MA1522 ASSIGNMENT 2 SOLUTIONS

- Do NOT upload this assignment problem set to any website.
- The assignment carries a total number of 30 marks. The marks for each question or part are as indicated.

In each of the following, assume $\mathbf{A}$ is an $n \times n$ matrix over real numbers. You do not need to show the steps of the row reductions.

(1) [1 mark for each question] Which of the following statements are true? Which are false? You do not need to justify your answers.

Summarized Answer: $\boxed{\text{F, T, F, T, F, T, T, F, T, T}}$.

(a) For any real number c, $\det(c\mathbf{A}) = c\det(\mathbf{A})$.

Answer: $\boxed{\text{False}}$. By lecture notes, $\det(c\mathbf{A}) = c^n\det(\mathbf{A})$.

(b) If $\mathbf{x}$ is a nonzero vector in $\mathbb{R}^n$ and $\mathbf{A}\mathbf{x} = \mathbf{0}$, then $\det(\mathbf{A}) = 0$.

Answer: $\boxed{\text{True}}$. Since the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has nontrivial solutions, $\mathbf{A}$ is singular. Hence $\det(\mathbf{A}) = 0$.

(c) The set $V = \{(x, y, z)^T : x + z = 1\}$ is a subspace of $\mathbb{R}^3$.

Answer: $\boxed{\text{False}}$. V does not contain the zero vector.

(d) The only subspaces of $\mathbb{R}^1$ are $\{0\}$ and $\mathbb{R}^1$.

Answer: $\boxed{\text{True}}$. If a subspace of $\mathbb{R}^1$ has a nonzero member x, $\{x\}$ will span $\mathbb{R}^1$, hence $V = \mathbb{R}^1$.

(e) The inner product of two vectors in $\mathbb{R}^n$ is also a vector in $\mathbb{R}^n$.

Answer: $\boxed{\text{False}}$. By lecture notes, the inner product of two vectors is a real number, not a vector.

(f) Any finite set of vectors that contains the zero vector must be linearly dependent.

Answer: $\boxed{\text{True}}$. Say the set is $\{\mathbf{0}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$, then it is linearly dependent, because

$$1\mathbf{0} + 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_k = \mathbf{0}.$$

(g) If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ span $\mathbb{R}^n$, then they are linearly independent.

Answer: $\boxed{\text{True}}$. Since $\dim\mathbb{R}^n = n$, by lecture notes, any n vectors spanning $\mathbb{R}^n$ will be a basis, hence independent.

(h) Let $k > 1$. There are linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$, such that

$$\operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k-1}).$$

Answer: $\boxed{\text{False}}$. If $\mathbf{x}_k \in \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k-1})$, then there are $c_1, \dots, c_{k-1} \in \mathbb{R}$ such that $\mathbf{x}_k = c_1\mathbf{x}_1 + \cdots + c_{k-1}\mathbf{x}_{k-1}$. Hence $c_1\mathbf{x}_1 + \cdots + c_{k-1}\mathbf{x}_{k-1} + (-1)\mathbf{x}_k = \mathbf{0}$, which shows $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are linearly dependent, contradiction.

(i) The solution set of the system $\mathbf{Ax} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$, where $\mathbf{0}$ is the zero vector of $\mathbb{R}^n$.

Answer: $\boxed{\text{True}}$. One can quote lecture notes or verify the three conditions of subspaces.

(j) Let V be a subspace of $\mathbb{R}^n$ and $V \neq \mathbb{R}^n$ then $\dim V < n$.

Answer: $\boxed{\text{True}}$. Let B be a basis of V. Then B can be extended to a basis B' of $\mathbb{R}^n$. Note the $|B'| = n$. Since $V \neq \mathbb{R}^n$, $B \subsetneq B'$. Hence $\dim V = |B| < n$.

(2) [1 mark for each question] Let $n > 5$ and $\mathbf{I}, \mathbf{B}, \mathbf{E}_1, \mathbf{E}_2$ and $\mathbf{E}_3$ be $n \times n$ matrices satisfying the following conditions: $\mathbf{I}$ is the identity matrix, $\det(\mathbf{B}) = 2025$, $\mathbf{E}_1$ is obtained from $\mathbf{I}$ by exchanging the first and second rows, $\mathbf{E}_2$ is obtained from $\mathbf{I}$ by multiplying Row 3 by $\frac{1}{2}$, and $\mathbf{E}_3$ is obtained by adding 3 times of the fourth row to the fifth row. Find

(i) $\det(\mathbf{E}_1\mathbf{B}^T)$.

(ii) $\det(\mathbf{E}_2^{-1}\mathbf{B})$.

(iii) $\det(\mathbf{B}\mathbf{E}_3)$.

(iv) $\det(\det(\mathbf{E}_2^{-1})\mathbf{B})$.

Answer: One can either use properties of elementary matrices to find the product first, or use the fact $\det(\mathbf{AB}) = \det(\mathbf{A})(\mathbf{B})$.

(i) $\boxed{\det(\mathbf{E}_1\mathbf{B}^T) = -2025}$, because $\det(\mathbf{E}_1) = -1$ and $\det(\mathbf{B}^T) = \det(\mathbf{B}) = 2025$.

(ii) $\boxed{\det(\mathbf{E}_2^{-1}\mathbf{B}) = 4050}$, because $\det(\mathbf{E}_2^{-1}) = 2$.

(iii) $\boxed{\det(\mathbf{B}\mathbf{E}_3) = 2025}$, because $\det(\mathbf{E}_3) = 1$.

(iv) $\boxed{\det(\det(\mathbf{E}_2^{-1})\mathbf{B}) = 2^n \cdot 2025}$, because $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$ and $\det(\mathbf{E}_2^{-1}) = 2$.

(3) [2 marks] Let $\mathbf{B} = (b_{ij})$ be the following $(n+1)\times(n+1)$ matrix:

$$\begin{pmatrix} 1 & a_1 & & & & & \\ -1 & 1-a_1 & a_2 & & & & \\ & -1 & 1-a_2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & a_{n-1} & \\ & & & & & 1-a_{n-1} & a_n \\ & & & & & -1 & 1-a_n \end{pmatrix}.$$

To be more precise, $b_{11} = 1$, $b_{ii} = 1 - a_{i-1}$ for $2 \le i \le n+1$, $b_{i(i+1)} = a_i$ for $1 \le i \le n$, $b_{(i+1)i} = -1$ for $1 \le i \le n$ and all other entries are zero. Find $\det(\mathbf{B})$ and explain how you derive your answer.

Answer: $\boxed{\det(\mathbf{B}) = 1}$. One can perform the elementary row operations $R_{i+1} := R_{i+1} + R_i$ sequentially from $i = 1$ to n. The resulting matrix will be upper triangular with only 1 on the main diagonal. Since those type of row operations does not change the determinant, $\det(\mathbf{B}) = 1$.

(4) [2 marks] Suppose $\det(\mathbf{A}) = 1$. Find $\operatorname{adj}(\operatorname{adj}(\mathbf{A}))$ and explain how you derive your answer.

Answer: $\boxed{\operatorname{adj}(\operatorname{adj}(\mathbf{A})) = \mathbf{A}}$. By lecture notes, for an invertible matrix $\mathbf{A}$, $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A})$. Since $\det(\mathbf{A}) = 1$, we have

$$\operatorname{adj}(\mathbf{A}) = \mathbf{A}^{-1}. \tag{1}$$

Next $\mathbf{A}^{-1}$ is also invertible, and in our case, $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} = 1$. We can apply (1) to $\mathbf{A}^{-1}$ and get

$$\operatorname{adj}(\mathbf{A}^{-1}) = (\mathbf{A}^{-1})^{-1}, \quad \text{which is,} \quad \operatorname{adj}(\operatorname{adj}(\mathbf{A})) = \mathbf{A}.$$

(5) [2 marks] Can you find three linearly independent vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{R}^3$ such that none of them is a linear combination of

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -3 \\ -11 \\ 7 \end{pmatrix}?$$

If so, give $\mathbf{x}, \mathbf{y}, \mathbf{z}$ explicitly and justify your answer; if not, explain why you can't.

Answer: $\boxed{\text{Yes}}$, for example, one can take the standard basis: $\boxed{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T}$ as $\mathbf{x}, \mathbf{y}, \mathbf{z}$. In fact, there are many other possible answers. Justification: First notice that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, in fact, by

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

we get $\mathbf{v}_3 = 2\mathbf{v}_1 - \frac{5}{2}\mathbf{v}_2$, consequently, $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ is a plane in $\mathbb{R}^3$. Now, we need to pick three vectors not on that plane. A natural starting point is the standard basis. Since

$$\begin{pmatrix} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 6 & -11 & 0 & 1 & 0 \\ 1 & -2 & 7 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 2 & 0 & \frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{10} & -\frac{1}{5} \\ 0 & 0 & 0 & 1 & -\frac{2}{5} & -\frac{1}{5} \end{pmatrix}$$

indeed, they are not in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(6) [2 marks] Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

spans the subspace

$$W = \left\{ \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Answer: First we check that $S \subseteq W$: $1 + 0 = 1$, $0 + 1 = 1$ and $1+1 = 2$. Since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, it suffices to show that $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = W$. Now

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 1 & 1 & x+y \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix}$$

that is, any vector $(x, y, x+y)^T \in W$ is equal to $x\mathbf{v}_1 + y\mathbf{v}_2$. The result follows.

(7) [2 marks] Find a maximal linearly independent subset of $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 3 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{u}_4 = \begin{pmatrix} 1 \\ 3 \\ 6 \\ 4 \end{pmatrix}, \quad \mathbf{u}_5 = \begin{pmatrix} 3 \\ 5 \\ 7 \\ 5 \end{pmatrix}.$$

Explain how you derive your answer.

Answer: One can take, for example, $\boxed{\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4\}}$. We begin with the augmented matrix corresponding to the homogenous equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_5\mathbf{u}_5 = \mathbf{0}$. Now

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 1 & 3 & 0 & 3 & 5 \\ 1 & 4 & -1 & 6 & 7 \\ 1 & 3 & 0 & 4 & 5 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The pivot columns tells us that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ are linearly independent. (if you remove the second and the fifth columns, the same reduction tells that the system only has trivial solutions $c_1 = c_2 = c_4 = 0$.) Furthermore, from the rref, one can see that $\mathbf{u}_3 = 3\mathbf{u_1} - \mathbf{u}_2$, and $\mathbf{u}_5 = -\mathbf{u_1} + 2\mathbf{u}_2$, hence it is maximal.

(8) Let

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix} \right\} \quad \text{and } T = \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix} \right\}.$$

(i) [2 marks] Show that both S and T are bases of $\mathbb{R}^3$.

Answer: Since

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

S is linearly independent. Since $\dim \mathbb{R}^3 = 3$, S is a basis. T is a basis by the same reason.

(ii) [2 marks] Find the transition matrix from T to S.

Answer: By lecture notes, we form the "augmented matrix" $(S|T)$ and do row operations as follows:

$$\begin{pmatrix} 1 & 4 & 7 & 1 & 2 & 3 \\ 2 & 5 & 8 & 4 & 5 & 6 \\ 3 & 6 & 10 & 7 & 8 & 10 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & \frac{11}{3} & \frac{10}{3} & 4 \\ 0 & 1 & 0 & -\frac{2}{3} & -\frac{1}{3} & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the transition matrix $\mathbf{P}$ is

$$\boxed{\begin{pmatrix} \frac{11}{3} & \frac{10}{3} & 4 \\ -\frac{2}{3} & -\frac{1}{3} & -2 \\ 0 & 0 & 1 \end{pmatrix}.}$$

(iii) [2 marks] Find a nonzero vector $\mathbf{v} = (x, y, z)^T$ in $\mathbb{R}^3$ which has the same coordinates relative to the basis S and T. Note that you need to find the values of x, y, z, not $[\mathbf{v}]_S$ nor $[\mathbf{v}]_T$.

Answer: Suppose $[\mathbf{v}]_S = [\mathbf{v}]_T = (a,b,c)^T$. By the definition of transition matrix, we have $\mathbf{P}[\mathbf{v}]_T = [\mathbf{v}]_S = [\mathbf{v}]_T$, thus

$$(\mathbf{P} - \mathbf{I})[\mathbf{v}]_T = \mathbf{0}.$$

Since

$$\begin{pmatrix} \frac{8}{3} & \frac{10}{3} & 4 \\ -\frac{2}{3} & -\frac{4}{3} & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

we have the general solution $a = t, b = -2t, c = t$, where $t \in \mathbb{R}$. We may set $t = 1$ and get $[\mathbf{v}]_T = (1,-2,1)^T$. Finally,

$$\mathbf{v} = 1\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Namely, $\boxed{x = 0, y = 0, z = 1}$.

Alternatively, suppose $[\mathbf{v}]_S = [\mathbf{v}]_T = (a,b,c)^T$. By the definition of coordinates, we have

$$a\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + c\begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix} = a\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + b\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}.$$

Solve, we have $a = t, b = -2t, c = t$ where $t \in \mathbb{R}$. Set $t = 1$, we have $a = 1, b = -2, c = 1$, and $\mathbf{v} = (0,0,1)^T$.

END OF ASSIGNMENT TWO