The LNM Institute of Information Technology
Department of Mathematics
Mathematics-1 & MTH102
Quiz-II

Time: 30 minutes | Date: 01/11/2018 | Maximum Marks: 10

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1 Examine the existence of the limit of following function at $(0,0)$

$$f(x,y) = \begin{cases} \dfrac{2x}{x^2+x+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

[3]

Solution: Along x axis, that is $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2} = \lim_{x\to 0} \frac{2x}{x^2+x}$$

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2} = \lim_{x\to 0} \frac{2}{x+1} = 2.$$

[1.5]

Along y axis, that is $0=0$

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2} = \lim_{y\to 0} \frac{0}{y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2} = \lim_{y\to 0} 0 = 0.$$

Limit is not unique or path dependent so Limit does not exist. **[1.5]**

3 For the function $f(x,y) = \sqrt{x^2+y^2}$, find directional derivatives in all possible directions at point $(0,0)$ (if it exists). **[2]**

Solution: Let $u = (u_1, u_2)$ be unit vector.

$$D_u f(0,0) = \lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t}$$

$$D_u f(0,0) = \lim_{t\to 0} \frac{\sqrt{t^2(u_1^2+u_2^2)}}{t}$$

[1]

$$D_u f(0,0) = \lim_{t\to 0} \frac{|t|}{t}.$$

Since

$$u_1^2 + u_2^2 = 1$$

Limit does not exist. Hence directional derivative does not exist. **[1]**

2 Find iterated(repeated) limit, $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$ where $f(x,y)$ is defined as

$$f(x,y) = \begin{cases} \dfrac{x+y}{x-y}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

[2]

Solution: For $y \neq 0$, we compute $\lim_{x\to 0} f(x,y)$.

Since we are interested in computing limit as $x \to 0$, for given any $y \neq 0$, our x will be such that $|x| < |y|$. Hence

$$f(x,y) = \frac{x+y}{x-y}, \text{ for } |x| < |y|.$$

This means

$$\lim_{x\to 0} \frac{x+y}{x-y} = \frac{0+y}{0-y} = -1$$

[**1**] Since for each $y \neq 0$, $\lim_{x\to 0} f(x,y) = -1$, therefore $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = -1$ [**1**]

4 Find all the points of local maxima and local minima for the function $f(x,y) = x^3 + y^3 - 3xy + 15$. [**3**]

Solution: Since f is a polynomial function, hence the critical points of f are exactly those points where $\nabla f(x,y) = (0,0)$.

$$f_x = 3x^2 - 3y = 0 \implies y = x^2$$
$$f_y = 3y^2 - 3x = 0 \implies y^2 = x$$
$$\implies (x^2)^2 = x \implies x(x^3-1) = 0 \implies x = 0, 1$$

So $(0,0)$ and $(1,1)$ are the only critical points of f. [**1**]

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3$$
$$\Delta f(x,y) = 36xy - 9$$
$$\Delta f(0,0) = -9 < 0 \implies (0,0) \text{ is a saddle point.}$$

[**1**]

$$\Delta f(1,1) = 36 - 9 > 0, f_{xx}(1,1) = 6 > 0 \implies (1,1) \text{ is a point of local minimum.}$$

[**1**]