NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 2**

1. Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ and $n \times p$ matrices respectively.

   (a) Suppose the homogeneous linear system $\mathbf{Bx} = \mathbf{0}$ has infinitely many solutions. How many solutions does the system $\mathbf{ABx} = \mathbf{0}$ have?

   Solution: Suppose $\mathbf{u}$ is a solution to $\mathbf{Bx} = \mathbf{0}$, that is, $\mathbf{Bu} = \mathbf{0}$. Premultiplying both sides of $\mathbf{Bu} = \mathbf{0}$ by $\mathbf{A}$, we have $\mathbf{ABu} = \mathbf{0}$, which shows that $\mathbf{u}$ is also a solution to $\mathbf{ABx} = \mathbf{0}$. Hence, $\mathbf{ABx} = \mathbf{0}$ has infinitely solutions too.

   (b) Suppose $\mathbf{Bx} = \mathbf{0}$ has only the trivial solution. Can we tell how many solutions are there for $\mathbf{ABx} = \mathbf{0}$.

   Solution: No, for example, let $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and consider two cases (i) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and (ii) $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Note that $\mathbf{Bx} = \mathbf{0}$ has only the trivial solution. For (i), $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{ABx} = \mathbf{0}$ has only the trivial solution while for (ii), $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $\mathbf{ABx} = \mathbf{0}$ has infinitely many solutions.

2. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Find a 4×3 matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{I}_3$.

   Hint: Write $\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix}$, where $\mathbf{x}_i$ is a 4×1 matrix, for $i = 1, 2, 3$.

   Solution: By block multiplication, we are solving for $\mathbf{Ax}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{Ax}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{Ax}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Solving them simultaneous,

$$\left(\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

   we get general solutions

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix},$$

$$\mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad s_1, s_2, s_3 \in \mathbb{R}.$$

Hence we may let

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Let $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Find a 3×4 matrix $\mathbf{Y}$ such that $\mathbf{YB} = \mathbf{I}_3$.

Solution: Consider solving for $\mathbf{B}^T\mathbf{Y}^T = (\mathbf{YB})^T = \mathbf{I}_3^T = \mathbf{I}_3$ instead. Then by part (a), we let $\mathbf{Y}^T = \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_1 & \mathbf{y}_3 \end{pmatrix}$, where $\mathbf{y}_i$ is a 4×1 matrix for $i = 1, 2, 3$, and we are solving for

$$\left(\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 & 1/2 \end{array}\right)$$

we get general solutions

$$\mathbf{y}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix},$$
$$\mathbf{y}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}, \quad s_1, s_2, s_3 \in \mathbb{R}.$$

So we may let

$$\mathbf{Y} = \begin{pmatrix} 1/2 & 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 1/2 & 0 \end{pmatrix}.$$

Remark. It is possible to use Part (a) to get (b) using the knowledge of elementary row and column matrices. We will give a sketch of the calculation. Interested students could work out the details for themselves. Indeed taking transpose in (a) gives $X^\top A^\top = I_3$. Consider the matrix A. After Interchanging Row 1 and Row 4, interchanging Row 2 and Row 3, interchanging Column 1 and Column 3, we get B. We perform corresponding column and row operations on $X^\top$ and we will get Y.

3. (i) Reduce the following matrices $\mathbf{A}$ to its reduced row-echelon form $\mathbf{R}$.

(ii) For each of the elementary row operation, write the corresponding elementary matrix.

(iii) Write the matrices $\mathbf{A}$ in the form $\mathbf{E}_1\mathbf{E}_2\ldots\mathbf{E}_n\mathbf{R}$ where $\mathbf{E}_1, \mathbf{E_2}, \ldots, \mathbf{E}_n$ are elementary matrices and $\mathbf{R}$ is the reduced row-echelon form of $\mathbf{A}$.

(a) $\mathbf{A} = \begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix}$.

Solution:

(i) $\mathbf{A} \xrightarrow{r_1: R_2 + \frac{2}{5}R_1} \xrightarrow{r_2: \frac{1}{5}R_1} \xrightarrow{r_3: 5R_2} \xrightarrow{r_4: R_1 + \frac{2}{5}R_2} \mathbf{R}$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ \frac{2}{5} & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}.$$

(b) $\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

Solution:

(i) $\mathbf{A} \xrightarrow{r_1: R_2 + 2R_1} \xrightarrow{r_2: R_3 - 4R_1} \xrightarrow{r_3: R_3 + R_2} \xrightarrow{r_4: -R_1} \xrightarrow{r_5: \frac{1}{10}R_2} \xrightarrow{r_6: R_1 + 3R_2} \mathbf{R}$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
$\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_6 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{19}{10} \\ 0 & 1 & -\frac{7}{10} \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

Solution:

(i) $\mathbf{A} \xrightarrow{r_1: R_2 - 2R_1} \xrightarrow{r_2: R_3 - R_1} \xrightarrow{r_3: R_2 \leftrightarrow R_3} \xrightarrow{r_4: \frac{1}{3}R_2} \xrightarrow{r_5: R_2 - R_3} \xrightarrow{r_6: R_1 + R_2} \mathbf{R}$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Determine if the following matrices are invertible. If the matrix is invertible, find its inverse.

(a) $\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}$.

Solution:

$$\left(\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_2+3R_1,\ -R_1,\ \frac{1}{7}R_2,\ R_1+3R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & \frac{3}{7} \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} \end{array}\right).$$

Hence the matrix is invertible and its inverse is $\frac{1}{7}\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

Solution:

$$\left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2+2R_1,\ R_3-4R_1,\ R_3+R_2} \left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right).$$

The matrix is not invertible.

5. Write down the conditions so that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ is invertible.

Solution:

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \xrightarrow[R_3-a^2R_1]{R_2-aR_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \xrightarrow{R_3-(b+a)R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

So we need $c \neq a$ and $c \neq b$ for the last row to be nonzero. Suppose so, we proceed,

$$\xrightarrow{\frac{1}{(c-a)(b-a)}R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1-R_3]{R_2-(c-a)R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $b \neq a$, then it is clear that the matrix can be reduced to the identity matrix. Thus the conditions are $a \neq b$, $b \neq c$, $c \neq a$, that is, they are distinct points.

Alternative: One can stop after the third ERO and note that the determinant of the resultant matrix is $(b-a)(c-a)(c-b)$, which is nonzero if and only if the 3 points are distinct. This determinant is called the *Vandermonde determinant* and we will revisit this type of determinants later in Question 4 in Extra Problems.

6. (a) Suppose $\mathbf{A}$ is a square matrix such that $\mathbf{A}^2 = \mathbf{0}$. Show that $\mathbf{I} - \mathbf{A}$ is invertible, with inverse $\mathbf{I} + \mathbf{A}$.

   Solution: To show that $\mathbf{I} - \mathbf{A}$, suffice to check that it has a left inverse. Indeed,

   $$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A}) = \mathbf{I}^2 - \mathbf{A}^2 = \mathbf{I}.$$

   (b) Suppose $\mathbf{A}^3 = \mathbf{0}$. Is $\mathbf{I} - \mathbf{A}$ invertible?

   Solution: Substituting $\mathbf{A}$ into the polynomial identity $(1-x)(1+x+x^2) = 1-x^3$, we get

   $$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2) = \mathbf{I} - \mathbf{A}^3 = \mathbf{I}.$$

   (c) A square matrix $\mathbf{A}$ is said to be *nilpotent* if there is a positive integer n such that $\mathbf{A}^n = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\mathbf{I} - \mathbf{A}$ is invertible.

   Solution: Substituting $\mathbf{A}$ into the polynomial identity $(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1-x^n$, we get

   $$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^{n-1}) = \mathbf{I} - \mathbf{A}^n = \mathbf{I}.$$

   Hence the inverse matrix of $\mathbf{I} - \mathbf{A}$ is $(\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^{n-1})$.
   Remark: The inverse could be derived from the formula for the sum of a geometric progression,

   $$\sum_{k=1}^{n} x^{k-1} = \frac{1-x^n}{1-x},$$

which is equivalent to $(1-x)\sum_{k=1}^{n} x^{k-1} = 1 - x^n$.
Extra: Show that every strictly upper or lower triangular matrix is nilpotent.

Extra problems

1. Show that a linear system $\mathbf{Ax} = \mathbf{b}$ has either no solution, only one solution or infinitely many solutions. (Hint: Suppose $\mathbf{Ax} = \mathbf{b}$ has two different solutions $\mathbf{u}$ and $\mathbf{v}$. Use $\mathbf{u}$ and $\mathbf{v}$ to construct infinitely many solutions.)

   Solution: We show that if $\mathbf{A}$ has more than one solution, then it has infinitely many solutions. Suppose $\mathbf{u}$ and $\mathbf{v}$ are distinct solutions, $\mathbf{Au} = \mathbf{b} = \mathbf{Av}$ and $\mathbf{u} \neq \mathbf{v}$. For any scalar t,
   $$\mathbf{A}(\mathbf{u} + t(\mathbf{u} - \mathbf{v})) = \mathbf{Au} + t(\mathbf{Au} - \mathbf{Av}) = \mathbf{b} - t(\mathbf{b} - \mathbf{b}) = \mathbf{b}.$$
   The above shows that $\mathbf{u} + t(\mathbf{u} - \mathbf{v})$ is a solution to $\mathbf{Ax} = \mathbf{b}$ too. Since $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$, we have constructed infinitely many solutions.

2. Determine which of the following statements are true. Justify your answer.

   (a) If $\mathbf{A}$ and $\mathbf{B}$ are diagonal matrices of the same size, then $\mathbf{AB} = \mathbf{BA}$.

   Solution: Write $\mathbf{A} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}$. Then
   $$\mathbf{AB} = \begin{pmatrix} a_{11}b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}b_{nn} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} & 0 & \cdots & 0 \\ 0 & b_{22}a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}a_{nn} \end{pmatrix} = \mathbf{BA}.$$

   (b) If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size, $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{B}^2 + 2\mathbf{AB}$.

   Solution: False. For example, let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Note that
   $$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2.$$

   (c) If $\mathbf{A}$ is a square matrix, then $\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric.

   Solution: True.
   $$\left(\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)\right)^T = \frac{1}{2}(\mathbf{A}^T + (\mathbf{A}^T)^T) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T).$$

(d) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the same size, then $\mathbf{A}-\mathbf{B}$ is symmetric.

Solution: True. Since $\mathbf{A}^T=\mathbf{A}$ and $\mathbf{B}^T=\mathbf{B}$, then

$$(\mathbf{A}-\mathbf{B})^T=\mathbf{A}^T-\mathbf{B}^T=\mathbf{A}-\mathbf{B}.$$

(e) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the same size, then $\mathbf{AB}$ is symmetric.

Solution: False. For example $\mathbf{A}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$. Then $\mathbf{AB}=\begin{pmatrix}0&-1\\0&0\end{pmatrix}$ is not symmetric.

(f) If $\mathbf{A}$ is a square matrix such that $\mathbf{A}^2=\mathbf{0}$, then $\mathbf{A}=\mathbf{0}$.

Solution: False. For example, let $\mathbf{A}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$.

(g) If $\mathbf{A}$ is an n by m matrix such that $\mathbf{AA}^T=\mathbf{0}$, then $\mathbf{A}=\mathbf{0}$.

Solution: True. Write $\mathbf{A}=\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\vdots\\\mathbf{a}_n\end{pmatrix}$, where $\mathbf{a}_i$ is the i-th row of $\mathbf{A}$, for $i=1,...,n$. Then $\mathbf{A}^T=\begin{pmatrix}\mathbf{a}_1^T & \mathbf{a}_2^T & \cdots & \mathbf{a}_n^T\end{pmatrix}$ and

$$\mathbf{AA}^T=\begin{pmatrix}\mathbf{a}_1\mathbf{a}_1^T & \mathbf{a}_1\mathbf{a}_2^T & \cdots & \mathbf{a}_1\mathbf{a}_n^T\\ \mathbf{a}_2\mathbf{a}_1^T & \mathbf{a}_2\mathbf{a}_2^T & \cdots & \mathbf{a}_2\mathbf{a}_n^T\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{a}_n\mathbf{a}_1^T & \mathbf{a}_n\mathbf{a}_2^T & \cdots & \mathbf{a}_n\mathbf{a}_n^T\end{pmatrix}=\begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix}.$$

The i-th diagonal entry of $\mathbf{AA}^T$ is

$$a_{i1}a_{i1}+a_{i2}a_{i2}+\cdots+a_{im}a_{im}=a_{i1}^2+a_{i2}^2+\cdots+a_{im}^2=0.$$

This is possible only if $a_{ik}=0$ for all k. Hence, we conclude that $a_{ik}=0$ for all i and k, which shows that $\mathbf{A}=\mathbf{0}$.

3. Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices of the same order. Prove that if $\mathbf{A}$ is singular, then $\mathbf{AB}$ and $\mathbf{BA}$ are singular. (Prove the statement without using determinant.)

Solution: Suppose to the contrary that $\mathbf{AB}$ is invertible. Let $\mathbf{C}$ be the inverse of $\mathbf{AB}$, that is, $\mathbf{I}=(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$. But this means that $\mathbf{A}$ is invertible with inverse $\mathbf{BC}$, a contradiction. Similarly, suppose to the contrary that $\mathbf{BA}$ is invertible. Let $\mathbf{D}$ be its inverse. Then $\mathbf{I} = \mathbf{D}(\mathbf{BA}) = (\mathbf{DB})\mathbf{A}$, which shows that $\mathbf{A}$ is invertible, with inverse $\mathbf{DB}$, which is a contradiction.

4. (**Polynomial Interpolation**)
Given any n points in the xy-plane that has distinct x-coordinates, it is known that there is a unique polynomial of degree $n-1$ or less whose graph passes through those points. A degree $n-1$ polynomial has the following expression

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

Suppose its graph passes through the points (x_1, y_1), (x_2, y_2),...,(x_n, y_n), it follows that the coordinates of the points must satisfy

$$\begin{array}{ccccccccc} a_0 & + & a_1x_1 & + & a_2x_1^2 & +\cdots+ & a_{n-1}x_1^{n-1} & = y_1 \\ a_0 & + & a_1x_2 & + & a_2x_2^2 & +\cdots+ & a_{n-1}x_2^{n-1} & = y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ a_0 & + & a_1x_n & + & a_2x_n^2 & +\cdots+ & a_{n-1}x_n^{n-1} & = y_n \end{array}$$

This is a linear system in the unknowns $a_0, a_1, ..., a_{n-1}$. The augmented matrix for the system is

$$\left(\begin{array}{ccccc|c} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{array}\right) \tag{V}$$

which has a unique solution whenever $x_1, x_2, ..., x_n$ are distinct.

(a) Find a cubic polynomial whose graph passes through the points

x	1	2	3	4
y	3	-2	-5	0

Solution: The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{array}\right)$$

Its RREF is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Hence the cubic polynomial is $x^3 - 5x^2 + 3x + 4$.

(b) **(MATLAB)** The coefficient matrix of the linear system (V) is called a *Vandermonde Matrix*. The function fliplr(vander(v)) returns the Vandermonde matrix such that its rows are powers of the vector v. For example,

```
>> v=[1;2;3;4;5;6;7;8];

>> A=fliplr(vander(v))
```

will generate the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 \\ 1 & 3 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 & 3^7 \\ 1 & 4 & 4^2 & 4^3 & 4^4 & 4^5 & 4^6 & 4^7 \\ 1 & 5 & 5^2 & 5^3 & 5^4 & 5^5 & 5^6 & 5^7 \\ 1 & 6 & 6^2 & 6^3 & 6^4 & 6^5 & 6^6 & 6^7 \\ 1 & 7 & 7^2 & 7^3 & 7^4 & 7^5 & 7^6 & 7^7 \\ 1 & 8 & 8^2 & 8^3 & 8^4 & 8^5 & 8^6 & 8^7 \end{pmatrix}$$

Use the Vandermonde matrix function to find a degree 7 polynomial that passes through

x	1	2	3	4	5	6	7	8
y	12	70	1244	10500	54268	205682	630540	1657024

Solution: `>> v=[1;2;3;4;5;6;7;8];`

`>> A=fliplr(vander(v))`

`>> b=[12;70;1244;10500;54268;205682;630540;1657024];`

`>> A\b` OR `>> rref([A b])`

which gives $a_0 = 8$, $a_1 = 7$, $a_2 = -6$, $a_3 = 5$, $a_4 = -4$, $a_5 = 3$, $a_6 = -2$, $a_7 = 1$. Thus, the polynomial is

$$x^7 - 2x^6 + 3x^5 - 4x^4 + 5x^3 - 6x^2 + 7x + 8$$