NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 10**

1. A population of ants is put into a maze with 3 compartments labeled a, b, and c. If the ant is in compartment a, an hour later, there is a 20% chance it will go to compartment b, and a 40% change it will go to compartment c. If it is in compartment b, an hour later, there is a 10% chance it will go to compartment a, and a 30% chance it will go to compartment c. If it is in compartment c, an hour later, there is a 50% chance it will go to compartment a, and a 20% chance it will go to compartment b. Suppose 100 ants has been placed in compartment a.

   (a) Find the transition probability matrix $\mathbf{A}$. Show that it is a stochastic matrix.

   Solution: $\begin{pmatrix} 0.4 & 0.1 & 0.5 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$. In fact, it is a doubly stochastic matrix, that is, the sum of the rows are also equal to 1.

   (b) By diagonalizing $\mathbf{A}$, find the number of ants in each compartment after 3 hours.

   Solution:

$$\begin{aligned} \begin{vmatrix} x-0.4 & -0.1 & -0.5 \\ -0.2 & x-0.6 & -0.2 \\ -0.4 & -0.3 & x-0.3 \end{vmatrix} &= x^3 - 1.3x + 0.26x + 0.04 \\ &= (x-1)(x+0.1)(x-0.4). \end{aligned}$$

   The eigenvalues are $\lambda = 1$, $\lambda = -0.1$, $\lambda = 0.4$.

   - Eigenspace E_1: $\begin{pmatrix} 1-0.4 & -0.1 & -0.5 \\ -0.2 & 1-0.6 & -0.2 \\ -0.4 & -0.3 & 1-0.3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$
     $\Rightarrow E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.
   - Eigenspace $E_{-0.1}$: $\begin{pmatrix} -0.1-0.4 & -0.1 & -0.5 \\ -0.2 & -0.1-0.6 & -0.2 \\ -0.4 & -0.3 & -0.1-0.3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
     $\Rightarrow E_{-0.1} = \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

- Eigenspace $E_{0.4}$: $\begin{pmatrix} 0.4-0.4 & -0.1 & -0.5 \\ -0.2 & 0.4-0.6 & -0.2 \\ -0.4 & -0.3 & 0.4-0.3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

  $\Rightarrow E_{0.4} = \text{span}\left\{ \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} \right\}.$

Hence $\mathbf{A} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0.4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$. Then

$$\begin{aligned} \mathbf{x}_3 &= \mathbf{A}^3\mathbf{x}_0 = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.1^3 & 0 \\ 0 & 0 & 0.4^3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 35 \\ 31.2 \\ 33.8 \end{pmatrix}. \end{aligned}$$

(c) **(MATLAB)** We can use MATLAB to diagonalize the matrix $\mathbf{A}$. Type

```
>> [P D]=eig(sym(A))
```

The matrix $\mathbf{P}$ will be an invertible matrix, and $\mathbf{D}$ will be a diagonal matrix. Compare the answer with what you have obtained in (b).

Solution: `>> A=[0.4 0.1 0.5; 0.2 0.6 0.2;0.4 0.3 0.3];`

```
>> [P D]=eig(A)
```

$$\mathbf{P} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/10 & 0 \\ 0 & 0 & 2/5 \end{pmatrix}.$$

MATLAB returns the same $\mathbf{P}$ and $\mathbf{D}$ as computed in (b)

(d) In the long run (assuming no ants died), where will the majority of the ants be?

Solution: As $n \to \infty$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & (-0.1)^n & 0 \\ 0 & 0 & 0.4^n \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So in the long run,

$$\begin{aligned} \mathbf{x}_\infty &= \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 33.33 \\ 33.33 \\ 33.33 \end{pmatrix}. \end{aligned}$$

(e) Suppose initially the numbers of ants in compartments a, b and c are α, β, and γ respectively. What is the population distribution in the long run (assuming no ants died)?

Solution:

$$\begin{aligned} &\begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} \alpha+\beta+\gamma \\ \alpha+\beta+\gamma \\ \alpha+\beta+\gamma \end{pmatrix}. \end{aligned}$$

This is always an equilibrium vector if $a + b + c \neq 0$.

2. By diagonalizing $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, find a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$.

Solution: The matrix $\mathbf{A}$ is a triangular matrix, so its eigenvalues are 1 and 4 with algebraic multiplicities $r_1 = 1$ and $r_4 = 2$.

- Eigenspace E_4: $\begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
  $\Rightarrow E_4 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.
  We have $\dim(E_4) = 2 = r_4$. Thus $\mathbf{A}$ is diagonalizable.
- Eigenspace E_1: $\begin{pmatrix} 0 & 0 & -3 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_1 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}.$

We conclude that $\mathbf{A} = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}^{-1}.$

Consider any of the 8 choices of $\mathbf{C} = \begin{pmatrix}\pm 2 & 0 & 0\\0 & \pm 2 & 0\\0 & 0 & 1\end{pmatrix}$. Then $\mathbf{C}^2 = \begin{pmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 1\end{pmatrix}$.

Therefore any choice of $\mathbf{B} = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}\pm 2 & 0 & 0\\0 & \pm 2 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}^{-1}$ will work.

3. For each of the following symmetric matrices $\mathbf{A}$, find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$.

   (a) $\mathbf{A} = \begin{pmatrix}3 & 1\\1 & 3\end{pmatrix}$.

   Solution: $\mathbf{P} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix}2 & 0\\0 & 4\end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix}2 & 2 & -2\\2 & -1 & 4\\-2 & 4 & -1\end{pmatrix}$.

   Solution: $\mathbf{P} = \begin{pmatrix}\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}}\\-\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}}\\\frac{2}{3} & 0 & \frac{5}{3\sqrt{5}}\end{pmatrix}$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix}-6 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{pmatrix}$.

4. (**MATLAB**) Let $\mathbf{A} = \begin{pmatrix}1 & -2 & 0 & 0\\-2 & 1 & 0 & 0\\0 & 0 & 1 & -2\\0 & 0 & -2 & 1\end{pmatrix}$.

   (a) Find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$, and compute $\mathbf{P}^T\mathbf{A}\mathbf{P}$.

   Solution: $\mathbf{P} = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0\\\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0\\0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{pmatrix}$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix}-1 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & 3\end{pmatrix}$.

(b) We will use MATLAB to orthogonally diagonalize $\mathbf{A}$. Type

```
>> A=[1 -2 0 0;-2 1 0 0;0 0 1 -2;0 0 -2 1];

>> [P D]=eig(A);

>> sym(P), sym(D)
```

Compare the result with your answer in (a).

Solution:

```
ans =

[-2^(1/2)/2,          0,          0, -2^(1/2)/2]
[-2^(1/2)/2,          0,          0,  2^(1/2)/2]
[         0, -2^(1/2)/2, -2^(1/2)/2,          0]
[         0, -2^(1/2)/2,  2^(1/2)/2,          0]


ans =

[-1,  0, 0, 0]
[ 0, -1, 0, 0]
[ 0,  0, 3, 0]
[ 0,  0, 0, 3]
```

The code results an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalize $\mathbf{A}$, and the diagonal matrix $\mathbf{D} = \mathbf{P}^T\mathbf{AP}$.

5. Find the SVD of the following matrices $\mathbf{A}$.

(a) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}$.

Solution:

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}, \quad \det(x\mathbf{I} - \mathbf{A}^T\mathbf{A}) = (x-9)(x-25).$$

The singular values are $\sigma_1 = 5 \geq \sigma_2 = 3$. Let $\mathbf{\Sigma} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$.

$$25\mathbf{I} - \mathbf{A}^T\mathbf{A} \xrightarrow{RREF} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \ \mathbf{u}_1 = \frac{1}{5}\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$9\mathbf{I} - \mathbf{A}^T\mathbf{A} \xrightarrow{RREF} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \ \mathbf{u}_2 = \frac{1}{3}\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 1/(3\sqrt{2}) \\ -1/(3\sqrt{2}) \\ 4/(3\sqrt{2}) \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/(3\sqrt{2}) & -1/(3\sqrt{2}) & 4/(3\sqrt{2}) \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} \Rightarrow \mathbf{u}_3 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}.$$

So, let $\mathbf{V} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} 1/\sqrt{2} & \sqrt{2}/6 & -2/3 \\ 1/\sqrt{2} & -\sqrt{2}/6 & 2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix}$. Hence,

$$\mathbf{A} = \begin{pmatrix} 1/\sqrt{2} & \sqrt{2}/6 & -2/3 \\ 1/\sqrt{2} & -\sqrt{2}/6 & 2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

(b) $\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

Solution: The matrix is the transpose of the matrix in Part (a). Instead of computing the SVD from scratch, we use (a) to help us get the answer.
Suppose $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, then $\mathbf{A}^T = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T)^T = \mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T$, and note that $\mathbf{V}$ and $\mathbf{U}^T$ are orthogonal matrices too. So,

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{2}/6 & -\sqrt{2}/6 & 2\sqrt{2}/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix}.$$

(c) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Solution: In this question, $\mathbf{A}$ is symmetric, and the eigenvalues are non-negative. Write $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where up to reordering the columns of $\mathbf{P}$, we can ensure that the diagonal of $\mathbf{D}$ is such that $d_{11} \geq d_{22} \geq \cdots \geq d_{nn}$. Then letting $\mathbf{\Sigma} = \mathbf{D}$ and $\mathbf{U} = \mathbf{V} = \mathbf{P}$ is an SVD of $\mathbf{A}$.

An orthogonal diagonalization of $\mathbf{A}$ is $\mathbf{P} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} & 0 & -1/\sqrt{3} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we may let $\mathbf{\Sigma} = \mathbf{D}$ and $\mathbf{U} = \mathbf{V} = \mathbf{P}$.

6. (**MATLAB**) Let $\mathbf{A} = \begin{pmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{pmatrix}$.

(a) Find a SVD of $\mathbf{A}$.

Solution: $\mathbf{U} = \begin{pmatrix} -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{pmatrix}$, $\mathbf{\Sigma} = \begin{pmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $\mathbf{V} = \begin{pmatrix} 2/5 & -4/5 & -2/5 & 1/5 \\ -4/5 & -2/5 & 1/5 & 2/5 \\ 1/5 & -2/5 & 4/5 & -2/5 \\ -2/5 & -1/5 & -2/5 & -4/5 \end{pmatrix}$.

(b) In MATLAB, type

```
>> [U S V]=svd(A)
```

Compare the result with your answer in (a).

Solution: They are the same.

Extra problems

1. Let $\mathbf{A}$ be a stochastic matrix. Prove that $\lambda = 1$ is an eigenvalue of $\mathbf{A}$.

   Solution: Write $\mathbf{A} = (a_{ij})_n$. Then by definition,

   $$a_{1j} + a_{2j} + \cdots + a_{nj} = 1, \text{ for all } j = 1, ..., n.$$

   Now $\lambda = 1$ is an eigenvalue of $\mathbf{A}$ if and only if it is an eigenvalue of $\mathbf{A}^T$. Consider the vector $\mathbf{v}$ with all entries equal to 1. Then

   $$\mathbf{A}^T\mathbf{v} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} + \cdots + a_{n1} \\ a_{12} + a_{22} + \cdots + a_{n2} \\ \vdots \\ a_{1n} + a_{2n} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \mathbf{v},$$

   which shows that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^T$ associated to eigenvalue 1.

   Alternatively, we will show that $\mathbf{I} - \mathbf{A}$ is singular.

   $$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1 - a_{nn} \end{pmatrix} \xrightarrow{R_1+R_2} \xrightarrow{R_1+R_3} \cdots \xrightarrow{R_1+R_n}$$

   $$\begin{pmatrix} 1 - \sum_{k=1}^n a_{k1} & 1 - \sum_{k=1}^n a_{k2} & \cdots & 1 - \sum_{k=1}^n a_{kn} \\ -a_{21} & 1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1 - a_{nn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1 - a_{nn} \end{pmatrix}$$

   and so its RREF cannot be the identity matrix.

2. Let $\mathbf{v}_1$ be an eigenvector of $\mathbf{A}$ associated to the eigenvalue λ_1 and $\mathbf{v}_2$ an eigenvector of $\mathbf{A}^T$ associated to eigenvalue λ_2. Suppose $\lambda_1 \neq \lambda_2$. Show that v_1 and v_2 are orthogonal.

   Solution: Taking transpose of the equation $\mathbf{A}^T\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ we obtain $\mathbf{v}_2^T\mathbf{A} = \lambda\mathbf{v}_2^T$. So,

   $$\lambda_2\mathbf{v}_2 \cdot \mathbf{v}_1 = \lambda_2\mathbf{v}_2^T\mathbf{v}_1 = (\mathbf{v}_2^T\mathbf{A})\mathbf{v}_1 = \mathbf{v}_2^T(\mathbf{A}\mathbf{v}_1) = \lambda_1\mathbf{v}_2^T\mathbf{v}_1 = \lambda_1\mathbf{v}_2 \cdot \mathbf{v}_1.$$

   In other words, $(\lambda_2 - \lambda_1)\mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{0}$. Since $(\lambda_2 - \lambda_1) \neq 0$, necessarily $\mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{0}$.

3. Let $\mathbf{A}$ be an $n \times n$ matrix. Show that there exists an orthogonal matrix $\mathbf{Q}$ such that

   $$\mathbf{A}\mathbf{A}^T = \mathbf{Q}^T\mathbf{A}^T\mathbf{A}\mathbf{Q}$$

Solution: Let $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ be a SVD of $\mathbf{A}$. Also, let $\mathbf{A}^T\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where up to reordering the columns of $\mathbf{P}$, we can ensure that the diagonal of $\mathbf{D}$ is such that $d_{11} \geq d_{12} \geq \cdots \geq d_{nn}$.

Then

$$\mathbf{A}\mathbf{A}^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T = \mathbf{U}\mathbf{D}\mathbf{U}^T.$$

Now let $\mathbf{Q} = \mathbf{P}\mathbf{U}^T$. Then

$$\mathbf{Q}^T\mathbf{A}^T\mathbf{A}\mathbf{Q} = \mathbf{U}\mathbf{P}^T\mathbf{P}\mathbf{D}\mathbf{P}^T\mathbf{P}\mathbf{U}^T = \mathbf{U}\mathbf{D}\mathbf{U}^T = \mathbf{A}\mathbf{A}^T.$$