PECOS

Predictive Engineering and Computational Sciences

Locally Conservative Discontinuous Petrov-Galerkin for Convection-Diffusion

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A Summary of DPG

Overview of Features

- Robust for singularly perturbed problems
- Stable in the preasymptotic regime
- Designed for adaptive mesh refinement

DPG is a minimum residual method:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} ||Bw_h - l||^2_{V'}$$

$$\Updownarrow$$

$$b(u_h, R_V^{-1} B\delta u_h) = l(R_V^{-1} B\delta u_h) \quad \forall \delta u_h \in U_h$$

where $v_{\delta u_h} := R_V^{-1} B\delta u_h$ are the optimal test functions.

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\begin{aligned} \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= g \\ \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= \mathbf{0} \end{aligned}$$

Multiply by test functions and integrate by parts over each element, K.

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + ((\boldsymbol{\beta} u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v)_{\partial K} &= (g, v)_K \\ \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - (u, \tau_n)_{\partial K} &= 0 \end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + (\hat{f}, v)_{\partial K} + \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K& \\ + (u, \nabla \cdot \boldsymbol{\tau})_K - (\hat{u}, \tau_n)_{\partial K} &= (g, v)_K \end{aligned}$$

Local Conservation

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{f} = \int_K g \,,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[1], we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \left|\left| R_V^{-1}(Bu_h - l) \right|\right|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{f}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K} \,,$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$.

[1] D. Moro, N.C. Nguyen, and J. Peraire. A hybridized discontinuous Petrov-Galerkin scheme for scalar conservation laws. *Int.J. Num. Meth. Eng.*, 2011. in print

Local Conservation

Finding the critical points of $L(u, \boldsymbol{\lambda})$, we get the following equations.

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B \delta u_h) - l(R_V^{-1} B \delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddlepoint problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{f}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable. Chan et al[2] developed the following robust norm for convection-diffusion.

$$||(v, \boldsymbol{\tau})||^2_{\mathbf{V},\Omega_h} = ||\nabla \cdot \boldsymbol{\tau}||^2 + \left|\left| \min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\} \boldsymbol{\tau} \right|\right|^2$$
$$+ \epsilon ||\nabla v||^2 + ||\boldsymbol{\beta} \cdot \nabla v||^2 \quad \underbrace{+ \left|\left| \min\left\{\sqrt{\frac{\epsilon}{|K|}}, 1\right\} v \right|\right|^2}_{\text{no longer necessary to make this a norm}}$$

[2] J. Chan, N. Heuer, T Bui-Thanh, and L. Demkowicz. Robust DPG method for convection-dominated diffusion problems ii: a natural inflow condition. Technical Report 21, ICES, 2012

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[2] J. Chan, N. Heuer, T Bui-Thanh, and L. Demkowicz. Robust DPG method for convection-dominated diffusion problems ii: a natural inflow condition. Technical Report 21, ICES, 2012

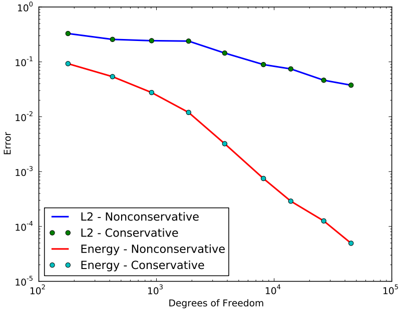
Erickson-Johnson Problem

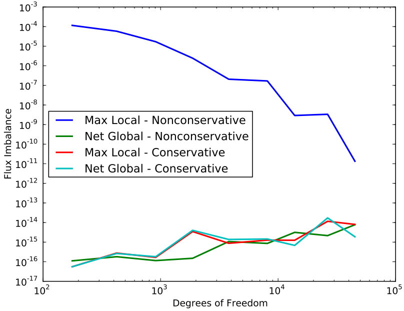
On domain $\Omega = [0,1]^2$, with $\beta = (1,0)^T$, $f = 0$ and boundary conditions

$$\hat{f} = u_0, \quad \beta_n \leq 0\,, \qquad \hat{u} = 0, \quad \beta_n > 0$$

Separation of variabes gives an analytic solution

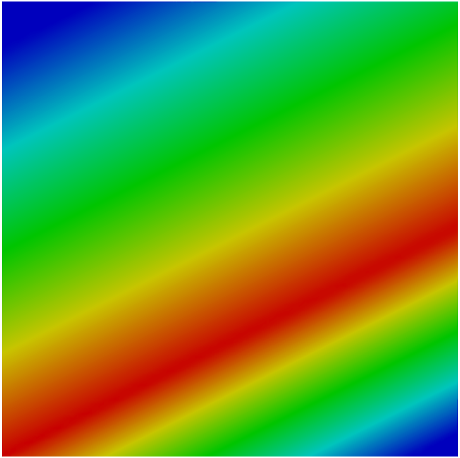
$$u(x,y) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\exp(r_2(x-1)) - \exp(r_1(x-1))}{r_1 \exp(-r_2) - r_2 \exp(-r_1)} \cos(n\pi y)$$

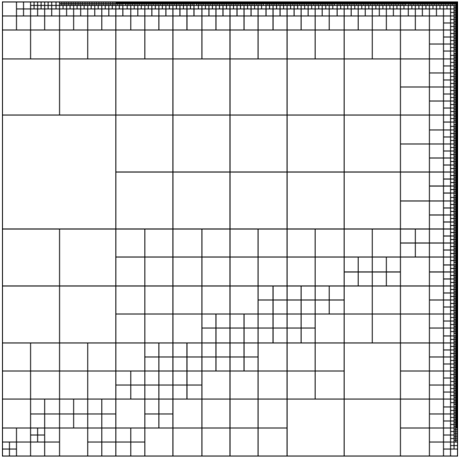




Skewed Convection-Diffusion Problem

After 8 refinement steps





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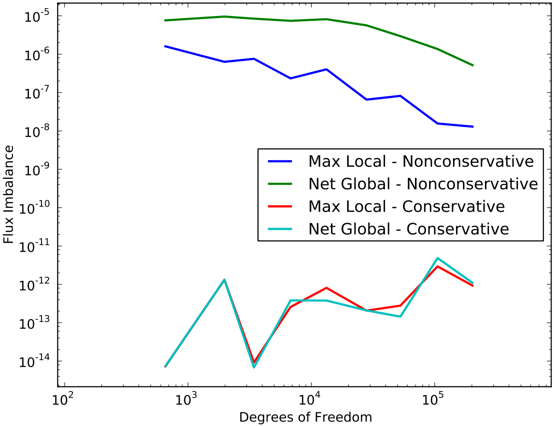
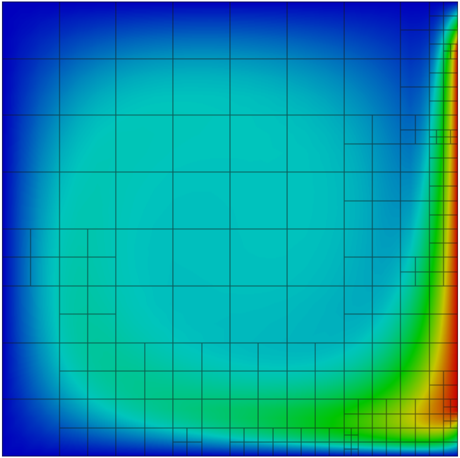
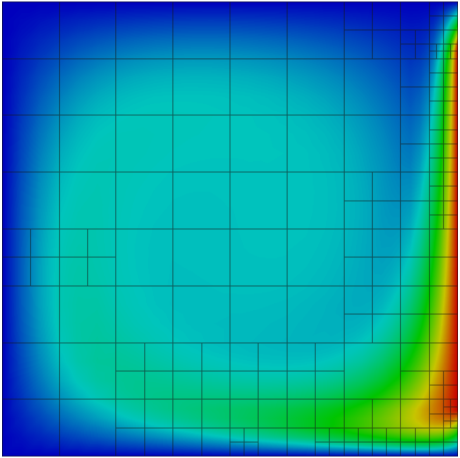


Double Glazing Problem

After 6 refinement steps

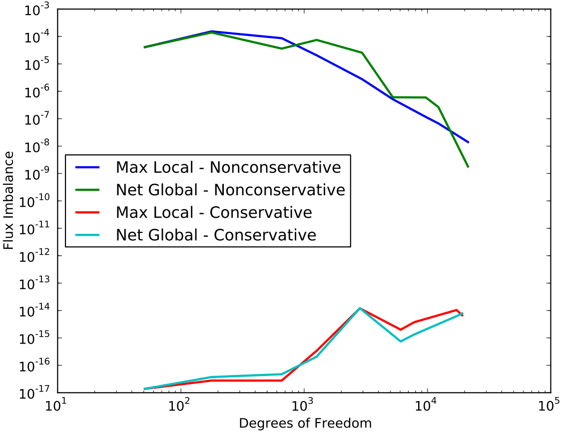


Without Local Conservation



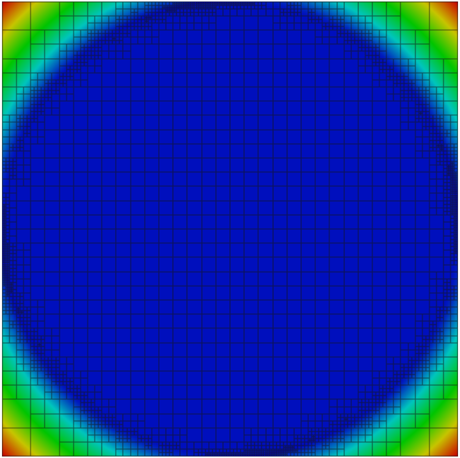
With Local Conservation

Double Glazing Problem

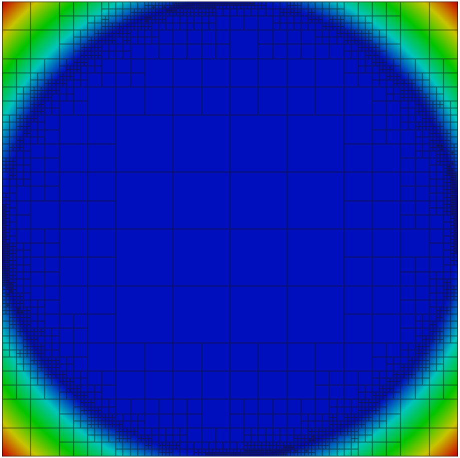


Vortex Problem

After 6 refinement steps

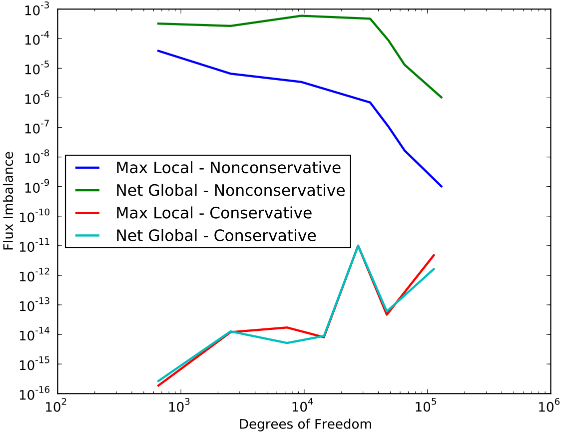


Without Local Conservation



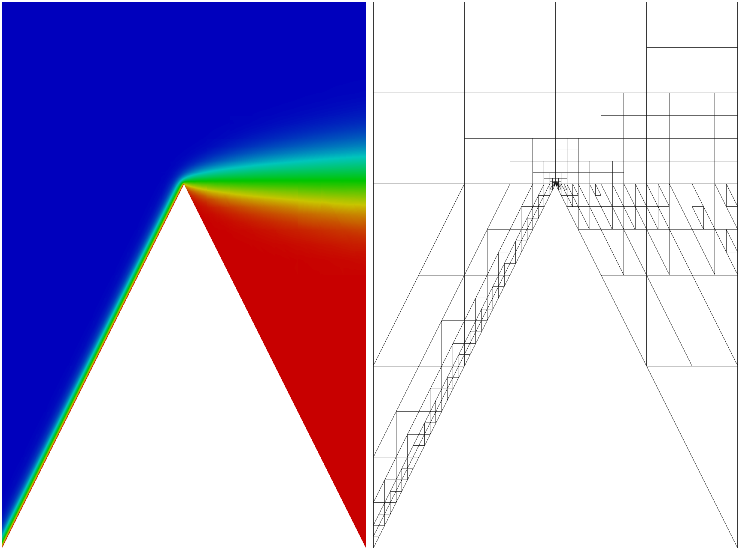
With Local Conservation

Vortex Problem

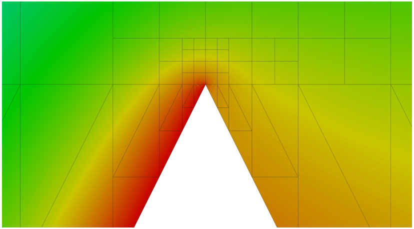


Wedge Problem

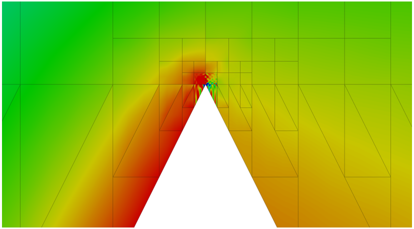
After 16 refinement steps



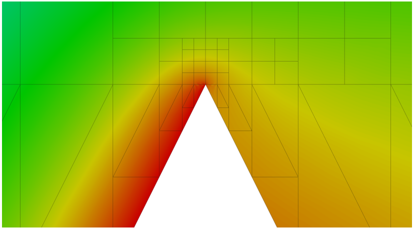
Wedge Problem



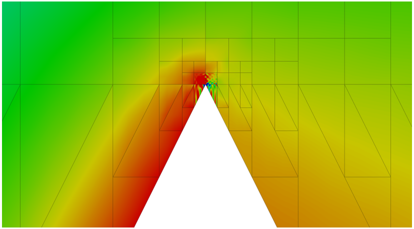
Nonconservative graph norm



Nonconservative robust norm

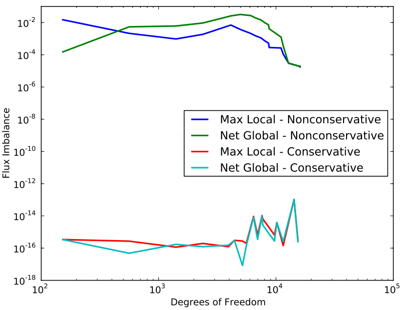


Conservative graph norm

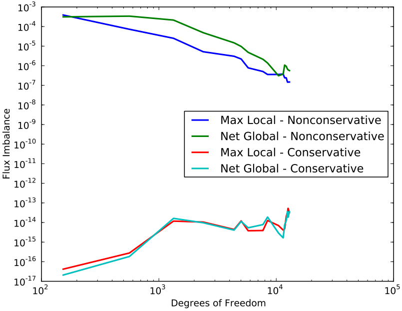


Conservative robust norm

Wedge Problem



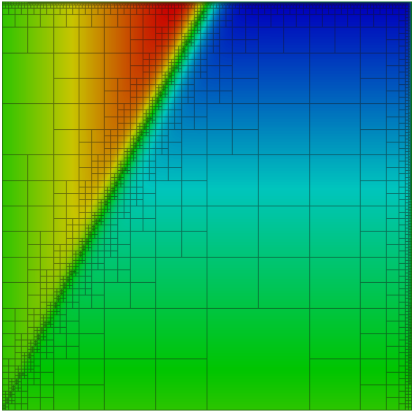
Graph norm

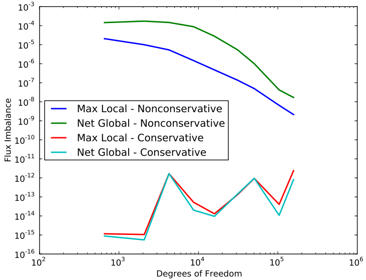


Robust norm

Discontinuous Source Problem

After 16 refinement steps





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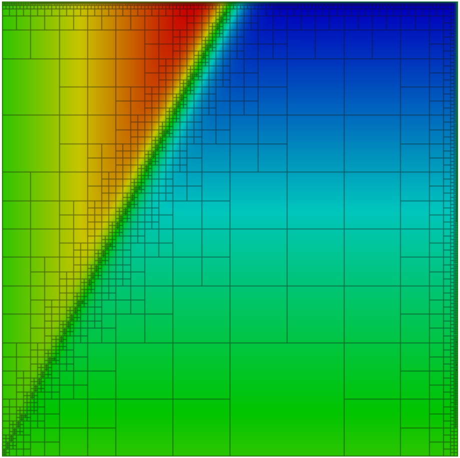

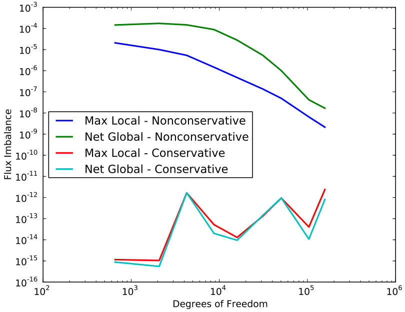

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