A Discontinuous Petrov-Galerkin Methodology for Incompressible Flow

Towards Automatic, Robust Mesh Adaptivity

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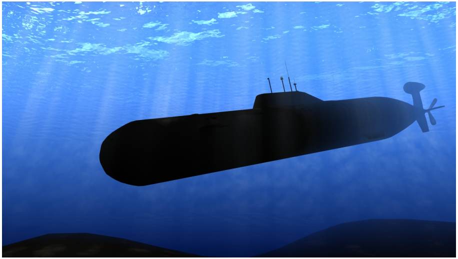
Motivation

Incompressible flows:

- arise in a variety of applications, from hydraulics to aerodynamics
- Navier-Stokes equations are of fundamental physical and mathematical interest:
  - believed to hold the key to understanding turbulence
  - precise conditions for existence and uniqueness of solutions remain unknown (Millennium Prize problem)



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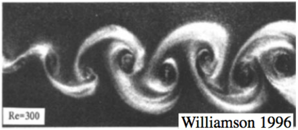


Typical solutions of incompressible flow problems involve both fine- and large-scale phenomena.

$$\Longrightarrow$$

A uniform finite element mesh of sufficient granularity is

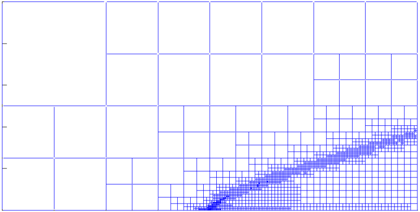
- wasteful of computational resources, or
- infeasible because of resource limitations.

Therefore, an adaptive mesh is required.

Motivation

In industry:

- adaptivity schemes used are ad hoc, requiring a domain expert to predict features of the solution,
- a badly chosen mesh may take considerably longer to converge or fail to converge, and
- typically, the Navier-Stokes solve will be just one component in an optimization loop $\implies$ any failure requiring human intervention is costly.



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By robust, we mean:

- always converges to a solution in predictable time, and
- the adaptive scheme is independent of the problem: no special expertise required for adaptivity.

Outline

The Abstract Problem and Minimization of the Residual

Take U, V Hilbert.

We seek $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V,$$

where b and l are linear in v. Define B by $Bu = b(u, \cdot) \in V'$; Bu is a linear functional on the test space V.

We seek to minimize the residual in the discrete space $U_h \subset U$:

$$u_h = \underset{w_h \in U_h}{\arg\min} \; \frac{1}{2} \left|\left| Bw_h - l \right|\right|_{V'}^2 .$$

The Abstract Problem and Minimization of the Residual

$$u_h = \underset{w_h \in U_h}{\arg\min} \; \frac{1}{2} \left|\left| B w_h - l \right|\right|_{V'}^2 .$$

Now, the dual space V' is not especially easy to work with; we would prefer to work with V itself. Recalling that the Riesz operator $R_V : V \to V'$ defined by

$$\langle R_V v, \delta v \rangle = (v, \delta v)_V, \quad \forall \delta v \in V,$$

is an *isometry*—$\left|\left| R_V v \right|\right|_{V'} = \left|\left| v \right|\right|_V$—we can rewrite the term we want to minimize as a norm in V:

$$\begin{aligned} \frac{1}{2} \left|\left| B w_h - l \right|\right|_{V'}^2 &= \frac{1}{2} \left|\left| R_V^{-1} \left(B w_h - l \right) \right|\right|_V^2 \\ &= \frac{1}{2} \left(R_V^{-1} \left(B w_h - l \right), R_V^{-1} \left(B w_h - l \right) \right)_V . \end{aligned}$$

The Abstract Problem and Minimization of the Residual

We seek to minimize

$$\frac{1}{2}\left(R_V^{-1}\left(Bw_h - l\right), R_V^{-1}\left(Bw_h - l\right)\right)_V .$$

The first-order optimality condition requires that the Gâteaux derivative be equal to zero for minimizer u_h; assuming B is linear, we have

$$\left(R_V^{-1}\left(Bu_h - l\right), R_V^{-1}B\delta u_h\right)_V = 0, \quad \forall \delta u_h \in U_h.$$

By the definition of R_V, this is equivalent to

$$\langle Bu_h - l, R_V^{-1}B\delta u_h\rangle = 0 \quad \forall \delta u_h \in U_h.$$

The Abstract Problem and Minimization of the Residual

We have:

$$\langle Bu_h - l, R_V^{-1} B\delta u_h \rangle = 0 \quad \forall \delta u_h \in U_h.$$

Now, if we identify $v_{\delta u_h} = R_V^{-1} B\delta u_h$ as a test function, we can rewrite this as

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}).$$

Thus, the discrete solution that minimizes the residual is exactly attained by testing the original equation with appropriate test functions. We call these optimal test functions.[1]

[1] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. Part II: Optimal test functions. *Numer. Meth. Part. D. E.*, 2010. in print

Evaluating Error

The derivation prompts the definition of an energy norm on the trial space:

$$\left|\left|u\right|\right|_E \stackrel{\text{def}}{=} \left|\left|Bu\right|\right|_{V'} = \sup_{v \in V} \frac{b(u,v)}{\left|\left|v\right|\right|_V}.$$

We can use this to define a measure of the error:

$$\begin{aligned} \left|\left|u - u_h\right|\right|_E &= \left|\left|Bu - Bu_h\right|\right|_{V'} \\ &= \left|\left|l - Bu_h\right|\right|_{V'} \\ &= \left|\left|R_V^{-1}\left(l - Bu_h\right)\right|\right|_V . \end{aligned}$$

Note that all the terms in the final expression are known, so that we can evaluate, not merely estimate, the error. This drives adaptivity.

From Strong-Form PDE to DPG Form

PDE

$$\triangle \phi = f$$

First-Order System

$$\nabla \cdot \psi = f$$

$$\psi - \nabla \phi = 0$$

Integration by Parts

$$(\psi \cdot n, v)_{\Gamma_h} - (\psi, \nabla v)_{\Omega_h} = (f, v)_{\Omega_h}$$

$$(\psi, q)_{\Omega_h} + (\phi, q \cdot n)_{\Gamma_h} - (\phi, \nabla \cdot q)_{\Omega_h} = 0$$

Ultraweak (DPG) Variational Formulation

$$(\widehat{\psi}_n, v)_{\Gamma_h} - (\psi, \nabla v)_{\Omega_h}$$

$$+(\psi, q)_{\Omega_h} + (\widehat{\phi}, q_n)_{\Gamma_h} - (\phi, \nabla \cdot q)_{\Omega_h} = (f, v)_{\Omega_h}$$

$$\text{“}b(u, v) = l(v)\text{”}$$

$$Au = f$$

$$(u, A^* v)_{\Omega_h} + \langle u, v \rangle_{\Gamma_h} = l(v)$$

$$(u, A^* v)_{\Omega_h} + \langle \widehat{u}, v \rangle_{\Gamma_h} = l(v)$$

Solving with DPG

Continuous Test Space

DPG Form

$$b(u_h, v) = l(v)$$

Optimal Test Functions

For each $u \in U_h$, find $v_u \in V : (v_u, w)_V = b(u, w) \forall w \in V$

Discrete Test Space

DPG Form

$$b(u_h, v_h) = l(v_h)$$

Optimal Test Functions

For each $u \in U_h$, find $v_u \in V_{p+\triangle p} : (v_u, w)_V = b(u, w)$ $\forall w \in V_{p+\triangle p}$

Stiffness Matrix

$$K_{ij} = b(e_i, v_{e_j}) = (v_{e_i}, v_{e_j})_V = (v_{e_j}, v_{e_i})_V = b(e_j, v_{e_i}) = K_{ji}$$

Error (for adaptivity)

$$||u - u_h||_E = \left|\left|R_V^{-1}(Bu_h - l)\right|\right|_V$$

Error (for adaptivity)

$$||u - u_h||_E \approx \left|\left|R_{V_{p+\Delta p}}^{-1}(Bu_h - l)\right|\right|_{V_{p+\Delta p}}$$

Graph Test Norm

For a strong operator A with formal adjoint A^*, the adjoint graph space is

$$H_{A^*} = \left\{ v \in L^2(\Omega) : A^* v \in L^2(\Omega) \right\}$$

and the (adjoint) graph norm on the test space V is given by

$$||v||_{\text{graph}} = ||v||_{H_{A^*}} = \left(||v||^2 + ||A^* v||^2 \right)^{1/2}.$$

E.g. if $A^* = \nabla$, then $H_{A^*} = H^1$, and $||v||_{H_{A^*}} = ||v||_{H^1}$.

Key Result: Well-posedness $\implies$ Optimal Convergence

Under modest technical assumptions (true for Stokes), we have[2]

$$||Au|| \geq \gamma\, ||u|| \implies \sup_{v \in H_{A^*}} \frac{b((u,\widehat{u}),v)}{||v||_{H_{A^*}}} \geq \gamma_{\mathrm{DPG}} \left(||u||^2 + ||\widehat{u}||^2_{\widehat{H}_A(\Gamma_h)} \right)^{1/2}$$

where $\gamma_{\mathrm{DPG}} = O(\gamma)$ is a mesh-independent constant, and $||\cdot||_{\widehat{H}_A(\Gamma_h)}$ is the minimum energy extension norm.

[2]Nathan V. Roberts, Tan Bui-Thanh, and Leszek F. Demkowicz. The DPG method for the Stokes problem. Technical Report 12-22, ICES, 2012

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By Babuška's Theorem,

$$\implies \left(||u - u_h||^2 + ||\widehat{u} - \widehat{u}_h||^2_{\widehat{H}_A(\Gamma_h)} \right)^{1/2} \leq \frac{M}{\gamma_{\mathrm{DPG}}} \, (\mathrm{B.A.E.}) \, .$$

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Suffices to show that $\|Au\| \geq \gamma \|u\|$ to prove optimal convergence rate!

[2] Nathan V. Roberts, Tan Bui-Thanh, and Leszek F. Demkowicz. The DPG method for the Stokes problem. Technical Report 12-22, ICES, 2012

Incompressible Flow Equations

Transient Incompressible Navier-Stokes

$$-\nabla p + \mu \Delta \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

assume steady

Steady Incompressible Navier-Stokes

$$-\nabla p + \mu \Delta \boldsymbol{u} = \boldsymbol{u} \cdot \nabla \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

neglect convective term

linearize convective term

Stokes

$$-\nabla p + \mu \Delta \boldsymbol{u} = \boldsymbol{0}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Oseen

$$-\nabla p + \mu \Delta \boldsymbol{u} = \boldsymbol{U} \cdot \nabla \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Classical Stokes Problem

The classical strong form of the Stokes problem in $\Omega \subset \mathbb{R}^2$ is given by

$$\begin{aligned} -\mu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{u}_D && \text{on } \partial\Omega, \end{aligned}$$

where μ is viscosity, p pressure, $\boldsymbol{u}$ velocity, and $\boldsymbol{f}$ a vector forcing function. Since by appropriate non-dimensionalization we can eliminate the constant μ, we take $\mu = 1$ throughout.

Stokes: Existing Approaches

Naive discretizations of Stokes can exhibit non-convergence or locking.[3]

Standard Galerkin discretizations:

- need to satisfy the LBB condition
- examples: the MINI element, Crouzeix-Raviart element, the class of $Q_k - P_{k-1}$ elements
- polynomial degree in the pressure discretization one lower than that for velocity $\implies$ pressure converges more slowly

[3] D. Boffi, F. Brezzi, and M. Fortin. Finite elements for the Stokes problem. In *Lecture Notes in Mathematics*, volume 1939, pages 45–100. Springer, 2008

Stokes: Existing Approaches

Local discontinuous Galerkin (LDG) method:[4]

- locally conservative
- allows pressure and velocity spaces to be chosen independently
- equal-order spaces may be used
- convergence *rate* for the pressure remains lower than that for the velocity

[4] B. Cockburn, G. Kanschat, D. Schotzau, and Ch. Schwab. Local Discontinuous Galerkin methods for the Stokes system. *SIAM J. on Num. Anal.*, 40:319–343, 2003

Stokes: Existing Approaches

Divergence-conforming B-splines:[5]

- pointwise divergence-free space $\implies$ locally conservative
- equal-order spaces
- optimal convergence rates for pressure and velocity
- non-conforming for multi-patch domains $\implies$ DG techniques for tangential continuity across patch interfaces

[5] J. Evans. *Divergence-free B-spline Discretizations for Viscous Incompressible Flows*. PhD thesis, University of Texas at Austin, 2011

DPG Applied to Stokes

To apply DPG, we need a first-order system. We introduce $\boldsymbol{\sigma} = \nabla \boldsymbol{u}$:

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{\sigma} - \nabla \boldsymbol{u} &= 0 && \text{in } \Omega. \end{aligned}$$

Testing with $(\boldsymbol{v}, q, \boldsymbol{\tau})$, and integrating by parts, we have

$$\begin{aligned} (\boldsymbol{\sigma} - p\boldsymbol{I}, \nabla \boldsymbol{v})_{\Omega_h} - \left\langle \widehat{\boldsymbol{t}}_n, \boldsymbol{v} \right\rangle_{\Gamma_h} &= (\boldsymbol{f}, \boldsymbol{v})_{\Omega_h} \\ (\boldsymbol{u}, \nabla q)_{\Omega_h} - \langle \widehat{\boldsymbol{u}} \cdot \boldsymbol{n}, q \rangle_{\Gamma_h} &= 0 \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\Omega_h} + (\boldsymbol{u}, \nabla \cdot \boldsymbol{\tau})_{\Omega_h} - \langle \widehat{\boldsymbol{u}}, \boldsymbol{\tau} \boldsymbol{n} \rangle_{\Gamma_h} &= \boldsymbol{0}, \end{aligned}$$

where traction $\boldsymbol{t}_n \stackrel{\text{def}}{=} (\boldsymbol{\sigma} - p\boldsymbol{I})\boldsymbol{n}$, and the hatted variables $\widehat{\boldsymbol{t}}_n$ and $\widehat{\boldsymbol{u}}$ are new unknowns representing the traces of the corresponding variables at the boundary.

DPG Applied to Stokes

DPG Formulation:

$$
\begin{aligned}
b(u,v) = &(\boldsymbol{\sigma} - p\boldsymbol{I}, \nabla \boldsymbol{v})_{\Omega_h} - \left\langle \widehat{\boldsymbol{t}}_n, \boldsymbol{v} \right\rangle_{\Gamma_h} \\
&+ (\boldsymbol{u}, \nabla q)_{\Omega_h} - \langle \widehat{\boldsymbol{u}} \cdot \boldsymbol{n}, q \rangle_{\Gamma_h} \\
&+ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\Omega_h} + (\boldsymbol{u}, \nabla \cdot \boldsymbol{\tau})_{\Omega_h} - \langle \widehat{\boldsymbol{u}}, \boldsymbol{\tau} \boldsymbol{n} \rangle_{\Gamma_h} = (\boldsymbol{f}, \boldsymbol{v})_{\Omega_h} = l(v).
\end{aligned}
$$

The natural spaces for the trial variables are then:

- fields: $p \in L^2(\Omega), \boldsymbol{u} \in \boldsymbol{L}^2(\Omega), \boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega)$,
- fluxes: $\widehat{\boldsymbol{t}}_n \in \boldsymbol{H}^{-1/2}(\Gamma_h)$,
- traces: $\widehat{\boldsymbol{u}} \in \boldsymbol{H}^{1/2}(\Gamma_h)$.

The natural norms for fluxes and traces are *minimum energy extension norms*.

Polynomial Space Choices

How to select polynomial spaces for u and $\widehat{u}$?

- Graph norm $\implies$ error bounded by best approximation error.

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- $\implies k+1$ for $H^{1/2}(\Gamma_h)$.
- $\implies k$ for $H^{-1/2}(\Gamma_h)$.

Graph Test Norm

The adjoint graph norm for our Stokes formulation is:[6]

$$||(\boldsymbol{\tau}, \boldsymbol{v}, q)||^2_{\text{graph}} = ||\nabla \cdot \boldsymbol{\tau} - \nabla q||^2 + ||\nabla \cdot \boldsymbol{v}||^2 + ||\boldsymbol{\tau} + \nabla \boldsymbol{v}||^2 + ||\boldsymbol{\tau}||^2 + ||\boldsymbol{v}||^2 + ||q||^2 .$$

[6] Nathan V. Roberts, Tan Bui-Thanh, and Leszek F. Demkowicz. The DPG method for the Stokes problem. Technical Report 12-22, ICES, 2012

Manufactured Solution

For our first numerical experiment (and following Cockburn et al.[7]), we consider a manufactured solution

$$
\begin{aligned}
u_1 &= -e^x(y\cos y + \sin y) \\
u_2 &= e^x y \sin y \\
p &= 2\mu e^x \sin y
\end{aligned}
$$

on domain $\Omega = (-1,1)^2$. We use this to determine appropriate boundary conditions for the DPG problem. We also perform an L^2 projection of the exact solution into the trial space to find the solution with the best approximation error.

[7] B. Cockburn, G. Kanschat, D. Schotzau, and Ch. Schwab. Local Discontinuous Galerkin methods for the Stokes system. *SIAM J. on Num. Anal.*, 40:319–343, 2003

Graph Test Norm: u_1 convergence

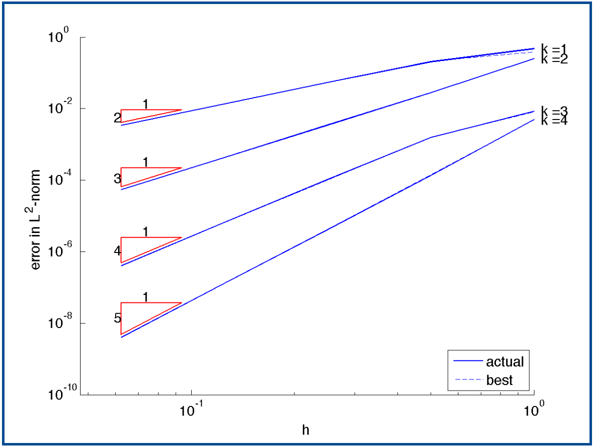

Figure: h-convergence with the graph norm: u_1. Dashed lines: best approximation error.

Graph Test Norm: u_2 convergence

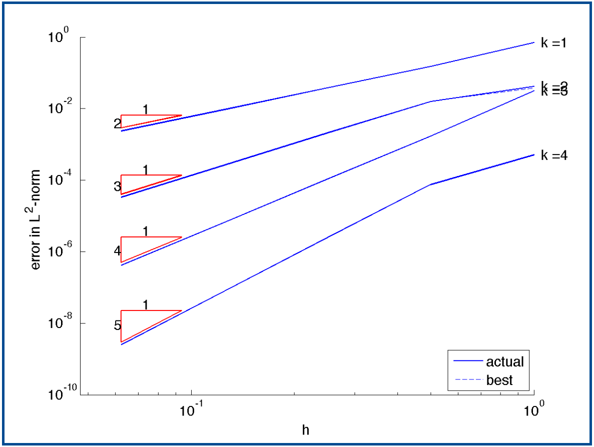

Figure: h-convergence with the graph norm: u_2. Dashed lines: best approximation error.

Graph Test Norm: p convergence

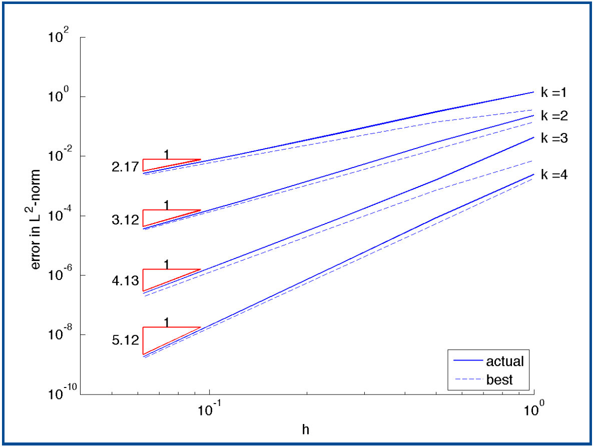

Figure: h-convergence with the graph norm: p. Dashed lines: best approximation error.

Naive Test Norm[8]

What if we don't use the graph norm, but a naive choice instead?

$$||(\boldsymbol{\tau}, \boldsymbol{v}, q)||^2_{\text{naive}} = ||\boldsymbol{\tau}||^2 + ||\nabla \cdot \boldsymbol{\tau}||^2 + ||\boldsymbol{v}||^2 + ||\nabla \boldsymbol{v}||^2 + ||q||^2 + ||\nabla q||^2 .$$

[8] N.V. Roberts, D. Ridzal, P.N. Bochev, L. Demkowicz, K.J. Peterson, and C. M. Siefert. Application of a discontinuous Petrov-Galerkin method to the Stokes equations. In *CSRI Summer Proceedings 2010*. Sandia National Laboratories, 2010

Naive Test Norm: u_1 convergence

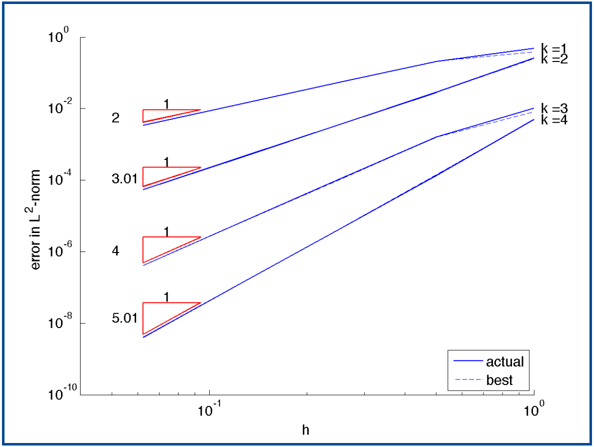

Figure: h-convergence with the naive norm: u_1. Dashed lines: best approximation error.

Naive Test Norm: u_2 convergence

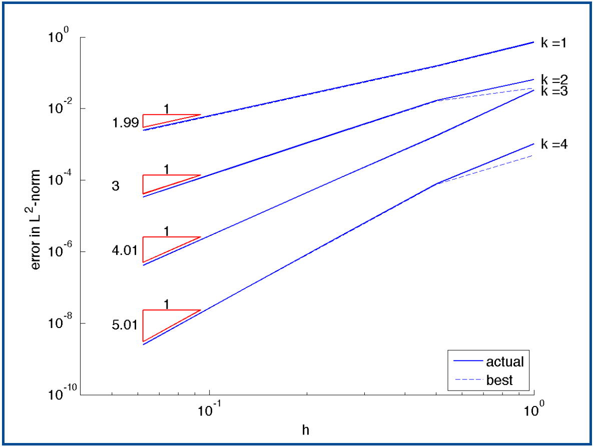

Figure: h-convergence with the naive norm: u_2. Dashed lines: best approximation error.

Naive Test Norm: p convergence

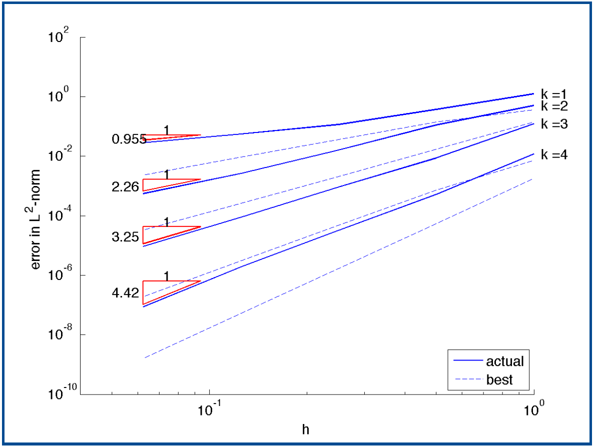

Figure: h-convergence with the naive norm: p. Dashed lines: best approximation error.

Graph vs. Naive Test Norm

What's the difference between the two norms? Why are the results better with the graph norm?

$$
\begin{aligned}
||(\boldsymbol{\tau}, \boldsymbol{v}, q)||^2_{\text{naive}} &= ||\nabla \cdot \boldsymbol{\tau}||^2 + ||\nabla \cdot \boldsymbol{v}||^2 + ||\nabla q||^2 + ||\boldsymbol{\tau}||^2 + ||\boldsymbol{v}||^2 + ||q||^2 \\
||(\boldsymbol{\tau}, \boldsymbol{v}, q)||^2_{\text{graph}} &= ||\nabla \cdot \boldsymbol{\tau} - \nabla q||^2 + ||\nabla \cdot \boldsymbol{v}||^2 + ||\boldsymbol{\tau} + \nabla \boldsymbol{v}||^2 \\
&\quad + ||\boldsymbol{\tau}||^2 + ||\boldsymbol{v}||^2 + ||q||^2
\end{aligned}
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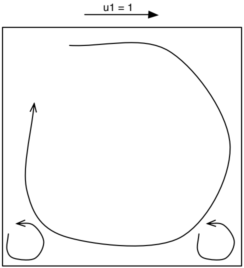
||(\boldsymbol{\tau},\boldsymbol{v},q)||^2_{\text{graph}} &= ||\nabla\cdot\boldsymbol{\tau} - \nabla q||^2 + ||\nabla\cdot\boldsymbol{v}||^2 + ||\boldsymbol{\tau} + \nabla\boldsymbol{v}||^2 \\
&\quad + ||\boldsymbol{\tau}||^2 + ||\boldsymbol{v}||^2 + ||q||^2
\end{aligned}
$$

The naive norm is stronger—e.g. it requires $\nabla\cdot\boldsymbol{\tau}\in L^2$ and $\nabla q\in L^2$, whereas the graph norm merely requires that $\nabla\cdot\boldsymbol{\tau} - \nabla q \in L^2$.

Lid-Driven Cavity Flow Problem

A classic test case for Stokes flow is the lid-driven cavity flow problem. Consider a square cavity with an incompressible, viscous fluid, with a lid that moves at a constant rate. The resulting flow will be vorticular; there will be Moffat eddies at the corners; in fact, the exact solution will have an infinite number of such eddies, visible at progressively finer scales.[9]



[9] H.K. Moffat. Viscous and resistive eddies near a sharp corner. *Journal of Fluid Mechanics*, 18(1):1–18, 1964

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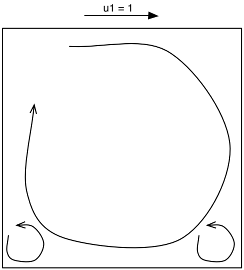


Because the BCs for the problem are discontinuous at the top corners, the exact solution lies outside of H^1. So we introduce a small “ramp” on either side, of width $\epsilon = \frac{1}{64}$.

Lid-Driven Cavity Flow Problem

Since the exact solution is unknown, we compare an overkill mesh to a series of adaptive and uniformly refined solutions.

- For all meshes, we use quadratic field variables ($k = 2$).
- Overkill mesh had 256×256 elements (5,576,706 dofs).
- Initial adaptive mesh had 2×2 elements.

Lid-Driven Cavity Flow: h-adaptivity

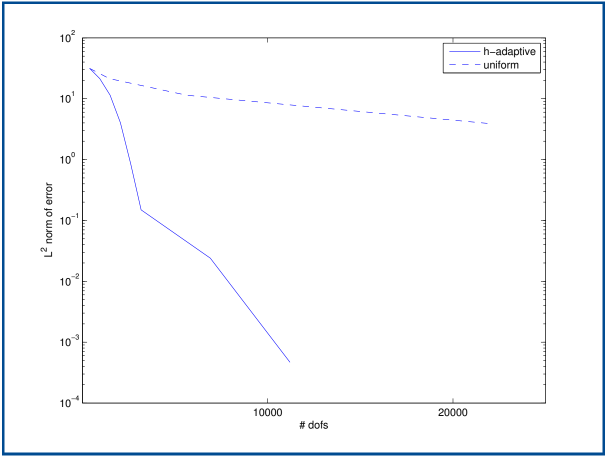
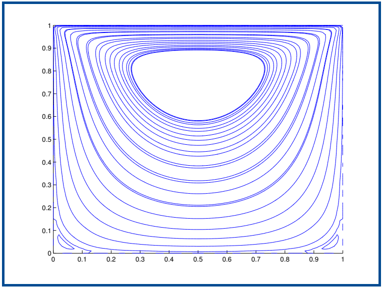
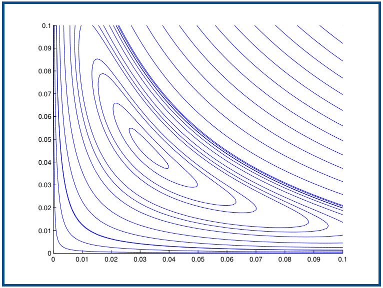


Figure: Euclidean norm of L^2 error in all field variables in h-adaptive mesh relative to an overkill mesh with 256×256 quadratic elements. The Euclidean norm of the overkill solution is 6.73.

Lid-Driven Cavity Flow: Streamlines

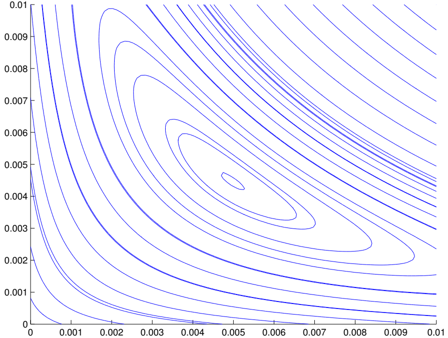




Streamlines for the full cavity and for the lower-left corner, on a quadratic mesh after 7 adaptive refinements. The lower-left corner shows the first Moffat eddy. The final mesh has 124 elements and 11,202 dofs.

Lid-Driven Cavity Flow: Streamlines

Using a cubic mesh and running 11 refinement steps, we can resolve the second Moffat eddy:



Streamlines for the lower-left corner on a cubic mesh after 11 adaptive refinements: the second Moffat eddy. The final mesh has 298 elements and 44,206 dofs.

Camellia

Design Goal: make DPG research and experimentation as simple as possible, without sacrificing too much by way of performance.

Existing FEM Software

- `deal.II`[10]
- libMesh[11]
- FEniCS[12]

[10] Wolfgang Bangerth and Guido Kanschat. Concepts for object-oriented finite element software – the deal.II library. In *Preprint 43, SFB 359*, 1999

[11] Benjamin S. Kirk, John W. Peterson, Roy H. Stogner, and Graham F. Carey. libMesh: a C++ library for parallel adaptive mesh refinement/coarsening simulations. *Eng. with Comput.*, 22(3):237–254, December 2006

[12] A. Logg, K.-A. Mardal, and G. N. Wells, editors. *Automated Solution of Differential Equations by the Finite Element Method*, volume 84 of *Lecture Notes in Computational Science and Engineering*. Springer, 2012

FEM Software: `deal.II` features

- flexible: possible to vary choices for FE spaces, spatial dimension, var. formulations, and linear solvers without too much effort
- easy to use through encapsulation: details of complex data structures hidden from user
- safe: runtime parameter checking allows many errors to be detected early in development
- extensive documentation
- hypercube topologies (lines, quads, hexahedra) supported (with several element types: CG and DG Lagrange, Nédélec, Raviart-Thomas)

FEM Software: `deal.II` features, continued

- h-, p-, and hp-adaptivity
- distributed stiffness matrix computation
- recently added: distributed mesh storage[13]

[13] Wolfgang Bangerth, Carsten Burstedde, Timo Heister, and Martin Kronbichler. Algorithms and data structures for massively parallel generic adaptive finite element codes. *ACM Transactions on Mathematical Software*, 38(2), 2011

FEM Software: libMesh and FEniCS

libMesh features

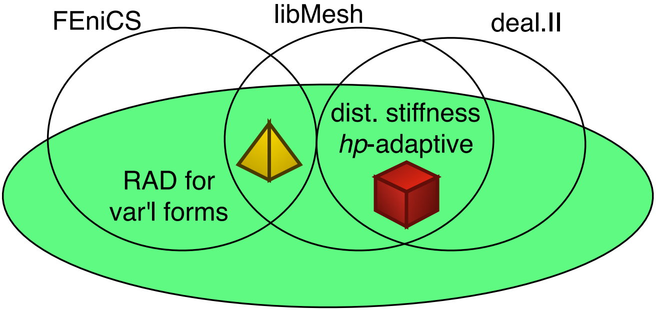
- inspired by `deal.II`
- `Element` class designed for subclassing
- many topologies provided: triangles, quads, hexahedra, tetrahedra, prisms, and pyramids
- h-, p-, and hp-adaptivity
- distributed stiffness matrix computation

FEniCS features

- aim: highly automated solution of FE problems
- emphasis is on simplicity, especially in specification of variational forms
- simplex topologies (intervals, triangles, tetrahedra) supported

Camellia[15]

Design Goal: make DPG research and experimentation as simple as possible, without sacrificing too much by way of performance.



Camellia (2D)—built atop Trilinos

[14] Michael A. Heroux et al. An overview of the Trilinos project. *ACM Trans. Math. Softw.*, 31(3):397–423, 2005

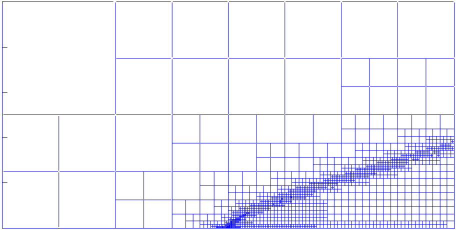
[15] Nathan V. Roberts, Denis Ridzal, Pavel B. Bochev, and Leszek F. Demkowicz. A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos. Technical Report SAND2011-6678, Sandia National Laboratories, 2011

Trilinos Support

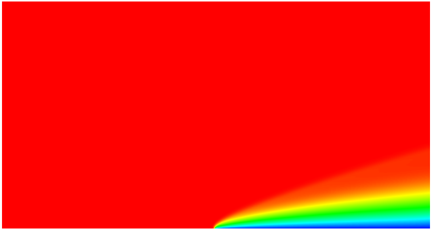
Feature	Trilinos Package
OO interface to MUMPS	Amesos
KLU solver	Amesos
conforming basis functions	Intrepid
pullbacks/Piola transforms	Intrepid
smart multidimensional arrays	Intrepid
distributed compressed row storage matrices	Epetra
cell topologies	Shards
reference-counted pointers	Teuchos
space-filling curves for spatially local mesh partitioning	Zoltan

Camellia: Other Users

- Jesse Chan: compressible Navier-Stokes
- Truman Ellis: compressible Navier-Stokes with turbulence



(a) Mesh

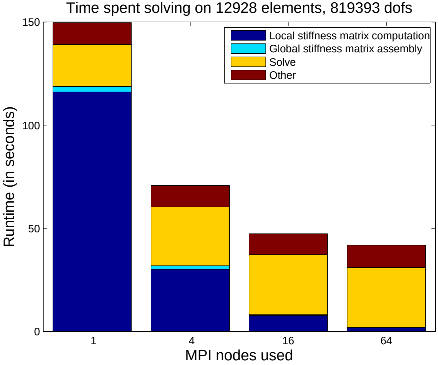


(b) u_1

Figure: Compressible Navier-Stokes for Carter flat plate problem, $\mathrm{Re} = 1000, \mathrm{Ma} = 3.$

Camellia: Stiffness Matrix Timing Test

- local stiffness matrix computation is embarrassingly parallel
- minimize assembly costs: spatially local mesh partitioning
- timing tests on Lonestar: solve convection-dominated diffusion



- collaborators working on parallel solvers: Kyungjoo Kim (shared memory architecture), Maciej Paszynski (distributed memory)

Camellia: Rapid Specification of Inner Products

Suppose we have a problem whose graph norm is

$$||(v, \boldsymbol{q})||^2_{\text{graph}} = ||v||^2 + ||\boldsymbol{q}||^2 + \left|\left|\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} + \nabla \cdot \boldsymbol{q}\right|\right|^2.$$

To specify this in Camellia, simply do:

```
VarFactory varFactory;
VarPtr v = varFactory.testVar("v", HGRAD);
VarPtr q = varFactory.testVar("q", HDIV);
IPPtr ip = Teuchos::rcp( new IP);
ip->addTerm(v);
ip->addTerm(q);
ip->addTerm(v->dx() - v->dy() + q->div());
```

The bilinear form can be specified similarly.

Navier-Stokes

The structure of Navier-Stokes is similar to Stokes:

- Naive discretizations have similar issues with non-convergence and locking.
- Can use same elements as for Stokes, but now need to deal with the nonlinearity.

Navier-Stokes

Existing approaches

- Classical pressure-correction methods (Chorin, Temam), velocity-correction methods (Orszag, Karniadakis), and consistent splitting schemes (Guermond and Shen), all of which split the nonlinear time step into a pair of elliptic equations.[16]
- Guermond and Minev's dimensional splitting, another projection method, which is cheap to compute, only requiring solution of a series of 1D boundary value problems. (Limited, for now, to axis-aligned parallelepiped meshes.)[17]

[16] J.L. Guermond, P. Minev, and Jie Shen. An overview of projection methods for incompressible flows. *Computer Methods in Applied Mechanics and Engineering*, 195:6011–6045, 2006

[17] J.L. Guermond and P.D. Minev. A new class of massively parallel direction splitting for the incompressible Navier–Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 200(23–24):2083 – 2093, 2011

Navier-Stokes

Existing approaches

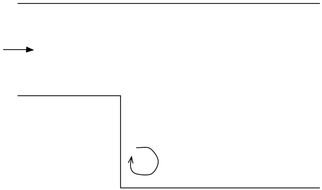
- Cockburn et al. applied LDG to Navier-Stokes, with success for Reynolds numbers up to 100.[18]
- Evans and Hughes applied divergence-conforming B-splines to Navier-Stokes, with success e.g. for driven cavity problem with Re up to 1000.[19]

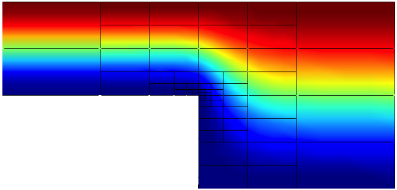
[18] Bernardo Cockburn, Guido Kanschat, and Dominik Schötzau. A locally conservative LDG method for the incompressible Navier-Stokes equations. *Math. Comp*, pages 1067–1095, 2004

[19] John Evans and Thomas J.R. Hughes. Isogeometric divergence-conforming B-splines for the steady Navier-Stokes equations. Technical Report 12-15, ICES, 2012

Backward-facing Step

A classical test case for incompressible flow problems is the backward-facing step problem, which has a recirculation region east of the step. We plan to perform numerical experiments using Stokes, Oseen, and Navier-Stokes.





Flow Around a Cylinder

A classical test case for incompressible Navier-Stokes problems is the flow around a cylinder.[20] We plan to perform numerical experiments using steady-state Navier-Stokes.

[20] L.S.G. Kovasznay. Hot-wire investigation of the wake behind cylinders at low Reynolds numbers. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 198(1053):174–190, 1949

Flow Around a Cylinder

A classical test case for incompressible Navier-Stokes problems is the flow around a cylinder.[20] We plan to perform numerical experiments using steady-state Navier-Stokes.

At around $\mathrm{Re} = 6$, the flow separates but there is still a steady solution, with symmetric vortices in the wake of the cylinder. Somewhere above $\mathrm{Re} = 40$, vortex shedding begins (and therefore there is no steady solution).

[20] L.S.G. Kovasznay. Hot-wire investigation of the wake behind cylinders at low Reynolds numbers. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 198(1053):174–190, 1949

Proposed Work: Area C

- Verify convergence rates for the Stokes, Oseen, and Navier-Stokes equations using manufactured solutions.
- Simulate the classical lid-driven cavity flow and backward-facing step problems using, in turn, the Stokes, Oseen, and Navier-Stokes equations.
- Simulate flow past a cylinder using the steady-state Navier-Stokes equations. Time permitting, we will do so with the transient equations as well.

Completed

- Verify convergence rates for the Stokes equations using manufactured solutions.
- Simulate lid-driven cavity flow using the Stokes equations.

Proposed Work: Area B

Design and develop a software toolbox (*Camellia*) for the investigation of DPG problems, with the following features:

Camellia Features (Completed)

- 2D meshes of triangles and quads of variable polynomial order,
- mechanisms for easy specification of DPG variational forms,
- h- and p- refinements, and
- distributed computation of the stiffness matrix.

Time permitting, add the following features:

- curvilinear elements,
- meshes of arbitrary spatial dimension,
- space-time elements, and
- distributed mesh and solution representation.

Proposed Work: Area A

- Completed: Pose several DPG formulations of the Stokes equations—the velocity-gradient-pressure (VGP), velocity-stress-pressure (VSP), and velocity-vorticity-pressure (VVP) formulations.
- Pose DPG formulations of the Oseen equations and the 2D incompressible Navier-Stokes equations.
- Completed: Prove the well-posedness of the VGP Stokes formulation for DPG, which has as consequence a guarantee of optimal convergence rates.
- Time permitting, complete similar proofs for the VSP and VVP formulations.
- Time permitting, complete similar proofs for the Oseen equations, including a study of *robustness*—analyzing the effects of an increasing Reynolds number.

Thank you!

Questions?

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Finite elements for the Stokes problem.
In *Lecture Notes in Mathematics*, volume 1939, pages 45–100. Springer, 2008.

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Zhiqiang Cai, Thomas A. Manteuffel, and Stephen F. McCormick.
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Math. Comp, pages 1067–1095, 2004.

L. Demkowicz and J. Gopalakrishnan.

A class of discontinuous Petrov-Galerkin methods. Part II: Optimal test functions.
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J. Evans.
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Isogeometric divergence-conforming B-splines for the steady Navier-Stokes equations.
Technical Report 12-15, ICES, 2012.

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J.L. Guermond, P. Minev, and Jie Shen.
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A new class of massively parallel direction splitting for the incompressible Navier–Stokes equations.
Computer Methods in Applied Mechanics and Engineering, 200(23–24):2083 – 2093, 2011.

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- Nathan V. Roberts, Tan Bui-Thanh, and Leszek F. Demkowicz.
  The DPG method for the Stokes problem.
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- Nathan V. Roberts, Denis Ridzal, Pavel B. Bochev, and Leszek F. Demkowicz.
  A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos.
  Technical Report SAND2011-6678, Sandia National Laboratories, 2011.
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  Application of a discontinuous Petrov-Galerkin method to the Stokes equations.
  In *CSRI Summer Proceedings 2010*. Sandia National Laboratories, 2010.

C. H. K. Williamson.
Vortex dynamics in the cylinder wake.
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FOSLS[21]

First-Order Systems Least Squares (FOSLS), like DPG:

- first-order system
- SPD stiffness matrix
- no need for stabilization terms or flux limiters
- discrete space only needs to be conforming (no need to satisfy LBB condition, e.g.)
- minimum-residual method
- *a posteriori* error measure (not estimator)

Like DPG, FOSLS provides optimal convergence rates, but it does so in a different norm (typically, the graph norm on the trial space).
Problem-specific approaches to get optimal rates in the L^2 norm, which DPG manages by virtue of Riesz inversion and a broken test space.

[21] Z. Cai, R. Lazarov, T. A. Manteuffel, and S. F. McCormick. First-order system least squares for second-order partial differential equations: part i. *SIAM J. Numer. Anal.*, 31(6):1785–1799, December 1994.

Minimization of the Nonlinear Residual

As before, we seek to minimize

$$\frac{1}{2}\left(R_V^{-1}\left(Bw_h - l\right), R_V^{-1}\left(Bw_h - l\right)\right)_V,$$

but now we allow B to be nonlinear. The first-order optimality condition then requires that

$$\left(R_V^{-1}\left(Bu_h - l\right), R_V^{-1} B'(u_h, \delta u_h)\right)_V = 0, \quad \forall \delta u_h \in U_h.$$

where $B'(u_h, \delta u_h)$ is the Gâteaux derivative of B at u_h in direction δu_h. Then

$$\langle Bu_h - l, R_V^{-1} B'(u_h, \delta u_h)\rangle = 0 \quad \forall \delta u_h \in U_h.$$

Minimization of the Nonlinear Residual

$$\langle Bu_h - l, R_V^{-1} B'(u_h, \delta u_h) \rangle = 0 \quad \forall \delta u_h \in U_h.$$

Now, we can again identify $v_{\delta u_h} = R_V^{-1} B'(u_h, \delta u_h)$ as a test function and note that

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

but notice that now $v_{\delta u_h}$ also depends on the solution u_h. Linearizing about $u_h + \Delta u_h$, we have

$$B(u_h + \Delta u_h) \approx Bu_h + B'(u_h, \Delta u_h)$$

and

$$B'(u_h + \Delta u_h, \delta u_h) \approx B'(u_h, \delta u_h) + B''(u_h, \delta u_h, \Delta u_h)$$

so that the optimality condition becomes

$$\begin{aligned} &\langle Bu_h + B'(u_h, \Delta u_h) - l, \\ &\quad R_V^{-1} \left(B'(u_h, \delta u_h) + B''(u_h, \delta u_h, \Delta u_h) \right) \rangle \rangle = 0 \quad \forall \delta u_h \in U_h. \end{aligned}$$

Minimization of the Nonlinear Residual

$$\begin{aligned}&\langle Bu_h + B'(u_h, \Delta u_h) - l,\\ &\quad R_V^{-1}\left(B'(u_h, \delta u_h) + B''(u_h, \delta u_h, \Delta u_h)\right)\rangle = 0 \quad \forall \delta u_h \in U_h.\end{aligned}$$

Dropping the terms that are second order in Δu_h and defining $v_{\delta u_h} = R_V^{-1} B'(u_h, \delta u_h)$, we have

$$\begin{aligned}&\langle Bu_h - l, v_{\delta u_h}\rangle\\ &+ \langle Bu_h - l, R_V^{-1} B''(u_h, \delta u_h, \Delta u_h)\rangle\\ &+ \langle B'(u_h, \Delta u_h), v_{\delta u_h}\rangle = 0.\end{aligned}$$

Rewrite the second term:

$$\begin{aligned}&\langle Bu_h - l, R_V^{-1} B''(u_h, \delta u_h, \Delta u_h)\rangle\\ &= \left(R_V^{-1}(Bu_h - l), R_V^{-1} B''(u_h, \delta u_h, \Delta u_h)\right)_V\\ &= \langle B''(u_h, \delta u_h, \Delta u_h), R_V^{-1}(Bu_h - l)\rangle.\end{aligned}$$

Minimization of the Nonlinear Residual

We then have:

$$\begin{aligned}&\langle Bu_h - l, v_{\delta u_h}\rangle \\ &+ \langle B''(u_h, \delta u_h, \Delta u_h), R_V^{-1}(Bu_h - l)\rangle \\ &+ \langle B'(u_h, \Delta u_h), v_{\delta u_h}\rangle = 0.\end{aligned}$$

Defining $v_{u_h} = R_V^{-1}(Bu_h - l)$, the linearized problem is then

$$b'(u_h, \Delta u_h; v_{\delta u_h}) + b''(u_h, \delta u_h, \Delta u_h; v_{u_h}) = -b(u_h, v_{\delta u_h}) + l(v_{\delta u_h}).$$

The solution to this problem minimizes the nonlinear residual $||Bu_h - l||_{V'}$.

Technical Assumptions (true for VGP Stokes)

Under modest technical assumptions (true for Stokes), we have[22]

$$\left|\left|Au\right|\right| \geq \gamma \left|\left|u\right|\right| \implies \sup_{v \in H_{A^*}} \frac{b((u,\widehat{u}),v)}{\left|\left|v\right|\right|_{H_{A^*}}} \geq \gamma_{\text{DPG}} \left(\left|\left|u\right|\right|^2 + \left|\left|\widehat{u}\right|\right|^2_{\widehat{H}_A(\Gamma_h)} \right)^{1/2}$$

where $\gamma_{\text{DPG}} = O(\gamma)$ is a mesh-independent constant, and $\left|\left|\cdot\right|\right|_{\widehat{H}_A(\Gamma_h)}$ is the minimum energy extension norm.

In the next slides, we detail the assumptions.

[22] Nathan V. Roberts, Tan Bui-Thanh, and Leszek F. Demkowicz. The DPG method for the Stokes problem. Technical Report 12-22, ICES, 2012

Technical Assumptions (true for VGP Stokes)

$$||Au|| \geq \gamma \, ||u|| \implies \sup_{v \in H_{A^*}} \frac{b((u,\widehat{u}),v)}{||v||_{H_{A^*}}} \geq \gamma_{\text{DPG}} \left(||u||^2 + ||\widehat{u}||^2_{\widehat{H}_A(\Gamma_h)} \right)^{1/2}$$

Define C as the operator arising from integration by parts:

$$(Au, v)_\Omega = (u, A^* v)_\Omega + \langle Cu, v \rangle.$$

We split C into C_1 and C_2 such that

$$\begin{aligned} \langle Cu, v \rangle &= \langle C_1 u, v \rangle + \langle C_2 u, v \rangle \\ &= \langle C_1 u, v \rangle + \langle u, C_2' v \rangle \end{aligned}$$

where $C_1 u = f_D$ corresponds to the Dirichlet BCs imposed.

Technical Assumptions (true for VGP Stokes)

$$||Au|| \geq \gamma\, ||u|| \implies \sup_{v \in H_{A^*}} \frac{b((u,\widehat{u}),v)}{||v||_{H_{A^*}}} \geq \gamma_{\text{DPG}} \left(||u||^2 + ||\widehat{u}||^2_{\widehat{H}_A(\Gamma_h)} \right)^{1/2}$$

Assumptions:

- Theorem Hypothesis: with homogeneous boundary condition $C_1 u = 0$ in place, operator A is bounded below in the L^2-orthogonal component of its null space.
- C_1 and C_2 are defined in such a way that

$$\left(\langle u, C_2' v \rangle = 0 \quad \forall u \;: C_1 u = 0 \right) \implies C_2' v = 0.$$

- A and A^* are surjective.
- Both graph spaces $H_A(\Omega)$ and $H_{A^*}(\Omega)$ admit corresponding trace spaces $\widehat{H}_A(\partial\Omega)$ and $\widehat{H}_{A^*}(\partial\Omega)$.
- The boundary term $\langle Cu, v \rangle$ arising from integration by parts is definite.