Review

linear transformation $\quad T : \underbrace{\mathbb{R}^n}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}^m}_{\text{codomain}}$

$$\alpha u + \beta v \longmapsto \alpha T(u) + \beta T(v) \qquad \forall\, \alpha, \beta \in \mathbb{R}$$

Assume $S = \{v_1, v_2, \cdots, v_n\}$ is a basis of $\mathbb{R}^n$. then

$$\forall\, v \in \mathbb{R}^n, \quad \exists!\ c_1, \cdots, c_n \in \mathbb{R} \quad \text{s.t.} \quad T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$$

$\longrightarrow$ T will be determined uniquely by $T(v_1), \cdots, T(v_n)$.

$$\| \qquad [T]_S [v]_S = \underbrace{\big(T(v_1) \ \cdots \ T(v_n)\big)}_{m \times n \text{ matrix } [T]_S} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

In particular, if one takes

$S = \{e_1, \cdots, e_n\} =: E$ the standard basis of $\mathbb{R}^n$,

$[T]_E = \big(T(e_1) \ \cdots \ T(e_n)\big)$ is called as the <u>standard matrix</u> of T.

In this case, $\forall\ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, we have $c_1 = x_1, \cdots, c_n = x_n$,

$$\Rightarrow \quad T(x) = \big(T(e_1) \ \cdots \ T(e_n)\big) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Assume $S' = \{u_1, \cdots, u_n\}$ is another basis, and P is the transition matrix from S to S'. Then $T(v) = [T]_{S'} [v]_{S'} = [T]_{S'} P [v]_S$

$$\Rightarrow \quad [T]_S = [T]_{S'} P$$

subspaces

Range $\quad \mathrm{Ran}(T) = T(\mathbb{R}^n) = \{ Tv \mid v \in \mathbb{R}^n \} \longrightarrow \mathrm{rank}(T) = \dim(\mathrm{Ran}(T))$

$\qquad = \{ u \in \mathbb{R}^m \mid u = T(v) \text{ for some } v \in \mathbb{R}^n \}$

Kernel $\quad \mathrm{Ker}(T) = \{ v \in \mathbb{R}^n \mid T(v) = 0 \} \longrightarrow \mathrm{nullity}(T) = \dim(\mathrm{Ker}(T))$

$\mathrm{Ran}(T) = \mathrm{Col}([T]_E) \qquad \mathrm{Ker}(T) = \mathrm{Null}([T]_E) \qquad$ standard basis E

Fix an arbitrary basis S. then
$$\mathrm{rank}(T) = \mathrm{rank}([T]_S)$$
$$\mathrm{nullity}(T) = \mathrm{nullity}([T]_S)$$

Injective $\iff \mathrm{nullity}(T) = 0 \qquad$ Surjective $\iff \mathrm{rank}(T) = m$

$\qquad\quad \iff \mathrm{rank}(T) = n$

The Dimension Theorem $\qquad n = \mathrm{rank}(T) + \mathrm{nullity}(T)$.

Geometry explanation for QR-factorization, diagonalization, orthogonal diagonalization and SVD (3rd cont.)

Now we suppose that, for any fixed matrix A.

$T_A : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation whose standard matrix is A.

As before, let $S = \{u_1, \cdots, u_n\}$ be another basis of $\mathbb{R}^n$,

and P be the transition matrix from S to E. (Note that P is an invertible matrix)

Then $P = (u_1 \cdots u_n)$.

Since $v = (u_1 \cdots u_n)[v]_S = P[v]_S$, we have

$$\begin{array}{ccc} T(v) & = & Av \\ \| & & \| \\ P[T(v)]_S & & AP[v]_S \end{array}$$

i.e. $[T(v)]_S = \boxed{P^{-1}AP}[v]_S$

By writing $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, we have implicitly chosen the standard basis E.

so $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ is in fact $[v]_E$.

May refer the remark in the revision part of Tutorial 7.

In this viewpoint,

A & $P^{-1}AP$ represent the same linear transformation T under different bases (E & S, respectively)

Matrix operations

Diagonalization $\longleftrightarrow$ finding a new basis S, s.t. the linear transformation T can be visualized as scalings on n directions

e.g. $T(u_1) = \lambda_1 u_1, \cdots, T(u_n) = \lambda_n u_n$

Orthogonal diagonalization $\longleftrightarrow$ ---- a new orthonormal basis S, s.t. ---

SVD $\longleftrightarrow$ $T : \mathbb{R}^n \to \mathbb{R}^m$, finding new orthonormal bases S_n, S_m for $\mathbb{R}^n$, $\mathbb{R}^m$ resp.

1. (i) Determine whether the following are linear transformations.

(ii) Write down the standard matrix for each other the linear transformations.

(iii) Find a basis for the range for each of the linear transformations.

(iv) Find a basis for the kernel for each of the linear transformations.

(a) $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

1.(a) ✓ $\quad T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A_1\begin{pmatrix} x \\ y \end{pmatrix}$

$\det(A_1) \neq 0$, A is invertible. $\Rightarrow$ T_1 is a bijection.

$\mathrm{Ran}(T_1) = \mathbb{R}^2 = \mathrm{Span}\{e_1, e_2\} \qquad \mathrm{Ker}(T_1) = \{0\}$

(b) $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2^x \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(b) X $\quad T_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0$

(c) $T_3\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(c) ✓ $\quad T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A_3\begin{pmatrix} x \\ y \end{pmatrix}$

$\mathrm{Ran}(T_3) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \qquad \mathrm{Ker}(T_3) = \mathrm{Null}(A_3) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

(d) $T_4\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_4\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y-x \\ y-z \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

(d) X $\quad T_4\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq 0$

(e) $T_5\colon \mathbb{R}^5 \to \mathbb{R}$ such that $T_5\left(\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}\right) = x_3 + 2x_4 - x_5$ for $\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^5$.

(e) $\quad T_5\left(\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = A_5\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}$

$\mathrm{Ran}(T_5) = \mathbb{R} \qquad \mathrm{Ker}(T_5) = \mathrm{Null}(A_5) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

(f) $T_6 : \mathbb{R}^n \to \mathbb{R}$ such that $T_6(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$.

(f) X $\quad T_6(e_1) = e_1 \cdot e_1 = 1.$

$T_6(2e_1) = (2e_1)\cdot(2e_1) = 4 \neq 2T_6(e_1)$

2. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $G : \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformations such that

$$F\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 - 3x_3 \\ 5x_2 - x_3 \end{pmatrix} \text{ and } G\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix},$$

and let $\mathbf{A}_F$ and $\mathbf{B}_G$ be the standard matrix of F and G, respectively.

(a) Find $\mathbf{A}_F$ and $\mathbf{B}_G$.

(b) Define

$$(F+G)(\mathbf{x}) := F(\mathbf{x}) + G(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathbb{R}^3.$$

Is $(F+G)$ a linear transformation? If it is, find its standard matrix.

(c) Write down the formula for $F(G(\mathbf{x}))$ and find its standard matrix.

(d) Find a linear transformation $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$H(G(\mathbf{x})) = \mathbf{x}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

$A_F + B_G \longleftrightarrow F + G$

$A_F B_G \longleftrightarrow F \circ G$

composite

2. (a) $$A_F = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 0 & 5 & -1 \end{pmatrix} \qquad B_G = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(b) Yes. $$(F+G)\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \underbrace{\begin{pmatrix} 0 & -2 & 1 \\ 6 & 2 & -3 \\ 1 & 6 & 0 \end{pmatrix}}_{A_F + B_G} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(c) $$F(G(x)) = F\left(\begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix}\right) = \begin{pmatrix} -11x_1 - 2x_2 + x_3 \\ x_1 - 2x_2 - 2x_3 \\ 24x_1 + 4x_2 - x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -11 & -2 & 1 \\ 1 & -2 & -2 \\ 24 & 4 & -1 \end{pmatrix}}_{A_F B_G} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Vert$$

$$F(B_G x) = A_F B_G x$$

(d) $$H(G(x)) = x \quad \forall \; x \in \mathbb{R}^3$$

$$\iff C_H B_G = I_3$$

$$\iff C_H = B_G^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ -5 & -2 & 5 \\ 4 & 1 & -1 \end{pmatrix}$$

i.e. $$H(x) = \frac{1}{3}\begin{pmatrix} x_1 + x_2 - x_3 \\ -5x_1 - 2x_2 + 5x_3 \\ 4x_1 + x_2 - x_3 \end{pmatrix}$$

3. For each of the following linear transformations, (i) determine whether there is enough information for us to find the formula of T; and (ii) find the formula and the standard matrix for T if possible.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$T\left(\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{v_1}\right) = \begin{pmatrix}1\\3\\0\\1\end{pmatrix},\ T\left(\underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{v_2}\right) = \begin{pmatrix}2\\2\\-1\\4\end{pmatrix},\ \text{and } T\left(\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{v_3}\right) = \begin{pmatrix}0\\4\\1\\6\end{pmatrix}.$$

3. (a) ✓ $\det(v_1\ v_2\ v_3) = 1 \neq 0 \Rightarrow \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$

i.e. $\mathrm{Span}\{v_1, v_2, v_3\} = \mathbb{R}^3$

$$T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = T\left(x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= xT(v_1) + yT(v_2) + zT(v_3)$$

$$= \begin{pmatrix}T(v_1) & T(v_2) & T(v_3)\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2 & 0\\3 & 2 & 4\\0 & -1 & 1\\1 & 4 & 6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{v_1}\right) = \begin{pmatrix}2\\0\end{pmatrix},\ T\left(\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{v_2}\right) = \begin{pmatrix}0\\2\end{pmatrix},\ \text{and } T\left(\underbrace{\begin{pmatrix}2\\0\end{pmatrix}}_{v_3}\right) = \begin{pmatrix}2\\2\end{pmatrix}.$$

(b) ✓ $\mathrm{Span}\{v_1, v_2, v_3\} = \mathbb{R}^2$

$T(e_1) = T(\frac{1}{2}v_3) = \frac{1}{2}T(v_3) = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad T(e_2) = \frac{1}{2}T(v_2) - \frac{1}{2}T(v_1) = \begin{pmatrix}-1\\1\end{pmatrix}$ $\rightsquigarrow$ consistent

$$T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}T(e_1) & T(e_2)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

(c) $T : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T\left(\underbrace{\begin{pmatrix}1\\-1\\0\end{pmatrix}}_{v_1}\right) = -1,\ T\left(\underbrace{\begin{pmatrix}0\\1\\-1\end{pmatrix}}_{v_2}\right) = 1 \text{ and } T\left(\underbrace{\begin{pmatrix}-1\\0\\1\end{pmatrix}}_{v_3}\right) = 0.$$

(c) ✗ $\det(v_1\ v_2\ v_3) = 0$, i.e. $\dim(\mathrm{Span}\{v_1, v_2, v_3\}) \leq 2$,

$\mathrm{Span}\{v_1\ v_2\ v_3\} \subsetneq \mathbb{R}^3$

There is no information to determine $T(v)$ for $v \notin \mathbb{R}^3 \setminus \mathrm{Span}\{v_1, v_2, v_3\}$

e.g. two possible L.T.s

$$T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = y \quad \text{or} \quad T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = -x - z$$

4. For each of the following linear transformations T, determine its rank and nullity, and whether it is one-to-one, and/or onto.

(a) $T\colon \mathbb{R}^4 \to \mathbb{R}^6$ such that the rank is 4.

(b) $T\colon \mathbb{R}^6 \to \mathbb{R}^4$ such that the nullity is 2.

(c) $T : \mathbb{R}^4 \to \mathbb{R}^6$ such that the reduce row-echelon form of its standard matrix has 3 nonzero rows.

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that T is one-to-one.

4. (a) $\text{nullity}(T) = 4 - \text{rank}(T) = 0$ one-to-one ✓ onto ✗

(b) $\text{rank}(T) = 6 - \text{nullity}(T) = 4$ one-to-one ✗ onto ✓

(c) $\text{rank}(T) = 3$ $\text{nullity}(T) = 4 - 3 = 1$ ✗ ✗

(d) $\text{nullity}(T) = 0$ $\text{rank}(T) = 3 - 0 = 3$ ✓ ✓

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{cases}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{pmatrix}
$$

$$
A \qquad x \quad = \quad b
$$

REF / RREF

elementary matrices

row-equivalent matrices

inverse of a matrix

determinant

adjoint of a matrix

LU-factorization

---

linear span

linear / vector space

affine space

linearly independent / dependent

basis

dimension

coordinates of a vector

transition matrix

column space

rank

row space

null space nullity

---

inner product

norm, angle

orthogonal / orthonormal

orthogonal / orthonormal basis

orthogonal projection

orthogonal matrix

QR-factorization

least square approximation

---

eigenvalue, eigenvector, eigenspace

characteristic polynomial

algebraic multiplicity

geometric multiplicity

diagonalization

orthogonal diagonalization

singular value

singular value decomposition

---

linear transformation

standard matrix

range, kernel

injection, surjection, bijection
