Locally Conservative Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

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Abstract

1 Intoduction

Verteeg and Malalasekera, in *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*[3, p. 110-113] cite three characteristics that they consider essential to any numerical discretization of convection-diffusion type problems: conservativeness, boundedness, and transportiveness.

Perot[2] argues

> Accuracy, stability, and consistency are the mathematical concepts that are typically used to analyze numerical methods for partial differential equations (PDEs). These important tools quantify how well the mathematics of a PDE is represented, but they fail to say anything about how well the physics of the system is represented by a particular numerical method. In practice, physical fidelity of a numerical solution can be just as important (perhaps even more important to a physicist) as these more traditional mathematical concepts. A numerical solution that violates the underlying physics (destroying mass or entropy, for example) is in many respects just as flawed as an unstable solution.

The discontinuous Petrov-Galerkin finite element method has been described as least squares finite elements with a twist. The key difference is that least square methods seek to minimize the residual of the solution in some Hilbert space norm, while DPG seeks the minimization in a dual norm through the inverse Riesz map. Exact mass conservation has been an issue that has plagued least squares finite elements for a long time. Several approaches have been used to try to adress this. Chang and Nelson[1] developed the 'restricted LSFEM'[1] be augmenting the least squares equations with a Lagrange multiplier explicitly enforcing mass conservation element-wise. Our conservative formulation of DPG takes a similar approach and both methods share similar negative of transforming a minimization method to a saddle-point problem.

The discontinuous Petrov-Galerkin finite element method has shown a lot of promise for convection-diffusion type problems including robustness in the face of singularly perturbed problems.

2 DPG is a Minimum Residual Method

We now proceed with a abstract derivation of the standard Discontinuous Petrov-Galerkin method. Suppose we have two Hilbert spaces, U and V, the trial and test spaces, respectively. And suppose we are trying to solve a well-posed variational problem $b(u,v) = l(v)$. We can rewrite this in operator form $Bu = l$, where $B : U \to V'$, where V' is the dual space to V. Then, for a discrete subspace $U_h \subset U$, we seek to find $u_h \in U_h$ that minimizes the error residual:

$$u_h = \underset{u_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bu_h - l \right\|_{V'}^2 . \tag{1}$$

Recalling that the Riesz operator $R_V : V \to V'$ is an isometry defined by

$$\langle R_V v, \delta v\rangle = (v, \delta v)_V\,, \quad \forall \delta v \in V,$$

we can use the Riesz inverse to minimize on V rather than its dual:

$$\frac{1}{2}\left\|Bu_h - l\right\|^2_{V'} = \frac{1}{2}\left\|R_V^{-1}(Bu_h - l)\right\|_V = \frac{1}{2}\left(R_V^{-1}(Bu_h - l), R_V^{-1}(Bu_h - l)\right)_V\,. \tag{2}$$

The first order optimality condition for (2) requires the Gâteaux derivative to be zero in all directions $\delta u \in U_h$, i.e.,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U.$$

By definition of the Riesz operator, this is equivalent to

$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h\right\rangle = 0 \quad \forall \delta u_h \in U_h\,. \tag{3}$$

Now, we can identify $v_{\delta u_h} := R_V^{-1}B\delta u_h$ as the optimal test function for trial function u_h and we can rewrite (3) as

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}). \tag{4}$$

The DPG method then is to solve (4) with optimal test functions $v_{\delta u_h} \in V$ that solve the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v\rangle = \langle B\delta u_h, \delta v\rangle = b(\delta u_h, \delta v)\,, \quad \forall \delta v \in V. \tag{5}$$

Using a continuous test basis would result in a global solve for every optimal test function. Therefore DPG uses a discontinuous test basis which makes each solve element-local and much more computationally tractable. Of course, (5) still requires the inversion of the infinite-dimensional Riesz map, but in approximating the V by finite dimensional V_h which is of a higher polynomial degree than U_h (hence "enriched space") works well in practice.

No assumptions have been made so far on the definition of the inner product on V. In fact, proper choice of $(\cdot,\cdot)_V$ can make the difference between a solid DPG method and one that suffers from robustness issues.

3 DPG Applied to Convection-Diffusion

Now that we have briefly outline the abstract DPG method, let us put it into practice with the Convection-Diffusion equation. The strong form of the steady Convection-Diffusion equation reads

$$\nabla\cdot(\boldsymbol{\beta} u) - \epsilon\Delta u = g\,,$$

where u is the property of interest, $\boldsymbol{\beta}$ is the convection vector, and g is the source term. Let us write this as an equivalent system of first order equations:

$$\begin{aligned} \nabla\cdot(\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= g \\ \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= \mathbf{0}\,. \end{aligned}$$

If we then multiply the top equation by some scalar test function v and the bottom equation by some vector-valued test function τ, we can integrate by parts over each element K:

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + ((\boldsymbol{\beta} u - \boldsymbol{\sigma})\cdot\mathbf{n}, v)_{\partial K} &= (g, v)_K \\ \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla\cdot\boldsymbol{\tau})_K - (u, \tau_n)_{\partial K} &= 0\,. \end{aligned} \tag{6}$$

The discontinuous Petrov-Galerkin method refers to the fact that we are using discontinuous optimal test functions that come from a space differing from the trial space. It does not specify our choice of trial space. Nevertheless, many considerations of DPG in the literature [] associate DPG with the so-called "ultra-weak formulation." We will follow the same derivation for the convection-diffusion equation, but we emphasize that other formulations are available. Thus, we seek field variables $u \in L^2(K)$ and $\boldsymbol{\sigma} \in \mathbf{L^2}(K)$. Mathematically, this leaves their traces on element boundaries undefined, and in a manner similar to the hybridized discontinuous Galerkin method, we define new unknowns for trace $\hat{u}$ and flux $\hat{f}$. Applying these definitions to (6) and adding the two equations together, we arrive at our desired bilinear form,

$$-(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + (\hat{f}, v)_{\partial K} + \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - (\hat{u}, \tau_n)_{\partial K} = (g, v)_K \tag{7}$$

References

[1] C. L. Chang and John J. Nelson. Least-squares finite element method for the stokes problem with zero residual of mass conservation. *SIAM J. Num. Anal.*, 34:480–489, 1997.

[2] J. B. Perot. Discrete conservation properties of unstructured mesh schemes. *Annual Review of Fluid Mechanics*, 43:299–318, 2011.

[3] H.K. Versteeg and W. Malalasekera. *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Prentice Hall, 2007.