LNMIIT/B.Tech/IC/2019-20/ODD/MTH102/MT

The LNM Institute of Information Technology
Department of Mathematics
Mathematics-1 & MTH102
Mid Term Exam(Solution)

Time: 90 Minutes | Date: 03/10/2019 | Maximum Marks: 30

Note: You should attempt all questions. **There are total eight questions.** Marks awarded are shown next to the question. Please make an index showing the question number and page number on the front page of your answer sheet in the following format. Without proper justification of proof, answer will not be considered. **Calculator is not allowed.**

Question No.				
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1. Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n-\log n}$. [4]

Solution: We have $a_n = \frac{1}{n-\log n}$.

$$lim_{n\to\infty} a_n = lim_{n\to\infty} \frac{1}{n-\log n} = lim_{n\to\infty} \frac{\frac{1}{n}}{1-\frac{\log n}{n}} = 0.$$

[1]

Remark: Marks are not given if any body write directly $lim_{n\to\infty}\frac{1}{n-\log n} = o.$

$$\frac{da_n}{dn} = -\frac{(n-1)}{n(n-\log n)^2}$$
$$\frac{da_n}{dn} \leq 0.$$

[1]

So a_n is decreasing. Hence by Leibnitz's test, the given series converges.

Remark: Marks are not given if any body write directly that a_n is decreasing.

Take $b_n = \left|\frac{(-1)^n}{n-\log n}\right|$ $b_n = \left|\frac{1}{n-\log n}\right|$

Take $c_n = \frac{1}{n}$,

$$lim_{n\to\infty} \frac{b_n}{c_n} = 1.$$

[1]

So by limit comparison test, $\sum_{n=1}^{\infty} b_n$ diverges as $\sum_{n=1}^{\infty} c_n$ diverges.

So, series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n - \log n}$. is not absolute convergence. [1]

2. Test the convergence/divergence of the following sequence: [3]

$$x_1 = 1, \; x_{n+1} = \frac{1}{2 + x_n} \quad \text{for} \quad n \geq 1.$$

Solution:

$$|x_{n+1} - x_n| = \left|\frac{1}{2 + x_n} - \frac{1}{2 + x_{n-1}}\right|$$

[1]

$$|x_{n+1} - x_n| = \left|\frac{x_n - x_{n-1}}{(2 + x_n)(2 + x_{n-1})}\right|$$

$$|x_{n+1} - x_n| \leq \frac{1}{4}|x_n - x_{n-1}|$$

[1]

$\alpha = \frac{1}{4}$. So by Cauchy Contractive condition, Given sequence is convergent. [1]

Remark: No marks of finding the limit.

3. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7, 0]$ such that $f(-7) = -3$ and $f'(x) \leq 2$ for all $x \in [-7, 0]$. Using Mean Value Theorem (MVT), find the largest possible value for $f(0)$. [4]

Solution:

As given in the statement of problem that the function satisfies the conditions of the Mean Value Theorem (MVT), (i.e., continuous and differentiable on the interval $[-7, 0]$,) so we apply the concept of MVT and gets

$f(0) - f(-7) = f'(c)(0 - (-7)) \Rightarrow f(0) + 3 = 7f'(c)$ [1.5 Marks]

Now solve for $f(0)$, we get $f(0) = 7f'(c) - 3$ [0.5 Marks]

Given in the problem statement that the maximum value of the derivative is 2, i.e., $f'(c) \leq 2$. Thus, plugging the maximum possible value of the derivative into $f'(c)$, we can get the maximum value of $f(0)$.

Doing this gives, $f(0) = 7f'(c) - 3 \leq 7(2) - 3 = 11$.

Therefore, largest possible value for $f(0)$ is 11. [2 Marks]

Or, written as an inequality $f(0) \leq 11$

4. Let $f(x) = x \sin(\frac{a}{bx})$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$, where a and b are non-zero fixed real numbers. Prove that $f(x)$ is continuous, but not differentiable at $x = 0$. [4]

Solution:

as $\lim\limits_{x\to 0} \dfrac{x\sin(\frac{a}{bx}) - 0}{x - 0} = \lim\limits_{x\to 0} \sin(\dfrac{a}{bx})$ [1 Marks]

As $\lim\limits_{x\to 0} \sin(\dfrac{a}{bx})$ does not exist, which we can easily see using sequential approach as

Take sequence $\frac{2a}{(2n+1)b\pi} \to 0$. However, $\sin(\frac{(2n+1)\pi}{2}) = (-1)^n$ does not converse. Thus, $\lim\limits_{x\to 0} \sin(\dfrac{a}{bx})$ does not exist for any non-zero constant a and b.

Hence, $f(x)$ is not differentiable at $x = 0$. [1 Marks]

Now check existence of $\lim\limits_{x\to 0} f(x)$.

Let $\epsilon > 0$, then $|f(x) - f(0)| = |x\sin(\frac{a}{bx}) - 0| \leq |x| = |x - 0|$ [1 Marks]

Choose a $\delta \leq \epsilon$ such that $|x - 0| < \delta$, then $|f(x) - f(0)| < \epsilon$ [1 Marks]

The above discussion shows that $f(x)$ is continuous, but not differentiable at $x = 0$.

Remark1: One may also prove using squeeze (or sandwich) theorem as $-|x| \leq |x\sin(\frac{a}{bx})| \leq |x|$ and if $-|x|$ and $|x|$ approach to 0, $f \to 0$.

Remark 2: One can also follow sequential approach to prove.

Remark 3: No marks are given for calculating for calculating right and left-hand limits. As proof is required to support your left/ right hand limit output claim using sequential or $\delta - \epsilon$ approach or sandwich theorem

5. Let $f : [0, 2] \to \mathbb{R}$ such that [4]

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x < 1 \\ 2, & \text{if } x = 1 \\ 1, & 1 < x \leq 2 \end{cases}$$

For each $\epsilon > 0$, find a partition P of $[0, 2]$ such that $U(P, f) - L(P, f) < \epsilon$.

Solution:

For each $\epsilon > 0$, find a partition $P := \{0, 1 - \frac{1}{N}, 1 + \frac{1}{N}, 2\}$ of $[0, 2]$ such that $\frac{8}{N} < \epsilon$. This follows from Archimedean property. [1]

Note that, $U(P, f) = 1(1 - \frac{1}{N}) + 3(1 + \frac{1}{N})$ [1]

and $L(P, f) = 1(1 + \frac{1}{N}) + 3(1 - \frac{1}{N})$. [1]

Therefore, $U(P, f) - L(P, f) = \frac{8}{N} < \epsilon$. [1]

6. Consider $a_n := \displaystyle\sum_{i=1}^{n} \frac{i}{n^2 + i^2}$ for $n \in \mathbb{N}$. Find $\lim\limits_{n\to\infty} a_n$. [3]

Solution:

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \displaystyle\sum_{i=1}^{n} \frac{1}{n} \cdot \frac{\frac{i}{n}}{1 + (\frac{i}{n})^2}$. [1]

Therefore using Riemann sum, $\lim_{n\to\infty} a_n = \int_0^1 \frac{x}{1+x^2} = \frac{ln2}{2}$. [2]

Remark: Q.6 cannot be solved using Squeeze/Sandwich Theorem because the limit on left hand side sequence $\frac{n(n+1)}{4n^2}$ is $\frac{1}{4}$ and limit on the right hand side sequence $\frac{n(n+1)}{2(n^2+1)}$ is $\frac{1}{2}$.

7. Let $f(x) := (x^2-1)/(x+2)$ for $x \in \mathbb{R}$ with $x \neq -2$. Determine the intervals where f is convex and concave. Also determine all the points of local maxima, local minima and point of inflection. [5]

Solution:

Local extrema: First Approach

$$f'(x) = \frac{x^2+4x+1}{(x+2)^2}, \quad x \neq 2$$
$$f''(x) = \frac{6}{(x+2)^3}, \quad x \neq 2$$

$f'(x) = 0 \iff x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$. **[1 Mark]**

Note that $f''(-2-\sqrt{3}) < 0$ and $f''(-2+\sqrt{3}) > 0$. Therefore f has a local maxima at $-2-\sqrt{3}$ and local minima at $-2+\sqrt{3}$. **[1 Mark]**

Local extrema: Second Approach

$$f'(x) = \frac{x^2+4x+1}{(x+2)^2}, \quad x \neq 2$$

$f'(x) = 0 \iff x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$. **[0.75 Mark]**

Note that

Interval	sign of f'
$(-\infty, -2-\sqrt{3})$	$+$
$(-2-\sqrt{3}, -2)$	$-$
$(-2, -2+\sqrt{3})$	$-$
$(-2+\sqrt{3}, \infty)$	$+$

Hence by first derivative test, f has a local maxima at $-2-\sqrt{3}$ and local minima at $-2+\sqrt{3}$. **[1.25 Mark]**

Local extrema: Third Approach

$$f'(x) = \frac{x^2+4x+1}{(x+2)^2}, \quad x \neq 2$$

$f'(x) = 0 \iff x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$. **[0.75 Mark]**

Note that $f'(x) > 0$ on $(-\infty, -2-\sqrt{3}) \cup (-2+\sqrt{3}, \infty)$ and $f'(x) < 0$ on $(-2-\sqrt{3}, -2) \cup (-2, -2+$

$\sqrt{3}$). Hence f is strictly increasing on $(-\infty, -2-\sqrt{3}) \cup (-2+\sqrt{3}, \infty)$ and strictly decreasing on $(-2-\sqrt{3}, -2) \cup (-2, -2+\sqrt{3})$. Since f is continuous everywhere in its domain, therefore f has a local maxima at $-2-\sqrt{3}$ and local minima at $-2+\sqrt{3}$. **[1.25 Mark]**

Concavity/Convexity:

$$f''(x) = \frac{6}{(x+2)^3}, \quad x \neq 2$$

Since $f''(x) < 0$ for $x \in (-\infty, -2)$ hence f is concave (or concave downward) on $(-\infty, -2)$ and $f''(x) > 0$ for $x \in (-2, \infty)$ hence f is convex (or concave upward) on $(-2, \infty)$.

Remark: There are total 2 marks for Concavity/Convexity. **There is no such term as convex downward and convex upward.**

Point of Inflection: Though f is concave on $(-\infty, -2)$ and convex on $(-2, \infty)$, but $x = -2$ is not in the domain of f hence there is no point of inflection. **[1 Mark]**

8. Find the greatest and the least values of $f : [0, 2] \to \mathbb{R}$ where $f(x) = 4x^3 - 8x^2 + 5x$. [3]

Solution:

First Approach: f is a polynomial function and $[0, 2]$ is a closed and bounded interval, hence f will attain its absolute maximum and absolute minimum values either on critical points of f or end points $\{0, 2\}$. [0.5]

$$f'(x) = 12x^2 - 16x + 5 = 0 \iff x = \frac{16 \pm \sqrt{256-240}}{24} = \frac{20}{24}, \frac{12}{24}. \qquad [0.5]$$

x	$f(x)$
0	0
1/2	1
5/6	0.92
2	10

[1]

Therefore, f attains its absolute maximum at $x = 2$, so largest value of f is 10 and f attains its absolute minimum at $x = 0$, least value of f is 0. [1]

Second Approach:

$$f'(x) = 12x^2 - 16x + 5 = 0 \iff x = \frac{16 \pm \sqrt{256-240}}{24} = \frac{20}{24}, \frac{12}{24}. \qquad [0.5]$$

interval	sign of f'
$[0, 1/2)$	$+$
$(1/2, 5/6)$	$-$
$(5/6, 2]$	$+$

[0.5 Mark]

Hence f is strictly increasing on $[0, 1/2) \cup (5/6, 2]$ and strictly decreasing on $(1/2, 5/6)$. **[0.5 Mark]**

By continuity of f, maximum of $\{f(1/2), f(2)\}$ will be absolute maximum of f and $\min\{f(0), f(5/6)\}$ will be absolute minimum. **[1 Mark]**

Therefore, f attains its absolute maximum at $x = 2$, so largest value of f is 10 and f attains its absolute minimum at $x = 0$, least value of f is 0. **[0.5 Mark]**

End of paper