

Globally and locally optimal test functions

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1 The ultra-weak variational formulation

1.1 The adjoint equation and L^2 optimality

Assuming boundedness below of A , as the extra L^2 term disappears, we get L^2 optimality. The ultra-weak variational form gives

$$b(u_h, v) = (u_h, A^*v) + \langle \hat{u}, v \rangle = \text{load and boundary conds} = b(u, A^*v)$$

If we use for our test functions solutions of the adjoint equation $A^*v = u$ (with appropriate adjoint boundary conditions), u_h is the best L^2 projection of the exact solution u onto U_h .

2 Test norms for the convection-diffusion problem

Introduce the graph norm here.

2.1 Local and global optimal test functions

Note that globally optimal test functions only produce boundary layers at the global inflow boundary.

Show representation of global test functions in local test functions using Jay's proof using orthogonal subspaces and flux test functions.

3 Numerical experiments

3.1 The effect of under-resolved boundary layers

Can we quantify the effect of under-resolving boundary layers?

$$b\left(\left(u, \sigma, \hat{u}, \hat{f}_n\right), (v, \tau)\right) = (u, \nabla_h \cdot \tau - \beta \cdot \nabla_h v)_{\Omega_h} + \left(\sigma, \frac{1}{\epsilon} \tau + \nabla_h v\right)_{\Omega_h} + \langle \hat{u}, \tau_n \rangle_{\Gamma_h} + \langle \hat{f}_n, v \rangle_{\Gamma_h}$$

Let \mathbf{U} denote the trial variables $\{u, \sigma, \hat{u}, \hat{f}_n\}$. For robustness in u , we can choose for (v, τ) the conforming solution to the adjoint equation such that

$$\begin{aligned}\nabla \cdot \tau - \beta \cdot \nabla v &= u \\ \frac{1}{\epsilon} \tau - \nabla v &= 0.\end{aligned}$$

By doing so, we recover the L^2 norm of u from the bilinear form, such that

$$\|u\|_{L^2(\Omega)}^2 = b(\mathbf{U}, (v, \tau)) = \frac{b(\mathbf{U}, (v, \tau))}{\|(v, \tau)\|_V} \|(v, \tau)\|_V \leq \|\mathbf{U}\|_E \|(v, \tau)\|_V$$

Let \lesssim denote a robust bound - if $\|(v, \tau)\|_V \lesssim \|u\|_{L^2(\Omega)}$, then we have that

$$\|u\|_{L^2(\Omega)} \lesssim \|\mathbf{U}\|_E$$

In other words, a necessary condition for a robust method is that the test norm measures solutions to the adjoint equations in a robust manner.

Can we conclude that the same thing needs to happen for approximate test functions?

If we assume that an underresolved test function behaves similarly to a test function under a larger viscosity, we expect that the term $\|\nabla \cdot \tau - \beta \cdot \nabla v\|_{L^2}$ still remains relatively bounded uniformly in ϵ . However, the term $\|\frac{1}{\epsilon} \tau - \nabla v\|_{L^2}$ does not — analytical calculations in 1D show that this term grows with the Peclet number in the element at the inflow, where the adjoint solution develops a boundary layer. As the boundedness of this term determines the robustness of σ , our energy norm is

$$\begin{aligned}\|\mathbf{U}\|_E &:= \sup_{(v, \tau)} \left(\frac{(u, \nabla_h \cdot \tau - \beta \cdot \nabla_h v)_\Omega}{\|(v, \tau)\|_V} + \frac{(\sigma, \frac{1}{\epsilon} \tau + \nabla_h v)_\Omega}{\|(v, \tau)\|_V} + \frac{\langle \hat{f}_n, v \rangle_\Gamma - \langle \hat{u}, \tau_n \rangle_\Gamma}{\|(v, \tau)\|_V} \right) \\ &\approx \|u\|_{L^2} + F(\text{Pe}) \|\sigma\|_{L^2} + \left\| \begin{pmatrix} \hat{u} \\ \hat{f}_n \end{pmatrix} \right\|\end{aligned}$$

where $F(\text{Pe})$ is some non-decreasing function of the Peclet number. Numerical experiments confirm that, for the quasi-optimal test norm, underresolution of the inflow boundary layer negatively affects only the robustness of u .

4 Conclusions

- There are three levels at which you should look - the continuous level (the adjoint problem for L^2 robustness), the global level (which links the adjoint equation to the test functions), and the localized test norm (which relates local test functions to global test functions).
- Your test norm may induce boundary layers even when they're not necessary for adjoint L^2 stability.
- If your test norm promises global boundary layers (that affect robustness), not approximating them leads to a loss of robustness.

You have to account for the boundary layer/ ϵ diffusion scale somehow - either by

- adjusting the test norm to account for robustness (weighted test norm), which pushes the ϵ into the inflow area,
- removing the continuous adjoint problem (“convection” inflow boundary condition), which removes the boundary layer for L^2 stability altogether,
- resolving the global boundary layers (demonstrated here and in Antii’s Shishkin meshes)