NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 3**

1. Let $\mathbf{A}$ be the 4×4 matrix obtained from $\mathbf{I}$ by the following sequence of elementary row operations:

$$\mathbf{I} \xrightarrow{\frac{1}{2}R_2} \xrightarrow{R_1 - R_2} \xrightarrow{R_2 \leftrightarrow R_4} \xrightarrow{R_3 + 3R_1} \mathbf{A}.$$

Write $\mathbf{A}^{-1}$ as a product of four elementary matrices.

Solution:

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. Find an LU factorization for the matrices $\mathbf{A}$, and solve the equation $\mathbf{Ax} = \mathbf{b}$.

(a) $\mathbf{A} = \begin{pmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$.

Solution:

$$\mathbf{A} \xrightarrow{R_2 + 3R_1,\ R_3 - 4R_1\ R_3 + R_2} \mathbf{U} = \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}$.

Solving for $\mathbf{Ly} = \mathbf{b}$, we have $\mathbf{y} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$.

Solving for $\mathbf{Ux} = \mathbf{y}$, we have $\mathbf{x} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$.

(b) $\mathbf{A} = \begin{pmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$.

Solution:

$$\mathbf{A} \xrightarrow{R_2 - 3R_1,\ R_3 + \frac{1}{2}R_1,\ R_3 + 2R_2} \mathbf{U} = \begin{pmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix},$$

and $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{pmatrix}$.

Solving for $\mathbf{Ly} = \mathbf{b}$, we have $\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$.

Solving for $\mathbf{Ux} = \mathbf{y}$, we have

$$\mathbf{x} = \begin{pmatrix} -1 + \frac{4}{3}s \\ -1 + \frac{5}{3}s \\ s \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}.$$

3. Let $\mathbf{A} = \begin{pmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{pmatrix}$.

(a) Find an LU factorization of $\mathbf{A}$.

Solution:

$$\mathbf{A} \sim \begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{pmatrix} \sim \mathbf{U} = \begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{pmatrix}$.

(b) We can find a LU factorization in MATLAB using the command `lu`. Enter the following codes.

```
>> A=[2 -6 6;-4 5 -7;3 5 -1;-6 4 -8;8 -3 9];

>> [L U]=lu(sym(A)).
```

Compare the results with the answer in (a).

Solution: MATLAB returns the same **L** and **U** found in (a).

4. Let $\mathbf{A} = \begin{pmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{pmatrix}$. Compute the determinant of **A** and find all the values of x such that **A** is singular.

Solution:

$$\begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{vmatrix} = -x^3 + 4x^2 - 5x + 2 = -(x-1)^2(x-2).$$

The matrix **A** is singular if and only if $\det \mathbf{A} = 0$ which is $x = 1$ or $x = 2$.

5. Show that $\begin{vmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix} = (1+x^3)\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}.$

Solution:

$$\begin{pmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{pmatrix} \xrightarrow{R_2 - xR_3} \begin{pmatrix} a+px & b+qx & c+rx \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{pmatrix} \xrightarrow{R_1 - xR_2}$$
$$\begin{pmatrix} a(1+x^3) & b(1+x^3) & c(1+x^3) \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{pmatrix}.$$

Now

$$\begin{vmatrix} a(1+x^3) & b(1+x^3) & c(1+x^3) \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{vmatrix} = (1+x^3)\begin{vmatrix} a & b & c \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{vmatrix}.$$

Finally,

$$\begin{pmatrix} a & b & c \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{pmatrix} \xrightarrow{R_2 + x^2R_1, R_3 - xR_1} \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix}$$

Since all the elementary row operation are adding a multiple of one row to another, and thus the determinant of the elementary matrices are 1. Hence,

$$\begin{aligned} \begin{vmatrix} a(1+x^3) & b(1+x^3) & c(1+x^3) \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{vmatrix} &= (1+x^3)\begin{vmatrix} a & b & c \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{vmatrix} \\ &= (1+x^3)\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}. \end{aligned}$$

6. Let $\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. Compute

(a) $\det(3\mathbf{A}^T)$;

(b) $\det(3\mathbf{A}\mathbf{B}^{-1})$; and

(c) $\det((3\mathbf{B})^{-1})$.

Solution: $\det(\mathbf{A}) = -2$ and $\det(\mathbf{B}) = 3$.

(a) $\det(3\mathbf{A}^T) = 3^4 \det(\mathbf{A}^T) = 3^4 \det(\mathbf{A}) = -162$

(b) $\det(3\mathbf{A}\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}) \det(\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}) \dfrac{1}{\det(\mathbf{B})} = -54$

(c) $\det((3\mathbf{B})^{-1}) = \dfrac{1}{\det(3\mathbf{B})} = \dfrac{1}{3^4 \det(\mathbf{B})} = \dfrac{1}{3^5} = \dfrac{1}{243}$

7. Use Cramer's rule to solve

$$\begin{cases} x + 5y + 3z = 1 \\ \phantom{x + {}} 2y - 2z = 2 \\ y + 3z = 0 \end{cases}$$

Solution: Let $\mathbf{A} = \begin{pmatrix} 1 & 5 & 3 \\ 0 & 2 & -2 \\ 0 & 1 & 3 \end{pmatrix}$, $\mathbf{A}_1 = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 2 & -2 \\ 0 & 1 & 3 \end{pmatrix}$, $\mathbf{A}_2 = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix}$, and $\mathbf{A}_3 = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix}$. Then

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} \det(\mathbf{A}_1) \\ \det(\mathbf{A}_2) \\ \det(\mathbf{A}_3) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -8 \\ 3 \\ -1 \end{pmatrix}.$$

8. Compute the adjoint of $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 6 \end{pmatrix}$, and use it to compute $\mathbf{A}^{-1}$.

Solution:

$$\begin{aligned}
\operatorname{adj}(\mathbf{A}) &= \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 12 & 6 & -5 \\ 3 & 0 & -1 \\ -6 & -3 & 2 \end{pmatrix}.
\end{aligned}$$

Therefore

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A}) = \frac{-1}{3} \begin{pmatrix} 12 & 6 & -5 \\ 3 & 0 & -1 \\ -6 & -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & -2 & 5/3 \\ -1 & 0 & 1/3 \\ 2 & 1 & -2/3 \end{pmatrix}.$$

Extra problems

1. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

   (a) Is it possible find an LU factorization of $\mathbf{A}$?

   Solution: First note that $\mathbf{A}$ is invertible. Suppose we write

   $$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} l_{11} & 0 & 0 \\ 1_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

   Observe that $0 = a_{11} = l_{11}u_{11}$, which means that either $u_{11} = 0$ or $l_{11} = 0$. Suppose $u_{11} = 0$, then $\mathbf{U}$ is singular, a contradiction to $\mathbf{A}$ being invertible. Similarly, if $l_{11} = 0$, then $\mathbf{L}$ is singular, and thus a contradiction to $\mathbf{A}$ being invertible. Hence, it is not possible to find an LU factorization of $\mathbf{A}$.

   (b) An $n \times n$ matrix $\mathbf{P}$ is a *permutation matrix* if there is exactly one 1 in each row and column, and 0 everywhere else, equivalent, a matrix obtained from the identity matrix using only row swap. Find a permutation matrix $\mathbf{P}$ such that $\mathbf{PA}$ has an LU factorization.

   Solution: Let $\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

   $$\mathbf{PA} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \xrightarrow{R_3 + R_2} \mathbf{U} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

   Then $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

   This question demonstrates that not every matrix can be written as a LU factorization. However, we can always find a permutation matrix $\mathbf{P}$ such that $\mathbf{PA}$ has an LU factorization.

2. Let $\mathbf{A}$ be a $n \times n$ invertible matrix.

   (a) Show that $\mathrm{adj}(\mathbf{A})$ is invertible and find its inverse.

   Solution: Since A is invertible, $\det(\mathbf{A}) \neq 0$. Using the identity $\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{I}$, we have

   $$\left(\frac{1}{\det(\mathbf{A})}\mathbf{A}\right) \mathrm{adj}(\mathbf{A}) = \mathbf{I}.$$

This shows that $\operatorname{adj}(\mathbf{A})$ is invertible whose inverse matrix is

$$\operatorname{adj}(\mathbf{A})^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{A}.$$

(b) Find $\det(\operatorname{adj}(\mathbf{A}))$.

Solution: Using $\operatorname{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{A}^{-1}$,

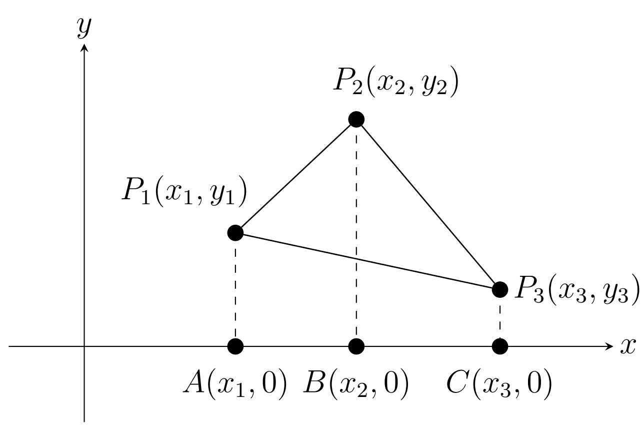
$$\det(\operatorname{adj}(\mathbf{A})) = \det(\det(\mathbf{A})\mathbf{A}^{-1}) = \det(\mathbf{A})^n \det(\mathbf{A})^{-1} = \det(\mathbf{A})^{n-1}.$$

(c) If $\det(\mathbf{A}) = 1$, show that $\operatorname{adj}(\operatorname{adj}(\mathbf{A})) = \mathbf{A}$.

Solution: We have $\operatorname{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{A}^{-1} = \mathbf{A}^{-1}$. Now $\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1} = 1$. By (a),

$$\operatorname{adj}(\operatorname{adj}(\mathbf{A})) = \operatorname{adj}(\mathbf{A}^{-1}) \overset{(a)}{=} \frac{1}{\det(\mathbf{A}^{-1})}(\mathbf{A}^{-1})^{-1} = (\mathbf{A}^{-1})^{-1} = \mathbf{A}.$$

3. Consider the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) as shown in the figure below.



We may compute the area of the triangle as

(area of trapezoid AP_1P_2B)+(area of trapezoid BP_2P_3C)−(area of trapezoid AP_1P_3C)

(a) Recall that the area of a trapezoid is $\frac{1}{2}$ the distance between the parallel sides of the trapezoid times the sum of the lengths of the parallel sides. Use this fact to show that the area of the triangle $P_1P_2P_3$ is

$$-\frac{1}{2}\left[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right].$$

Solution: The area of triangle $P_1P_2P_3$ is

$$\begin{aligned}
&= \frac{1}{2}(x_2-x_1)(y_1+y_2)+\frac{1}{2}(x_3-x_2)(y_2+y_3)-\frac{1}{2}(x_3-x_1)(y_1+y_3)\\
&= \frac{1}{2}x_2y_1-\frac{1}{2}x_1y_2+\frac{1}{2}x_3y_2-\frac{1}{2}x_2y_3-\frac{1}{2}x_3y_1+\frac{1}{2}x_1y_3\\
&= -\frac{1}{2}\left[(x_2y_3-x_3y_2)-(x_1y_3-x_3y_1)+(x_1y_2-x_2y_1)\right].
\end{aligned}$$

(b) Show that the expression in the square brackets obtained in part (a) is the determinant of the following matrix

$$\mathbf{A}=\begin{pmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{pmatrix}.$$

Solution: By cofactor expansion along the third column of $\mathbf{A}$, we have

$$\det(\mathbf{A})=\begin{vmatrix} x_2 & y_2\\ x_3 & y_3\end{vmatrix}-\begin{vmatrix} x_1 & y_1\\ x_3 & y_3\end{vmatrix}+\begin{vmatrix} x_1 & y_1\\ x_2 & y_2\end{vmatrix}=(x_2y_3-x_3y_2)-(x_1y_3-x_3y_1)+(x_1y_2-x_2y_1).$$

(c) Explain why we need to take the absolute value of $\det(\mathbf{A})$ before concluding that the area of the triangle is

$$\frac{1}{2}|\det(\mathbf{A})|.$$

Solution: The determinant may be positive or negative, depending on the location of the points and how they are labeled. For example, compare the determinant of

$$\begin{pmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 & y_1 & 1\\ x_3 & y_3 & 1\\ x_2 & y_2 & 1\end{pmatrix}.$$

(d) Find the area of the following quadrilaterals with the given vertices.

(i) P with vertices $(2,3)$, $(5,3)$, $(4,5)$, $(7,5)$.

(ii) Q with vertices $(-2,3)$, $(1,4)$, $(3,0)$, $(-1,-3)$.

Solution:

(i) The quadrilateral is divided into two triangles P_1 and P_2 with vertices $(2,3)$, $(5,3)$, $(4,5)$ and $(4,5)$, $(5,3)$, $(7,5)$ respectively. The area of the quadrilatral is:

$$\frac{1}{2}\begin{vmatrix} 2 & 3 & 1\\ 5 & 3 & 1\\ 4 & 5 & 1\end{vmatrix}+\frac{1}{2}\begin{vmatrix} 4 & 5 & 1\\ 5 & 3 & 1\\ 7 & 5 & 1\end{vmatrix}=\frac{1}{2}(6)+\frac{1}{2}(6)=6.$$

(ii) The quadrilateral is divided into two triangles P_1 and P_2 with vertices $(-2,3)$, $(1,4)$, $(3,0)$ and $(-2,3)$, $(-1,-3)$, $(3,0)$ respectively. The area of the quadrilateral is:

$$\frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ -2 & 3 & 1 \\ 3 & 0 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -2 & 3 & 1 \\ -1 & -3 & 1 \\ 3 & 0 & 1 \end{vmatrix} = \frac{1}{2}(14) + \frac{1}{2}(27) = 20.5.$$

4. (**Approximate integration**)
There are some integrals that are not possible to solve by finding the antiderivate. The integral can be approximated by using an interpolating polynomial to approximate the integrand and integrating the approximating polynomial. For example, suppose we want to evaluate the integral

$$\int_0^1 e^{-x^2} dx$$

We begin by picking a few points between the limits, say

$$x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,\ x_4 = 1$$

and evaluate the integrand $f(x) = e^{-x^2}$ at these points. They are approximately

$$f(0) = 1, f(0.25) = 0.9394, f(0.5) = 0.7788, f(0.75) = 0.5698, f(1) = 0.3679$$

The interpolating polynomial is (check it!)

$$p(x) = 0.0416x^4 + 0.4882x^3 - 1.1846x^2 + 0.0226x + 1$$

and

$$\int_0^1 p(x)dx \approx 0.7468$$

which, up to the fourth decimal place, is exactly the integral $\int_0^1 e^{-x^2} dx$. The more points we pick, and thus the higher the degree of the polynomial, the more accurate the approximation becomes.

By using MATLAB, we can easily approximate the integral using interpolating polynomial. Suppose we want an interpolating polynomial of degree n. First create a vector whose entries are $n+1$ regular steps between a to b

```
>> v=[a:(b-a)/n:b]';
```

Then we create the Vandermonde matrix

```
>> A=fliplr(vander(v));
```

Let $\mathbf{b}$ be the vector whose entries are the evaluation of e^{-x^2} at the points of $\mathbf{v}$

```
>> b=exp(-v.^2);
```

Then we obtain the coefficient of the interpolating polynomial of degree n

```
>> A\b
```

Use an interpolating polynomial of degree 10 to approximate the integral $\int_0^1 e^{-x^2}dx$.

Solution: `>> v=[0:1/10:1]';`

The interpolating polynomial is

$$\begin{aligned}&0.0043x^{10} - 0.0345x^9 + 0.0893x^8 - 0.0405x^7\\ &- 0.1440x^6 - 0.0084x^5 + 0.502x^4 - 0.0003x^3 - x^2 + 1\end{aligned}$$

whose antiderivative is

$$\begin{aligned}&\frac{0.0043}{11}x^{11} - \frac{0.0345}{10}x^10 + \frac{0.0893}{9}x^9 - \frac{0.0405}{8}x^8 - \frac{0.1440}{7}x^7\\ &- \frac{0.0084}{6}x^6 + \frac{0.502}{5}x^5 - \frac{0.0003}{4}x^4 - \frac{1}{3}x^3 + x.\end{aligned}$$

Thus, the integral is approximately

$$\frac{0.0043}{11} - \frac{0.0345}{10} + \frac{0.0893}{9} - \frac{0.0405}{8} - \frac{0.1440}{7} - \frac{0.0084}{6} + \frac{0.502}{5} - \frac{0.0003}{4} - \frac{1}{3} + 1 = 0.7468.$$