Application of a Discontinuous Petrov-Galerkin method to compressible flow problems

Jesse Chan
Supervisors: Leszek Demkowicz, Robert Moser

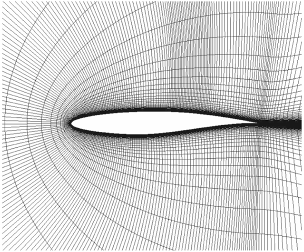
Institute for Computational Engineering and Sciences

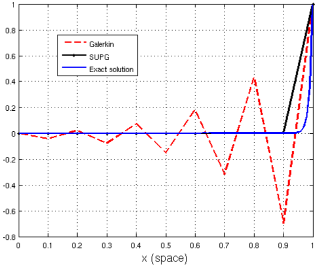
Sept 12, 2012

Compressible Navier-Stokes equations

Numerical difficulties:

- Resolving solution features (sharp, localized viscous-scale phenomena)
    - Shocks
    - Boundary layers - resolution needed for drag/load
- Stability of numerical schemes
    - Coarse/adaptive grids
    - Higher order
- Stabilized methods
    - Stabilization parameters
    - Higher-order





DPG: a minimum residual method via optimal testing

Given a trial space U and Hilbert test space V,

$$b(u,v) = \ell(v) \iff Bu = \ell, \qquad \left.\begin{array}{cl} \langle Bu, v\rangle_V & := b(u,v) \\ \langle \ell, v\rangle_V & := \ell(v). \end{array}\right\}$$

We seek to minimize the dual residual over $U_h \subset U$

$$J(u_h) = \frac{1}{2}\|Bu_h - \ell\|_{V'}^2 = \frac{1}{2}\left\|R_V^{-1}(Bu_h - \ell)\right\|_V^2.$$

where $R_V : V \to V'$ is the isometric Riesz map

$$\langle R_V v, \delta v\rangle_V := (v, \delta v)_V, \quad \forall \delta v \in V.$$

Conditions for minimization of the convex functional give

$$b(u_h, v_{\delta u}) = \ell(v_{\delta u}), \quad \forall \delta u \in U_h, \quad v_{\delta u} := R_V^{-1} B\delta u.$$

Practical details of DPG

Computation of $v_{\delta u} := R_V^{-1} B\delta u$ is **global** and **infinite-dimensional**.

- By choosing a broken test space V and localizable norm $\|v\|_V^2 = \sum_K \|v\|_{V(K)}^2$, test functions can be determined locally.
- In practice, we use an enriched space $V_h \subset V$, where $\dim(V_h) > \dim(U_h)$ elementwise, and optimal test functions are approximated by computing $v_{\delta u} := R_{V_h}^{-1} B\delta u \in V_h$ through[1]

$$(v_{\delta u}, \delta v)_V = b(\delta u, \delta v), \quad \delta u \in U_h, \quad \forall \delta v \in V_h$$

Typically, if $U_h = \mathcal{P}^p(\mathbb{R}^n)$, $V_h = \mathcal{P}^{p+\triangle p}(\mathbb{R}^n)$, where $\triangle p \geq n$.[2]

---

[1] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. II. Optimal test functions. *Num. Meth. for Partial Diff. Eq*, 27:70–105, 2011

[2] J. Gopalakrishnan and W. Qiu. An analysis of the practical DPG method. Technical report, IMA, 2011. Submitted

Properties of DPG

- Stiffness matrices are symmetric positive-definite. For trial/test bases $\{\phi_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$, with $B_{ji} = b(\phi_j, v_i)$ and $l_i = \ell(v_i)$. DPG solves

$$\left(B^T R_V^{-1} B\right) u = \left(B^T R_V^{-1}\right) l,$$

For localizable norms and discontinuous testing, R_V is block diagonal.

- DPG provides the best approximation in the energy norm

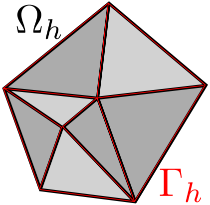
$$\|u\|_E = \|Bu\|_{V'} = \sup_{\|v\|_V = 1} |b(u,v)| .$$

- The energy error is computable through the error representation function e defined through $(e, \delta v)_V = \ell(v) - b(u_h, \delta v)$ for all $\delta v \in V$.

$$\|u - u_h\|_E = \|B(u-u_h)\|_{V'} = \left\|R_V^{-1}(l - Bu_h)\right\|_V = \|e\|_V$$

Ultra-weak formulation for convection-diffusion

The first order convection-diffusion system:



$$A(u,\sigma) := \begin{bmatrix} \nabla \cdot (\beta u - \sigma) \\ \frac{1}{\epsilon}\sigma - \nabla u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

The variational formulation is

$$b\left(\left(u,\sigma,\widehat{u},\widehat{f}_n\right),(v,\tau)\right) = \left(u, \nabla_h \cdot \tau - \beta \cdot \nabla_h v\right)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla_h v\right)_{\Omega_h}$$
$$- \left\langle [\![\tau \cdot n]\!], \widehat{u}\right\rangle_{\Gamma_h} + \left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h},$$

where $\widehat{f}_n := \beta_n u - \sigma_n$ and $\left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h}$ is defined

$$\left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h} := \sum_K \int_{\partial K} \mathrm{sgn}(\vec{n})\, \widehat{f}_n v.$$

Graph norm under convection-diffusion

The graph norm[3] for convection-diffusion gives exceptional stability.

$$\|(v,\tau)\|_{V(K)}^2 = \|\nabla\cdot\tau - \beta\cdot\nabla v\|_{L^2(K)}^2 + \|\epsilon^{-1}\tau + \nabla v\|_{L^2(K)}^2 + \|v\|_{L^2(K)}^2.$$

Problem with this test norm: approximability of test functions.

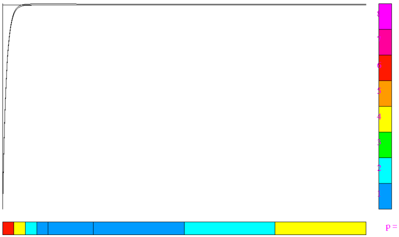


Figure: Components of optimal test functions for flux $\widehat{f}_n$ on the *right-hand* side of a unit element for $\epsilon = 0.01$.

[3]T. Bui-Thanh, L. Demkowicz, and O. Ghattas. A unified discontinuous Petrov-Galerkin method and its analysis for Friedrichs' systems. *Submitted to SIAM J. Numer. Anal.*, 2011. Also ICES report 11-34, November 2011

Determining an alternative test norm

$$b(\mathbf{U},\mathbf{V}) = (u, \nabla\cdot\tau - \beta\cdot\nabla v)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla v\right)_{\Omega_h} + \text{boundary terms}$$

Recover $\|u\|^2_{L^2(\Omega)}$ with conforming (v,τ) satisfying the *adjoint equations*

$$\begin{aligned} \nabla\cdot\tau - \beta\cdot\nabla v &= u \\ \frac{1}{\epsilon}\tau + \nabla v &= 0 \end{aligned}\;, \quad \langle [\![\tau\cdot n]\!], \widehat{u}\rangle_\Gamma, \left\langle \widehat{f}_n, [\![v]\!]\right\rangle_\Gamma = 0$$

"Necessary" conditions for robustness —

$$\|u\|^2_{L^2(\Omega)} = b\left(\mathbf{U},(v,\tau)\right) = \frac{b\left(\mathbf{U},(v,\tau)\right)}{\|(v,\tau)\|_V}\|(v,\tau)\|_V \leq \|\mathbf{U}\|_E\,\|(v,\tau)\|_V$$

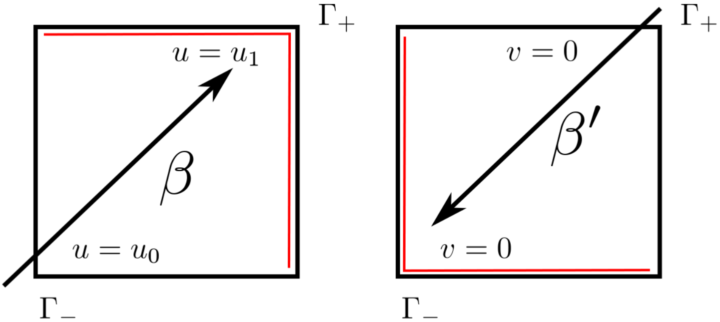
Let $\lesssim$ denote a robust bound - if $\|(v,\tau)\|_V \lesssim \|u\|_{L^2(\Omega)}$, then we have that

$$\|u\|_{L^2(\Omega)} \lesssim \|\mathbf{U}\|_E$$

Main idea: the test norm should measure adjoint solutions robustly.

Dirichlet inflow condition: issues as $\epsilon \to 0$

Standard choice of boundary condition: $u = u_0$ on inflow boundary Γ_{in}, induces boundary layers in adjoint problems, $\|\beta \cdot \nabla v\|_{L^2} = O(\epsilon^{-1})$.

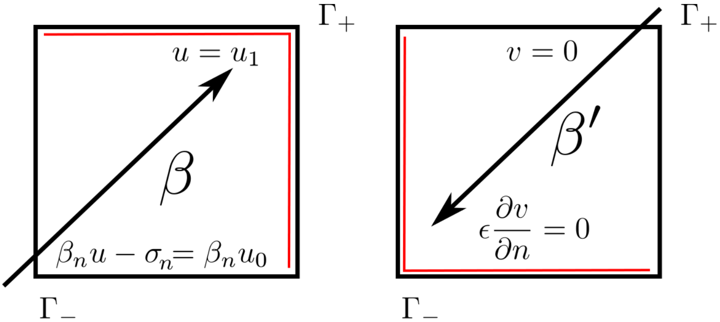


(a) Primal problem (b) Adjoint problem

Figure: For the standard Dirichlet inflow condition, the solution to the adjoint problem can develop strong boundary layers at the outflow of the adjoint problem.

Solution: New inflow boundary condition on $\widehat{f}_n$

Non-standard choice of boundary condition: $\widehat{f}_n = \beta_n u_0$ on $\Gamma_{\rm in}$, induces smoother adjoint problems, $\|\beta \cdot \nabla v\|_{L^2} = O(1)$.



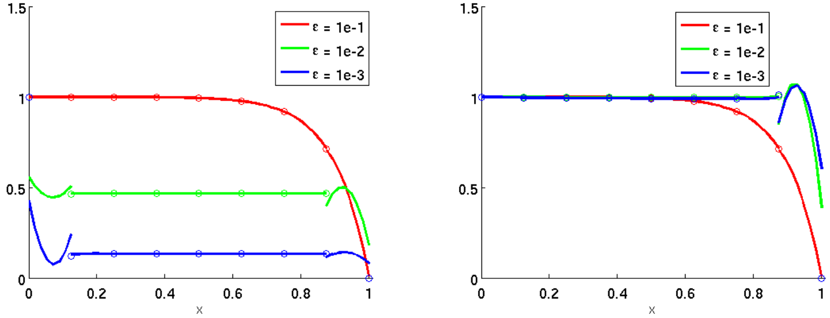
(a) Primal problem

(b) Adjoint problem

Figure: Under the new inflow condition, the wall-stop boundary condition is relaxed to a zero-stress condition at the outflow boundary of the adjoint problem.

Test norms and adjoint solutions

Intuition: the effectiveness of DPG under a test norm is governed by how a **specific test norm** measures the **solutions of the adjoint problem**.



(a) Dirichlet inflow

(b) "Convection" inflow

Figure: DPG solutions to convection-diffusion for both inflow conditions using an H^1 test norm.

Mesh-scaled test norms

For solutions (v,τ) of the adjoint equations, we derive quantities that are robustly bounded from above by $\|u\|_{L^2(\Omega)}$. Our test norm, as defined over a single element K, is now

$$\|(v,\tau)\|_{V,K}^2 = \min\left\{\frac{\epsilon}{|K|},1\right\}\|v\|^2 + \epsilon\|\nabla v\|^2 + \|\beta\cdot\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \min\left\{\frac{1}{\epsilon},\frac{1}{|K|}\right\}\|\tau\|^2.$$

which induces the proven *robust* bound[4]

$$\|u\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} + \epsilon\|\widehat{u}\| + \sqrt{\epsilon}\left\|\widehat{f}_n\right\| \lesssim \left\|\left(u,\sigma,\widehat{u},\widehat{f}_n\right)\right\|_E.$$

[4] J. Chan, N. Heuer, T. Bui Thanh, and L. Demkowicz. Robust DPG method for convection-diffusion problems II: natural inflow conditions. Technical Report 12-21, ICES, June 2012. Submitted

Numerical verification: Erikkson-Johnson problem

Experiments done using Camellia[5] and a boundary layer solution.

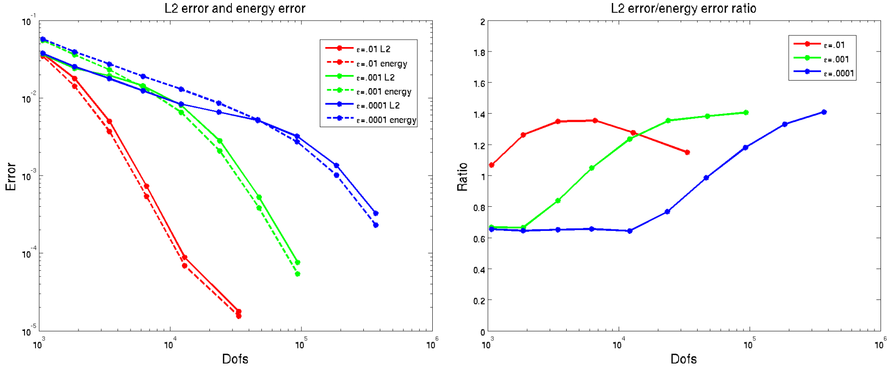


Figure: L^2/energy errors and their ratio for $\epsilon = .01, .001, .0001$.

[5] N. Roberts, D. Ridzal, P. Bochev, and L. Demkowicz. A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos. Technical Report SAND2011-6678, Sandia National Laboratories, 2011

2D test case: Burgers equation

$$\frac{\partial\left(u^2/2\right)}{\partial x} + \frac{\partial u}{\partial y} + \epsilon \Delta u = f$$

Burgers equation can be written with $\beta(u) = (u/2, 1)$

$$\nabla \cdot (\beta(u)u - \sigma) = f$$
$$\frac{1}{\epsilon}\sigma - \nabla u = 0.$$

i.e. nonlinear convection-diffusion on domain $[0,1]^2$.

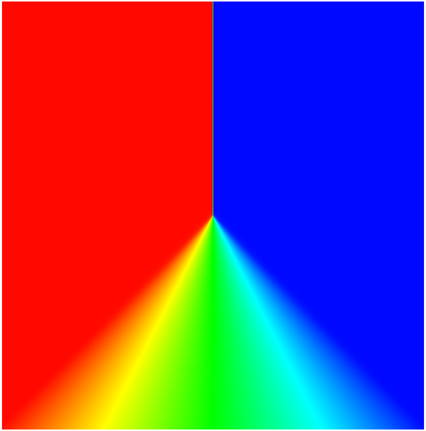
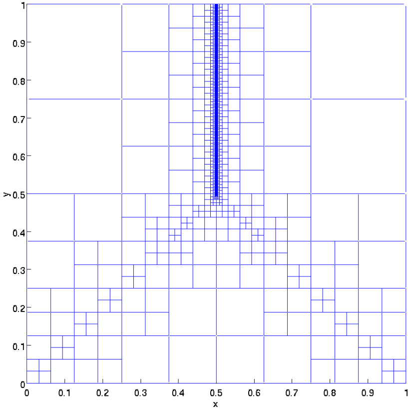


Figure: Shock solution for Burgers' equation, $\epsilon = 1e-4$, using Newton-Raphson.

Adaptivity begins with a cubic 4×4 mesh.



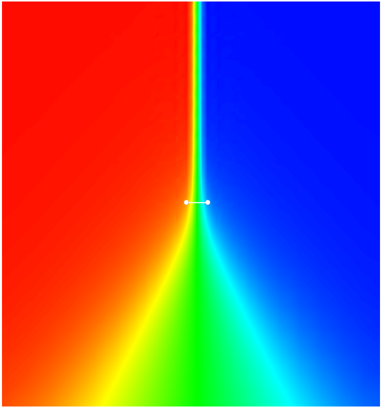


Figure: Adaptive mesh after 9 refinements, and zoom view at point (.5,.5) with shock formation and $1e-3$ width line for reference.

2D Compressible Navier-Stokes equations (ideal gas)

Given density ρ, velocities $\mathbf{u} = (u_1, u_2)$ and temperature T,

$$
\begin{aligned}
\nabla \cdot \begin{bmatrix} \rho u_1 \\ \rho u_2 \end{bmatrix} &= 0, \\
\nabla \cdot \left(\begin{bmatrix} \rho u_1^2 + p \\ \rho u_1 u_2 \end{bmatrix} - \boldsymbol{\sigma}_1 \right) &= 0, \\
\nabla \cdot \left(\begin{bmatrix} \rho u_1 u_2 \\ \rho u_2^2 + p \end{bmatrix} - \boldsymbol{\sigma}_2 \right) &= 0, \\
\nabla \cdot \left(\begin{bmatrix} ((\rho e) + p) u_1 \\ ((\rho e) + p) u_2 \end{bmatrix} - \boldsymbol{\sigma} \mathbf{u} + \vec{q} \right) &= 0, \\
\frac{1}{2\mu} \boldsymbol{\sigma} - \frac{\lambda}{4\mu(\mu + \lambda)} \mathrm{tr}(\boldsymbol{\sigma}) \mathbf{I} &= \nabla \mathbf{u} - \mathrm{Re}\, \mathbf{w}, \\
\kappa^{-1} \vec{q} &= \nabla T,
\end{aligned}
$$

where $\mathbf{w}$ represents the antisymmetric part of ∇u

Extrapolation of test norms

Convection-diffusion:

$$\nabla \cdot (\beta u - \sigma) = f$$
$$\frac{1}{\epsilon}\sigma - \nabla u = 0.$$

Navier-Stokes: defining vector variables $U = \{\rho, u_1, u_2, T\}$ and $\Sigma = \{\boldsymbol{\sigma}, \mathbf{q}, w\}$,

$$\nabla \cdot (A_{\text{invisc}} U - A_{\text{visc}} \Sigma) = R_{\text{conserv}}(U, \Sigma)$$
$$E_{\text{visc}} \Sigma - \nabla U = R_{\text{constit}}(U, \Sigma)$$

where $R_{\text{conserv}}(U, \Sigma)$ and $R_{\text{constit}}(U, \Sigma)$ are the conservation/constitutive residuals.

Extrapolation of test norms

Convection-diffusion:

$$\nabla \cdot (\beta u - \sigma) = f$$
$$\frac{1}{\epsilon}\sigma - \nabla u = 0.$$

Navier-Stokes: defining vector variables $U = \{\rho, u_1, u_2, T\}$ and $\Sigma = \{\boldsymbol{\sigma}, \mathbf{q}, w\}$,

$$\nabla \cdot (A_{\text{invisc}} U - A_{\text{visc}} \Sigma) = R_{\text{conserv}}(U, \Sigma)$$
$$E_{\text{visc}} \Sigma - \nabla U = R_{\text{constit}}(U, \Sigma)$$

where $R_{\text{conserv}}(U, \Sigma)$ and $R_{\text{constit}}(U, \Sigma)$ are the conservation/constitutive residuals.

Carter's flat plate problem

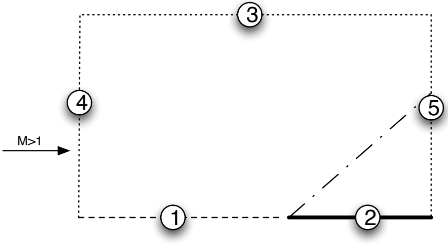
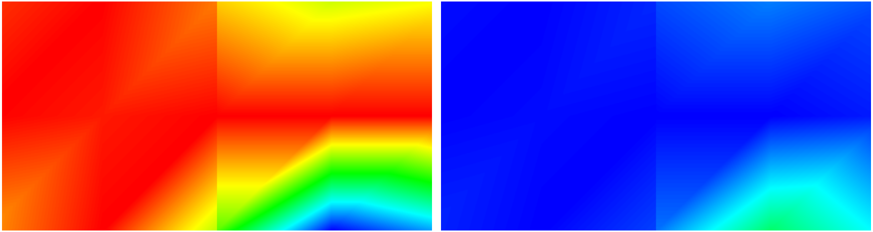


Figure: Carter flat plate problem on domain $[0,2] \times [0,1]$. Plate begins at $x = 1$, $\mathrm{Re} = 1000$.

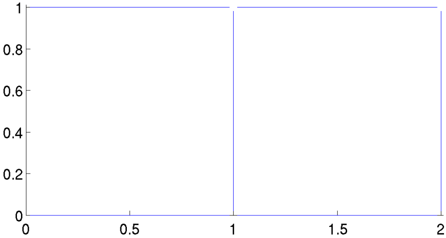
1. Symmetry boundary conditions.
2. Prescribed temperature and wall stagnation conditions.
3. Symmetry boundary conditions.
4. Inflow: conserved quantities specified using far-field values.
5. No outflow condition set.

Stress/heat flux boundary conditions are set in terms of the momentum and energy fluxes.

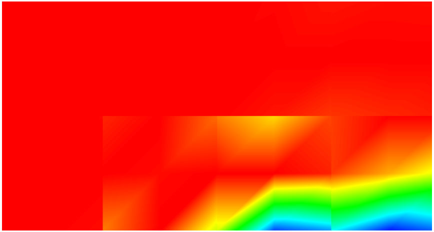
Refinement level 0



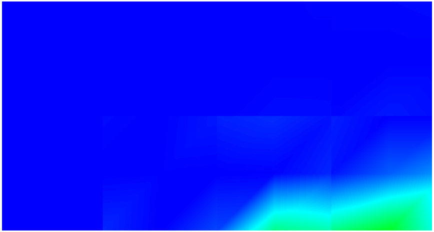
(a) u_1 (b) T



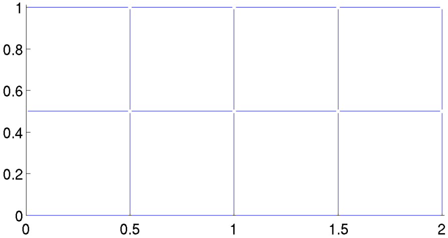
Refinement level 1



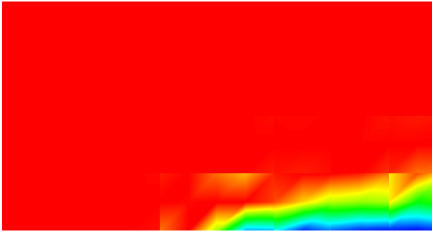
(a) u_1



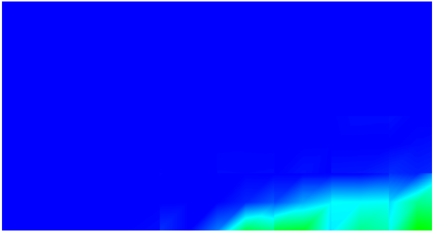
(b) T



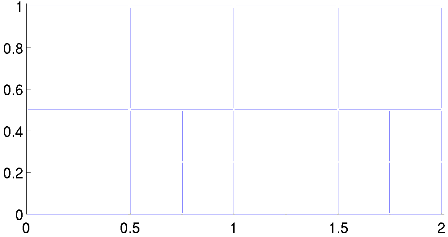
Refinement level 2



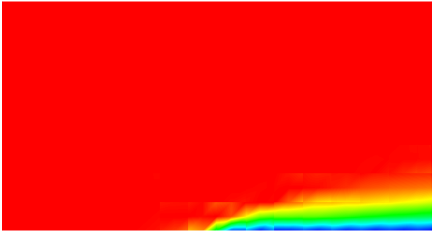
(a) u_1



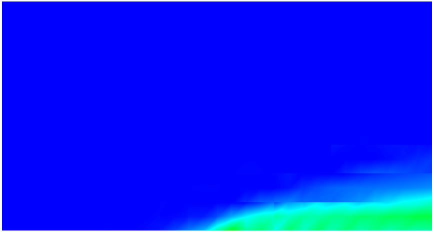
(b) T



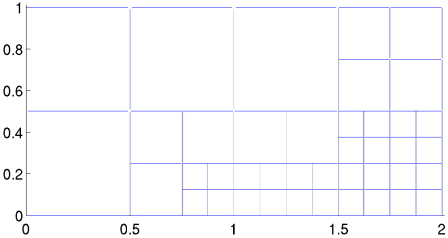
Refinement level 3



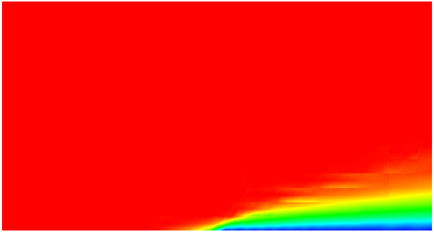
(a) u_1



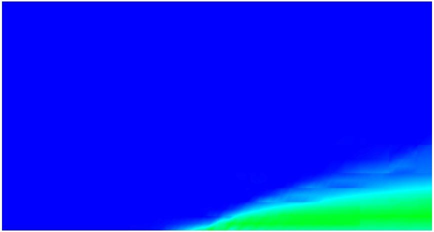
(b) T



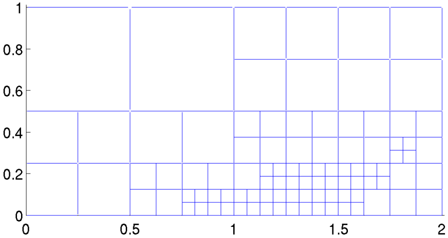
Refinement level 4



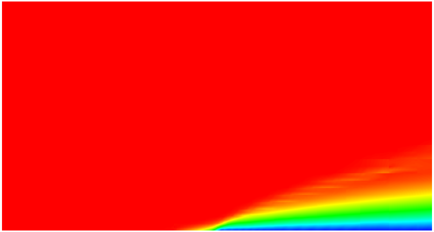
(a) u_1



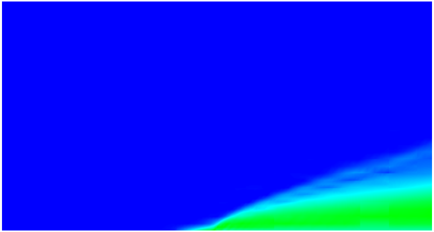
(b) T



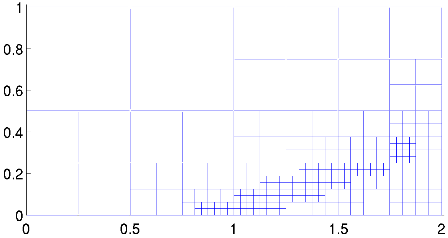
Refinement level 5



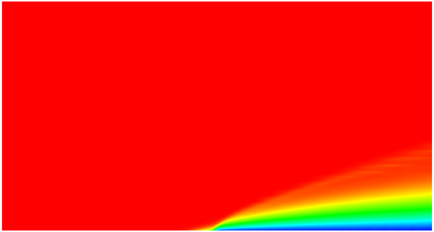
(a) u_1



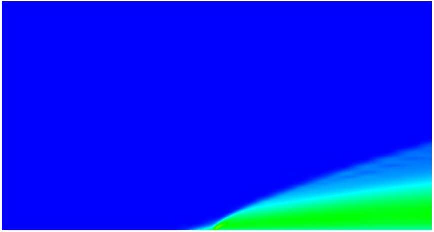
(b) T



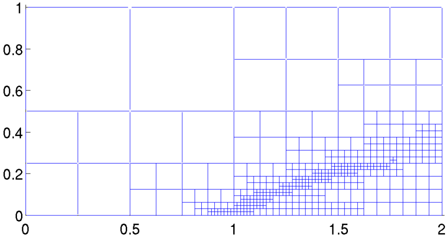
Refinement level 6



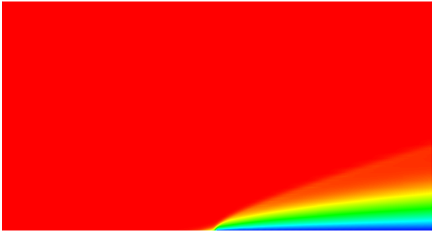
(a) u_1



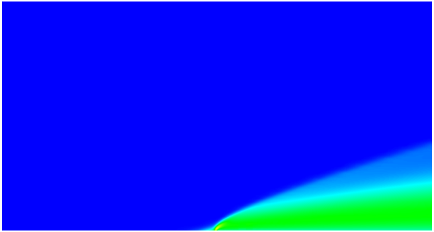
(b) T



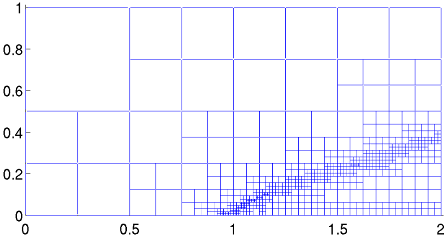
Refinement level 7



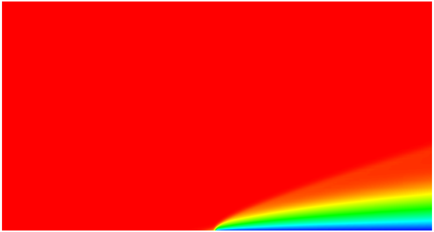
(a) u_1



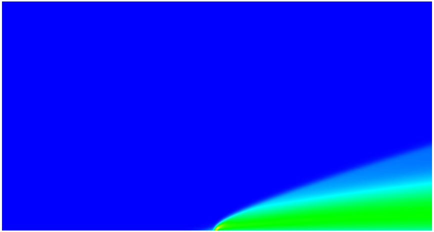
(b) T



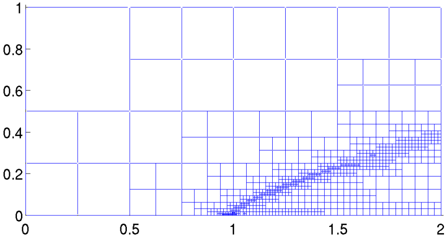
Refinement level 8



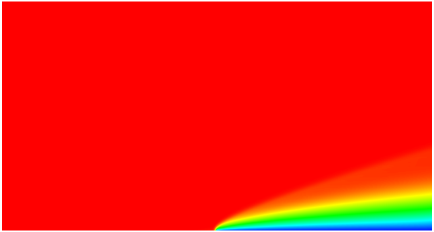
(a) u_1



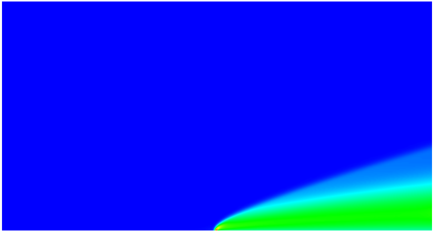
(b) T



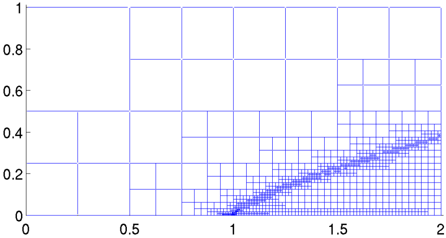
Refinement level 9



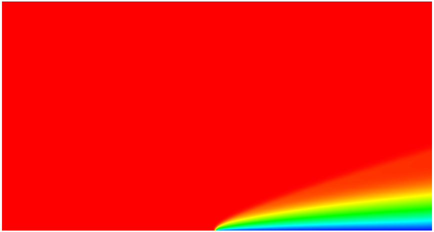
(a) u_1



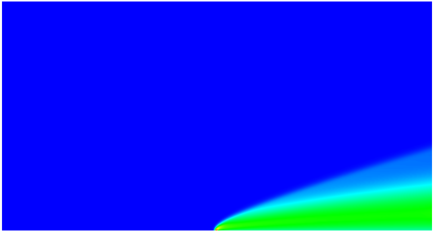
(b) T



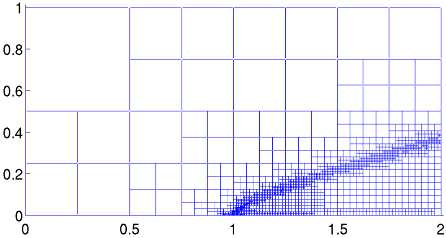
Refinement level 10



(a) u_1

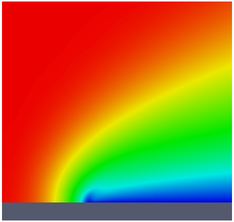


(b) T

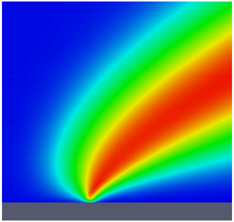


Zoomed solutions at plate/stagnation point

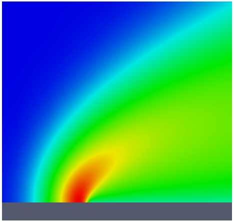
(a) ρ



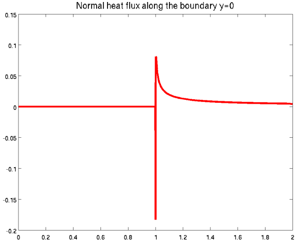
(b) u_1



(c) u_2



(d) T



(e) q_n

Thank you!

Questions?

F. Brezzi, B. Cockburn, L.D. Marini, and E. Süli.
Stabilization mechanisms in discontinuous Galerkin finite element methods.
Computer Methods in Applied Mechanics and Engineering, 195(25–28):3293 – 3310, 2006.

A. Brooks and T. Hughes.
Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations.
Comp. Meth. Appl. Mech. Engr, 32:199–259, 1982.

T. Bui-Thanh, L. Demkowicz, and O. Ghattas.
A unified discontinuous Petrov-Galerkin method and its analysis for Friedrichs' systems.
Submitted to SIAM J. Numer. Anal., 2011.
Also ICES report 11-34, November 2011.

J. Chan, L. Demkowicz, R. Moser, and N. Roberts.
A New Discontinuous Petrov-Galerkin Method with Optimal Test Functions. Part V: Solution of 1D Burgers' and Navier-Stokes Equations.
Technical Report 10-25, ICES, June 2010.

J. Chan, L. Demkowicz, and M. Shashkov.
Space-time DPG for shock problems.
Technical Report LA-UR 11-05511, LANL, September 2011.

J. Chan, N. Heuer, T. Bui Thanh, and L. Demkowicz.
Robust DPG method for convection-diffusion problems II: natural inflow conditions.
Technical Report 12-21, ICES, June 2012.
Submitted.

B. Cockburn, J. Gopalakrishnan, and R. Lazarov.
Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems.
SIAM J. Numer. Anal., 47(2):1319–1365, February 2009.

L. Demkowicz and J. Gopalakrishnan.
Analysis of the DPG method for the Poisson equation.
SIAM J. Numer. Anal., 49(5):1788–1809, September 2011.

L. Demkowicz and J. Gopalakrishnan.
A class of discontinuous Petrov-Galerkin methods. II. Optimal test functions.
Num. Meth. for Partial Diff. Eq, 27:70–105, 2011.

L. Demkowicz, J. Gopalakrishnan, and A. Niemi.
A class of discontinuous Petrov-Galerkin methods. Part III: Adaptivity.
Appl. Numer. Math., 62(4):396–427, April 2012.

J. Gopalakrishnan and W. Qiu.
An analysis of the practical DPG method.
Technical report, IMA, 2011.
Submitted.

L. Mansfield.
On the Conjugate Gradient Solution of the Schur Complement System Obtained from Domain Decomposition.
SIAM Journal on Numerical Analysis, 27(6):pp. 1612–1620, 1990.

N. Roberts, D. Ridzal, P. Bochev, and L. Demkowicz.

A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos.
Technical Report SAND2011-6678, Sandia National Laboratories, 2011.