Chapter 22 Gauss's Law

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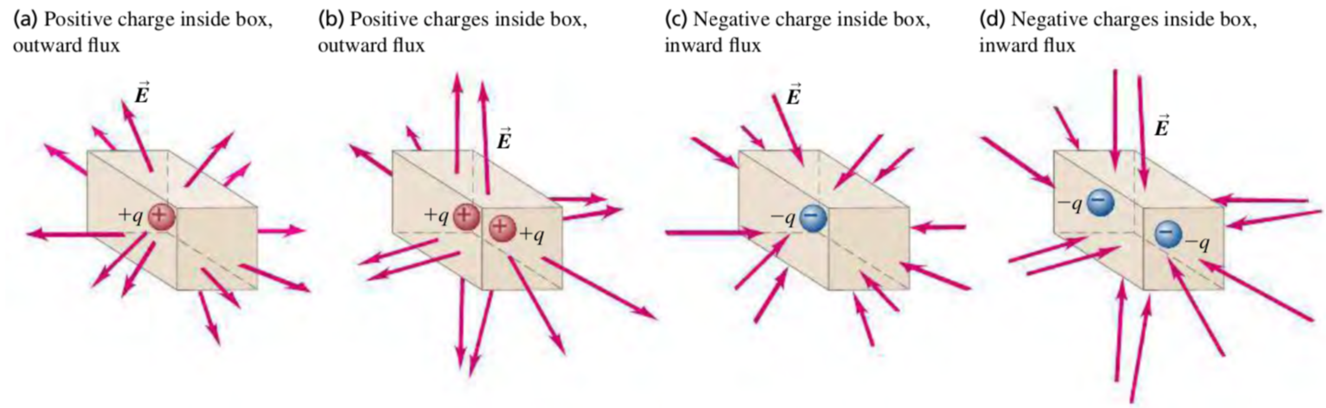
1 Charge and Electric flux

- Gauss's Law fundamentally, is, given a general distribution of charge, we can surround it with an imaginary surface that encloses the charge.
- Gauss's law is a relationship between the field at all the points on the surface and the total charge enclosed within the surface.
- When you have the imaginary surface, you can move a test charge q_0 around the vicinity of the surface.
- By measuring the force $\vec{\mathbf{f}}$ experienced by the test charge, you make a three dimensional map of the electric field $\vec{\mathbf{E}} = \frac{\vec{\mathbf{f}}}{q_0}$

1.1 Electric flux and Enclosed Charge

- the electric-field vectors point out of the surface, we say that there is an outward electric flux
- This can be thought of similar to how fluid moves and has flux within regions, even though electric fields don't technically move, they can be thought of as a flow
- The following images show a simple relationship. Positive charge inside the box goes with an outward electric flux through the box's surface, and negative charge inside goes with an inward electric flux.



- If the positive and negative charges cancel each other out, then the amount of charge entering or leaving the system is 0, so the flux is zero
- The magnitude of the electric flux decreases with distance to $\frac{1}{r^2}$

2 Calculating Electric flux

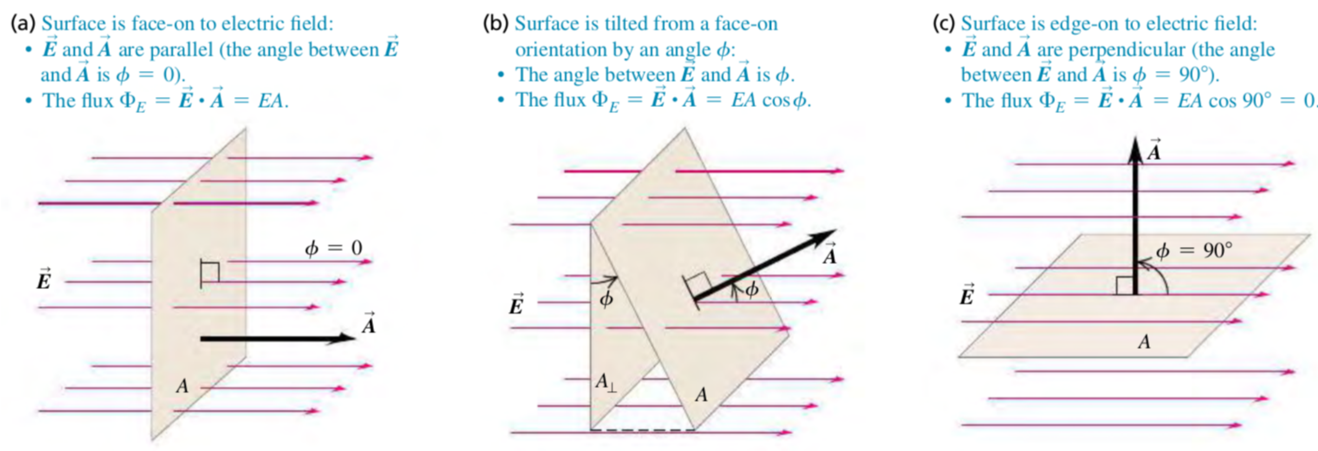
The net electric flux through a closed surface is directly proportional to the net charge inside that surface. To calculate this, use the analogy of the field of velocity vectors $\vec{\mathbf{v}}$ and the electric field $\vec{\mathbf{E}}$

2.1 flux of a Uniform Electric field

- We define the electric flux through this area to be the product of the field magnitude E and the area A

$$\Phi_E = EA \tag{1}$$

- flux can be thought of as the number of lines passing through A, so the more lines passing through A, the larger the magnitude of Φ_E
- The electric flux Φ_E through the surface equals the scalar product of the electric field $\vec{\mathbf{E}}$ and the area vector $\vec{\mathbf{A}}$



- When A is flat, but not perpendicular to the field $\vec{\mathbf{E}}$, then fewer field lines pass through it. The area that counts is the area $A_\perp$ and is equal to

$$\Phi_E = EA\cos\phi \tag{2}$$

- Since $E\cos\phi$ is a component of $\vec{\mathbf{E}}$, we can rewrite this as:

$$\Phi_E = E_\perp A \tag{3}$$

- In terms of the vector area $\vec{\mathbf{A}}$, we can write the electric flux as the scalar product of $\vec{\mathbf{E}}$ and $\vec{\mathbf{A}}$

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} \tag{4}$$

- We can represent the direction of the vector area $\vec{\mathbf{A}}$ by using the unit vector $\hat{\mathbf{n}}$, then:

$$\vec{\mathbf{A}} = A\hat{\mathbf{n}} \tag{5}$$

2.2 flux of a Nonuniform Electric field

- What happens when $\vec{\mathbf{E}}$ is not uniform at all points in the area? divide A into small elements dA, then calculate the electric flux through each element and integrate the results to obtain the total flux:

$$\Phi_E = \int E\cos\phi dA = \int E_\perp dA = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \tag{6}$$

  where ϕ is the angle between $\vec{\mathbf{E}}$ and the normal to the surface, dA is an element of the surface, and $d\vec{\mathbf{A}}$ is the vector element of the surface area

- We call this integral the **surface integral** of the component $E_\perp$

3 Gauss's Law

Gauss's law is an alternative to Coulomb's law. While completely equivalent to Coulomb's law, Gauss's law provides a different way to express the relationship between electric charge and electric field

3.1 Point Charge Inside a Spherical Surface

- Gauss's Law states that the total electric flux through any closed surface is proportional to the total (net) electric charge inside the surface
- Start with the field with the single positive point charge q. The field lines radiate out equally in all directions
- If we place this charge at the center of an imaginary sphere with radius R, the magnitude E of the electric field at any position on the surface would be:

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2} \tag{7}$$

- At each point on the surface, $\vec{\mathbf{E}}$ is perpendicular to the surface.
- The total electric flux is the product of the field magnitude E and the total area $A = 4\pi R^2$ of the sphere:

$$\Phi_E = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\epsilon_0} \tag{8}$$

- Note that the flux is independent of the radius R of the sphere. It depends on only the charge q enclosed by the sphere.

3.2 Point Charge Inside a Nonspherical Surface

- This concept can also be applied to non-spherical surfaces. Outside the sphere with radius R, consider any irregular shape.
- The total flux through the sphere must be the same as the total flux through the irregular surface, thus for an irregular surface:

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{1}{\epsilon_0} \tag{9}$$

- For a closed surface enclosing no charge, we have that:

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0 \tag{10}$$

3.3 General Form of Gauss's Law

- Suppose the surface encloses several charges $q_1, q_2, q_3, ...$ them the total (resultant) electric field $\vec{\mathbf{E}}$ at any point is the vector sum of the $\vec{\mathbf{E}}$ fields of the individual charges.
- Let $\vec{\mathbf{E}}$ be the total field at the position by the surface area element $d\vec{\mathbf{A}}$, let $E_\perp$ be the component perpendicular to the plane of that element. The we write the equation for Gauss's Law:

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{encl}}{\epsilon_0} \tag{11}$$

- The various forms of Gauss's law can thus be written as:

$$\Phi_E = \oint E \cos\phi dA = \oint E_\perp dA = \oint \vec{\mathbf{E}} d\vec{\mathbf{A}} = \frac{Q_{encl}}{\epsilon_0} \tag{12}$$

- For a spherical Gaussian surface of radius r with a positive charge $+q$, the electric field points out of the Gaussian surface so at every point in $\vec{\mathbf{E}}$ is the same direction as $d\vec{\mathbf{A}}$, $\phi = 0$
- Since E is the same at all points of the surface, you only have to consider the integral for $\int dA = A = 4\pi r^2$, so Gauss's law applied is:

$$\Phi_E = \oint E_\perp dA = \oint \left(\frac{q}{4\pi\epsilon_0 r^2}\right) dA = \frac{q}{4\pi\epsilon_0 r^2} \oint dA = \frac{q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{q}{\epsilon_0} \tag{13}$$

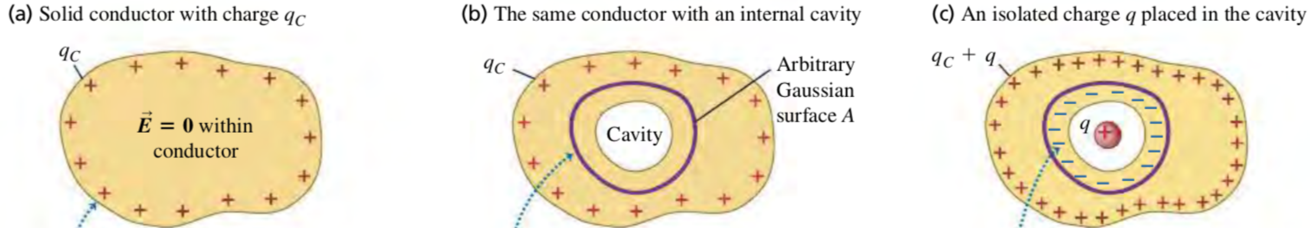
4 Applications of Gauss's Law

- Gauss's law is valid for any distribution of charges and for any closed surface.
- If we know the charge distribution, we can use it to find the field, or if we know the field, we can use Gauss's Law to find the charge distribution
- In many practical problems, we often encounter situations in which we want to know the electric field caused by a charge distribution on a conductor. When excess charge is placed on a solid conductor and is at rest, it resides entirely on the surface, not in the interior of the materia

5 Charges On Conductors

- We know that the electric field at every point within a conductor is zero and any excess charge on a solid conductor is located entirely on its surface
- Finding the electric field within a charged conductor.

(a) Solid conductor with charge q_C **(b)** The same conductor with an internal cavity **(c)** An isolated charge q placed in the cavity



The charge q_C resides entirely on the surface of the conductor. The situation is electrostatic, so $\vec{E} = \mathbf{0}$ within the conductor.

Because $\vec{E} = \mathbf{0}$ at all points within the conductor, the electric field at all points on the Gaussian surface must be zero.

For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.