1 Introduction

1.1 Introduction to DPG

The idea of optimal test functions was introduced by Demkowicz and Gopalakrishnan in [2]. Conceptually, these optimal test functions are the result of a minimum residual method applied to the operator form of a variational equation. Given Hilbert spaces U and V and a the variational problem $b(u,v) = l(v)$, $\forall u \in U, v \in V$, we can identify $B : U \to V'$ and $l \in V'$

$$\left. \begin{array}{c} b(u_h, v) = \langle Bu_h, v \rangle \\ l(v) = \langle l, v \rangle \end{array} \right\} \iff Bu_h = l$$

We seek the minimization of the residual in the dual norm

$$\min_{u_h \in U_h} J(u_h) = \frac{1}{2} \|Bu_h - l\|_{V'}^2 = \frac{1}{2} \sup_{v \in V} \|b(u_h, v) - l(v)\|_V^2$$

Recall R_V, the Riesz operator $\langle R_V v, \delta v \rangle = (v, \delta v)_V$ identifying elements of Hilbert space with elements of the dual. As R_V and its inverse are isometries, $\|f\|_{V'} = \|R_V^{-1} f\|_V$, and

$$\min_{u_h \in U_h} J(u_h) = \frac{1}{2} \|Bu_h - l\|_{V'}^2 = \frac{1}{2} \|R_V^{-1}(Bu_h - l)\|_V^2$$

First order optimality conditions require the Gateux derivative to be zero in all directions $u_h \in U_h$.

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1} B \delta u_h \right)_V = 0, \quad \forall \delta u_h \in U_h$$
$$\to \langle Bu_h - l, v_h \rangle = 0, \quad v_h = R_V^{-1} B \delta u_h$$

which returns a standard variational equation $b(u_h, v_h) - l(v_h) = 0$, for the specific choice of test functions $v_h = R_V^{-1} B \delta u_h$. We identify the *trial-to-test* operator $T = R_V^{-1} B$, which maps a trial function u_h to its corresponding *optimal* test function $v_h = T u_h$. These test functions can be solved for by solving the auxiliary variational problem

$$(v_h, \delta v)_V = b(u_h, v)$$

Finally, while the above variational problem is a global operation for standard continuous test spaces, the use of discontinuous test spaces reduces it to an element-local problem. In practice, this variational problem can rarely be solved exactly and it is solved approximately using the standard Bubnov-Galerkin method and an "enriched" subspace of V. We assume the corresponding error in approximation of the optimal test functions is negligible for the scope of this paper. Further work concerning the effect of approximation error in the computation of optimal test functions can be found in [4].

Under the standard conditions for well-posedness of the continuous variational problem, the discrete DPG method delivers the best approximation error in the "energy norm" $\|u\|_E = \sup \frac{b(u,v)}{\|v\|_V}$. Additionally, the actual energy error $\|u - u_h\|_E$ is computable through $\|e_h\|_V$, where

$$(e_h, \delta v)_V = b(u - u_h, v) = b(u_h, v) - l(v)$$

This is simply a consequence of the least-squares nature of DPG; the energy error is simply the measure of the residual in the proper norm.

1.2 Variational form

The DPG (discontinuous Petrov-Galerkin) method is the combination of the concept of computable optimal test functions with the so-called "ultra-weak formulation", where every equation is relaxed. Moreover, to maintain conformity while seeking an L^2 setting on interior "field" variables, boundary terms are identified as additional new unknowns. The result is a hybrid DG method with locally supported optimal test functions. For a given operator equation $Au = f$, the ultra-weak formulation is

$$b\left((\widehat{u}, u), v\right) - l(v) = \langle \widehat{u}, v \rangle - (u, A^* v) - (f, v)$$

2 Choice of test norm

Up to now, we've neglected discussion of the proper choice of inner product/norm on the space V. This choice is important, as the choice of norm on V determines the

Under the assumption of injectivity of the adjoint of our variational operator B, we can define the optimal test norm

$$\|v\|_{\text{opt}} = \sup_{u \in U} \frac{b(u,v)}{\|u\|_U}$$

If the solution is in U, then choosing $\|v\| = \|v\|_{\text{opt}}$ implies

$$\|u\|_E = \sup_{v \in V} \frac{b(u,v)}{\|v\|_V} = \sup_{v \in V} \sup_{w \in V} \frac{b(u,v)}{b(w,v)} \|w\|_U = \sup_{w \in V} \frac{b(u,v^*)}{b(w,v^*)} \|w\|_U = \|u\|_U$$

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This norm is non-localizable, however - even with discontinuous test functions, solving for optimal test functions under this optimal norm will result in global problems as well, due to the coupling given by the boundary terms in our bilinear form. The "quasi-optimal" test norm is created by removing such boundary terms and replacing them with scaled L^2 norms of the field variables.

2.1 Singular perturbation problems and robustness

Standard Bubnov-Galerkin methods tend to perform poorly for the class of partial-differential equations known as singularly perturbed problems, where a given parameter ϵ may approach either 0 or ∞ in the context of physical problems. This poor performance is captured by the error bound

$$\|u - u_h\|_E \leq C(\epsilon) \inf_{w_h} \|u - w_h\|_U$$

The growth of C with ϵ is referred to as a loss of robustness, where the bound on the error in the finite element solution by the best approximation error. Intuitively, as our singular perturbation parameter changes, our finite element error is bounded more and more loosely to the best approximation error, allowing for degradation of the solution

A well-known example is the growth of spurious oscillations in the finite element solution of the 1D convection-diffusion problem $u' - \epsilon u'' = f$ (with Dirichlet boundary conditions) as $\epsilon \to 0$. Another class of examples are wave propagation problems, in which the singular perturbation parameter is the wavenumber, $k \to \infty$. The lack of robustness in this case manifests as "pollution" error, a phenomenon where the finite element solution degrades over many wavelengths (commonly manifesting as a phase error between the FE solution and the exact solution). The DPG method is currently being analyzed for both acoustic and elastic waves in context of the Helmholtz equation [7] and equations of linear elasticity. Numerically, DPG appears to yield a "pollution-free" method for these problems under a specific regularization of the "quasi-optimal" test norm.

3 Model problem and robustness using DPG

We consider the model convection-diffusion problem on domain Ω with boundary Γ

$$\nabla \cdot (\beta u - \sigma) = f$$
$$\frac{1}{\epsilon}\sigma - \nabla u = 0$$

with the corresponding variational form

$$b\left((u,\sigma,\widehat{u},\widehat{\sigma}_n),(\tau,v)\right) = (u, \nabla \cdot \tau - \beta \cdot \nabla v) + \left(\sigma, \epsilon^{-1}\tau + \nabla v\right) - \langle [\tau_n], \widehat{u} \rangle + \langle \widehat{f}_n, [v] \rangle$$

where the above inner products are taken element-wise over the finite element mesh Ω_h, and the duality pairings are taken on Γ_h, the mesh "skeleton", or union of edges of elements K in Ω_h. The functional setting is now well understood as well (see [1] for details) - $u, \sigma \in L^2(\Omega)$, $v \in H^1(\Omega_h)$, $\tau \in H(\mathrm{div}, \Omega_h)$, where $H^1(\Omega_h)$ and $H(\mathrm{div}, \Omega_h)$ are element-wise "broken" Sobolev spaces. By duality, $\widehat{u}$ lives in the trace space of $H^1(\Omega_h)$, while $\widehat{f}_n$ comes from the normal trace space of $H(\mathrm{div}, \Omega_h)$.

To analyze the robustness of DPG, we note that

$$b\left((u, \sigma, \widehat{u}, \widehat{\sigma}_n), (\tau, v)\right) = \|u\|_{L^2}^2$$

for the specific continuous test functions τ and v satisfying

$$\nabla \cdot \tau - \beta \cdot \nabla v = g$$
$$\frac{1}{\epsilon}\tau + \nabla v = f$$

with boundary conditions such that the boundary terms in the bilinear form vanish. Then, for the above choice of (τ, v), with $g = u$ and $f = 0$,

$$\|u\|_{L^2}^2 = b\left((u, \sigma, \widehat{u}, \widehat{\sigma}_n), (\tau, v)\right) = \frac{b\left((u, \sigma, \widehat{u}, \widehat{\sigma}_n), (\tau, v)\right)}{\|(\tau, v)\|_V}\|(\tau, v)\|_V \leq \|u\|_E\|(\tau, v)\|_V$$

We will use the notation $\lesssim$ to denote an ϵ-independent bound. If we can show $\|(\tau, v)\|_V \lesssim \|u\|_{L^2}$ for any choice of $u \in L^2$, then dividing through by $\|u\|_{L^2}$ gives

$$\|u\|_{L^2} \lesssim \|u\|_E$$

which results in in robust control of the L^2 error by the error in the energy norm, in which DPG is optimal. The study of robustness for the DPG method has now been reduced to proving stability estimates for the adjoint problem.

For convection-diffusion, the quasi-optimal norm is

$$\|(\tau, v)\|_V^2 = \|\nabla \cdot \tau - \beta \cdot \nabla v\|_{L^2}^2 + \|\epsilon^{-1}\tau + \nabla v\|_{L^2}^2 + C_1\|v\|_{L^2}^2 + C_2\|\tau\|_{L^2}^2$$

for some choice of constants C_1 and C_2. We note that the optimal test norm will automatically guarantee the robust bound $\|(\tau, v)\|_V \leq \|u\|_{L^2}$. However, use of this norm for the convection-diffusion problem is difficult, as the quasi-optimal test norm will induce strong boundary layers of width ϵ/h^2 (in comparison, in wave propagation problems, the mesh size tends to be on the order of the wavelength/singular perturbation parameter, resulting in optimal test functions that are much easier to approximate over an element). As the relevant range of ϵ for physical problems is $1e-7$, solving on (at least) partially under-resolved meshes is unavoidable. Resolving such boundary layers for an underresolved mesh has been investigated numerically using specially designed subgrid meshes by Niemi, Collier, and Calo in [6]. We approach this from a different perspective, looking instead for a robust test norm which does not induce boundary layers in the approximation of optimal test functions.

3.1 Boundary conditions

We split the boundary Γ into three portions

$$\Gamma_- := \{x \in \Gamma; \beta_n(x) < 0\} \quad \text{(inflow)}$$
$$\Gamma_+ := \{x \in \Gamma; \beta_n(x) > 0\} \quad \text{(outflow)}$$
$$\Gamma_0 := \{x \in \Gamma; \beta_n(x) = 0\}$$

Demkowicz and Heuer proved in [3] that for Dirichlet boundary conditions everywhere on Γ, robustness as $\epsilon \to 0$ is achieved by the test norm

$$\| (\tau, v) \|_{V,w}^2 = \|v\| + \epsilon\|\nabla v\| + \|\beta \cdot \nabla v\|_{w+\epsilon} + \|\nabla \cdot \tau\|_{w+\epsilon} + \frac{1}{\epsilon}\|\tau\|_{w+\epsilon}$$

where $\| \cdot \|_{w+\epsilon}$ is a weighted L^2 norm, where the weight $w \in (0,1)$ is required to vanish on Γ_-. The need for the weight is intuitively explained by the induced adjoint problem. The bilinear form for the Dirichlet case is

$$b\left((u, \sigma, \widehat{u}, \widehat{\sigma}_n), (\tau, v)\right) = (u, \nabla \cdot \tau - \beta \cdot \nabla v) + \left(\sigma, \epsilon^{-1}\tau + \nabla v\right) + \langle \widehat{f}_n, [v] \rangle$$

Choosing a continuous conforming test function v removes the contribution of the term $\langle \widehat{f}_n, [v] \rangle$ on the mesh skeleton; however, it is necessary to assume in addition $v = 0$ on Γ to have $b\left((u, \sigma, \widehat{u}, \widehat{\sigma}_n), (\tau, v)\right) = \|u\|_{L^2}^2$. Intuitively, the adjoint problem is similar to the primal problem with the direction of inflow reversed, such that the inflow becomes the outflow, and outflow inflow. The need for the weight arises due to the presence of the Dirichlet boundary condition on v near the inflow; this induces strong boundary layers at the inflow such that $\nabla v \approx O(\epsilon^{-1})$. It may be of interest to note that, for sufficiently small ϵ, these issues manifest themselves in numerical experiments as additional refinements near the inflow.

We improve upon the result of Demkowicz and Heuer by adopting a new inflow boundary condition, given by Hesthaven et al in [5], where we set

$$\beta_n u - \sigma_n = f_n = u_0$$

on Γ_-, and continue with the wall boundary condition $u = 0$ on Γ_+. For our model problem, as for most problems of interest in CFD, we expect ∇u to be small near the inflow, and that the solution using $u = u_0$ and $\beta_n u - \sigma_n = f_n = u_0$ should converge to each other for sufficiently small ϵ.

Under the new inflow conditions, the induced bilinear form is now

$$b\left((u, \sigma, \widehat{u}, \widehat{\sigma}_n), (\tau, v)\right) = (u, \nabla \cdot \tau - \beta \cdot \nabla v) + \left(\sigma, \epsilon^{-1}\tau + \nabla v\right) - \langle [\tau_n], \widehat{u} \rangle_{\Gamma_-} + \langle \widehat{f}_n, [v] \rangle_{\Gamma_+}$$

The corresponding adjoint problem now has boundary conditions

$$\tau_n = 0, \quad x \in \Gamma_- \tag{1}$$

$$v = 0, \quad x \in \Gamma_+ \tag{2}$$

and it can be shown that DPG is robust using an unweighted version of the test norm in [3]

$$\| (\tau, v) \|_V^2 = \|v\| + \epsilon\|\nabla v\| + \|\beta \cdot \nabla v\| + \|\nabla \cdot \tau\| + \frac{1}{\epsilon}\|\tau\|$$

3.2 Stability of the adjoint problem (bound from below)

We reduce the adjoint problem to the scalar second order equation

$$-\epsilon\Delta v - \beta \cdot \nabla v = g - \epsilon\nabla \cdot f \tag{3}$$

with boundary conditions

$$\tau_n = -\epsilon\nabla v \cdot n = f \cdot n, \quad x \in \Gamma_-$$
$$v = 0, \quad x \in \Gamma_+$$

and treat the cases $f = 0$, $g = 0$ separately. The above boundary conditions reduce down to (1) and (2) when requesting L^2 robustness in u ($f = 0$); these more general boundary conditions are necessary for bounds involving $f \neq 0$.

Additionally, the $\nabla\cdot$ operator is understood now in the weak sense, as the dual operator of $-\nabla : H_0^1(\Omega) \to L^2(\Omega)$, such that $\nabla \cdot f \in \left(H_0^1(\Omega)\right)'$. $f \cdot n$ is understood as a limit; smooth functions are dense in $H^1(\Omega)$, and we define $f \cdot n$ as the limit of $f_i \cdot n$, where $f_i \in C^\infty(\Omega)$, $f_i \to f$.

For this analysis, it will be necessary to assume conditions on β. For each proof, we require $\beta \in C^2(\bar{\Omega})$ and $\beta, \nabla \cdot \beta = O(1)$. Additionally, we will assume one or both of the following assumptions

$$\nabla \times \beta = 0, \quad 0 < C \le |\beta|^2 + \frac{1}{2}\nabla \cdot \beta, \quad C = O(1) \tag{4}$$

$$\nabla\beta + \nabla\beta^T - \nabla \cdot \beta I = O(1) \tag{5}$$

Lemma 1. *Assuming* (4) *and* (3) *hold, for sufficiently small* ϵ,

$$\epsilon\|\nabla v\|^2 + \|v\|^2 \lesssim \|g\| + \epsilon\|f\|$$

Proof. Since $\nabla \times \beta = 0$, and Ω is simply connected, there exists a scalar potential ψ, $\nabla\psi = \beta$ such that $e^\psi = O(1)$. Take the transformed function $w = e^\psi v$; following (2.26) in [3], we substitute w into the the left hand side of equation (3), arriving at the relation

$$-\epsilon\Delta w - (1 - 2\epsilon)\beta \cdot \nabla w + \left((1 - \epsilon)|\beta|^2 + \epsilon\nabla \cdot \beta\right)w = e^\psi(g - \epsilon\nabla \cdot f)$$

Multiplying by w and integrating over Ω gives

$$-\epsilon\int_\Omega \Delta w w - (1 - 2\epsilon)\int_\Omega \beta \cdot \nabla w w + \int_\Omega \left((1 - \epsilon)|\beta|^2 + \epsilon\nabla \cdot \beta\right)w^2 = \int_\Omega e^\psi(g - \epsilon\nabla \cdot f)w$$

Integrating by parts gives

$$-\epsilon\int_\Omega \Delta w w - (1-2\epsilon)\int_\Omega \beta\cdot\nabla w w = \epsilon\left(\int_\Omega |\nabla w|^2 - \int_\Gamma w\nabla w \cdot n\right) + \frac{(1 - 2\epsilon)}{2}\left(\int_\Omega \nabla \cdot \beta w^2 - \int_\Gamma \beta_n w^2\right)$$

Noting that $w = 0$ on Γ_+ reduces the boundary integrals over Γ to just the inflow Γ_-. Furthermore, we have the relation $\nabla w = e^\psi(\nabla v + \beta v)$. Applying the above and boundary conditions on Γ_-, the first boundary integral becomes

$$\int_{\Gamma_-} w\nabla w \cdot n = \int_{\Gamma_-} we^\psi(\nabla v + \beta v) \cdot n = \int_{\Gamma_-} we^\psi(f \cdot n + \beta_n v)$$

Noting $\int_{\Gamma_-} \beta_n w^2 \le 0$ through $\beta_n < 0$ on the inflow gives

$$\epsilon\int_\Omega |\nabla w|^2 + \int_\Omega \left((1 - \epsilon)|\beta|^2 + \frac{1}{2}\nabla \cdot \beta\right)w^2 - \epsilon\int_{\Gamma_-} we^\psi f \cdot n \le \int_\Omega e^\psi(g - \epsilon\nabla \cdot f)w$$

assuming ϵ is sufficiently small. Our assumptions on β allow us to bound $\left((1 - \epsilon)|\beta|^2 + \frac{1}{2}\nabla \cdot \beta\right) \lesssim 1$ and $e^\psi \lesssim 1$. We can then bound the left hand side from below by

$$\epsilon\|\nabla w\|^2 + \|w\|^2 - \epsilon\int_{\Gamma_-} we^\psi f \cdot n \lesssim \epsilon\int_\Omega |\nabla w|^2 + \int_\Omega \left((1 - \epsilon)|\beta|^2 + \frac{1}{2}\nabla \cdot \beta\right)w^2 - \epsilon\int_{\Gamma_-} we^\psi f \cdot n$$

Interpreting $\nabla \cdot f$ as a functional, the right hand gives

$$\int_\Omega e^\psi(g - \epsilon\nabla \cdot f)w = \int_\Omega e^\psi g + \int_\Omega \epsilon f \cdot \nabla(e^\psi w) - \int_\Gamma \epsilon f \cdot ne^\psi w$$

The boundary integral on Γ reduces to Γ_-, which is then nullified by the same term on the left hand side, leaving us with

$$\epsilon\|\nabla w\|^2 + \|w\|^2 \lesssim \int_\Omega e^\psi g + \int_\Omega \epsilon f \cdot \nabla(e^\psi w) = \int_\Omega e^\psi g + \int_\Omega \epsilon f \cdot (\beta w + \nabla w)$$

From here, the proof is identical to [3]; an application of Peter-Paul Young's inequality to the right hand side and bounds on $\|v\|, \|\nabla v\|$ by $\|w\|, \|\nabla w\|$ complete the estimate. $\qquad\square$

Lemma 2. *Assume v satisfies (3) and β satisfies (4) and (5). If $\nabla \cdot f = 0$ and ϵ is sufficiently small,*

$$\|\beta \cdot \nabla v\| \lesssim \|f\|$$

Proof. Define $v_\beta = \beta \cdot \nabla v$. Multiplying by v_β and integrating over Ω gives

$$\|v_\beta\|^2 = -\int_\Omega g v_\beta - \epsilon \int_\Omega \Delta v v_\beta$$

Note that

$$-\int_\Omega \beta \cdot \nabla v \Delta v = -\int_\Omega \beta \cdot \nabla v \nabla \cdot \nabla v$$

Integrating this by parts, we get

$$-\int_\Omega \beta \cdot \nabla v \nabla \cdot \nabla v = \int_\Omega \nabla(\beta \cdot \nabla v) \cdot \nabla v - \int_\Gamma n \cdot \nabla v \beta \cdot \nabla v$$

Note that

$$\nabla(\beta \cdot \nabla v) = \nabla \beta \cdot \nabla v + \beta \cdot \nabla \nabla v$$

where $\nabla \beta$ and $\nabla \nabla v$ are understood to be tensors. Then,

$$\int_\Omega \nabla(\beta \cdot \nabla v) \cdot \nabla v = \int_\Omega (\nabla \beta \cdot \nabla v) \cdot \nabla v + \int_\Omega \beta \cdot \nabla \nabla v \cdot \nabla v$$

Noting that $\nabla v \cdot \nabla \nabla v = \nabla \frac{1}{2}(\nabla v \cdot \nabla v)$, if we integrate by parts again, we get

$$-\int_\Omega \Delta v v_\beta = -\int_\Gamma n \cdot \nabla v \beta \cdot \nabla v + \frac{1}{2}\int_\Gamma \beta_n(\nabla v \cdot \nabla v) - \frac{1}{2}\int_\Omega \nabla \cdot \beta(\nabla v \cdot \nabla v) + \int_\Omega (\nabla \beta \cdot \nabla v) \cdot \nabla v$$

$$= -\int_\Gamma n \cdot \nabla v \beta \cdot \nabla v + \frac{1}{2}\int_\Gamma \beta_n(\nabla v \cdot \nabla v) + \int_\Omega \nabla v \left(\nabla \beta - \frac{1}{2}\nabla \cdot \beta I\right)\nabla v$$

Finally, substituting this into our adjoint equation multiplied by v_β, we get

$$\|v_\beta\|^2 = -\int_\Omega g\beta \cdot \nabla v + \epsilon \int_\Gamma \left(-n \cdot \nabla v \beta + \frac{1}{2}\beta_n \nabla v\right) \cdot \nabla v + \epsilon \int_\Omega \nabla v \left(\nabla \beta - \frac{1}{2}\nabla \cdot \beta I\right)\nabla v$$

The last term can be bounded by our assumption on $\|\nabla \beta - \frac{1}{2}\nabla \cdot \beta I\|^2 \leq C$.

$$\epsilon \int_\Omega \nabla v \left(\nabla \beta - \frac{1}{2}\nabla \cdot \beta I\right)\nabla v \leq C\frac{\epsilon}{2}\|\nabla v\|^2$$

For the boundary terms, on Γ_-, $\nabla v \cdot n = 0$, reducing the above to $\beta_n |\nabla v|^2 \leq 0$. On Γ_+, $v = 0$ implies $\nabla v \cdot \tau = 0$, where τ is a tangential direction. An orthogonal decomposition yields $\nabla v = (\nabla v \cdot n)n$, reducing the above to

$$\epsilon \int_\Gamma -\frac{1}{2}|\beta_n|(\nabla v \cdot n)^2 \leq 0$$

leaving us with the estimate

$$\|v_\beta\|^2 \leq -\int_\Omega g\beta \cdot \nabla v + C\frac{\epsilon}{2}\|\nabla v\|^2$$

Since $C = O(1)$, an application of Young's inequality and Lemma 1 complete the estimate.

$\square$

3.3 Energy norm equivalence (bound from above)

4 Numerical experiments

In our numerical experiments, we will be interested

4.1 Erickson model problem

4.2 Setting just u_0, not $u_0 - \sigma_n$

4.3 Near discontinuous hat

5 Conclusions

References

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