THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
MATHEMATICS-1 & MTH102
END TERM EXAM

Timing: 8:30 AM-11:30 AM   Date: 05/12/2019   Maximum Marks: 50

Note: You should attempt all questions. **There are total ten Questions.** Marks awarded are shown next to the question. There are no Marks just for writing the final answer and obtaining the answer without using proper method or giving proper justification. **You can not use any method outside of course content. Calculator is not allowed.** Please make an index showing the question number and page number on the front page of your answer sheet in the following format.

Question No.				
Page No.				

1. (a) Show that $\sqrt{p}$ is irrational number where p is a prime number. [3]

Solution: Assume $\sqrt{p}$ is rational then $\sqrt{p} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}, n \neq 0$. [0.5]
Without loss of generality we may assume that $(m, n) = 1$ and $n \in \mathbb{N}$. [0.5]

Then $pn^2 = m^2 \implies p|m^2 \implies p|m$, that is $m = kp$ for some $k \in \mathbb{Z}$. [1]
Hence $pn^2 = (pk)^2$. We get $n^2 = pk^2 \implies p|n^2 \implies p|n$. [0.5]
This is contradiction to $(m, n) = 1$. [0.5]

(b) Let $f(x) = x^2$ for all $x \in [a, b]$, where $a, b, \in \mathbb{R}, a < b$. Show that f is continuous on $[a, b]$ using $\epsilon - \delta$ definition. [3]

Solution: Let $c \in [a, b]$ and $\epsilon > 0$ be given. [0.5]
Define $R = \max\{|a|, |b|\}$. Choose δ such that $0 < \delta < \frac{\epsilon}{2R}$. [0.75]
Let $x \in [a, b]$ be such that $|x - c| < \delta$. Then $|x + c| \leq |x| + |c| < R + R$. [0.5]
Hence for all $x \in [a, b]$ with $|x - c| < \delta$ we have

$$\begin{aligned} |f(x) - f(c)| &= |x^2 - c^2| \\ &= |x + c||x - c| \\ &\leq (|x| + |c|)|x - c| \\ &< 2R|x - c| \\ &< 2R\delta \\ &< \epsilon. \end{aligned}$$

[0.5]
Thus, f is continuous at $c \in [a, b]$. and hence on $[a, b]$, since c was an arbitrary element of $[a, b]$. [0.25]

Aliter Soln: Let $c \in [a, b]$ and $\epsilon > 0$ be given. [0.5]
Choose δ such that $0 < \delta < \sqrt{c^2 + \epsilon} - |c|$. Then $(\delta + |c|)^2 < c^2 + \epsilon \implies \delta^2 + 2|c|\delta < \epsilon$ [0.75]

Let $x \in [a,b]$ be such that $|x-c| < \delta$. Then $|x+c| = |x-c+2c| \leq |x-c| + 2|c| < \delta + 2|c|$. [0.5]
Hence for all $x \in [a,b]$ with $|x-c| < \delta$ we have

$$\begin{aligned}|f(x) - f(c)| &= |x^2 - c^2| \\ &= |x+c||x-c| \\ &\leq (\delta + 2|c|)\delta \\ &= \delta^2 + 2|c|\delta \\ &< \epsilon.\end{aligned}$$

[0.5]

Thus, f is continuous at $c \in [a,b]$. and hence on $[a,b]$, since c was an arbitrary element of $[a,b]$. [0.25]

2. (a) Suppose that $0 < \alpha < 1$ and that $(x_n)_{n\geq 1}$ is a sequence which satisfies the following condition:

$$|x_{n+1} - x_n| \leq \alpha^n \quad n = 1, 2, 3, \ldots$$

Then prove that (x_n) is a Cauchy sequence. [3]

Solution: Let $\epsilon > 0$ be given. [0.25]

Since $0 < \alpha < 1$, hence $\alpha^m \to 0$ as $m \to \infty$, i.e., there exist n_0 such that for all $m \geq n_0$, $\alpha^m < \epsilon(1-\alpha)$. [1]

Let $n > m$. Then

$$\begin{aligned}|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \ldots + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{m+1} - x_m| \\ &\leq \alpha^{n-1} + \alpha^{n-2} + \ldots + \alpha^m = \alpha^m[1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1+m}] \\ &\leq \alpha^m[1 + \alpha + \alpha^2 + \ldots] \\ &= \frac{\alpha^m}{1-\alpha}\end{aligned}$$

[1]

Hence for $n > m \geq n_0$ we get

$$|x_n - x_m| < \epsilon.$$

[0.25]

Interchanging the n and m in above we get

$$n, m \geq n_0 \implies |x_n - x_m| < \epsilon.$$

[0.5]

(b) Using Cauchy's mean value theorem show that [3]

$$1 - \frac{x^2}{2!} \leq \cos x \text{ for all } x \in \mathbb{R}.$$

Solution: For $x = 0$, the equality holds. [0.25]

Let $f(x) = 1 - \cos x$ and $g(x) = \frac{x^2}{2}$. Let $x > 0$, then f, g are continuous on $[0, x]$ and differentiable on $(0, x)$. So by CMVT there exists $c \in (0, x)$ such that

$$c(1 - \cos x - 0) = \sin c(x^2/2 - 0) \implies 1 - \cos x = \left(\frac{\sin c}{c}\right) \frac{x^2}{2} \le \frac{x^2}{2}.$$

[1.75]

where the ineqality follows using $\dfrac{\sin x}{x} \le 1$ for all $x \ne 0$.

If $x < 0$, then $-x > 0$ hence there exists $c' \in (0, -x)$ such that

$$c'(1 - \cos(-x) - 0) = \sin c'((-x)^2/2 - 0) \implies 1 - \cos x = \left(\frac{\sin c'}{c'}\right) \frac{x^2}{2} \le \frac{x^2}{2}.$$

[1]

3. (a) Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if} \quad x = \frac{1}{2} \\ 0 & \text{if} \quad x \in [0, 1] \setminus \left\{\dfrac{1}{2}\right\} \end{cases}.$$

Is f integrable on $[0, 1]$? If yes, find $\displaystyle\int_0^1 f(x)dx$. [3]

Solution: For any partition $P = \{0 = x_0, \cdots, x_n = 1\}$ of $[0, 1]$, $m_i(f) = 0$ for $i = 1, 2, \cdots, n$. So $L(P, f)$ is zero and hence lower integral is 0. [0.5]

Let $P = \{0 = x_0, \cdots, x_n = 1\}$ be any partition of $[0, 1]$ and $\frac{1}{2} \in [x_i, x_{i+1}]$ for some i. [0.5]

Then $U(P, f) = \Delta x_{i+1}$. [0.5]

Since we can choose a partition P of $[0, 1]$ such that Δx_{i+1} is as small as possible and 0 is the infimum of set $[0, 1]$. Hence, upper integral $\int^{-} f(x)dx = \inf_P U(P, f) = 0$. [0.5]

Since 0 is the infimum of $[0, 1]$. So f is integrable. [0.5]

$\int_0^1 f(x)dx = 0$. [0.5]

Remark : (If any student write as): Since f is discontinuous only one point so f is integrable and $\int_0^1 f(x)dx = 0$. [1]

(b) Investigate the convergence of the improper integral using the limit comparison test [3]

$$I = \int_0^1 \frac{dx}{\sqrt{1 - x^3}}.$$

Solution: Note that, Given $f(x) = 1 - x^3 = (1-x)(1+x+x^2)$. Let us compare the given function with $g(x) = \frac{1}{\sqrt{1-x}}$. [0.5]

$$\lim_{x\to 1}\frac{f(x)}{g(x)} = \lim_{x\to 1}\frac{1/\sqrt{1-x^3}}{1/\sqrt{1-x}} = \lim_{x\to 1}\frac{\sqrt{1-x}}{\sqrt{1-x^3}} = \lim_{x\to 1}\frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{3}} \quad \text{non-zero finite number.}$$

[1]

Hence, by Limit Comparison test, Both $\int_0^1 f(x)$ and $\int_0^1 g(x)$ either converges or diverges. But,

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{t\to 1}\int_0^t \frac{dx}{\sqrt{1-x}} = \lim_{t\to 1}\left[-2\sqrt{1-t}+2\right] = 2.$$

[1]

So, $\int_0^1 g(x)$ converges. Hence, $\int_0^1 f(x)$ also converges. [0.5]

4. (a) Find the repeated(iterated) limits $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]$, $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$ and simultaneous limit $\lim_{(x,y)\to(0,0)} f(x,y)$ of the following function [4]

$$f(x,y) = \begin{cases} 1, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0. \end{cases}$$

Solution: Repeated Limit:

$$\lim_{y\to 0} f(x,y) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

[0.5]

$$\lim_{x\to 0}(\lim_{y\to 0} f(x,y)) = 1.$$

[0.5]

Similarly,

$$\lim_{x\to 0} f(x,y) = \begin{cases} 1, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

[0.5]

$$\lim_{y\to 0}(\lim_{x\to 0} f(x,y)) = 1.$$

[0.5]

Hence, repeated limit exist.

Simultaneous Limit:

Let $\epsilon = \frac{1}{2} > 0$ (say, other constant is possible but less than 1) be given. [0.5]

$|f(x,y) - f(0,0)| = 1 \nleq \epsilon$ for all points (excluding points on x-axis or y-axis) in any neighborhood of $(0,0)$. [1]

So, simultaneous limit does not exist. [0.5]

Alternate solution: Along x-axis(y =0) or y-axis (x = 0)

$$\lim_{(x,y)\to(0,0)} = 0$$

[1]

Along y-axis (where $x \neq 0$ and $y \neq 0$) or any other curve with $xy \neq 0$

$$\lim_{(x,y)\to(0,0)} = 1$$

[1]

So, simultaneous limit does not exist.

(b) Approximate $\sqrt{2}$ by using Taylor's one degree polynomial around 1 in the interval $[1,2]$. [3]

Solution **(Interval Estimate)** Let $f : [1,2] \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$.

Taylor's theorem for $n = 1$ gives us

$$f(x) = f(1) + f^{'}(1)(x-1) + \frac{f^{''}(c)}{2}(x-1)^2$$

for some $c \in (1,2)$. [1]
We have
Put $x = 2$

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{8c\sqrt{c}}$$

for some $c \in (1,2)$. [1]

Since $c \in (1,2)$

$$1 + \frac{1}{2} - \frac{1}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16\sqrt{2}}.$$

$$1 + \frac{1}{2} - \frac{1}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16 * \frac{3}{2}}.$$

since $\sqrt{2} < \frac{3}{2}$

$\frac{11}{8} < \sqrt{2} < \frac{35}{24}$

or $1.375 < \sqrt{2} < 1.4583$. [1]

Solution **(Point Estimate)** Let $f : [1, 2] \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$.

The Taylor's one degree polynomial around 1 is given by

$P_1(x) = f(1) + f'(1)(x - 1)$ [1]

Now $f'(1) = \frac{1}{2}$. Hence $P_1(2) = 1 + \frac{1}{2}(2 - 1) = 3/2$. [2]

5. (a) The plane $x + y + z = 12$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the point on the ellipse that is closest to the origin using the method of Lagrange Multipliers. [4]

Solution: As the square of the distance from the point (x, y, z) to the origin is also minimize the distance, minimizing the distance to the origin is equivalent to minimizing $f(x, y, z) = x^2 + y^2 + z^2$.

Now, the constraints may be written as $g(x, y, z) = x+y+z-12 = 0$ and $h(x, y, z) = x^2+y^2-z = 0$.

By method of Lagrange Multipliers, we have $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$

It gives $(2x, 2y, 2z) = \lambda(1, 1, 1) + \mu(2x, 2y, -1)$ [0.5marks]

Together with the constraint equations, we now have the system of equations

$$2x = \lambda + 2\mu x \tag{1}$$

$$2y = \lambda + 2\mu y \tag{2}$$

$$2z = \lambda - \mu \tag{3}$$

$$x + y + z - 12 = 0 \tag{4}$$

$$x^2 + y^2 - z = 0. \tag{5}$$

Equations (1) and (2) give

$\lambda = 2x(1 - \mu)$

$\lambda = 2y(1 - \mu)$

$\Rightarrow x(1 - \mu) = y(1 - \mu)$ [0.5marks]

This expression gives either $\mu = 1$ (i,e, $\lambda = 0$) or $x = y$

However, if $\mu = 1$ and $\lambda = 0$, we have from (3), $z = -\frac{1}{2}$, which contradicts (5), i.e., $x^2 + y^2 = -\frac{1}{2}$.

Consequently, the only possibility is to have $x = y$. [1.5marks]

Using (5), we get $z = 2x^2$. Substituting this into (4) gives us

$x + y + z - 12 = x + x + 2x^2 - 12 = 2(x + 3)(x - 2) = 0,$

It gives $x = -3$ or $x = 2$. Since $y = x$ and $z = 2x^2$, we have $(2, 2, 8)$ and $(-3, -3, 18)$ are the only candidates for extrema. [1marks]

Finally, since $f(2, 2, 8) = 72$ and $f(-3, -3, 18) = 342$, the closest point on the intersection of the two surfaces to the origin is (2, 2, 8). [0.5marks]

Remark **In Q.5(a)** if you have not discussed the case $\mu = 1$ and directly divided equations by $1 - \mu$, then you will not get 1.5 marks alloted for the case $\mu = 1$.

(b) Examine the function $f(x, y)$ for local maxima, local minima and saddle point, where

$$f(x, y) = (x^2 + y^2)e^{(y^2 - x^2)}.$$

[3]

Solution: Here $f_x = e^{(y^2-x^2)}(2x - 2x^3 - 2xy^2) = e^{(y^2-x^2)}(2x(1 - x^2 - y^2))$ and
$f_y = e^{(y^2-x^2)}(2y + 2yx^2 + 2y^3) = e^{(y^2-x^2)}(2y(1 + x^2 + y^2))$.
Therefore $f_x = 0$ and $f_y = 0$ gives $2x(1-x^2-y^2) = 0$ and $2y(1+x^2+y^2) = 0$. Since $1+x^2+y^2 \neq 0$, we get critical points $(0, 0)$, $(1, 0)$ and $(-1, 0)$. [0.5marks]
Now $f_{xx} = e^{(y^2-x^2)}(2 - 10x^2 - 2y^2 + 4x^4 + 4x^2y^2)$,
$f_{xy} = e^{(y^2-x^2)}(8xy - 4yx^3 - 4xy^3)$,
$f_{yy} = e^{(y^2-x^2)}(10y^2 + 4y^2x^2 + 4y^4 + 2 + 2x^2)$. [0.5marks]
Therefore $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = (2)(2) - (0)^2 = 4 > 0$ and $f_{xx}(0, 0) = 2 > 0$ gives $(0, 0)$ is a local minima.
Moreover $f_{xx}(\pm 1, 0)f_{yy}(\pm 1, 0) - f_{xy}^2(\pm 1, 0) = (\frac{-4}{e})(\frac{4}{e}) - (0)^2 = \frac{-16}{e^2} < 0$ gives $(\pm 1, 0)$ are saddle points. [2marks]

6. (a) Evaluate the iterated integral $\displaystyle\int_0^1 \left[\int_x^1 e^{y^2} dy\right] dx$. [2]

Solution: Apply the change of order, we get $\displaystyle\int_0^1 \int_0^y e^{y^2} dxdy$ [1]

Solving it, we get $\displaystyle\int_0^1 ye^{y^2}\ dy$ [0.5]

Finally, $\displaystyle\int_0^1 ye^{y^2}\ dy = \frac{1}{2}(e - 1)$ [0.5]

(b) Evaluate the double integral $\displaystyle\iint_R \sin\left(\frac{x - y}{x + y}\right) dA$, where R is region bounded by the square with vertices $(1, 0)$, $(2, 1)$, $(1, 2)$ and $(0, 1)$. [4]

Solution: Apply the change of variables to solve this problem.

In the given problem the region (square with vertices $(1, 0)$, $(2, 1)$, $(1, 2)$ and $(0, 1)$) is bounded by the lines $x = y - 1, x = y + 1, x + y = 1, x + y = 3$.

Applying the transformation $x - y = u$ and $x + y = v$, which gives $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$. [1]

We get region bounded by $u = 1, u = -1, v = 1, v = 3$, which is a rectangular region in xy-plane.

Now, $J(u, v) = 1/2$ [1]

Then applying the method of change of variables, we get

$$\begin{aligned}
\iint_R \sin(\frac{x-y}{x+y})dA &= \int_1^3 \int_{-1}^1 \sin(\frac{u}{v})|J(u,v)|dudv \\
&= \frac{1}{2}\int_1^3 \int_{-1}^1 \sin(\frac{u}{v})dudv \\
&= \frac{1}{2}\int_1^3 [v\cos(\frac{v}{u})]_{-1}^1 du \\
&= \frac{1}{2}\int_1^3 0du \\
&= 0 \qquad [2]
\end{aligned}$$

7. Evaluate the integral $\iiint_D 30xy\ dV$, where D is the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $2x + y + z = 4$. [3]

Solution: Notice that each point in the solid lies above the triangular region R in the xy-plane indicated that R as forming the base of the solid. Notice that for each fixed point $(x, y) \in R, z$ ranges from $z = 0$ up to $z = 4 - 2x - y$. Notice that for each fixed $x \in [0, 2], y$ ranges from 0 up to $y = 4 - 2x$.

$$\begin{aligned}
\iiint_D 30xy\ dV &= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} 30xy\ dzdydx \qquad [1.5m] \\
&= 30\int_0^2 \int_0^{4-2x} [xyz]_0^{4-2x-y}\ dydx \\
&= 30\int_0^2 \int_0^{4-2x} xy(4-2x-y)\ dydx \qquad [0.5m] \\
&= 30\int_0^2 [4x\frac{y^2}{2} - 2x^2\frac{y^2}{2} - x\frac{y^3}{3}]_0^{4-2x} dx \\
&= 5\int_0^2 [12x(4-2x)^2 - 6x^2(4-2x)^2 - 2x(4-2x)^3]dx \qquad [0.5m] \\
&= 64 \qquad [0.5m]
\end{aligned}$$

Remark: One may use different order of integration, then accoringly limits of integration will change.

$$\iiint_D 30xy\ dV = \int_0^4 \int_0^{4-y} \int_0^{\frac{4-y-z}{2}} 30xy\ dxdzdy$$

$$\iiint_D 30xy\ dV = \int_0^4 \int_0^{4-z} \int_0^{\frac{4-y-z}{2}} 30xy\ dxdydz$$

$$\iiint_D 30xy\ dV = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-z} 30xy\ dydzdx$$

Please note that $dxdydz$ means first integrate w.r.t. x, then w.r.t. y and then w.r.t. z. If You have not written this carefully you loose marks.

8. Find the absolute maxima and minima of the function $f(x,y) = x^2 + 2y^2 - x$ on the unit disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. [3]

Solution: $\nabla f = (f_x, f_y) = (2x-1, 4y) = (0,0)$ gives critical point $(\frac{1}{2}, 0)$. [1mark]

For other critical points, we use Lagrange method for f on the unit circle $x^2 + y^2 = 1$. Let $g(x,y) := x^2 + y^2 - 1$. Then $\nabla f = \lambda \nabla g$ and $g(x,y) = 0$ gives equations $-1 + 2x = 2\lambda x$, $4y = 2\lambda y$ and $x^2 + y^2 = 1$. Therefore we get critical points $(\frac{-1}{2}, \frac{\sqrt{3}}{2})$, $(\frac{-1}{2}, \frac{-\sqrt{3}}{2})$, $(1,0)$, and $(-1,0)$. [0.5 + 0.5 Marks]

The absolute maxima is $\frac{9}{4}$ at $(\frac{-1}{2}, \frac{\pm\sqrt{3}}{2})$, and absolute minima is $f(\frac{1}{2}, 0) = \frac{-1}{4}$. [0.5 + 0.5 Marks]

Alternate solution: $\nabla f = (f_x, f_y) = (2x-1, 4y) = (0,0)$ gives critical point $(\frac{1}{2}, 0)$. [1mark]

Let $g(x) = 2 - x^2 - x$. Therefore $g'(x) = -2x - 1$. For critical points $g'(x) = 0$ gives $x = \frac{-1}{2}$ and in this case $y = \frac{\pm\sqrt{3}}{2}$. [0.5marks]
Boundary points are $x = \pm 1$ and the corresponding value of y on the unit circle is $y = 0$. [0.5marks]

Note that $f(\frac{1}{2}, 0) = \frac{-1}{4}$, $f(\frac{-1}{2}, \frac{\pm\sqrt{3}}{2}) = \frac{9}{4}$, $f(1,0) = 0$, and $f(-1,0) = 1$.
Therefore absolute maxima is $\frac{9}{4}$ at $(\frac{-1}{2}, \frac{\pm\sqrt{3}}{2})$, and absolute minima is $f(\frac{1}{2}, 0) = \frac{-1}{4}$. [0.5 + 0.5 marks]

9. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(0,0) = 0$ and $f(x,y) = \dfrac{x^2 y^2}{x^2 + y^2}$. Prove that f is continuous at $(0,0)$ using sequential definition of continuity. [3]

Solution: Note that $f(0,0) = 0$. Therefore to prove f is continuous at $(0,0)$ we need to show that whenever $(x_n, y_n) \to (0,0)$, $f(x_n, y_n) = \frac{x_n^2 y_n^2}{x_n^2 + y_n^2} \to 0$. Since $(x_n, y_n) \to (0,0)$, we have $x_n \to 0$ and $y_n \to 0$. [1mark]

Using AM-GM inequality ($\frac{x^2+y^2}{2} \geq \sqrt{x^2 y^2}$) or $x^2 \leq x^2 + y^2$, we get

$0 \leq \frac{x_n^2 y_n^2}{x_n^2 + y_n^2} \leq \frac{x_n^2}{2}$. [1mark]

Since $x_n \to 0$, we have $\frac{x_n^2}{2} \to 0$, and hence $f(x_n, y_n) \to 0$ by Squeeze/Sandwich theorem. [1mark]

Remark1: There is no marks if you have used any other technique (i.e., $\epsilon - \delta$) to prove the result. Marks are awarded only if sequential definition of continuity is properly used.

Remark1: Marks are deducted if without defining the sequence (x_n, y_n) approach is followed.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(0,0) = 0$ and $f(x,y) = \dfrac{x^2 y}{x^2 + y^2}$ for $(x,y) \neq (0,0)$. Let $u = (u_1, u_2)$ be a unit vector. Prove that whenever u_1 and u_2 both are nonzero, we get [3]

$$D_u f(0,0) \neq \nabla f(0,0) \cdot u.$$

Solution: $D_u f(0,0) := \lim_{t\to 0} \dfrac{f(tu_1, tu_2) - f(0,0)}{t} = \lim_{t\to 0} \dfrac{t^3 u_1^2 u_2}{t(t^2 u_1^2 + t^2 u_2^2)} = u_1^2 u_2.$ [1.5mark]

Now $f_x(0,0) := \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^2\times 0}{h^2+0^2} - 0}{h} = 0$ and
$f_y(0,0) := \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{k\times 0}{k^2+0^2} - 0}{k} = 0.$ [1marks]
Therefore $\nabla f(0,0) = (0,0)$ and hence $\nabla f(0,0) \cdot u = 0$.
Therefore whenever u_1 and u_2 are nonzero, we get

$$D_u f(0,0) \neq \nabla f(0,0) \cdot u.$$

[0.5marks]

Remark **In Q. 10,** 1 mark is not given, if $f_x(0,0)$ and $f_y(0,0)$ are not computed.

End of The Paper