MA1522: Linear Algebra for Computing

Chapter 3: Euclidean Vector Spaces

3.1 Euclidean Vector Spaces

Vectors

Recall that a (real) *n-vector* (or *vector*) is a collection of n ordered real numbers,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \text{ where } v_i \in \mathbb{R} \text{ for } i = 1, ..., n.$$

Here the entry v_i is also known as the i-th coordinate.

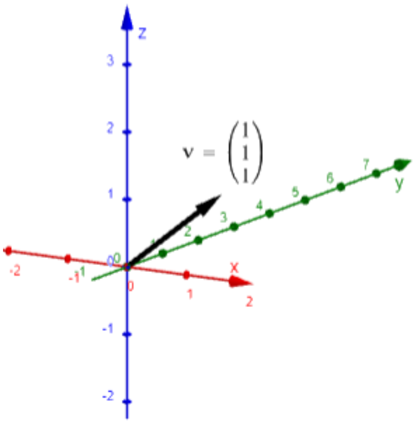
Definition

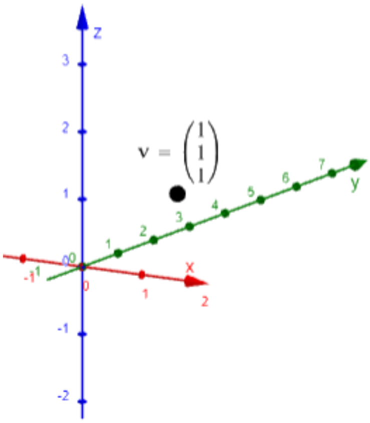
The *Euclidean n-space*, denoted as $\mathbb{R}^n$, is the collection of all n-vectors

$$\mathbb{R}^n = \left\{ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| \, v_i \in \mathbb{R} \text{ for } i = 1, ..., n. \right\}.$$

Geometric Interpretation of Vectors

Geometrically, a vector $\mathbf{v}$ can be interpreted as an arrow, with the tail placed at the origin $\mathbf{0}$, and the head of the arrow at $\mathbf{v}$, or it could represent a position in the Euclidean n-space. For example, the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$ represents both the point and the arrow.





Geometric Interpretation of Vector Algebra

Since vectors are matrices, we are able to apply the matrix algebra on vectors. These operations have geometrical interpretations.

1. Adding $\mathbf{u}$ to $\mathbf{v}$ is visualized as putting the tail of $\mathbf{v}$ at the head of $\mathbf{u}$, and the head of $\mathbf{v}$ is the resultant,

2. Scalar multiple of a vector is scaling the vector,

Vectors Algebra

The following properties follows from properties of matrix algebra. However, try using the geometrical interpretations to prove the following properties.

Theorem

Let $\mathbb{R}^n$ be a Euclidean vector space. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$ and a, b be some real numbers.

(i) *The sum $\mathbf{u} + \mathbf{v}$ is a vector in $\mathbb{R}^n$.*

(ii) *(Commutative) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.*

(iii) *(Associative) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.*

(iv) *(Zero vector) $\mathbf{0} + \mathbf{v} = \mathbf{v}$.*

(v) *The negative $-\mathbf{v}$ is a vector in $\mathbb{R}^n$ such that $\mathbf{v} - \mathbf{v} = \mathbf{0}$.*

(vi) *(Scalar multiple) $a\mathbf{v}$ is a vector in $\mathbb{R}^n$.*

(vii) *(Distribution) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.*

(viii) *(Distribution) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.*

(ix) *(Associativity of scalar multiplication) $(ab)\mathbf{u} = a(b\mathbf{u})$.*

(x) *If $a\mathbf{u} = \mathbf{0}$, then either $a = 0$ or $\mathbf{u} = \mathbf{0}$.*

Abstract Vector Spaces

Some of these properties of Euclidean vector space tells us that it is a (an abstract) vector space.

Definition

A set V equipped with addition and scalar multiplication is said to be a *vector space* over $\mathbb{R}$ if it satisfies the following axioms.

1. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.
2. (Commutative) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. (Associative) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
4. (Zero vector) There is a vector $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$ in V.
5. (Negative) For any vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. For any scalar a in $\mathbb{R}$ and vector $\mathbf{v}$ in V, $a\mathbf{v}$ is a vector in V.
7. (Distribution) For any scalar a in $\mathbb{R}$ and vectors $\mathbf{u}, \mathbf{v}$ in V, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
8. (Distribution) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
9. (Associativity of scalar multiplication) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $a(b\mathbf{u}) = (ab)\mathbf{u}$.
10. For any vector $\mathbf{u}$ in V, $1\mathbf{u} = \mathbf{u}$.

Challenge

1. Show that the set of all degree at most n polynomials with real coefficients is a vector space with the usual addition and scalar multiplication,
   (i) $b(a_n x^n + \cdots + a_1 x + a_0) = ba_n x^n + \cdots + ba_1 x + ba_0$,
   (ii) $(a_n x^n + \cdots + a_1 x + a_0) + (b_n x^n + \cdots + b_1 x + b_0) = (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + (a_0 + b_0)$.

2. Show that the set of all $n \times m$ real-valued matrices is a vector space, with the usual matrix addition and scalar multiplication. The set of all $n \times m$ real-valued matrices is sometimes denoted as $\mathbb{R}^{n \times m}$.

3.2 Dot Product, Norm, Distance

Discussion

Matrix addition and scalar multiplication can be applicable directly to vectors. However, how do we, if it's even possible, define the multiplication of vectors?

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be two (column) vectors. The multiplication

$$\underset{(n \times 1)}{\mathbf{u}} \quad \underset{(n \times 1)}{\mathbf{v}}$$

is undefined.

Multiplying Vectors

We are able to multiply if we transpose one of the vectors.

1. (**Outer Product**) $\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^T = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & \cdots & u_n v_n \end{pmatrix} = (u_i v_j)_n$ (Not part of syllabus)
2. (**Inner Product**) $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$. Also known as *dot product*.

Definition

The *inner product* (or *dot product*) of vectors $\mathbf{u} = (u_i)$ and $\mathbf{v} = (v_i)$in $\mathbb{R}^n$ is defined to be

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example

1. $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = (1)(2) + (2)(2) + (-1)(2) = 4.$

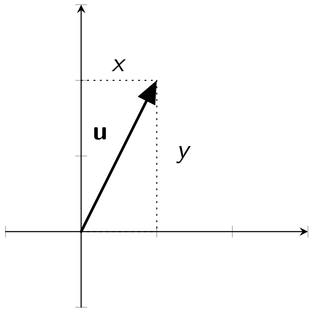
2. $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (1)(1) + (0)(1) + (-1)(1) = 0.$

3. $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = (2)(1) + (3)(-2) = -4.$

Norm in $\mathbb{R}^2$

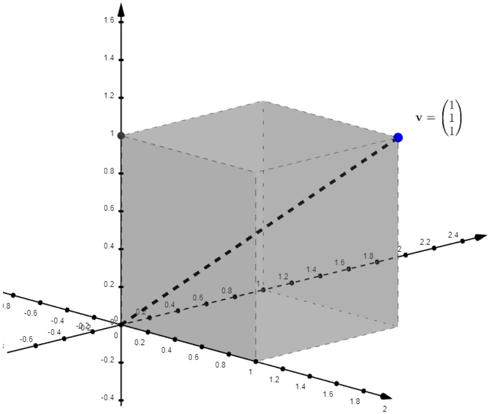
The distance between the point $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and the origin in $\mathbb{R}^2$ is given by

$$\text{distance} = \sqrt{x^2 + y^2}.$$



Question

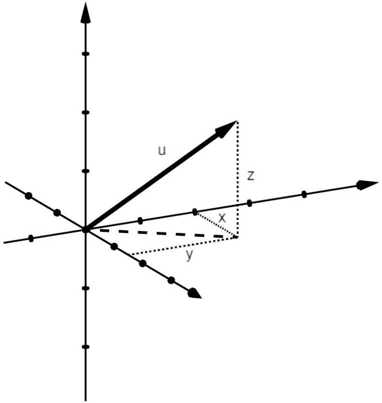
What is the length of the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?



Norm in $\mathbb{R}^3$

The distance between the point $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the origin in $\mathbb{R}^3$ is given by

$$\text{distance} = \sqrt{(\sqrt{x^2+y^2})^2 + z^2} = \sqrt{x^2+y^2+z^2}.$$



Norm in $\mathbb{R}^n$

Definition

The *norm* of a vector $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} = (u_i)$, is the square root of the inner product of $\mathbf{u}$ with itself, and is denoted as $\|\mathbf{u}\|$,

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

This is also known as the *length* or *magnitude* of the vector.

Properties of Inner Product and Norm

Theorem

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$, and a, b, c be some scalars.

(i) *(Symmetric)* $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(ii) *(Scalar multiplication)* $c\mathbf{u} \cdot \mathbf{v} = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$.

(iii) *(Distribution)* $\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{u} \cdot \mathbf{w}$.

(iv) *(Positive definite)* $\mathbf{u} \cdot \mathbf{u} \geq 0$ *with equality if and only if* $\mathbf{u} = \mathbf{0}$.

(v) $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$.

Partial Proof.

Proof for (iv) only. The rest are left as exercise. Let $\mathbf{u} = (u_i)_{n\times 1}$. Since $\mathbf{u}_i \in \mathbb{R}$, $u_i^2 \geq 0$ for all $i = 1, ..., n$. Therefore,

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0.$$

Note also that this is a sum of nonnegative numbers, which is equal to 0 if and only if all the $u_i^2 = 0$, which is equivalent to $u_i = 0$ for all $i = 1, ..., n$. $\square$

Unit Vectors

Definition

A vector $\mathbf{u}$ in $\mathbb{R}^n$ is a *unit vector* if its norm is 1,

$$\|\mathbf{u}\| = 1$$

Example

1. Let $\mathbf{e}_i$ denote the i-th column of the $n \times n$ identity matrix $\mathbf{I}_n$. Then $\mathbf{e}_i$ is a unit vector for all $i = 1, 2, ..., n$.
2. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is a unit vector.
3. $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is not a unit vector; $\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is a unit vector pointing in the same direction.

Normalizing a Vector

Let $\mathbf{u}$ be a nonzero vector $\mathbf{u} \neq \mathbf{0}$. By multiplying by the reciprocal of the norm, we get a unit vector,

$$\mathbf{u} \longrightarrow \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Indeed, $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ is a unit vector,

$$\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) \cdot \left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{\mathbf{u} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} = 1.$$

This is called *normalizing* $\mathbf{u}$.

Distance Between Vectors

By Pythagorous theorem, the distance between $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ in $\mathbb{R}^2$ is

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \left\| \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\|.$$

Similarly in $\mathbb{R}^3$, the distance between $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ is

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \left\| \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\|.$$

Definition

The *distance* between two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $d(\mathbf{u}, \mathbf{v})$, is defined to be

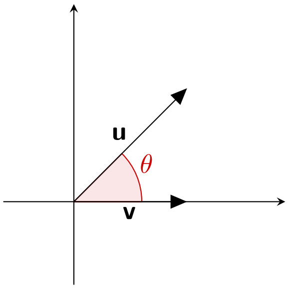
$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Angle

Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$.

The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is

$$\cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x x_0}{\|\mathbf{u}\| x_0} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

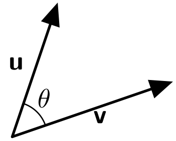


Definition

Define the *angle* θ between two nonzero vectors, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ to be such that

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

Note that $0 \leq \theta \leq \pi$.



3.3 Linear Combinations and Linear Spans

Linear Combinations

Definition

Let $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. A *linear combination* of the vectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ is

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

for some $c_1, c_2, ..., c_k \in \mathbb{R}$. The scalars $c_1, c_2, ..., c_k$ are called *coefficients*.

Think of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ as the directions, and $c_1, c_2, ..., c_k$ as the amount of units to walk in the respective directions.

Example

Consider the vectors $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$.

Click on the following link `https://www.geogebra.org/m/qzhtjwcc`. Adjust the different values of c_1 and c_2 to visualize the linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(i) When $c_1 = c_2 = 1$, $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(ii) When $c_1 = 2$ and $c_2 = -1$, $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

(iii) When $c_1 = 3/2$ and $c_2 = 1/2$, $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} 5/2 \\ 2 \end{pmatrix}$.

(iv) When $c_1 = -1$ and $c_2 = 3$, $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$.

Linear Span

Definition

Let $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. The *span* of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ is the subset of $\mathbb{R}^n$ containing all the linear combinations of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$,

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \left\{ \ c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \ \middle| \ c_1, c_2, ..., c_k \in \mathbb{R} \ \right\}.$$

That is every vector $\mathbf{v}$ in the set $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$,

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

for some scalars $c_1, c_2, ..., c_k$.

Example

Click on the following link https://www.geogebra.org/m/n7ypnzsn. This activity will demonstrate the span of the 2 vectors $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$.

- Click on the play button besides c_1 and c_2 to see the different linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$.
- The collection of all these linear combination is the orange plane.
- Consider the vector $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Is it in the span of $\mathbf{u}_1$ and $\mathbf{u}_2$?

Example

Consider the vectors $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Is the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$? Equivalently, is $\mathbf{v}$ in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$?

$\mathbf{v}$ is in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if and only if there exists coefficients c_1, c_2, and c_3 such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, that is,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

This is a vector equation, which when written as a matrix equation gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Example

This is a linear system. Solving it, we have

$$\left(\begin{array}{ccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{v} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{array}\right).$$

Since the system is consistent, we can conclude that $\mathbf{v}$ is in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Moreover, the solution of the system tells us that $c_1 = 6, c_2 = -2, c_3 = -3$, that is,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Example

Now consider $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Is $\mathbf{v}$ in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$?

Find c_1, c_2, and c_3 such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & -1 & 1 & 3 \end{array}\right) \xrightarrow{R_3 - R_1} \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right).$$

The system is inconsistent. Hence, $\mathbf{v}$ is not in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

In fact, if you plot the span of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in `https://geogebra.org/3d`, you will see that the span is a plane and $\mathbf{v}$ is outside the plane.

Algorithm to Check for Linear Combination

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

- Form the $n \times k$ matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ whose columns are the vectors in S.
- Then a vector $\mathbf{v}$ in $\mathbb{R}^n$ is in span$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ if and only if the system $\mathbf{Ax} = \mathbf{v}$ is consistent.
- If the system is consistent, then the solutions to the system are the possible coefficients of the linear combination. That is, if $\mathbf{u} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ is a solution to $\mathbf{Ax} = \mathbf{v}$, then

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

Explicitly, $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ if and only if $\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array}\right)$ is consistent.

Question

Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

1. Is $\mathbf{v}$ in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$?

2. If it is, write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$,

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

3. Are the coefficients c_1, c_2, c_3 unique?

Question

Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Find a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that is not in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

When will $\text{span}(S) = \mathbb{R}^n$?

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ...\mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. Now instead of checking if a specific vector $\mathbf{v}$ is in $\text{span}(S)$, we may ask if every vector is in the span, that is, whether $\text{span}(S) = \mathbb{R}^n$.

Example

1. $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\}$. Now we check if every $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in $\text{span}(S)$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 1 & 2 & 3 & y \\ 1 & 1 & 2 & z \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2x - y \\ 0 & 1 & 1 & -x + y \\ 0 & 0 & 0 & -x + z \end{array} \right).$$

The system is consistent if and only if $z - x = 0$. This show that not every vector in $\mathbb{R}^3$ is in $\text{span}(S)$, that is, $\text{span}(S) \neq \mathbb{R}^3$. For example, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is not in the span.

When will span$(S) = \mathbb{R}^n$?

2. Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$. Is span$(S) = \mathbb{R}^3$?

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & -1 & 2 & y \\ 1 & 0 & 1 & z \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -x-y+3z \\ 0 & 1 & 0 & x-z \\ 0 & 0 & 1 & x+y-2z \end{array} \right).$$

The system is always consistent regardless of any choice of x, y, z. This show that span$(S) = \mathbb{R}^3$. In fact, given any $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-x-y+3z) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (x-z) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + (x+y-2z) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Discussion

Consider now a vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$. Observe that elementary row operations would not make any entries zero; every entry would still be a linear combination of $x_1, x_2, \ldots, x_n$.

Example

1. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix}$
2. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xrightarrow{R_3 - aR_1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 - ax_1 \end{pmatrix}$
3. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xrightarrow{cR_2} \begin{pmatrix} x_1 \\ cx_2 \\ x_3 \end{pmatrix}$, for some $c \neq 0$.

Discussion

This means that in the reduction of $\left(\begin{array}{cccc|c} & & & & x_1 \\ & & & & x_2 \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \vdots \\ & & & & x_n \end{array}\right)$, the entries in the last column will never be 0, but some linear combination of $x_1, x_2, ..., x_n$. In this case, the system is consistent if and only if the reduced row-echelon form of $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ does not have any zero row.

Algorithm to check if $\text{span}(S) = \mathbb{R}^n$.

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

- Form the $n \times k$ matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ whose columns are the vectors in S.
- Then $\text{span}(S) = \mathbb{R}^n$ if and only if the system $\mathbf{Ax} = \mathbf{v}$ is consistent for all $\mathbf{v}$.
- This is equivalent to the reduced row-echelon form of $\mathbf{A}$ having no zero rows.

Explicitly, $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \mathbb{R}^n$ if and only if the reduced row-echelon form of $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ has no zero rows.

Example

The $n \times n$ identity matrix $\mathbf{I}_n$ is in reduced row-echelon form and does not have any zero rows. Hence, its columns span $\mathbb{R}^n$.

Indeed, let $\mathbf{e}_i$ denote the i-th column of $\mathbf{I}_n$ for $i = 1, ..., n$. Then for any vector $\mathbf{w}$,

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = w_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + w_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Hence, $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\} = \mathbb{R}^n$. This set is called the *standard basis* of $\mathbb{R}^n$.

Example

Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\mathbf{u}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Putting the vectors as columns of a matrix and reducing, $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$, we can conclude that $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \mathbb{R}^3$. Indeed, given any $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$,

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & x \\ 1 & -1 & 2 & 0 & y \\ 1 & 0 & 1 & 1 & z \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3z-y-x \\ 0 & 1 & 0 & 0 & x-z \\ 0 & 0 & 1 & -1 & x+y-2z \end{array}\right)$$

tells us that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (3z-y-x-2s)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (x-z)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + (x+y-2x+s)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ for any $s \in \mathbb{R}$.

Example

Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. Then $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ tells us that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \neq \mathbb{R}^3$. Indeed,

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 1 & -1 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \xrightarrow{R_3 - \frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 0 & -2 & -2 & y - x \\ 0 & 0 & 0 & z - y/2 - x/2 \end{array}\right)$$

tells us that whenever $z - y/2 - x/2 \neq 0$, the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is not in the span, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Question

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of k vectors in $\mathbb{R}^n$.

1. Show that if $k < n$ then $\text{span}(S) \neq \mathbb{R}^n$.

2. If $k > n$, can we make any conclusion?

Properties of Linear Spans

Theorem (Properties of Linear Spans)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

(i) *The zero vector $\mathbf{0}$ is in $\text{span}(S)$.*

(ii) *The span is closed under scalar multiplication, that is, for any vector $\mathbf{u}$ in $\text{span}(S)$ and scalar α, the vector $\alpha\mathbf{u}$ is a vector in $\text{span}(S)$.*

(iii) *The span is closed under addition, that is, for any vectors $\mathbf{u}, \mathbf{v}$ in $\text{span}(S)$, the sum $\mathbf{u} + \mathbf{v}$ is a vector in $\text{span}(S)$.*

Proof.

We will only provide the main idea of the proof, the details are left to the readers.

(i) $\mathbf{0} = 0\mathbf{u}_1 + 0\mathbf{u}_2 + \cdots + 0\mathbf{u}_k$.

(ii) Write $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$. Then $\alpha\mathbf{v} = (\alpha c_1)\mathbf{u}_1 + (\alpha c_2)\mathbf{u}_2 + \cdots + (\alpha c_k)\mathbf{u}_k$.

(iii) Write $\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ and $\mathbf{v} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k$. Then $\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{u}_1 + (c_2 + d_2)\mathbf{u}_2 + \cdots + (c_k + d_k)\mathbf{u}_k$.

$\square$

Properties of Linear Spans

Remark

Properties (ii) and (iii) can be combined together into one property (ii'):
The span is closed under linear combinations, that is, if $\mathbf{u}, \mathbf{v}$ are vectors in $\text{span}(S)$ and α, β are any scalars, then the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is a vector in $\text{span}(S)$.

Observe that property (ii') implies that $\text{span}(S)$ is closed under linear combination. That is, suppose $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are vectors in $\text{span}(S)$, then for any scalars $c_1, c_2, ..., c_m$, the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ is also in the span. For by property (ii'), $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is in $\text{span}(S)$, and thus by property (ii') again, we have $(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + c_3\mathbf{v}_3$ is in $\text{span}(S)$ too. Thus, by induction, we can conclude that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ is in $\text{span}(S)$.
Since this is true for any scalars $c_1, c_2, ..., c_m$, we have arrived at the following corollary.

Corollary (Linear span is closed under linear combinations)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *be a set of vectors in* $\mathbb{R}^n$. *For any vectors* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *in* $\text{span}(S)$, *the span of* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *is a subset of* $\text{span}(S)$,

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\} \subseteq \text{span}(S).$$

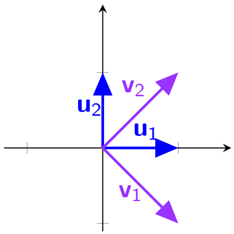
Example

Let $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. It is easy to see that the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ are in span(S).

By the corollary, given any c_1, c_2, the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{pmatrix} c_1 + c_2 \\ c_2 - c_1 \\ 0 \end{pmatrix}$ is in span(S).

Indeed,

$$\begin{pmatrix} c_1 + c_2 \\ c_2 - c_1 \\ 0 \end{pmatrix} = (c_1+c_2)\mathbf{u}_1+(c_2-c_1)\mathbf{u}_2 = (c_1+c_2)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+(c_2-c_1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$



In fact, observe that in this case, span$\{\mathbf{v}_1, , \mathbf{v}_2\} = $ span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

Algorithm to check for Set Relations between Spans

Now suppose we are given 2 sets of vectors $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.

- By the corollary, if $\mathbf{v}_i \in \text{span}(S)$ for $i = 1, ..., m$, we can conclude that $\text{span}(T) \subseteq \text{span}(S)$.
- Recall that to check if $\mathbf{v}_i \in \text{span}(S)$, we check that the system $\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_i \end{array}\right)$ is consistent for all $i = 1, ..., m$.
- There are in total m such linear systems to check. However, since they have the same coefficient matrix, we may combine and check them together, that is, check that

$$\left(\begin{array}{cccc|c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \end{array}\right)$$

is consistent.

Algorithm to check for Set Relations between Spans

Theorem

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *and* $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ *be sets of vectors in* $\mathbb{R}^n$. *Then* $\text{span}(T) \subseteq \text{span}(S)$ *if and only if* $\left(\begin{array}{cccc|c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \end{array}\right)$ *is consistent.*

So, to check if $\text{span}(S) = \text{span}(T)$, we check that

- $\text{span}(S) \subseteq \text{span}(T)$, that is,

$$\left(\begin{array}{c|c} \text{"}T\text{"} & \text{"}S\text{"} \end{array}\right) = \left(\begin{array}{cccc|c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{array}\right) \text{ is consistent, and}$$

- $\text{span}(T) \subseteq \text{span}(S)$, that is,

$$\left(\begin{array}{c|c} \text{"}S\text{"} & \text{"}T\text{"} \end{array}\right) = \left(\begin{array}{cccc|c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \end{array}\right) \text{ is consistent.}$$

Example

Let $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$,and $T = \left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

The augmented matrix

$$\left(\begin{array}{cc|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{v}_1 & \mathbf{v}_2 \end{array} \right) = \left(\begin{array}{cc|c|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

is already in reduced row-echelon form, and since the system is consistent, we can conclude that $\mathrm{span}(T) \subseteq \mathrm{span}(S)$.

On the other hand,

$$\left(\begin{array}{cc|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u}_1 & \mathbf{u}_2 \end{array} \right) = \left(\begin{array}{cc|c|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{cc|c|c} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

is consistent too. This shows that $\mathrm{span}(S) \subseteq \mathrm{span}(T)$ too.

Therefore we conclude that $\mathrm{span}(S) = \mathrm{span}(T)$.

Example

Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\}$ and $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

To check if $\text{span}(S) \subseteq \text{span}(T)$,

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 & 1 & 2 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right) \text{ is consistent.}$$

To check if $\text{span}(T) \subseteq \text{span}(S)$,

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & -1 & 2 \\ 1 & 1 & 2 & 1 & 0 & 1 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right) \text{ is not consistent.}$$

This shows that $\text{span}(T) \not\subseteq \text{span}(S)$. In particular, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \notin \text{span}(S)$.

Question

Let S and T be the sets given in the previous example.

1. Observe the left hand side of the augmented matrix in the reduction

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 & 1 & 2 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right).$$

   What can you conclude about span(T)?

2. Observe the left hand side of the augmented matrix in the reduction

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & -1 & 2 \\ 1 & 1 & 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

   What can you conclude about span(S)?

Challenge

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. Referring to the properties of a spanning set or otherwise, show that the set $V = \text{span}(S)$ is a (abstract) vector space. That is, it satisfies the 10 axioms of the definition of vector spaces.

3.4 Subspaces

Solution Sets to a Linear system

Recall that the set of solutions to a linear system $\mathbf{Ax} = \mathbf{b}$ is a subset in $\mathbb{R}^n$ (it is the empty set if the system is inconsistent). We may express this set *implicitly* as

$$V = \left\{ \mathbf{u} \in \mathbb{R}^n \;\middle|\; \mathbf{Au} = \mathbf{b} \right\},$$

or *explicitly* as

$$V = \left\{ \mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k \;\middle|\; s_1, s_2, ..., s_k \in \mathbb{R} \right\},$$

where $\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k$, $s_1, s_2, ..., s_k \in \mathbb{R}$ is the general solution.

Example

Consider the linear system

$$\left\{ \begin{array}{ccccccc} x & + & y & & & = & 0 \\ & & & & z & = & 1 \end{array} \right.$$

It can be written implicitly as

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x = -y, z = 1 \right\}$$

or explicitly as

$$V = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \middle| \; s \in \mathbb{R} \right\}.$$

Example

Consider the linear system

$$\begin{array}{rcrcrcl} 3x & + & 2y & - & z & = & 1 \\ & & y & - & z & = & 0 \end{array}$$

Implicitly, it can be written as

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; 3x + 2y - z = 1, y - z = 0 \right\}.$$

The general solution is

$$x = \frac{1}{3}(1 - s), \quad y = s, \quad z = s, \quad s \in \mathbb{R}.$$

So, explicitly, the solution set is

$$\left\{ \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}.$$

Solution Sets to Linear Systems

Write the implicit expression of the following solution set

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}.$$

$$\begin{array}{rclrclrcl} x & = & 1 - 2s + t, & y & = & 2 + s, & z & = & t - 1 \\ x & = & 1 - 2(y-2) + z + 1, & s & = & y - 2, & t & = & z + 1 \end{array} \Rightarrow x + 2y - z = 6$$

So, implicitly, the set has the expression

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x + 2y - z = 6 \right\}.$$

Discussion

Recall that the general solution of a homogeneous system $\mathbf{Ax} = \mathbf{0}$ has the form

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k, \quad s_1, s_2, \ldots, s_k \in \mathbb{R}.$$

Explicitly, the solution set is

$$V = \left\{ \ s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k \ \middle| \ s_1, s_2, \ldots, s_k \in \mathbb{R} \ \right\}.$$

Observe however that this is just $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$,

$$V = \left\{ \ s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k \ \middle| \ s_1, s_2, \ldots, s_k \in \mathbb{R} \ \right\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

By the properties of a linear span, this would mean that the solution set to a homogeneous system is a vector space that is a subset of the Euclidean vector space. We call a vector space nested inside another vector space a *subspace*.

Example

Let V be the solution set to the system

$$x \quad - \quad y \quad + \quad z \quad = \quad 0 \ .$$

Explicitly,

$$V = \left\{ \ s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \ \middle| \ s, t \in \mathbb{R} \ \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Subspace

It turns out that for a subset V of the Euclidean space $\mathbb{R}^n$ to satisfy all 10 axioms of being a vector space, suffice for it to satisfies only 3 of them.

Definition

A subset V of $\mathbb{R}^n$ is a *subspace* if it satisfies the following properties.

(i) V contains the zero vector $\mathbf{0} \in V$.

(ii) V is closed under scalar multiplication. For any vector $\mathbf{v}$ in V and scalar α, the vector $\alpha\mathbf{v}$ is in V.

(iii) V is closed under addition. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.

Remark

(i) Property (i) can be replaced with property (i'): V is nonempty.

(ii) Properties (ii) and (iii) is equivalent to property (ii'):
V is closed under linear combination. For any $\mathbf{u}, \mathbf{v}$ in V, and scalars α, β, the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is in V.

Solution Space of Homogeneous System

Theorem

The solution set $V = \left\{ \mathbf{u} \,\middle|\, \mathbf{A}\mathbf{u} = \mathbf{b} \right\}$ *to a linear system* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *is a subspace if and only if* $\mathbf{b} = \mathbf{0}$*, that is, the system is homogeneous.*

Proof.

($\Rightarrow$) Suppose $V = \left\{ \mathbf{u} \,\middle|\, \mathbf{A}\mathbf{u} = \mathbf{b} \right\}$ is a subspace. By property (i), it must contain the origin, which means that $\mathbf{0}$ must be a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$. Hence,

$$\mathbf{0} = \mathbf{A}\mathbf{0} = \mathbf{b} \quad \Rightarrow \quad \mathbf{b} = \mathbf{0}.$$

($\Leftarrow$) Suppose $\mathbf{b} = \mathbf{0}$, that is, $V = \left\{ \mathbf{u} \,\middle|\, \mathbf{A}\mathbf{u} = \mathbf{0} \right\}$ the solution set to a homogeneous system.

- Clearly $\mathbf{0} \in V$
- For any $\mathbf{v} \in V$, that $\mathbf{A}\mathbf{v} = \mathbf{0}$, and any $\alpha \in \mathbb{R}$, $\mathbf{A}(\alpha\mathbf{v}) = \alpha\mathbf{A}\mathbf{v} = \alpha\mathbf{0} = \mathbf{0} \Rightarrow \alpha\mathbf{v} \in V$.
- Suppose $\mathbf{u}, \mathbf{v} \in V$, that is $\mathbf{A}\mathbf{u} = \mathbf{0}$ and $\mathbf{A}\mathbf{v} = \mathbf{0}$. Then $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow \mathbf{u} + \mathbf{v} \in V$.

$\square$

Definition

The solution set to a homogeneous system is call a *solution space*.

Examples

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ x + y + z = 0 \right\}$. Since it is a solution set of a homogeneous system, it is a subspace. We will also show that it satisfies the 3 criteria.

(i) Clearly $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in V.

(ii) Suppose $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V$, that is $x + y + z = 0$. Then for any $\alpha \in \mathbb{R}$, $\alpha x + \alpha y + \alpha z = \alpha(x + y + z) = \alpha(0) = 0$.

(iii) Suppose $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ are in V. Then

$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = (0) + (0) = 0.$$

Example

Is the set $V = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$ a subspace?

It is not a subspace since it does not contain $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Challenge

Prove that if a subset V of $\mathbb{R}^n$ satisfies the 3 criteria of a subspace, then it satisfies all 10 axioms of a vector space.

Equivalent Definition for Subspaces

Theorem

A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if is a linear span, $V = span(S)$, for some finite set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.

Proof.

($\Leftarrow$) This follows from the property of linear span.

($\Rightarrow$) Only present a sketch, details are left as exercise.
Since V is a subspace, it is nonempty. Take a $\mathbf{u}_1 \in V$. If $\text{span}(\mathbf{u}_1) = V$, let $S = \{\mathbf{u}_1\}$. Otherwise, there is a $\mathbf{u}_2 \in V \backslash \text{span}(\mathbf{u}_1)$. If $\text{span}(\mathbf{u}_1, \mathbf{u}_2) = V$, let $S = \{\mathbf{u}_1, \mathbf{u}_2\}$. Otherwise, continue this process to define $\mathbf{u}_i \in V \backslash \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{i-1}\}$. Eventually, the process must stop, that is, there is a $k \in \mathbb{Z}$ such that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = V$ (why?). $\square$

Remarks

1. To show that a set V is a subspace, we can either
   (a) find a spanning set, that is find a set S such that $V = \text{span}(S)$, or

   (b) show that V satisfies the 3 conditions of being a subspace.

2. To show that a subset V is not a subspace, we can either
   (i) show that it does not contain the zero vector, $\mathbf{0} \notin V$,

   (ii) find a vector $\mathbf{v} \in V$ and a scalar $\alpha \in \mathbb{R}$ such that $\alpha\mathbf{v} \notin V$, or

   (iii) find vectors $\mathbf{u}, \mathbf{v} \in V$ such that the sum is not in V, $\mathbf{u} + \mathbf{v} \notin V$.

Example

1. $V = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a subspace.

2. $V = \left\{ \begin{pmatrix} x + y \\ x - y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ is a subspace.

Example

3. $V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; ab = cd \right\}$ is not a subspace because $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ belong to to V, but $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ does not.
4. $V = \left\{ \begin{pmatrix} s \\ s^2 \\ t \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$ is not a subspace since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ belongs to V, but $2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ does not.

Question

1. Show that the set containing the zero vector $\{\mathbf{0}\}$ is a subspace.

2. Construct a set V such that it satisfies condition (i) and (ii) but not (iii); that is, V contains the origin and is closed under scalar multiplication, but not closed under addition.

Question

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ a subset of V, $S \subseteq V$. Show that the span of S is contained in V, $\text{span}(S) \subseteq V$.

Subspaces of $\mathbb{R}^2$

(i) Zero space: $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ This is a point.

(ii) Lines, $L = \text{span}\left\{ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\}$ for some fixed $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, These are lines, which looks like $\mathbb{R}^1$.

(iii) Whole $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^3$

(i) Zero space: $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ This is a point.

(ii) Lines: $L = \text{span} \left\{ \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \right\}$ for some fixed $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. These are lines, which looks like $\mathbb{R}^1$.

(iii) Planes, $P = \text{span} \left\{ \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\}$ for some $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ that are not a scalar multiple of each other, These are planes, which looks like $\mathbb{R}^2$.

(iv) Whole $\mathbb{R}^3$.

Solution Set to Non-homogeneous System

Recall that

$$\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k, s_1, s_2, ..., s_k \in \mathbb{R}$$

is a general solution to a consistent non-homogeneous system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$ if and only if

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k, s_1, s_2, ..., s_k \in \mathbb{R}$$

is a general solution to the homogeneous system $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{u}$ is a particular solution to the non-homogeneous system $\mathbf{Ax} = \mathbf{b}$.

Theorem (Affine Space)

The solution set $W = \left\{ \mathbf{w} \mid \mathbf{Aw} = \mathbf{b} \right\}$ *of a non-homogeneous linear system* $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$ *is given by*

$$\mathbf{u} + V := \left\{ \mathbf{u} + \mathbf{v} \mid \mathbf{v} \in V \right\},$$

where $V = \left\{ \mathbf{v} \mid \mathbf{Av} = \mathbf{0} \right\}$ *is the solution space to the associated homogeneous system and* $\mathbf{u}$ *is a particular solution,* $\mathbf{Au} = \mathbf{b}$*.*

That is, vectors in $\mathbf{u} + V$ are of the form $\mathbf{u} + \mathbf{v}$ for some $\mathbf{v}$ in V.

Example

Let $\mathbf{A} = \begin{pmatrix} -1 & 1 & 2 & 1 \\ 3 & 3 & 6 & 9 \\ 3 & -1 & -2 & 1 \end{pmatrix}$.

$$\left(\begin{array}{cccc|c} -1 & 1 & 2 & 1 & 0 \\ 3 & 3 & 6 & 9 & 0 \\ 3 & -1 & -2 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

tells us that the solution set to the homogeneous system is

$$V = \left\{ s\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The solution set V is a subspace.

Example

Let $\mathbf{A} = \begin{pmatrix} -1 & 1 & 2 & 1 \\ 3 & 3 & 6 & 9 \\ 3 & -1 & -2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$.

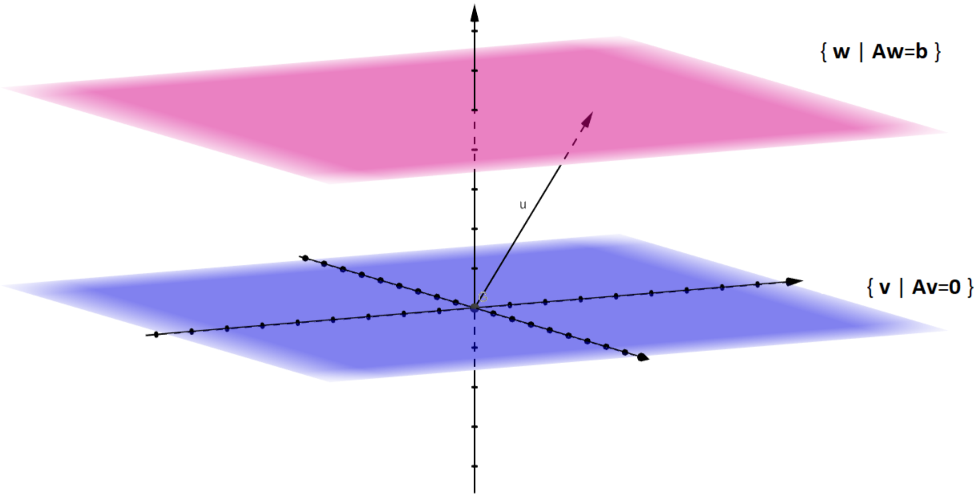
$$\left(\begin{array}{cccc|c} -1 & 1 & 2 & 1 & 3 \\ 3 & 3 & 6 & 9 & 3 \\ 3 & -1 & -2 & 1 & -5 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

tells us that the solution set to the non-homogeneous system is

$$W = \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \text{span}\left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

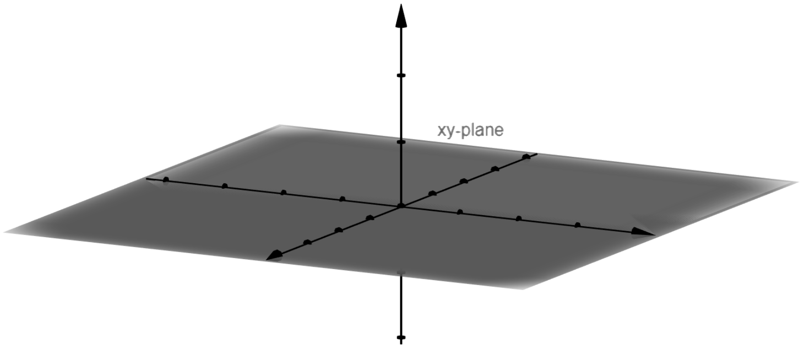
The solution set V is not a subspace as it does not contain the origin. It is shifted away from the origin via the vector $\mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$. Observe that W and V are parallel planes.

Solution Set to Linear System



Question

Is $\mathbb{R}^2 \subseteq \mathbb{R}^3$?



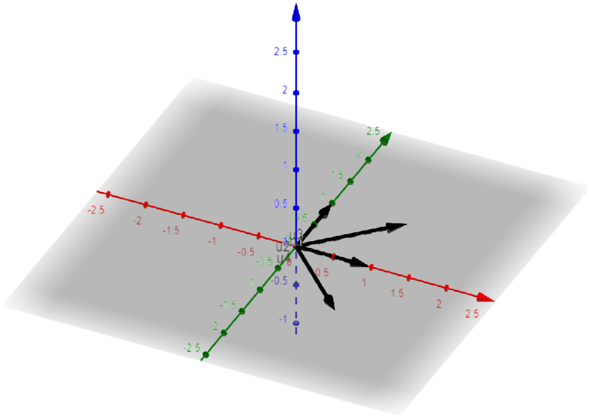
3.5 Linear Independence

Motivation

Consider

$$\mathbf{u}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{u}_4 = \begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

https://www.geogebra.org/m/w2avu5ft



Observe that $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\} \neq \text{span}\{\mathbf{u}_1\}$. This shows that the set $\{\mathbf{u}_1, \mathbf{u}_2\}$ is "optimal"; that is, it is the minimal set to span V. This is because we may use $\mathbf{u}_1 + \mathbf{u}_2$ in place of $\mathbf{u}_3$, and $\mathbf{u}_1 - \mathbf{u}_2$ in place of $\mathbf{u}_4$. Hence, we might say that $\mathbf{u}_3$ and $\mathbf{u}_4$ are "redundant" since they are linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Example

Consider the set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}$.

- Observe that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ tells us that $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ is "redundant".
- But manipulating the equation, we have $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, which tells us that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ are equally "redundant".
- So instead, we might put all the vectors to the left side of the equation and write it as

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Discussion

Now given a set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.

- A vector $\mathbf{u}_i$ is a redundant vector in the span if it is linearly dependent on the others,

$$\mathbf{u}_i = c_1\mathbf{u}_1 + \cdots + c_{i-1}\mathbf{u}_{i-1} + c_{i+1}\mathbf{u}_{i+1} + \cdots + c_k\mathbf{u}_k.$$

- To check for redundancy, we have to check if the system

$$c_1\mathbf{u}_1 + \cdots + c_{i-1}\mathbf{u}_{i-1} + c_{i+1}\mathbf{u}_{i+1} + \cdots + c_k\mathbf{u}_k = \mathbf{u}_i$$

  is consistent for each $i = 1, ..., k$. This is very tedious.

- However, if $\mathbf{u}_i$ is linearly dependent on the other vectors, then we have

$$c_1\mathbf{u}_1 + \cdots + c_{i-1}\mathbf{u}_{i-1} - \mathbf{u}_i + c_{i+1}\mathbf{u}_{i+1} + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

- This is a nontrivial solution, and this checks for all $i = 1, .., k$ simultaneously!

Discussion

- For if suppose we are able to find some $c_1, c_2, \ldots, c_k$ not all zero such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

- Without lost of generality, say $c_k \neq 0$. Manipulating the equation, we have

$$\frac{c_1}{-c_k}\mathbf{u}_1 + \frac{c_2}{-c_k}\mathbf{u}_2 + \cdots + \frac{c_{k-1}}{-c_k}\mathbf{u}_{k-1} = \mathbf{u}_k,$$

  Then we conclude that $\mathbf{u}_k$ is linearly dependent on $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$.
- If none of the vector is linearly dependent on the others, or that the vectors are linearly independent if we cannot find $c_1, c_2, \ldots, c_k$ not all zero such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

Linearly Independent

Definition

A set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is *linearly independent* if the only coefficients $c_1, c_2, ..., c_k$ satisfying the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0},$$

are $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, we say that the set is *linearly dependent*.

Example

Let $\mathbf{e}_i$ be the i-th column of the $n \times n$ identity matrix $\mathbf{I}_n$. Then

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ if and only if } c_1 = 0, c_2 = 0, ..., c_n = 0.$$

Hence the standard basis is linearly independent.

Example

Consider the set $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}$. Suppose $c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Convert the set into a matrix equation, we are solving for $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & -1 & 0 & 0 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

The system has nontrivial solutions. Hence, the set is linearly dependent.

Example

Is the set $S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ linearly independent?

Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Writing it as a matrix equation, we are asking if the homogeneous system $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has nontrivial solutions.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - R_2} \xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

tells us that the homogeneous system has only the trivial solution, and hence, S is linearly independent.

Algorithm to Check for Linear Independence

Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

- $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ is linearly independent if and only if the homogeneous system $(\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_k)\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The homogeneous system has only the trivial solution if and only if the reduce row-echelon form of $\begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k\end{pmatrix}$ has no non-pivot column.

Theorem

A subset $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ of $\mathbb{R}^n$ is linearly independent if and only if the reduced row-echelon form of $\mathbf{A} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k\end{pmatrix}$ has no non-pivot columns.

Examples

1. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$

   $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is already in RREF. Since it has a nonpivot column, S is linearly dependent.

2. $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}$

   $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Since the RREF has a nonpivot column, S is linearly dependent.

Question

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. Let

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_1 + \mathbf{u}_2, \\
\mathbf{v}_3 &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3.
\end{aligned}$$

Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent too.

Question

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in of $\mathbb{R}^n$. Show that if $k > n$, then S is linearly dependent.

Special Cases

1. $\{\mathbf{0}\}$, where $\mathbf{0} \in \mathbb{R}^n$ is the zero vector is always linearly dependent.

   Take say, $c_1 = 1$, then we have $(1)\mathbf{0} = \mathbf{0}$. Alternatively, the matrix $(\mathbf{0})$ is in RREF and the only column is a non-pivot column.

2. If $\mathbf{v} \neq \mathbf{0}$, then $\{\mathbf{v}\} \in \mathbb{R}^n$ is linearly independent.

   The only solution to $c\mathbf{v} = \mathbf{0}$ is $c = 0$. Alternatively, $(\mathbf{v})$ reduces to the matrix with 1 in the first entry and zero otherwise, and the only column is a pivot column.

3. $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if one is a scalar multiple of the other, $\alpha\mathbf{v}_1 = \mathbf{v}_2$ or $\mathbf{v}_1 = \beta\mathbf{v}_2$.

   $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent if and only if c_1 or $c_2 \neq 0$. Say $c_1 \neq 0$. Then $\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2$. The argument for $c_2 \neq 0$ is analogous.

4. The empty set $\{\} = \varnothing$ is linearly independent.

   Vacuously true since there are no vector to check.

Linear Dependency and Adding or Removing Vectors

Theorem

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *is linearly dependent set of vectors in* $\mathbb{R}^n$. *Then for any vector* $\mathbf{u}$ *in* $\mathbb{R}^n$*,*

$$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$$

is linearly dependent.

Since the set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent, we can find a say $c_i \neq 0$ such that

$$c_1\mathbf{u}_1 + \cdots + c_i\mathbf{u}_i + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

Hence, by adding $0\mathbf{u}$, that is, let $c = 0$, we have

$$c_1\mathbf{u}_1 + \cdots + c_i\mathbf{u}_i + \cdots + c_k\mathbf{u}_k + c\mathbf{u} = \mathbf{0},$$

where not all $c, c_1, ..., c_i, ..., c_k$ are zero.

Hence, any set $\{\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{0}\}$ containing the zero vector is linearly dependent.

Linear Dependency and Adding or Removing Vectors

Theorem

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *is linearly independent set of vectors in* $\mathbb{R}^n$ *and* $\mathbf{u}$ *is not a linearly combination of* $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$*. Then the set* $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ *is linearly independent.*

i.e. $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *linearly independent and* $\mathbf{u} \notin \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \Rightarrow \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ *linearly independent.*

Here is a heuristic explanation. Readers may refer to the appendix for the proof.

Since $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly independent, the RREF of $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ has no non-pivot column. Now since $\mathbf{u} \notin \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, the last column of the RREF of $\left(\begin{array}{ccc|c} \mathbf{u}_1 & \cdots & \mathbf{u}_k & \mathbf{u} \end{array} \right)$ is a pivot column. But observe that the LHS of the RREF of $\left(\begin{array}{ccc|c} \mathbf{u}_1 & \cdots & \mathbf{u}_k & \mathbf{u} \end{array} \right)$ is the RREF of $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$. Hence, every column in the RREF of $\left(\begin{array}{ccc|c} \mathbf{u}_1 & \cdots & \mathbf{u}_k & \mathbf{u} \end{array} \right) = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{u} \end{pmatrix}$ is a pivot column. This shows that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ is linearly independent.

Linear Dependency and Adding or Removing Vectors

Theorem

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *is linearly independent set of vectors in* $\mathbb{R}^n$*. Then any subset of* $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *is linearly independent.*

If $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ has no redundancy, then it is clear that any subset cannot have redundancy. Readers may refer to the appendix for the proof.

3.6 Basis and Coordinates

Motivation

Consider the set $E = \left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. It is clear that any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$ can be unique written as a linear combination of the vectors in E,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In fact, we call x, y, z the coordinates of the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. However, the set E is not the only set that enjoys this property.

Motivation

Consider the set $B = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. Now let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a vector in $\mathbb{R}^3$. Then

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & (x+y-z)/2 \\ 0 & 1 & 0 & (x-y+z)/2 \\ 0 & 0 & 1 & (y-x+z)/2 \end{array} \right)$$

tells us that the linear combination

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{x+y-z}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{x-y+z}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{-x+y+z}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

is unique.

Motivation

On the other hand, consider the set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. The vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is not a linear combination of the vectors in S,

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

This shows that $\operatorname{span}(S) \neq \mathbb{R}^3$.

Motivation

Consider another set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Check that the span of S is indeed the whole $\mathbb{R}^3$, $\text{span}(S) = \mathbb{R}^3$. However, the linear combination is not unique. For example, consider the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$,

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right)$$

tells us that

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = (1-s) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (1+s) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - (1+s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for any $s \in \mathbb{R}$. Observe that this is because the set S is not linearly independent.

Motivation

Consider now the solution space $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x + y - 2z = 0 \right\}$. Since it is a subspace of $\mathbb{R}^3$, it is a vector space itself. Explicitly, we have

$$V = \left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Open GeoGebra: `https://geogebra.org/3d`.

1. Type in $x + y - 2z = 0$, enter.
2. Type in $u1 = (-1, 1, 0)$ and hit enter, and $u2 = (2, 0, 1)$ and hit enter.
3. It is easy to see that every vector in V can be written uniquely as a linear combination of the $\mathbf{u}_1$ and $\mathbf{u}_2$.

Motivation

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; y - z = 0 \right\} = \left\{ s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \middle| \; s, t, \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. Check that the set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ spans V. However, the vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ in V can be written as as linear combination of vectors in S in more than one way,

$$\begin{aligned} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Observe that the set S is linearly dependent.

Basis

Definition

Let V be a subspace of $\mathbb{R}^n$. A set $S = \{\mathbf{u}_1, \cdots \mathbf{u}_k\} \subseteq V$ is a *basis* for V if

(i) S spans V, $\text{span}(S) = V$, and

(ii) S is linearly independent.

Theorem

Suppose S is a basis for V. then every vectors $\mathbf{v} \in V$ can be written as a linear combination of vectors in S uniquely.

Proof.

(i) $\text{span}(S) = V$ tells us that every vector $\mathbf{v} \in V$ can be written as a combination of vectors in S.

(ii) S is linearly independent tells us that if $\mathbf{v}$ is a linear combination of vectors in S, the coefficient is unique.

$$
\begin{aligned}
& \mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = d_1\mathbf{u}_1 + \cdots + d_k\mathbf{u}_k \\
\Leftrightarrow \quad & (c_1 - d_1)\mathbf{u}_1 + \cdots + (c_k - d_k)\mathbf{u}_k = \mathbf{0} \\
\Leftrightarrow \quad & c_1 = d_1, \quad \cdots \quad c_k = d_k
\end{aligned}
$$

$\square$

Example

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ x + y - z = 0 \right\}$.

- The general solution to the linear system is $s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $s, t \in \mathbb{R}$.
- This shows that every vector $\mathbf{v}$ in the solution space V is a linear combination of the vectors in $S = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. Hence, S spans V.
- Since S contains only 2 vectors which are not a multiple of each other, S is linearly independent too.
- Therefore, S is a basis for V .

Basis for Solution Set of Homogeneous System

Let $V = \{ \ \mathbf{u} \mid \mathbf{Au} = \mathbf{0} \ \}$ be the solution space to the homogeneous system $\mathbf{Ax} = \mathbf{0}$. Suppose

$$s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, ..., s_k \in \mathbb{R}$$

is the general solution. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for the subspace $V = \{ \ \mathbf{u} \mid \mathbf{Au} = \mathbf{0} \ \}$.

Example

Let V be the solution set to

$$\begin{cases} x_1 + x_2 + 2x_4 + x_5 = 0 \\ 2x_1 - x_2 + 3x_3 + 3x_5 = 0 \\ x_1 - 2x_2 + 3x_3 - 2x_4 + 2x_5 = 0 \\ 2x_1 - x_2 + 3x_3 + 3x_5 = 0 \end{cases}$$

Solving the system, $\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 0 \\ 2 & -1 & 3 & 0 & 3 & 0 \\ 1 & -2 & 3 & -2 & 2 & 0 \\ 2 & -1 & 3 & 0 & 3 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2/3 & 4/3 & 0 \\ 0 & 1 & -1 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$, we conclude that $S = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ -4/3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans V. Using the last 3 coordinates, we can also conclude that S is linearly independent (details left to readers). Hence, S is a basis for V.

Example

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, x + y - z = 0 \right\}$. It was shown that $T = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for V. Show that $S = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for V.

1. First we show that $\text{span}(S) = V = \text{span}(T)$.

   (i) $\left(\begin{array}{cc|cc} -1 & 1 & -1 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ shows that $\text{span}(S) \subseteq \text{span}(T)$.

   (ii) $\left(\begin{array}{cc|cc} -1 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|cc} 1 & 0 & 2/3 & -1/3 \\ 0 & 1 & -1/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$ shows that $\text{span}(T) \subseteq \text{span}(S)$.

   Therefore, $\text{span}(S) = \text{span}(T) = V$.
2. Next, since S contains 2 vectors that are not a multiple of each other, S is linearly independent.

Hence, S is a basis for V too. This also shows that basis for a subspace may not be unique.

Basis for the zero space $\{\mathbf{0}\}$

Recall that the zero space $\{\mathbf{0}\}$ is a subspace. Find a basis for $\{\mathbf{0}\}$

The basis for the zero space $\{\mathbf{0}\}$ is the empty set $\{\}$ or $\varnothing$.

- Firstly, $\text{span}\{\mathbf{0}\} = \{\mathbf{0}\}$ but the set $\{\mathbf{0}\}$ is not linearly independent.
- However, if S is a set that contains any nonzero vector, then $\text{span}(S)$ will be strictly bigger than the zero space, $\{\mathbf{0}\} \subsetneq \text{span}(S)$.
- The empty set is linearly independent vacuously.
- However, $\text{span}\{\}$ does not make sense.
- The real definition of the span of S is the smallest subspace V such that $S \subseteq V$. That is $V = \text{span}(S)$ if $V \subseteq W$ for all subspaces W containing S.
- Since the zero space is the smallest subspace containing the empty set, span of the empty set is the zero space.

Question

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ a subset of vectors in V. Which of the following statements is/are true?

1. If S is linearly independent, then S spans V.

2. If S is linearly dependent, then S does not span V.

3. If S spans V, then S is linearly independent.

4. If S does not span V, then S is linearly dependent.

Basis for $\mathbb{R}^n$ and Invertibility

A priori, there is no relationship between linear independence and spanning a subspace. However, in the special case when the subset S of $\mathbb{R}^n$ contains exactly n vectors, then linear independence is equivalent to spanning $\mathbb{R}^n$.

Theorem

A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the columns are linearly independent.

Proof.

Write $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$ and let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ be the set containing the columns of $\mathbf{A}$. Then $\mathbf{A}$ is invertible if and only if the reduce row-echelon form is the identity matrix. But we have also seen that S is linearly independent if and only if the reduce row-echelon form of $\mathbf{A}$ has no non-pivot columns, which for a square matrix, must mean that the reduce row-echelon form is the identity matrix. $\square$

Theorem

A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the columns spans $\mathbb{R}^n$.

Proof.

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ be the set containing the columns of $\mathbf{A}$. Then S spans $\mathbb{R}^n$ if and only if the reduced row-echelon form of $\mathbf{A}$ do not have any nonzero row, which for a square matrix, would mean that the reduce row-echelon form is the identity matrix. This is equivalent to $\mathbf{A}$ being invertible. $\square$

Basis for $\mathbb{R}^n$ and Invertibility

Corollary

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ be a subset of $\mathbb{R}^n$ containing n vectors. Then S is linearly independent if and only if S spans $\mathbb{R}^n$.

Proof.

Let $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$ be the matrix whose columns are the vectors in S. Then $\mathbf{A}$ is a square matrix. Then by the two theorems, S is linearly independent if and only if $\mathbf{A}$ is invertible, if and only if S spans $\mathbb{R}^n$. $\square$

Corollary

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a subset of $\mathbb{R}^n$ and $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ be the matrix whose columns are vectors in S. Then S is a basis for $\mathbb{R}^n$ if and only if $k = n$ and $\mathbf{A}$ is an invertible matrix.

Proof.

($\Rightarrow$) If $k < n$, then S cannot span $\mathbb{R}^n$. If $k > n$, then S cannot be linearly independent. Hence, if S is a basis, S must have exactly n vectors, and by the previous theorem, $\mathbf{A}$ must be invertible.
($\Leftarrow$) Conversely, if $k = n$ and $\mathbf{A}$ is invertible, then S is a basis by the previous theorem. $\square$

Basis for $\mathbb{R}^n$ and Invertibility

Theorem

A $n \times n$ square matrix $\mathbf{A}$ invertible if and only if the rows of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

Theorem

A square matrix $\mathbf{A}$ of order n is invertible if and only if the rows of $\mathbf{A}$ are linearly independent.

The proofs of the 2 theorems follow from the fact that $\mathbf{A}$ is invertible if and only if $\mathbf{A}^T$ is, and the rows of $\mathbf{A}$ are the columns of $\mathbf{A}^T$.

Equivalent Statements for Invertibility

Theorem

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) $\mathbf{A}$ *is invertible.*

(ii) $\mathbf{A}^T$ *is invertible.*

(iii) *(left inverse) There is a matrix* $\mathbf{B}$ *such that* $\mathbf{BA} = \mathbf{I}$.

(iv) *(right inverse) There is a matrix* $\mathbf{B}$ *such that* $\mathbf{AB} = \mathbf{I}$.

(v) *The reduced row-echelon form of* $\mathbf{A}$ *is the identity matrix.*

(vi) $\mathbf{A}$ *can be expressed as a product of elementary matrices.*

(vii) *The homogeneous system* $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution.*

(viii) *For any* $\mathbf{b}$, *the system* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

(ix) *The determinant of* $\mathbf{A}$ *is nonzero,* $\det(\mathbf{A}) \neq 0$.

(x) *The columns/rows of* $\mathbf{A}$ *are linearly independent.*

(xi) *The columns/rows of* $\mathbf{A}$ *spans* $\mathbb{R}^n$.

Challenge

Recall that the set of 2×2 matrices, $\mathbb{R}^{2\times 2}$, is a vector space. Show that the set

$$\left\{ \mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^{2\times 2}$.

Introduction to Coordinates Relative to a Basis

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; z = 0 \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Observe that any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ identifies with a unique vector $x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ in V.

Let $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$, it is also a basis for V.

- Now a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ defines a vector $x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \\ 0 \end{pmatrix}$ in V.
- Conversely, a vector $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \frac{x+y}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{x-y}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ in V defines a vector $\begin{pmatrix} (x+y)/2 \\ (x-y)/2 \end{pmatrix}$ in $\mathbb{R}^2$.

Introduction to Coordinates Relative to a Basis

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x + y - z = 0 \right\} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. Then we have the unique correspondence

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \quad \longleftrightarrow \quad x \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y - x \\ x \\ y \end{pmatrix} \in V.$$

- These examples demonstrate that a subspace V of $\mathbb{R}^n$ can be identified with some $\mathbb{R}^k$.
- That is, instead of giving a vector in V in terms of its coordinates in $\mathbb{R}^n$, we may represent it with a vector in $\mathbb{R}^k$ for some $k \leq n$.
- This identification depends on the choice of basis of V.
- Explicitly, let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a basis for V, a subspace of $\mathbb{R}^n$. Then we have a unique correspondence

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{R}^k \longleftrightarrow \mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \in V.$$

Coordinates Relative to a Basis

Definition

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for V, a subspace of $\mathbb{R}^n$ and

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

be the unique expression of a vector $\mathbf{v}$ in V in terms of the basis S. The vector in $\mathbb{R}^k$ defined by the coefficients of the linear combination is called the *coordinates of $\mathbf{v}$ relative to basis S*, and is denoted as

$$[\mathbf{v}]_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

Examples

1. Let $E = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. For any $\mathbf{w} = (w_i) \in \mathbb{R}^n$,

$$\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + \cdots + w_n\mathbf{e}_n.$$

$\Rightarrow [\mathbf{w}]_E = \mathbf{w}$

Example

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]_E = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Examples

2. $S = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x + y - z = 0 \right\}$. Let $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. To compute the coordinates of $\mathbf{v}$ relative to basis S, find c_1, c_2 such that $c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} -1 & 1 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

So,

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = (-1) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad [\mathbf{v}]_S = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Remarks

- Even though $\mathbf{v} \in V \subseteq \mathbb{R}^n$ has n coordinates, its coordinates relative to basis S, $[\mathbf{v}]_S$, has k coordinates if the basis S has k vectors.
- Note that the correspondence is unique only if S is a basis. If S is not linearly independent, a few vectors in $\mathbb{R}^k$ can map to the same $\mathbf{v} \in V$.
- The relative coordinates depend on the ordering of the basis. If $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $T = \left\{ \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, then for $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$,

$$[\mathbf{v}]_S = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \end{pmatrix} = [\mathbf{v}]_T.$$

Algorithm for Computing Relative Coordinate

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for V.

- Let $\mathbf{v}$ be a vector in V. To find $[\mathbf{v}]_S$, we must find the coefficients $c_1, c_2, ..., c_k$ such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

- Converting it to a matrix equation, we have

$$\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots \mathbf{u}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{v},$$

- which is equivalent to solving the linear system

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right).$$

Example

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \; x_1 - 2x_2 + x_3 = 0, x_2 + x_3 - 2x_4 = 0 \right\}. \text{ Basis: } S = \left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Find the coordinates of $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in V$ relative to S.

$$\left(\begin{array}{cc|c} -3 & 4 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \Rightarrow \quad [\mathbf{v}]_S = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Question

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for a subspace $V \subseteq \mathbb{R}^5$. Let $\mathbf{v} \in V$ be such that

$$\left(\begin{array}{ccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{v} \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Which of the following is $[\mathbf{v}]_S$?

$$(i) \begin{pmatrix} 1 \\ -5 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (ii) \begin{pmatrix} 1 \\ -5 \\ 0 \\ 0 \end{pmatrix}, \quad (iii) \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} \quad (iv) \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

Question

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a set of vectors in $\mathbb{R}^n$ and V a subspace. Let $\mathbf{v}$ be a vector in V.

(i) Suppose there is a non-pivot column in the left side of the reduced row-echelon form of

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right).$$

What can you conclude?

(ii) Suppose

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right)$$

is inconsistent. What can you conclude?

Properties of Coordinates Relative to a Basis

Theorem

Let V be a subspace of $\mathbb{R}^n$ and B a basis for V.

(i) *For any vectors $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} = \mathbf{v}$ if and only if $[\mathbf{u}]_B = [\mathbf{v}]_B$.*

(ii) *For any $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m \in V$,*

$$[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m]_B = c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + \cdots + c_m[\mathbf{v}_m]_B.$$

Proof.

Exercise. $\square$

3.7 Dimensions

Introduction

- Intuitively, we say that $\mathbb{R}^3$ is 3-dimensional, and $\mathbb{R}^2$ is 2-dimensional.
- Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| z = 0 \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. By the discussion in coordinates relative to a basis, we can identify V with $\mathbb{R}^2$, and hence intuitively say that V is 2-dimensional.
- However, the identification of V with $\mathbb{R}^k$ depends on the choice of the basis of V.
- Recall that bases for any nonzero subspace $V \neq \{\mathbf{0}\}$ is not unique.
- So suppose now $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ are bases for a subspace V. Using S, we identify V with $\mathbb{R}^k$ and using T, we identify V with $\mathbb{R}^m$. Then do we say that V is k-dimensional, or m-dimensional?
- Ideally, we want $m = k$, which is in fact true!

More Properties of Coordinates Relative to a Basis

Theorem

Let V be a subspace of $\mathbb{R}^n$ and B a basis for V. Suppose B contains k vectors, $|B| = k$. Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ be vectors in V. Then

(i) $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *is linearly independent (respectively, dependent) if and only if* $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, ..., [\mathbf{v}_m]_B$ *is linearly independent (respectively, dependent) in* $\mathbb{R}^k$; *and*

(ii) $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ *spans* V *if and only if* $\{[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, ..., [\mathbf{v}_m]_B\}$ *spans* $\mathbb{R}^k$.

The proof is given in the appendix, we will provide a heuristic of the proof. By the properties of coordinates of a vector relative to a basis, we have that the linear system $\left(\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m & \mathbf{u} \end{array}\right)$ has the exact same properties as the linear system $\left(\begin{array}{cccc|c} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \cdots & [\mathbf{v}_m]_S & [\mathbf{u}]_S \end{array}\right)$. So, let $\mathbf{u} = \mathbf{0}$ to proof property (i). For property (ii), let $[\mathbf{u}]_S$ be a vector in $\mathbb{R}^k$ to prove ($\Rightarrow$), and let $\mathbf{u}$ be a vector in V to prove ($\Leftarrow$).

Dimension

Corollary

Let V be a subspace of $\mathbb{R}^n$ and B a basis for V. Suppose B contains k vectors, $|B| = k$.

(i) *If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ is a subset of V with $m > k$, then S is linearly dependent.*

(ii) *If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ is a subset of V with $m < k$, then S is cannot span V.*

Proof.

Consider the set $T = \{[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, ..., [\mathbf{v}_m]_B\}$ in $\mathbb{R}^k$. If $m > k$, then T is linearly dependent, and hence by the previous theorem, so is S. If $m < k$, then T cannot span $\mathbb{R}^k$, and so by the previous theorem, S cannot span V. $\square$

Dimension

Corollary

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *and* $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ *are bases for a subspace* $V \subseteq \mathbb{R}^n$*. Then* $k = m$*.*

Proof.

Exercise. □

Definition

Let V be a subspace of $\mathbb{R}^n$. The *dimension* of V, denoted by $\dim(V)$, is defined to be the number of vectors in any basis of V.

In other words, V is k-dimensional if and only if V identifies with $\mathbb{R}^k$ using coordinates relative to any basis B of V.

Example

1. The dimension of the Euclidean n-space, $\mathbb{R}^n$ is n, since the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ has n vectors.

2. $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; z = 0 \right\}$ is 2-dimensional since the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ has 2 vectors.

3. $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| \; a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \right\}$ is $n-1$-dimensional if not all $a_i = 0$. This is called a hyperplane in $\mathbb{R}^n$.

Dimension of the Zero Space $\{\mathbf{0}\}$

We will provide a intuitive reasoning of why the empty set is the basis for the zero space $\{\mathbf{0}\}$ in $\mathbb{R}^n$.

- Intuitively, the dimension is the independent degree of freedom of movement: In a 3 dimensional space, we can travel forwards backwards, side ways, and up and down; in a 2-dimensional space, we can travel forwards backwards, as well as side ways; in a 1-dimensional space, we can only walk forward or backwards.

- So, since we have no freedom of movement in the zero space, the zero space should be 0-dimensional.

- But this would tell us that by definition, the basis for the zero space must have no vectors, that is, it must be the empty set.

Dimension of Solution Space

Recall that the vectors in the general solution of a homogeneous system form a basis for the solution space. This means that the dimension of the solution space is equal to the number of parameters in the general solution. This is in turn equal to the number of non-pivot columns in the reduce row-echelon form of the coefficient matrix.

Theorem

Let $\mathbf{A}$ *be a* $m \times n$ *matrix. The number of non-pivot columns in the reduced row-echelon form of* $\mathbf{A}$ *is the dimension of the solution space*

$$V = \{ \ \mathbf{u} \in \mathbb{R}^n \ \middle| \ \mathbf{Au} = \mathbf{0} \ \}.$$

Let $s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ be the general solution to the homogeneous system $\mathbf{Ax} = \mathbf{0}$. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for V and so by definition, $\dim(V) = k$. But this means that the reduced row-echelon form of $\mathbf{A}$ has k non-pivot columns.

Example

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; 2x - 3y + z = 0 \right\}$. This is a hyperplane in $\mathbb{R}^3$, so $\dim(V) = 2$. We can see this also from the fact that the coefficient matrix $\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}$ has 2 non-pivot columns.

Now consider the set $S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\}$. Check that S is a subset of V. Since S contains 3 vectors and $\dim(V) = 2 < 3$, S must be linearly dependent. Indeed,

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -5 & 1 & -2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example

Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_1 + x_2 + x_3 + x_4 = 0 \right\}$. It is a hyperplane in $\mathbb{R}^4$, hence $\dim(V) = 3$.

Consider the set $S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$. Check that S is a subset of V. Since S only contains 2 vectors, it cannot span the whole of V. For example, the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$ is in V, but $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{array}\right)$ is inconsistent.

Spanning Set Theorem

Theorem

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *be a subset of vectors in* $\mathbb{R}^n$*, and let* $V = \text{span}(S)$*. Suppose* V *is not the zero space,* $V \neq \{\mathbf{0}\}$*. Then there must be a subset of* S *that is a basis for* V*.*

Proof.

If S is linearly independent, then S is a basis for V. Otherwise, one of the vectors $\mathbf{u}_i$ in S can be written as a linear combination of the other. Without lost of generality (rearranging if necessary), say

$$\mathbf{u}_k = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_{k-1}\mathbf{u}_{k-1}$$

for some coefficients $c_1, c_2, ..., c_{k-1}$. We claim that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{k-1}\}$ still spans V. For if $\mathbf{v}$ is a vector in V, we have

$$\begin{aligned}
\mathbf{v} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k \\
&= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_{k-1}\mathbf{u}_{k-1}) \\
&= (a_1 + a_k c_1)\mathbf{u}_1 + (a_2 + a_k c_2)\mathbf{u}_2 + \cdots + (a_{k-1} + a_k c_{k-1})\mathbf{u}_{k-1}
\end{aligned}$$

which shows that $\mathbf{v}$ is a linear combination of vectors in $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{k-1}\}$. If $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{k-1}\}$ is linearly independent, then it is a basis for V. Otherwise, continue the process of throwing away some redundant vectors, we can conclude that there must be a subset of S that is a basis for V. □

Linear Independence Theorem

Theorem

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ a linearly independent subset of V, $S \subseteq V$. Then there must be a set T containing S, $S \subseteq T$ such that T is a basis for V.

Proof.

If $\mathrm{span}(S) = V$, then S is a basis for V. Otherwise, since $\mathrm{span}(S) \subsetneqq V$, there must be a vector in V that is not contained in $\mathrm{span}(S)$, $\mathbf{u}_{k+1} \in V \setminus \mathrm{span}(S)$. Note that since $\mathbf{u}_{k+1} \notin \mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, the set $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is linearly independent and $\dim(\mathrm{span}(S_1)) = k + 1$. If $\mathrm{span}(S_1) = V$, we are done. Otherwise, repeating the argument above, we can find $\mathbf{u}_{k+2}$ in V such that $S_2 = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}_{k+1}, \mathbf{u}_{k+2}\}$ is a linearly independent subset of V. Continue inductively, this process must stop when the number of vectors in S_m is equal to the dimension of V, for otherwise, if $|S_m| > \dim(V)$, then S_m cannot be linearly independent. So let $T = S_m$ when $|S_m| = \dim(V)$. $\square$

Challenge

Let V be a k-dimensional subspace of $\mathbb{R}^n$. Using the dimension of V (instead of proving using equivalent statements of invertibility), prove that a subset S in V containing k vectors, $|S| = k$, is linearly independent if and only if it spans V.

Discussion

Recall that for a set S to be a basis for a subspace V in $\mathbb{R}^n$, we must check that

(i) $\text{span}(S) = V$, and

(ii) S is linearly independent.

However, if we know the dimension of V and if the number of vectors in the set S is equal to the dimension of V, $|S| = \dim(V)$, then it suffice to check one of the above criteria.

Dimension and Subspaces

Theorem

Let U and V be subspaces of $\mathbb{R}^n$.

(i) *If U is a subset of V, $U \subseteq V$, then the dimension of U is no greater than the dimension of V, $\dim(U) \leq \dim(V)$.*

(ii) *If U is a strict subset of V, $U \subsetneqq V$, then the dimension of U is strictly smaller than V, $\dim(U) < \dim(V)$.*

i.e. $U \subseteq V$, then $\dim(U) \leq \dim(V)$ with equality $\Leftrightarrow U = V$.

Sketch of Proof.

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for U. Then $\dim(U) = k$. Since U is a subset of V, S is a linearly independent subset of V. So necessary $\dim(V) \geq k$. If $U \neq V$, then we can find a set T strictly bigger than S, $S \subsetneqq T$ such that T is a basis for V. Hence, $\dim(V) = |T| > |S| = k = \dim(U)$. $\square$

Equivalent ways to check for Basis

Theorem (B1)

Let V be a k-dimensional subspace of $\mathbb{R}^n$, $\dim(V) = k$. Suppose S is a linearly independent subset of V containing k vectors, $|S| = k$. Then S is a basis for V.

Proof.

Let $U = \text{span}(S)$. Since S is linearly independent, S is a basis for U, and hence, $\dim(U) = k$. Since $S \subseteq V$, $U \subseteq V$. Also, $\dim(U) = k = \dim(V)$. Therefore, $U = V$, and so S is a basis for V. $\square$

Theorem (B2)

Let V be a k-dimensional subspace of $\mathbb{R}^n$, $\dim(V) = k$. Suppose S is a set containing k vectors, $|S| = k$, such that $V \subseteq \text{span}(S)$. Then S is a basis for V.

Proof.

Let $U = \text{span}(S)$, then $V \subseteq U$. So, $k = \dim(V) \leq \dim(U) \leq k$ which shows that $k = \dim(U)$ and hence $V = U = \text{span}(S)$. Next, observe that S must be linearly independent. For if S is linearly dependent, then $k = \dim(U) = \dim(\text{span}(S)) < k$, a contradiction. $\square$

Equivalent ways to check for basis

In summary

Definition	**(B1)**	**(B2)**
(1) $\text{span}(S) = V$ (2) S is L.I.	(1) $\lvert S\rvert = \dim(V)$ (2) $S \subseteq V$ (3) S is Linearly independent	(1) $\lvert S\rvert = \dim(V)$ (2) $V \subseteq \text{span}(S)$

- Using (B1), we do not need to check that $\text{span}(S) = V$.
- Using (B2), we do not need to check that S is linearly independent.

Example

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x + y - z = 0 \right\}$. Show that $S = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for V.

- Check that $S \subseteq V$

$$(-1) + (2) - (1) = 0, \quad (1) + (1) - (2) = 0.$$

- Check that S is linearly independent. But this is clear since the 2 vectors in S cannot be a multiple of each other.
- $\dim(V) = 2$ since the RREF, which is just the coefficient matrix, has 2 non-pivot columns.
- Hence, $\text{span}(S) \subseteq V$ and $\dim(\text{span}(S)) = \dim(V)$ tells us that $\text{span}(S) = V$.

Example

Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \; x_1 - 2x_2 + x_3 = 0, x_2 + x_3 - 2x_4 = 0 \right\}$. Show that $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ is a basis for V.

- Check that $S \subseteq V$

$$\begin{aligned} (1) - 2(1) + (1) = 0, \quad (1) + (1) - 2(1) = 0 \quad &\Rightarrow \quad (1,1,1,1) \in V \\ (3) - 2(1) + (-1) = 0, \quad (1) + (-1) - 2(0) = 0 \quad &\Rightarrow \quad (3,1,-1,0) \in V \end{aligned}$$

- Check that S is linearly independent. But this is clear since the 2 vectors in S cannot be a multiple of each other. Hence, $\dim(\operatorname{span}(S)) = 2$.
- $\dim(V) = 2$ since the RREF, which is just the coefficient matrix, has 2 non-pivot columns.
- Hence, $\operatorname{span}(S) \subseteq V$ and $\dim(\operatorname{span}(S)) = \dim(V)$ tells us that $\operatorname{span}(S) = V$.

Example

Let $T = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ and $V = \text{span}(T)$.

Show that $S = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 1 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis for V.

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 4 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 6 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -1 & 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1/4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

It is clear that T is linearly independent. So, $\dim(\text{span}(T)) = 4$. The augmented matrix above shows that $\text{span}(T) \subseteq \text{span}(S)$, and the LHS of the augmented matrix shows that S is linearly independent too. Hence, $\dim(\text{span}(S)) = 4$ too. Therefore $\text{span}(S) = \text{span}(T)$.

3.8 Transition Matrices

Introduction

Let $V \subseteq \mathbb{R}^n$ be a subspace. Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for V. For a $\mathbf{v} \in V$, how are $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$ related?

$$\begin{tikzcd}
{[\mathbf{v}]_S \in \mathbb{R}^k} \arrow[r, leftrightarrow] \arrow[rr, leftrightarrow, bend right] & \mathbf{v} \in V \subseteq \mathbb{R}^n \arrow[r, leftrightarrow] & {[\mathbf{v}]_T \in \mathbb{R}^k}
\end{tikzcd}$$

$$[\mathbf{v}]_S = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \quad \longleftrightarrow \quad c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = \mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k \quad \longleftrightarrow \quad [\mathbf{v}]_T = \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix}$$

Example

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for a subspace V of $\mathbb{R}^n$. Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be such that

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_1 + \mathbf{u}_2, \\
\mathbf{v}_3 &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3.
\end{aligned}$$

Check that T is a basis for V too. Suppose now $\mathbf{v}$ is a vector in V with $[\mathbf{v}]_T = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. This means that

$$\begin{aligned}
\mathbf{v} &= \mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 \\
&= \mathbf{u}_1 + 2(\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) \\
&= 4\mathbf{u}_1 + 3\mathbf{u}_2 + \mathbf{u}_3
\end{aligned}$$

This means that $[\mathbf{v}]_S = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$.

Example

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for a subspace V of $\mathbb{R}^n$. Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be such that

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1, \\ \mathbf{v}_2 &= \mathbf{u}_1 + \mathbf{u}_2, \\ \mathbf{v}_3 &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3. \end{aligned}$$

In general write $[\mathbf{v}]_S = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and $[\mathbf{v}]_T = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. Then

$$\begin{aligned} d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + d_3\mathbf{u}_3 = \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= c_1\mathbf{u}_1 + c_2(\mathbf{u}_1 + \mathbf{u}_2) + c_3(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) \\ &= (c_1 + c_2 + c_3)\mathbf{u}_1 + (c_2 + c_3)\mathbf{u}_2 + (c_3)\mathbf{u}_3 \end{aligned}$$

Since S is a basis, the coefficients of the linear combination must be equal, that is

$$\left\{ \begin{array}{ccccccc} d_1 & = & c_1 & + & c_2 & + & c_3 \\ d_2 & = & & & c_2 & + & c_3 \\ d_3 & = & & & & & c_3 \end{array} \right. \quad \text{which can be expressed as } [\mathbf{v}]_S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} [\mathbf{w}]_T.$$

Example

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for a subspace V of $\mathbb{R}^n$. Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be such that

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1, \\ \mathbf{v}_2 &= \mathbf{u}_1 + \mathbf{u}_2, \\ \mathbf{v}_3 &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3. \end{aligned}$$

Further, let us write the coordinates of the vectors in T relative to the basis S,

$$[\mathbf{v}_1]_S = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad [\mathbf{v}_2]_S = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad [\mathbf{v}_3]_S = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Observe that these vectors are the columns of the matrix above. That is, if $[\mathbf{v}]_S = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and $[\mathbf{v}]_T = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, then

$$[\mathbf{v}]_S = \left([\mathbf{v}_1]_S \quad [\mathbf{v}_2]_S \quad [\mathbf{v}_3]_S\right)[\mathbf{w}]_T.$$

Transition Matrix

Definition

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for the subspace V. Define the *transition matrix* from T to S to be

$$\mathbf{P} = \left([\mathbf{v}_1]_S \quad [\mathbf{v}_2]_S \quad \cdots \quad [\mathbf{v}_k]_S\right),$$

the matrix whose columns are the coordinates of the vectors in T relative to the basis S.

Theorem

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for the subspace V. Let $\mathbf{P}$ be the transition matrix from T to S. Then for any vector $\mathbf{w}$ in V,

$$[\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T.$$

Heuristic of the Proof

- Let $\mathbf{e}_i$ denote the i-th column of the $k \times k$ identity matrix $\mathbf{I}_k$.
- Recall that for a $m \times k$ matrix $\mathbf{A}$, the product $\mathbf{A}\mathbf{e}_i$ is the i-th column of $\mathbf{A}$,
$$\mathbf{A}\mathbf{e}_i = \mathbf{a}_i.$$
- Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for a subspace V of $\mathbb{R}^n$ and let $\mathbf{P}$ be the transition matrix from T to S.
- Note that since $\mathbf{v}_i = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + \mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_k$, $[\mathbf{v}_i]_T = \mathbf{e}_i$.
- Hence,
$$[\mathbf{v}_i]_S = \mathbf{P}[\mathbf{v}_i]_T = \mathbf{P}\mathbf{e}_i \text{ is the } i\text{-th column of } \mathbf{P}.$$

Algorithm to find Transition Matrix

Let $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be basis for a subspace V in $\mathbb{R}^n$.

- To find $\mathbf{P}$, the transition matrix from T to S, we need to find $[\mathbf{v}_i]_S$ for $i = 1, 2, ..., k$.
- This is equivalent to solving $\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_i \end{array} \right)$ for $i = 1, 2, ..., k$.
- Since these linear systems have the same coefficient matrix, we can solve them simultaneously,

$$\left(\begin{array}{cccc|cccc} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{array} \right).$$

- Now since S is a basis, the system must have a unique solution, and the reduced row-echelon form of the augmented matrix above will be of the form

$$\left(\begin{array}{c|cccc} \mathbf{I}_k & [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \cdots & [\mathbf{v}_k]_S \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0} & \mathbf{0} & & \mathbf{0} \end{array} \right) = \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{(n-k)\times k} \end{array} \right)$$

  where $\mathbf{P}$ is the transition matrix from T to S.

In summary,

$$\left(\begin{array}{c|c} \text{“}S\text{”} & \text{“}T\text{”} \end{array} \right) = \left(\begin{array}{cccc|cccc} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{array} \right) \xrightarrow{rref} \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{(n-k)\times k} \end{array} \right),$$

Example

Suppose $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$, $T = \left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

The transition matrix from T to S is

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 1 \end{array} \right) \xrightarrow{rref} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & -1/2 & 3/2 \\ 0 & 0 & 1 & 3/2 & -1/2 & -1/2 \end{array} \right) \Rightarrow \mathbf{P} = \begin{pmatrix} -1/2 & 3/2 & -1/2 \\ -1/2 & -1/2 & 3/2 \\ 3/2 & -1/2 & -1/2 \end{pmatrix}.$$

Let $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 1 & 2 \end{array} \right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 2 \end{array} \right) \Rightarrow [\mathbf{w}]_T = \begin{pmatrix} 3/2 \\ 3/2 \\ 2 \end{pmatrix}$.

$$\begin{pmatrix} 1/2 \\ 3/2 \\ 1/2 \end{pmatrix} = [\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T = \begin{pmatrix} -1/2 & 3/2 & -1/2 \\ -1/2 & -1/2 & 3/2 \\ 3/2 & -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 3/2 \\ 3/2 \\ 2 \end{pmatrix}$$

Indeed, $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{array} \right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \end{array} \right) \Rightarrow [\mathbf{w}]_S = \begin{pmatrix} 1/2 \\ 3/2 \\ 1/2 \end{pmatrix}$.

Question

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - 2y + z = 0 \right\}$. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $T = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$ are bases for V (check).

Given that

$$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which statement is true?

(a) The transition matrix from T to S is $\mathbf{P} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}$.

(b) The transition matrix from S to T is $\mathbf{P} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}$.

(c) None of the other options are true.

Inverse of Transition Matrix

Theorem

Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from T to S. Then $\mathbf{P}^{-1}$ is the transition matrix from S to T.

Proof.

Exercise. Note that you cannot assume that $\mathbf{P}$ is invertible. $\square$

Appendix

Linear Dependency and Adding Vectors

Theorem

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ *is linearly independent set of vectors in* $\mathbb{R}^n$ *and* $\mathbf{u}$ *is not a linearly combination of* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$. *Then the set* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}\}$ *is linearly independent.*

Proof.

Let $c_1, c_2, \ldots, c_k, c$ be coefficients satisfying the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k + c\mathbf{u} = \mathbf{0}.$$

If $c \neq 0$, then manipulating the equation gives

$$-\frac{c_1}{c}\mathbf{u}_1 - \frac{c_2}{c}\mathbf{u}_2 - \cdots - \frac{c_k}{c}\mathbf{u}_k = \mathbf{u},$$

a contradiction to $\mathbf{u}$ not being a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$. So, necessarily $c = 0$. Then

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k\mathbf{0}$$

tells us that $c_1 = c_2 = \cdots = c_k = 0$ by the independence of S. Therefore only the trivial coefficients satisfy the equation above, which proves that the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}\}$ is linearly independent $\square$

Linear Dependency and Adding or Removing Vectors

Theorem

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly independent set of vectors in $\mathbb{R}^n$. Then any subset of $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly independent.

Proof.

Let $\{\mathbf{u}_{i_1}, \mathbf{u}_{i_2}, ..., \mathbf{u}_{i_l}\}$ be a subset of $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$. Relabel index, or rearranging the vectors in the set, we may assume that the subset is $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_l\}$ for some $l \leq k$. Suppose $c_1, c_2, ..., c_l$ are coefficients satisfying the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_l\mathbf{u}_l = \mathbf{0}.$$

Pad the equation by 0, we have

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_l\mathbf{u}_l + 0\mathbf{u}_{l+1} + \cdots + 0\mathbf{u}_k = \mathbf{0}.$$

This is a linear combination of the vectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, and since the set is indepenendent, necessary the coefficients are 0. In particular, $c_1 = c_2 = \cdots = c_l = 0$, which proves that the set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_l\}$ is independent. $\square$

More Properties of Coordinates Relative to a Basis

Theorem

Let B be a basis for V containing k vectors, $|B| = k$. Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ be vectors in V. Then

(i) $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *is linearly independent (respectively, dependent) if and only if* $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, ..., [\mathbf{v}_m]_B$ *is linearly independent (respectively, dependent) in* $\mathbb{R}^k$*; and*

(ii) $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ *spans* V *if and only if* $\{[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, ..., [\mathbf{v}_m]_B\}$ *spans* $\mathbb{R}^k$.

Proof.

(i) Follows from the properties of coordinates relative to a basis, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_m = \mathbf{0}_{n\times 1}$ if and only if

$$\mathbf{0}_{k\times 1} = [\mathbf{0}_{n\times 1}]_B = [c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_m]_B = c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + \cdots + c_k[\mathbf{v}_m]_B.$$

More Properties of Coordinates Relative to a Basis

Continue of Proof.

(ii) ($\Leftarrow$) Suppose $\{[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, ..., [\mathbf{v}_m]_B\}$ spans $\mathbb{R}^k$. Given any $\mathbf{v} \in V$, $[\mathbf{v}]_B \in \mathbb{R}^k$ and so can find $c_1, ..., c_m$ such that $[\mathbf{v}]_B = c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + \cdots + c_m[\mathbf{v}_m]_B$ in $\mathbb{R}^k$. Then $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$, which proves that $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ spans V.

($\Rightarrow$) Let $B = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ spans V. Any vector $\mathbf{w} = (w_1, w_2, ..., w_k) \in \mathbb{R}^k$ defines a vector $\mathbf{v} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k$ in V, and so can write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$. Then

$$\mathbf{w} = [\mathbf{v}]_B = c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + \cdots + c_m[\mathbf{v}_m]_B$$

shows that $\{[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, ..., [\mathbf{v}_m]_B\}$ spans $\mathbb{R}^k$.

$\square$

Transition Matrix

Theorem

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for the subspace V. Let $\mathbf{P}$ be the transition matrix from T to S. Then for any vector $\mathbf{w}$ in V,

$$[\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T.$$

Proof.

Let $\mathbf{v}_j = a_{1j}\mathbf{u}_1 + a_{2j}\mathbf{u}_2 + \cdots + a_{kj}\mathbf{u}_k = \sum_{i=1}^k a_{ij}\mathbf{u}_i$. Then $[\mathbf{v}_j]_S = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{kj} \end{pmatrix}$. Write $[\mathbf{w}]_S = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{pmatrix}$ and $[\mathbf{w}]_T = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$.

Then

$$\begin{aligned} d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k = \mathbf{w} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \\ &= c_1(a_{11}\mathbf{u}_1 + a_{21}\mathbf{u}_2 + \cdots + a_{k1}\mathbf{u}_k) + c_2(a_{12}\mathbf{u}_1 + a_{22}\mathbf{u}_2 + \cdots + a_{k2}\mathbf{u}_k) \\ &\quad + \cdots + c_k(a_{1k}\mathbf{u}_1 + a_{2k}\mathbf{u}_2 + \cdots + a_{kk}\mathbf{u}_k) \\ &= (c_1a_{11} + c_2a_{12} + \cdots + c_ka_{1k})\mathbf{u}_1 + (c_1a_{21} + c_2a_{22} + \cdots + c_ka_{2k})\mathbf{u}_2 \\ &\quad + \cdots + (c_1a_{k1} + c_2a_{k2} + \cdots + c_ka_{kk})\mathbf{u}_k \end{aligned}$$

Transition Matrix

Continue.

Since S is a basis, the coefficients must be unique. So comparing the coefficients, we get

$$
\begin{aligned}
[\mathbf{w}]_S = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{pmatrix} &= \begin{pmatrix} c_1 a_{11} + c_2 a_{12} + \cdots + c_k a_{1k} \\ c_1 a_{21} + c_2 a_{22} + \cdots + c_k a_{2k} \\ \vdots \\ c_1 a_{k1} + c_2 a_{k2} + \cdots + c_k a_{kk} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \\
&= \begin{pmatrix} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \cdots & [\mathbf{v}_k]_S \end{pmatrix} = \mathbf{P}[\mathbf{w}]_T
\end{aligned}
$$

$\square$