1 TSC envelope

To find the midplane density of the TSC envelope model, we have to estimate The TSC density structure is given by

$$\rho(r,\theta) = \rho_0 \left(\frac{r}{R_c}\right)^{-3/2} \left(1 + \frac{\mu}{\mu_0}\right)^{-1/2} \left(\frac{\mu}{\mu_0} + \frac{2\,\mu_0^2\,R_c}{r}\right)^{-1} \tag{1}$$

Writing $\gamma \equiv r/R_c$, this is

$$\rho(r) = \rho_0\,\gamma^{-3/2} \left(1 + \frac{\mu}{\mu_0}\right)^{-1/2} \left(\frac{\mu}{\mu_0} + \frac{2\,\mu_0^2}{\gamma}\right)^{-1} \tag{2}$$

We now have to find μ_0 for $\mu = 0$. The streamline equation is:

$$\mu_0^3 + \mu_0\,(\gamma - 1) - \mu\,\gamma = 0 \tag{3}$$

If $\mu = 0$, this gives

$$\mu_0\,[\mu_0^2 + (\gamma - 1)] = 0 \tag{4}$$

This has solution $\mu_0 = 0$ and $\mu_0^2 = (1 - \gamma)$.

1.1 Outside R_c

For $\gamma > 1$, neither of these are satisfactory (since μ/μ_0 will be undefined for the former, and μ_0 is imaginary for the latter. The solution *is* 0, but we need to compute μ/μ_0 by some other means. We can multiply the streamline equation by μ^2/μ_0^3, giving

$$\mu^2 + (\mu/\mu_0)^2\ (\gamma - 1) - (\mu/\mu_0)^3\ \gamma = 0 \tag{5}$$

For $\mu = 0$, this is

$$(\mu/\mu_0)^2\ (\gamma - 1) - (\mu/\mu_0)^3\ \gamma = 0 \tag{6}$$

which can be factorized as

$$(\mu/\mu_0)^2\ [(\gamma-1)-(\mu/\mu_0)\ \gamma]=0 \tag{7}$$

This has two unique solutions, $\mu/\mu_0=0$ and $\mu/\mu_0=(\gamma-1)/\gamma$. The former is not acceptable, because combined with $\mu_0=0$, this gives an undefined density. Therefore, we have:

$$\begin{cases} \mu_0=0 \\ \mu/\mu_0=\dfrac{\gamma-1}{\gamma} \end{cases} \tag{8}$$

Which gives

$$\rho(r,\theta)=\rho_0\,\gamma^{-3/2}\left(\frac{2\gamma-1}{\gamma}\right)^{-1/2}\left(\frac{\gamma-1}{\gamma}\right)^{-1} \tag{9}$$

This can be simplified:

$$\rho(r,\theta)=\rho_0\,(2\gamma-1)^{-1/2}\,(\gamma-1)^{-1} \tag{10}$$

1.2 Inside R_c

For $\gamma<1$, The ratio equation only has one real positive solution, which is 0. However, both solutions for μ, i.e. 0 and $1-\gamma$ are valid. Since the ratio is 0, this means the only solution which does not give an undefined density is $1-\gamma$. Therefore,

$$\begin{cases} \mu_0^2=1-\gamma \\ \mu/\mu_0=0 \end{cases} \tag{11}$$

Which gives

$$\rho(r)=\rho_0\,\gamma^{-3/2}\left[\frac{2(1-\gamma)}{\gamma}\right]^{-1} \tag{12}$$

i.e.

$$\rho(r)=\frac{\rho_0}{2}\frac{\gamma^{-1/2}}{1-\gamma} \tag{13}$$

In cumulative terms, this is

$$\Sigma(r)=\frac{\rho_0}{2}\times\left\{\ln\left[\frac{\sqrt{\gamma_1}+1}{1-\sqrt{\gamma_1}}\right]-\ln\left[\frac{\sqrt{\gamma_0}+1}{1-\sqrt{\gamma_0}}\right]\right\} \tag{14}$$