THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
MATHEMATICS-III & MTH213
ENDTERM

Time: 3 hours | Date: 27/11/2017 | Maximum Marks: 100

Note: Usual notations are used. Attempt all questions. Your writing should be legible and neat. Marks awarded are shown next to the question. **Start a new question on a new page and answer all its parts in the same place**. Please make an index showing the question number and page number on the front page of your answer sheet in the following format.

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1. (a) If $H(z)$ and $K(z)$ are continuous at $z = z_0$, then prove that the following functions are also continuous at $z = z_0$ using the definition of continuity ($\delta - \varepsilon$ definition)

$$G(z) = 5H(z)K(z)$$

Answer: (a) By definition of continuity, for$\epsilon > 0$, there exist $\delta > 0$ such that $|H(z) - H(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$, Now take $\delta_1 > 0$ and $\epsilon < 1$, we get
$|H(z) - H(z_0)| < 1$ for $|z - z_0| < \delta_1$, then $|H(z)| - |H(z_0)| < |H(z) - H(z_0)| < 1$, i.e., $|H(z)| < |H(z_0)| + 1 = A$, where A is a positive constant.
For $\epsilon > 0$, there exist $\delta_2,\ \delta_3$ for which $|H(z) - H(z_0)| < \frac{\epsilon}{10(1+|K(z_0)|)}$ for $|z - z_0| < \delta_2$, and $|K(z) - K(z_0)| < \frac{\epsilon}{10A}$ for $|z - z_0| < \delta_3$.
Let $\delta =$ Minimum $(\delta_1, \delta_2, \delta_3)$.
For all z satisfying $|z - z_0| < \delta$,
$|5H(z)K(z) - 5H(z_0)K(z_0)| = 5|H(z)K(z) - H(z)K(z_0) + H(z)K(z_0) - H(z_0)K(z_0)|$
$\leq 5|H(z) - K(z_0)||H(z)| + 5|K(z_0)||H(z) - H(z_0)|$ (by triangle inequity)
$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ [6m]

(b) **Prove or Disprove**: The following complex valued functions are analytic at z = 0 [4+4]

i. $f(z) = e^{-y}\sin x - ie^{-y}\cos x$
ii. $f(z) = |z|^2 + 3i$

Answer (b) One can use the necessary conditions of differentiability (i.e. the Cauchy-Riemann equations) to show that $f(z)$ is not differentiable other than $z = 0$. However, to show that $f(z)$ is analytic at $z = 0$, needs to show that $f(z)$ is differentiable in some nbd of $z = 0$. To prove the differentiability, Cauchy-Riemann equations must satisfies and continuity of partial derivatives.

(i) Write $f(z) = u(x, y) + iv(x, y)$, gives $u(x, y) = e^{-y}\sin x$ and $v(x, y) = -e^{-y}\cos x$. $u_x = e^{-y}\cos x$ equals $v_y = e^{-y}\cos x$, and $u_y = -e^{-y}\sin x$ equals $v_x = e^{-y}\sin x$ everywhere. This shows that u and v satisfies C-R equations in the neighborhood (nbd) of $z = 0$. [2m]

Now the partial derivatives are $u_x = e^{-y}\cos x$, $v_y = e^{-y}\cos x$, $u_y = -e^{-y}\sin x$ and $v_x = e^{-y}\sin x$.

$e^{-y}, \sin x, \cos x$ all are continuous functions and the product of continuous functions are also continuous. Thus, the first-order partial derivatives of $u(x,y)$ and $v(x,y)$ are continuous everywhere.

As $u(x,y)$ and $v(x,y)$ satisfies C-R equations and have continuous partial derivatives everywhere, $f(z)$ is differentiable at $z=0$ and also in the nbd of $z=0$. Hence, It is analytic at $z=0$. [2m]

(ii) Write $f(z) = u(x,y) + iv(x,y)$, gives $u(x,y) = x^2 + y^2$ and $v(x,y) = 3$.

Clearly, $u_x = 2x$ equals $v_y = 0$, and $u_y = 2y$ equals $v_x = 0$ if and only if $x = y = 0$. [2m]

$2x, 2y, 0$ are all are continuous functions. Thus, the first-order partial derivatives of $u(x,y)$ and $v(x,y)$ are continuous everywhere.

As $u(x,y)$ and $v(x,y)$ satisfies C-R equations only at z=0 and have continuous partial derivatives everywhere, $f(z)$ is differentiable only at $z=0$ and not in some nbd of $z=0$. Hence, It is not analytic at $z=0$. [2m]

Note: If you use only C-R equations, but not the continuous partial derivatives to describe function differentiability (analyticity) only 2 marks are awarded.

2. (a) Prove the identity $z = \tan\left[\frac{1}{i}\log\left(\frac{iz+1}{iz-1}\right)^{\frac{1}{2}}\right]$. **[3]**

Solution: $\tan\left[\frac{1}{i}\log\left(\frac{iz+1}{iz-1}\right)^{\frac{1}{2}}\right] = \tan\left[\frac{1}{2i}\log\left(\frac{iz+1}{iz-1}\right)\right] = \tan\left[\frac{1}{2i}\log\left(\frac{z-i}{z+i}\right)\right]$ **[1]**

$= \tan\left(\int \frac{dz}{z^2 - i^2}\right)$ **[1]**

$= \tan\left(\int \frac{dz}{z^2+1}\right) = \tan(\tan^{-1} z) = z$ **[1]**

(b) Justify whether the function $\sin z$ is a bounded function or not in $\mathbb{C}$. **[3]**

Solution: If possible, let, $\sin z$ is a bounded function in $\mathbb{C}$. We know that $\sin z$ is entire functiom. **[1]**

Then by Liouville's theorem, $\sin z$ must be constant. **[1]**

But it is certainly not a constant function. Hence, $\sin z$ is not a bounded function in $\mathbb{C}$. **[1]**

(c) Determine the number of zeros, counting multiplicities, of the polynomial $2z^5 - 6z^2 + z + 1$ in the annulus $1 \le |z| < 2$. **[4]**

Solution: Let $f(z) = 6z^2$ and $g(z) = 2z^5 + z + 1$. Then observe that $|f(z)| = 6|z|^2 = 6$ and $|g(z)| \le 2|z|^5 + |z| + 1 = 4 < |f(z)|$ when $|z| = 1$. The conditions in Rouches theorem are thus satisfied. Consequently, since $f(z)$ has two zeros, counting multiplicities, inside the circle $|z| = 1$, so does $f(z) + g(z)$. That is, $2z^5 - 6z^2 + z + 1$ has two roots inside $|z| = 1$. **[1.5]**

Again, let $f_1(z) = 2z^5$ and $g_1(z) = 6z^2 + z + 1$. Then observe that $|f_1(z)| = 2|z|^5 = 64$ and $|g(z)| \le 6|z|^2 + |z| + 1 = 27 < |f_1(z)|$ when $|z| = 2$. The conditions in Rouches theorem

are thus satisfied. Consequently, since $f_1(z)$ has five zeros, counting multiplicities, inside the circle $|z| = 2$, so does $f_1(z) + g_1(z)$. That is, $2z^5 - 6z^2 + z + 1$ has five roots inside $|z| = 2$. **[1.5]**
Thus $2z^5 - 6z^2 + z + 1$ has 3 roots in the annulus $1 \le |z| < 2$. **[1]**

(d) Find the value of the complex integration $I = \oint_C \frac{Log(z+3) + \cos z}{(z+1)^2} dz$, where C is the closed curve $|z| = 2$ traversed in the counter-clockwise direction. **[4]**

Solution: $\frac{Log(z+3) + \cos z}{(z+1)^2}$ has singular points only at $z = -1$ which is a pole of order 2 and $x \le -3$ which are branch point and branch line of $\log(z+3)$ i.e. points of discontinuity of discontinuity $Log(z+3)$. **[1]**
However, the closed curve $C : |z| = 2$ contains only one singular point $z = -1$. Hence by the Derivative of Analytic Function Theorem,
$I = \oint_C \frac{Log(z+3) + \cos z}{(z+1)^2} dz = 2\pi i f'(-1), \; f(z) = Log(z+3) + \cos z$ **[1.5]**
i.e., $I = 2\pi i(\frac{1}{2} + \sin 1) = \pi i(1 + 2\sin 1)$ **[1.5]**

3. (*a*) Find the radius of convergence of the power series $\sum a_n(z+1)^n$ where a_n is $\frac{1}{n2^n}$. **[6]**
Radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad [2]$$
$$= \lim_{n\to\infty} \left| \frac{(n+1)2^{n+1}}{n2^n} \right| \quad [2]$$
$$= 2. \quad [2]$$

(*b*) For the function $f(z) = \frac{\pi \cot \pi z}{(z+\frac{1}{2})^2}$, determine its isolated singular points and weather these points are poles, removable singular points or essential singular points. Further evaluate $\int_C f(z)dz$, where C is the positively oriented circle centred at 0 with radius $\pi/2$. **[8]**

Sol. Given function

$$f(z) = \frac{\pi \cot \pi z}{(z+\frac{1}{2})^2} = \frac{\pi \cos \pi z}{\sin \pi z (z+\frac{1}{2})^2},$$

[1]

$f(z)$ is nonanalytic at those points where $\sin \pi z(z+\frac{1}{2})^2 = 0$, *i.e.*, $z = -\frac{1}{2}$ and $z = n$, $n \in \mathbb{Z}$ are singular points and these are isolated. [2]
$\lim_{z\to-\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z) = -\pi^2$, which is finite and non-zero $\Rightarrow z = -1/2$ is simple pole. [1]
$\lim_{z\to n}(z-n)f(z) = \frac{1}{\left(n+\frac{1}{2}\right)^2}$, which is finite and non-zero for any integer $n \in \mathbb{Z} \Rightarrow z = n$ is also simple pole. [1]

Out of these singular points, $z = -1/2, -1, 0, 1$ lie inside the circle $C = \{z : |z| = \pi/2\}$.[1]
Therefore, By Cauchy residue theorem

$$\begin{aligned}\int_C f(z)dz &= 2\pi i \left[\text{Res}_{z=-1/2}f(z) + \text{Res}_{z=-1}f(z) + \text{Res}_{z=0}f(z) + \text{Res}_{z=1}f(z)\right] \\ &= 2\pi i \left[-\pi^2 + \frac{1}{\left(-1+\frac{1}{2}\right)^2} + \frac{1}{\left(0+\frac{1}{2}\right)^2} + \frac{1}{\left(1+\frac{1}{2}\right)^2}\right]\end{aligned}$$

[2]

4. (*a*) If isolated singular point z_0 of a function $f(z)$ is a pole of order m, then determine the residue of $f(z)$ at $z = z_0$. **[7]**

Sol. Suppose that $z = z_0$ is a pole of order m of f, then $f(z)$ has a Laurent series expansion for some positive R:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}, \quad b_m \neq 0,\ 0 < |z-z_0| < R. \text{ [2]}$$

$$\Rightarrow (z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^{n+m} + b_1(z-z_0)^{m-1} + \cdots + b_{m-1}(z-z_0) + b_m, \qquad \text{[1]}$$

which is power series expansion of the function $(z-z_0)^m f(z)$ throughout the entire disk $|z-z_0| < R$, thus analtyic there. [1]

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\} = (n+m)(n+m-1)\cdots(n+2)\sum_{n=0}^{\infty} a_n(z-z_0)^{n+1} + b_1\,(m-1)\,! \quad \text{[2]}$$

$$\Rightarrow b_1 = \frac{1}{(m-1)\,!}\left[\frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\}\right]_{z=z_0}. \qquad \text{[1]}$$

(*b*) Use calculus of residue to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$, $(a > b > 0)$. **[8]**

Sol. We find the real part of the integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx$. For this we take the corresponding complex function as $e^{iz}f(z)$, where, $f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)}$. [1]
Now, We consider a simple closed contour consisting of line segment $z = x\,(-R \leq x \leq R)$ on real axis and the upper half C_R of the circle $|z| = R$ from $z = -R$ to $z = R$, where $R > a$. [1]

Thus, we have

$$\int_C e^{iz}f(z)dz = \int_{-R}^{R} e^{ix}f(x)dx + \int_{C_R} e^{iz}f(z)dz, \qquad \text{[1]}$$

The function $e^{iz}f(z)$ has isolated singular points $z = \pm ai, \pm bi$, which are simple poles, but only two poles ai, bi lie inside C and [1]

$$\begin{aligned}
\int_C e^{iz} f(z)dz &= 2\pi i \left[\mathrm{Res}_{z=i}\{e^{iz} f(z)\} + \mathrm{Res}_{z=2i}\{e^{iz} f(z)\}\right] \\
&= 2\pi i \left[\lim_{z\to ai} \frac{e^{iz}}{(z+ai)(z^2+b^2)} + \lim_{z\to bi} \frac{e^{iz}}{(z^2+a^2)(z+bi)}\right] \\
&= 2\pi i \left[-\frac{e^{-a}}{2ai(-a^2+b^2)} + \frac{e^{-b}}{2bi(-b^2+a^2)}\right] \\
&= \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).
\end{aligned}$$

[2]

Further, we have $\lim_{R\to\infty} \int_{C_R} e^{iz} f(z) = 0$. [1]

Thus letting $R \to \infty$, we have

$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$.

Equating the real parts from both sides, we have

$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$. [1]

5. (*a*) Show that $(x-a)^2+(y-b)^2+z^2 = 1$ is a complete integral of $z^2(1+p^2+q^2) = 1$. Further, find the singular integrals of the given PDE. **[6]**

Sol. Differentiating $(x-a)^2 + (y-b)^2 + z^2 = 1$ with respect to x and y, then solving for p and q, we get

$$p = \frac{a-x}{z}, \; q = \frac{b-y}{z}$$

by substituting the value of z, p, q in the given PDE, it satisfies the identity. Hence, $(x-a)^2 + (y-b)^2 + z^2 = 1$ is a complete integral of the given PDE. **[2]**

Now writing $z = F(x, y, a, b)$, we get $F(x, y, a, b) = \pm\sqrt{1-(x-a)^2} - (y-b)^2$.

$F_a = 0$ implies $a = \pm x$ and $F_b = 0$ implies $b = \pm y$. **[2]**

By substituting the value of a and b in the given complete integral we get following singular integrals:

(i) $a = x\&b = y$ implies $z^2 = 1$

(ii) $a = x\&b = -y$ implies $4y^2 + z^2 = 1$

(iii) $a = -x\&b = y$ implies $4x^2 + z^2 = 1$

(iv) $a = -x\&b = -y$ implies $4x^2 + 4y^2 + z^2 = 1$ **[2]**

(*b*) Classify the following second order PDE and reduce the equation to canonical form and hence solve it: **[10]**

$$(n-1)^2 u_{xx} - y^{2n} u_{yy} = ny^{2n-1} u_y,$$

when n is an positive integer.

Sol. For $n = 1$ the given PDE reduces to

$$\begin{aligned} -y^2 u_{yy} &= y u_y, \\ u_{yy} &= -\frac{1}{y} u_y, \end{aligned}$$

which is of parabolic type and already in cannocical form.

Since $u_{yy} = -\frac{1}{y} u_y$ can be written as $\frac{\partial}{\partial y}(y u_y) = 0$. By integrating w.r.t. y we get $u_y = \frac{1}{y} f(x)$. Integrating once again w.r.t. y, we get the solution as

$$u(x, y) = f(x) \log y + g(x).$$

For $n > 1$, the given PDE is of Hyperbolic type.

Two real characteristic curves are $\xi = x + y^{1-n} = c_1$ and $\eta = x - y^{1-n}$.

The given PDE reduces to the cannonical form $u_{\xi\eta} = 0$.

By integrating we get the solution as $u(x, y) = f(\xi) + g(\eta) = f(x + y^{1-n}) + g(x - y^{1-n})$, where f and g are C^2 functions.

6. (*a*) Find the solution of the following problem: **[5]**

$$\begin{aligned} u_{tt} - u_{xx} - x + t &= 0, \qquad -\infty < x < \infty, t > 0, \\ u(x, 0) = x^3, \quad u_t(x, 0) &= \cos x \quad -\infty \leq x \leq \infty \end{aligned}$$

Sol. Rewriting the PDE we have $u_{tt} - c^2 u_{xx} = x - t = F(x, t)$

Consider the following decompostion of the given problem: $(P1)$:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad -\infty < x < \infty, t > 0, \\ u(x, 0) = x^3, \quad u_t(x, 0) &= \cos x \quad -\infty \leq x \leq \infty \end{aligned}$$

and $(P2)$:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= x - t, \qquad -\infty < x < \infty, t > 0, \\ u(x, 0) = 0, \quad u_t(x, 0) &= 0 \quad -\infty \leq x \leq \infty \end{aligned}$$

Using de-Alembert's formula the solution $u_1(x, t)$ of $(P1)$ is

$$\begin{aligned} u_1(x, t) &= \frac{1}{2}[(x - t)^3 + (x + t)^3] - \frac{1}{2} \int_{x-t}^{x+t} \cos s ds \\ &= \frac{1}{2}[(x - t)^3 + (x + t)^3] - \frac{1}{2}[\sin(x + t) - \sin(x - t)] \qquad \textbf{[2]} \end{aligned}$$

Using Duhamel principle we get the solution $u_2(x,t)$ of $(P2)$ is

$$\begin{aligned} u_2(x,t) &= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} (r-s)drds \\ &= -\frac{t^3}{6} + \frac{t^2 x}{2}. \qquad [\mathbf{2}] \end{aligned}$$

Using method of superposition principle, we get the solution $u(x,t)$ of given PDE as

$$u(x,t) = u_1(x,t) + u_2(x,t). \qquad [\mathbf{1}]$$

(b) Classify the following PDE and then solve the problem: **[8]**

$$\begin{aligned} u_{tt} &= 9u_{xx} - u, \quad 0 < x < \pi,\ t > 0 \\ u(x,0) = x + \sin 2x, u_t(x,0) &= 0, \quad 0 \le \pi \le \pi, \\ u(0,t) = u(\pi,t) &= 0, \quad t \ge 0. \end{aligned}$$

Sol. Use separation of variable technique. Let $u(x,t) = F(x)G(t)$. Then substituting into the governing equations, we get

$$\frac{\ddot{G}}{9G} = \frac{F''}{F} - \frac{1}{9} = \lambda.$$

Therefore the given PDE reduced to two second order ODEs

$$\begin{aligned} F'' - (\lambda + 1/9) &= 0 \\ \ddot{G} - 9\lambda G = 0. &\qquad [\mathbf{1}] \end{aligned}$$

Now

$$F(0) = F(\pi) = 0 \Rightarrow \lambda + \frac{1}{9} = -\left(\frac{n\pi}{\pi}\right)^2, \quad n = 1, 2, 3, \cdots \qquad [\mathbf{2}]$$

Thus $F_n(x) = \sin(nx)$. **[1]**

$$\ddot{G} = -\left(1 + \left(\frac{3n\pi}{\pi}\right)^2\right) G = -(1 + 9n^2)G = -\gamma_n^2 G,$$

where, $\gamma_n = \sqrt{1+n^2}$. Solving for G yields

$$G_n(t) = A_n \cos(\gamma_n t) + B_n \sin(\gamma_n t) \qquad [\mathbf{2}]$$

Thus using superposition principle we get

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(\gamma_n t) + B_n \sin(\gamma_n t)] \sin nx.$$

Using the condition $u_t(x,0) = 0 \Rightarrow B_n = 0$. Also $u(x,0) = x + \sin 2x$ gives

$$A_n = \frac{2}{\pi}\int_0^{\pi} (x + \sin 2x) \sin nx dx = \begin{cases} 0 & n = 2 \\ (-1)^n & n \ne 2. \end{cases}$$

So the solution of the given PDE is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\gamma_n t) \sin nx. \qquad [\mathbf{2}]$$

7. (*a*) Prove that the solution of the following problem, if it exists, is unique: **[6]**

$$\begin{aligned} u_t - ku_{xx} &= F(x,t), \qquad 0 < x < l, t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le l \\ u(0,t) &= 0, \; u(l,t) = 0 \qquad t \ge 0. \end{aligned}$$

Sol. Assume u_1 and u_2 are two solutions of the given Heat equation. Then $v = u_1 - u_2$ satisfies the following heat equation

$$\begin{aligned} v_t - kv_{xx} = 0 \qquad 0 < x < L, t > 0, \\ v(x,0) = 0, \quad 0 \le x \le L, \\ v(0,t) = v(L,t) = 0, \quad t \ge 0 \qquad [\mathbf{1}] \end{aligned}$$

Now consider a function $E(t)$ of t defined as

$$E(t) = \frac{1}{2k} \int_0^L v^2 dx. \qquad [\mathbf{1}]$$

So, $E(t) \ge 0 \forall t \ge 0$. Since v is differentiable we can differentiate $E(t)$ with respect to t to get

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2k} \int_0^L 2vv_t dx. = \frac{1}{k} \int_0^L vv_t dx \\ &= \int_0^L vv_{xx} dx = vv_x|_0^L - \int_0^L v_x^2 dx \end{aligned}$$

Since $v(0,t) = v(L,t) = 0, \quad t \ge 0$, we get

$$\frac{dE}{dt} = -\int_0^L v_x^2 dx \le 0 \qquad [\mathbf{2}]$$

Therefore, $E(t)$ is a positive and decresing function. Since $v(x,0) = 0$ implies $E(0) = 0$. Hence $E(t) \ge 0 \; \forall \; t \ge 0$ implies $E(t) \equiv 0$ for all $t \ge 0$.
Therefore, $v(x,t) \equiv 0$ for all $0 \le x \le L$ and $t \ge 0$ and $u_1 = u_2$. **[2]**

(*b*) Classify the following PDE and then find the solution: **[8]**

$$\begin{aligned} u_t &= 4u_{xx}, \qquad 0 < x < \pi, t > 0, \\ u(x,0) &= \sin x, \quad 0 \le x \le \pi \\ u(0,t) &= 0, \; u(l,t) = 1 \qquad t \ge 0. \end{aligned}$$

Sol. Let the $u_p(x,t) = cx + d$ be the particular solution satisfying the nonhomogeneous BCs.

BCs implies $c = \frac{1}{\pi}$ and $d = 0$. So $u_p(x,t) = \frac{x}{\pi}$. **[2]**

Now the correspoding PDE with homogeneous BCs is:

$$\begin{aligned} v_t &= 4v_{xx}, \qquad 0 < x < \pi, t > 0, \\ v(x,0) &= \sin x - u_p(x,0) = \sin x - \frac{x}{\pi} = F(x,t), \quad 0 \leq x \leq \pi \\ v(0,t) &= 0,\ v(\pi,t) = 0 \qquad t \geq 0. \qquad [\mathbf{1}] \end{aligned}$$

The solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n \exp(-4n^2 t) \sin nx,$$

[1]

where

$$a_n = \frac{2}{\pi} \int_0^\pi \left(\sin x - \frac{x}{\pi} \sin nx \right) dx.$$

[1]

By solving the integral

$$a_n = \begin{cases} -1, & n = 1 \\ \frac{(-1)^n 2}{n\pi} & n \neq 1. \end{cases} \qquad [\mathbf{2}]$$

The solution of the given PDE is $u(x,t) = v(x,t) + u_p(x,t)$. **[1]**