Time Value of Money

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Agenda

1. Future Values and Compound Interest
2. Present Values
3. Multiple Cash Flows
4. Level Cash Flows: Perpetuities and Annuities
5. Relevant Spreadsheet Formulae
6. Effective Annual Interest Rates
7. Inflation & The Time Value of Money

Future Values and Compound Interest

- Future Value
  - Amount to which an investment will grow after earning interest
- Compound Interest
  - Interest earned on interest
- Simple Interest
  - Interest earned only on the original investment

Future Values and Compound Interest

Simple Interest Example

Interest earned at a rate of 6% for five years on a principal balance of $100.

Interest earned per year = 100 × .06 = $6

Period	**0**	**1**	**2**	**3**	**4**	**5**
Interest Earned		6	6	6	6	6
Value	100	106	112	118	124	130

Value at the end of Year 5 = $130

Future Values and Compound Interest

Compound Interest Example

Interest earned at a rate of 6% for five years on the previous year's balance.

Interest earned per year = prior year balance × .06

Period	**0**	**1**	**2**	**3**	**4**	**5**
Interest Earned		6	6.36	6.74	7.15	7.57
Value	100	106	112.36	119.10	126.25	133.82

Value at the end of Year 5 = $133.82

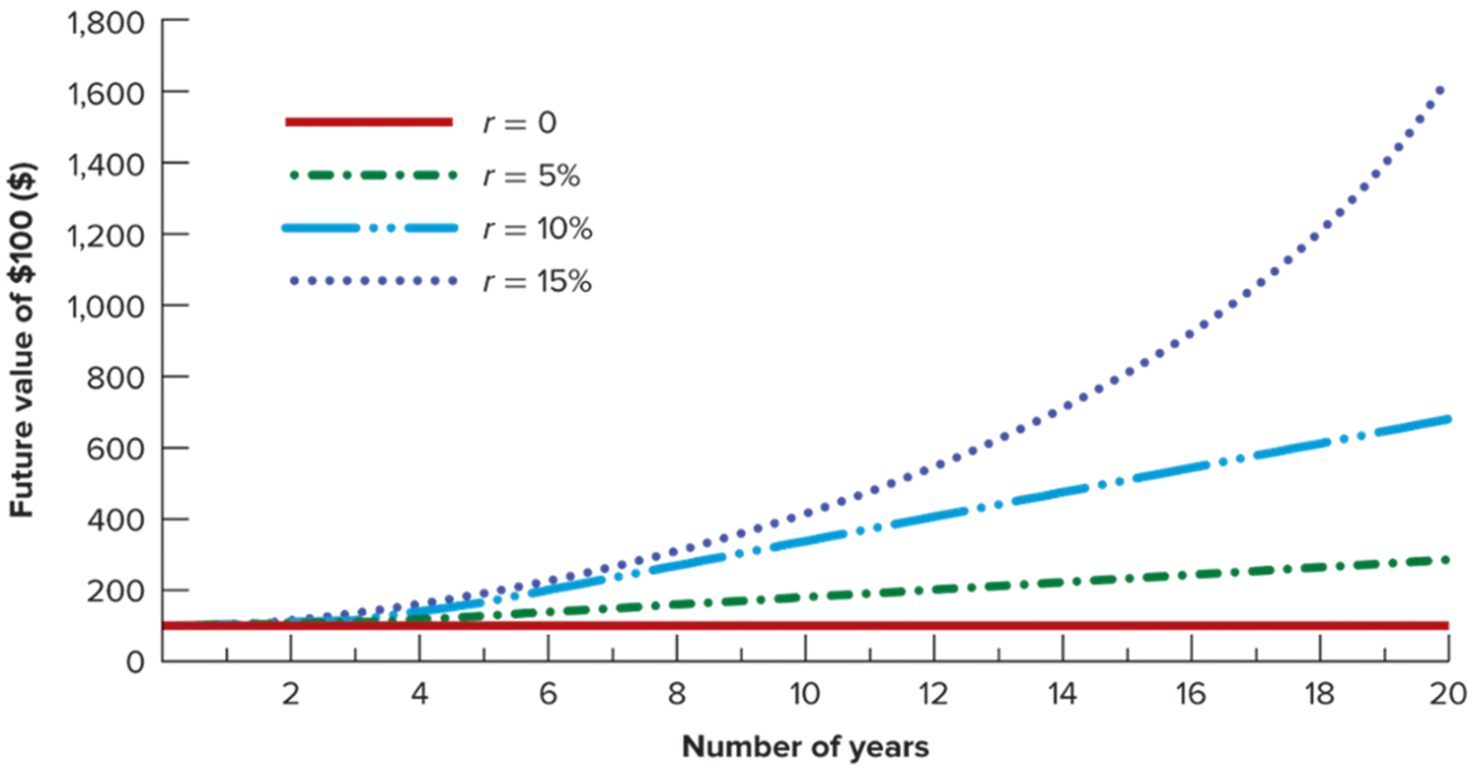
Future Values and Compound Interest

Future Value = FV

$$\boxed{FV = \$100 \times (1 + r)^t}$$

$$FV = \$100 \times (1 + .06)^5 = \$133.82$$

Future Values and Compound Interest



Example

In 1973 Gordon Moore, one of Intel's founders, predicted that the number of transistors that could be placed on a single silicon chip would double every 18 months, equivalent to an annual growth of 59% (i.e., 1.59^1.5=2.0). The first microprocessor was built in 1971 and had 2,250 transistors. By 2016, high-end Intel chips contained 7.2 billion transistors, 3.2 million times the number of transistors 45 years earlier. What has been the annual compound rate of growth in processing power? How does it compare with the prediction of Moore's law?

Example Solution

- Call g the annual growth rate of transistors over the 45-year period between 1971 and 2016. Then

$$
\begin{aligned}
2{,}250 \times (1+g)^{45} &= 7{,}200{,}000{,}000 \\
(1+g)^{45} &= 3{,}200{,}000 \\
1+g &= 3{,}200{,}000{,}000^{1/45} = 1.395
\end{aligned}
$$

- So the actual growth rate was g = .395, or 39.5%, not quite as high as Moore's prediction, but not so shabby either.

Present Values

Present Value

Value today of a future cash flow

Discount Factor

Present value of a $1 future payment

Discount Rate

Interest rate used to compute present values of future cash flows

Present Values

Present Value = PV

$$PV = \frac{\text{Future value after t periods}}{(1 + r)^t}$$

Present Values

1. Discounted Cash Flow (DCF): Method of calculating present value by discounting future cash flows.
2. Discount Factor (DF): Present Value of $1, and can be used to compute the present value of any cash flow.

$$DF = \frac{1}{(1 + r)^t}$$

Example

You just bought a new computer for $3,000. The payment terms are 2 years same as cash. If you can earn 8% on your money, how much money should you set aside today in order to make the payment when due in two years?

Example Solution

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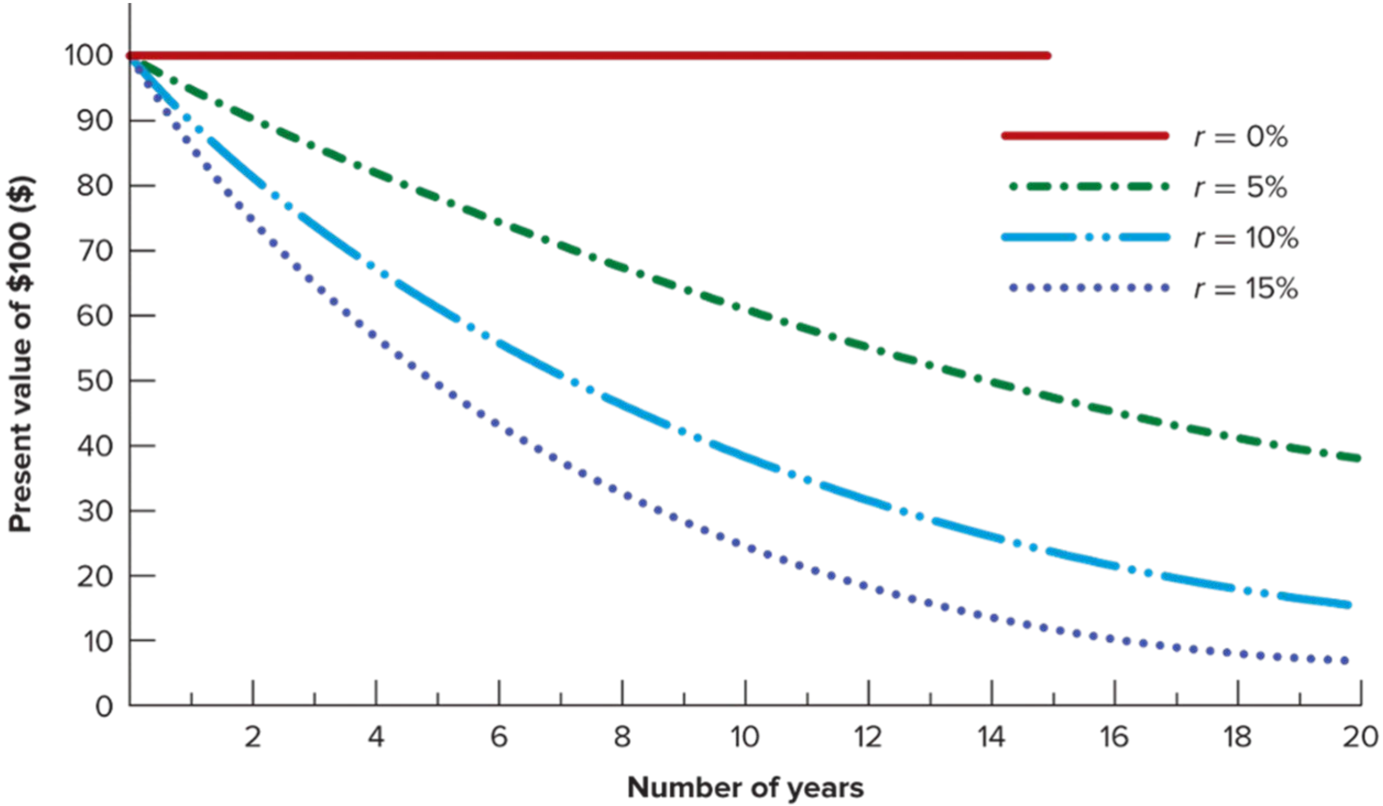
$$PV = \frac{3{,}000}{(1.08)^2} = \$2{,}572$$

Present Values



Example

Kangaroo Autos is offering free credit on a \$20,000 car. You pay \$8,000 up front and then the balance at the end of 2 years. Turtle Motors next door does not offer free credit but will give you \$1,000 off the list price. If the interest rate is 10%, which company is offering the better deal?

Example Solution

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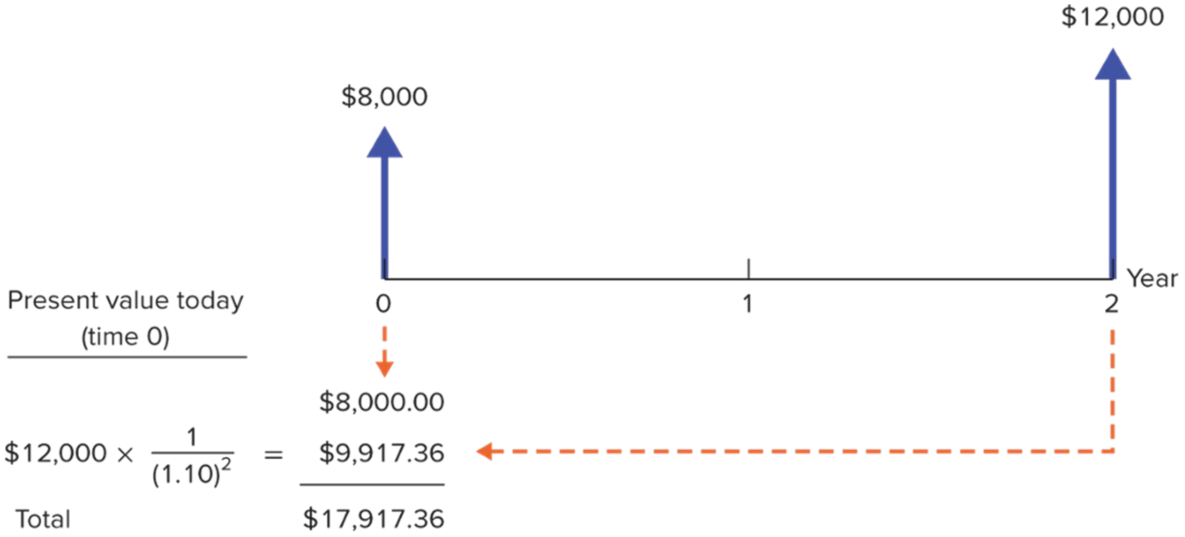
Let's draw a timeline.



Example

How many times have you heard of an investment adviser who promises to double your money? Is this really an amazing feat? That depends on how long it will take for your money to double. With enough patience, your funds eventually will double even if they earn only a very modest interest rate. Suppose your investment adviser promises to double your money in 8 years. What interest rate is implicitly being promised?

Example Solution

The adviser is promising a future value of $2 for every $1 invested today. Therefore, we find the interest rate by solving for r as follows

$$\text{Future value (FV)} = \text{PV} \times (1 + r)^t$$

$$\$2 = \$1 \times (1 + r)^8$$

$$1 + r = 2^{1/8} = 1.0905$$

$$r = .0905, \text{ or } 9.05\%$$

Example

You are able to put $1,200 in the bank now, and another $1,400 in 1 year. If you earn an 8% rate of interest, how much will you be able to spend on a computer in 2 years?

Example Solution

You are able to put \$1,200 in the bank now, and another \$1,400 in 1 year. If you earn an 8% rate of interest, how much will you be able to spend on a computer in 2 years?

$$FV_1 = 1{,}400 \times (1.08) = 1{,}512.00$$

$$FV_2 = 1{,}200 \times (1.08)^2 = 1{,}399.68$$

$$\text{Total FV} = \$2{,}911.68$$

Example

Your auto dealer gives you the choice to pay $15,500 cash now, or make three payments: $8,000 now and $4,000 at the end of the following two years. If your cost of money is 8%, which do you prefer?

Example Solution

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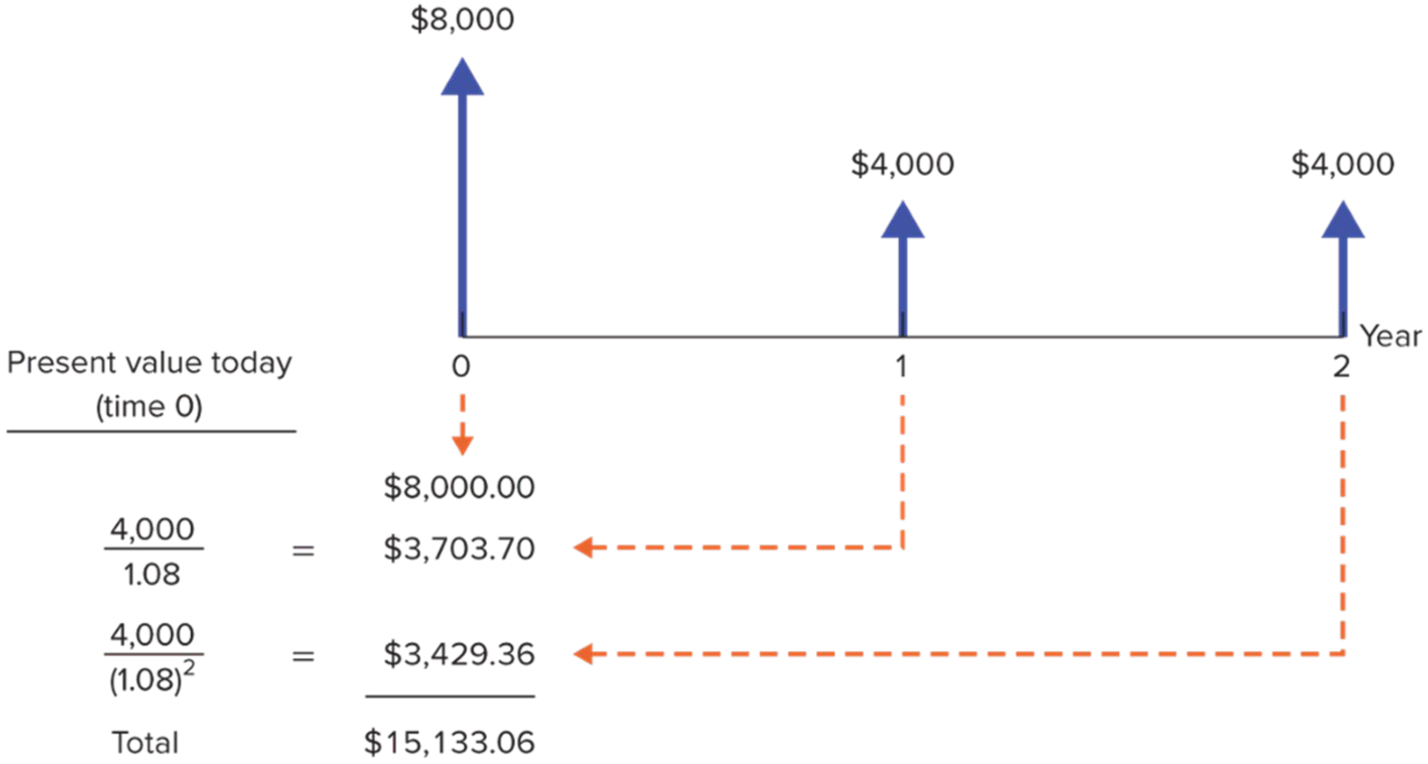
Immediate payment 8,000.00

$$PV_1 = \frac{4{,}000}{(1 + .08)^1} = 3{,}703.70$$

$$PV_2 = \frac{4{,}000}{(1 +. 08)^2} = 3{,}429.36$$

Total PV = $15,133.06

Example Solution



Example

In order to avoid estate taxes, your rich aunt Frederica will pay you $10,000 per year for 4 years, starting 1 year from now. What is the present value of your benefactor's planned gifts? The interest rate is 7%. How much will you have 4 years from now if you invest each gift at 7%?

Example Solution

Gift at Year	Present Value
1	$10{,}000/(1.07) = \$9{,}345.79$
2	$10{,}000/(1.07)^2 = 8{,}734.39$
3	$10{,}000/(1.07)^3 = 8{,}162.98$
4	$10{,}000/(1.07)^4 = 7{,}628.95$
	$33,872.11

Gift at Year	Future Value
1	$10{,}000 \times (1.07)^3 = \$12{,}250.43$
2	$10{,}000 \times (1.07)^2 = 11{,}449$
3	$10{,}000 \times (1.07) = 10{,}700$
4	$10{,}000 = 10{,}000$
	$44,399.43

Perpetuities and Annuities

- Perpetuity
  - A stream of level cash payments that never ends
- Annuity
  - Level stream of cash flows at regular intervals with a finite maturity

Perpetuities and Annuities

PV of Perpetuity Formula

$$\boxed{PV = \frac{C}{r}}$$

C: Cash Payment

r: Interest Rate

Example

In order to create an endowment, which pays $100,000 per year forever, how much money must be set aside today in the rate of interest is 10%?

Example Solution

In order to create an endowment, which pays $100,000 per year forever, how much money must be set aside today in the rate of interest is 10%?

$$PV = \frac{100{,}000}{.10} = \$1{,}000{,}000$$

Example (Continued)

If the first perpetuity payment will not be received until four years from today, how much money needs to be set aside today?

Example Solution (Continued)

If the first perpetuity payment will not be received until four years from today, how much money needs to be set aside today?

$$PV = \frac{1{,}000{,}000}{(1 + .10)^3} = \$751{,}315$$

Perpetuities and Annuities

PV of Annuity Formula

$$PV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right]$$

C = cash payment

r = interest rate

t = Number of years cash payment is received

Perpetuities and Annuities

PV Annuity Factor (PVAF): The present value of $1 a year for each of t years

$$PVAF = \left[\frac{1}{r} - \frac{1}{r(1+r)^t} \right]$$

r = interest rate

t = Number of years cash payment is received

Example

You are purchasing a car. You are scheduled to make 3 annual installments of $8,000 per year. Given a rate of interest of 10%, what is the price you are paying for the car (i.e., what is the PV)?

Example Solution

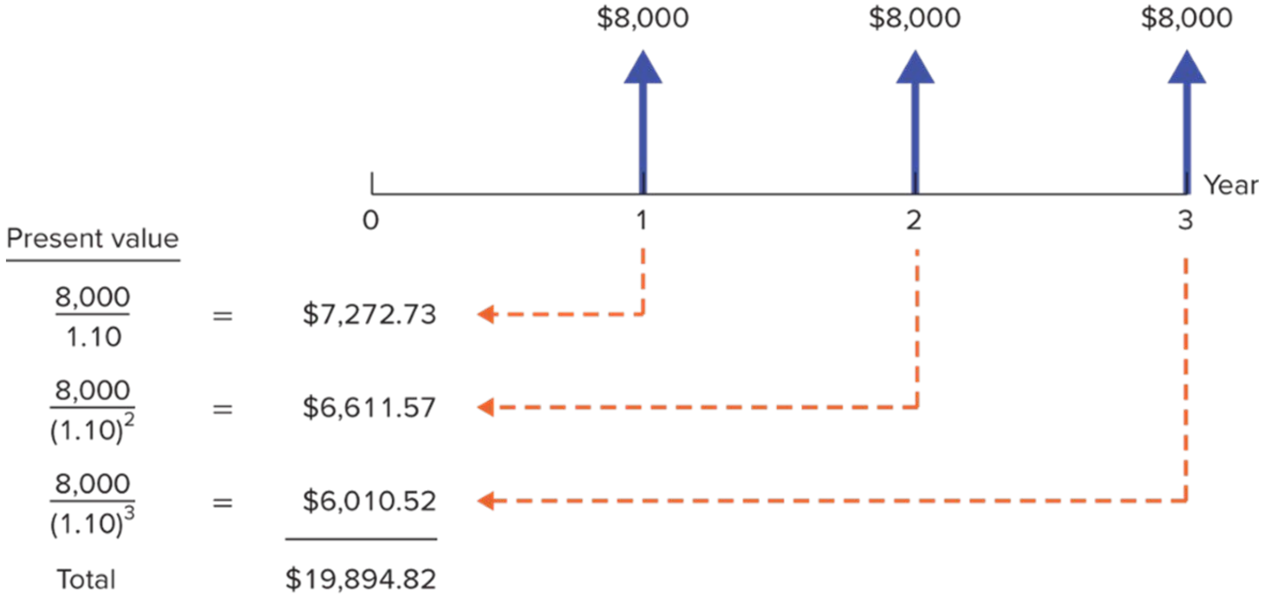
You are purchasing a car. You are scheduled to make 3 annual installments of \$8,000 per year. Given a rate of interest of 10%, what is the price you are paying for the car (i.e., what is the PV)?

$$PV = 8{,}000 \times \left[\frac{1}{.10} - \frac{1}{.10(1 + .10)^3} \right]$$

$$PV = \$19{,}894.82$$

Example Solution (Alternative Approach)

You are purchasing a car. You are scheduled to make 3 annual installments of $8,000 per year. Given a rate of interest of 10%, what is the price you are paying for the car (i.e., what is the PV)?



Perpetuities and Annuities

Future Value of Annuity Payment

$$\boxed{FV = [C \times PVAF] \times (1 + r)^t}$$

Example

You plan to save $3,000 every year for 4 years. Given an 8% rate of interest, what will be the FV of your account?

Example Solution

You plan to save \$3,000 every year for 4 years. Given an 8% rate of interest, what will be the FV of your account?

$$PV = 3{,}000 \times \left[\frac{1}{.08} - \frac{1}{.08(1 + .08)^4} \right]$$

$$PV = \$9{,}936$$

$$FV = \$9{,}936 \times (1.08)^4 = \$13{,}518$$

Example

You are purchasing a home and are scheduled to make 30 annual installments of $10,000 per year. Given an interest rate of 5%, what is the price you are paying for the house (i.e. what is the present value)?

Example Solution

You are purchasing a home and are scheduled to make 30 annual installments of $10,000 per year. Given an interest rate of 5%, what is the price you are paying for the house (i.e. what is the present value)?

$$PV = \$10{,}000\left[\frac{1}{.05} - \frac{1}{.05(1+.05)^{30}}\right]$$

$$PV = \$153{,}724.51$$

Example

You plan to save for 50 years and then retire. Given a 10% rate of interest, if you desire to have $500,000 at retirement, how much must you save each year?

Example Solution

You plan to save for 50 years and then retire. Given a 10% rate of interest, if you desire to have $500,000 at retirement, how much must you save each year?

$$500{,}000 = \textit{Annual Savings} \times \left[\frac{1}{.10} - \frac{1}{.10(1 + .10)^{50}} \right] \times (1 + .10)^{50}$$

Annual Savings = $429.59

Annuities Due

- Annuity Due
  - Level stream of cash flows starting immediately
- How does it differ from an ordinary annuity?

$$PV_{\text{Annuity Due}} = PV_{\text{Annuity}} \times (1 + r)$$

- How does the future value differ from an ordinary annuity?

$$FV_{\text{Annuity Due}} = FV_{\text{Annuity}} \times (1 + r)$$

Example

Suppose you invest $429.59 annually at the beginning of each year at 10% interest. After 50 years, how much would your investment be worth?

Example Solution

Suppose you invest $429.59 annually at the beginning of each year at 10% interest. After 50 years, how much would your investment be worth?

$$FV_{AD} = FV_{Annuity} \times (1 + r)$$

$$FV_{AD} = 429.59 \times \left[\frac{1}{.10} - \frac{1}{.10(1 + .10)^{50}} \right] \times (1 + .10)^{50} \times 1.10$$

$$= 550{,}000$$

Home Mortgages

Sometimes you may need to find the series of cash payments that would provide a given value today. For example, home purchasers typically borrow the bulk of the house price from a lender. The most common loan arrangement is a 30-year loan that is repaid in equal monthly installments. Suppose that a house costs $125,000 and that the buyer puts down 20% of the purchase price, or $25,000, in cash, borrowing the remaining $100,000 from a mortgage lender such as the local savings bank. What is the appropriate monthly mortgage payment?

The borrower repays the loan by making monthly payments over the next 30 years (360 months). The savings bank needs to set these monthly payments so that they have a present value of $100,000. Thus

$$\begin{aligned}\text{Present value} &= \text{mortgage payment} \times \text{360-month annuity factor}\\ &= \$100{,}000\\ \text{Mortgage payment} &= \frac{\$100{,}000}{\text{360-month annuity factor}}\end{aligned}$$

Suppose that the interest rate is 1% a month. Then

$$\text{Mortgage payment} = \frac{\$100{,}000}{\left[\dfrac{1}{.01} - \dfrac{1}{.01(1.01)^{360}}\right]} = \frac{\$100{,}000}{97.218} = \$1{,}028.61$$

Amortization Schedule

An example of an amortizing loan. If you borrow $1,000 at an interest rate of 10%, you would need to make an annual payment of $315.47 over 4 years to repay the loan with interest.

Year	Beginning-of-Year Balance	Year-End Interest Due on Balance	Year-End Payment	Amortization of Loan	End-of-Year Balance
1	$1,000.00	$100.00	$315.47	$215.47	$784.53
2	784.53	78.45	315.47	237.02	547.51
3	547.51	54.75	315.47	260.72	286.79
4	286.79	28.68	315.47	286.79	0

Relevant Spreadsheet Formulae

Present Value = PV(rate, nper, pmt, [fv], [type])

Future Value = FV(rate,nper,pmt,[pv],[type])

Interest Rate = RATE(nper, pmt, pv, [fv], [type], [guess])

Annuity Payments = PMT(rate, nper, pv, [fv], [type])

Number of Periods = NPER(rate,pmt,pv,[fv],[type])

type :

- **0 or omitted when payments begin at the end of the period**
- **1 when payments begin at the beginning of the period**

Effective Interest Rates

- Effective Annual Interest Rate
  - Interest rate that is annualized using compound interest

$$EAR = (1 + MR)^{12} - 1$$

- Annual Percentage Rate
  - Interest rate that is annualized using simple interest

$$APR = MR \times 12$$

*where MR is the monthly interest rate.

Example

Given a monthly rate of 1%, what is the Effective Annual Rate(EAR)? What is the Annual Percentage Rate (APR)?

Example Solution

Given a monthly rate of 1%, what is the Effective Annual Rate(EAR)? What is the Annual Percentage Rate (APR)?

$$EAR = (1 + .01)^{12} - 1 = r$$

$$EAR = (1 + .01)^{12} - 1 = .1268 \text{ or } 12.68\%$$

$$APR = .01 \times 12 = .12 \text{ or } 12.00\%$$

Inflation

- Inflation
    - Rate at which prices as a whole are increasing
- Nominal Interest Rate
    - Rate at which money invested grows
- Real Interest Rate
    - Rate at which the purchasing power of an investment increases

Inflation

Accurate Formula,

$$1 + \text{real interest rate} = \frac{1 + \text{nominal interest rate}}{1 + \text{inflation rate}}$$

Approximation Formula,

$$\text{Real int. rate} \approx \text{nominal int. rate} - \text{inflation rate}$$

Example

If the interest rate on one year govt. bonds is 6.0% and the inflation rate is 2.0%, what is the real interest rate?

Example Solution

If the interest rate on one year govt. bonds is 6.0% and the inflation rate is 2.0%, what is the real interest rate?

$$1 + \text{real interest rate} = \frac{1+.06}{1+.02}$$

$$1 + \text{real interest rate} = 1.039$$

$$1 + \text{real interest rate} = .039 \text{ or } 3.9\%$$

$$\text{Approximation} = .06 - .02 = .04 \text{ or } 4.0\%$$

Inflation

- Current dollar cash flows must be discounted by the nominal interest rate
- Real cash flows must be discounted by the real interest rate

Example

Good news: You will almost certainly be a millionaire by the time you retire in 50 years. Bad news: The inflation rate over your lifetime will average about 3%.

a. What will be the real value of $1 million by the time you retire in terms of today's dollars?
b. What real annuity (in today's dollars) will $1 million support if the real interest rate at retirement is 2% and the annuity must last for 20 years?

Example Solution

Good news: You will almost certainly be a millionaire by the time you retire in 50 years. Bad news: The inflation rate over your lifetime will average about 3%.

a. What will be the real value of $1 million by the time you retire?
b. What real annuity will $1 million support if the real interest rate at retirement is 2% and the annuity must last for 20 years?

a. $1 million will have a real value of $1 million/$(1.03)^{50}$ = $228,107.

b. At a real rate of 2%, this can support a real annuity of:

$$C \times \left[\frac{1}{0.02} - \frac{1}{0.02 \times (1.02)^{20}} \right] = \$228{,}107 \Rightarrow C = \text{PMT} = \$13{,}950$$

Example

You plan to retire in 30 years and want to accumulate enough by then to provide yourself with $30,000 a year for 15 years.

a. If the interest rate is 10%, how much must you accumulate by the time you retire?
b. How much must you save each year until retirement in order to finance your retirement consumption?
c. Now you remember that the annual inflation rate is 4%. If a loaf of bread costs $1 today, what will it cost by the time you retire?
d. You really want to consume $30,000 a year in real dollars during retirement and wish to save an equal real amount each year until then. What is the real amount of savings that you need to accumulate by the time you retire?
e. Calculate the required preretirement real annual savings necessary to meet your consumption goals.
f. What is the nominal value of the amount you need to save during the first year? (Assume the savings are put aside at the end of each year.)
g. What is the nominal value of the amount you need to save during the 30th year?

Example Solution

The PV here refers to PV 30 years from now. This is the amount needed in that year's dollars.

a. $\text{PV} = \$30{,}000 \times \left[\dfrac{1}{0.10} - \dfrac{1}{0.10 \times (1.10)^{15}}\right] = \$228{,}182.39$

b. The present value of the retirement goal is:

$\$228{,}182.39/(1.10)^{30} = \$13{,}076.80$

The present value of your 30-year savings stream must equal this present value.
Therefore, we need to find the payment for which:

$$C \times \left[\frac{1}{0.10} - \frac{1}{0.10 \times (1.10)^{30}}\right] = \$13{,}076.80 \Rightarrow C = \text{PMT} = \$1{,}387.18$$

You must save $1,387.18 per year.

c. $1.00 \times (1.04)^{30} = \$3.24$

d. We repeat part (a) using the real interest rate: $(1.10/1.04) - 1 = 0.0577$, or 5.77%.

The retirement goal in real terms is:

$$\text{PV} = \$30{,}000 \times \left[\frac{1}{0.0577} - \frac{1}{0.0577 \times (1.0577)^{15}}\right] = \$295{,}796.61$$

Example Solution

e. The future value of your 30-year savings stream must equal \$295,796.61. Therefore, we solve for payment (PMT) in the following equation:

$$C \times \left[\frac{1.0577^{30} - 1}{0.0577} \right] = \$295{,}796.61 \Rightarrow C = \text{PMT} = \$3{,}895.66$$

Therefore, we find that you must save \$3,895.66 per year in real terms. This value is much higher than the result found in part (b) because the rate at which purchasing power grows is less than the nominal interest rate, 10%.

f. If the *real* amount saved is \$3,895.66 and prices rise at 4% per year, then the amount saved at the end of 1 year, in nominal terms, will be:

$$\$3{,}895.66 \times 1.04 = \$4{,}051.49$$

The 30th year will require nominal savings of:

$$3{,}895.66 \times (1.04)^{30} = \$12{,}635.17$$

References

Much of this presentation is derived from the course textbook: Fundamentals of Corporate Finance by Richard A. Brealey, Stewart C. Myers and Alan J. Marcus, 10th edition, McGraw Hill Education.