Chapter 14 Periodic Motion

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Contents

1 Describing Oscillation

Oscillation always occurs if the force is a restoring force that tends to return the system to equilibrium.

- In the case of springs, the x component of acceleration is

$$a_x = \frac{F_x}{m}$$

- This force is called the restoring force

1.1 Amplitude, Period, Frequency, and Angular Frequency

- The amplitute of motion (A), is the maximum displacement from the equilibrium and is always positive.
- The period T represents the time to complete one cycle
- The frequency, f, is the number of cycles in a unit of time.
- The angular frequency ω, is 2π times the frequency

$$\omega = 2\pi f \tag{1}$$

- Additionally, another way of representing period and frequency is as reciprocals of eaceh other

$$f = \frac{1}{T} \tag{2}$$

2 Simple Harmonic Motion

- The simplest kind of oscillation occurs when the restoring force Fx is directly proportional to the displacement from equilibrium x.
- In an ideal spring, the restoring force F_x is equal to $-kx$ where x is the displacement and k is the force constant
- When the restoring force is proportional to the displacement from equilibrium, the oscillation is called simple harmonic motion.
- The standard equation for simple harmonic motion is given by:

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{3}$$

  Where here the $\frac{d^2x}{dt^2}$ is the second derivative of displacement
- For many many systems, the restoring force is approximately proportional to the displacement, these systems can still be modeled as simple harmonic motion

2.1 Circular motion and the equations of SHM

- In order to explore properties of simple harmonic motion, must express x of the oscillating body in terms of time $x(t)$
- Phasors are vectors that follow around a body moving in circular motion, these have the same angular speed ω as the rotating body
- The x-component of the phasor at time t is just the x-coordinate of the reference point Q:

$$x = Acos\theta \tag{4}$$

  Note that Q is essentially at the origin of the xy plane in this example
- The acceleration of Q, denoted as $\vec{a}_Q$ and is constant. It is defined in terms of the angular speed and the radius of the circle

$$a_Q = \omega^2 A \tag{5}$$

- The equation of angular speed ω can then be given by

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}} \tag{6}$$

- Note that we use ω for both the angular speed at point Q and for the angular velocity at point P, this is because these quantities are identical If point Q makes one complete revolution in time T, then point P goes through one complete cycle of oscillation in the same time;
- During time T, Q moves 2π radians, so the angular speed is $\omega = 2\pi T$

2.2 Period and amplitude in SHM

- In simple harmonic motion the period and frequency do not depend on the amplitude A. The reason for this is that the time of one complete oscillation is the same no matter what, regardless on the value of A.
- For a larger A, the body reaches $|x|$ and is thus subjected to larger restoring forces, increasing the speed of the body over the cycle, which perfectly compensates for the larger travel path

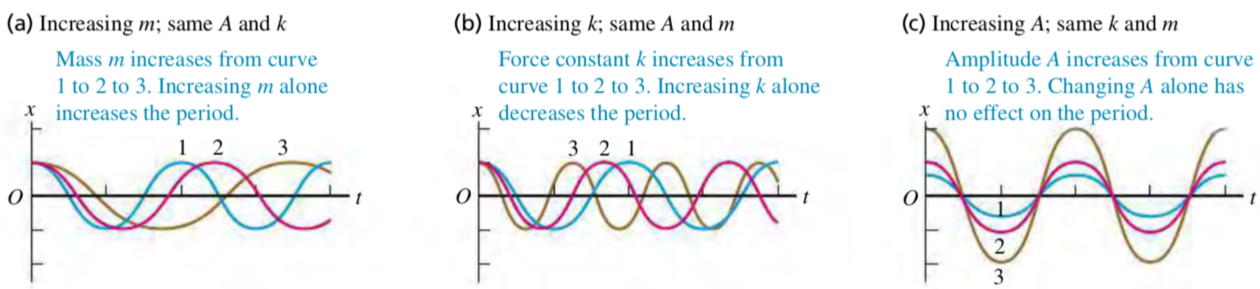
2.3 Displacement, velocity, and acceleration in SHM

- The displacement of simple harmonic motion can be characterized by

$$x = A\cos(\omega t + \phi) \tag{7}$$

Where ϕ is the phase angle, and the angular frequency is $\sqrt{\frac{k}{m}}$

- Variations of simple harmonic motion. All cases shown have $\phi = 0$



- If we start at time $t = 0$, then the time T to complete one cycle can be given by

$$\omega T = \sqrt{\frac{k}{m}}T = 2\pi \quad or \quad T = 2\pi\sqrt{\frac{m}{k}} \tag{8}$$

- The velocity and acceleration functions can be calculated as follows

$$v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \tag{9}$$

$$a_x = \frac{d^2x}{dt^2} = -\omega^2 A \sin(\omega t + \phi) \tag{10}$$

- We can then see that from the previous equations:

$$a_x = -\omega^2 x = -\frac{k}{m}x \tag{11}$$

- When the body is passing through the equilibrium at x = 0, the velocity equals either vmax or -vmax (depending on which way the body is moving) and the acceleration is zero.
- When the body is at maximum displacement, $x \pm A$, then the velocity is 0 and the magnitute of the acceleration is the largest, and at these points the restoring force is $F_x = -kx$
- The way to determine the amplitude A and phase angle f for an oscillating body when given its initial displacement x_0 and initial velocity v_{0x} is to notice that

$$v_{0x} = -\omega A \sin\phi \tag{12}$$

To find ϕ, divide the two equations:

$$\frac{v_{0x}}{x_0} = \frac{-\omega A \sin\phi}{A\cos\phi} = -\omega \tan\phi \tag{13}$$

Which then lets us see that

$$\phi = \arctan\left(-\frac{v_0 x}{\omega x_0}\right) \tag{14}$$

- It is also easy to find the amplitude in the previous example, with the following equation

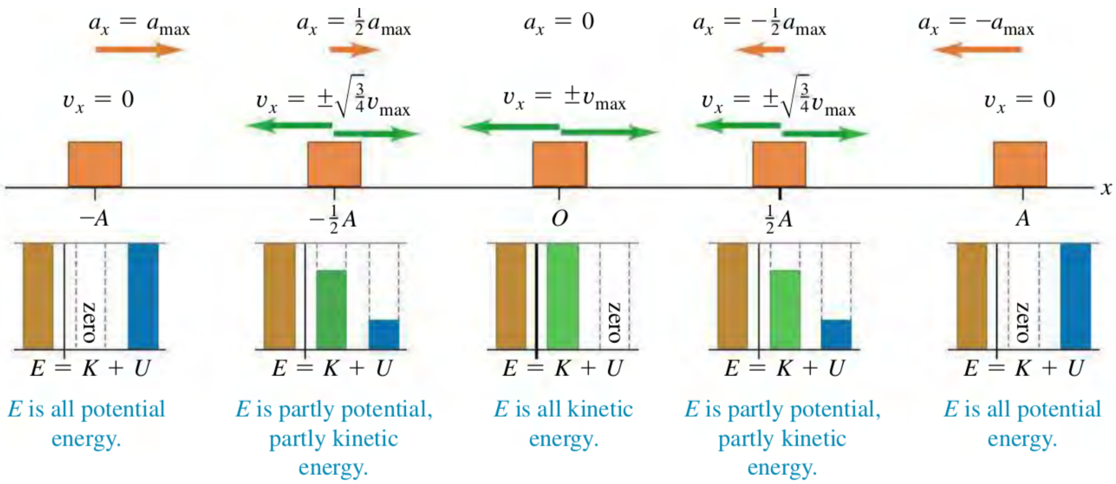
$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}} \tag{15}$$

3 Energy in Simple Harmonic Motion

- In the rotating body, The vertical forces do no work, so the total mechanical energy of the system is conserved.
- The kinetic energy of the body is $K = \frac{1}{2}mv^2$ and the potential energy of the spring is $U = \frac{1}{2}kx^2$ So the total mechanical energy is just given by $E = K + U$
- Whent the item reaches the point $x = A$, then $v_x = 0$ at this point, so we have that

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = constant \tag{16}$$

- Graphs of E, K, and U versus displacement in SHM.



- This equation can also be verified by plugging in x and v_x as well as using $\omega^2 = \frac{k}{m}$

$$\begin{aligned} E &= \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}m[-\omega A\sin(\omega t + \phi)]^2 + \frac{1}{2}k[A\cos(\omega t + \phi)]^2 \\ &= \frac{1}{2}kA^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi) = \frac{1}{2}A^2 \end{aligned}$$

- We can also solve for velocity v_x of the body at the given displacement of x

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \tag{17}$$

- The maximum speed v_{max} occurs at $x = 0$. Using the previous equations and $\omega = \sqrt{\frac{k}{m}}$, we have

$$v_{max} = \sqrt{\frac{k}{m}}A = \omega A \tag{18}$$

  Which agrees that v_x oscillates between $\pm\omega A$

4 Applications of Simple Harmonic Motion

As of now, the only application used has been a body attached to an ideal horizontal spring. There are however, many different applications of this phenomena, with different restoring forces for different situations

4.1 Vertical SHM

- Suppose there is now a vertical spring that is suspended from the ceiling with a body attached to it. This spring's equilibrium will be shifted by Δl such that

$$k\Delta l = mg = F_g \tag{19}$$

* When the body is x distance above the equilibrium, the spring is extended $\Delta I - x$. Then the upward force exerted is $k(\Delta I - x)$ and the net force is

$$F_{net} = k(\Delta I - x) + (-mg) = -kx \tag{20}$$

4.2 Angular SHM

- A mechanical watch which keeps time with a mechanical wheel is an example of angular simple harmonic motion. It has a moment of inertia I and a restoring torque τ_z
- We can write the equation of torque $\tau_z = -k\theta$
- Using Newton's second law for a rigid body, we have that $\sum \tau_z = I\alpha_z = I\frac{d^2\theta}{dt^2}$ and we see that

$$-\kappa\theta = I\alpha \quad or \quad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \tag{21}$$

- This is exaclty the same equation as for simple harmonic motion but with x replaced by θ and k/m replaced by κ/I
- the angular displacement θ as a function of time is given by

$$\theta = \Theta\cos(\omega t + \phi) \tag{22}$$

Where Θ is the angular amplitude

5 The Simple Pendulum

- A simple pendulum is an idealized model consisting of a point mass suspended by a massless, unstretchable string
- The path of the point mass is the arc of a circle with radius L equal to the length of the string
- The restoring force F_θ is the tangential component of the net force

$$F_\theta = -mg\sin\theta \tag{23}$$

- Gravity provides the restoring force F_θ; the tension T merely acts to make the point mass move in an arc,
- Since F_θ is proportional to $\sin\theta$, not to θ, motion is not simple harmonic, but for small θ, $\theta \approx \sin\theta$ so it is approximately harmonic
- With this approximation, it follows that

$$F_\theta = -mg\theta = -mg\frac{x}{L} = -\frac{mg}{L}x \tag{24}$$

- Here the force constant k can be given by $k = mg/L$ and the angular frequency ω with a small amplitude is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}} \tag{25}$$

- The corresponding frequency and period relationships are

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{26}$$

- The motion of the pendulum is only approximately harmonic, for large values of angular displacement, Θ, the departures can be modeled as

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \cdots\right) \tag{27}$$

6 The Physical Pendulum

- The physical pendulum is any real pendulum extended body, as contrasted to the idealized simple pendulum with all of its mass concentrated at a point.
- In equilibrium, the center of gravity is directly below the pivot. When the body is displaced as shown, the weight mg causes the restoring torque

$$\tau_z = -(mg)(d\sin\theta) \tag{28}$$

- With a small $\theta \approx \sin\theta$, we have that the torque is

$$\tau_z = -(mgd)\theta \tag{29}$$

- With Newton's Second law, we have that $\sum \tau_z = I\alpha_z$, so

$$-(mgd)\theta = I\alpha_z = I\frac{d^2\theta}{dt^2} \tag{30}$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta \tag{31}$$

- The angular frequency for this is then,

$$\omega = \sqrt{\frac{mgd}{I}} \tag{32}$$

  Where $f = (1/2\pi)\omega$ and $T = 1/f$

$$T = 2\pi\sqrt{\frac{I}{mgd}} \tag{33}$$

7 Damped Oscillations

- All these oscillations descibed so far however, are frictionless, whereas in the real world this is very rarely the case
- The decrease in amplitude caused by dissapatitve forces is called damping, which comes as a new force $F_x = -bv_x$ where $v_x = dx/dt$ is the velocity and b is the damping constant
- The net force on the body is then

$$\Sigma F_x = -kx - bv_x \tag{34}$$

  with Newton's second law for the system

$$-kx - bv_x = ma_x \quad or \quad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \tag{35}$$

- If the damping force is relatively small, the motion is described by

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi) \tag{36}$$

- The angular frequency of these damped oscillations is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{37}$$

- When $b = 2\sqrt{km}$ the system is said to be critically damped, the system will return to equilibrium without oscillating. When $b > 2\sqrt{km}$, the system is said to be overdamped, leading to very slow return to equilibrium without oscillating. The solution to this system then becomes.

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t} \tag{38}$$

When $b < 2\sqrt{km}$, the system is overdamed, leading to high oscillation with steadily decreasing amplitude

7.1 Energy in Damped Oscillations

- In damped oscillations, the damping force is non-conservative, so the mechanical energy will decrease continuously, eventually reaching 0. So start with the derivative of energy:

$$\frac{dE}{dt} = mv_x \frac{dv_x}{dt} + kv\frac{dx}{dt} \tag{39}$$

But $dv_x/dt = a_x$ and $dx/dt = v_x$ so

$$\frac{dE}{dt} = v_x(ma_x + kx) \tag{40}$$

Since $ma_x + kx = -bdx/dt = -bv_x$ so

$$\frac{dE}{dt} = v_x(-bv_x) = -bv_x^2 \tag{41}$$

8 Forced Oscillations and Resonance

- A damped oscillator will eventually stop moving, but its amplitude can be maintained if there is a force applied to it in a periodic way. This additional force is called a driving force.
- The amplitude of a forced oscillator can be found as a function of frequency

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \tag{42}$$

So from this we can see that A is at its maximum when $\omega_s = \sqrt{k/m}$