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The LNM Institute of Information Technology, Jaipur
Department of Mathematics
Mathematics-III MTH213
Mid Term

Duration: 90 mins. September 26, 2019 Max.Marks: 30

Name: ________________ Roll No.: ________________

1. (a) Prove that for any complex number $z \neq 1$, we have **[3]**

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z},$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)}, \qquad 0 < \theta < 2\pi.$$

Solution Consider $S = 1 + z + z^2 + \cdots + z^n$.
Then $zS = z + z^2 + \cdots + z^n + z^{n+1}$.
Then $S - zS = 1 - z^{n+1}$.
Hence $1 + z + z^2 + \cdots + z^n = \dfrac{1 - z^{n+1}}{1 - z}$, $z \neq 1$. Substitute $z = e^{i\theta}$ in the above equation and using $(e^{i\theta})^k = e^{ik\theta}$, we get
$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \dfrac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}, z \neq 1.$
$= \dfrac{e^{i\frac{n+1}{2}\theta}\left(e^{-i(\frac{n+1}{2})\theta} - e^{i(\frac{n+1}{2})\theta}\right)}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}\theta} - e^{i\frac{\theta}{2}}\right)}$
$= e^{i\frac{n\theta}{2}}\frac{\sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})}.$
Comparing the real parts, we get
$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{\cos(\frac{n\theta}{2})\sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})}.$
Hence $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)}, \qquad 0 < \theta < 2\pi..$

(b) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function such that $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that f is constant. **[3]**

Solution Consider $g(z) = \frac{1}{f(z)}$ $\forall z$.
Since $|f(z)| \geq 1$, $f(z) \neq 0$ for any z.
Since f is entire and $f(z)$ is never zero for any z, g is entire.
Also, $|f(z)| \geq 1$ implies $|g(z)| \leq 1$.
Hence g is entire and bounded.
By Liouville's theorem g is constant.
Hence f is constant.

2. (a) Show that the function $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $(z \neq 0)$, $f(0) = 0$ satisfy CR equations at $(0,0)$ but are not differentiable there. **[3]**

Solution: Here $u(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ and $v(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$
CR-equations are $u_x = v_y$ and $u_y = -v_x$ at $(0,0)$.
$u_x(0,0) = v_y(0,0) = 1$ and $u_y(0,0) = -v_x(0,0) = -1$.
Hence CR-equations are satisfied at $(0,0)$.
To check the differentiability, consider the limit
$\lim\limits_{\Delta z \to 0} \dfrac{f(0+\Delta z) - f(0)}{\Delta z}$
Along y-axis, $\Delta x = 0$, $\Delta y \to 0$, we have
$\lim\limits_{\Delta y \to 0} \dfrac{-\Delta y^3(1-i)}{\Delta y^3} = -1 + i.$
Along $\Delta x = \delta y$, , $\Delta x \to 0$, we have
$\lim\limits_{\Delta x \to 0} \dfrac{\Delta x^3(1+i)}{\Delta x^3} = 1 + i.$
So along different paths we are getting different limits. So $f'(0)$ does not exist.

(b) Suppose f is an analytic function on a domain D. Show that if $|f|^2$ is harmonic, then f is constant on D. **[3]**

Solution $|f|^2 = u^2 + v^2$. Suppose $|f|$ is harmonic, then $|f|_{xx} + |f|_{yy} = 0$.
$\Rightarrow 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) + 2v(v_{xx} + v_{yy}) + 2(v_x^2 + v_y^2) = 0$
Since f is analytic, u and v are harmonic. Hence terms in first and third bracket are equal to zero. Hence
$\Rightarrow u_x^2 + u_y^2 + v_x^2 + v_y^2 = 0$
$\Rightarrow u_x = u_y = v_x = v_y = 0$
Hence $f(z) = u_x + iv_x = 0$ on a domain D. Hence $f(z)$ is constant.

3. (a) Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$. Conclude that the cosine function is unbounded in $\mathbb{C}$. **[3]**

Solution $|\cos z|^2 = |\cos(x+iy)|^2$
$=|\cos x \cos(iy) - \sin x \sin(iy)|^2$
$=|\cos x \cosh y - i \sin x \sinh y|^2$
$= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$
$= \cos^2 x \cosh^2 y + (1-\cos^2 x)\sinh^2 y$
$= \cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y$
$= \cos^2 x + \sinh^2 y.$
Since $\cos^2 x$ is non-negative and $\sinh^2 y$ is unbounded, $\cos z$ is unbounded.

(b) Let $f(z) = \dfrac{1}{z^2}$. Show that $\int_C f(z)dz = 0$ where C is any closed contour not passing through 0. **[3]**

Solution Since the antiderivative of $fz) = \dfrac{1}{z^2}$ is $F(z) = -\dfrac{1}{z}$ in the domain, $|z|>0$. Since the curve C lies entirely in the domain $|z|>0$, so $\int_C f(z)dz = 0$.

4. (a) Let C_R denote the upper half circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Find an upperbound of

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right|.$$

Hence show that $\lim\limits_{R\to\infty} \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz = 0.$ **[3]**

Solution On C_R, $|f(z)| \leq \dfrac{2R^2+1}{(R^2-1)(R^2-4)}$. In the ML-inequality, we take $M = \frac{2R^2+1}{(R^2-1)(R^2-4)}$. Here $L = \pi R$. hence by ML inequality, we have $\left|\int_{C_R} \dfrac{2z^2-1}{z^4+5z^2+4}dz\right| \leq \dfrac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$. Since the R.H.S tends to zero as R tends to ∞, we have $\lim\limits_{R\to\infty} \int_{C_R} \dfrac{2z^2-1}{z^4+5z^2+4}dz = 0.$

(b) Find all the possible Laurent series expansion of $f(z) = \dfrac{2z}{z^2-9}$ about $z = 3$ and using that find the residue of $f(z)$ at $z = 3$. **[3]**

Solution $f(z) = \dfrac{1}{z-3} + \dfrac{1}{z+3}$
We will consider the domains $D_1 : 0 < |z-3| < 6$ and $D_2 : |z-3| > 6$.
On D_1,

$$f(z) = \frac{1}{z-3} + \frac{1}{z+3} = \frac{1}{z-3} + \frac{1}{6}\frac{1}{\frac{z-3}{6}+1} = \frac{1}{z-3} + \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-3}{6}\right)^n.$$

$Res_{z=3}f(z) = 1$
On D_2,

$$f(z) = \frac{1}{z-3} + \frac{1}{z+3} = \frac{1}{z-3} + \frac{1}{z-3}\frac{1}{\frac{6}{z-3}+1} = \frac{1}{z-3} + \frac{1}{z-3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{6}{z-3}\right)^n.$$

5. (a) Evaluate $\int_c z^2 e^{\frac{5}{z}}$ where C is any closed contour in counterclockwise direction with $z = 0$ inside it. **[2]**

Solution The function $f(z) = z^2 e^{\frac{5}{z}}$ has only one singularity and the Laurent series expansion of $f(z)$ is
$f(z) = z^2(1 + \frac{5}{z} + \frac{25}{z^2.2!} + \frac{125}{z^3.3!} + \cdots) = 1 + 5z + \frac{25}{2!} + \frac{125}{z.3!} + \cdots$.
So residue of $f(z)$ at 0 is $\frac{125}{6}$.
Hence $\int_c z^2 e^{\frac{5}{z}} = \frac{125}{3}\pi i$

(b) Using contour integral, evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$. **[4]**

Solution Here $p(x) = 1$ and $q(x) = (x^2 + 1)^2$ Consider $f(x) = \frac{1}{(z^2+1)}$.

The function $f(z)$ is not analytic at $z = \pm i$.
$z = i$ lies in the upper half plane. $z = i$ is a pole of order 2. Choose $R > 1$, let C_R denote the semicircle centered at 0 and radius R in the upper half plane. Let L be the line on the x-axis joining $-R$ to R. Let $C = C_R + L$ in the counterclockwise direction. Since the degree of $q(x)$ is 2 greater than the degree of $p(x)$, $\int_{C_R} f(z) = 0$.
Substitute the parametric equation $z = x$, $-R \le x \le R$ in the integral $\int_L f(z)dz = \int_{-R}^{R} f(x)dx$.
By residue theorem for any $R > 1$, we have $\int_C f(z)dz = 2\pi i Res_{z=i} f(z) = \frac{1}{2}\pi$.
Thus $\lim_{R\to\infty} \int_C f(z)dz = \lim_{R\to\infty} \int_{C_R} f(z)dz + \lim_{R\to\infty} \int_L f(z)dz$
$\Rightarrow \frac{1}{2}\pi = 0 + \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$.. Since the function is even, $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\pi$.