Review

	A	P (or U, V)	D (or Σ)
① Diagonalizable $A = PDP^{-1}$	some square matrices	invertible	diagonal
② Orthogonally diagonalizable $A = PDP^T$	all symmetric matrices	orthogonal	diagonal
③ Singular value decomposition (SVD) $A = U\Sigma V^T$	all matrices	orthogonal (U, V)	$\begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix}$ D' diagonal positive

Finding an orthogonal diagonalization for any symmetric matrix:

Same for ①

- Compute eigenvalues (characteristic polynomial)
- Solve for a basis of each eigenspace

Note: if A is sym. the eigenspaces from distinct eigenvalues are orthogonal to each other.

Compare the algebraic & geometric multiplicities for every eigenvalues (No need here. they are always the same)

additional step for ②

- Convert the previously found basis of each eigenspace into an orthonormal basis (Gram-Schmidt process + normalization)

Collect these vectors in every orthonormal basis together $\longrightarrow P$

Write the corresponding eigenvalue for each column vector of $P \longrightarrow D$

Finding a SVD for an arbitrary matrix $A \in M_{m\times n}$:

Orthogonally diagonalize A^TA (a symmetric matrix) $\quad A^TA = PDP^T$

with $D = \mathrm{diag}(d_{11}, d_{22}, \cdots, d_{nn})$

$d_{11} \geq d_{22} \geq \cdots \geq d_{rr} > 0 = d_{r+1,r+1} = \cdots = d_{nn} = 0$

Set $V = P$

$$\Sigma = \begin{pmatrix} D'_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}$$

where $D' := \mathrm{diag}(\sqrt{d_{11}}, \cdots, \sqrt{d_{rr}})$

singular values $\sigma_1 = \sqrt{d_{11}}$, $\vdots$, $\sigma_r = \sqrt{d_{rr}}$ (strictly positive)

Let $u_1 = \frac{1}{\sigma_1} A v_1$

$\vdots$

$u_r = \frac{1}{\sigma_r} A v_r$

Note: $A = U \Sigma V^T$ $U = (u_1 \cdots u_m)$ $V = (v_1 \cdots v_n)$

then $U\Sigma = AV$

$$\underbrace{(\sigma_1 u_1 \cdots \sigma_r u_r \ 0_{m\times(n-r)})}_{\parallel} \qquad \underbrace{(Av_1 \cdots Av_n)}_{\parallel}$$

1 - 1 correspondence

Solve for $u_{r+1}, \cdots, u_n$. i.e.

find an orthonormal basis of $\mathrm{Null}\left((u_1 \cdots u_r)^T\right)$

$\Leftrightarrow$ - - - - - - - of $\left(\mathrm{Span}\{u_1, \cdots, u_r\}\right)^{\perp} \subseteq \mathbb{R}^m$

$\Leftrightarrow$ extend $\{u_1, \cdots, u_m\}$ to be an orthonormal basis of $\mathbb{R}^m$

Rmk: The chosen $\{u_1, \cdots, u_r\}$ has already been orthonormal.

Topics on Markov Chain

a sequence of probability vectors together with a stochastic matrix P

$$x_0, \quad x_1 = Px_0, \quad x_2 = Px_1 = P^2 x_0, \quad \cdots, \quad x_k = P x_{k-1} = P^k x_0, \quad \cdots$$

where probability vectors $v = (v_i)_n$ satisfying $v_i \geq 0 \quad \forall i$ & $\sum_i v_i = 1$

stochastic matrices $P \in M_{n\times n}$ whose columns are probability vectors.

(This Markov chain is well-defined, since Pv is still a probability vector

for any stochastic matrix P & probability vector v.

Check: $\sum_i (Pv)_i = \sum_{i,j} P_{ij} v_j = \sum_j \left[v_j\left(\sum_i P_{ij}\right)\right] = \sum_j v_j = 1.$)

Assume P is diagonalizable and has positive entries

then $P = QDQ^{-1}$, where $D = \mathrm{diag}(I_r, d_{r+1,r+1}, \cdots, d_{nn})$ (☆)

$$1 > d_{r+1,r+1} \geq \cdots \geq d_{nn} > -1$$

$\Rightarrow x_n = P^n x_0 = QD^n Q^{-1} x_0$

Let $n \to \infty$, then $D^n \longrightarrow \mathrm{diag}(I_r, 0) = \begin{pmatrix} I_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{pmatrix} =: D_\infty$

$x_n \longrightarrow x_\infty = QD_\infty Q^{-1} x_0$ converges to a unique vector

Moreover, $Px_\infty = QDQ^{-1}QD_\infty Q^{-1}x_0 = QD_\infty Q^{-1}x_0 = x_\infty$

i.e. x_∞ is an eigenvector associated to eigenvalue 1.

Remark: 1 is always an eigenvalue for stochastic matrices

Moreover, all eigenvalues for stochastic matrices are between -1 and 1.

If we further assume that the stochastic matrix has positive entries,

then -1 is not an eigenvalue.

Therefore, (☆) is valid.

Proof: Since P & P^T have the same eigenvalues, it suffices to show for P^T.

① Let $v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, then $P^T v = v$

$\Rightarrow$ v is an e.ve. of P^T associated to e.va. 1.

② Now suppose $P^T u = \lambda u$, for $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$. Let $|u_k| \geq |u_i|$, $\forall i$ for some k.

Then $\lambda u_k = \sum_j p_{jk} u_j$ $\quad |\lambda||u_k| \leq \sum_j p_{jk} |u_j| \leq |u_k| \sum_j p_{jk} = |u_k|$

$\Rightarrow$ $|\lambda| \leq 1$.

③ The above equalities hold iff $u_1 = u_2 = \cdots = u_n$

we have used $p_{ij} > 0$ here

Let $\tilde{u} = \frac{1}{u_1} u = \frac{1}{u_2} u = \cdots = \frac{1}{u_n} u = v$

then $|\lambda| = 1$ iff v is an eigenvalue associated to λ.

But v is an eigenvector from eigenvalue 1, thus $\lambda = 1$.

1. A population of ants is put into a maze with 3 compartments labeled a, b, and c. If the ant is in compartment a, an hour later, there is a 20% chance it will go to compartment b, and a 40% change it will go to compartment c. If it is in compartment b, an hour later, there is a 10% chance it will go to compartment a, and a 30% chance it will go to compartment c. If it is in compartment c, an hour later, there is a 50% chance it will go to compartment a, and a 20% chance it will go to compartment b. Suppose 100 ants has been placed in compartment a.

   (a) Find the transition probability matrix **A**. Show that it is a stochastic matrix.

   (b) By diagonalizing **A**, find the number of ants in each compartment after 3 hours.

   (c) **(MATLAB)** We can use MATLAB to diagonalize the matrix **A**. Type

```
>> [P D]=eig(sym(A))
```

   The matrix **P** will be an invertible matrix, and **D** will be a diagonal matrix. Compare the answer with what you have obtained in (b).

1. (a)

$$A = \begin{pmatrix} 0.4 & 0.1 & 0.5 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$$

stochastic matrix ✓

(sums of rows are all 1)

(b) $A = PDP^{-1}$, with $P = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 1 & & \\ & -0.1 & \\ & & 0.4 \end{pmatrix}$

$$x_3 = A^3 x_0 = PD^3P^{-1}x_0 = \begin{pmatrix} 35 \\ 31.2 \\ 33.8 \end{pmatrix}$$

   (d) In the long run (assuming no ants died), where will the majority of the ants be?

   (e) Suppose initially the numbers of ants in compartments a, b and c are α, β, and γ respectively. What is the population distribution in the long run (assuming no ants died)?

(d) $x_n = A^n x_0 = PD^nP^{-1}x_0$

$n \to \infty$, $D^n \to \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$, $x_n \to P\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}P^{-1}x_0 = \frac{1}{3}\begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$

(e) Set $x_0 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$, then $x_n \to \frac{1}{3}\begin{pmatrix} \alpha+\beta+\gamma \\ \alpha+\beta+\gamma \\ \alpha+\beta+\gamma \end{pmatrix}$

2. By diagonalizing $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, find a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$.

2. By the argument shown in the Tutorial 9, we conclude

$$A = PDP^{-1} \quad \text{with} \quad P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & & \\ & 4 & \\ & & 1 \end{pmatrix}$$

Then $A = PDP^{-1} = P\sqrt{D}\cdot\sqrt{D}P^{-1} = \underbrace{(P\sqrt{D}P^{-1})}_{B}(P\sqrt{D}P^{-1}) = B^2$

where $\sqrt{D} := \operatorname{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$ for diagonal matrix $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ $(\lambda_i \geq 0)$

Here $\sqrt{D} = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 1 \end{pmatrix}$, $B = P\sqrt{D}P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

3. For each of the following symmetric matrices $\mathbf{A}$, find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$.

(a) $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

(b) $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$.

3. (a) $\det(\lambda I - A) = (\lambda - 2)(\lambda - 4) \longrightarrow$ distinct eigenvalues

Solve $(2I - A) = 0 \Rightarrow$ basis of $E_2 = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$(4I - A) = 0 \Rightarrow$ basis of $E_4 = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$\xrightarrow{\text{normalization}}$ $P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ $\qquad P^TAP = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

(b) $\det(\lambda I - A) = (\lambda + 6)(\lambda - 3)^2 \qquad \lambda_1 = -6 \qquad \lambda_2 = 3$

Solve $(-6I - A)x = 0 \Rightarrow$ basis of $E_{-6} = \left\{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right\}$

$(3I - A)x = 0 \Rightarrow$ basis of $E_3 = \left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\Big\downarrow$ G-S process

$\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}\right\}$

$\xrightarrow{\text{normalization}}$

$$P = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{pmatrix} \qquad P^TAP = \begin{pmatrix} -6 & & \\ & 3 & \\ & & 3 \end{pmatrix}$$

4. **(MATLAB)** Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$.

   (a) Find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$, and compute $\mathbf{P}^T\mathbf{AP}$.

   (b) We will use MATLAB to orthogonally diagonalize $\mathbf{A}$. Type

```
>> A=[1 -2 0 0;-2 1 0 0;0 0 1 -2;0 0 -2 1];

>> [P D]=eig(A);

>> sym(P), sym(D)
```

   Compare the result with your answer in (a).

4. (a)

$$P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \qquad P^TAP = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$$

5. Find the SVD of the following matrices $\mathbf{A}$.

   (a) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}$. A_a

5. (a) $A^TA = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$ $\qquad \det(\lambda I - A^TA) = (\lambda-9)(\lambda-25)$

singular values $\sigma_1 = 5$, $\sigma_2 = 3$. $\quad \Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$

Orthogonally diagonalize $A^TA = V\begin{pmatrix} 25 & \\ & 9 \end{pmatrix}V^T \quad V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = (v_1 \ \ v_2)$

$A = U\Sigma V^T \qquad U = (u_1 \ \ u_2 \ \ u_3)$

$(5u_1 \ \ 3u_2) = U\Sigma = AV = (Av_1 \ \ Av_2)$ $\quad \Rightarrow \quad u_1 = \frac{1}{5}Av_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad u_2 = \frac{1}{3}Av_2 = \begin{pmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ 4/3\sqrt{2} \end{pmatrix}$

u_3 satisfies $\begin{cases} u_1 \cdot u_3 = u_2 \cdot u_3 = 0 \\ u_3 \cdot u_3 = 1 \end{cases}$ i.e. $\begin{pmatrix} u_1^T \\ u_2^T \end{pmatrix} u_3 = 0 \quad \Rightarrow \quad u_3 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & -2/3 \\ 1/\sqrt{2} & -1/3\sqrt{2} & 2/3 \\ 0 & 4/3\sqrt{2} & 1/3 \end{pmatrix}$$

   (b) $\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$. A_b

(b) in (a) $A_a = U\Sigma V^T$.

so $A_b = A_a^T = V\Sigma^T U^T \qquad \Sigma^T = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(c) A is symmetric. $A = PDP^T$ $\quad P = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \end{pmatrix}$ $\quad D = \begin{pmatrix} 3 & & \\ & 1 & \\ & & 0 \end{pmatrix}$

$U = V = P$ $\qquad \Sigma$ $\quad (d_{11} \geq d_{22} \geq \cdots)$

6. (**MATLAB**) Let $\mathbf{A} = \begin{pmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{pmatrix}$.

(a) Find a SVD of **A**.

(b) In MATLAB, type

```
>> [U S V]=svd(A)
```

Compare the result with your answer in (a).

6. (a) Similar as in 5,

$A = U\Sigma V^T$ with $U = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\Sigma = \mathrm{diag}(40, 20, 10, 0)$

$V = \begin{pmatrix} \frac{2}{5} & -\frac{4}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{4}{5} & -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} & -\frac{2}{5} & -\frac{4}{5} \end{pmatrix}$