The LNM Institute of Information Technology
Jaipur, Rajasthan

End Semester Exam

MATH-II, 28^{th} April 2015
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Time: 3 Hours, Maximum Marks: 60

Note: Attempt all the questions. Usual notations are used. Make an index showing the question number attempted on the page number in top of the answer sheet in the following format, otherwise you may be penalized by **2 marks**. Read the questions carefully.

Question No.				
Page No.				

Usual notations are used (e.g. $I \subseteq \mathbb{R}$ is an interval, $W(y_1, y_2)$=Wronskian of two functions y_1, y_2,
$\delta_{mn} = \begin{cases} 0, & \text{if; } m \neq n \\ 1, & \text{if } m = n. \end{cases}$)

1. (*a*) If a $n \times n$ real matrix A satisfies the relation $AA^T = 0$ then show that $A = 0$. Is the same true if A is a complex matrix? Justify your answer. **[4]**

   Sol. Since $AA^T = 0$, trace $(AA^T) = 0$ which implies $\sum_{i,j} a_{ij}^2 = 0$.
   Therefore, $a_{ij} = 0$ for all $i, j = 1, 2, \cdots n$.
   This is not true for complex matrix for example for $A = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$, $AA^T = 0$ but A itself is not zero matrix.

   (*b*) Prove or disprove that there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which is onto but not one-one.
   . **[4]**

   Sol. Assume that there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which is onto. Therefore, range$(T) = 3$.
   By rank nullity theorem nullity of T=0.
   Hence, T has to be one-one. So, there is no linear transformation T exists which is onto but not one-one.

   (*c*) Prove or disprove that a matrix A is invertible if 0 is not an eigen value of A. **[4]**

   Sol. Assume A is not invertible.
   Then, $Ax = 0$ has non-trivial solution $x \neq 0$.
   So, $Ax = 0 = 0x$ and hence 0 is an eigen value of A which is a contradiction.
   Therefore, A is not invertible.

2. (*a*) Let A be an $n \times n$ real matrix. Then show that $Ax = 0$ has a non-trivial solution if and only if rank$(A) < n$. **[6]**

   Sol. Assume $Ax = 0$ has non-trivial solution $x_0 \neq 0$. On contrary, assume r=rank$(A) = n$.
   So,n =rank(A)=rank([A 0]) $= r_a$. Since, $r_a = r = n$, $Ax = 0$ has precisely one solution.
   $A0 = 0$ implies 0 is the only solution of $Ax = 0$ which contradicts the fact that $Ax = 0$ has non-trivial solution.
   Coversely, assume that r=rank$(A) < n$. So r_a =rank([A 0])=rank$(A) < n$.
   Therefore, $Ax = 0$ has infinite solutions and we can choose one non-trivial solution among them.

   (*b*) Let V be an inner product space. Then show that $| < u, v > | \leq \|u\| \|v\|$ for all $u, v \in V$. Under what condition on the vectors u and v the equality holds ? Justify your answer. **[6]**

   Sol. If $u = 0$, then the result is obviously true. Assume $u \neq 0$. Consider the real valued function $f(t) =< v + ut, v + ut > \ \geq 0 \ \forall t$.
   $f(t) =< v + ut, v + ut >=< v, v > +2t < u, v > +t^2 < u, u >$ (Assuming Field is $\mathbb{R}$). Thus $f(t)$ is a polynomial in t with real coefficients.
   $f'(t) = 2 < u, v > +2t < u, u >$. t_0 is an extremum point of f only if $f'(t_0) = 0$ i.e. only if $< u, v > +t_0 < u, u >= 0$. This suggest that we choose $t_0 = -\dfrac{< u, v >}{< u, u >}$.
   By double derivative test $f(t_0)$ is a minimum i.e. $0 \leq f(t_0) \leq f(t) \forall t$. Since $f(t) \geq 0 \ \forall t$, $f(t_0) \geq 0$ and $< v, v > -2\dfrac{< u, v >}{< u, u >} < u, v > + \dfrac{| < u, v > |^2}{< u, u >} \geq 0$.
   It follows that $< v, v > \ \geq \dfrac{| < u, v > |^2}{< u, u >}$ or, $| < u, v > | \leq \|u\| \|v\|$.

If $u \neq 0$, then the equality hold iff $tu+v=0$ for $t=-\dfrac{<u,v>}{<u,u>}$ i.e. $v=\dfrac{<u,v>}{<u,u>}u$ that is u and v are linearly dependent.

3. (a) Show that $\displaystyle\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn}$. **[6]**

Ans: (ii) Let $f(x)$ be any function with at least n continuous derivatives in $[-1,1]$. Consider the integral $I=\int_{-1}^{1} f(x)P_n(x)\,dx = (1/2^n n!)\int_{-1}^{1} f(x)(d^n/dx^n)(x^2-1)^n\,dx$.
Repetition of integration by parts gives $I=(-1)^n/(2^n n!)\int_{-1}^{1} f^n(x)(x^2-1)^n\,dx$.
If $m \neq n$, without any loss of generality we take $f=P_m$, $m<n$ and then $f^n(x)=0$ and $I=0$.
If $f(x)=P_n(x)$, then $f^n(x)=(1/2^n n!)(d^{2n}/dx^{2n})(x^2-1)^n=(2n!)/(2^n n!)$.
Thus,

$$I=(2n!)/(2^{2n}(n!)^2)\int_{-1}^{1}(1-x^2)^n\,dx = 2(2n!)/(2^{2n}(n!)^2)\int_{0}^{1}(1-x^2)^n\,dx.$$

Substitute $x=\sin\theta$ to get

$$I=\frac{2(2n!)}{2^{2n}(n!)^2}\int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta = \frac{2(2n!)}{2^{2n}(n!)^2}I_n.$$

Since

$$\int\cos^{2n+1}d\theta = \frac{1}{2n+1}\cos^{2n}\theta\sin\theta + \frac{2n}{2n+1}\int\cos^{2n-1}\theta\,d\theta,$$

we find

$$I_n=\frac{2n}{2n+1}I_{n-1}=\frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}I_0.$$

Now

$$I_0=\int_0^{\pi/2}\cos\theta\,d\theta=1.$$

Hence,

$$I_n=\frac{2^n n!}{1\cdot3\cdot5\cdots(2n-1)(2n+1)}=\frac{2^{2n}(n!)^2}{(2n!)(2n+1)}$$

Thus, we get the required result.

(b) Find the trigonometric Fourier series of $f(x)=\begin{cases}-k, & \text{if } -\pi<x<0\\ k, & \text{if } 0<x<\pi\end{cases}$ and $f(x+2\pi)=f(x)$. **[5]**

Ans: $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=0$

$$\begin{aligned}a_n =&\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)cosnxdx=\frac{1}{\pi}[\int_{-\pi}^{0}(-k)cosnxdx+\int_0^{\pi}kcosnxdx\\ =&\frac{1}{\pi}[-k\frac{\sin nx}{n}]_{-\pi}^{0}+k\frac{\sin nx}{n}]_0^{\pi}=0\end{aligned}$$

$$\begin{aligned}b_n =&\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)cosnxdx=\frac{1}{\pi}[\int_{-\pi}^{0}(-k)cosnxdx+\int_0^{\pi}kcosnxdx\\ =&\frac{1}{\pi}[k\frac{\cos nx}{n}]_{-\pi}^{0}-k\frac{\cos nx}{n}]_0^{\pi}\end{aligned}$$

So $b_n=\dfrac{k}{n\pi}[\cos 0-\cos(-n\pi)-\cos n\pi+\cos 0]=\dfrac{2k}{n\pi}(1-\cos n\pi)$.

Now $1-\cos n\pi=\begin{cases}2, & \text{for ; } nodd\\ 0, & \text{for } n \text{ even.}\end{cases}$
Hence the Fourier coefficients b_n of our function are

$$b_1=\frac{4k}{\pi}, b_2=0, b_3=\frac{4k}{3\pi}, b_4=0, b_5=\frac{4k}{5\pi},...$$

Since the a_n are zero, the Fourier series of f is

$$\frac{4k}{\pi}(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...)$$

(*c*) Find the eigenvalues and eigenfunctions of the following Strum-Liouville problem:
$\left(e^{2x}y'\right)' + (\lambda+1)e^{2x}y = 0, \quad y(0) = y(\pi) = 0.$ [Substitute $y = e^{-x}u$] **[6]**

Ans: (ii) Using the above transformations, we get $u'' + \lambda u = 0, \quad u(0) = u(\pi) = 0.$
For $\lambda \leq 0$ trivial solution.
Thus, $\lambda = p^2 > 0$ and $u = c_1 \cos px + c_2 \sin px$.
The transformed BCs are $u(0) = u(\pi) = 0$ and thus $c_1 = 0$ and $p = n,\ n = 1, 2, 3, \cdots$.
Thus, $\lambda_n = n^2$ and since $y_n = e^{-x}u_n$ and so $y_n = e^{-x}\sin nx$.

4. (*a*) Two solutions y_1, y_2 of $y'' + p(x)y' + q(x)y = 0,\ x \in I$, where $p(x)$, $q(x)$ are continuous functions on I are Linearly Dependent if and only if $W(y_1, y_2) = 0$ at certain point $x_0 \in I$. Show that $y = x$ and $y = e^x$ are not solutions of the linear homogeneous equation on I. **[7]**

Proof: Let y_1, y_2 be LD. Thus, there exists a, b not both zero such that

$$ay_1(x) + by_2(x) = 0 - -(*)$$

We can differentiate (*) once and obtain

$$ay_1'(x) + by_2'(x) = 0 - -(**)$$

Now (*) and (**) can be viewed as linear homogeneous equations in two unknowns a and b. Since the solution is nontrivial, the determinant must be zero. Thus $W(y_1, y_2) = 0,\ \forall x \in \mathcal{I}$. Hence, $W(y_1, y_2)$ must be zero at $x_0 \in \mathcal{I}$.
Conversely, suppose $W(y_1, y_2) = 0$ at $x_0 \in \mathcal{I}$. Now consider

$$ay_1(x_0) + by_2(x_0) = 0 - -(***)$$

and

$$ay_1'(x_0) + by_2'(x_0) = 0 - - - (@*)$$

Now the determinant of the system of linear equations (in unknowns a, b) of (***) and (@$*$) is the Wronskian $W(y_1, y_2)$ at x_0. Since, this is zero, we can find nontrivial solution for a and b. Take these nontrivial a and b and form

$$y(x) = ay_1(x) + by_2(x)$$

By (***) and (4$*$), we find $y(x_0) = y'(x_0) = 0$. Hence, by uniqueness theorem $y(x) \equiv 0$, i.e. for nontrivial a and b

$$ay_1(x) + by_2(x) = 0, \qquad x \in \mathcal{I}$$

Hence y_1, y_2 are LD.

Now $y_1 = x$ and $y_2 = e^x$ are LI. So they must be fundamental solutions of the linear homogeneous ODE. Thus $W(y_1, y_2)$ must never be zero. But $W(y_1, y_2) = 0$ at $x = 1$, a contradiction.

(*b*) By using Frobenius method find two Linearly Independent solutions of
$x^2y'' + xy' + (x^2 - 1/9)y = 0.$——(*)

Ans: (*We find the L.I solutions for* $x > 0$. For $x < 0$, we substitute $t = -x$ and carry out similar procedure for $t > 0$. Give full mark if they don't write this also)

Since $x = 0$ is a regular singular point. A Frobenius series solution exists for the larger root say r_1. The solution of (*) can be represented by an extended power series

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, a_0 \neq 0. - - - -(**)$$

Now from (**), we find

$$x^2y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}, \quad xy'(x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}.$$

Substituting into (*), we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}\right) + (x^2 - 1/9)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0.$$

After some manipulation

$$\sum_{n=0}^{\infty} \rho(n+r)a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0, \quad \rho(r) = r^2 - 1/9.$$

$\Big($**Alternative:** since $p(0) = 1, q(0) = -1/9$, the indicial equation is given by

$$r^2 - 1/9 = 0.$$

Hence, $r_1 = 1/3, r_2 = -1/3$ and $r_1 - r_2 = 2/3$. Since $x = 0$ is a regular singular point. A Frobenius series solution exists for the larger root $r_1 = 1/3$ and can be represented by an extended power series

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

$a_0 \neq 0$. After some manipulation we get

$$\sum_{n=0}^{\infty} \rho(n+r)a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

and rest follows like further, $\Big)$

This equation is rearranged as

$$\rho(r)a_0 + \rho(r+1)a_1 x + \sum_{n=2}^{\infty} \Big(\rho(n+r)a_n + a_{n-2}\Big)x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \ \rho(r+1)a_1 = 0, \ \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.$$

From the relation $\rho(r) = 0$, we get $r_1 = 1/3, r_2 = -1/3$.

Now with the larger root $r_1 = 1/3$ we find

$$a_1 = 0, \quad a_n = -\frac{a_{n-2}}{n(n+2/3)}, \quad n \geq 2.$$

Iterating we find (by induction),

$$a_{2n+1} = 0, \quad a_{2n} = (-1)^n \frac{1}{2^{2n} n!(1/3+1)(1/3+2)\cdots(1/3+n)} a_0, \quad n \geq 1.$$

Hence

$$y_1(x) = a_0 x^{1/3} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1/3+1)(1/3+2)\cdots(1/3+n)}\right).$$

Choose

$$a_0 = \frac{1}{2^{1/3}\Gamma(1/3+1)}.$$

First solution :

$$J_{1/3}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1/3+1)} \left(\frac{x}{2}\right)^{2n+1/3}.$$

Second independent solution:

we note $r_1 - r_2 = 2/3$ is not a nonnegative integer, i.e. $r_1 - r_2 \notin \{0, 1, 2, 3....\}$ We know that a second Frobenius series solution for $r_2 = -1/3$ exist.

By similar calculation the resulting series is given by (4) with $1/3$ replaced by $-1/3$. Hence, the second solution is given by

$$J_{-1/3}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-1/3+1)} \left(\frac{x}{2}\right)^{2n-1/3}.$$

(c) Apply Laplace transform to solve the initial value problem $y'' + 2y' + y = e^{-t},\ y(0) = -1, y'(0) = 1.$

Ans: By applying Laplace transform both sides of the equation and using initial conditions we get

$$L(y'') + L(y') + L(y) = L(e^{-t}),$$

Denote $Y = L(y),$ and we have $\quad (L(y'') = s^2L(y) - sy(0) - y'(0), L(y') = sL(y) - y(0)),$

$$(s^2Y + s - 1) + 2(sY + 1) + Y = \frac{1}{s+1}$$

$$(s^2 + 2s + 1)Y = (s+1)^2Y = -s - 1 + \frac{1}{s+1}$$

$$Y = \frac{-s-1}{(s+1)^2} + \frac{1}{(s+1)^3} = \frac{-1}{(s+1)} + \frac{1}{(s+1)^3}$$

$$y(t) = \left(\frac{1}{2}t^2 - 1\right)e^{-t}.$$

End of paper