LNMIIT/B.Tech/C/IC/2017-18/EVEN/MTH221/M

THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
PROBABILITY AND STATISTICS (SOLUTION)& MTH221
MID TERM

Time: 90 minutes   Date: 20/02/2018   Maximum Marks: 25

Note: You should attempt all questions. Your writing should be legible and neat. Marks awarded are shown next to the question. Please make an index showing the question number and page number on the front page of your answer sheet in the following format, otherwise you may be penalized by the deduction of **2 marks**.

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1. In the game of cards, the entire deck of 52 cards is dealt out to 4 players what is the probability that [2 + 2]

   (a) one of the players receives all 13 spades;

   (b) each player receives 1 ace?

   Proof: (a) Let E_i be the event that hand i has all 13 spades, then

   $$P(E_i) = \frac{1}{{}^{52}C_{13}}, i = 1, 2, 3, 4.$$

   Because the events E_i, i = 1, 2, 3, 4, are mutually exclusive, the probability that one of the hands is dealt all 13 spades is

   $$P(\bigcup_{i=1}^{4} E_i) = \sum_{i=1}^{4} P(E_i) = \frac{4}{{}^{52}C_{13}}.$$

   (b) To determine the number of outcomes in which each of the distinct players receives exactly 1 ace, put aside the aces. Now number of possible ways that 48 cards are divided into 4 equally groups $= \frac{48!}{12!12!12!12!}$.

   Because there are 4! ways of dividing the 4 aces so that each player receives 1, we see that the number of possible outcomes in which each player receives exactly 1 ace is $4!(\frac{48!}{12!12!12!12!})$.

   $$\text{So desired probability} = \frac{4!(\frac{48!}{12!12!12!12!})}{(\frac{52!}{13!13!13!13!})} \approx 0.1055$$

2. Let each of N men throws his hat at a place. The hats are mixed up, and then each man randomly select from among their own N hats. What is the probability that exactly k of the N men have matches? [4]

proof: Let E denote the event that k people have a match, and letting G be the event that none of the other $N-k$ people have a match, we have $P(E \cap G) = P(E)P(G|E)$.

Let F_i, $i = 1, \cdots, k$, be the event that the ith member of the set has a match. Then,

$$\begin{aligned} P(E) &= P(F_1 \cap F2 \cap \cdots \cap Fk) = P(F1)P(F2|F1)P(F3|F1F2)\cdots P(Fk|F1Fk-1) \\ &= \frac{1}{N}\frac{1}{N-1}\frac{1}{N-2}\frac{1}{N-3}\cdots\frac{1}{N-(k-1)} \\ &= \frac{(N-k)!}{N!} \end{aligned}$$

So, Probability that each of the k person select his hat $= P(E) = \frac{(N-k)!}{N!}$. [1 marks]

The probability that at lest one match when N people randomly select from among their own N hats, is given by

$P(\bigcup_{r=1}^{N} F_i) = \sum_{r=1}^{N} \frac{(-1)^r}{r!}$

Probability that none of the remaining $(N-k)$ person select his hat $= 1 - P(\bigcup_{r=1}^{(N-k)} F_r) = \sum_{r=o}^{(N-k)} \frac{(-1)^r}{r!}$.

[1 marks]

Hence, the probability that a specified set of k people have matches and no one else does is

$P(E \cap G) = P(E)P(G|E) = \frac{(N-k)!}{N!} \sum_{r=o}^{(N-k)} \frac{(-1)^r}{r!}$

As there will be exactly k matches and if the preceding is true for any of the ${}^{N}C_K$ sets of k individuals, the desired probability is

P(exactly k matches) $= {}^{N}C_k \frac{1}{N(N-1)(N-2)...(N-k+1)} \sum_{r=o}^{(N-k)} \frac{(-1)^r}{r!} = \frac{1}{k!} \sum_{r=o}^{(N-k)} \frac{(-1)^r}{r!}$ [2 marks]

3. State the Bayes theorem and using it solve the following problem: [4]

   Suppose that there are two websites, A and B, for renting books. The site A receives 60% of all orders. Among the orders placed on site A, 75% arrive on time. Among the orders placed on site B, 90% arrive on time. Given that an order arrived on time, find the probability that the order was placed on site B.

Ans Bayes Theorem:

Let A and B be events with non-zero probability. Then $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ [1 marks]

Let us denote the events,
A = the order is placed on web site A

B = the order is placed on web site B
T = the order is received on time

We have given that $P(A) = 0.6, P(B) = 0.4, P(T|A) = 0.75, P(T|B) = 0.9$. Now we have to compute $P(B|T)$

By the Bayes Rule, $P(B|T) = \dfrac{P(T|B)P(B)}{P(T)}$

Where using Law of Total Probability, we calculate

$P(T) = P(T|A)P(A) + P(T|B)P(B) = (0.75)(0.6) + (0.9)(0.4) = 0.81$ [1 marks]

Then, $P(B|T) = \dfrac{(0.9)(0.4)}{0.81} = \dfrac{4}{9}$ [2 marks]

4. Show that Poisson random variable may be used as an approximation for a binomial random variable with parameter (n, p), when n is large, p is small enough so that $\lambda = np$ is moderate size. Compute the expected value and variance of poisson random variable. [5]

Ans To see this, suppose that X is a binomial random variable with parameters (n, p), and let $\lambda = np$. Then,

$$\begin{aligned} P\{X = i\} &= \frac{n!}{(n.i)!i!}p^i(1-p)^{n-i} \\ &= \frac{n!}{(n.i)!i!}(\frac{\lambda}{n})^i(1-\frac{\lambda}{n})^{n-i} \\ &= \frac{n(n-1)...(n-i+1)}{n^i}\frac{\lambda^i}{i!}\frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^i} \end{aligned}$$

Now, for n large and λ moderate,

$P\{X = i\} \approx e^{-\lambda}\dfrac{\lambda}{i!}$ [2 marks]

Expectation $E[X] = \displaystyle\sum_{i=0}^{\infty}\frac{ie^{-\lambda}\lambda^i}{i!} = \lambda\sum_{i=1}^{\infty}\frac{e^{-\lambda}\lambda^{(i-1)}}{(i-1)!} = \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{(i-1)}}{(i-1)!}$

Now take $j = i - 1$, we get

$E[X] = \lambda e^{-\lambda}\displaystyle\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda$ [1 marks]

Now, $E[X^2] = \displaystyle\sum_{i=0}^{\infty}\frac{i^2e^{-\lambda}\lambda^i}{i!} = \lambda\sum_{i=1}^{\infty}\frac{ie^{-\lambda}\lambda^{(i-1)}}{(i-1)!}$

Now taking $i - 1 = j$, we get

$E[x^2] = \lambda\displaystyle\sum_{j=0}^{\infty}\frac{(j+1)e^{-\lambda}\lambda^j}{j!} = \lambda\left[\sum_{j=0}^{\infty}j\frac{e^{-\lambda}\lambda^j}{j!} + \sum_{j=0}^{\infty}\frac{e^{-\lambda}\lambda^j}{j!}\right] = \lambda(\lambda+1)$

Variance $Var(X) = E[X^2] - (E[X])^2 = \lambda$ [2 marks]

5. Let X be continuous random variable denotes the required time (in years) to develop a software. Suppose that X has the following probability density function: [4]

$$f(x) = \begin{cases} b(4x - 2x^2), & 0 < x < 2; \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of b

(b) Compute the probability that it takes more than one year to develop the software.

(c) Compute the expected number of years it takes to develop a software.

Ans (a) As f is a probability density function, we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

So, $\int_0^2 b(4x - 2x^2)dx = 1$

$b = \frac{3}{8}$ [2 marks]

Hence,

$$f(x) = \begin{cases} \frac{3}{8}(4x - 2x^2), & 0 < x < 2. \\ 0, & \text{otherwise.} \end{cases}$$

(b) probability that it takes more than one year to develop the software $= PX > 1 = \int_1^{\infty} f(x)dx = \frac{3}{8}\int_1^2 (4x - 2x^2)dx = \frac{1}{2}$ [1 marks]

(c) expected number of years it takes to develop a software $= E(X) = \int_{-\infty}^{\infty} xf(x)dx = \frac{3}{8}\int_0^2 x(4x - 2x^2)dx = 1$ [1 marks]

6. Let X be an exponential random variable. Find the distribution function of X and X^2. [2 + 2]

proof: We know that the pdf of exponential random variable for some $\lambda > 0$ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

The cumulative distribution function $F_X(a)$ of an exponential random variable X is given by

$$F_X(a) = PX \leq a$$

$$F_X(a) = \int_0^a \lambda e^{-\lambda x} dx$$

$$F_X(a) = 1 - e^{-\lambda a}, a \geq 0.$$

We know that

$$F_{X^2}(x) = \begin{cases} F(\sqrt{x}) - F(-\sqrt{x}), & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

$$F_{X^2}(x) = \begin{cases} 1 - e^{-\lambda\sqrt{x}} + 0, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

$$F_{X^2}(x) = \begin{cases} 1 - e^{-\lambda\sqrt{x}}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$