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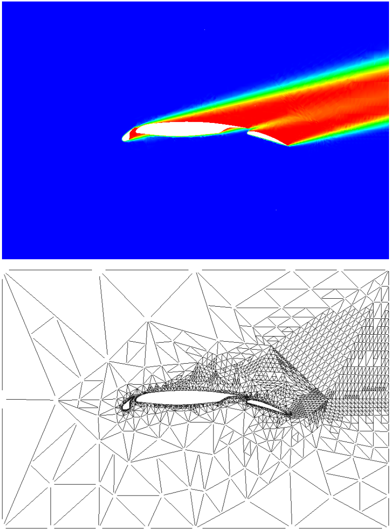

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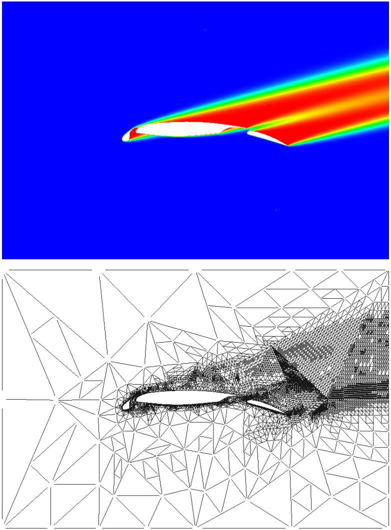
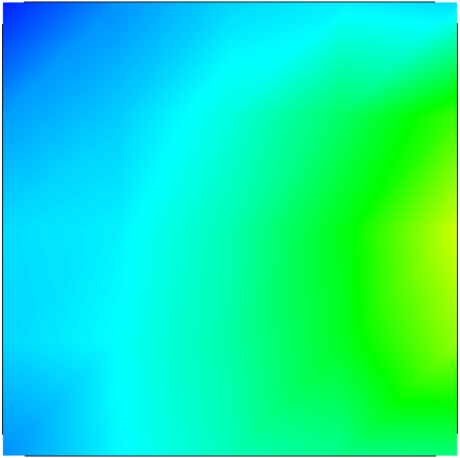


Without Local Conservation

Double Glazing Problem



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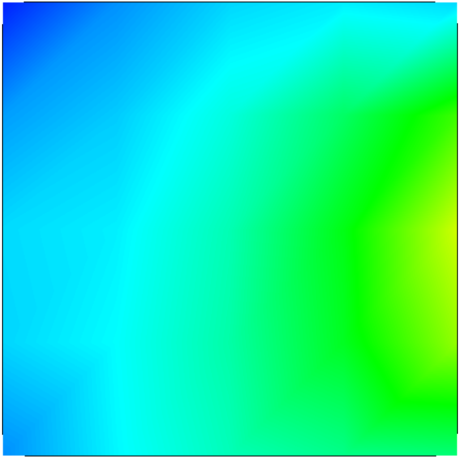

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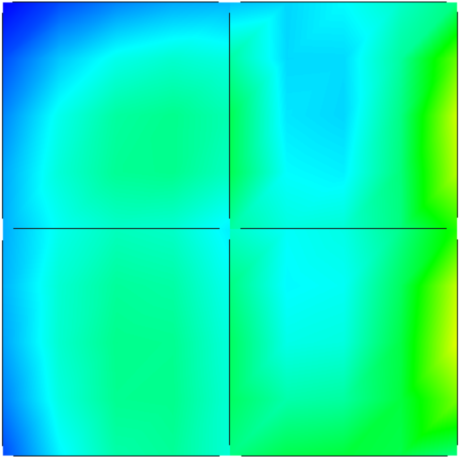

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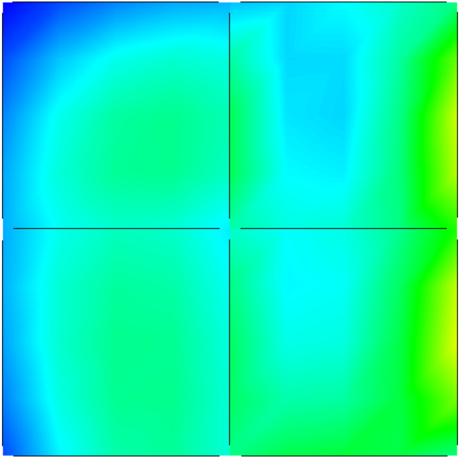

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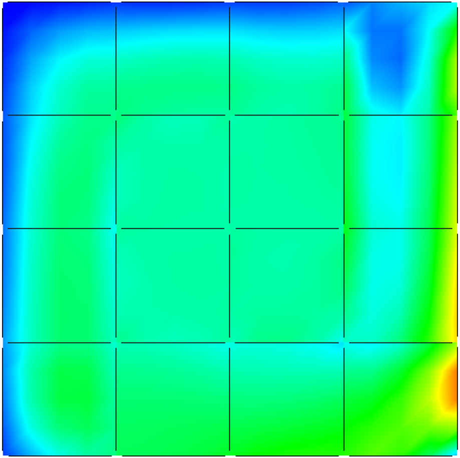

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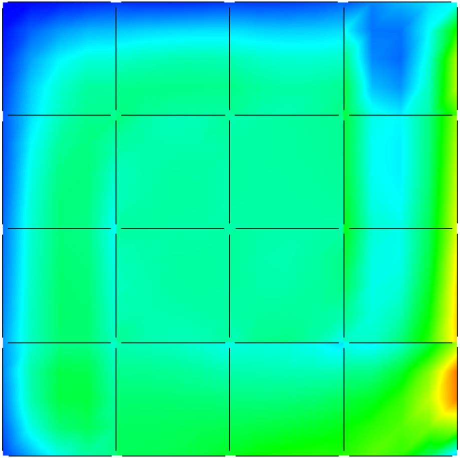

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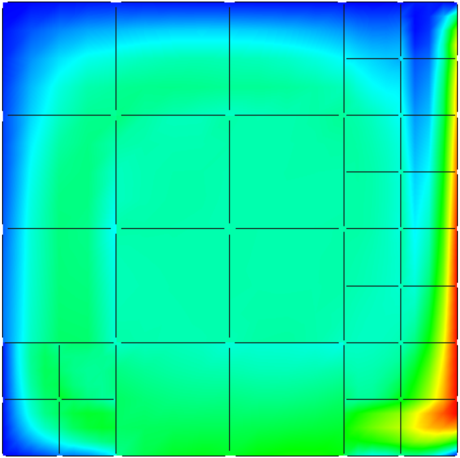

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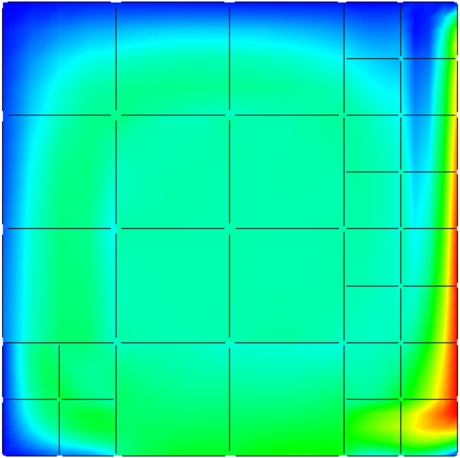

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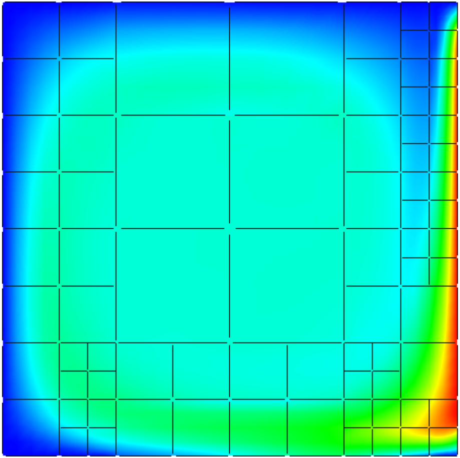

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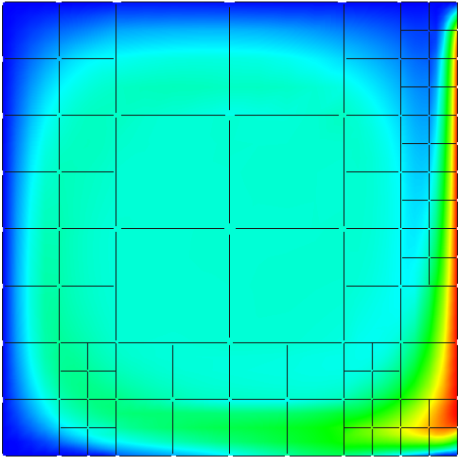

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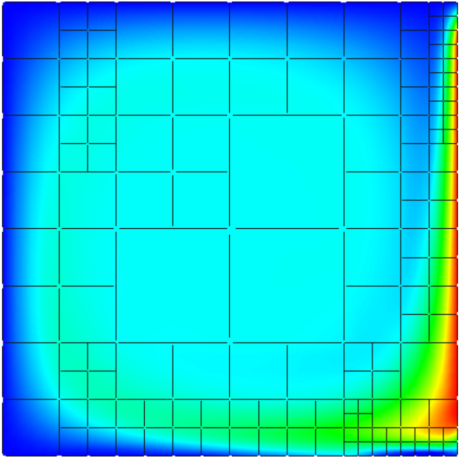

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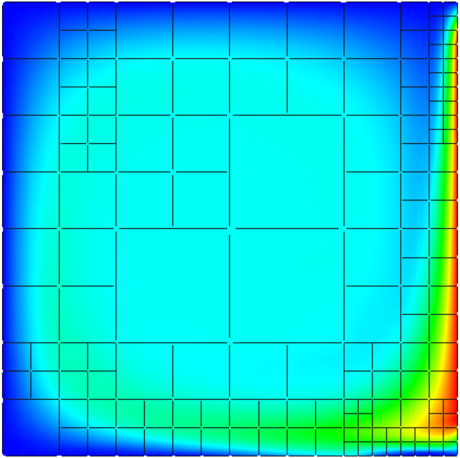

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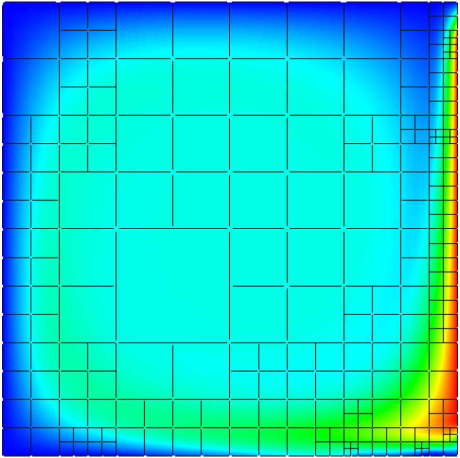

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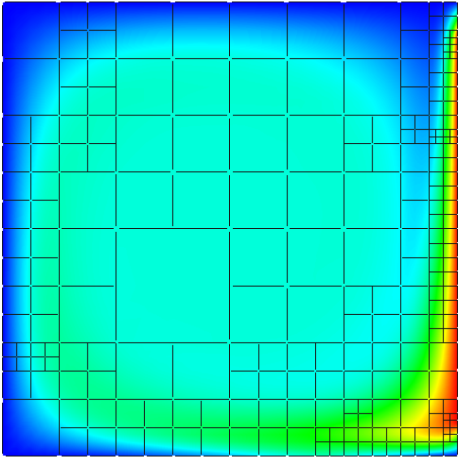

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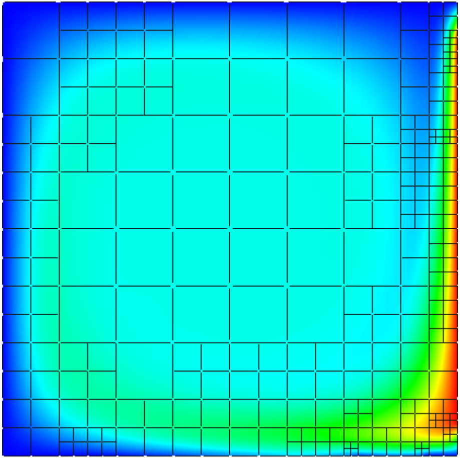

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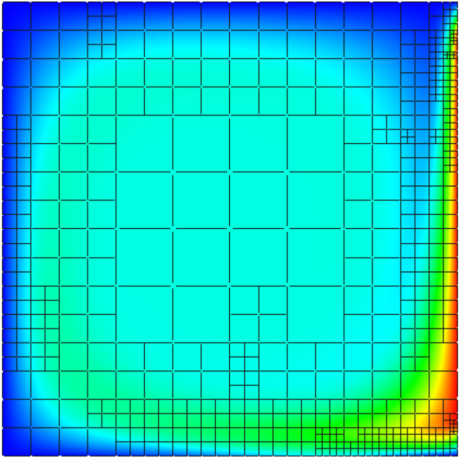

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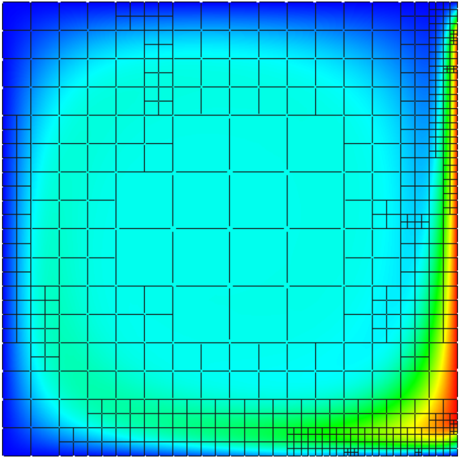

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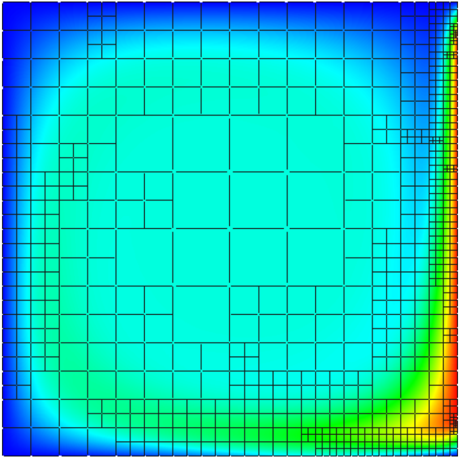
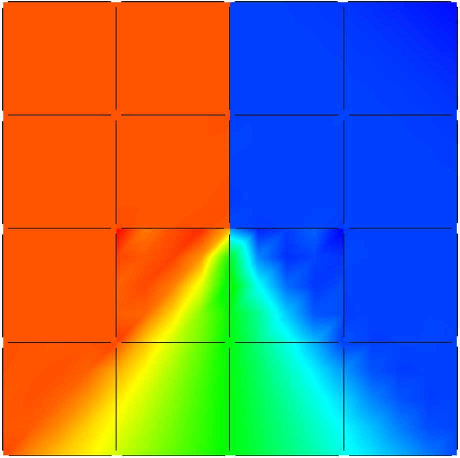


Without Local Conservation

Inviscid Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \Leftrightarrow \quad \nabla_{x,t} \cdot \begin{pmatrix} \frac{u^2}{2} \\ u \end{pmatrix} = 0$$



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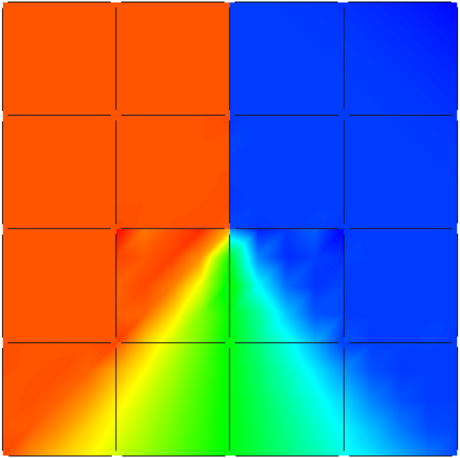

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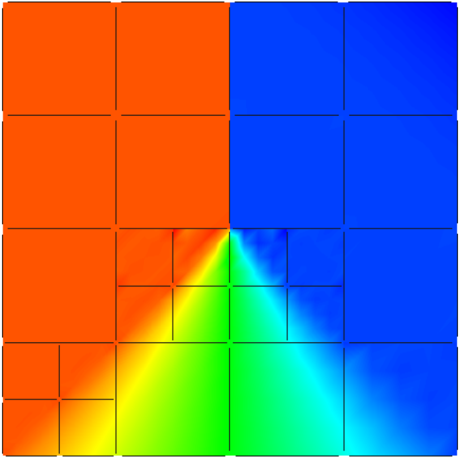
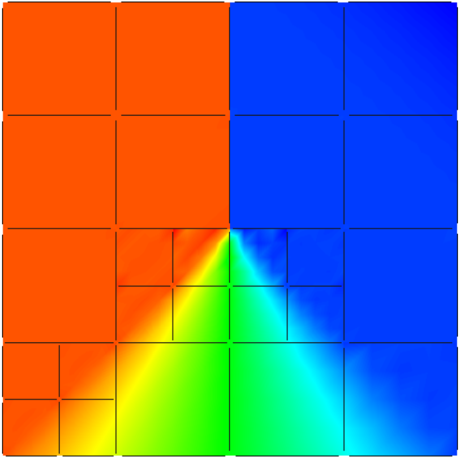


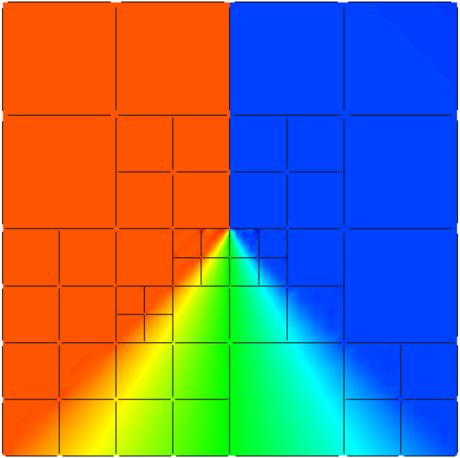
With Local Conservation



Without Local Conservation

Inviscid Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \Leftrightarrow \quad \nabla_{x,t} \cdot \begin{pmatrix} \frac{u^2}{2} \\ u \end{pmatrix} = 0$$



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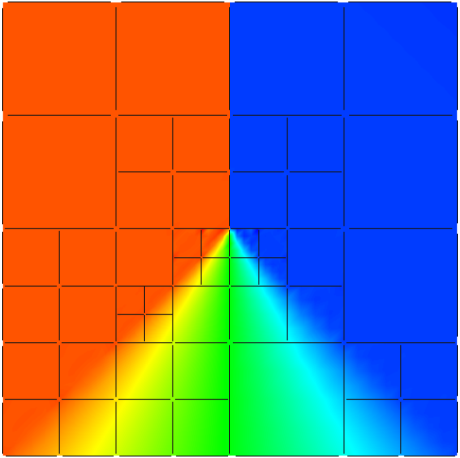

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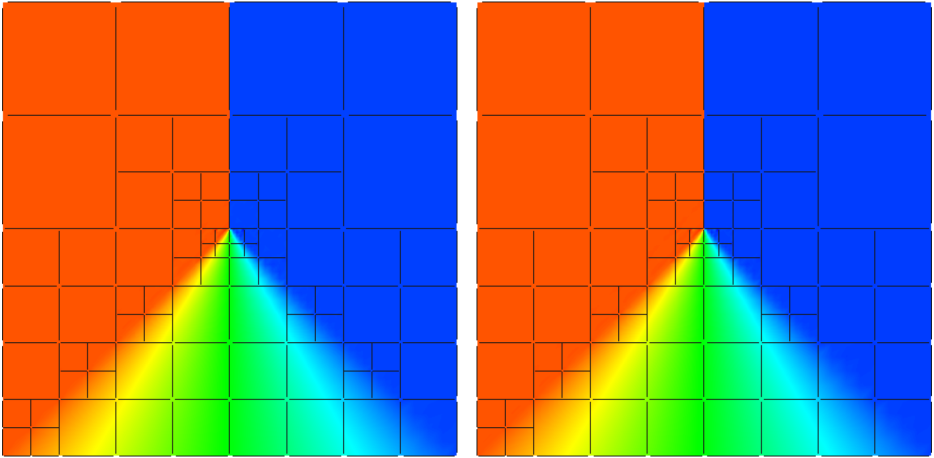

With Local Conservation | Without Local Conservation

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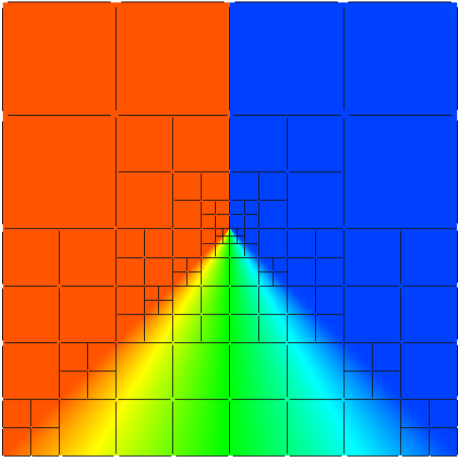

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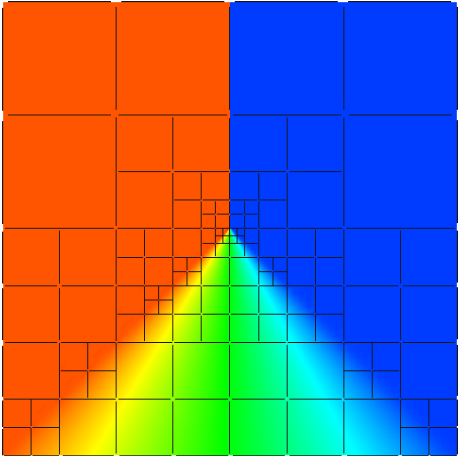

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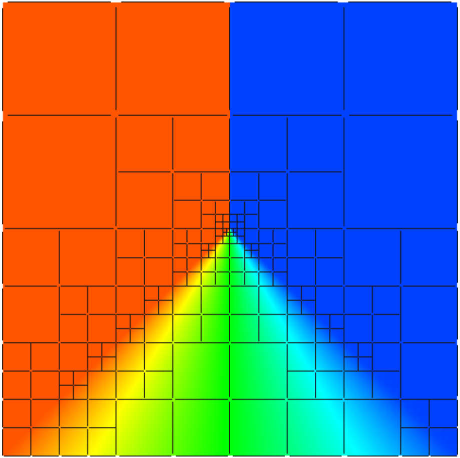

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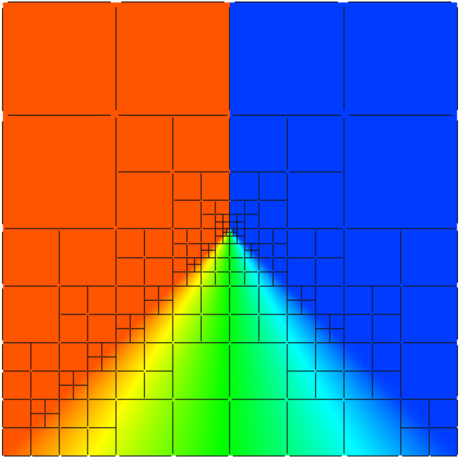

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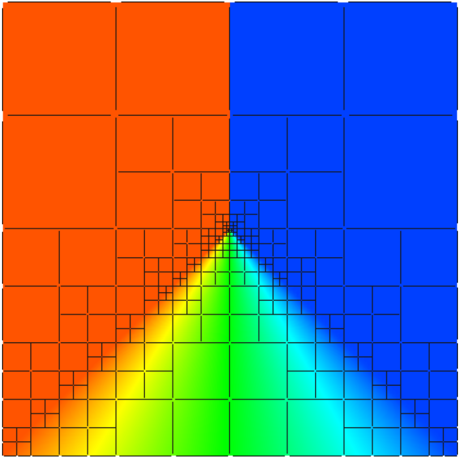

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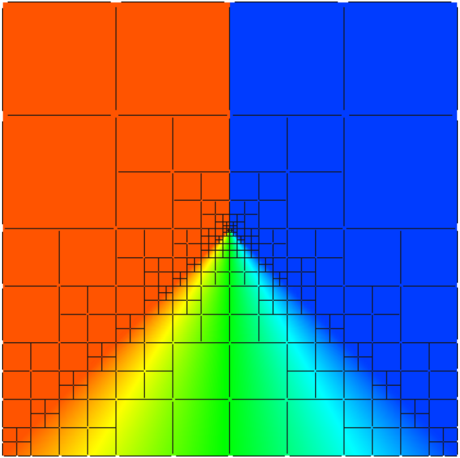

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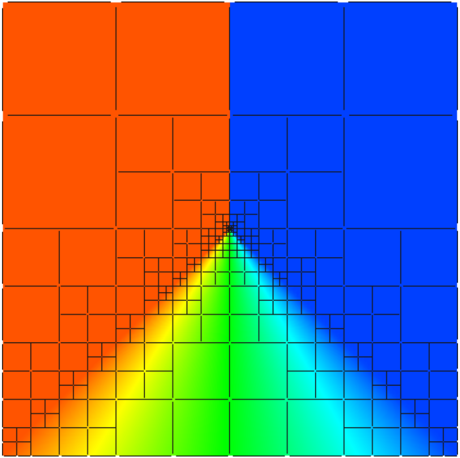

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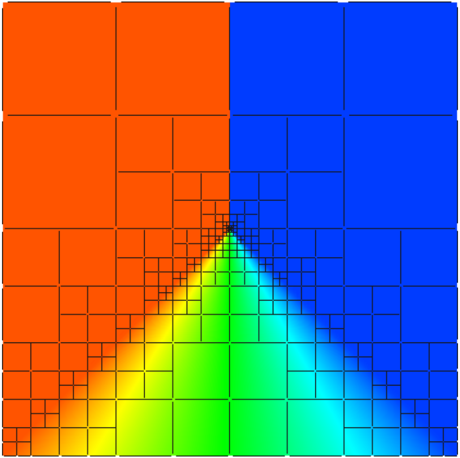

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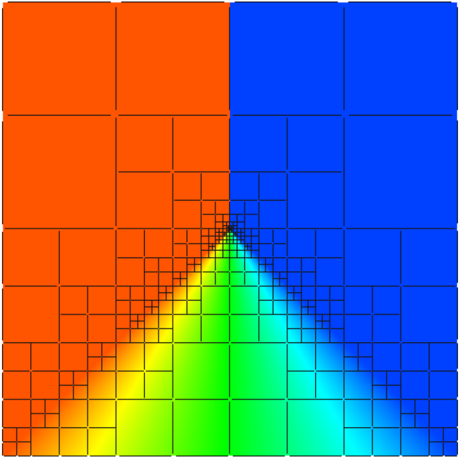

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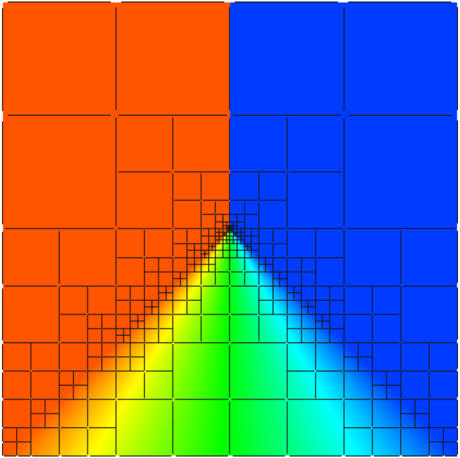

Without Local Conservation

Summary

What have we done?

- We've turned our minimization problem into a saddlepoint problem.
- The change is computationally feasible.
- Mathematically, it gets rid of troublesome term.

Does it make a difference?

- Enforcement changes refinement strategy.
- Standard DPG is nearly conservative in practice.

	Local Flux Imbalance		Global Flux Imbalance	
	Before	After	Before	After
Vortex	10^{-8}	10^{-15}	10^{-5}	10^{-16}
Hemker	10^{-6}	10^{-14}	10^{-4}	10^{-14}
High Lift Airfoil	10^{-7}	10^{-14}	10^{-5}	10^{-14}
Double Glazing	10^{-8}	10^{-15}	10^{-9}	10^{-15}
Inviscid Burgers	10^{-7}	10^{-17}	10^{-6}	10^{-16}

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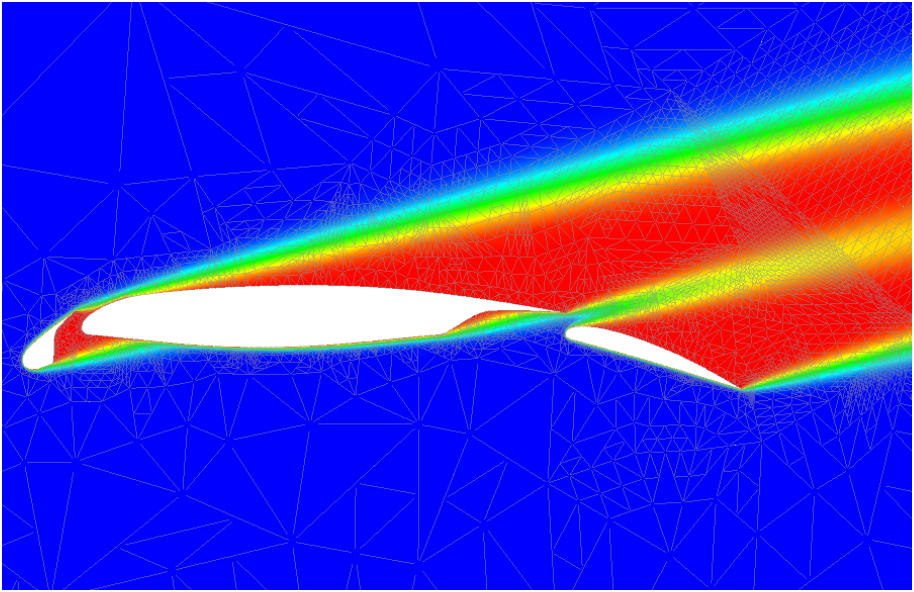
We need to study the effect on real fluid dynamics.

Questions?



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