# MA1522 Linear Algebra for Computing

## Lecture 13: Linear Transformations

Yang Yue

Department of Mathematics
National University of Singapore

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# Outline

# Question in Section 7.2

What are the rank and nullity of the following linear transformation?

1. $T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$

2. $T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$

# Slides 31 and 32: Kernel of Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

### Definition

The *kernel* of $T$ is defined by $\ker(T) = \{\ \mathbf{u} \in \mathbb{R}^n \mid T(\mathbf{u}) = \mathbf{0}\ \}$.

### Theorem

*The kernel of $T$ is a subspace.*

### Definition

The *nullity* of $T$ is the dimension of the kernel of $T$,

$$\text{nullity}(T) = \dim(\ker(T)).$$

Let $\mathbf{A}$ be the standard matrix of $T$. Then

$$\text{nullity}(T) = \dim(\ker(T)) = \dim(\text{Null}(\mathbf{A})) = \text{nullity}(\mathbf{A}).$$

# Slides 29 and 30: Range of Linear Transformation

## Definition

The *range* of $T$ is

$$\mathrm{R}(T) = T(\mathbb{R}^n) = \left\{ \mathbf{v} \in \mathbb{R}^m \;\middle|\; \mathbf{v} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in \mathbb{R}^n \right\}.$$

## Theorem

*The range of $T$ is a subspace. In fact, it is the column space of its standard matrix.*

## Definition

The *rank* of $T$ is the dimension of the range of $T$

$$\mathrm{rank}(T) = \dim(\mathrm{R}(T)).$$

In fact, $\mathrm{rank}(T) = \dim(\mathrm{R}(T)) = \dim(\mathrm{Col}(\mathbf{A})) = \mathrm{rank}(\mathbf{A})$.

# Answer to Question in Section 7.2 (part 1)

What are the rank and nullity of the following linear transformation?

1. $T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Answer: By observation $\ker(T) = \mathbb{R}^n$ and $\mathrm{R}(T) = \{\mathbf{0}\}$, we have $\mathrm{rank}(T) = 0$ and $\mathrm{nullity}(T) = n$.

(You can also use the standard matrix $\mathbf{A} = \mathbf{0}_n$ to get the same conclusion.)

# Answer to Question in Section 7.2 (part 2)

What are the rank and nullity of the following linear transformation?

2. $$T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Answer: By observation $\ker(T) = \{\mathbf{0}\}$ and $\mathrm{R}(T) = \mathbb{R}^n$, we have $\mathrm{rank}(T) = n$ and $\mathrm{nullity}(T) = 0$.

(You can also use the standard matrix $\mathbf{A} = \mathbf{I}_n$ to get the same conclusion.)

# Exercise one in Section 7.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if $T$ is injective, then necessarily $n \leq m$.

Recall: On Slide 35,

## Definition

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is *injective*, or *one-to-one* if whenever $T(\mathbf{u}_1) = T(\mathbf{u}_2)$, then $\mathbf{u}_1 = \mathbf{u}_2$

## Theorem

*A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is injective if and only if the kernel is trivial,* $\ker(T) = \{\mathbf{0}\}$.

# Slide 36: Full Rank Equals Number of Columns

## Theorem

*Suppose* $\mathbf{A}$ *is an* $m \times n$ *matrix. The following statements are equivalent.*

(i) $\mathbf{A}$ *is full rank, where the rank is equal to the number of columns,* $\text{rank}(\mathbf{A}) = n$.

(ii) *The rows of* $\mathbf{A}$ *spans* $\mathbb{R}^n$, $\text{Row}(\mathbf{A}) = \mathbb{R}^n$.

(iii) *The columns of* $\mathbf{A}$ *are linearly independent.*

(iv) *The homogeneous system* $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution, that is,* $\text{Null}(\mathbf{A}) = \{\mathbf{0}\}$.

(v) $\mathbf{A}^T\mathbf{A}$ *is an invertible matrix of order* $n$.

(vi) $\mathbf{A}$ *has a left inverse.*

(vii) *The linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ *defined by* $\mathbf{A}$ *is injective.*

# Answer to Exercise one in Section 7.2

Q: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if $T$ is injective, then necessary $n \leq m$.

Answer: Using the "(i) $\Leftrightarrow$ (vii)" in the Theorem on Slide 36, we have $n = \text{rank}(\mathbf{A}) \leq m$ (because $\mathbf{A}$ is $m \times n$, so $\text{rank}(\mathbf{A}) \leq \min\{n, m\}$).

(You can also use the injectivity of $T$ to get $\text{nullity}(T) = 0$ and $\text{rank}(T) = \text{rank}(\mathbf{A}) = n$.)

# Exercise two in Section 7.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if $T$ is surjective, then necessarily $n \geq m$.

Recall: On Slide 39,

### Definition

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *surjective* or *onto* if for every $\mathbf{v}$ in the codomain $\mathbb{R}^m$, there exists a $\mathbf{u}$ in the domain $\mathbb{R}^n$ such that $T(\mathbf{u}) = \mathbf{v}$.

# Slide 40: Full Rank Equals Number of Rows

**Theorem**

*Suppose $\mathbf{A}$ is an $m \times n$ matrix. The following statements are equivalent.*

(i) *$\mathbf{A}$ is full rank, where the rank is equal to the number of rows,* $\text{rank}(\mathbf{A}) = m$.

(ii) *The columns of $\mathbf{A}$ spans $\mathbb{R}^m$,* $\text{Col}(\mathbf{A}) = \mathbb{R}^m$.

(iii) *The rows of $\mathbf{A}$ are linearly independent.*

(iv) *The linear system $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$.*

(v) *$\mathbf{AA}^T$ is an invertible matrix of order $m$.*

(vi) *$\mathbf{A}$ has a right inverse.*

(vii) *The linear transformation $T$ defined by $\mathbf{A}$ is surjective.*

# Answer to Exercise two in Section 7.2

Q: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if $T$ is surjective, then necessary $n \geq m$.

Answer: Using the "(i) $\Leftrightarrow$ (vii)" in the Theorem on Slide 40, we have $m = \text{rank}(\mathbf{A}) \leq n$ (because $\mathbf{A}$ is $m \times n$, so $\text{rank}(\mathbf{A}) \leq \min\{n, m\}$).

(You can also use the surjectivity of $T$ to get $m = \text{rank}(T) = \text{rank}(\mathbf{A})$.)

# Exercise three in Section 7.2

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if it is both injective and surjective.

Show that $T : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if and only if there is a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

# Slide 44: Equivalent Statements of Invertibility

Let $\mathbf{A}$ be a square matrix of order $n$. The following are equivalent.

(i) $\mathbf{A}$ is invertible.

(ii) $\mathbf{A}^T$ is invertible.

(iii) (left inverse) There is a matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{I}$.

(iv) (right inverse) There is a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$.

(v) The reduced row-echelon form of $\mathbf{A}$ is the identity matrix.

(vi) $\mathbf{A}$ can be expressed as a product of elementary matrices.

(vii) The homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

(viii) For any $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

(ix) The determinant of $\mathbf{A}$ is nonzero, $\det(\mathbf{A}) \neq 0$.

(x) The columns/rows of $\mathbf{A}$ are linearly independent.

(xi) The columns/rows of $\mathbf{A}$ spans $\mathbb{R}^n$.

(xii) $\text{rank}(\mathbf{A}) = n$ ($\mathbf{A}$ has full rank).

(xiii) $\text{nullity}(\mathbf{A}) = 0$.

(xiv) 0 is not an eigenvalue of $\mathbf{A}$.

(xv) The linear transformation $T$ defined by $\mathbf{A}$ is injective.

(xvi) The linear transformation $T$ defined by $\mathbf{A}$ is surjective.

# Answer to Exercise three in Section 7.2

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if it is both injective and surjective.

Show that $T : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if and only if there is a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Proof: ($\Rightarrow$) Let $\mathbf{A}$ be the standard matrix for $T$. Using either item (xv) or item (xvi), we know (i) holds, i.e., $\mathbf{A}$ is invertible. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $S(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x}$. Then

$$T(S(\mathbf{x})) = \mathbf{A}\mathbf{A}^{-1}\mathbf{x} = \mathbf{x},$$

and similarly,

$$S(T(\mathbf{x})) = \mathbf{x}.$$

# Answer to Exercise three in Section 7.2 (conti.)

Q: Show that $T : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if and only if there is a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Proof: ($\Leftarrow$) Let us show that $T$ is injective. Suppose that $T(\mathbf{u}_1) = T(\mathbf{u}_2)$. Then $S(T(\mathbf{u}_1)) = S(T(\mathbf{u}_2))$. By assumption that $ST =$ Identity, $\mathbf{u}_1 = \mathbf{u}_2$.

Next we show that $T$ is surjective. Let $\mathbf{v}$ be an arbitrary vector in $\mathbb{R}^n$. Define $\mathbf{u} = S(\mathbf{v})$. Then $T(\mathbf{u}) = T(S(\mathbf{v})) = \mathbf{v}$. We are done.

# Practice Problem 1

Let $\mathbf{A} = \mathbf{L}\begin{pmatrix} -1 & 1 & 7 & -1 \\ 0 & 4 & 12 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ be an LU factorization of a $3 \times 4$ matrix $\mathbf{A}$.

(a) Given $\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ 17 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$, find $\mathbf{A}$.

(b) It is given that $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a least squares solution to the system $\mathbf{Ax} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Use your answer in (a) or otherwise, find all the least squares solutions to the system.

## Answer to Problem 1 (part a)

Q: Given LU factorization $\mathbf{A} = \mathbf{L}\begin{pmatrix} -1 & 1 & 7 & -1 \\ 0 & 4 & 12 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and

$\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ 17 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$. Find $\mathbf{A}$.

Answer:

$$\begin{pmatrix} 4 \\ 14 \\ 17 \end{pmatrix} = \mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{L}\begin{pmatrix} -1 & 1 & 7 & -1 \\ 0 & 4 & 12 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{L}\begin{pmatrix} 4 \\ 10 \\ -1 \end{pmatrix}.$$

Write $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix}$.

# Answer to Problem 1 (part a) conti.

Then

$$\begin{pmatrix} 4 \\ 4x+10 \\ 4y+10z-1 \end{pmatrix} = \mathbf{L}\begin{pmatrix} 4 \\ 10 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ 17 \end{pmatrix}$$

Also,

$$\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} = \mathbf{A}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{L}\begin{pmatrix} -1 & 1 & 7 & -1 \\ 0 & 4 & 12 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{L}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 2 \\ 2x+2 \\ 2y+2z+1 \end{pmatrix} = \mathbf{L}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}.$$

# Answer to Problem 1 (part a) conti.

Hence, $x = 1$, and solving

$$\begin{cases} 4y + 10z = 18 \\ 2y + 2z = 6 \end{cases}$$

gives $y = 2$, $z = 1$. Hence,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

Therefore,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 7 & -1 \\ 0 & 4 & 12 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 7 & -1 \\ -1 & 5 & 19 & 1 \\ -2 & 6 & 26 & -1 \end{pmatrix}.$$

# Answer to Problem 1 (part b)

Q: Given LU factorization $\mathbf{A} = \mathbf{L}\begin{pmatrix} -1 & 1 & 7 & -1 \\ 0 & 4 & 12 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a least squares solution to the system $\mathbf{Ax} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Use your answer in (a) or otherwise, find all the least squares solutions to the system.

Answer: Since the rank of $\mathbf{A}$ is 3, the column space is the whole $\mathbb{R}^3$. Hence, for any $\mathbf{b} \in \mathbb{R}^3$, $\mathbf{Ax} = \mathbf{b}$ is consistent. Therefore, least squares solutions are actual solutions (See Slide 15 in Week 10's slide).

# Answer to Problem 1 (part b) conti.

$$\left( \begin{array}{c|c} \mathbf{A} & \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} \end{array} \right) = \left( \begin{array}{cccc|c} -1 & 1 & 7 & -1 & 1 \\ -1 & 5 & 19 & 1 & 1 \\ -2 & 6 & 26 & -1 & 2 \end{array} \right) \xrightarrow{RREF} \left( \begin{array}{cccc|c} 1 & 0 & -4 & 0 & -1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right).$$

The general solution is

$$\begin{pmatrix} -1+4s \\ -3s \\ s \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}.$$

(We skip Problem 2, as it is about Applications of Least Squares Approximation.)

# Problem 3 (part a)

Q: A $4 \times 3$ matrix $\mathbf{A}$ has SVD decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T,$$

where $\mathbf{V} = \begin{pmatrix} 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$, and such that the characteristic polynomial of $\mathbf{A}^T\mathbf{A}$ is $(x-3)(x-6)(x-10)$.

(a) Find $\Sigma$.

Answer: The eigenvalues of $\mathbf{A}^T\mathbf{A}$ are 3, 6, and 10. Hence, the singular values are (in descending order) $\sqrt{10}$, $\sqrt{6}$, $\sqrt{3}$.

$$\Sigma = \begin{pmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}.$$

# Problem 3 (part b)

(b) It is given that

$$\frac{1}{2\sqrt{5}}\mathbf{A}\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \frac{1}{6}\mathbf{A}\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{pmatrix},\quad \frac{1}{3}\mathbf{A}\begin{pmatrix}-1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Find $\mathbf{U}$. Give exact answer.

Answer: Let $\mathbf{U}=\begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4\end{pmatrix}$. Then

$$\mathbf{u}_1=\frac{1}{\sqrt{10}}\mathbf{A}\begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}=\frac{1}{2\sqrt{5}}\mathbf{A}\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix};$$

(Recall: $\mathbf{u}_1=\frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1$.)

# Problem 3 (part b), Answer conti.

$$\mathbf{u}_2 = \frac{1}{\sqrt{6}}\mathbf{A}\begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} = \frac{1}{6}\mathbf{A}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix};$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{3}}\mathbf{A}\begin{pmatrix} -1\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \frac{1}{3}\mathbf{A}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

# Problem 3 (part b) Answer conti.

Finally, $\mathbf{u}_4$ is a unit vector orthogonal to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, which tell us $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}^T \mathbf{u}_4 = 0$. Thus,

$$\mathbf{u}_4 = \pm \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad \text{and } \mathbf{U} = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & \pm 1/\sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & \mp 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(You may take either positive or negative.)

# Answer to Problem 3 (part c)

(c) Use the information given in (b) to find $\mathbf{A}$. Give exact answer.

Answer: Since $\mathbf{V}$ is given, we may use $\mathbf{U}$ in (b) (you may choose either of them) and $\Sigma$ in (a) to find $\mathbf{A}$:

$$\begin{aligned}
\mathbf{A} &= \mathbf{U}\Sigma\mathbf{V}^T \\
&= \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}^T \\
&= \begin{pmatrix} \sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{5} & \sqrt{5} \\ \sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ -1 & -1 & 1 \end{pmatrix}.
\end{aligned}$$

# Problem 4

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - y - 2z \\ -2x + 2y + 2z \\ -y - z \end{pmatrix}.$$

(a) Write down the standard matrix of the transformation $T$.

(b) Find a nonzero vector $\mathbf{u}$ in $\mathbb{R}^3$ such that $T(\mathbf{u}) = \mathbf{u}$. Explain how you derive your answer.

(c) Find a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$. Explain how you derive your answer.

# Answer to Problem 4 (part a)

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - y - 2z \\ -2x + 2y + 2z \\ -y - z \end{pmatrix}.$$

(a) Write down the standard matrix of the transformation $T$.

Answer: $\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ -2 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}$.

# Answer to Problem 4 (part b)

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - y - 2z \\ -2x + 2y + 2z \\ -y - z \end{pmatrix}.$$

(b) Find a nonzero vector $\mathbf{u}$ in $\mathbb{R}^3$ such that $T(\mathbf{u}) = \mathbf{u}$. Explain how you derive your answer.

Answer: Since $T(\mathbf{u}) = \mathbf{u}$, we have $(\mathbf{A} - \mathbf{I})(\mathbf{u}) = \mathbf{0}$.

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Any nonzero multiple of $\mathbf{u} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ works.

## Answer to Problem 4 (part c)

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - y - 2z \\ -2x + 2y + 2z \\ -y - z \end{pmatrix}.$$

(c) Find a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$. Explain how you derive your answer.

Answer: $\mathbf{Au} = T(\mathbf{u}) = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$. Just form the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 5 \\ -2 & 2 & 2 & -4 \\ 0 & -1 & -1 & 1 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right).$$

Hence, $\mathbf{u} = (1, 2, -3)^T$.

# Alternative Answer to Problem 4 (part c)

You may also observe (or show) that $\mathbf{A}$ is invertible, and then

$$\mathbf{A}\mathbf{u} = T(\mathbf{u}) = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} \text{ if and only if } \mathbf{u} = \mathbf{A}^{-1} \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -1 & -2 \\ -2 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.$$