# 2022 National College Entrance Examination I

Mathematics

**Instructions:**

1. Before starting, make sure to write your name and admission ticket number on the answer sheet.
2. When answering multiple-choice questions, after selecting the answer for each question, use a 2B pencil to darken the corresponding option number on the answer sheet. If you need to change your answer, erase it cleanly and then darken another option number. Answers written on the exam paper are invalid.
3. For non-multiple-choice questions, use a black pen or fountain pen with black ink. The answers must be written in the designated area corresponding to each question on the answer sheet.
4. Maintain the cleanliness of the answer sheet. After the exam, submit both the exam paper and the answer sheet.

## 1 Multiple Choice Questions: This section contains 8 questions. Each question is worth 5 points, totaling 40 points. Among the four options provided for each question, only one is correct.

1. If the sets $M = \{x \mid \sqrt{x} < 4\}$ and $N = \{x \mid 3x \geqslant 1\}$, then $M \cap N =$

   a) $\{x \mid 0 \leqslant x < 2\}$ b) $\{x \mid \frac{1}{3} \leqslant x < 2\}$ c) $\{x \mid 3 \leqslant x < 16\}$ d) $\{x \mid \frac{1}{3} \leqslant x < 16\}$

2. If $i(1 - z) = 1$, then $z + \overline{z} =$

a) $-2$ b) $-1$ c) $1$ d) $2$

3. In $\triangle ABC$, point $D$ lies on side $AB$ such that $BD = 2DA$. Let $\overrightarrow{CA} = \mathbf{m}$ and $\overrightarrow{CD} = \mathbf{n}$. Then $\overrightarrow{CB} =$

   a) $3\mathbf{m} - 2\mathbf{n}$ b) $-2\mathbf{m} + 3\mathbf{n}$ c) $3\mathbf{m} + 2\mathbf{n}$ d) $2\mathbf{m} + 3\mathbf{n}$

4. The South-to-North Water Diversion Project alleviated water shortages in some northern regions by storing water in a reservoir. It is known that when the water level of the reservoir is at an elevation of 148.5m, the corresponding water surface area is 140.0 $\text{km}^2$; when the water level is at an elevation of 157.5m, the corresponding water surface area is 180.0 $\text{km}^2$. Assuming the reservoir's shape between these two water levels is a frustum (truncated cone), the approximate increase in the volume of water when the water level rises from 148.5m to 157.5m is (where $\sqrt{7} \approx 2.65$).

   a) $1.0 \times 10^9$ m$^3$ b) $1.2 \times 10^9$ m$^3$ c) $1.4 \times 10^9$ m$^3$ d) $1.6 \times 10^9$ m$^3$

5. From the integers 2 to 8 (inclusive), two different numbers are randomly selected. The probability that these two numbers are coprime is

   a) $\frac{1}{6}$ b) $\frac{1}{3}$ c) $\frac{1}{2}$ d) $\frac{2}{3}$

6. Let $f(x) = \sin\left(\omega x + \frac{\pi}{4}\right) + b$ $(\omega > 0)$ have a minimum positive period of $T$. Given that $\frac{2\pi}{3} < T < \pi$, and the graph of $y = f(x)$ is centrally symmetric about the point $\left(\frac{3\pi}{2}, 2\right)$, then $f\left(\frac{\pi}{2}\right) =$

   a) $1$ b) $\frac{3}{2}$ c) $\frac{5}{2}$ d) $3$

7. Let $a = 0.1e^{0.1}$, $b = \frac{1}{9}$, and $c = -\ln 0.9$. Then

   a) $a < b < c$ b) $c < b < a$ c) $c < a < b$ d) $a < c < b$

8. Given a right quadrilateral pyramid with lateral edge length $l$, all its vertices lie on the same sphere. If the volume of the sphere is $36\pi$, and $3 \leqslant l \leqslant 3\sqrt{3}$, then the range of possible volumes for the quadrilateral pyramid is

   a) $[18, \frac{81}{4}]$
   b) $[\frac{27}{4}, \frac{81}{4}]$
   c) $[\frac{27}{4}, \frac{64}{3}]$
   d) $[18, 27]$

## 2 Multiple Choice Questions: This section contains 4 questions. Each question is worth 5 points, totaling 20 points. Among the four options provided for each question, multiple options may be correct. Selecting all correct options yields 5 points, partially correct selections yield 2 points, and any incorrect selection yields 0 points.

9. Given a cube $ABCD - A_1B_1C_1D_1$, then

   a) The line $BC_1$ makes a $90°$ angle with $DA_1$.
   b) The line $BC_1$ makes a $90°$ angle with $CA_1$.
   c) The line $BC_1$ makes a $45°$ angle with the plane $BB_1D_1D$.
   d) The line $BC_1$ makes a $45°$ angle with the plane $ABCD$.

10. Given the function $f(x) = x^3 - x + 1$, then

   a) $f(x)$ has two extremum points.
   b) $f(x)$ has three zeros.
   c) The point $(0, 1)$ is the center of symmetry of the curve $y = f(x)$.
   d) The line $y = 2x$ is a tangent to the curve $y = f(x)$.

11. Given $O$ as the origin, and point $(1, 1)$ lies on the parabola $C : x^2 = 2py$ $(p > 0)$. A line passing through point $B(0, -1)$ intersects $C$ at points $P$ and $Q$, respectively. Then

   a) The directrix of $C$ is $y = -1$.
   b) The line $AB$ is tangent to $C$.

c) $|OP| \cdot |OQ| > |OA|^2$. d) $|BP| \cdot |BQ| > |BA|^2$.

12. Given the function $f(x)$ and its derivative $f'(x)$ are both defined for all $x \in \mathbb{R}$. Let $g(x) = f'(x)$. If both $f\left(\frac{3}{2} - 2x\right)$ and $g(2+x)$ are even functions, then

a) $f(0) = 0$ b) $g\left(-\frac{1}{2}\right) = 0$ c) $f(-1) = f(-4)$ d) $g(-1) = g(2)$

# 3 Fill in the Blanks: This section contains 4 questions. Each question is worth 5 points, totaling 20 points.

13. In the expansion of $\left(1 - \frac{y}{x}\right)(x+y)^8$, the coefficient of $x^2y^6$ is ________ (Answer using numbers).

14. Write the equation of a straight line that is tangent to both the circle $x^2 + y^2 = 1$ and the circle $(x-3)^2 + (y-4)^2 = 16$: ______________.

15. If the curve $y = (x+a)e^x$ has two tangents passing through the origin, then the range of $a$ is ________.

16. Given the ellipse $C: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$, let $A$ be the upper vertex of $C$, and $F_1$, $F_2$ be the two foci. The eccentricity is $\frac{1}{2}$. Let a line passing through $F_1$ and perpendicular to $AF_2$ intersect $C$ at points $D$ and $E$, with $|DE| = 6$. The perimeter of $\triangle ADE$ is ________.

# 4 Answer Questions: This section contains 6 questions, totaling 70 points. Provide detailed explanations, proofs, or calculation steps for your answers.

17. **(10 points)**

Let $S_n$ denote the sum of the first $n$ terms of the sequence $\{a_n\}$. Given that $a_1 = 1$, and the sequence $\left\{\frac{S_n}{a_n}\right\}$ forms an arithmetic sequence with a common difference of $\frac{1}{3}$.

1. Find the general formula for $\{a_n\}$.
2. Prove that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < 2$.

18. **(12 points)**

In $\triangle ABC$, let the angles opposite sides $a$, $b$, and $c$ be $A$, $B$, and $C$ respectively. Given that:

$$\frac{\cos A}{1+\sin A} = \frac{\sin 2B}{1+\cos 2B}$$

1. If $C = \frac{2\pi}{3}$, find $B$.
2. Find the minimum value of $\frac{a^2+b^2}{c^2}$.

19. **(12 points)**

In the right triangular prism $ABC - A_1B_1C_1$, the volume is 4, and the area of $\triangle A_1BC$ is $2\sqrt{2}$.

1. Find the distance from point $A$ to the plane $A_1BC$.
2. Let $D$ be the midpoint of $A_1C$, $AA_1 = AB$, and the plane $A_1BC$ is perpendicular to the plane $ABB_1A_1$. Find the sine of the dihedral angle $\angle ABDC$.

20. **(12 points)**

A medical team is studying the relationship between a local endemic disease and the hygiene habits of the residents (hygiene habits are classified as "Good" and "Not Good"). In a random survey, they investigated 100 cases in the disease group and 100 individuals in the control group. The data obtained is as follows:

| | Not Good | Good |
|---|---|---|
| Disease Group | 40 | 60 |
| Control Group | 10 | 90 |

1. Can we be 99% confident that there is a difference in hygiene habits between the disease group and the control group?

2. From the population, a person is randomly selected. Let event $A$ denote "the person has not good hygiene habits", and event $B$ denote "the person has the disease". The ratio $\frac{P(B|A)}{P(\overline{B}|A)}$ to $\frac{P(B|\overline{A})}{P(\overline{B}|\overline{A})}$ is a measure indicator $R$ of how not good hygiene habits affect the risk of having the disease.

   (i) Prove that:

$$R = \frac{P(A \mid B)}{P(\overline{A} \mid B)} \cdot \frac{P(\overline{A} \mid \overline{B})}{P(A \mid \overline{B})}$$

   (ii) Using the survey data, provide estimates for $P(A \mid B)$ and $P(A \mid \overline{B})$, and then use part (i) to estimate $R$.

**Note:** $K^2 = \frac{n(ab-bc)^2}{(a+b)(c+d)(a+c)(b+d)}$,

| $P(K^2 \geq k)$ | 0.050 | 0.010 | 0.001 |
|---|---|---|---|
| $k$ | 3.841 | 6.635 | 10.828 |

21. **(12 points)**

Point $A(2,1)$ lies on the hyperbola $C : \frac{x^2}{a^2} - \frac{y^2}{a^2-1} = 1$ $(a > 1)$. A straight line $l$ intersects $C$ at points $P$ and $Q$. The sum of the slopes of lines $AP$ and $AQ$ is zero.

1. Find the slope of line $l$.
2. If $\tan \angle PAQ = 2\sqrt{2}$, find the area of $\triangle PAQ$.

22. **(12 points)**

Given the functions $f(x) = e^x - ax$ and $g(x) = ax - \ln x$ have the same minimum value.

1. Find $a$.
2. Prove that there exists a horizontal line $y = b$ that intersects both curves $y = f(x)$ and $y = g(x)$ at three distinct points, and the $x$-coordinates of these intersection points form an arithmetic sequence from left to right.