NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 6**

1. (a) Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$. Show that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ forms a basis for $\mathbb{R}^3$.

Solution: We have

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ -1 & 1 & 3 \end{vmatrix} = 7 \neq 0.$$

Thus $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathbb{R}^3$.

(b) Suppose $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find the coordinate vector of $\mathbf{w}$ relative to S.

Solution: We have

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 2 & -1 & 1 \\ -1 & 1 & 3 & 1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/7 \\ 0 & 0 & 1 & 5/7 \end{array}\right) \quad \Rightarrow \quad [\mathbf{w}]_S = \begin{pmatrix} 1 \\ -1/7 \\ 5/7 \end{pmatrix}.$$

(c) Let $T = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be another basis for $\mathbb{R}^3$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$. Find the transition matrix from T to S.

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 2 & 2 & -1 & 5 & 3 & 2 \\ -1 & 1 & 3 & 4 & 7 & 4 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right)$$

So, the transition matrix $\mathbf{P}$ from T to S is $\mathbf{P} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

(d) Find the transition matrix from S to T.

Solution:

$$\mathbf{Q} = \mathbf{P}^{-1} = \begin{pmatrix} 3/4 & 1/2 & -3/4 \\ -1/2 & 0 & 1/2 \\ -1/8 & -1/4 & 5/8 \end{pmatrix}.$$

(e) Use the vector $\mathbf{w}$ in Part (b). Find the coordinate vector of $\mathbf{w}$ relative to T.

Solution:

$$[\mathbf{w}]_T = \mathbf{Q}[\mathbf{w}]_S = \begin{pmatrix} 3/4 & 1/2 & -3/4 \\ -1/2 & 0 & 1/2 \\ -1/8 & -1/4 & 5/8 \end{pmatrix} \begin{pmatrix} 1 \\ -1/7 \\ 5/7 \end{pmatrix} = \begin{pmatrix} 1/7 \\ -1/7 \\ 5/14 \end{pmatrix}.$$

2. Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for a subspace V. Define $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3,\ \mathbf{v}_2 = \mathbf{u}_2 + \mathbf{u}_3 \text{ and } \mathbf{v}_3 = \mathbf{u}_2 - \mathbf{u}_3.$$

(a) Show that T is a basis for V.

Solution: By construction, $\mathbf{v}_i \in V$ for $i = 1, 2, 3$, and hence, $\mathrm{span}(T) \subseteq V$. Next, $|T| = 3 = \dim(V)$. Finally, suppose

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= c_1(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) + c_2(\mathbf{u}_2 + \mathbf{u}_3) + c_3(\mathbf{u}_2 - \mathbf{u}_3) \\ &= c_1\mathbf{u}_1 + (c_1 + c_2 + c_3)\mathbf{u}_2 + (c_1 + c_2 - c_3)\mathbf{u}_3 \end{aligned}$$

Then since S is linearly independent,

$$\begin{cases} c_1 & & & & & = 0 \\ c_1 & + & c_2 & + & c_3 & = 0 \\ c_1 & + & c_2 & - & c_3 & = 0 \end{cases}$$

which has only the trivial solution $c_1 = c_2 = c_3 = 0$. Hence, T is linearly independent. Thus, T is a basis.

(b) Find the transition matrix from S to T.

Solution: Observe that by construction,

$$[\mathbf{v}_1]_S = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad [\mathbf{v}_2]_S = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad [\mathbf{v}_3]_S = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Hence, the transition matrix $\mathbf{P}$ from T to S is

$$\mathbf{P} = \begin{pmatrix} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & [\mathbf{v}_3]_S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Hence, the transition matrix $\mathbf{Q}$ from S to T is

$$\mathbf{Q} = \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{pmatrix}.$$

3. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Is $\mathbf{b}$ in the column space of $\mathbf{A}$? If it is, express it as a linear combination of the columns of $\mathbf{A}$.

Solution:
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$
Thus $\mathbf{b}$ is not a linear combination of the columns of $\mathbf{A}$.

(b) Let $\mathbf{A} = \begin{pmatrix} 1 & 9 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5, 1, -1 \end{pmatrix}$. Is $\mathbf{b}$ in the row space of $\mathbf{A}$? If it is, express it as a linear combination of the rows of $\mathbf{A}$.

Solution: Note that $\mathbf{b}$ is in the row space of $\mathbf{A}$ if and only if $\mathbf{b}^T$ is in the column space of $\mathbf{A}^T$. Hence we are solving for
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right).$$
We get $\mathbf{b} = (5, 1, -1) = (1, 9, 1) - 3(-1, 3, 1) + (1, 1, 1)$.

(c) Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}$. Is the row space and column space of $\mathbf{A}$ the whole $\mathbb{R}^4$?

Solution:
$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Thus the column space of $\mathbf{A}$ is the whole $\mathbb{R}^4$. Since $\mathbf{A}$ is invertible if and only if $\mathbf{A}^T$ is, the row space must also be the whole $\mathbb{R}^4$.

4. For each of the following matrices $\mathbf{A}$,

(i) Find a basis for the row space of $\mathbf{A}$.

(ii) Find a basis for the column space of $\mathbf{A}$.

(iii) Find a basis for the nullspace of $\mathbf{A}$.

(iv) Hence determine rank($\mathbf{A}$), nullity($\mathbf{A}$) and verify the dimension theorem for matrices.

(v) Is $\mathbf{A}$ full rank?

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 1 & -4 & -1 & -9 \\ -1 & 0 & -3 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$

Solution:

$$\mathbf{A} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) A basis for the row space is $\{(1, 0, 3, -1), (0, 1, 1, 2)\}$.

(ii) A basis for the column space is $\{(1, 1, -1, 2, 0)^T, (2, -4, 0, 1, 1,)^T\}$.

(iii) A basis for the nullspace is $\{(-3, -1, 1, 0)^T, (1, -2, 0, 1)^T\}$.

(iv) $\text{rank}(\mathbf{A}) = 2$, $\text{nullity}(\mathbf{A}) = 2$. Since $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 2 + 2 = 4$ which is the number of columns of $\mathbf{A}$, we have verified the dimension theorem for matrices.

(v) $\text{rank}(\mathbf{A}) = 2 < \min\{4, 5\}$. $\mathbf{A}$ is not full rank.

(b) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{pmatrix}.$

Solution:

$$\mathbf{A} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) A basis for the row space is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

(ii) A basis for the column space is $\{(1, 2, 3, 2, 1)^T, (3, 1, -5, -2, 1)^T, (7, 8, -1, 2, 5)^T\}$.

(iii) The basis for the nullspace is the empty set.

(iv) $\text{rank}(\mathbf{A}) = 3$, $\text{nullity}(\mathbf{A}) = 0$. Since $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 3 + 0 = 3$ which is the number of columns of $\mathbf{A}$, we have verified the dimension theorem for matrices.

(v) $\text{rank}(\mathbf{A}) = 3 = \min\{3, 5\}$. $\mathbf{A}$ is full rank.

5. Let W be a subspace of $\mathbb{R}^5$ spanned by the following vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 \begin{pmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 5 \\ 15 \\ 10 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} 2 \\ 1 \\ 15 \\ 8 \\ 6 \end{pmatrix}.$$

(a) Find a basis for W.

Solution:

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 1 & 15 & 8 & 6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 6 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ forms a basis for W.

(b) What is $\dim(W)$?

Solution: From (a), $\dim(W) = 3$

(c) Extend the basis W found in (a) to a basis for $\mathbb{R}^5$.

Solution: From (a), $\left\{ \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ form a basis for $\mathbb{R}^5$.

6. Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 5 \\ 12 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \\ 4 \end{pmatrix} \right\}$ and $V = \text{span}(S)$. Find a subset $S' \subseteq S$ such that S' forms a basis for V.

Solution:

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -1 & 3 & 1 & -1 \\ 1 & 0 & 5 & 2 & 1 \\ 3 & 1 & 12 & 5 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 5 & 2 & 1 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $S' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Extra problems

1. Suppose $\mathbf{A}$ and $\mathbf{B}$ are two matrices such that $\mathbf{AB} = \mathbf{0}$. Show that the column space of $\mathbf{B}$ is contained in the nullspace of $\mathbf{A}$.

   Solution: Write $\mathbf{B} = \begin{pmatrix}\mathbf{b}_1 \ \mathbf{b}_2 \cdots \mathbf{b}_n\end{pmatrix}$, where $\mathbf{b}_i$ is the i-th column of $\mathbf{B}$. Then

   $$\mathbf{0} = \mathbf{AB} = \mathbf{A}\begin{pmatrix}\mathbf{b}_1 \ \mathbf{b}_2 \cdots \mathbf{b}_n\end{pmatrix} = \begin{pmatrix}\mathbf{Ab}_1 \ \mathbf{Ab}_2 \cdots \mathbf{Ab}_n\end{pmatrix}$$

   By comparing the columns, we conclude that $\mathbf{Ab}_i = \mathbf{0}$ for all $i = 1, ..., n$. Hence, $\mathbf{b}_i \in \mathrm{Null}(\mathbf{A})$ for all $i = 1, ..., n$, that is, $\{\mathbf{b}_1, ..., \mathbf{b}_n\} \subseteq \mathrm{Null}(\mathbf{A})$. Since the nullspace of $\mathbf{A}$ is a subspace space and is thus closed under linear combinations, this shows that $\mathrm{Col}(\mathbf{B}) = \mathrm{span}\{\mathbf{b}_1, ..., \mathbf{b}_n\} \subseteq \mathrm{Null}(\mathbf{A})$.

2. Let $\mathbf{A}$ be a $n \times m$ matrix and $\mathbf{P}$ an $n \times n$ matrix.

   (a) If $\mathbf{P}$ is invertible, show that $\mathrm{rank}(\mathbf{PA}) = \mathrm{rank}(\mathbf{A})$.

   Solution: Since $\mathbf{P}$ is invertible, we can write $\mathbf{P} = \mathbf{E}_k \cdots \mathbf{E}_1$, for some elementary matrices $\mathbf{E}_1, ..., \mathbf{E}_k$. This means that $\mathbf{PA} = \mathbf{E}_k \cdots \mathbf{E}_1\mathbf{A}$, which shows that $\mathbf{PA}$ is row equivalent to $\mathbf{A}$, and therefore they have the same row space, $\mathrm{Row}(\mathbf{PA}) = \mathrm{Row}(\mathbf{A})$. Thus,

   $$\mathrm{rank}(\mathbf{PA}) = \dim(\mathrm{Row}(\mathbf{PA})) = \dim(\mathrm{Row}(\mathbf{A})) = \mathrm{rank}(\mathbf{A})$$

   <u>Alternative Solution.</u> It is a fact that for an a by b matrix $\mathbf{X}$ and a b by c matrix $\mathbf{Y}$, we have

   $$\mathrm{rank}(\mathbf{XY}) \leq \mathrm{rank}(\mathbf{Y}). \qquad (*)$$

   Let $\mathbf{B} = \mathbf{PA}$. Since $\mathbf{P}$ is invertible, we have $\mathbf{P}^{-1}\mathbf{B} = \mathbf{A}$. Using the above fact (*), we get

   $$\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{P}^{-1}\mathbf{B}) \leq \mathrm{rank}(\mathbf{B}) = \mathrm{rank}(\mathbf{PA}) \leq \mathrm{rank}(\mathbf{A}).$$

   Hence all the inequalities are equalities and we get $\mathrm{rank}(\mathbf{PA}) \leq \mathrm{rank}(\mathbf{A})$.

   (b) Given an example such that $\mathrm{rank}(\mathbf{PA}) < \mathrm{rank}(\mathbf{A})$.

   Solution: Let $\mathbf{A} = \mathbf{I}_n$ the $n \times n$ identity matrix and $\mathbf{P} = \mathbf{0}_n$ the $n \times n$ zero matrix, for some $n \geq 1$. Then

   $$\mathrm{rank}(\mathbf{PA}) = 0 < \mathrm{rank}(\mathbf{A}) = n.$$

   (c) If $\mathrm{rank}(\mathbf{PA}) = \mathrm{rank}(\mathbf{A})$. Can we conclude that $\mathbf{P}$ is invertible? Justify your answer.

   Solution: No. For example, let $\mathbf{P} = \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\mathbf{PA} = \mathbf{A}$ and so $\mathrm{rank}(\mathbf{PA}) = \mathrm{rank}(\mathbf{A})$.

3. Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$ be row equivalent $m \times n$ matrices, where $\mathbf{a}_i$ and $\mathbf{b}_i$ are the i-th column of $\mathbf{A}$ and $\mathbf{B}$, respectively, for $i = 1, ..., n$. Show that for any $c_1, c_2, ..., c_n \in \mathbb{R}$,

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

if and only if

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}.$$

Solution: Since $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, we can write $\mathbf{B} = \mathbf{E}_k \cdots \mathbf{E}_1\mathbf{A} = \mathbf{PA}$, where $\mathbf{P} = \mathbf{E}_k \cdots \mathbf{E}_1$, for some elementary matrices $\mathbf{E}_1, ..., \mathbf{E}_k$. Then $\mathbf{P}$ is invertible. Moreover,

$$\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix} = \mathbf{B} = \mathbf{PA} = \mathbf{P}\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} = \begin{pmatrix} \mathbf{Pa}_1 & \mathbf{Pa}_2 & \cdots & \mathbf{Pa}_n \end{pmatrix},$$

which shows that $\mathbf{b}_i = \mathbf{Pa}_i$ for $i = 1, 2, ..., n$. Since $\mathbf{P}$ is invertible, we have $\mathbf{a}_i = \mathbf{P}^{-1}\mathbf{b}_i$ for $i = 1, 2, ..., n$. We set $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$. Then

$$\begin{aligned} c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n &= \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{Ac}, \\ c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n &= \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{Bc} = \mathbf{PAc}. \end{aligned}$$

We have

$$\begin{aligned} & c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0} \\ \Leftrightarrow\quad & \mathbf{Ac} = \mathbf{0} \\ \Leftrightarrow\quad & \mathbf{PAc} = \mathbf{P0} \text{ (because } \mathbf{P} \text{ is invertible.)} \\ \Leftrightarrow\quad & \mathbf{Bc} = \mathbf{0} \\ \Leftrightarrow\quad & c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}. \end{aligned}$$

4. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{u}$ be a vector in V and let c be a scalar. Prove the following:

   (a) $[\mathbf{u} + \mathbf{v}]_S = [\mathbf{u}]_S + [\mathbf{v}]_S$.

Solution: We first write $\mathbf{u}$ and $\mathbf{v}$ in terms of the basis vectors, say

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n \text{ and } \mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots + d_n\mathbf{v}_n.$$

Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \ldots + (c_n + d_n)\mathbf{v}_n$$

which implies

$$[\mathbf{u} + \mathbf{v}]_S = \begin{pmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \vdots \\ c_n + d_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = [\mathbf{u}]_S + [\mathbf{v}]_S .$$

(b) $[c\mathbf{u}]_S = c\,[\mathbf{u}]_S$.

Solution: Similarly,

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \ldots + (cc_n)\mathbf{v}_n \Rightarrow [c\mathbf{u}]_S = \begin{pmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_n \end{pmatrix} = c\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c\,[\mathbf{u}]_S .$$

(c) Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are vectors in V. Note that for each $i = 1, 2, \ldots, k$, $[\mathbf{u}_i]_S$ is a vector in $\mathbb{R}^n$. By induction and using (a) and (b), it follows that if $c_1, c_2, \ldots, c_k \in \mathbb{R}$, then

$$[c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k]_S = c_1\,[\mathbf{u}_1]_S + c_2\,[\mathbf{u}_2]_S + \ldots + c_k\,[\mathbf{u}_k]_S .$$

Prove that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent in V if and only if $\{[\mathbf{u}_1]_S, [\mathbf{u}_2]_S, \ldots, [\mathbf{u}_k]_S\}$ is linearly independent in $\mathbb{R}^n$.

Solution: Assume that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent in V. Consider the equation $c_1\,[\mathbf{u}_1]_S + \ldots + c_k\,[\mathbf{u}_k]_S = \mathbf{0}$ which is a vector equation in $\mathbb{R}^n$. By part (c), the equation above can be rewritten as $[c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k]_S = \mathbf{0}$. So the coordinates of $c_1\mathbf{u}_1 + \ldots + c_k\mathbf{u}_k$ with respect to the basis S are all zero, that is, $c_1\mathbf{u}_1 + \ldots + c_k\mathbf{u}_k = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_n = \mathbf{0}$. As $\{\mathbf{u}_1, \ldots \mathbf{u}_k\}$ is linearly independent, the equation above implies $c_1 = c_2 = \ldots = c_k = 0$, so $\{(\mathbf{u}_1)_S, \ldots, (\mathbf{u}_k)_S\}$ is linearly independent in $\mathbb{R}^n$.

Conversely, assume $\{[\mathbf{u}_1]_S, \ldots, [\mathbf{u}_k]_S\}$ is a linearly independent set in $\mathbb{R}^n$. Consider the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k = \mathbf{0}$$

which implies

$$[c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k]_S = [\mathbf{0}]_S = \mathbf{0},$$

Using the result in part (c), we have

$$c_1 \left[\mathbf{u}_1\right]_S + \ldots + c_k \left[\mathbf{u}_k\right]_S = \mathbf{0}$$

and this implies that $c_1 = c_2 = \ldots = c_k = 0$ because $\{\left[\mathbf{u}_1\right]_S, \ldots, \left[\mathbf{u}_k\right]_S\}$ is linearly independent. Thus we have shown that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent set in V.