APPLICATION OF A DISCONTINUOUS PETROV-GALERKIN METHOD TO THE STOKES EQUATIONS

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Abstract. The discontinuous Petrov-Galerkin finite element method proposed by L. Demkowicz and J. Gopalakrishnan [5, 6] guarantees the optimality of the solution in what they call the *energy norm*. An important choice that must be made in the application of the method is the definition of the inner product on the test space. In this paper, we apply the DPG method to the Stokes problem in two dimensions, analyzing it to determine appropriate inner products, and perform a series of numerical experiments.

1. Introduction. Recently, L. Demkowicz and J. Gopalakrishnan have proposed a new class of discontinuous Petrov-Galerkin (DPG) methods [5, 6, 7, 10, 3], which compute test functions that are adapted to the problem of interest to produce stable discretization schemes. An important choice that must be made in the application of the method is the definition of the inner product on the test space. In this paper, we apply the method to the Stokes problem in two dimensions, analyzing it to determine appropriate inner products, and perform numerical experiments to test these inner products.

Whereas traditional Galerkin methods use the same space for test and trial spaces, Petrov-Galerkin methods allow the test and trial spaces to differ. The DPG approach computes test functions that are *optimal*, in a sense that we make precise in Section 2. One consequence of this choice of test functions is that the stiffness matrix for a continuous, weakly coercive variational formulation is symmetric (hermitian, for complex-valued problems) and positive definite. Of course, the determination of test functions is an extra step compared with traditional methods; it is important that these can be determined cheaply. By using discontinuous Galerkin (DG) formulations, DPG achieves this, reducing the computation of the test functions to a local problem. Our method bears some resemblance to the MDG method [9] in that a local problem is solved on each element. The key difference with that paper is that in MDG the local problem is restriction of the original equations whereas in DPG the local problem is implied by the selected test space inner product. Furthermore, in MDG the local problem is used to express DG degrees of freedom in terms of continuous degrees of freedom, i.e., to effect static condensation on the element.

Our primary goal is the application of the method to the Stokes problem in two dimensions. The strong form of the problem is

$$-2\mu\nabla\cdot\underline{\boldsymbol{\epsilon}}+\nabla p=\boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u}=0 \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}=\boldsymbol{g_D} \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega\subset\mathbb{R}^2$, μ is viscosity, $\underline{\boldsymbol{\epsilon}}=\nabla^{\text{sym}}\boldsymbol{u}$ is strain, p is pressure, u velocity, and $\boldsymbol{f}$ a vector forcing function.

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The paper is structured as follows. In Section 2, we give an introduction to the basic features of the DPG method. In Section 3, we derive the weak formulation of the problem. In Section 4, we motivate the choice of inner product on the test space with reference to an argument for the continuity of the bilinear form. In Section 5, we present the numerical results. We conclude in Section 6.

2. DPG Method. Here, we sketch some of the main features of the DPG method. For details, we refer the reader to a series of papers by Demkowicz et al., in particular the second ICES Report [6], from which most of this section is derived. We begin with theoretical definitions and results, and then describe the approach to practical realization. Consider the abstract variational boundary-value problem:

$$\text{Find } u \in U : b(u,v) = l(v) \quad \forall v \in V. \tag{2.1}$$

We take U and V to be real Hilbert spaces. We assume $b(\cdot,\cdot)$ is continuous, i.e.

$$|b(u,v)| \leq M\, ||u||_U\, ||v||_V\,, \tag{2.2}$$

for some real M. We assume also that $b(\cdot,\cdot)$ is weakly coercive, that is

$$\inf_{||u||_U=1} \sup_{||v||_V=1} b(u,v) > \gamma, \tag{2.3}$$

for some $\gamma > 0$. If we additionally assume that

$$\{v \in V : b(u,v) = 0 \quad \forall u \in U\} = \{0\}, \tag{2.4}$$

then it is well known that the problem (2.1) has a unique solution provided that $l \in V'$, the dual of V.

2.1. Energy Norm. We define an alternate norm, called the *energy norm*, on the trial space U by

$$||u||_E \stackrel{\text{def}}{=} \sup_{||v||_V=1} b(u,v). \tag{2.5}$$

This norm is the one in which the optimality is guaranteed by the selection of optimal test functions. It is an equivalent norm to the standard norm on U, i.e.

$$\gamma\, ||u||_U \leq ||u||_E \leq M\, ||u||_U \quad \forall u \in U. \tag{2.6}$$

2.2. Optimal Test Functions. We are now prepared to give a definition of the optimal test functions. Define a map $T : U \to V$ from the trial space to the test space by: For $u \in U$, define Tu, the *optimal test function* corresponding to u, as the unique solution to

$$(Tu, v)_V = b(u,v) \quad \forall v \in V.$$

By the Riesz representation theorem, T is well-defined. Note that

$$||u||_E = \sup_{||v||_V=1} b(u,v) = \sup_{||v||_V=1} (Tu,v)_V = \frac{1}{||Tu||_V}(Tu,Tu)_V = ||Tu||_V\,.$$

Thus the energy norm is generated by the inner product on V, i.e.

$$(u,u)_E \stackrel{\text{def}}{=} (Tu,Tu)_V. \tag{2.7}$$

In practice, we approximate T by a discrete operator T_n, described in Section 2.4.

2.3. Optimal Test Space for U_n. Take a finite-dimensional trial space $U_n \subset U$. Define the *optimal test space* for U_n as $V_n = \text{span}\{Te_j : j = 1, \cdots, n\}$, where the e_j form a basis for U_n.

Solve the discrete problem

$$\text{Find } u_n \in U_n : b(u_n, v) = l(v)\ \forall v \in V_n. \tag{2.8}$$

Then the error is the best approximation error in the energy norm,

$$||u - u_n||_E = \inf_{w_n \in U_n} ||u - w_n||_E\,, \tag{2.9}$$

and this is the sense in which the test space is *optimal*.

2.4. Practical Realization. The method involves two steps: first, find the optimal test functions; second, use the optimal test functions to solve the discrete problem 2.8. The optimal test functions are not in general polynomials. In practice, we approximate them with an "enriched" polynomial space — a space of polynomials of slightly higher degree than the trial space. This is done to provide a higher-fidelity approximation to the continuous space of optimal test functions. The best choice for the amount of "enrichment" is determined experimentally for each problem.

In general, we apply the following procedure:

1. Given a boundary value problem, develop mesh-dependent $b(\cdot,\cdot)$ with test space V that allows inter-element discontinuities (hence *Discontinuous* Petrov-Galerkin). We develop this in Section 3.
2. Choose trial space U_n (in particular the norm of interest in U_n), and the inner product on V, which will be motivated by the choice of trial space. We detail this process for the Stokes problem in Section 4.
3. Compute optimal test functions. Approximate T by $T_n : U_n \to \widetilde{V}_n \subset V$. We use an enriched space of piecewise polynomials for $\widetilde{V}_n$. Defining $t_j = T_n e_j$, we solve
$$(t_j, \tilde{e}_i)_V = b(e_j, \tilde{e}_i)$$
for t_j, where the $\tilde{e}_i$ form the basis for $\widetilde{V}_n$.
4. Use the optimal test functions to solve the problem on $U_n \times \widetilde{V}_n$. We note that the stiffness matrix here is symmetric positive definite (hermitian, for a complex-valued problem),
$$\begin{aligned} b(e_j, t_i) = (T_n e_j, t_i)_V = (T_n e_j, T_n e_i)_V = \overline{(T_n e_i, T_n e_j)_V} \\ = \overline{(T_n e_i, t_j)_V} = \overline{b(e_i, t_j)}. \end{aligned}$$
Also, note that this means that we may compute the stiffness matrix in terms of the inner product on the test space V, without explicit recourse to the bilinear form.

3. Stokes Formulation. Our general approach to variational formulation in DPG is as follows. First, rewrite the strong form of the problem as a system of first-order partial differential equations. Then, multiply by test functions and integrate by parts, moving all derivatives to the test functions. We thus arrive at the *ultra-weak* form of the problem, a formulation in which all solution variables are in L^2.

Starting with the strong formulation defined in equations (1.1)-(1.3), introduce stress σ and vorticity ω by

$$\underline{\boldsymbol{\sigma}} = 2\mu\underline{\boldsymbol{\epsilon}} - p\underline{\boldsymbol{I}}$$
$$\underline{\boldsymbol{\omega}} = \frac{1}{2}(\nabla\boldsymbol{u} - \nabla\boldsymbol{u}^T)$$

so that equation (1.1) becomes simply $-\nabla\cdot\underline{\boldsymbol{\sigma}} = \boldsymbol{f}$. We also have

$$\underline{\boldsymbol{\epsilon}} = \frac{1}{2\mu}(\underline{\boldsymbol{\sigma}} + p\underline{\boldsymbol{I}}).$$

Since $\underline{\boldsymbol{\epsilon}} = \nabla^{\mathrm{sym}}\boldsymbol{u} = \nabla\boldsymbol{u} - \underline{\boldsymbol{\omega}}$, the entire system is

$$\begin{aligned}
\frac{1}{2\mu}(\underline{\boldsymbol{\sigma}} + p\underline{\boldsymbol{I}}) - \nabla\boldsymbol{u} + \underline{\boldsymbol{\omega}} &= \underline{\boldsymbol{0}} && \text{in } \Omega,\\
-\nabla\cdot\underline{\boldsymbol{\sigma}} &= \boldsymbol{f} && \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g_D} && \text{on } \partial\Omega.
\end{aligned}$$

Note that the antisymmetric part of the first equation recovers the definition of $\underline{\boldsymbol{\omega}}$, so that it need not enter the system separately. Define scalar $\omega = \omega_{21} = \frac{1}{2}(u_{1,2} - u_{2,1})$. Our strong formulation is

$$\begin{aligned}
\frac{1}{2\mu}\begin{pmatrix}\sigma_{11}+p\\ \sigma_{21}\end{pmatrix} - \nabla u_1 + \begin{pmatrix}0\\ \omega\end{pmatrix} &= \boldsymbol{0} && \text{in } \Omega,\\
\frac{1}{2\mu}\begin{pmatrix}\sigma_{12}\\ \sigma_{22}+p\end{pmatrix} - \nabla u_2 - \begin{pmatrix}\omega\\ 0\end{pmatrix} &= \boldsymbol{0} && \text{in } \Omega,\\
-\nabla\cdot\begin{pmatrix}\sigma_{11}\\ \sigma_{21}\end{pmatrix} &= f_1 && \text{in } \Omega,\\
-\nabla\cdot\begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix} &= f_2 && \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g_D} && \text{on } \partial\Omega.
\end{aligned}$$

Multiplying the first two equations by vector test functions $\boldsymbol{q}_i$ and the following three by scalar test functions v_i, and integrating by parts over an element K, we obtain

$$\begin{aligned}
&\int_K \left(\frac{1}{2\mu}\begin{pmatrix}\sigma_{11}+p\\ \sigma_{21}\end{pmatrix} + \begin{pmatrix}0\\ \omega\end{pmatrix}\right)\cdot\boldsymbol{q}_1 + \int_K u_1\nabla\cdot\boldsymbol{q_1} &&- \int_{\partial K}\widehat{u}_1\boldsymbol{q_1}\cdot\nu &&= \boldsymbol{0}\\
&\int_K \left(\frac{1}{2\mu}\begin{pmatrix}\sigma_{12}\\ \sigma_{22}+p\end{pmatrix} - \begin{pmatrix}\omega\\ 0\end{pmatrix}\right)\cdot\boldsymbol{q}_2 + \int_K u_2\nabla\cdot\boldsymbol{q_2} &&- \int_{\partial K}\widehat{u}_2\boldsymbol{q_2}\cdot\nu &&= \boldsymbol{0}\\
&\int_K \begin{pmatrix}\sigma_{11}\\ \sigma_{21}\end{pmatrix}\cdot\nabla v_1 &&- \int_{\partial K}\widehat{\boldsymbol{\sigma}}_1 v_1\cdot\nu &&= \int_K f_1 v_1\\
&\int_K \begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix}\cdot\nabla v_2 &&- \int_{\partial K}\widehat{\boldsymbol{\sigma}}_2 v_2\cdot\nu &&= \int_K f_2 v_2\\
&-\int_K \boldsymbol{u}\cdot\nabla v_3 &&+ \int_{\partial K}\widehat{\boldsymbol{u}} v_3\cdot\nu &&= 0,
\end{aligned}$$

where the "hatted" variables ($\widehat{u}_1$, e.g.) are the *fluxes* introduced by relaxing the continuity requirement at element boundaries. These differ from the *numerical fluxes* that appear in other DG methods, in that they are not constructed a priori, but simply enter the variational problem as additional unknowns. We solve for them at the same time as we solve the rest of the unknowns. As in other DG methods, the fluxes will approach the corresponding "unhatted" solution variables as the latter approach the exact solution.

4. Inner Product Determination. As discussed in Section 2, the optimality proof depends on the continuity and weak coercivity of the bilinear form. In this section, we use continuity to motivate a particular choice of inner product on V.

We seek to show that $|b(U,v)| \leq M\,||U||_U\,||v||_V$, for some constant M, for spaces U and V to be specified. The norm on U should be specified in such a way that minimizing the error in this norm will produce the results we want. We define

$$||U||_U^2 = \sum_{i=1}^{7}\left(\frac{||u_i||_{L_2(\Omega)}}{\alpha_i}\right)^2 + \sum_{i=1}^{2}\left(\frac{\left|\left|\widehat{F}_i\right|\right|_{H^{1/2}(\partial\Omega)}}{\widehat{\alpha}_i}\right)^2 + \sum_{i=3}^{5}\left(\frac{\left|\left|\widehat{F}_i\right|\right|_{H^{-1/2}(\partial\Omega)}}{\widehat{\alpha}_i}\right)^2,$$

where u_1 and u_2 are as above, $u_3 = \sigma_{11}, u_4 = \sigma_{12} = \sigma_{21}, u_5 = \sigma_{22}, u_6 = \omega$, and $u_7 = p$, $\widehat{F}_i$ is the flux corresponding to the ith equation (that is, $\widehat{F}_1 = \widehat{u}_1, \widehat{F}_2 = \widehat{u}_2, \widehat{F}_3 = \widehat{\boldsymbol{\sigma}}_1 \cdot \nu, \widehat{F}_4 = \widehat{\boldsymbol{\sigma}}_2 \cdot \nu$, and $\widehat{F}_5 = \widehat{\boldsymbol{u}} \cdot \nu$), and the α_i and $\widehat{\alpha}_i$ are positive weights that allow us to emphasize specific components. The reason we use the $H^{1/2}$ norm on the fluxes corresponding to $H(\mathrm{div})$ test functions is that $\boldsymbol{q} \in H(\mathrm{div}) \implies \mathrm{tr}(\boldsymbol{q}) \in H^{-1/2}$. Thus for $\int_{\partial\Omega}\widehat{F}_i\boldsymbol{q}_i\cdot\nu$ to make sense mathematically, we require $\widehat{F}_i \in H^{1/2}$. A similar argument establishes that the fluxes corresponding to H^1 test functions should lie in $H^{-1/2}$. Let us consider the first equation of our bilinear form,

$$\begin{aligned} b_1(U,v) &= \int_\Omega \begin{pmatrix} \frac{\sigma_{11}+p}{2\mu} \\ \frac{\sigma_{21}}{2\mu}+\omega \end{pmatrix} \cdot \boldsymbol{q}_1 + \int_\Omega u_1 \nabla\cdot\boldsymbol{q}_1 - \int_{\partial\Omega}\widehat{u}_1\boldsymbol{q}_1\cdot\nu \\ &= \left(\frac{\sigma_{11}+p}{2\mu}, q_{11}\right)_\Omega + \left(\frac{\sigma_{21}}{2\mu}+\omega, q_{12}\right)_\Omega + (u_1, \nabla\cdot\boldsymbol{q}_1)_\Omega - (\widehat{u}_1, \boldsymbol{q}_1\cdot\nu)_{\partial\Omega} \end{aligned}$$

Now, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |b_1(U,v)| \leq &\frac{1}{2\mu}\left(||\sigma_{11}||_0 + ||p||_0\right)||q_{11}||_0 + \left(\frac{1}{2\mu}||\sigma_{21}||_0 + ||\omega||_0\right)||q_{12}||_0 \\ &+ ||u_1||_0\,||\nabla\cdot\boldsymbol{q}_1||_0 + ||\widehat{u}_1||_{H^{1/2}(\partial\Omega)}\,||\boldsymbol{q}_1\cdot\nu||_{H^{-1/2}(\partial\Omega)} \end{aligned} \tag{4.1}$$

Applying the finite-dimensional Cauchy-Schwarz inequality, we have

$$\begin{aligned} |b_1(U,v)| \leq &\left(\left(\frac{||\sigma_{11}||_0}{2\mu}\right)^2 + \left(\frac{||p||_0}{2\mu}\right)^2 + \left(\frac{||\sigma_{21}||_0}{2\mu}\right)^2 + ||\omega||_0^2 + ||u_1||_0^2 + ||\widehat{u}_1||_0^2\right)^{1/2} \\ &\cdot\left(||q_{11}||_0^2 + ||q_{12}||_0^2 + ||\nabla\cdot\boldsymbol{q}_1||_0^2 + ||\boldsymbol{q}_1\cdot\nu||_{H^{-1/2}(\partial\Omega)}^2\right)^{1/2} \end{aligned}$$

Note that for a particular choice of weights, namely $\alpha_3 = \alpha_4 = \alpha_7 = \frac{1}{2\mu}, \alpha_6 = \alpha_1 = \widehat{\alpha}_1 = 1$, we then immediately have

$$|b_1(U,v)| \leq ||U||_U\left(||q_{11}||_0^2 + ||q_{12}||_0^2 + ||\nabla\cdot\boldsymbol{q}_1||_0^2 + ||\boldsymbol{q}_1\cdot\nu||_{H^{-1/2}(\partial\Omega)}^2\right)^{1/2}$$

motivating a norm

$$||\boldsymbol{q}_1||_{V_1} = \left(||q_{11}||_0^2 + ||q_{12}||_0^2 + ||\nabla\cdot\boldsymbol{q}_1||_0^2 + ||\boldsymbol{q}_1\cdot\nu||^2_{H^{-1/2}(\partial\Omega)}\right)^{1/2}.$$

However, one of our purposes in defining a weighted norm $||U||$ was to gain some control over scale equivalence in computation of the test space inner product, and the argument above in providing a tight bound has separated the weights from the test space terms. Instead, let us return to the inequality (4.1), and note that by the definition of $||\cdot||_U$, $||u_i||_{L_2(\Omega)} \le \alpha_i ||U||_U$; similarly, $\left|\left|\widehat{F}_i\right|\right|_{H^{1/2}(\Omega)} \le \widehat{\alpha}_i ||U||_U$ for $i = 1, 2$ and $\left|\left|\widehat{F}_i\right|\right|_{H^{-1/2}(\Omega)} \le \widehat{\alpha}_i ||U||_U$ for $i = 3, 4, 5$. Thus we have

$$\begin{aligned} |b_1(U,v)| \le \Bigg(&\frac{\alpha_3+\alpha_7}{2\mu}||q_{11}||_0 + \left(\frac{\alpha_4}{2\mu}+\alpha_6\right)||q_{11}||_0 \\ &+ \alpha_1 ||\nabla\cdot\boldsymbol{q}_1||_0 + \widehat{\alpha}_1 ||\boldsymbol{q}_1\cdot\nu||_{H^{1/2}(\partial\Omega)}\Bigg)||U||_U, \end{aligned}$$

motivating the norm

$$||\boldsymbol{q}_1||_{V_1} = \frac{\alpha_3+\alpha_7}{2\mu}||q_{11}||_0 + \left(\frac{\alpha_4}{2\mu}+\alpha_6\right)||q_{12}||_0 + \alpha_1 ||\nabla\cdot\boldsymbol{q}_1||_0 + \widehat{\alpha}_1 ||\boldsymbol{q}_1\cdot\nu||_{H^{1/2}(\partial\Omega)}.$$

For convenience, we implement a similar norm given by

$$\begin{aligned} ||\boldsymbol{q}_1||^2_{V_1} \stackrel{\text{def}}{=} &\left(\frac{\alpha_3+\alpha_7}{2\mu}\right)^2 ||q_{11}||_0^2 + \left(\frac{\alpha_4}{2\mu}+\alpha_6\right)^2 ||q_{12}||_0^2 + \alpha_1^2 ||\nabla\cdot\boldsymbol{q}_1||_0^2 \\ &+ \widehat{\alpha}_1^2 ||\boldsymbol{q}_1\cdot\nu||^2_{L_2(\Omega)}. \end{aligned}$$

Similarly, for $\boldsymbol{q}_2$ we implement

$$\begin{aligned} ||\boldsymbol{q}_2||^2_{V_2} \stackrel{\text{def}}{=} &\left(\frac{\alpha_4}{2\mu}+\alpha_6\right)^2 ||q_{21}||_0^2 + \left(\frac{\alpha_5}{2\mu}+\alpha_7\right)^2 ||q_{22}||_0^2 + \alpha_2^2 ||\nabla\cdot\boldsymbol{q}_2||_0^2 \\ &+ \widehat{\alpha}_2^2 ||\boldsymbol{q}_2\cdot\nu||^2_{L_2(\Omega)}. \end{aligned}$$

For $b_3(U,v)$, we have

$$\begin{aligned} b_3(U,v) &= \int_\Omega \boldsymbol{\sigma}_1\cdot\nabla v_1 - \int_{\partial\Omega}\widehat{\boldsymbol{\sigma}}_1 v_1\cdot\nu \\ &= \int_\Omega \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}\cdot\nabla v_1 - \int_{\partial\Omega}\widehat{F}_3 v_1. \end{aligned}$$

Thus

$$|b_3(U,v)| \le ||U|| \left(\left|\left|\begin{pmatrix}\alpha_3\\ \alpha_4\end{pmatrix}\cdot\nabla v_1\right|\right|_0 + \widehat{\alpha}_3||v_1||_{H^{1/2}(\partial\Omega)}\right),$$

and the norm we implement for v_1 is

$$||v_1||^2_{V_3} \stackrel{\text{def}}{=} \alpha_3^2\left|\left|\frac{\partial v_1}{\partial x}\right|\right|_0^2 + \alpha_4^2\left|\left|\frac{\partial v_1}{\partial y}\right|\right|_0^2 + \widehat{\alpha}_3^2||v_1||_0^2.$$

Similarly,

$$||v_2||^2_{V_4} \stackrel{\text{def}}{=} \alpha_4^2 \left|\left|\frac{\partial v_2}{\partial x}\right|\right|^2_0 + \alpha_5^2 \left|\left|\frac{\partial v_2}{\partial y}\right|\right|^2_0 + \widehat{\alpha}_4^2 \, ||v_2||^2_0$$

and

$$||v_3||^2_{V_5} \stackrel{\text{def}}{=} \alpha_1^2 \left|\left|\frac{\partial v_3}{\partial x}\right|\right|^2_0 + \alpha_2^2 \left|\left|\frac{\partial v_3}{\partial y}\right|\right|^2_0 + \widehat{\alpha}_5^2 \, ||v_2||^2_0 .$$

Now, we can define a general norm on the test space by

$$\begin{aligned} ||(\boldsymbol{q}_1, \dots, \boldsymbol{q}_L; v_1, \dots, v_M)||^2 = \\ & \int_\Omega \left(\sum_{i=1}^{L} (a_i \nabla \cdot \boldsymbol{q}_i)^2 + \sum_{i=i}^{L} ((b_{i1} q_{i1})^2 + (b_{i2} q_{i2})^2) \right. \\ & \left. + \sum_{i=1}^{M} \left(\left(c_{i1} \frac{\partial v_i}{\partial x} \right)^2 + \left(c_{i2} \frac{\partial v_i}{\partial y} \right)^2 \right) + \sum_{i=1}^{M} (d_i v_i)^2 \right) \\ & + \int_{\partial\Omega} \left(\sum_{i=1}^{L} (e_i \boldsymbol{q}_i \cdot \nu)^2 + \sum_{i=1}^{M} (f_i v_i)^2 \right) . \end{aligned} \tag{4.2}$$

Based on the analysis above, we require

$$\begin{aligned} a_1 &= \alpha_1, & a_2 &= \alpha_2 \\ b_{11} &= \frac{\alpha_3 + \alpha_7}{2\mu}, & b_{12} &= \frac{\alpha_4}{2\mu} + \alpha_6 \\ b_{21} &= \frac{\alpha_4}{2\mu} + \alpha_6, & b_{22} &= \frac{\alpha_5 + \alpha_7}{2\mu} \\ c_{11} &= \alpha_3, & c_{12} &= \alpha_4 \\ c_{21} &= \alpha_4 & c_{22} &= \alpha_5 \\ c_{31} &= \alpha_1 & c_{32} &= \alpha_2 \\ d_i &= 0 \\ e_1 &= \widehat{\alpha}_1, e_2 = \widehat{\alpha}_2 & f_1 &= \widehat{\alpha}_3, f_2 = \widehat{\alpha}_4, f_3 = \widehat{\alpha}_5. \end{aligned} \tag{4.3}$$

4.1. Choice of α values. How do we determine appropriate weights α_i and $\widehat{\alpha}_i$ for the norm of U? Our choice is motivated by considerations of *norm equivalence* arising, for instance, in the least-squares finite element literature, see [1, Sec. 4.5]. For simplicity, we apply a similar guideline, which we call *scale equivalence*. Let us consider a mesh with elements of size h. In a least-squares method, one would motivate the choice of weights by examining the factors of h entering the stiffness matrix through derivatives in the bilinear form. One would then select weights so that each term of the bilinear form had the same h-factor, thereby ensuring that no single term dominates the least-squares functional as $h \to 0$.

Recall that in DPG the optimality is expressed in terms of the energy norm in equation (2.9), which in turn is defined by the inner product on V in equation (2.7). As in least-squares methods, there is an underlying optimization principle (equations (2.9) and (2.5)), and thus it makes sense to have all terms in the test space inner product equally weighted in the discrete setting. In this section, therefore, we aim to determine weights α_i and $\widehat{\alpha}_i$ that will allow this.

Computing the optimal test functions involves the solution of a problem of the form

$$(t_j, \tilde{e}_i)_V = b(e_j, \tilde{e}_i),$$

where the $\tilde{e}_i$ form the basis for the enriched polynomial space $\widetilde{V}_n$ used to represent the test functions, and t_j is the optimal test function corresponding to $e_j \in U$. Thus the matrix for determining the optimal test functions is generated by computing inner products $(\tilde{e}_k, \tilde{e}_i)_V$. The goal is to keep the summands entering this matrix of the same order of magnitude in h.

We assume a partition of Ω into quadrilateral elements. Since the various components (e.g. $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$) of the test function do not interact, we can examine each separately. Suppose that the element has dimensions (h_1, h_2) and $\boldsymbol{q}_1 = \binom{xy}{xy}$. Then

$$\begin{aligned}(\boldsymbol{q}_1, \boldsymbol{q}_1)_V &= \int_K \left(a_1^2(y^2+2xy+x^2) + (b_{11}^2+b_{12}^2)x^2y^2\right) + \int_{\partial K} e_1^2 \left(\begin{pmatrix} xy \\ xy \end{pmatrix} \cdot \nu\right)^2 \\ &= O\left(a_1^2(h_1h_2^3 + h_1^2h_2^2 + h_1^3h_2) + (b_{11}^2+b_{12}^2)h_1^3h_2^3 + e_1^2(h_1^2h_2^3 + h_1^3h_2^2)\right)\end{aligned}$$

Clearly, no choice of weights will make all summands of the same order in both h_1 and h_2; the best we can do is to make the h_1 and h_2 orders of each summand differ by no more than 2, and make the sum of the h_1 and h_2 orders the same across all summands. This can be accomplished by setting $a_1^2 = h_1h_2, b_{11}^2 = b_{12}^2 = 1$, and $e_1^2 = \sqrt{h_1h_2}$.

The computation with $\boldsymbol{q}_2$ is identical. Now, consider $v_1 = xy$. We have

$$\begin{aligned}(v_1, v_1)_V &= \int_K \left(c_{11}^2y^2 + c_{12}^2x^2 + d_1^2x^2y^2\right) + \int_{\partial K} f_1^2x^2y^2 \\ &= O\left(c_{11}^2h_1h_2^3 + c_{12}^2h_1^3h_2 + (b_{11}^2+d_1^2)h_1^3h_2^3 + f_1^2(h_1^2h_2^3 + h_1^3h_2^2)\right)\end{aligned}$$

Again, we cannot choose the coefficients to get each summand to have the same order in both h_1 and h_2, but here at least the c_{11} and c_{12} coefficients can be chosen so that their respective terms match precisely. We would like to have $c_{11}^2 = h_1^2$, $c_{12}^2 = h_2^2, d_1^2 = 1$, and $f_1^2 = \sqrt{h_1h_2}$, and similarly for v_2, we'd like $c_{21}^2 = h_1^2$, etc. This cannot be fully achieved because of the way the α_i enter the inner product; specifically, $\alpha_4 = c_{12} = c_{21}$. Instead, we arrive at the following weights:

$$\alpha_1 = \alpha_2 = \sqrt{h_1h_2} \tag{4.4}$$
$$\alpha_3 = h_1 \tag{4.5}$$
$$\alpha_4 = \sqrt{h_1h_2} \tag{4.6}$$
$$\alpha_5 = h_2 \tag{4.7}$$
$$\alpha_6 = \alpha_7 = 1 \tag{4.8}$$
$$\widehat{\alpha}_i = \sqrt{\sqrt{h_1h_2}}\,. \tag{4.9}$$

We detail numerical results for this inner product in Section 5.3. In Section 5.4, we present a version where $\alpha_3 = \alpha_4 = \alpha_5 = 1$, with very similar results.

5. Numerical Results. We solve the Stokes problem on the domain $(-1,1) \times (-1,1)$, with $\mu = 1$. We follow the choice of manufactured solution employed in a paper by Cockburn et al. [4], in which they apply the LDG method to Stokes. We compare our convergence rates to theirs; the L^2 error measurements are not strictly comparable because they employ a triangular mesh, whereas we use a quadrilateral mesh. As stated in Section 2.4, the space for V is an "enriched" polynomial space. The numerical results presented below were produced with test functions of degree one higher than that of the trial space.

Although the differences between our meshes and those in Cockburn et al. mean that our error measurements are not strictly comparable, we still would expect to attain similar *rates* of convergence, and for the L^2 error values in each component to be within an order of magnitude or so. The rates of convergence we would expect in a velocity-stress-pressure (VSP) least-squares context would be $k+1$ for the velocity components u_1 and u_2, and k for the pressure p, where k denotes the polynomial degree of the trial space [1, p. 269]. We have yet to carry out the convergence analysis for DPG.

Following Cockburn et al., we use

$$\begin{aligned} u_1 &= -e^x(y\cos y + \sin y) \\ u_2 &= e^x y \sin y \\ p &= 2\mu e^x \sin y \end{aligned}$$

as our manufactured solution. We impose the constraint $p(0,0) = p_0$ in order to establish the uniqueness of the solution.

We try four inner products, the first two as baselines, and the latter two as suggested by our analysis. As expected, the choice of the inner product makes a great deal of difference to the rate of convergence.

5.1. Generic Inner Product. As a baseline to show the importance of a good inner product on the test function space, the results in this section are produced using a test space inner product unrelated to our analysis. In the general form of the norm specified in equation (4.2), let $a_i = b_{ij} = c_{ij} = d_i = e_i = f_i = 1$.

As can be seen in Table A.1, although our convergence rates generally start out near the asymptotic rates predicted, they fall off quickly. The rate for pressure with quadratic elements is particularly poor. The L^2 error values in $\boldsymbol{u}$ are perhaps not too bad, within an order of magnitude or so of the LDG results. However, the pressure error values are extremely poor, off by up to three orders of magnitude.

As an experiment, we tried enriching the fluxes, using polynomials of degree $k+1$ to represent the solution fluxes; at the same time, we enriched the test function space further, using polynomials of degree $k+2$. As shown in Table A.2, this uniformly reduces the error, particularly in the pressure, and improves the convergence rate observed in the pressure for quadratic elements. Although cubic elements also saw uniformly reduced error, the convergence rates observed were somewhat worse.

5.2. "All Ones" Inner Product. In Section 4.1, we derived weights for the inner product so as to weight all terms in the determination of the optimal test functions equally. To see the impact of our choice of those weights in relief, we try an inner product in which $\alpha_i = \widehat{\alpha}_j = 1$. Compared with the generic inner product employed in the previous section, this inner product takes account of the continuity argument.

As can be seen in Table B.1, with this choice of inner product, DPG performs slightly better than with the generic inner product, but the rates of convergence in p are quite poor, especially for quadratic elements. For the 64×64 mesh, we even see regression in the p error compared with the 32×32 mesh, suggesting that some terms in the inner product dominate as $h \to 0$, preventing convergence.

As in Section 5.1, we tried enriching the fluxes, using polynomials of degree $k+1$ to represent the solution fluxes; at the same time, we enriched the test function space further, using polynomials of degree $k+2$. As shown in Table B.2, this uniformly reduces the error, particularly in the pressure, and improves the convergence rate observed in the pressure for quadratic elements. Although cubic elements also saw uniformly reduced error, the convergence *rates* observed for pressure were somewhat worse.

5.3. Mesh-Dependent Inner Product. In this inner product, we choose the α_i values as derived in Section 4.1 and specified in equations (4.4)-(4.9). There, we aimed to achieve scale equivalence in the determination of the optimal test functions while selecting an inner product that allowed our argument for the continuity of $b(\cdot,\cdot)$ to remain intact.

As can be seen in Table C.1, with this inner product, we have far superior convergence compared with either of the previous two inner products we have considered. Here, the convergence rates for both velocity and pressure are very close to those predicted by theory, and the L^2 error for $\boldsymbol{u}$ is within a factor of 3 of the LDG results. However, our L^2 error in pressure remains more than an order of magnitude worse than that LDG was able to achieve.

We again tried enriching the flux space; the results are in Table C.2. Here, however, the results are merely comparable to those in the enriched flux space experiments using the previous two inner products. With cubic elements, enriching the fluxes made for slightly worse results, perhaps due to round-off errors. All told, it appears that whatever is lost to scale inequivalence in the previous two inner products is regained through higher-fidelity flux approximation.

5.4. Mesh-Dependent Inner Product, Least-Squares Compromise $\alpha_3 = \alpha_4 = \alpha_5 = 1$. Finally, we tried an inner product with weights just as in Section 5.3, except $\alpha_3 = \alpha_4 = \alpha_5 = 1$. The rationale was that, in the norm of U, these weights are applied to the tensor $\underline{\boldsymbol{\sigma}}$, which is a derivative, so the natural norm for U in a least-squares approach (arising from concern for scale equivalence of terms within the form $b(\cdot,\cdot)$) would have an extra factor of h on u_1 and u_2 compared to the components of $\underline{\boldsymbol{\sigma}}$.

As can be seen in Table D.1, the results are almost identical to those reported in the previous section. The only exception is the error in the pressure on a cubic, 64×64 mesh, for which the present inner product produced an error about half that produced by the previous inner product.

We again tried enriching the flux space; the results are in Table D.2. The enriched fluxes again give us lower error, but as we refine, the advantage this gives us appears to become less significant; for cubic elements, the error values for the 32×32 mesh are nearly identical to those we attained for this inner product without enriching the fluxes.

6. Conclusions and Future Work. A robust application of the DPG method requires a test space inner product that simultaneously allows a proof of coercivity and continuity of the variational form and achieves scale equivalence in both the inner product matrix used to compute the optimal test functions, and in the stiffness matrix used to compute the solution. In this paper, we have applied the DPG method to the Stokes problem, comparing several inner product choices. The two inner products that did not account for scale equivalence within the inner product matrix both demonstrated substantially poorer performance; while those that did account for it achieved optimal convergence rates.

The fact that our L^2 errors in pressure were substantially worse than those for the LDG method suggests that there may be a better choice of inner product; it may be that an examination of coercivity (here absent) would suggest a better choice. The strategy we intend to employ in the future is to use test norms motivated by examining the *optimal test norm* studied in previous DPG efforts (see [10, Sec. 2]); this is a norm on the *global* test space for which $||\cdot||_E = ||\cdot||_U$.

Enriching the flux space erased most of the distinctions between our various inner products, and greatly reduced the errors observed in the pressure. It appears that the benefit of using a better inner product is that we can save the computational cost associated with the enriched flux space!

In the future, we plan to investigate the *hp*-adaptive solution of Stokes equations using DPG, which offers stability independent of discretization parameters. We also plan to use DPG to solve Stokes on polygons and polyhedra.

The work presented here was completed using L. Demkowicz's *hp*-adaptive code; we are presently implementing a DPG framework using Intrepid [2] and Trilinos [8].

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Appendix A. Numerical Results: Generic Inner Product.

Table A.1
L^2 error and h-convergence rates for generic inner product selected without reference to continuity argument, as defined in Section 5.1: comparison with LDG [4].

Quadratic Elements								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	3.6e-2	-	-	-	5.9e-1	-	-	-
4 × 4	5.1e-3	2.82	-	-	2.0e-1	1.58	-	-
8 × 8	8.1e-4	2.73	2.0e-4	-	1.7e-1	0.92	5.1e-4	-
16 × 16	1.6e-4	2.59	2.4e-5	3.06	9.6e-2	0.81	1.2e-4	2.09
32 × 32	3.9e-5	2.46	2.9e-6	3.05	5.1e-2	0.81	3.0e-5	2.04
64 × 64	9.8e-6	2.36	-	-	2.6e-2	0.83	-	-
Cubic Elements								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	2.7e-3	-	-	-	2.7e-1	-	-	-
4 × 4	1.7e-4	3.97	5.8e-5	-	2.8e-2	3.27	2.4e-4	-
8 × 8	1.2e-5	3.88	3.6e-6	4.01	3.8e-3	3.08	3.9e-5	2.62
16 × 16	1.0e-6	3.79	2.2e-7	4.02	5.1e-4	3.01	5.3e-6	2.75
32 × 32	1.0e-7	3.68	-	-	7.1e-5	2.96	-	-
64 × 64	1.3e-8	3.55	-	-	3.6e-5	2.67	-	-

Table A.2
L^2 error and h-convergence rates for generic inner product selected without reference to continuity argument with enriched fluxes ($k_{flux} = k+1, k_{test} = k+2$), as defined in Section 5.1: comparison with LDG [4].

Quadratic Elements - Enriched Fluxes								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	3.5e-2	-	-	-	7.6e-1	-	-	-
4 × 4	4.4e-3	3.01	-	-	1.4e-1	2.48	-	-
8 × 8	5.4e-4	3.02	2.0e-4	-	1.9e-2	2.67	5.1e-4	-
16 × 16	6.6e-5	3.03	2.4e-5	3.06	3.5e-3	2.61	1.2e-4	2.09
32 × 32	8.2e-6	3.02	2.9e-6	3.05	7.0e-4	2.54	3.0e-5	2.04
Cubic Elements - Enriched Fluxes								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	2.6e-3	-	-	-	3.2e-2	-	-	-
4 × 4	1.6e-4	3.96	5.8e-5	-	9.4e-3	1.78	2.4e-4	-
8 × 8	1.0e-5	4.00	3.6e-6	4.01	1.2e-3	2.35	3.9e-5	2.62
16 × 16	6.0e-7	4.02	2.2e-7	4.02	1.6e-4	2.58	5.3e-6	2.75
32 × 32	3.7e-8	4.02	-	-	3.3e-5	2.57	-	-

Appendix B. Numerical Results: "All Ones" Inner Product.

Table B.1
L^2 error and h-convergence rates for an inner product for which $\alpha_i = \widehat{\alpha}_j = 1$ (i.e. with weights selected without concern for scale equivalence) as defined in Section 5.2: comparison with LDG [4].

Quadratic Elements								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	3.6e-2	-	-	-	5.6e-1	-	-	-
4 × 4	5.0e-3	2.84	-	-	1.7e-1	1.70	-	-
8 × 8	7.7e-4	2.77	2.0e-4	-	1.4e-1	1.00	5.1e-4	-
16 × 16	1.5e-4	2.64	2.4e-5	3.06	8.2e-2	0.87	1.2e-4	2.09
32 × 32	3.4e-5	2.51	2.9e-6	3.05	4.3e-2	0.85	3.0e-5	2.04
64 × 64	8.5e-6	2.40	-	-	2.2e-2	0.86	-	-
Cubic Elements								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	2.7e-3	-	-	-	2.7e-1	-	-	-
4 × 4	1.7e-4	3.99	5.8e-5	-	2.7e-2	3.30	2.4e-4	-
8 × 8	1.2e-5	3.92	3.6e-6	4.01	3.7e-3	3.09	3.9e-5	2.62
16 × 16	9.2e-7	3.84	2.2e-7	4.02	4.8e-4	3.03	5.3e-6	2.75
32 × 32	9.0e-8	3.73	-	-	6.3e-5	2.99	-	-
64 × 64	1.4e-8	3.55	-	-	2.8e-4	2.25	-	-

Table B.2
L^2 error and h-convergence rates for an inner product for which $\alpha_i = \widehat{\alpha}_j = 1$ (i.e. with weights selected without concern for scale equivalence) with enriched fluxes ($k_{flux} = k+1, k_{test} = k+2$), as defined in Section 5.2: comparison with LDG [4].

Quadratic Elements - Enriched Fluxes								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	3.5e-2	-	-	-	7.6e-1	-	-	-
4 × 4	4.4e-3	3.00	-	-	1.3e-1	2.50	-	-
8 × 8	5.4e-4	3.02	2.0e-4	-	1.9e-2	2.66	5.1e-4	-
16 × 16	6.6e-5	3.02	2.4e-5	3.06	3.6e-3	2.60	1.2e-4	2.09
32 × 32	8.2e-6	3.02	2.9e-6	3.05	7.1e-4	2.53	3.0e-5	2.04
Cubic Elements - Enriched Fluxes								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	2.5e-3	-	-	-	3.5e-2	-	-	-
4 × 4	1.6e-4	3.95	5.8e-5	-	8.6e-3	2.02	2.4e-4	-
8 × 8	1.0e-5	3.98	3.6e-6	4.01	1.2e-3	2.46	3.9e-5	2.62
16 × 16	6.1e-7	4.01	2.2e-7	4.02	1.5e-4	2.65	5.3e-6	2.75
32 × 32	3.7e-8	4.02	-	-	3.3e-5	2.59	-	-

Appendix C. Numerical Results: Mesh-Dependent Inner Product.

Table C.1

L^2 error and h-convergence rates for a mesh-dependent inner product with weights as specified in equations (4.4)-(4.9) and discussed in Section 5.3, an inner product that represents our best compromise between the continuity argument and concerns for scale equivalence in the determination of the optimal test functions.: comparison with LDG [4].

Quadratic Elements								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	3.6e-2	-	-	-	5.6e-1	-	-	-
4 × 4	4.7e-3	2.91	-	-	1.1e-1	2.39	-	-
8 × 8	6.0e-4	2.94	2.0e-4	-	2.8e-2	2.15	5.1e-4	-
16 × 16	7.6e-5	2.96	2.4e-5	3.06	7.3e-3	2.07	1.2e-4	2.09
32 × 32	9.5e-6	2.97	2.9e-6	3.05	1.8e-3	2.05	3.0e-5	2.04
64 × 64	1.2e-6	2.98	-	-	4.3e-4	2.04	-	-
Cubic Elements								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	2.7e-3	-	-	-	2.7e-1	-	-	-
4 × 4	1.6e-4	4.06	5.8e-5	-	2.4e-2	3.47	2.4e-4	-
8 × 8	9.9e-6	4.04	3.6e-6	4.01	3.1e-3	3.22	3.9e-5	2.62
16 × 16	6.1e-7	4.03	2.2e-7	4.02	3.9e-4	3.13	5.3e-6	2.75
32 × 32	3.8e-8	4.02	-	-	4.9e-5	3.08	-	-
64 × 64	2.4e-9	4.02	-	-	4.3e-6	3.13	-	-

Table C.2

L^2 error and h-convergence rates for a mesh-dependent inner product with weights as specified in equations (4.4)-(4.9) and discussed in Section 5.3, with enriched fluxes ($k_{flux} = k+1, k_{test} = k+2$): comparison with LDG [4].

Quadratic Elements - Enriched Fluxes								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	3.5e-2	-	-	-	7.6e-1	-	-	-
4 × 4	4.4e-3	3.00	-	-	1.2e-1	2.62	-	-
8 × 8	5.4e-4	3.01	2.0e-4	-	1.9e-2	2.64	5.1e-4	-
16 × 16	6.7e-5	3.01	2.4e-5	3.06	4.7e-3	2.46	1.2e-4	2.09
32 × 32	8.4e-6	3.01	2.9e-6	3.05	1.1e-3	2.35	3.0e-5	2.04
64 × 64	1.0e-6	3.01	-	-	2.8e-4	2.27	-	-
Cubic Elements - Enriched Fluxes								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	2.5e-3	-	-	-	3.5e-2	-	-	-
4 × 4	1.6e-4	3.99	5.8e-5	-	5.4e-3	2.70	2.4e-4	-
8 × 8	1.0e-5	3.99	3.6e-6	4.01	6.1e-4	2.92	3.9e-5	2.62
16 × 16	6.2e-7	4.00	2.2e-7	4.02	1.3e-4	2.74	5.3e-6	2.75
32 × 32	3.8e-8	4.00	-	-	2.5e-5	2.63	-	-

Appendix D. Numerical Results: Mesh-Dependent Inner Product, Least-Squares Compromise.

TABLE D.1

L^2 error and h-convergence rates for an inner product with weights as described in Section 5.4, an inner product that brings concern for scale equivalence in the stiffness matrix into our compromise between the continuity argument and concerns for scale equivalence in the determination of optimal test functions: comparison with LDG [4].

Quadratic Elements								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	3.6e-2	-	-	-	5.6e-1	-	-	-
4 × 4	4.6e-3	2.97	-	-	1.2e-1	2.23	-	-
8 × 8	5.8e-4	2.97	2.0e-4	-	2.7e-2	2.19	5.1e-4	-
16 × 16	7.4e-5	2.97	2.4e-5	3.06	6.6e-3	2.14	1.2e-4	2.09
32 × 32	9.4e-6	2.97	2.9e-6	3.05	1.7e-3	2.10	3.0e-5	2.04
64 × 64	1.2e-6	2.98	-	-	4.1e-4	2.07	-	-
Cubic Elements								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	2.7e-3	-	-	-	2.7e-1	-	-	-
4 × 4	1.6e-4	4.10	5.8e-5	-	2.4e-2	3.47	2.4e-4	-
8 × 8	9.7e-6	4.05	3.6e-6	4.01	3.1e-3	3.22	3.9e-5	2.62
16 × 16	6.1e-7	4.03	2.2e-7	4.02	3.9e-4	3.12	5.3e-6	2.75
32 × 32	3.8e-8	4.02	-	-	4.9e-5	3.08	-	-
64 × 64	2.4e-9	4.02	-	-	2.3e-6	3.26	-	-

TABLE D.2

L^2 error and h-convergence rates for an inner product with weights as described in Section 5.4, an inner product that brings concern for scale equivalence in the stiffness matrix into our compromise between the continuity argument and concerns for scale equivalence in the determination of optimal test functions, with enriched fluxes ($k_{flux} = k + 1, k_{test} = k + 2$): comparison with LDG [4].

Quadratic Elements - Enriched Fluxes								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	3.5e-2	-	-	-	7.6e-1	-	-	-
4 × 4	4.2e-3	3.08	-	-	4.2e-2	4.18	-	-
8 × 8	5.2e-4	3.04	2.0e-4	-	1.0e-2	3.11	5.1e-4	-
16 × 16	6.5e-5	3.02	2.4e-5	3.06	2.7e-3	2.64	1.2e-4	2.09
32 × 32	8.2e-6	3.02	2.9e-6	3.05	7.3e-4	2.40	3.0e-5	2.04
Cubic Elements - Enriched Fluxes								
	DPG Error		LDG Error		DPG Error		LDG Error	
Mesh Size	$\boldsymbol{u}$	rate	$\boldsymbol{u}$	rate	p	rate	p	rate
2 × 2	2.5e-3	-	-	-	3.5e-2	-	-	-
4 × 4	1.5e-4	4.06	5.8e-5	-	4.5e-3	2.96	2.4e-4	-
8 × 8	9.5e-6	4.04	3.6e-6	4.01	2.0e-3	2.07	3.9e-5	2.62
16 × 16	5.9e-7	4.02	2.2e-7	4.02	3.6e-4	2.10	5.3e-6	2.75
32 × 32	3.7e-8	4.02	-	-	4.9e-5	2.26	-	-