Chapter 15 Periodic Motion

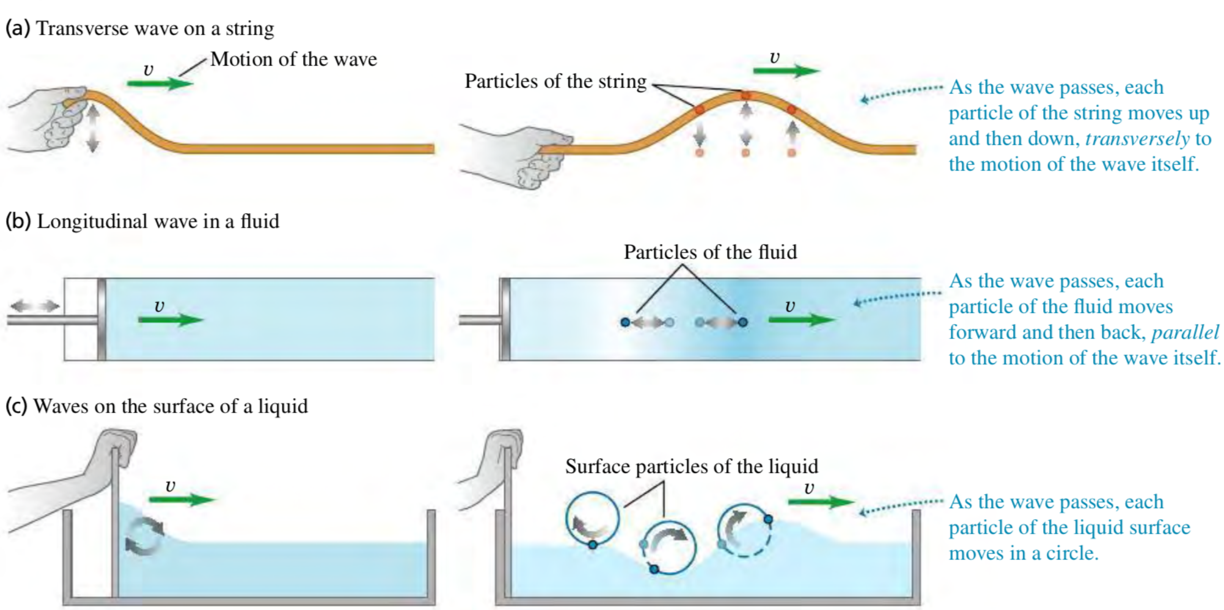
Apostolos Delis

June 6, 2019

Contents

1 Types of Mechanical Waves

- A mechanical wave is a disturbance that travels through some material or substance called the medium for the wave.
- There are three varieties of mechanical waves



- Waves that have displacements of the medium perpendicular to to their direction of travel are called transverse waves while waves that have displacement in the direction of motion are called longitudinal waves
- Waves transport energy, not matter from one region to another

2 Periodic Waves

2.1 Periodic Tranverse Waves

- Suppose you move a string up and down with simple harmonic motion. The wave that results is a symmetric sequence of crests and troughs. These waves are called sinusoidal waves.
- for a periodic wave, the shape of the string at any instant is a repeating patterm, where the length of that pattern is λ
- The speed of the wave is thus given by $v = \lambda/T$

2.2 Periodic Longitudinal Waves

- Longitudinal waves have periodic oscillation in the direction of motion, ex: pistons pressurizing water, where the regions of increased density are called compressions
- The fundamental equation $v = \lambda/T$ stil holds

3 Mathematical Description of a Wave

We call $y = y(x,t)$ the wave function. During wave motion a particle with equilibrium position x is displaced some distance y in the direction perpendicular to the x-axis.

3.1 Wave Function of Sinusoidal Wave

- Particles that lag behind one another but follow the same path are said to be phase shifted
- An example of a particle that moves from the farthest left point $(x = 0)$ will oscillate with SHM as follows: (Note that $\omega = 2\pi f$)
$$y(x = 0, t) = A\cos\omega t = A\cos 2\pi f t \tag{1}$$
- This can be generalized for a motion at x at the earlier time $t - x/v$
$$y(x,t) = A\cos\left[\omega\left(t - \frac{x}{v}\right)\right] \tag{2}$$
- This function can also be written in terms of period $T = 1/f$ and wavelength $\lambda = v/f = 2\pi v/w$
$$y(x,t) = A\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \tag{3}$$
- Note that $\omega = vk$
- It is often convenient to quantify the wave number $k = \frac{2\pi}{\lambda}$, then the wave equation can be rewritten as:
$$y(x,t) = A\cos(kx - \omega t) \tag{4}$$
- Note that if the wave is going in the opposite direction $(-x)$ then the whole equation becomes:
$$y(x,t) = A\cos\left[\omega\left(t + \frac{x}{v}\right)\right] = A\cos\left[2\pi\left(\frac{x}{\lambda} + \frac{t}{T}\right)\right] = A\cos(kx + \omega t) \tag{5}$$
- The quantity $(kx \pm \omega t)$ is called the phase. Wave speed is the speed with which a particle must move along the wave to keep alongside a point of a given phase. taking the derivative with respect to t yields
$$\frac{dx}{dt} = \frac{\omega}{k} \tag{6}$$

3.2 Particle Velocity and Acceleration in a Sinusoidal Wave

- The wave function can be used to get an expression for the transpose velocity of any particle in the transverse wave
- Given the wave function $y(x,t) = A\cos(kx - \omega t)$ then the derivative $v_y(x,t)$ is

$$v_y(x,y) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t) \tag{7}$$

- The acceleration of any particle is the second partial derivative of y(x,y) with respect to t:

$$a_y(x,y) = \frac{\partial^2 y(x,y)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t) \tag{8}$$

- The second partial derivative with respect to x tells us the curvature of the string

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x,t) \tag{9}$$

- Given the relationship that $\omega = vk$ we can see that

$$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2 \tag{10}$$

- then the wave equation can be written as a second order partial differential equation of the form:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} \tag{11}$$

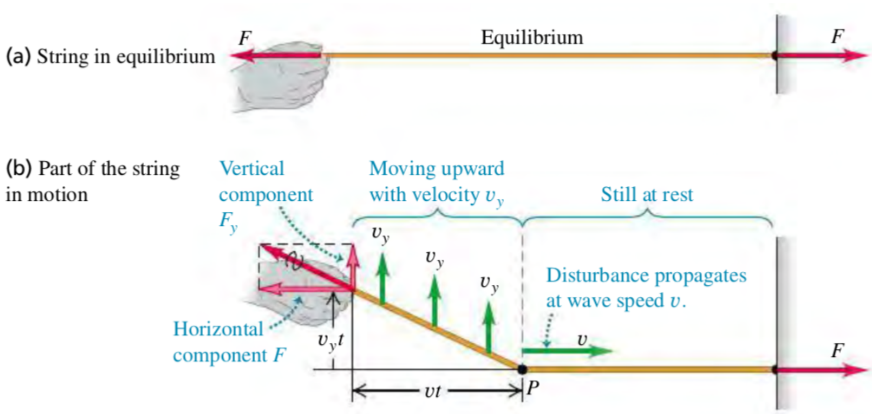
- For longitudinal waves, the wave function $y(x,t)$ still measures displacement of a particle of the medium from equilibrium. But the displacement this time is parallel to the x-axis

4 Speed of a Transverse Wave

What determines the speed of a transverse wave is the tension of the string and the mass per unit length. Increased tension leads to an increase in restoring forces which increases speed. While increasing the mass per unit makes the motion more sluggish and thus slowing down the speed.

4.1 Wave Speed of a String: First Method

- Consider a flexible string, with tensions F and a linear mass density of μ, when Starting at $t = 0$, if this was a point mass, then the end would move with constant acceleration. But here, the effect of F_y is to set more and more mass in motion.



- Since the $F_y t$ is the impulse which is equal to the transverse momentum, we have that.

$$F_y t = m v_y \tag{12}$$

- To derive an expression for the wave speed v, we note that

$$\frac{F_y}{F} = \frac{v_y t}{vt} \Rightarrow F = F\frac{v_y}{v} \tag{13}$$

and we have that transverse impulse equals $F_y t = F\frac{v_y}{v}t$

- Then the transverse momentum $= mv_y = (\mu vt)v_y = F\frac{v_y}{v}t$
- Then the wave speed v is given by

$$v = \sqrt{\frac{F}{\mu}} \tag{14}$$

Where F is tension of the string and μ was mass per unit length

- Note that the second method for wave speed can be found using Newton's second law, where $\sum \vec{F} = m\vec{a}$

4.2 The Speed of Mechanical Waves

It turns out that for many types of mechanical waves, the expression of the wave speed canbe given the same general form:

$$v = \sqrt{\frac{\text{Restoring force returning the system equilibrium}}{\text{Inertia resisting the return to equilibrium}}} \tag{15}$$

5 Energy In Wave Motion

- Every wave motion has energy associated with it. For example, a string transfers enery from what part to the next during propagation.
- When point a moves in the y-direction, the force F_y does *work* on a. The corresponding power P at a is thus

$$P(x,t) = F_y(x,t)v_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}\frac{\partial y(x,t)}{\partial t} \tag{16}$$

- For a sinusoidal wave we have that

$$y(x,t) = A\cos(kx - \omega t) \tag{17}$$

$$\frac{\partial y(x,t)}{\partial x} = -kA\sin(kx - \omega t) \tag{18}$$

$$\frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx - \omega t) \tag{19}$$

$$P(x,t) = Fk\omega A^2\sin^2(kx - \omega t) \tag{20}$$

- Using the relationships $\omega = vk$ and $v^2 = F/\mu$ you can express $P(x,t)$ also as:

$$P(x,t) = \sqrt{\mu F}\omega^2 A^2\sin^2(kx - \omega t) \tag{21}$$

- The maximum value of instantaneous power occurs when the $\sin^2 = 1$

$$P_{max} = \sqrt{\mu F}\omega^2 A^2 \tag{22}$$

- Note that the average power is hence given by

$$P_{av} = \frac{1}{2}\sqrt{\mu F}\omega^2 A^2 \tag{23}$$