MA1522: Linear Algebra for Computing

Tutorial 7

Revision

Inner/Dot Product

The inner(or dot) product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i .$$

The norm of a vector $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} = (u_i)$ is defined to be the square root of the inner product of $\mathbf{u}$ with itself, and is denoted as $\|\mathbf{u}\|$,

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

The distance between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u} = (u_i)$, $\mathbf{v} = (v_i)$ is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} = \|\mathbf{u} - \mathbf{v}\|.$$

We define the angle θ between two nonzero vectors, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ to be such that

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

Properties of Dot Product and Norm

Theorem

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors and $a, b, c \in \mathbb{R}$ be scalar.

(i) *(Symmetric)* $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(ii) *(Scalar multiplication)* $c\mathbf{u} \cdot \mathbf{v} = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$.

(iii) *(Distribution)* $\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{u} \cdot \mathbf{w}$.

(iv) *(Positive definite)* $\mathbf{u} \cdot \mathbf{u} \geq 0$ *with equality if and only if* $\mathbf{u} = \mathbf{0}$.

(v) $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$.

A vector $\mathbf{u}$ is a unit vector if $\|\mathbf{u}\| = 1$. A nonzero vector can be normalize by multiplying it by the reciprocal of its norm,

$$\mathbf{u} \longrightarrow \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Orthogonally

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$. If $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \subseteq \mathbb{R}^n$ of vectors is orthogonal if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for every $i \neq j$, that is, vectors in S are pairwise orthogonal.

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \subseteq \mathbb{R}^n$ of vectors is orthonormal if for all $i, j = 1, ..., k$,

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

That is, S is orthogonal, and all the vectors are unit vectors.

Theorem

An orthogonal set of nonzero vectors is linearly independent.

Theorem

Every orthonormal set is linearly independent.

Orthogonal and Orthonormal Basis

A basis S for a subspace V is an orthogonal (orthonormal) basis if it an orthogonal (orthonormal) set.

Theorem

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be an orthogonal basis for a subspace $V \subseteq \mathbb{R}^n$. Then for any $\mathbf{v} \in V$,

$$\mathbf{v} = \left(\frac{\mathbf{v}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 + \left(\frac{\mathbf{v}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right)\mathbf{u}_2 + \cdots + \left(\frac{\mathbf{v}\cdot\mathbf{u}_k}{\|\mathbf{u}_k\|^2}\right)\mathbf{u}_k$$

If further S is an orthonormal basis, then

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{u}_2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_k)\,\mathbf{u}_k.$$

$$\text{i.e. } S \text{ orthogonal, } [\mathbf{v}]_S = \begin{pmatrix} \frac{\mathbf{v}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2} \\ \frac{\mathbf{v}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2} \\ \vdots \\ \frac{\mathbf{v}\cdot\mathbf{u}_k}{\|\mathbf{u}_k\|^2} \end{pmatrix}, \; S \text{ orthonormal, } [\mathbf{v}]_S = \begin{pmatrix} \mathbf{v}\cdot\mathbf{u}_1 \\ \mathbf{v}\cdot\mathbf{u}_2 \\ \vdots \\ \mathbf{v}\cdot\mathbf{u}_k \end{pmatrix}.$$

Tutorial 7 Solutions

Question 1(a)

Let $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ be a linear equation. Express this linear system as $\mathbf{a} \cdot \mathbf{x} = b$ for some (column) vectors $\mathbf{a}$ and $\mathbf{x}$.

$$b = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Geometrically, the solutions $\mathbf{x}$ are vectors whose projection onto the line spanned by $\mathbf{a}$ is $\frac{b}{\|\mathbf{a}\|}$.

Question 1(b)

Find the solution set of the linear system

$$\begin{array}{ccccccccc} x_1 & + & 3x_2 & - & 2x_3 & & & = 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & = 0 \\ & & & + & 5x_3 & + & 10x_4 & = 0 \end{array}$$

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>> rref([1 3 -2 0;2 6 -5 -2;0 0 5 10])
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$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 0 & 0 & 5 & 10 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the solution set is

$$\left\{ s \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}.$$

Question 1(c)

Find a nonzero vector $\mathbf{v} \in \mathbb{R}^4$ such that $\mathbf{a}_1 \cdot \mathbf{v} = 0$, $\mathbf{a}_2 \cdot \mathbf{v} = 0$, and $\mathbf{a}_3 \cdot \mathbf{v} = 0$, where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 6 \\ -5 \\ -2 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 10 \end{pmatrix}.$$

From (a), we are solving for $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ such that

$$\left\{\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{v} & = & 0 \\ \mathbf{a}_2 \cdot \mathbf{v} & = & 0 \\ \mathbf{a}_3 \cdot \mathbf{v} & = & 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcrcrcrl} v_1 & + & 3v_2 & - & 2v_3 & & & = 0 \\ 2v_1 & + & 6v_2 & - & 5v_3 & - & 2v_4 & = 0 \\ & & & + & 5v_3 & + & 10v_4 & = 0 \end{array}\right.$$

We have solved the system in (b), may choose $s = 1$, $t = 0$.

Question 1 Remarks

This exercise demonstrates the fact that if $\mathbf{A}$ is a $m \times n$ matrix, then the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ consist of all the vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of $\mathbf{A}$.

Also, $\mathbf{v}$ is orthogonal to the set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ if and only if $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}^T \mathbf{v} = \mathbf{0}$.

Question 2(a)

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set. Suppose

$$\mathbf{x} = \mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3 \quad \text{and} \quad \mathbf{y} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Determine the value of $\mathbf{x} \cdot \mathbf{y}$.

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3) \cdot (2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3) = 2(\mathbf{v}_1 \cdot \mathbf{v}_1) + 6(\mathbf{v}_2 \cdot \mathbf{v}_2) - 2(\mathbf{v}_3 \cdot \mathbf{v}_3) = 2 + 6 - 2 = 6.$$

Now let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then S is an orthonormal basis for $V = \text{span}(S)$.

$$[\mathbf{x}]_S = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \quad [\mathbf{y}]_S = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \Rightarrow [\mathbf{x}]_S \cdot [\mathbf{y}]_S = 2 + 6 - 2 = 6 = \mathbf{x} \cdot \mathbf{y}.$$

Question 2(a)

Alternative solution.

$$\begin{aligned} \mathbf{x} &= \mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \mathbf{Q}[\mathbf{x}]_S \\ \mathbf{y} &= 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \mathbf{Q}[\mathbf{y}]_S. \end{aligned}$$

where $\mathbf{Q} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$. Then

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = (\mathbf{Q}[\mathbf{x}]_S)^T \mathbf{Q}[\mathbf{y}]_S = [\mathbf{x}]_S^T \mathbf{Q}^T \mathbf{Q}[\mathbf{y}]_S = [\mathbf{x}]_S^T [\mathbf{y}]_S = [\mathbf{x}]_S \cdot [\mathbf{y}]_S,$$

where we use the fact that $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$.

Question 2(b)

Determine the value of $||\mathbf{x}||$ and $||\mathbf{y}||$.

$$\begin{aligned} ||\mathbf{x}|| &= \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{(\mathbf{v}_1\cdot\mathbf{v}_1)+4(\mathbf{v}_2\cdot\mathbf{v}_2)+4(\mathbf{v}_3\cdot\mathbf{v}_3)} = \sqrt{1+4+4} = 3 \\ ||\mathbf{y}|| &= \sqrt{\mathbf{y}\cdot\mathbf{y}} = \sqrt{4(\mathbf{v}_1\cdot\mathbf{v}_1)+9(\mathbf{v}_2\cdot\mathbf{v}_2)+(\mathbf{v}_3\cdot\mathbf{v}_3)} = \sqrt{4+9+1} = \sqrt{14} \end{aligned}$$

Also,

$$\begin{aligned} ||[\mathbf{x}]_S|| &= \sqrt{1+4+4} = 3 = ||\mathbf{x}|| \\ ||[\mathbf{y}]_S|| &= \sqrt{4+9+1} = \sqrt{14} = ||\mathbf{y}|| \end{aligned}$$

Question 2(c)

Determine the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||||\mathbf{y}||} = \frac{6}{3\sqrt{14}} \Rightarrow \theta = \cos^{-1} \frac{2}{\sqrt{14}} = 57.69^\circ.$$

Observe that if α is the angle between $[\mathbf{x}]_S$ and $[\mathbf{y}]_S$ in $\mathbb{R}^3$, then

$$\cos(\alpha) = \frac{\mathbf{x}]_S \cdot [\mathbf{y}]_S}{||[\mathbf{x}]_S||||[\mathbf{y}]_S||} = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||||\mathbf{y}||} = \cos(\theta) \ \Rightarrow \ \theta = \alpha,$$

that is, the angle between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is the same as the angle between $[\mathbf{x}]_S$ and $[\mathbf{y}]_S$ in $\mathbb{R}^3$.

Question 2 Remarks

This exercise demonstrates that if $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ is an orthonormal basis for a subspace V in $\mathbb{R}^n$, then for any vectors $\mathbf{u}, \mathbf{v} \in V$, we may compute the inner product, norm, angle, distance between $\mathbf{u}, \mathbf{v}$ using $[\mathbf{u}]_S$ and $[\mathbf{v}]_S$.

Note that this only works when S is an orthonormal basis.

Question 3(a)

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$. Compute $\mathbf{v}_1 \cdot \mathbf{v}_1$, $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_2 \cdot \mathbf{v}_1$ and $\mathbf{v}_2 \cdot \mathbf{v}_2$.

```
>> v1=[1;2;-1];v2=[1;0;1]; v1'*v1,v1'*v2,v2'*v2
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$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 1 + 4 + 1 = 6, \qquad \mathbf{v}_1 \cdot \mathbf{v}_2 = 1 + 0 - 1 = 0,$$

$$\mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \qquad \mathbf{v}_2 \cdot \mathbf{v}_2 = 1 + 0 + 1 = 2.$$

Question 3(b)

Compute $\mathbf{V}^T\mathbf{V}$. What does the entries of $\mathbf{V}^T\mathbf{V}$ represent?

$\mathbf{V}^T\mathbf{V} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$.

Since $\mathbf{V} = (\mathbf{v}_1 \ \mathbf{v}_2)$, $\mathbf{V}^T = \begin{pmatrix} \mathbf{v}^T \\ \mathbf{v}_2^T \end{pmatrix}$. Hence

$$\mathbf{V}^T\mathbf{V} = \begin{pmatrix} \mathbf{v}_1^T\mathbf{v}_1 & \mathbf{v}_1^T\mathbf{v}_2 \\ \mathbf{v}_2^T\mathbf{v}_1 & \mathbf{v}_2^T\mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{pmatrix}.$$

The (i, j)-entry of $\mathbf{V}^T\mathbf{V}$ is $\mathbf{v}_i \cdot \mathbf{v}_j$.

Question 4(a) and (b)

Let W be a subspace of $\mathbb{R}^n$. The *orthogonal complement* of W, denoted as $W^\perp$, is defined to be

$$W^\perp := \{ \ \mathbf{v} \in \mathbb{R}^n \ \middle| \ \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W \ \}.$$

Let $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}$, and $\mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, and $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent and orthogonal.

```
>> w1=[1;1;1;1;1];w2=[1;2;-1;-2;0];w3=[1;-1;1;-1;0];A=[w1 w2 w3]; A'*A
```

Let $\mathbf{A} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{pmatrix}$. From question 3, to show that S is orthogonal, suffice to show that $\mathbf{A}^T\mathbf{A}$ is a diagonal matrix. $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ which shows that S is an orthogonal set of nonzero vectors.

An orthogonal set of nonzero vectors is linearly independent. Therefore S is linear independent.

Question 4(c)

Show that $W^\perp$ is a subspace of $\mathbb{R}^5$ by showing that it is a span of a set. What is the dimension?

By Question 1, $W^\perp$ is the nullspace of $\mathbf{A}^T$, and hence a subspace.

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>> rref(A')
```

$$\mathbf{A}^T \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 & -1/4 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 2 & 3/4 \end{pmatrix}$$

The nullspace of $\mathbf{A}^T$ is spanned by $\left\{ \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -3 \\ 0 \\ 4 \end{pmatrix} \right\}$. This shows that $W^\perp$ is a subspace of $\mathbb{R}^5$ of dimension 2.

Question 4(d)

Obtain an orthonormal set T by normalizing $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

The diagonal entries of $\mathbf{A}^T\mathbf{A}$ are the norm squared of $\mathbf{w}_i$, hence, dividing by the square root of the diagonals, we have

$$T = \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

Question 4(e)

Let $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. Find the projection of $\mathbf{v}$ onto W.

The projection is

```
>> v=[2;0;1;1;-1];vproj=(v'*w1)/(w1'*w1)*w1+(v'*w2)/(w2'*w2)*w2+(v'*w3)/(w3'*w3)*w3
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$$\frac{\mathbf{v}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{v}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\mathbf{w}_2 + \frac{\mathbf{v}\cdot\mathbf{w}_3}{\mathbf{w}_3\cdot\mathbf{w}_3}\mathbf{w}_3 = \frac{1}{10}\begin{pmatrix} 10 \\ -1 \\ 12 \\ 3 \\ 6 \end{pmatrix}.$$

Question 4(f)

Let $\mathbf{v}_W$ be the projection of $\mathbf{v}$ onto W. Show that $\mathbf{v} - \mathbf{v}_W$ is in $W^\perp$.

```
>> A'*(v-vproj)
```

This shows that $(\mathbf{v} - \mathbf{v}_W)$ is in the nullspace of $\mathbf{A}^T$, which is $W^\perp$.

This exercise demonstrated the fact that every vector $\mathbf{v}$ in $\mathbb{R}^5$ can be written as $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_W^\perp$, for some $\mathbf{v}_W$ in W and $\mathbf{v}_W^\perp$ in $W^\perp$. In other words, $W + W^\perp = \mathbb{R}^5$. See Extra Problems Question 3.

Question 5(a)

Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ where

$\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{u_3} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$, and $\mathbf{u_4} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$. Check that S is an orthogonal basis for $\mathbb{R}^4$.

```
>> u1=[1;2;2;-1];u2=[1;1;-1;1];u3=[-1;1;-1;-1];u4=[-2;1;1;2];U=[u1 u2 u3 u4]; U'*U
```

Hence, S is an orthogonal set. Since it is an orthogonal set of nonzero vectors, it is linearly independent. Moreover it contains 4 vectors so it is a basis of $\mathbb{R}^4$. Alternatively, since the product $\mathbf{U}^T\mathbf{U}$ is invertible, $\mathbf{U}$ is invertible. Hence the columns form a basis for $\mathbb{R}^4$.

Question 5(b)

Is it possible to find a nonzero vector $\mathbf{w}$ in $\mathbb{R}^4$ such that $S \cup \{\mathbf{w}\}$ is an orthogonal set?

No. Recall that an orthogonal set of nonzero vectors is linearly independent. Recall also that if a set contains $k > n$ vectors in $\mathbb{R}^n$, it must be linearly dependent. So, if $\mathbf{w}$ exists, then $S \cup \{\mathbf{w}\}$ is an orthogonal set of nonzero vectors, and thus is linearly independent. This is a contradiction since now there exists 5 linearly independent vectors in $\mathbb{R}^4$.

Question 5(c)

Obtain an orthonormal set T by normalizing $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$.

From the answers in (a), we have

$$T = \left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Question 5(d)

Let $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$. Find $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

Recall that S is an orthogonal basis, and T is an orthonormal basis obtained from normalizing S. So,

$$[\mathbf{v}]_S = \begin{pmatrix} \frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1} \\ \frac{\mathbf{v}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2} \\ \frac{\mathbf{v}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3} \\ \frac{\mathbf{v}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4} \end{pmatrix} = \begin{pmatrix} 3/10 \\ 1/2 \\ -1 \\ 9/10 \end{pmatrix}.$$

Write $T = \{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3', \mathbf{u}_4'\}$,

$$[\mathbf{v}]_T = \begin{pmatrix} \mathbf{v}\cdot\mathbf{u}_1' \\ \mathbf{v}\cdot\mathbf{u}_2' \\ \mathbf{v}\cdot\mathbf{u}_3' \\ \mathbf{v}\cdot\mathbf{u}_4' \end{pmatrix} = \begin{pmatrix} 3/\sqrt{10} \\ 1 \\ -2 \\ 9/\sqrt{10} \end{pmatrix}.$$

Question 5(e)

Suppose $\mathbf{w}$ is a vector in $\mathbb{R}^4$ such that $[\mathbf{w}]_S = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Find $[\mathbf{w}]_T$.

Let us compute $\mathbf{P}$, the transition matrix from S to T.

$$\mathbf{P} = \left([\mathbf{u}_1]_T \quad [\mathbf{u}_2]_T \quad [\mathbf{u}_3]_T \quad [\mathbf{u}_4]_T\right) = \begin{pmatrix} \sqrt{10} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{10} \end{pmatrix}.$$

Hence,

$$[\mathbf{w}]_T = \mathbf{P}[\mathbf{w}]_S = \begin{pmatrix} \sqrt{10} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{10} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{10} \\ 4 \\ 2 \\ \sqrt{10} \end{pmatrix}.$$