The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-II ■ Solutions Mid Sem-I

Q1. (i) Curves of electric force are the orthogonal trajectories to the equipotential lines. Differentiating $x^2 + y^2 = c$, we get $y' = \frac{-x}{y}$. [02] marks
Replace y' by $-\frac{1}{y'}$ to get curves of electric force as $y = cx$. [02] marks
(ii) Given that $\frac{1}{Ny-Mx}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y}\right) = g(z) = g(xy)$.
Let μ be an integrating factor of $M(x,y)dx + N(x,y)dy = 0$. Then the equation $\mu M(x,y)dx + \mu N(x,y)dy = 0$ is exact. Using the condition of exactness we get

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial y}.$$ [02] marks

If μ is a function of $z = xy$ the this reduces to

$$\frac{\mu'}{\mu} = \frac{1}{Ny - Mx}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y}\right) = g(xy) = g(z).$$

Solving this we get $\mu = e^{\int g(z)}$. [03] marks

Q2. (i) Since $F(x,y)$ is given to be continuously differentiable therefore F and F' are continuous and so $\frac{\partial F}{\partial y}$ exists and is continuous. [02] marks
Now since $\frac{\partial F}{\partial y}$ is continuous therefore we can always for a rectangle containing a neighbourhood of the origin in which $\frac{\partial F}{\partial y}$ is bounded and so by uniqueness theorem the IVP has atmost one solution. [03] marks
(ii) Euler's method: $y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, \ldots$ [01] mark
Here $f(x,y) = x + y, x_0 = 0, y_0 = y(0) = 1$. Therefore we get

$$\begin{aligned} y_1 &= 1 + h \\ y_2 &= 1 + 2h + 2h^2 \\ y_3 &= 1 + 3h + 6h^2 + 2h^3 \quad \text{etc.} \end{aligned}$$ [01] mark

Now if $h = 10^{-10}$ then $y_1 = 1 + 10^{-10} = 1.0000000001 = 1$ on a single precision computer. This will in turn imply $y_n = 1(constant)$ for all $n \geq 1$ on an ordinary precision computer. But, the solution to the problem is not constant. [02] marks

Q3. (i) On the contrary, suppose there is no zero of y_2 between two consecutive zeros of y_1 at $x = a, b$. Without loss of generality, suppose $y_2(x) > 0$ for $x \in [a,b]$ and $y_1'(a) > 0, y_1'(b) < 0$. Then $W(y_1, y_2) < 0$ at $x = a$ and $W(y_1, y_2) > 0$ at $x = b$. This implies $W(y_1, y_2) = 0$ at some point between $x = a$ and $x = b$. This contradicts that y_1, y_2 are fundamental solutions. [03] marks

To show the uniqueness, let there exist two zeros between $x = a$ and $x = b$. Using same argument (reversing the role of y_1, y_2), we conclude that y_1 has a zero between zeros of y_2 and hence in (a, b), which is a contradiction. [02] marks

To prove the remaining part consider the Wronskian $W(y_1, y_2) = ad - bc$, where $y_1 = a\sin x + b\cos x, y_2 = c\sin x + d\cos x$. Then y_1 and y_2 are linearly independent provided $ad \neq bc$. The result now follows from the above proof. [03] marks

(ii) Let $u_1(x)$ and $u_2(x)$ be continuously differentiable functions (to be determined) such that

$$y_p = u_1 y_1 + u_1 y_2, x \in I \tag{1}$$

is a particular solution of $y'' + p(x)y' + q(x)y = r(x)$. Differentiating (1) we obtain

$$y_p = u_1 y_1' + u_1 y_2' + u_1' y_1 + u_2' y_2. \tag{2}$$

We choose u_1 and u_2 so that

$$u_1' y_1 + u_2' y_2 = 0. \tag{3}$$

Substituting (3) in (2), we have

$$y_p = u_1 y_1' + u_1 y_2'. \tag{4}$$

Now

$$y_p'' = u_1 y_1'' + u_1 y_2'' + u_1' y_1' + u_2' y_2'. \tag{5}$$

Since y_p is a particular solution of $y'' + p(x)y' + q(x)y = r(x)$, substitution of (1),(4) and (5) in $y'' + p(x)y' + q(x)y = r(x)$, gives us

$$u_1(y_1'' + p(x)y_1' + q(x)y_1) + u_2(y_2'' + p(x)y_2' + q(x)y_2) + u_1'y_1' + u_2'y_2' = r(x). \tag{6}$$

As y_1 and y_2 are solutions of the homogeneous equation (A), we obtain the condition

$$u_1' y_1' + u_2' y_2' = r(x). \tag{7}$$

Solving (3) and (7) for u_1' and u_2', we get

$$u_1' = -\frac{y_2 r(x)}{W(y_1, y_2)} \quad u_2' = \frac{y_1 r(x)}{W(y_1, y_2)}$$

where the Wronskian, $W \neq 0$ for any $x \in I$. Integrating this we get

$$u_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx \quad u_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx$$

and hence

$$y_p = -y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$ [06] marks

Q4. (i) Given that $x = e^t$. Therefore, we have $\frac{dt}{dx} = \frac{1}{e^t} = \frac{1}{x}$. Now

$$xd(y) = x\frac{dy}{dx} = x\frac{dy}{dt}\frac{dt}{dx} = xD(y(t))\frac{1}{x} = Dy(t) \qquad \text{[02] marks}$$

(ii) For $n = 1$ the result follows from part (i). Let us assume that the result is true for $n = k$ i.e.

$$x^k d^k y = (D(D-1)\cdots(D-k+1))y(t) \tag{8}$$

Now consider

$$\begin{aligned}
x^{k+1}d^{k+1}y &= x^{k+1}\frac{d^{k+1}y}{dx^{k+1}} = x^{k+1}\frac{d}{dx}\left(\frac{d^k y}{dx^k}\right) = x^{k+1}\frac{d}{dx}\left(d^k y\right) && \text{[02] marks} \\
&= x^{k+1}\frac{d}{dx}\left(\frac{1}{x^k}D(D-1)\cdots(D-k+1)y(t)\right) \quad \text{(using(8))} && \text{[01] mark} \\
&= -k\Big\{D(D-1)\cdots(D-k+1)y(t)\Big\} + D\Big\{D(D-1)\cdots(D-k+1)y(t)\Big\} \\
&= (D-k)\Big\{D(D-1)\ldots(D-k+1)y(t)\Big\} \\
&= D(D-1)\ldots(D-k+1)(D-k)y(t)
\end{aligned}$$

Therefore the result is true for $n = k + 1$. Hence by mathematical induction the result is true for any $n \geq 1$. [03] marks