INTRODUCTION TO FOURIER TRANSFORMS FOR IMAGE PROCESSING

BASIS FUNCTIONS:

The Fourier Transform (in this case, the 2D Fourier Transform) is the series expansion of an image function (over the 2D space domain) in terms of "cosine" image (orthonormal) basis functions.

The definitons of the transform (to expansion coefficients) and the inverse transform are given below:

```
        F(u,v) = SUM{ f(x,y)*exp(-j*2*pi*(u*x+v*y)/N) }
    and
        f(x,y) = SUM{ F(u,v)*exp(+j*2*pi*(u*x+v*y)/N) }

    where u = 0,1,2,...,N-1 and v = 0,1,2,...,N-1
        x = 0,1,2,...,N-1 and y = 0,1,2,...,N-1
        j = SQRT( -1 )
        and SUM means double summation  over proper
        x,y or u,v ranges
```

First we will investigate the "basis" functions for the Fourier Transform (FT). The FT tries to represent all images as a summation of cosine-like images. Therefore images that are pure cosines have particularly simple FTs.
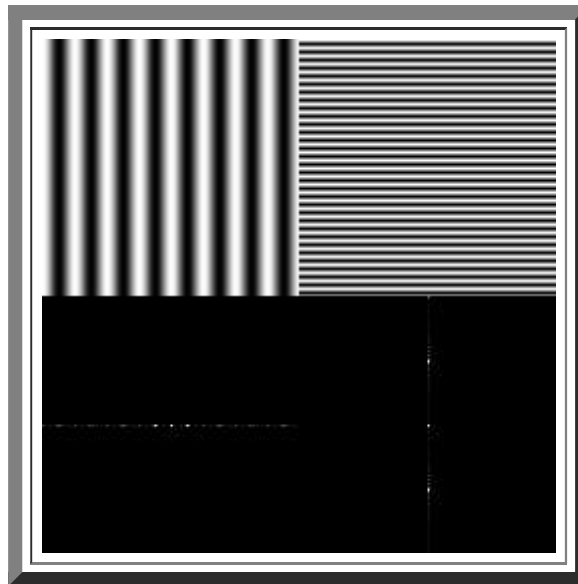
This shows 2 images with their Fourier Transforms directly underneath. The images are a pure horizontal cosine of 8 cycles and a pure vertical cosine of 32 cycles. Notice that the FT for each just has a single component, represented by 2 bright spots symmetrically placed about the center of the FT image. The center of the image is the origin of the frequency coordinate system. The u-axis runs left to right through the center and represents the horizontal component of frequency. The v-axis runs bottom to top through the center and represents the vertical component of frequency. In both cases there is a dot at the center that represents the (0,0) frequency term or average value of the image. Images usually have a large average value (like 128) and lots of low frequency information so FT images usually have a bright blob of components near the center. Notice that high frequencies in the vertical direction will cause bright dots away from the center in the vertical direction. And that high frequencies in the horizontal direction will cause bright dots away from the center in the horizontal direction.
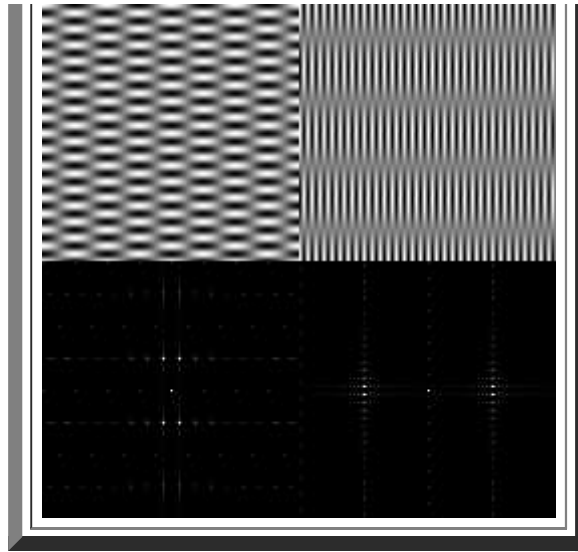
Here are 2 images of more general Fourier components. They are images of 2D cosines with both horizontal and vertical components. The one on the left has 4 cycles horizontally and 16 cycles vertically. The one on the right has 32 cycles horizontally and 2 cycles vertically. (Note: You see a gray band when the function goes through gray = 128 which happens twice/cycle.) You may begin to notice there is a lot of symmetry. For all REAL (as opposed to IMAGINARY or COMPLEX) images, the FT is symmetrical about the origin so the 1st and 3rd quadrants are the same and the 2nd and 4th quadrants are the same. If the image is symmetrical about the x-axis (as the cosine images are) 4-fold symmetry results.

MAGNITUDE VS. PHASE:

Recall that the definition of the Fourier Transform is:

```
        F(u,v) = SUM{ f(x,y)*exp(-j*2*pi*(u*x+v*y)/N) }
    and
        f(x,y) = SUM{ F(u,v)*exp(+j*2*pi*(u*x+v*y)/N) }

    where u = 0,1,2,...,N-1 and v = 0,1,2,...,N-1
          x = 0,1,2,...,N-1 and y = 0,1,2,...,N-1
          and SUM means double summation  over proper
          x,y or u,v ranges
```

Note that f(x,y) is the image and is REAL, but F(u,v) (abbreviate as F) is the FT and is, in general, COMPLEX. Generally, F is represented by its MAGNITUDE and PHASE rather that its REAL and IMAGINARY parts, where:

```
        MAGNITUDE(F) = SQRT( REAL(F)^2+IMAGINARY(F)^2 )
        PHASE(F) = ATAN( IMAGINARY(F)/REAL(F) )
```

Briefly, the MAGNITUDE tells "how much" of a certain frequency component is present and the PHASE tells "where" the frequency component is in the image. To illustrate this consider the following.
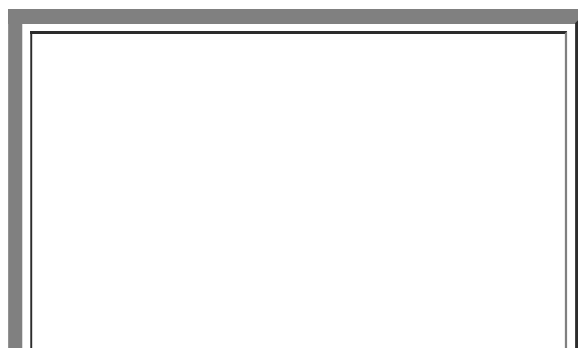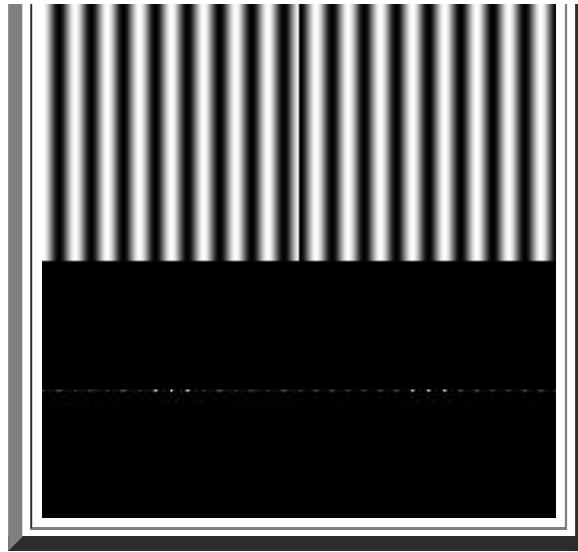
Note that the FT images we look at are just the MAGNITUDE images. The images displayed are horizontal cosines of 8 cycles, differing only by the fact that one is shifted laterally from the other by 1/2 cycle (or by PI in phase). Note that both have the same FT MAGNITUDE image. The PHASE images would be different, of course. We generally do not display PHASE images because most people who see them shortly thereafter succomb to hallucinogenics or end up in a Tibetan monastery. Nevertheless, it is wise to remember that when one looks at a common FT image and thinks about "high" frequency power and "low" frequency power, this is only the MAGNITUDE part of the FT.

By the way, you may have heard of the FFT and wondered if was different from the FT. FFT stands for "Fast" Fourier Transform and is simply a fast algorithm for computing the Fourier Transform.
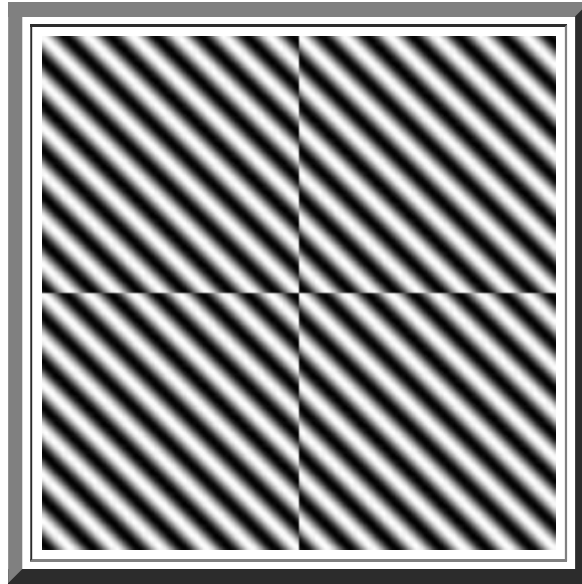
ROTATION AND EDGE EFFECTS:

In general, rotation of the image results in equivalent rotation of its FT. To see that this is true, we will take the FT of a simple cosine and also the FT of a rotated version of the same function. The results can be seen by:
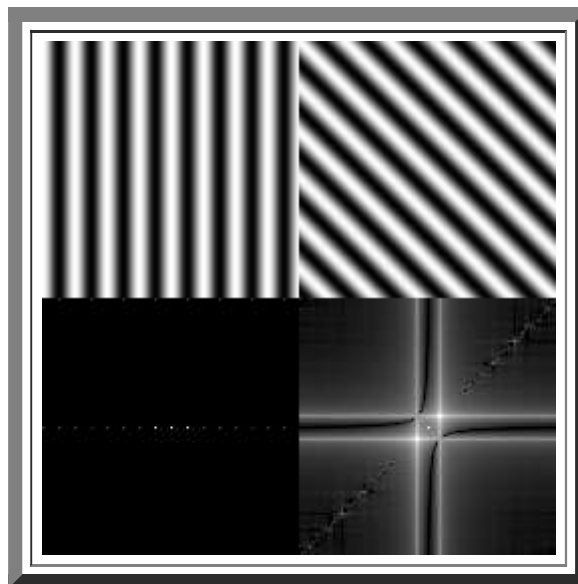
At first, the results seem rather surprising. The horizontal cosine has its normal, very simple FT. But the rotated cosine seems to have an FT that is much more complicated, with strong diagonal components, and also strong "plus sign" shaped horizontal and vertical components. The question is, where did these horizontal and vertical components come from? The answer is that the FT always treats an image as if it were part of a periodically

replicated array of identical images extending horizontally and vertically to infinity. And there are strong edge effects between the neighbors of such a periodic array as can be seen by:


Thus, what we see as the FT in the "slant" image (lower right of the image before last) is actually the combination of the actual FT of the cosine function and that caused by the edge effects of looking at a finite part of the image. These edge effects can be significantly reduced by "windowing" the image with a function that slowly tapers off to a medium gray at the edge. The result can be seen by:
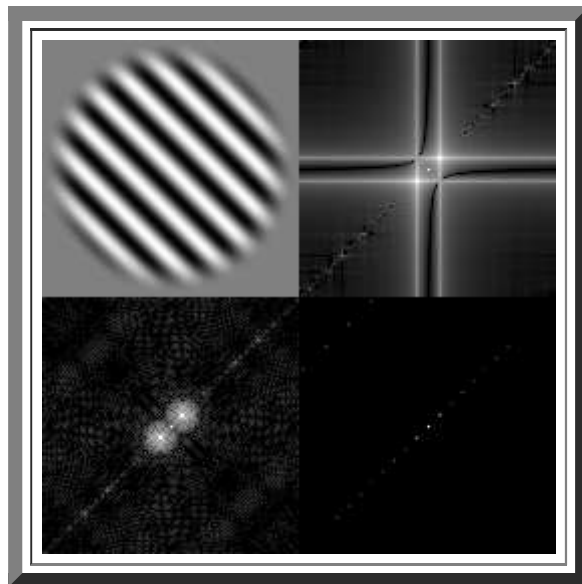
The windowed image is shown in the upper left. Its FT is shown in the lower left. The non-windowed FT is shown in the upper right and the actual, true FT of a cosine is shown in the lower right. These images are all scaled differently and the comparison is only qualitative, but it can be seen that the windowed image FT is much closer to the true FT and eliminates many of the edge effects.

SOME IMAGE TRANSFORMS:

Now, with the above introduction, the best way to become familiar with Fourier Transforms is to see lots of images and lots of their FTs. First, an interesting pair of images, one sharp and clear, and the other blurred and noisy.
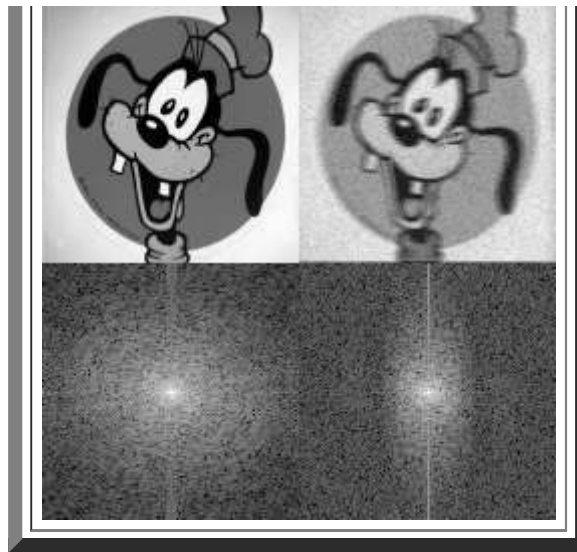
There are 2 images, goofy and the degraded goofy, with FTs below each. Notice that both suffer from edge effects as evidenced by the strong vertical line through the center. The major effect to notice is that in the transform of the degraded goofy the high frequencies in the horizontal direction have been significantly attenuated. This is due to the fact that the degraded image was formed by smoothing only in the horizontal direction. Also, if you look carefully you can see that the degraded goofy has a slightly larger background noise level at high frequencies. This is difficult to see and perhaps not even meaningful because the images are scaled differently, but if really there, it is due to the random noise added to the degraded goofy. Notice also that it is difficult to make much sense out of the low frequency information. This is typical of real life images.

The next images show the effects of edges in images:
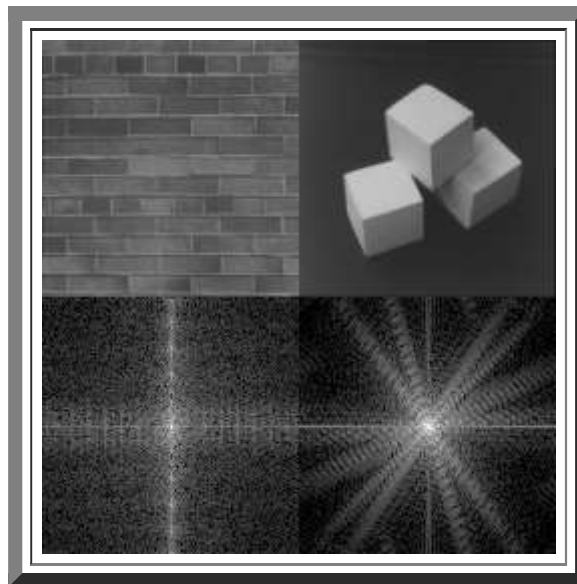
Notice the strong periodic component, especially in the vertical direction for the bricks image. Horizontal components appear closer together in the FT. In the blocks image, notice a bright line going to high frequencies perpendicular to the strong edges in the image. Anytime an image has a strong-contrast, sharp edge the gray values must change very rapidly. It takes lots of high frequency power to follow such an edge so there is usually such a line in its magnitude spectrum.
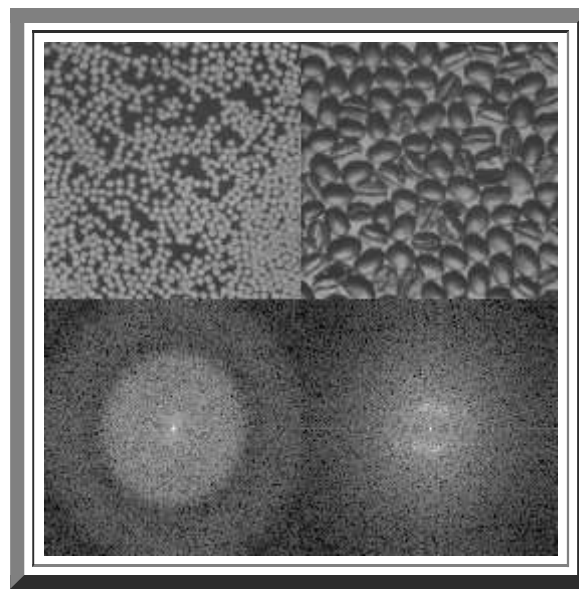
Now lets look at a bunch of different shapes and their FTs.
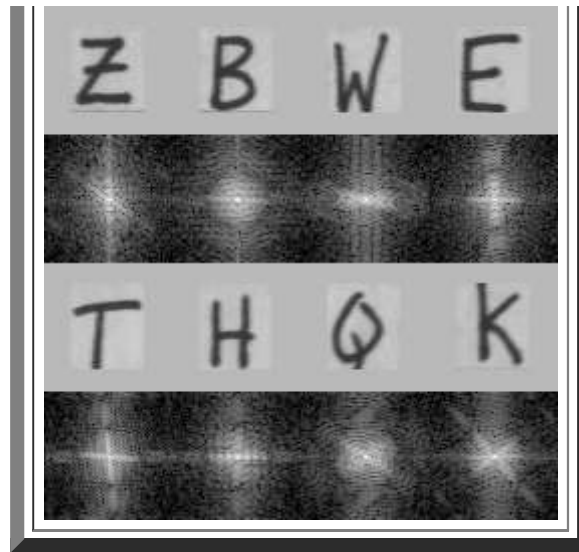
Notice that the letters have quite different FTs, especially at the lower frequencies. The FTs also tend to have bright lines that are perpendicular to lines in the original letter. If the letter has circular segments, then so does the FT.

Now lets look at some collections of similar objects:

Notice the concentric ring structure in the FT of the white pellets image. It is due to each individual pellet. That is, if we took the FT of just one pellet, we would still get this pattern. Remember, we are looking only at the magnitude spectrum. The fact that there are many pellets and information about exactly where each one is is contained mostly in the phase. The coffee beans have less symmetry and are more variably colored so they do not show the same ring structure. You may be able to detect a faint "halo" in the coffee FT. What do you think this is from?
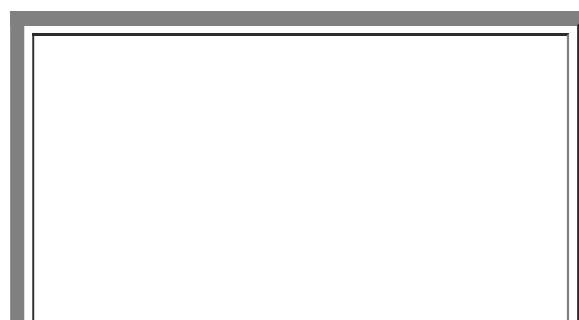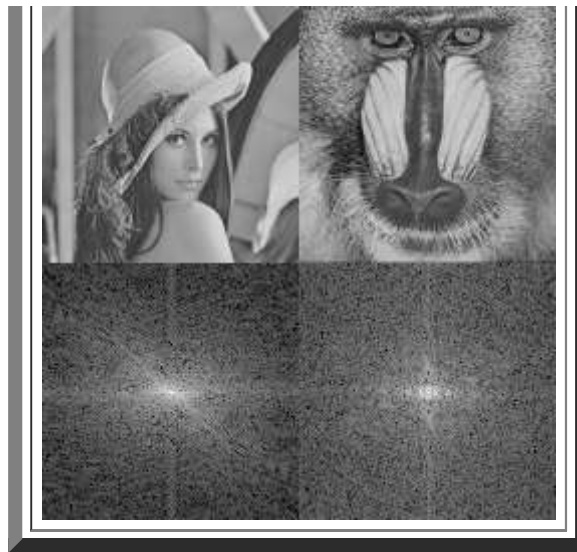
Here are our first truly general images. Notice there is very little structure. You can see a top left to bottom right slanting line in the girl image FT. It is probably due to the edge between her hat and her hair. There are also some small edge effects in both images. The mandril image appears to have more high frequency power, probably due to the hair.
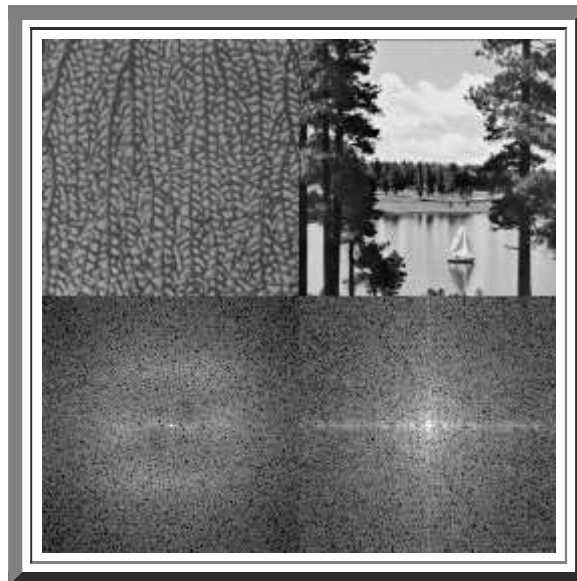
The seafan image has a lot of little holes that are about the same size and somewhat randomly oriented. The size of the holes is about 2 pixels wide so that corresponds to frequency components about 1/2 way out to the maximum. The strong horizontal components in the lake image is probably due to the tree trunk edges.

Now, here is your first quiz. Consider an image that is all black except for a single pixel wide stripe from the top left to the bottom right. What is its FT? Also, consider an image that is totally random. That is, every pixel is some random value, independent of all other pixels. What is its FT?
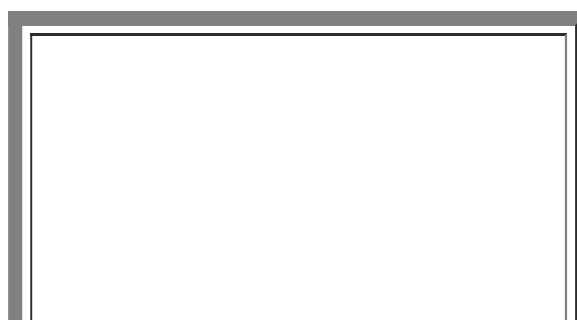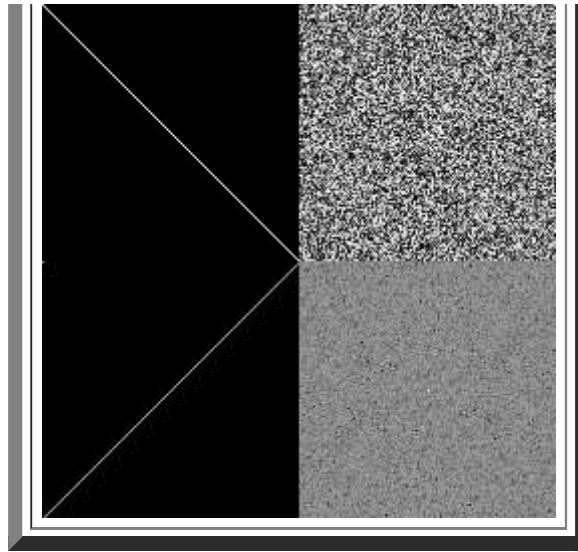
Do you believe it? If not, you can check it yourself. By the way, notice the single bright dot in the middle of the noise FT image. Why is it there? Why does the noise FT look dark gray?

SOME FILTERS:

Now we start to illustrate the use of some filters on the girl image. The first is a lowpass filter. The upper left is the original image. The lower left is produced by:

```
fft2d 128 < girlimage > girlfft
mag2d 128 < girlfft > girlmag
```
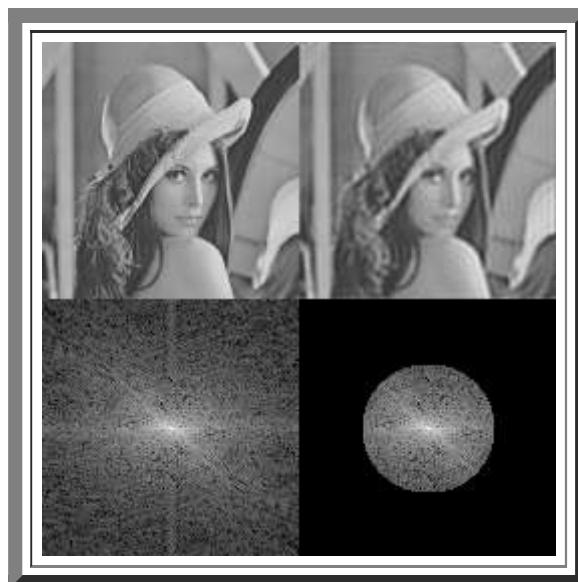
The lower right is then produced by:

```
fftfilt 128 low ideal 50 < girlfft > lpgirlfft
mag2d 128 < lpgirlfft > lpgirlmag
```

Finally, the upper right is produced by:

```
ifft2d 128 < lpgirlfft > lpgirl
```

To see the results:

The left side of the image we have seen before. In the lower right, notice how sharply the high frequencies are cut off by the "ideal" lowpass filter. Notice also that not very much power is being thrown away beyond the circle that is cut off. In the upper right, the reconstructed image is obviously blurrier due to the loss of high frequencies. Overall contrast is still pretty good due to that fact that not too much power was thrown away. Notice also that there are obvious "ringing" artifacts in the reconstructed image. This is due to the very sharp cutoff of the "ideal" filter. A Butterworth or Exponential filter with reasonably low order would not cause these.

Now we will do a highpass filter. The following image is produced in the same way as the previous one except:

```
fftfilt 128 high butter 50 < girlfft > hpgirlfft
```

In other words, a butterworth filter of 1st order is used.

Notice in the lower right that this filter does not cut off sharply at the 50% point as the lowpass did. However, the center bright spot, which accounts for most of the power in the image, is clearly gone. The image in the upper right, which looks totally black, in fact is not totally black. If you use the colormap capability of "dym" to stretch the gray values from 0-20 out over the entire range, you can see that this highpass filter has preserved the image information where there are very rapid changes in gray level. Such a process is frequently what is desired in an edge detector. However, it is not an improvement in the image. There are 2 problems. First, it is too dark. This can be fixed by rescaling or re-contrast- stretching the image after filtering. This is commonly done and is easy. Second, and harder, is the fact that too much of the low frequency tonal information is gone.

Image sharpening requires a "sharpening" filter or high frequency emphasis filter. This kind of filter preserves some of the low frequency information but relatively boosts the higher frequencies. To do such a thing, we will construct our own filter which will be piecewise-linear. The filter will be circularly symmetrical and will have coefficients as follows:

```
    0      0.5
   96      4.0
  127      4.0
```

In other words, Fourier coefficients of frequency-distance 0 from the origin will be multiplied by 0.5. As you go away from the origin or zero frequency, out to frequency-distance 96, the multiplier will be interploated between 0.5 and 4.0. From then outward, the multiplier will be 4.0. So higher frequency coefficients are multiplied by values greater than 1.0 and lower frequency coefficients are multiplied by values less thatn 1.0. The overall net effect on the image power is that it is unchanged. The above values are in a file called "filter_coeffs". To apply the filter, the following steps are carried out:

```
filttabler < filter_coeffs > filter_file
fftfilt 128 file filterfile < girlfft > mfgirlfft
```

The rest of the image is constructed as before. To see the result:

Notice the relative brightness at high frequencies in the lower right image. Which upper image is sharper? Which upper image looks better? Portraits are one of the few contradictions to the general principal that sharper is better.

Filtering can also be used to reduce noise. It is particularly effective when the noise is confined to just a few frequencies:
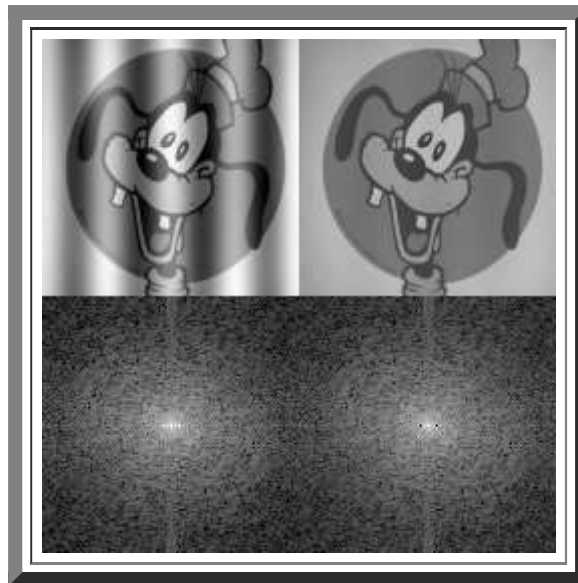
The image on the upper left is goofy with a superimposed cosine added to it, representing noise. In the lower left, notice the strong cosine "dots" just to the left and right of the origin. In the lower right, these "dots" have been removed (I actually did it with the "trace" capability in dym). The resulting magnitude file is then used with the "filter" command to filter the Fourier coefficients. The file of coefficients is then inverse FT'd to get the upper right image. The cosine "noise" is gone.

Life is not always this easy as is shown in the next example:
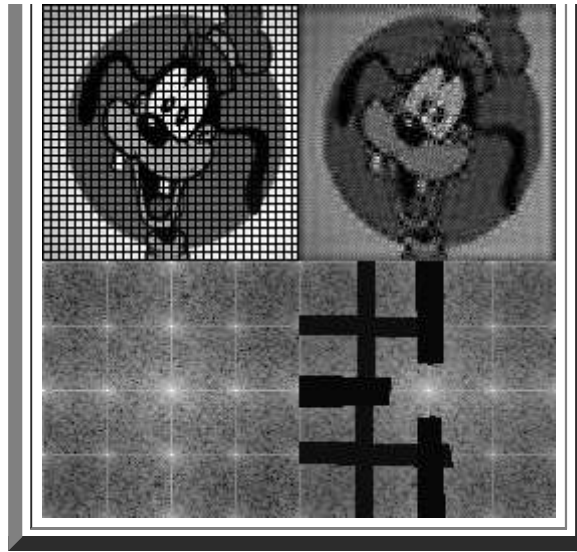
In this case, a grid has been placed over goofy. The lower left shows the resulting FT. Notice that the grid is quite sharp so it has lots of high frequencies so its impact on the frequency domain is very spread out. Dym was again used to "paint" out the grid frequencies as much as possible. The right half of the lower right image is not painted because it is the symmetric reflection of the left half and is not used by the filter.

```
YOUR ASSIGNMENT: (SHOULD YOU CHOOSE TO ACCEPT IT)

(1) Pick an image.
(2) FFT it and find the magnitude spectrum.
        see man for fft2d and mag2d
(3) Do something to the spectrum or the fft.
        ex:     filter
                fftfilt
                something like:
                        cm
                        double
                        multiply by alternating +1,-1
                        take phase only
                        take magnitude only
(4) Reconstruct an image by inverse fft.
        see man for ifft2d
(5) Put the results together like the above images using "group"
        see man for group
(6) Explain your results (1-2 pages).
    More credit will be given to the imagination of what you do than
    to the correctness of your explanation.
```