Introduction to Turbulence Modeling

or Why Turbulence Modeling is Black Magic

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Reynolds Averaged Navier Stokes

Let $\phi = \bar{\phi} + \phi'$, where

$$\bar{\phi} = \lim_{T\to\infty} \int_{t_0}^{t_0+T} \phi(t)\, dt$$

Interested in solving for $\bar{u}_i$. Nondimensionalizing such that $\rho = 1$,

Continuity Equation

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\Downarrow$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

Momentum Equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_i}$$

$$\Downarrow$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[-\overline{u_i' u_j'} + \frac{1}{Re}\frac{\partial \bar{u}_i}{\partial x_j}\right]$$

But the Reynolds stress, $\overline{u_i' u_j'}$ is an unclosed quantity.

Modeling the Reynolds Stress

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[-\overline{u_i' u_j'} + \frac{1}{Re}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right]$$

Eddy Viscosity Hypothesis

$$\overline{u_i' u_j'} = -\nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) + \frac{2k}{3}\delta_{ij}$$

where $k = \frac{1}{2}\overline{u_i' u_i'}$.

This assumption is fundamentally flawed, but useful.

We still need a model for ν_T and k.

Spalart-Allmaras Model

- Model the eddy viscosity
- Assume k is negligible
- Surprisingly decent for the types of flows it was designed for (external aerodynamics)

$$\frac{\partial \tilde{\nu}}{\partial t} + \bar{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = C_{b1}[1 - f_{t2}]\tilde{S}\tilde{\nu} + \frac{1}{\sigma}\left\{\nabla \cdot [(\nu + \tilde{\nu})\nabla\tilde{\nu}] + C_{b2}\left|\nabla\tilde{\nu}\right|^2\right\}$$
$$- \left[C_{w1} f_w - \frac{C_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + f_{t1}\Delta U^2$$

$$\nu_T = \tilde{\nu} f_{v1}, \quad f_{v1} = \frac{\chi^3}{\chi^3 + v_{v1}^3}, \quad \chi = \frac{\tilde{\nu}}{\nu} \quad \cdots$$

$k - \epsilon$ Exact Equations

Turbulence Kinetic Energy Equation

$$\frac{\partial k}{\partial t} + \bar{u}_i \frac{\partial k}{\partial x_i} = -\overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial}{\partial x_i} \left(\frac{1}{2} \overline{u_j' u_j' u_i'} + \overline{p' u_i'} \right) + \nu \frac{\partial^2 k}{\partial x_i \partial x_i} - \epsilon$$

Dissipation Equation

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_i \frac{\partial \epsilon}{\partial x_i} = \nu \frac{\partial^2 \epsilon}{\partial x_i \partial x_i} + P_\epsilon + D_\epsilon - \Phi_\epsilon$$

$$P_\epsilon = -2\nu \left[\overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}} \frac{\partial \bar{u}_i}{\partial x_k} + \overline{\frac{\partial u_j'}{\partial x_i} \frac{\partial u_j'}{\partial x_k}} \frac{\partial \bar{u}_i}{\partial x_k} + \overline{u_k' \frac{\partial u_i'}{\partial x_j}} \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_j} + \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_k'}{\partial x_m}} \right]$$

$$D_\epsilon = -\frac{\partial}{\partial x_k} \left(\overline{u_k' \epsilon'} + 2\nu \overline{\frac{\partial p'}{\partial x_m} \frac{\partial u_k'}{\partial x_m}} \right)$$

$$\Phi_\epsilon = 2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m} \frac{\partial^2 u_i'}{\partial x_k \partial x_m}}$$

$k - \epsilon$ Approximations

Turbulence Kinetic Energy Equation

$$\frac{1}{2}\overline{u_j'u_j'u_i'} + \overline{p'u_i'} \approx -\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_i}$$

Dissipation Equation

From dimensional analysis

$$\nu_T \approx C_\mu \frac{k^2}{\epsilon}$$

$$P_\epsilon \approx -C_{\epsilon 1}\frac{\epsilon}{k}\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}$$

$$\Phi_\epsilon \approx C_{\epsilon 2}\frac{\epsilon^2}{k}$$

From gradient transport model

$$D_\epsilon \approx \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_i}\right)$$

Channel Flow Predictions

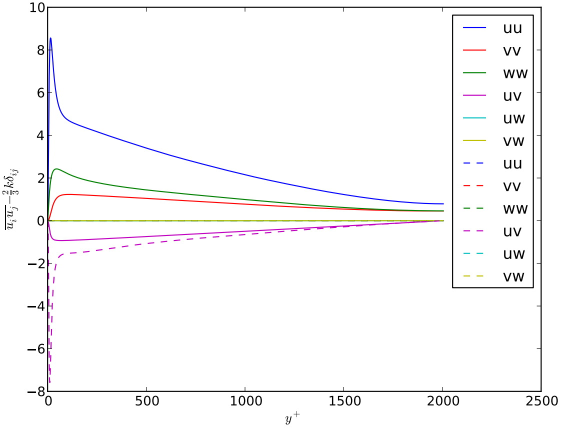


Figure: Reynolds Stress Components

$$\overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij}$$

$$\approx -\nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

Channel Flow Predictions

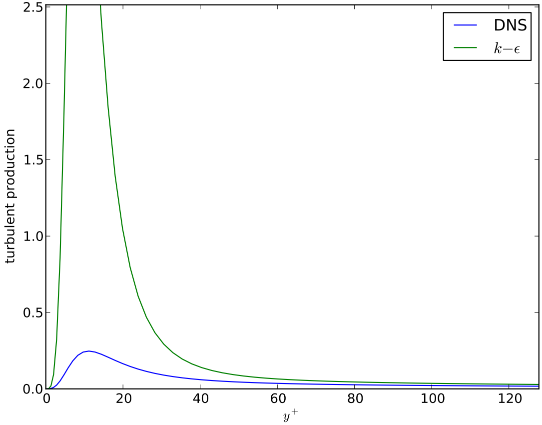


Figure: Production of Turbulent Kinetic Energy

$$\overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} \approx$$

$$C_\mu \frac{k^2}{\epsilon} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j}$$

Channel Flow Predictions

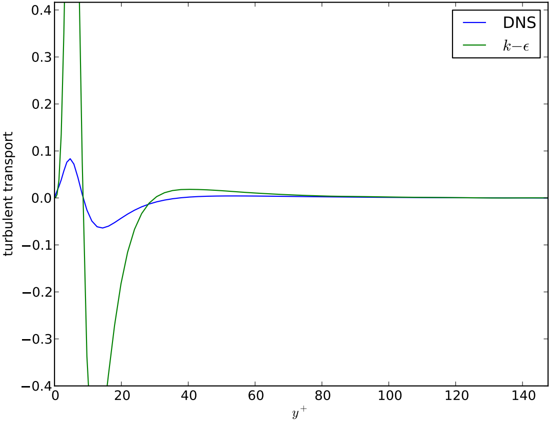


Figure: Transport of Turbulent Kinetic Energy

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}\overline{u'_k u'_k u'_i} + \overline{p' u'_i}\right)$$
$$\approx -\frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_i}\right)$$

Notes on $k - \epsilon$

Fixes near the wall

- Wall functions
- Two-layer models
- SST $k - \omega$
- $\overline{v^2} - f$ four equation model (f is an elliptic relaxation function)

Boundary conditions

- BC on ϵ is not obvious
- Enforce both $k = 0$ and $\frac{\partial k}{\partial n} = 0$ at walls.

Popularity

- One of the earliest models
- Physically motivated derivation
- Insensitive to freestream conditions on k and ϵ

Wilcox (1993) $k - \omega$ Model

Turbulence Kinetic Energy Equation

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = 2\nu_T \left|S\right|^2 - C_\mu k\omega + \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right)$$

Specific Dissipation Equation

$$\frac{\partial \omega}{\partial t} + \bar{u}_j \frac{\partial \omega}{\partial x_j} = 2C_{\omega 1} \left|S\right|^2 - C_{\omega 2}\omega^2 + \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_T}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right)$$

$$\omega \equiv \frac{\epsilon}{C_\mu k}$$

$$\nu_T = \frac{k}{\omega}$$

Notes on $k - \omega$

Wall Treatment

- Produces decent results at the wall
- Formally, ω is singular at a perfectly smooth wall
- Numerically, this is fixed by assuming a finite roughness
  - $\omega_{wall} = \frac{40000\nu_{wall}}{k_s^2}$, need $\frac{u_\tau k_s}{\nu} < 5$
- Can perform the same trick with two BCs on k

Other Notes

- Early models were very sensitive to freestream conditions on ω
- Menter propsed *shear stress transport* model to overcome this shortcoming
- Possible to write a $k - \epsilon$ model with $k - \omega$ “physics”

Other Ideas – Unsteady RANS

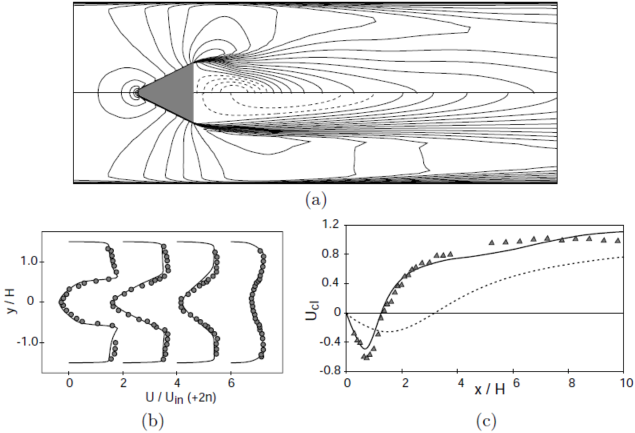
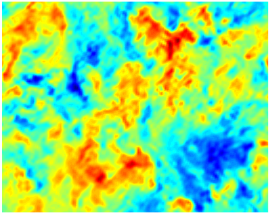


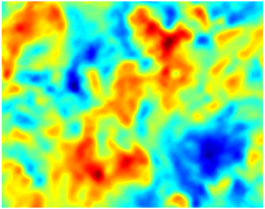
Figure 7.72 *Vortex shedding from a triangular cylinder. (a) composite showing time-average contours of U in the upper half, versus a steady solution in the lower half. The dashed lines indicate negative velocity. (b) time-averaged velocity profiles in the wake. (c) velocity along centerline: time-average, ——— ; steady computation, - - - - (Durbin, 1995).*

Other Ideas

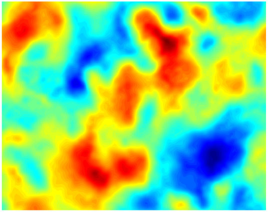
- Large Eddy Simulation
- Reynolds Stress Models
  - Elliptic relaxation model solves 18 coupled, highly nonlinear PDEs
- Variational Multiscale
- Direct Numerical Simulation



DNS Velocity Field



Filtered Velocity Field
$\Delta = L/32$



Filtered Velocity Field
$\Delta = L/16$