THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
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Second Mid Semester Examination — MATHEMATICS-II, 11^{th} MARCH 2014

TIME: 60 MINUTES, MAXIMUM MARKS: 40

Note: Attempt all the questions. Make an index showing the question number attempted on the page number in top of the answer sheet in the following format, otherwise you may be penalized by **2 marks**. Read the questions carefully. No query will be entertained after **15** minutes of Examination.

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1. (a) Find a solution for the differential equation $(1+x^2)y'' - 4xy' + 6y = 0$ such that $y = a_0 y_1(x) + a_1 y_2(x)$, where y_1 and y_2 are power series. **[5]**

Solution: Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into $(1+x^2)y'' - 4xy' + 6y = 0$, we get $a_{n+2} = \frac{(n-2)(n-3)}{(n+1)(n+2)} a_n$; $n \geq 0$: Iterating we get $a_2 = -3a_0$; $a_3 = -\frac{1}{3}a_1$; $a_n = 0$; $n \geq 4$. Thus, $y = a_0(1-3x^2) + a_1(x - \frac{1}{3}x^3) = a_0 y_1(x) + a_1 y_2(x)$.

(b) Prove the Rodrigues's formula: $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}[(x^2-1)^n]$. **[5]**

Solution: Let $V(x) = (x^2-1)^n$. Then $V'(x) = 2nx(x^2-1)^{n-1}$ or $(x^2-1)V'(x) = 2nxV(x)$. Now differentiating $(n+1)$ times (by the use of the Leibniz rule for differentiation), we get $(x^2-1)V^{(n+2)}(x) + 2(n+1)xV^{(n+1)}(x) + \frac{2n(n+1)}{2}V^{(n)}(x) - 2nxV^{(n+1)}(x) - 2n(n+1)V^{(n)}(x) = 0$. By denoting, $U(x) = V^{(n)}(x)$, we have or $(1-x^2)U'' - 2xU' + n(n+1)U = 0$. This tells us that $U(x)$ is a solution of the Legendre Equation. Also, let us note that $\frac{d^n}{dx^n}(x^2-1)^n = \frac{d^n}{dx^n}(x-1)(x+1)^n = n!(x+1)^n +$ terms containing a factor of $(x-1)$. Therefore, $\frac{d^n}{dx^n}(x^2-1)^n \Big|_{x=1} = 2^n n!$.

So, $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)^n]$

2. (a) Reduce $x^2y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution. **[5]**

Solution: Here $p = \frac{1}{x}$; $q = 1 - \frac{1}{4x^2}$. So $v = e^{-\frac{1}{2}\int p\,dx} = \frac{1}{\sqrt{x}}$. Using the transformation $y = uv$, we get $u'' + u = 0$ and its general solution is $u = A\sin x + B\cos x$. For the original equation, the general solution is $y = \frac{1}{\sqrt{x}}(A\sin x + B\cos x)$.

(b) Prove that between each pair of consecutive positive zeros of $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa. **[5]**

Solution: Let a and b be two consecutive positive zeros of $J_{\nu+1}$. Let $f(x) = x^{\nu+1}J_{\nu+1}(x)$. Then $f(a) = f(b) = 0$. Thus there exists c between a and b such that $f'(c) = 0$. So $f'(c) = (c^{\nu+1}J_{\nu+1}(c))' = c^{\nu+1}J_\nu(c) = 0$. Thus $J_\nu(c) = 0$.

Similarly, let a and b be two consecutive positive zeros of J_ν. Let $f(x) = x^{-\nu} J_\nu(x)$. Then $f(a) = f(b) = 0$. Thus there exists c between a and b such that $f'(c) = 0$. So $f'(c) = (c^{-\nu} J_\nu(c))' = -c^{-\nu} J_{\nu+1}(c) = 0$. Thus $J_{\nu+1}(c) = 0$.

To prove uniqueness, let there exist two zeros c and d of $J_\nu(x)$ between consecutive zeros a and b of $J_{\nu+1}(x)$ This implies that there exist a zero of $J_{\nu+1}(x)$ between c and d. Consequently, there is a zero of $J_{\nu+1}(x)$ between a and b, Which contradicts the fact that a and b are consecutive zeros of $J_{\nu+1}(x)$.

3. (*a*) Find all the nontrivial solutions of the boundary value problem $y'' + \lambda y = 0$ for $x \in (0, \frac{\pi}{2})$ with $y(0) = 0, y'\left(\frac{\pi}{2}\right) = 0$. For any piecewise smooth function $f(x)$ defined on $[0, \frac{\pi}{2}]$, find c_m in terms of $f(x)$ such that

$$f(x) = \sum_{m=1}^{\infty} c_m y_m(x),$$

where $y_m(x)$ is a solution of given boundary value problem. **[6]**

Sol. Case I: $\lambda = 0$ gives $y(x) = Ax + B$. Boundary Condition implies $y \equiv 0$.
Case II: Let $\lambda < 0$ and $\lambda = -n^2$. The solution is $y(x) = Ae^{nx} + Be^{-nx}$. Boundary condition implies $y \equiv 0$.
Case III: Let $\lambda > 0$ and $\lambda = n^2$. The solution is $y(x) = A\cos nx + B \sin nx$.
$y(0) = 0 \Rightarrow A = 0$. $y'(\pi/2) \Rightarrow Bn\cos\frac{n\pi}{2} = 0$. $B = 0$ implies $y \equiv 0$, so assume $B \neq 0$.
Hence, $cos\frac{n\pi}{2} = 0$ and $\frac{n\pi}{2} = \frac{2m-1}{2}\pi, m = 0, \pm1, \pm2, \cdots$.
The eigen values are $\lambda_m = n^2 = (2m-1)^2, m = 1, 2, \cdots$.
By taking $B = 1$, the corresponding eigen functions are

$$y_m(x) = \sin(2m-1)x, m = 1, 2, 3, \cdots$$

For any piecewise smooth function $f(x)$ defined on $[0, \frac{\pi}{2}]$, we can represent $f(x)$ as $f(x) = \sum_{m=1}^{\infty} c_m y_m(x)$.
Multiply y_m both sides and integrate from 0 to $\pi/2$ w.r.t. x. Since eigenfunctions are orthogonal we get

$$c_m = \frac{\int_0^{\pi/2} f(x)\sin(2m-1)x dx}{\int_0^{\pi/2} \sin^2(2m-1)x dx} = \frac{4}{\pi}\int_0^{\pi/2} f(x)\sin(2m-1)x dx.$$

(*b*) Let $\{y_m\}_{m=1}^{\infty}$ be the eigen functions of the following Sturm-Liouville problem:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [\lambda q(x) + r(x)]y = 0, \;\; a < x < b$$

with boundary condition:

$$c_1 y(a) + c_2 y'(a) = 0 \qquad \text{and} \qquad d_1 y(b) + d_2 y'(b) = 0,$$

where c_1 or $c_2 \neq 0$ and d_1 or $d_2 \neq 0$, $p(x) > 0$ and $q(x) > 0$. Show that eigen functions $\{y_m\}_{m=1}^{\infty}$ are orthogonal with respect to the weight function $q(x)$. **[4]**

Sol. Assume y_m and y_n are two distinct eigen functions corresponding to two distinct eigen values λ_m and λ_n. This implies

$$[py_m']' + [\lambda_m q + r]y_m = 0, \tag{1}$$

and

$$[py_n']' + [\lambda_n q + r]y_n = 0. \tag{2}$$

Multiplying y_n into (1), y_m into (2) and subtracting we get

$$y_n[py_m']' - y_m[py_n']' + (\lambda_m - \lambda_n)qy_my_n = 0. \tag{3}$$

Integrating (3) from a to b w.r.t x, we obtain

$$(\lambda_m - \lambda_n)\int_a^b qy_my_ndx = p(b)W(b) - p(a)W(a).$$

Since y_m and y_n satisfies the boundary conditions, we get

$$c_1y_m(a) + c_2y_m'(a) = 0 \qquad \text{and} \qquad c_1y_n(a) + c_2y_n'(a) = 0.$$

c_1 and c_2 can't be zero simultaneously, so it has a nontrivial solution c_1 and c_2 which implies the determinant of coefficient matrix must be zero. Therefore, $W(a)$=0. A similar argument shows that $W(b) = 0$.

Since $\lambda_m \neq \lambda_n$, $\int_a^b qy_my_ndx = 0$ and hence eigen functions are orthogonal.

4. (*a*) Using Laplace transformation determine the hammerblow response of the following damped mass-spring system under the unit impulse at time $t = 1$: **[5]**

$$y'' + 2y' - 3y = \delta(t-1),$$
$$y(0) = 0, y'(0) = 2.$$

Sol. Let $L\{y(t)\} = Y(s)$.
$L\{y'(t)\} = sY - y(0) = sY$ and $L\{y''(t)\} = s^2Y - sy(0) - y'(0) = s^2Y - 2$
Taking Laplace transformation of the given DE, we get

$$s^2Y - 2 + 2sY - 3Y = e^{-s}.$$

Solving for Y, we get

$$Y = \frac{e^{-s}+2}{(s-1)(s+3)}.$$

Using partial fraction we obtain

$$Y = \frac{e^{-s}}{4(s-1)} - \frac{e^{-s}}{4(s+3)} + \frac{1}{2(s-1)} - \frac{1}{2(s+3)}.$$

Applying second shifting theorem we find the inverse Laplace transformation as

$$y(t) = \frac{1}{4}u(t-1)e^{(t-1)} - \frac{1}{4}u(t-1)e^{-3(t-1)} + \frac{1}{2}u(t)e^{t} - \frac{1}{2}u(t)e^{-3t}$$

which is equal to

$$f(t) = \begin{cases} \frac{1}{2}(e^t - e^{-3t}) & 0 < t \leq 1 \\ \frac{1}{2}(e^t - e^{-3t}) + \frac{1}{4}(e^{t-1} - e^{-3(t-1)}) & t > 1. \end{cases}$$

(*b*) Represent $f(t) = \frac{1}{2}te^{3t}, 0 < t < \frac{\pi}{2}$ in terms of unit step function and then find the Laplace transorm of the resulting function. **[5]**
Sol. Unit step representation of $f(t)$ is $\frac{1}{2}[u(t) - u(t-\pi/2)]te^{3t}$.

Using first and second shifting theorem, we get

$$L\{u(t)te^{3t}\} = \frac{1}{(s-3)^2}. \tag{4}$$

$$L\{u(t-\pi/2)te^{3t}\} = e^{-\frac{\pi}{2}s}L\{(t+\pi/2)e^{3(t+\pi/2)}\} \tag{5}$$

Since

$$L\{(t+\pi/2)e^{3(t+\pi/2)}\} = L\{e^{\frac{3\pi}{2}}te^{3t} + \frac{\pi}{2}e^{\frac{3\pi}{2}}e^{3t}\},$$

applying first shifting theorem, we find

$$L\{e^{\frac{3\pi}{2}}te^{3t} + \frac{\pi}{2}e^{\frac{3\pi}{2}}e^{3t}\} = \frac{e^{\frac{3\pi}{2}}}{(s-3)^2} + \frac{\pi}{2}\frac{e^{\frac{3\pi}{2}}}{(s-3)}$$

and hence

$$L\{u(t-\pi/2)te^{3t}\} = e^{-\frac{\pi s}{2}}\left(\frac{e^{\frac{3\pi}{2}}}{(s-3)^2} + \frac{\pi}{2}\frac{e^{\frac{3\pi}{2}}}{(s-3)}\right).$$

Therefore, from (4)-(5) the laplace transformation of given function is:

$$L\{f(t)\} = \frac{1}{2}\left[\frac{1}{(s-3)^2} - e^{-\frac{\pi s}{2}}\left(\frac{e^{\frac{3\pi}{2}}}{(s-3)^2} + \frac{\pi}{2}\frac{e^{\frac{3\pi}{2}}}{(s-3)}\right)\right].$$

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