Locally Conservative Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

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Abstract

1 Intoduction

Verteeg and Malalasekera, in *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*[5, p. 110-113] cite three characteristics that they consider essential to any numerical discretization of convection-diffusion type problems: conservativeness, boundedness, and transportiveness.

Perot[4] argues

> Accuracy, stability, and consistency are the mathematical concepts that are typically used to analyze numerical methods for partial differential equations (PDEs). These important tools quantify how well the mathematics of a PDE is represented, but they fail to say anything about how well the physics of the system is represented by a particular numerical method. In practice, physical fidelity of a numerical solution can be just as important (perhaps even more important to a physicist) as these more traditional mathematical concepts. A numerical solution that violates the underlying physics (destroying mass or entropy, for example) is in many respects just as flawed as an unstable solution.

The discontinuous Petrov-Galerkin finite element method has been described as least squares finite elements with a twist. The key difference is that least square methods seek to minimize the residual of the solution in some Hilbert space norm, while DPG seeks the minimization in a dual norm through the inverse Riesz map. Exact mass conservation has been an issue that has plagued least squares finite elements for a long time. Several approaches have been used to try to adress this. Chang and Nelson[2] developed the 'restricted LSFEM'[2] be augmenting the least squares equations with a Lagrange multiplier explicitly enforcing mass conservation element-wise. Our conservative formulation of DPG takes a similar approach and both methods share similar negative of transforming a minimization method to a saddle-point problem.

The discontinuous Petrov-Galerkin finite element method has shown a lot of promise for convection-diffusion type problems including robustness in the face of singularly perturbed problems.

2 DPG is a Minimum Residual Method

We now proceed with a abstract derivation of the standard Discontinuous Petrov-Galerkin method. Suppose we have two Hilbert spaces, U and V, the trial and test spaces, respectively. And suppose we are trying to solve a well-posed variational problem $b(u,v) = l(v)$. We can rewrite this in operator form $Bu = l$, where $B : U \to V'$, where V' is the dual space to V. Then, for a discrete subspace $U_h \subset U$, we seek to find $u_h \in U_h$ that minimizes the error residual:

$$u_h = \underset{u_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bu_h - l \right\|_{V'}^2 . \tag{1}$$

Recalling that the Riesz operator $R_V : V \to V'$ is an isometry defined by

$$\langle R_V v, \delta v \rangle = (v, \delta v)_V\,, \quad \forall \delta v \in V,$$

we can use the Riesz inverse to minimize on V rather than its dual:

$$\frac{1}{2}\left\|Bu_h - l\right\|_{V'}^2 = \frac{1}{2}\left\|R_V^{-1}(Bu_h - l)\right\|_V^2 = \frac{1}{2}\left(R_V^{-1}(Bu_h - l), R_V^{-1}(Bu_h - l)\right)_V\,. \tag{2}$$

The first order optimality condition for (2) requires the Gâteaux derivative to be zero in all directions $\delta u \in U_h$, i.e.,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U.$$

By definition of the Riesz operator, this is equivalent to

$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h \right\rangle = 0 \quad \forall \delta u_h \in U_h\,. \tag{3}$$

Now, we can identify $v_{\delta u_h} := R_V^{-1}B\delta u_h$ as the optimal test function for trial function u_h and we can rewrite (3) as

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}). \tag{4}$$

The DPG method then is to solve (4) with optimal test functions $v_{\delta u_h} \in V$ that solve the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v)\,, \quad \forall \delta v \in V. \tag{5}$$

Using a continuous test basis would result in a global solve for every optimal test function. Therefore DPG uses a discontinuous test basis which makes each solve element-local and much more computationally tractable. Of course, (5) still requires the inversion of the infinite-dimensional Riesz map, but in approximating the V by finite dimensional V_h which is of a higher polynomial degree than U_h (hence "enriched space") works well in practice.

No assumptions have been made so far on the definition of the inner product on V. In fact, proper choice of $(\cdot,\cdot)_V$ can make the difference between a solid DPG method and one that suffers from robustness issues.

3 DPG Applied to Convection-Diffusion

Now that we have briefly outline the abstract DPG method, let us put it into practice with the Convection-Diffusion equation. The strong form of the steady Convection-Diffusion equation reads

$$\nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u = g\,,$$

where u is the property of interest, $\boldsymbol{\beta}$ is the convection vector, and g is the source term. Let us write this as an equivalent system of first order equations:

$$\begin{aligned} \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= g \\ \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= \mathbf{0}\,. \end{aligned}$$

If we then multiply the top equation by some scalar test function v and the bottom equation by some vector-valued test function τ, we can integrate by parts over each element K:

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + ((\boldsymbol{\beta} u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v)_{\partial K} &= (g, v)_K \\ \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - (u, \tau_n)_{\partial K} &= 0\,. \end{aligned} \tag{6}$$

The discontinuous Petrov-Galerkin method refers to the fact that we are using discontinuous optimal test functions that come from a space differing from the trial space. It does not specify our choice of trial space. Nevertheless, many considerations of DPG in the literature [] associate DPG with the so-called "ultra-weak formulation." We will follow the same derivation for the convection-diffusion equation, but we emphasize that other formulations are available. Thus, we seek field variables $u \in L^2(K)$ and $\boldsymbol{\sigma} \in \mathbf{L^2}(K)$. Mathematically, this leaves their traces on element boundaries undefined, and in a manner similar to the hybridized discontinuous Galerkin method, we define new unknowns for trace $\hat{u}$ and flux $\hat{f}$. Applying these definitions to (6) and adding the two equations together, we arrive at our desired bilinear form,

$$-(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + (\hat{f}, v)_{\partial K} + \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - (\hat{u}, \tau_n)_{\partial K} = (g, v)_K \,. \tag{7}$$

A cursory look at (7) tells us which function spaces to look in for each variable:

$$\begin{array}{lll} u \in L^2(K) & \hat{u} \in H^1(K)|_{\partial K} & v \in H^1(K) \\ \boldsymbol{\sigma} \in \mathbf{L}^2(K) & \hat{f} \in \boldsymbol{H}(\mathbf{div}, K)|_{\partial K} & \boldsymbol{\tau} \in H(div, K)\,. \end{array}$$

All that is left to pin down this problem is a definition of our test norm so we can invert the Riesz operator and calculate our optimal test functions. Within each element, we perform a Bubnov-Galerkin solve for the optimal test functions. Define finite-dimensional subspaces $\mathbf{U}_h \subset \mathbf{U} := L^2(\Omega_h) \times \mathbf{L}^2(\Omega_h) \times H^{1/2}(\Gamma_h) \times H^{-1/2}(\Gamma_h)$ the trial space and $\mathbf{V}_h \subset \mathbf{V} := H^1(\Omega_h) \times H(div, \Omega_h)$ the "enriched" test space. For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{f}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V_h}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

As mentioned earlier the choice of test norm on V can have profound influence on the robustness of our method. Unfortunately, the structure of the optimal test norm makes it non-localizable. For many problems, it suffices to use the so-call quasi-optimal test norm which is based on the adjoint of the B operator, but for convection-diffusion type equations, the adjoint develops boundary layers which make solving for the optimal test functions much more difficult for small diffusion. In an earlier consideration of DPG for convection-diffusion problems, Chan *et al.* developed the more robust test norm[1],

$$\|(v, \boldsymbol{\tau})\|^2_{\mathbf{V},\Omega_h} = \|\nabla \cdot \boldsymbol{\tau}\|^2 + \left\| \min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\} \boldsymbol{\tau} \right\|^2 + \epsilon \|\nabla v\|^2 + \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \left\| \min\left\{\sqrt{\frac{\epsilon}{|K|}}, 1\right\} v^2 \right\| \,. \tag{8}$$

Unfortunately, this test norm also has a few issues. For one, some of the assumptions that went into its development break down as the flow field degenerates to zero. In such cases the final L^2 term on v can register higher error levels and trigger unnecessary refinements in smooth regions, see ref.

4 Locally Conservative Formulation

A simple control volume analysis will tell us that a locally conservative method must enforce that

$$\int_{\partial K} \hat{f} = \int_K g \,, \quad \forall K \in \Omega_h \,, \tag{9}$$

which is equivalent to having the set $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ for $K = 1, \ldots, N$ (N is the number of mesh elements) in the test space, where each 1_K has value one on element K and zero elsewhere. In fact, if we insert this test function into (7), all of the τ and ∇v terms vanish and we are left exactly with this condition. Numerical experiments imply that local conservation occurs in one dimension, but the standard DPG method is not exactly locally conservative for higher dimensional problems.

Following Moro *et al.* [3], we can explicitly augment our test space with constants through the use of Lagrange multipliers. Going back to (2), we can define our Lagrange function,

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \left\| R_V^{-1}(Bu_h - l) \right\|^2_V - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{f}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K} \,, \tag{10}$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$. We then proceed as before and find the critical points of (10),

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B \delta u_h) - l(R_V^{-1} B \delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \,, \quad \forall \delta u_h \in U_h \tag{11}$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \,, \quad \forall K \,. \tag{12}$$

Equation (11) is just (4) with the extra Lagrange terms. As usual, the second equation just enforces the constraint. As a consequence, we now explicitly have constants in our test space and should enforce local conservation to machine precision. The negative side is that we have added an additional unknown associated with every mesh element and turned our well-behaved minimization into a saddle point problem.

This change has further consequences to how we compute our optimal test functions. Since constants are now explicitly represented in the test space, we only need to search for optimal test functions in the orthogonal complement of constants. The final term in (8) is somewhat troublesome, but becomes unnecessary when solving in the orthogonal complement of constants. Instead, we can replace it with a much nicer zero mean term. Thus, (8) becomes

$$\|(v, \boldsymbol{\tau})\|^2_{\mathbf{V},\Omega_h} = \|\nabla \cdot \boldsymbol{\tau}\|^2 + \left\| \min\left\{ \frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}} \right\} \boldsymbol{\tau} \right\|^2 + \epsilon \|\nabla v\|^2 + \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \left(\int_K v \right)^2 . \tag{13}$$

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