Introduction to Large Eddy Simulation

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Filtered Navier-Stokes

For filter kernel G,

$$\overline{u}(x) = \int_\Omega G(x, x')u(x')dx'$$

Filtering the incompressible Navier-Stokes equations:

$$\frac{\partial \overline{u_i}}{\partial t} = -\overline{\frac{\partial u_i u_j}{\partial x_j}} - \overline{\frac{\partial p}{\partial x_i}} + \nu \overline{\frac{\partial^2 u_i}{\partial x_j \partial x_j}}$$

If $G(x, x') = G(x - x')$, we can commute differentiation and filtering

$$\frac{\partial \overline{u_i}}{\partial t} = -\frac{\partial \overline{u_i u_j}}{\partial x_j} - \frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j}$$

Identify “sub-grid stress” analogous to Reynolds stress,

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$$

Common Filters

Explicit Filters:

- Fourier cut-off filter

$$\hat{G}(k) = \begin{cases} 1 & |k| < k_c \\ 0 & |k| > k_c \end{cases}$$

  Filter width $\Delta = \pi/k_c$.

- Top-hat filter

$$G(x) = \begin{cases} 1/\Delta & |x| < \Delta/2 \\ 0 & |x| > \Delta/2 \end{cases}$$

- Gaussian filter

$$G(x) = \frac{e^{-x^2/2\Delta^2}}{\Delta\sqrt{2\pi}}$$

Implicit Filters:

- The numerical method is the filter – very common, but suspect
- Define your filter as some projection of the Navier-Stokes solution onto some finite dimensional function space

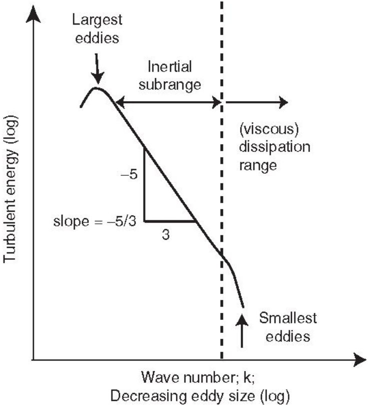
The Closure Problem

Filtered Navier-Stokes

$$\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}$$

- Smaller scales can be modeled as isotropic homogeneous
- Homogeneous isotropic turbulence is **much** easier to model
- Primarily need to capture energy cascade effect with extra dissipation
- Smagorinsky model

$$\tau_{ij} = -(C_s \Delta)^2 (2 \overline{S}_{lm} \overline{S}_{lm})^{1/2} \overline{S}_{ij} + \frac{1}{3} \tau_{kk} \delta_{ij}$$



The Dynamic Model

Let $C = \sqrt{2}C_s^2$, then Germano's identity gives a process by which to solve for C. Least squares solution gives

$$C = \frac{\mathcal{L}_{ij} M_{ij}}{M_{kl} M_{kl}}$$

where

$$M_{ij} = \Delta^2 \left(\widetilde{|\overline{S}|\overline{S}_{ij}} - \alpha^2 |\tilde{\overline{S}}|\tilde{\overline{S}}_{ij} \right)$$

This fixed several longstanding issues with LES

- Adjustable constants are eliminated.
- The model turns off (i.e. $C = 0$) in well resolved or laminar regions.
- The need for damping near the wall is eliminated.

Outstanding Problems – Wall Modeling

Near the wall, the “large” dynamicly important scales of turbulence are actually quite small. Thus, to resolve the large scales, the LES must resolve these small structures, and these structures get smaller relative to the thickness of the shear layer as the Reynolds number increases. This makes LES increasingly expensive with increasing Reynolds number. Wall modeling techniques that do not require the near-wall structures to be resolved are needed if LES is to be useful at high Reynolds number.

Outstanding Problems – Numerical Errors

It is common practice in LES for the filter width Δ to be the grid size in the numerical simulation. The dynamic procedure uses the behavior of scales between $\alpha\Delta$ and Δ is size to set the constants, but these scales near the grid size have large numerical errors. Thus, the dynamic model is working on data that is polluted by numerical errors. In fact, Ghosal showed that even with high order numerical methods, the numerical truncation errors can be much larger than the subgrid terms. There are three possible solutions to this problem:

- Make Δ much larger than the grid size (expensive).
- Use exceptionally accurate numerics, i.e. spectral methods (not generally applicable).
- Account for the numerical error in the formulation of the LES models (no one knows how to do this).

Outstanding Problems – Embedded laminar features

In some flows, not everything is turbulent, and the features of the laminar flow can be too small to resolve. Good examples of this are the separation shear layers in the cross flow past a circular cylinder. These laminar features cannot be resolved, but the models are not valid for them either.

Outstanding Problems – What did you compute?

In an LES, one does not compute the velocity field, rather one computes a filtered velocity, and therefore one needs to be careful in interpreting the results. For example, if one is interested in the statistical quantity $\langle u^2 \rangle$, one cannot determine this exactly from the LES. In the LES, one can only determine $\left\langle \overline{u}^2 \right\rangle \neq \left\langle u^2 \right\rangle$, unless the energy in the subgrid turbulence is negligible. In general one can expect to estimate such statistical quantities as $\langle u_i(x + r) u_j(x) \rangle$ accurately provided $|r| \gg \Delta$. However, since the vorticity is strictly a small-scale quantity (the subgrid contribution to enstrophy is not small), one cannot expect to reproduce vorticity statistics in an LES.

Outstanding Problems – Commutation Error

Filters that are used in actual LES simulations are not generally homogeneous, that is $G(x, x') \neq G(x - x')$, so that filtering no longer commutes with differentiation. Then, most significantly,

$$\overline{\frac{\partial u_i u_j}{\partial x_j}} \neq \frac{\partial \overline{u_i u_j}}{\partial x_j},$$

and the difference between these two terms is generally called the commutation error. This commutation error plays an important role in the equations. Suppose that a flow domain is divided into two regions, one in which the filter scale Δ is larger than in the other region. If trubulence passes from the small filter scale region to the large, then not all the scales of the turbulence that could be represented in the finer region can be represented in the coarser region. Therefore, some of the smaller scales must be removed as the turbulence passes from one region to the other. When turbulence is passing the the opposite direction, scales that were not represented become representable, and must be added to the LES solution.

Variational Multiscale LES

- Use variational projections in place of the traditional filtered equations
- Focus on modeling the fine-scale equations
- Avoids filters eliminates:
  - Inhomogeneous non-commutative filters necessary for wall-bounded flows
  - Use of complex filtered quantities in compressible flows
- Retains numerical consistency in the coarse-scale equations permitting full rate-of-convergence
- Newer approaches attempt to capture all scales consistently and to avoid use of eddy viscosities altogether