The LNM Institute of Information Technology, Jaipur

End Sem Examination 2011

Mathematics II

Date: 4^{th} May 2011 | Full Mark 100 | Duration: 3 hours

1. (i) Consider the initial value problem (4)

$$y' = \sqrt{|y|}, \qquad y(0) = 0.$$

Show that this problem doesn't have a unique solution. Discuss with reference to the existence and uniqueness theorem.

(ii) Solve the non-linear (Bernoulli) differential equation (5)

$$y' - Ay = -By^2,$$

where A, B are positive constants.

(iii) Show that the curves $y^2 = 4c(x + c)$ are self-orthogonal. (4)

2. (i) One solution of the linear ODE (8)

$$y'' + \left(-2 - \frac{2}{x}\right) y' + \frac{4}{x} y = 0$$

is $y_1(x) = e^{2x}$. Let $y_2(x)$ be another solution and $W(y_1, y_2)(x)$ be the Wronskian of y_1 and y_2. Then prove that

$$W'(x) = \left(2 + \frac{2}{x}\right) W(x).$$

Find a solution $W(x)$ of this differential equation? For this solution $W(x)$, find a particular solution of the first order linear ODE

$$y_1(x)v' - y_1'(x)v = W(x)$$

using the method of undetermined coefficients.

(ii) Let $p(x)$ and $q(x)$ be continuous on an interval I. Show that a pair of solutions of the equation

$$y'' + p(x)y' + q(x)y = 0$$

are linearly dependent if they have a maxima/minima at the same point $x_0 \in I$. (4)

3. (**Population Model**) The population of long tailed weasels and meadow voles has been studied by MIT biologists. They measure the populations relative to a baseline and established the following relationship between weasels and voles:

$$\begin{aligned} \frac{dx}{dt} &= 0.5x(t) + y(t) \\ \frac{dy}{dt} &= -2.25x(t) + 0.5y(t), \end{aligned}$$

where $x(t)$ and $y(t)$ respectively denote the population of weasels and voles at any time t. [Note: The natural growth rate for each species is 0.5, and cross terms show that more voles are good for weasels but more weasels are very harmful for voles!]
What are the eigen values of the matrix associated with this model? For one of the eigen values find the solution of $\frac{d\mathbf{X}}{dt} = \mathbf{AX}, \mathbf{X} = (x, y)^{\mathrm{T}}$. Then take the real and imaginary part to produce two real solutions. Suppose that at $t = 0$ there are 100 more weasels then the voles and the vole population is the baseline value (say, 0). What is the solution to the initial value problem? (7)

4. (i) Prove the following relation (7)

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}.$$

Here $P_n(x)$ denote the Legendre polynomial of degree n. (Hint: One may use here Rodrigues's formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$)

(ii) Consider the Strum-Liouville problem (5)

$$(p(x)y')' + [q(x) + \lambda r(x)]y = 0$$

with $p(x) > 0$ in $[a, b]$ and $y(a) \neq y(b), y'(a) \neq y'(b)$. Show that any two eigen functions corresponding to an eigenvalue are unique except for a constant factor.

(iii) Use the Laplace transform to analyse Bessel's equation (6)

$$xy'' + y' + xy = 0$$

with the single initial condition $y(0) = 1$.

5. (i) Find the Fourier series expansion of the function (4)

$$f(x) = x^2/2 \qquad -\pi < x < \pi.$$

Further, show that

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdot \cdot\cdot = \frac{\pi^2}{12}.$$

(ii) Find the Fourier integral representation of the function (5)

$$f(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

and hence show that

$$\int_0^\infty \frac{\sin w}{w} dw = \frac{\pi}{2}.$$

6. (i) Consider the quasi-linear PDE and the initial condition (5)

$$\begin{aligned} u_t + 2uu_x &= -3u, \qquad t > 0, -\infty < x < \infty \\ u(x, 0) &= b \sin x, \qquad -\infty < x < \infty \end{aligned}$$

where $b > 0$ is a constant. Show that the solution is given by

$$\sin\left(x - \frac{2}{3}u(e^{3t} - 1)\right) = \frac{1}{b}ue^{3t}.$$

(ii) Determine the region of the plane where the PDE (5)

$$xu_{xx} + 2xu_{xy} + (x - 1)u_{yy} = 0$$

is hyperbolic and determines its canonical form.

7. The acoustic pressure in an organ pipe obeys the 1-D wave equation (in physical variable) (10)

$$p_{tt} = c^2 p_{xx}$$

where c is the speed of sound in air. Each organ pipe is closed at one end and open at the other end. Given initial condition $p(x, 0) = f(x)$ and $p_t(x, 0) = g(x)$. Find the pressure for the organ pipe using separation of variables.
(Hint: At the closed end the boundary condition is $p_x(0, t) = 0$ and at the open end the boundary condition is $p(l, t) = 0$.)

8. Solve the heat problem with the boundary conditions (5)

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\ u(0, t) &= 0 = u(1, t), u(x, 0) = f_\alpha(x) \end{aligned}$$

for $t > 0, 0 \leq x \leq 1$ and

$$f_\alpha(x) = \begin{cases} 0, & \text{if } 0 < x < 1/2 - \alpha/2 \\ \frac{u_0}{\alpha}, & \text{if } 1/2 - \alpha/2 < x < 1/2 + \alpha/2 \\ 0, & \text{if } 1/2 + \alpha/2 < x < 1 \end{cases}$$

where u_0 is an arbitrary constant.

(Hint: You can directly use the formula discussed in the class.)

(a) Show that the temperature at the mid point of the rod when $t = \dfrac{1}{\pi^2}$ (dimensionless) is approximated by (3)

$$u\left(\frac{1}{2}, \frac{1}{\pi^2}\right) \simeq \frac{2u_0}{e}\left(\frac{\sin\frac{\pi\alpha}{2}}{\pi\alpha/2}\right).$$

(b) Can you distinguish between a pulse with width $\alpha = \frac{1}{1000}$ from one with $\alpha = \frac{1}{2000}$ by measuring $u\left(\frac{1}{2}, \frac{1}{\pi^2}\right)$. (2)

9. (i) Let $f(x, y)$ be a real valued, twice continuously differentiable function on a planner region U. Suppose that f and f^2 both are harmonic. Prove that $f(x, y)$ must be constant. (3)

(ii) Find the potential inside a sphere S of radius $R = 1$, when the potential on the surface of the sphere is $f(\varphi) = \cos\varphi$. What is the potential at the North pole, South pole and the equator? (8)

(Hint: Solve Laplace equation on a sphere using spherical co-ordinates.)

End of paper