ASSIGNMENT 2

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1 Godel's incompleteness theorem

Godel's incompleteness theorems are two theorems of mathematical logic that are concerned with the limits of provability in formal axiomatic theories. These are important both in mathematical logic and in the philosophy of mathematics. The first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an effective procedure (i.e., an algorithm) is capable of proving all truths about the arithmetic of natural numbers. For any such consistent formal system, there will always be statements about natural numbers that are true, but that are unprovable within the system. The second incompleteness theorem, states that the system cannot demonstrate its own consistency.

- Formal systems: completeness, consistency, and effective axiomatization

There are several properties that a formal system may have, including completeness, consistency, and the existence of an effective axiomatization.

- Effective axiomatization

A formal system is said to be effectively axiomatized if its set of theorems is a recursively enumerable set.

- Completeness

A set of axioms is complete if, for any statement in the axioms' language, that statement or its negation is provable from the axioms. This is the notion relevant for Gödel's first Incompleteness theorem. In his completeness theorem, Gödel proved that first order logic is semantically complete

- Consistency

A set of axioms is (simply) consistent if there is no statement such that both the statement and its negation are provable from the axioms, and inconsistent otherwise.

- First incompleteness theorem

First Incompleteness Theorem: "Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F."

- Proof sketch for first theorem

- Statements in the system can be represented by natural numbers (known as Gödel numbers). The significance of this is that properties of statements—such as their truth and falsehood—will be equivalent to determining whether their Gödel numbers have certain properties, and that properties of the statements can therefore be demonstrated by examining their Gödel numbers. This part culminates in the construction of a formula expressing the idea that "statement S is provable in the system" (which can be applied to any statement "S" in the system).

- In the formal system it is possible to construct a number whose matching statement, when interpreted, is self-referential and essentially says that it (i.e. the statement itself) is unprovable. This is done using a technique called "diagonalization"

- Within the formal system this statement permits a demonstration that it is neither provable nor disprovable in the system, and therefore the system cannot in fact be -consistent. Hence the original assumption that the proposed system met the criteria is false.

- Second incompleteness theorem

Second Incompleteness Theorem: "Assume F is a consistent formalized system which contains elementary arithmetic. Then $F \nvdash Cons(F) F \nvdash Cons(F)$."

Proof sketch for the second theorem:

Let p stand for the undecidable sentence constructed above, and assume that the consistency of the system can be proved from within the system itself. The demonstration above shows that if the system is consistent, then p is not provable. The proof of this implication can be formalized within the system, and therefore the statement "p is not provable", or "not P(p)" can be proved in the system.

But this last statement is equivalent to p itself (and this equivalence can be proved in the system), so p can be proved in the system. This contradiction shows that the system must be inconsistent.