The LNM Institute of Information Technology
Jaipur, Rajasthan

Mid Semester Examination — Mathematics-II, 24^{th} Feb 2015

Time: 90 Minutes, Maximum Marks: 50

Note: Attempt all the questions. Make an index showing the question number attempted on the page number in top of the answer sheet in the following format, otherwise you may be penalized by **2 marks**. Read the questions carefully. No query will be entertained after **15** minutes of Examination.

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1. (a) Let A and B be two $n \times n$ invertible matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$. **[3]**
Sol. $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$. So $(B^{-1}A^{-1})$ is left inverse of AB.
Hence $(AB)^{-1} = B^{-1}A^{-1}$. (Note that for a square matrix with either a left or right inverse is invertible)

(b) Let A be an $n \times n$ nilpotent matrix of order 5 (i.e. $A^5 = 0$, and $A^m \neq 0$ for $m \leq 4$). Find the inverse of $I + A$. **[4]**
Sol. Since $A^m \neq 0$ for $m \leq 4$, A^m is invertible and the matrix $B = I - A + A^2 - A^3 + A^4$ is invertible.
Now verify that B is inverse (left or right) of $I + A$.

2. (*a*) Consider the set $V = \{(x_1, x_2, x_3, x_4)\colon x_1 + x_2 + x_3 + x_4 = 0, x_i \in \mathbb{R},\ i = 1, 2, 3, 4\}$. Verify that V is a subspace of $\mathbb{R}^4$. Find a basis and determine the dimension for V. **[6]**
Sol. Let $x, y \in V$. Then $x = (x_1, x_2, x_3, x_4)$ such that $x_1 + x_2 + x_3 + x_4 = 0$ and $y = (y_1, y_2, y_3, y_4)$ such that $y_1 + y_2 + y_3 + y_4 = 0$.
For $\alpha \in \mathbb{R}$, $\alpha x + y = \alpha(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (\alpha x_1 + y_1, \alpha x_2 + y_2, \alpha x_3 + y_3, \alpha x_4 + y_4)$. Now since $x_1 + x_2 + x_3 + x_4 = 0$, $y_1 + y_2 + y_3 + y_4 = 0$, we have $\alpha x_1 + y_1 + \alpha x_2 + y_2 + \alpha x_3 + y_3 + \alpha x_4 + y_4 = 0$ and hence $\alpha x + y \in V$. Therefore, V is a subspace of $\mathbb{R}^4$.
$V = \{(x_1, x_2, x_3, x_4)\colon x_1 + x_2 + x_3 + x_4 = 0, x_i \in \mathbb{R},\ i = 1, 2, 3, 4\} = \{(x_1, x_2, x_3, x_4)\colon x_4 = -x_1 - x_2 - x_3, x_i \in \mathbb{R},\ i = 1, 2, 3\}$. By taking $x_1 = s, x_2 = t, x_3 = u \in \mathbb{R}$, we have
$$\begin{bmatrix} s \\ t \\ u \\ -s-t-u \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$
Since $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}$ is LI, it forms a basis for V and hence dimension of V is 3.

(*b*) Consider the following system of linear equations:
$$\begin{aligned} x + 2y + z &= 3 \\ ay + 5z &= 10 \\ 2x + 7y + az &= b. \end{aligned}$$
Find the values of a and b for which the system has unique solution, more than one solutions and no solution. **[7]**

Sol. By elementary row operations we get a upper triangular form of the augmented matrix as $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 0 & a^2-2a-15 & ab-6a-30 \end{bmatrix}$

The system has uniqe solution if rank of coefficient matrix= rank of augmented matrix $= 3$ (No. of unknowns). This is the case when $a^2 - 2a - 15 \neq 0$ or, $(a-5)(a+3) \neq 0$ i.e. $a \neq 5$ and $a \neq -3$.

The system has more than one solution if rank of coefficient matrix= rank of augmented matrix < 3. This is possible when $a^2 - 2a - 15 = ab - 6a - 30 = 0$. Solving this we get the pairs $(a, b) = (5, 12)$ and $(-3, -4)$ for which the system has more than one solutions.

The system has no solution if rank of coefficient matrix $<$ rank of augmented matrix. This is possible if $a^2 - 2a - 15 = 0$ and $ab - 6a - 30 \neq 0$ i.e. if $a = 5$ and $b \neq 12$, or $a = -3$ and $b \neq -4$.

3. (*a*) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation given by $T(x, y, z) = (x + y - z, x + z)$. Let $B_1 = \{(1, 0, 2), (0, 2, 1), (3, 0, 0)\}$ and $B_2 = \{(1, 1), (2, 1)\}$ be ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Then find the matrix of T with respect to the given ordered bases. **[5]**

Sol.

$$\begin{aligned} T(1,0,2) &= (-1,3) = \alpha(1,1) + \beta(2,1) = 7(1,1) + (-3)(2,1) \\ T(0,2,1) &= (1,1) = \alpha(1,1) + \beta(2,1) = 1(1,1) + 0(2,1) \\ T(3,0,0) &= (3,3) = \alpha(1,1) + \beta(2,1) = 3(1,1) + 0(2,1) \end{aligned}$$

Therefore, the matrix associated with T with respect to the given ordered bases is $\begin{bmatrix} 7 & 1 & 3 \\ -3 & 0 & 0 \end{bmatrix}$.

(*b*) Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V. Denote $N(T)$= null space of T. Then show that T is one one if and only if $N(T) = \{\mathbf{0}\}$. **[4]**

Sol. Assume T is one one. Let $v \in N(T)$. Then $T(v) = 0$. For any linear trasformation $T(0) = 0$. Thus $T(v) = T(0)$. T is one one implies $v = 0$ i.e. $N(T) = \{\mathbf{0}\}$.

Conversely, assume $N(T) = \{\mathbf{0}\}$. Let $u, v \in V$ such that $T(u) = T(v)$. By linearity $T(u - v) = \mathbf{0}$ and that implies $u - v \in N(T) = \{\mathbf{0}\}$ and hence $u = v$. Therefore, T is one one.

4. (*a*) For given linearly independent set $\{(1, 0, 1), (1, 0, 0), (2, 1, 0)\}$ find an orthonormal basis of $\mathbb{R}^3$ using Gram-Schmidt process. **[5]**

Sol. Let $u_1 = (1, 0, 1), u_2 = (1, 0, 0), u_3 = (2, 1, 0)$. Applying Gram-Schmidt process, we get $v_1 = u_1 = (1, 0, 1)$

$v_2 = u_2 - \frac{<u_2,v_1>}{\|v_1\|^2} v_1 = \frac{1}{2}(1, 0, -1)$.

$v_3 = u_3 - \frac{<u_3,v_1>}{\|v_1\|^2} v_1 - \frac{<u_3,v_2>}{\|v_2\|^2} v_2 = (0, 1, 0)$.

Normalizing we get an orthonormal basis of $\mathbb{R}^3$ as

$\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\} = \{\frac{1}{\sqrt{2}}(1, 0, 1), \frac{1}{\sqrt{2}}(1, 0, -1), (0, 1, 0)\}$.

(*b*) Verify whether the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ is diagonalizable ? If diagonalizable then, find a matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. **[6]**

Sol. The characteristic equation is $-(\lambda - 1)(\lambda + 2)^2 = 0$ and the eigen values of A are 1,-2,-2.

Corresponding to the eigen value $\lambda = 1$ the solution space of the homogeneous sytem $(A - I)x = 0$ is

$\{(x, y, z) \colon x = z, y = -z, z \in \mathbb{R}\}$ (Note that z is free variable here). The nonzero elements are eigenvector corresponding to $\lambda = 1$. In particular if we take $z = 1$, then $(1, -1, 1)^t$ is an eigen vector.

Corresponding to the eigen value $\lambda = -2$ the solution space of the homogeneous sytem $(A + 2I)x = 0$ is $\{(x, y, z) \colon x = -y - z,\ y, z \in \mathbb{R}\}$ (Note that y, z are free variables here). The nonzero elements are eigenvector corresponding to $\lambda = -2$. In particular, take $y = 0, z = 1$, then $(-1, 0, 1)^t$ is an eigen vector. $y = 1, z = 0$ implies $(-1, 1, 0)^t$ is an another eigenvector. Since, $\{(-1, 0, 1)^t, (-1, 1, 0)^t\}$ is LI, it forms a basis for eigen space corresponding to the eigen value $\lambda = -2$.

Now, take three LI eigenvectors as a column vectcor of Q i.e. $Q = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ with $Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

5. (*a*) Verify the exactness of the differential equation $(\sin x \sin y + 1)y' = (\cos x \cos y - \cot x)$. Find the general solution if it is exact. **[5]**

Sol. Here $M(x, y) = \cos x \cos y - \cot x$ and $N(x, y) = -(\sin x \sin y + 1)$.
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\cos x \sin y$, the equation is exact.
Now, $u = \int M(x, y)dx + g(y) = \sin x \cos y - \log \sin x + g(y)$.
Since, $\frac{\partial u}{\partial y} = N$, we get $g' = -1$ and hence $g = -y + c_1$. Hence the solution is $\sin x \cos y - \log \sin x - y = c$.

(*b*) Assuming that the differential equation $M(x, y)dx + N(x, y)dy = 0$ has an integrating factor which is a function of $x + y^2$, find the relation to be satisfied by M and N. Hence, find an integrating factor of $(3y^2 - x) + 2y(y^2 - 3x)y' = 0$ which is a function of $(x + y^2)$. . **[5]**

Sol. Assume $F(x + y^2)$ is an integrating factor of the given differential equation. Hence, $F(x + y^2)M(x, y)dx + F(x + y^2)N(x, y)dy = 0$ is exact.
Using condition of exactness, we get $\dfrac{F'}{F} = \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N - 2yM}$.
Now for $(3y^2 - x) + 2y(y^2 - 3x)y' = 0$, $M = 3y^2 - x$ and $N = 2y(y^2 - 3x)$. From the above relation between M and N, we get $(x + y^2)F' = -3F$. Solving for F, we find the integrating factor $F = \dfrac{1}{(x + y^2)^3}$.

End of paper