Probability and Statistics

Quiz-II 6 April, 2018

Time: 30 Minutes Maximum Marks: 10

Q.1 Given that $E(X) = 5$, $E(X^2) = 27$, $E(Y) = 7$, $E(Y^2) = 51$ and $\text{var}(X+Y) = 8$, find $\text{cov}(X+Y, X+2Y)$. **[5 marks]**

Solution:

$$\begin{aligned}
\text{cov}(X+Y, X+2Y) &= \text{cov}(X, X+2Y) + \text{cov}(Y, X+2Y) \\
&= \text{cov}(X, X) + \text{cov}(X, 2Y) + \text{cov}(Y, X) + \text{cov}(Y, 2Y) \\
&= \text{var}(X) + 2\text{cov}(X, Y) + \text{cov}(X, Y) + 2\text{cov}(Y, Y) \\
&= \text{var}(X) + 3\text{cov}(X, Y) + 2\text{var}(Y)
\end{aligned}$$

In order to determine $\text{cov}(X, Y)$ we write

$$\begin{aligned}
8 = \text{var}(X+Y) = \text{cov}(X+Y, X+Y) &= \text{var}(X) + 2\text{cov}(X, Y) + \text{var}(Y) \\
\implies \text{cov}(X, Y) &= \frac{8 - \text{var}(X) - \text{var}(Y)}{2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\text{cov}(X+Y, X+2Y) &= \text{var}(X) + \frac{3}{2}(8 - \text{var}(X) - \text{var}(Y)) + 2\text{var}(Y) \\
&= 12 - \frac{1}{2}\text{var}(X) + \frac{1}{2}\text{var}(Y) = 12 - (27-25)/2 + (51-49)/2 = 12
\end{aligned}$$

Alternative Sol.: $Cov(X+Y, X+2Y) = E((X+Y)(X+2Y)) - E(X+Y)E(X+2Y)$.
Using the properties of expectation and the given data, we get

$$\begin{aligned}
E(X+Y)E(X+2Y) &= (E(X) + E(Y))(E(X) + 2E(Y)) \\
&= (5+7)(5+2\times 7) \\
&= 12 \times 19 \\
&= 228. \\
\text{And} \quad E((X+Y)(X+2Y)) &= E(X^2 + 3XY + 2Y^2) \\
&= E(X^2) + 3E(XY) + 2E(Y^2) \\
&= 27 + 3E(XY) + 2 \times 51 \\
&= 129 + 3E(XY).
\end{aligned}$$

Now, to find $E(XY)$, we will use the relation $Var(X+Y) = 8$.

$$\begin{aligned}
8 = Var(X+Y) &= E(X+Y)^2 - (E(X+Y))^2 \\
&= E(X^2) + 2E(XY) + E(Y^2) - (E(X) + E(Y))^2 \\
&= 27 + 2E(XY) + 51 - (5+7)^2 \\
&= -66 + 2E(XY) \Rightarrow E(XY) = 37.
\end{aligned}$$

Thus, $Cov(X+Y, X+2Y) = 129 + 3 \times 37 - 228 = \boxed{12}$.

Q.2 The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{e^{-\frac{x}{y}} e^{-y}}{y}, & x > 0,\ y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Compute conditional density function of X given $Y = y$, i.e., $f_{X|Y}(x|y)$ for all $y \in \mathbb{R}$ and hence find $P(X > 1|Y = y)$ for all $y \in \mathbb{R}$. **[5 marks]**

Solution: If $y \leq 0$, then joint pdf is 0, therefore $f_Y(y) = 0$. For $y > 0$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^{\infty} \frac{e^{-\frac{x}{y}} e^{-y}}{y} dx = \frac{e^{-y}}{y} \int_0^{\infty} e^{-\frac{x}{y}} dx = \frac{e^{-y}}{y} \left[\frac{1}{-\frac{1}{y}} e^{-\frac{x}{y}} \right]_{x=0}^{x=\infty} = -e^{-y}[e^{-\infty} - e^0] = e^{-y}$$

Therefore

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{e^{-\frac{x}{y}}}{y}, & x > 0,\ y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now by total probability theorem, for $y > 0$ we have

$$P(X > 1|Y = y) = \int_1^{\infty} f_{X|Y}(x|y)dx = \int_1^{\infty} \frac{e^{-\frac{x}{y}}}{y} dx = \frac{1}{y} \left[\frac{1}{-\frac{1}{y}} e^{-\frac{x}{y}} \right]_{x=1}^{x=\infty} = -[e^{-\infty} - e^{-\frac{1}{y}}] = e^{-\frac{1}{y}}$$

If $y \leq 0$ then $P(X > 1|Y = y) = 0$.