

# Locally Conservative Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

Truman Ellis, Leszek Demkowicz, and Jesse Chan

Institute for Computational Engineering and Sciences,  
The University of Texas at Austin,  
Austin, TX 78712

## Abstract

## 1 Introduction

Verteeg and Malalasekera, in *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*[3, p. 110-113] cite three characteristics that they consider essential to any numerical discretization of convection-diffusion type problems: conservativeness, boundedness, and transportiveness.

Perot[2] argues

Accuracy, stability, and consistency are the mathematical concepts that are typically used to analyze numerical methods for partial differential equations (PDEs). These important tools quantify how well the mathematics of a PDE is represented, but they fail to say anything about how well the physics of the system is represented by a particular numerical method. In practice, physical fidelity of a numerical solution can be just as important (perhaps even more important to a physicist) as these more traditional mathematical concepts. A numerical solution that violates the underlying physics (destroying mass or entropy, for example) is in many respects just as flawed as an unstable solution.

The discontinuous Petrov-Galerkin finite element method has been described as least squares finite elements with a twist. The key difference is that least square methods seek to minimize the residual of the solution in some Hilbert space norm, while DPG seeks the minimization in a dual norm through the inverse Riesz map. Exact mass conservation has been an issue that has plagued least squares finite elements for a long time. Several approaches have been used to try to adress this. Chang and Nelson[1] developed the 'restricted LSFEM'[1] by augmenting the least squares equations with a Lagrange multiplier explicitly enforcing mass conservation element-wise. Our conservative formulation of DPG takes a similar approach and both methods share similar negative of transforming a minimization method to a saddle-point problem.

The discontinuous Petrov-Galerkin finite element method has shown a lot of promise for convection-diffusion type problems including robustness in the face of singularly perturbed problems.

## 2 DPG is a Minimum Residual Method

We now proceed with a abstract derivation of the standard Discontinuous Petrov-Galerkin method. Suppose we have two Hilbert spaces,  $U$  and  $V$ , the trial and test spaces, respectively. And suppose we are trying to solve a well-posed variational problem  $b(u, v) = l(v)$ . We can rewrite this in operator form  $Bu = l$ , where  $B : U \rightarrow V'$ , where  $V'$  is the dual space to  $V$ . Then, for a discrete subspace  $U_h \subset U$ , we seek to find  $u_h \in U_h$  that minimizes the error residual:

$$u_h = \arg \min_{u_h \in U_h} \frac{1}{2} \|Bu_h - l\|_{V'}^2. \quad (1)$$

Recalling that the Riesz operator  $R_V : V \rightarrow V'$  is an isometry defined by

$$\langle R_V v, \delta v \rangle = (v, \delta v)_V, \quad \forall \delta v \in V,$$

we can use the Riesz inverse to minimize on  $V$  rather than its dual:

$$\frac{1}{2} \|Bu_h - l\|_{V'}^2 = \frac{1}{2} \|R_V^{-1}(Bu_h - l)\|_V^2 = \frac{1}{2} (R_V^{-1}(Bu_h - l), R_V^{-1}(Bu_h - l))_V. \quad (2)$$

The first order optimality condition for (2) requires the Gâteaux derivative to be zero in all directions  $\delta u \in U_h$ , i.e.,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u)_V = 0, \quad \forall \delta u \in U.$$

By definition of the Riesz operator, this is equivalent to

$$\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle = 0 \quad \forall \delta u_h \in U_h. \quad (3)$$

Now, we can identify  $v_{\delta u_h} := R_V^{-1}B\delta u_h$  as the optimal test function for trial function  $u_h$  and we can rewrite (3) as

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}). \quad (4)$$

The DPG method then is to solve (4) with optimal test functions  $v_{\delta u_h} \in V$  that solve the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v), \quad \forall \delta v \in V. \quad (5)$$

Using a continuous test basis would result in a global solve for every optimal test function. Therefore DPG uses a discontinuous test basis which makes each solve element-local and much more computationally tractable. Of course, (5) still requires the inversion of the infinite-dimensional Riesz map, but in approximating the  $V$  by finite dimensional  $V_h$  which is of a higher polynomial degree than  $U_h$  (hence “enriched space”) works well in practice.

No assumptions have been made so far on the definition of the inner product on  $V$ . In fact, proper choice of  $(\cdot, \cdot)_V$  can make the difference between a solid DPG method and one that suffers from robustness issues.

### 3 DPG Applied to Convection-Diffusion

Now that we have briefly outline the abstract DPG method, let us put it into practice with the Convection-Diffusion equation. The strong form of the steady Convection-Diffusion equation reads

$$\nabla \cdot (\beta u) - \epsilon \Delta u = g,$$

where  $u$  is the property of interest,  $\beta$  is the convection vector, and  $g$  is the source term. Let us write this as an equivalent system of first order equations:

$$\begin{aligned} \nabla \cdot (\beta u - \sigma) &= g \\ \frac{1}{\epsilon} \sigma - \nabla u &= \mathbf{0}. \end{aligned}$$

## References

- [1] C. L. Chang and John J. Nelson. Least-squares finite element method for the stokes problem with zero residual of mass conservation. *SIAM J. Num. Anal.*, 34:480–489, 1997.
- [2] J. B. Perot. Discrete conservation properties of unstructured mesh schemes. *Annual Review of Fluid Mechanics*, 43:299–318, 2011.
- [3] H.K. Versteeg and W. Malalasekera. *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Prentice Hall, 2007.