MA1522 Linear Algebra for Computing

Lecture 5: Determinants

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10 February, 2025

Outline

Question in Section 2.8

Q: Suppose a square matrix $\mathbf{A}$ has a zero row or column, what can you conclude about $\det(\mathbf{A})$?

What is the determinant of the following matrix?

$$\begin{pmatrix} 1 & 1 & 3 & 0 & 5 & -2 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 7 & 2 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 & 2 & 1 \\ 1 & -5 & 7 & 0 & 8 & 0 \end{pmatrix}$$

Slide 136: Determinant by Cofactor Expansion

The *determinant* of $\mathbf{A}$ is defined to be

$$\det(\mathbf{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{k=1}^{n} a_{ik}A_{ik} \tag{1}$$

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{k=1}^{n} a_{kj}A_{kj} \tag{2}$$

This is called the *cofactor expansion* along $\left\{ \begin{array}{ccc} \text{row} & i & (1) \\ \text{column} & j & (2) \end{array} \right.$.

Answer: If we expand along the zero row, we immediately get $\det(\mathbf{A}) = 0$.

Question in Section 2.9.

Q: Suppose a square matrix $\mathbf{A}$ has 2 equal rows (or columns). What can you conclude about its determinant?

On Slide 149, we have

Theorem

Let $\mathbf{A}$ be a $n \times n$ square matrix. Suppose $\mathbf{B}$ is obtained from $\mathbf{A}$ via a single elementary row operation. Then the determinant of $\mathbf{B}$ is obtained as such.

$\mathbf{A} \xrightarrow{R_i + aR_j} \mathbf{B}$	$\det(\mathbf{B}) = \det(\mathbf{A})$
$\mathbf{A} \xrightarrow{cR_i} \mathbf{B}$	$\det(\mathbf{B}) = c\det(\mathbf{A})$
$\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B}$	$\det(\mathbf{B}) = -\det(\mathbf{A})$

Answer to Question in Section 2.9.

Q: Suppose a square matrix $\mathbf{A}$ has 2 equal rows (or columns). What can you conclude about its determinant?

Answer: We conclude that $\det(\mathbf{A}) = 0$. Suppose that the equal rows are rows R_i and R_j with $i \neq j$. By Slide 149, we can perform $R_j - R_i$ to make R_j a zero row without changing the determinant. The result follows.

Question one in Section 2.10

Let $\mathbf{A}$ be a LU factorizable square matrix with LU factorization

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & * & \cdots & * \\ 0 & u_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}.$$

What is the determinant of $\mathbf{A}$?

Slide 155: Determinant of Product of Matrices

Theorem

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be square matrices of the same size. Then*

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}).$$

By induction, we get

$$\det(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k) = \det(\mathbf{A}_1)\det(\mathbf{A}_2)\cdots\det(\mathbf{A}_k).$$

Answer to Question one in Section 2.10

Q: Let $\mathbf{A}$ be a LU factorizable square matrix with LU factorization

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & * & \cdots & * \\ 0 & u_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}.$$

What is the determinant of $\mathbf{A}$?

Answer: Since $\mathbf{L}$ is unit lower triangular, $\det(\mathbf{L}) = 1$. Thus

$$\det(\mathbf{A}) = \det(\mathbf{L})\det(\mathbf{U}) = u_{11} \cdot \ldots \cdot u_{nn}.$$

Examples in Section 2.10

Let $\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

$$\mathbf{A} \xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

So $\det(\mathbf{A}) = -2$ and $\det(\mathbf{B}) = 3$.

1. $\det(3\mathbf{A}^T) = 3^4 \det(\mathbf{A}^T) = 3^4 \det(\mathbf{A}) = -162$
2. $\det(3\mathbf{A}\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}) \det(\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}) \frac{1}{\det(\mathbf{B})} = -54$
3. $\det((3\mathbf{B})^{-1}) = \frac{1}{\det(3\mathbf{B})} = \frac{1}{3^4 \det(\mathbf{B})} = \frac{1}{3^5} = \frac{1}{243}$.

Question two in Section 2.10

1. Show that a square matrix $\mathbf{A}$ is singular if and only if $\mathbf{A}$ times its adjoint is the zero matrix, $\mathbf{A}(\mathrm{adj}(\mathbf{A})) = \mathbf{0}$.

2. Is it true that $\mathbf{A}$ is singular if and only if the adjoint of $\mathbf{A}$ times $\mathbf{A}$ is the zero matrix, $(\mathrm{adj}(\mathbf{A}))\mathbf{A} = \mathbf{0}$?

Slide 161: Adjoint

Definition

Let $\mathbf{A}$ be a $n \times n$ square matrix. The *adjoint* of $\mathbf{A}$, denoted as $\text{adj}(\mathbf{A})$, is the $n \times n$ square matrix whose (i, j) entry is the (j, i)-cofactor of $\mathbf{A}$,

$$\text{adj}(\mathbf{A}) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

Adjoint Formula

Theorem

Let $\mathbf{A}$ *be a square matrix and* $\text{adj}(\mathbf{A})$ *its adjoint. Then*

$$\mathbf{A}(\text{adj}(\mathbf{A})) = \det(\mathbf{A})\mathbf{I},$$

where $\mathbf{I}$ *is the identity matrix.*

Corollary (Adjoint Formula for Inverse)

Let $\mathbf{A}$ *be an invertible matrix. Then the inverse of* $\mathbf{A}$ *is given by*

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\text{adj}(\mathbf{A}).$$

The corollary follows immediately from the previous theorem, and the fact that $\det(\mathbf{A}) \neq 0$.

Answer to Question two in Section 2.10, part 1

Q: Show that a square matrix $\mathbf{A}$ is singular if and only if $\mathbf{A}$ times its adjoint is the zero matrix, $\mathbf{A}(\mathrm{adj}(\mathbf{A})) = \mathbf{0}$.

Answer: $\mathbf{A}$ is singular iff $\det(\mathbf{A}) = 0$ iff $\mathbf{A}(\mathrm{adj}(\mathbf{A})) = \mathbf{0}$ by Adjoint Formula $\mathbf{A}(\mathrm{adj}(\mathbf{A})) = \det(\mathbf{A})\mathbf{I}$.

A Theorem

Prove that

$$\mathrm{adj}(\mathbf{A})^T = adj(\mathbf{A}^T).$$

Proof. We have used the notations $\mathbf{M}_{ij}$ for (i, j) matrix minor of $\mathbf{A}$ and A_{ij} for (i, j) cofactor of $\mathbf{A}$. Note $A_{ij} = (-1)^{i+j} \det(\mathbf{M}_{ij})$.

Let's use $\mathbf{N}_{ij}$ for (i, j) matrix minor of $\mathbf{A}^{\mathbf{T}}$ and A^T_{ij} for (i, j) cofactor of $\mathbf{A}^{\mathbf{T}}$.

Key observation:

$$\mathbf{M}_{ij} = \mathbf{N}^T_{ji}.$$

Proof (continued)

$$
\begin{aligned}
& \text{The } (i,j) \text{ entry of } \mathrm{adj}(\mathbf{A})^T \\
= \;& (j,i) \text{ entry of } \mathrm{adj}(\mathbf{A}) \\
= \;& (i,j) \text{ cofactor of } \mathbf{A} \\
= \;& (-1)^{i+j} \det(\mathbf{M}_{ij}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \text{the } (i,j) \text{ entry of } \mathrm{adj}(\mathbf{A}^T) \\
= \;& (j,i)\text{-cofactor of } \mathbf{A}^T \\
= \;& (-1)^{j+i} \det(\mathbf{N}_{ji}) \\
= \;& (-1)^{j+i} \det(\mathbf{M}_{ij}^T).
\end{aligned}
$$

Since determinant is invariant under transpose (Slide 137), the (i,j) entry of $adj(\mathbf{A})^T = (i,j)$ entry of $adj(\mathbf{A}^T)$.

A Corollary

Using the Theorem, we can show:

$$\mathrm{adj}(\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathbf{I}$$

Proof. For any matrix $\mathbf{B}$, we have

$$(\mathbf{B}\,\mathrm{adj}(\mathbf{B}))^T = (\det(\mathbf{B})\mathbf{I})^T \quad \Rightarrow \quad \mathrm{adj}(\mathbf{B})^T\mathbf{B}^T = \det(\mathbf{B})\mathbf{I} = \det(\mathbf{B}^T)\mathbf{I}.$$

But by previous results, $\mathrm{adj}(\mathbf{B})^T = \mathrm{adj}(\mathbf{B}^T)$. Hence,

$$\mathrm{adj}(\mathbf{B}^T)\mathbf{B}^T = \det(\mathbf{B}^T)\mathbf{I}.$$

Now replace $\mathbf{B}$ with $\mathbf{A}^T$,

$$\mathrm{adj}(\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathbf{I}.$$

Question two in Section 2.10, part 2

Q: Is it true that $\mathbf{A}$ is singular if and only if the adjoint of $\mathbf{A}$ times $\mathbf{A}$ is the zero matrix, $(\text{adj}(\mathbf{A}))\mathbf{A} = \mathbf{0}$?

Answer: By the corollary, $\text{adj}(\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathbf{I}$. Thus $\mathbf{A}$ is singular iff $\det(\mathbf{A}) = 0$ iff $(\text{adj}(\mathbf{A}))\mathbf{A} = \mathbf{0}$.

Introductory Remarks

- In MA1522, we only study the Euclidean spaces $\mathbb{R}^n$.
- But there are "structures" which are not $\mathbb{R}^n$, but share the "essential" properties of $\mathbb{R}^n$.
- These structures are the *Vector Spaces*. Of course, $\mathbb{R}^n$ will be the typical examples.
- These "essential" properties are extracted from $\mathbb{R}^n$ and serve as defining "axioms".
- (Axiomatical methods. Example, equivalence relations.)
- Vector spaces only extract the "basic" properties of $\mathbb{R}^n$. One can add "metric" or "inner product" to pursue different objectives.

Definition of Abstract Vector Spaces

A set V equipped with addition and scalar multiplication is said to be a *vector space* over $\mathbb{R}$ if it satisfies the following axioms.

1. (Commutative) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. (Associative) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
3. (Zero vector) There is a vector $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$ in V.
4. (Negative) For any vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. (Distribution) For any scalar a in $\mathbb{R}$ and vectors $\mathbf{u}, \mathbf{v}$ in V, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
6. (Distribution) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
7. (Associativity of scalar multiplication) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $a(b\mathbf{u}) = (ab)\mathbf{u}$.
8. For any vector $\mathbf{u}$ in V, $1\mathbf{u} = \mathbf{u}$.

Remark

"Equipped with addition and scalar multiplication" means:

For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V; and

For any scalar a in $\mathbb{R}$ and vector $\mathbf{v}$ in V, $a\mathbf{v}$ is a vector in V.

Challenge

1. Show that the set of all degree at most n polynomials with real coefficients is a vector space with the usual addition and scalar multiplication,
   (i) $(a_n x^n + \cdots + a_1 x + a_0) + (b_n x^n + \cdots + b_1 x + b_0) = (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + (a_0 + b_0)$,
   (ii) $b(a_n x^n + \cdots + a_1 x + a_0) = ba_n x^n + \cdots + ba_1 x + ba_0$.

2. Show that the set of all $n \times m$ real-valued matrices is a vector space, with the usual matrix addition and scalar multiplication. The set of all $n \times m$ real-valued matrices is sometimes denoted as $\mathbb{R}^{n \times m}$.

We only show 1.

Definition of P_n

In part one, the set V is

$$\{a_0 + a_1x + \cdots + a_nx^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\},$$

where $n \geq 0$ is a fixed given integer (and we allow $a_n = 0$). Every $f(x)$ in V will be called a vector.

The addition of vectors are defined by

$$\begin{aligned} & (a_nx^n + \cdots + a_1x + a_0) + (b_nx^n + \cdots + b_1x + b_0) \\ = \; & (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + (a_0 + b_0). \end{aligned}$$

Note that the sum is still in V (it may have degree $< n$.)

The scalar multiplication is defined by

$$b(a_nx^n + \cdots + a_1x + a_0) = ba_nx^n + \cdots + ba_1x + ba_0,$$

where $b \in \mathbb{R}$.

Again notice that scalar product is still in V.

Verifying P_n is a Vector Spaces (I)

We verify the properties in the definition one by one:

1. (Commutative) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

   Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ and $g(x) = b_n x^n + \cdots + b_1 x + b_0$ be two vectors in P_n. Then

$$\begin{aligned} & f(x) + g(x) \\ = \; & (a_n x^n + \cdots + a_1 x + a_0) + (b_n x^n + \cdots + b_1 x + b_0) \\ = \; & (a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0) \\ = \; & (b_n + a_n) x^n + \cdots + (b_1 + a_1) x + (b_0 + a_0) \\ = \; & (b_n x^n + \cdots + b_1 x + b_0) + (a_n x^n + \cdots + a_1 x + a_0) \\ = \; & g(x) + f(x). \end{aligned}$$

2. (Associative) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

   (Skipped)

Verifying P_n is a Vector Spaces (II)

We verify the remaining properties in the definition one by one:

3. (Zero vector) There is a vector $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$ in V.

   Just take $z(x)$ to be 0. Then for any $f(x) \in P_n$,

   $$z(x) + f(x) = f(x).$$

4. (Negative) For any vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

   For any $f(x) \in P_n$, say $f(x) = a_n x^n + \cdots + a_1 x + a_0$. Let $-f(x) = -a_n x^n - \cdots - a_1 x - a_0$. Then $f(x) + (-f(x)) = z(x)$.

Verifying P_n is a Vector Spaces (III)

5. (Distribution) For any scalar a in $\mathbb{R}$ and vectors $\mathbf{u}, \mathbf{v}$ in V, $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

   Let $a \in \mathbb{R}$, $f(x) = a_n x^n + \cdots + a_1 x + a_0$ and $g(x) = b_n x^n + \cdots + b_1 x + b_0$ be two vectors in P_n. Then

$$\begin{aligned}
& a(f(x) + g(x)) \\
= \; & a[(a_n + b_n)x^n + \cdots + (a_1 + b_1)x + (a_0 + b_0)] \\
= \; & (aa_n + ab_n)x^n + \cdots + (aa_1 + ab_1)x + (aa_0 + ab_0) \\
= \; & af(x) + ag(x).
\end{aligned}$$

6. (Distribution) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

   (Skipped)

Verifying P_n is a Vector Spaces (IV)

7. (Associativity of scalar multiplication) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $a(b\mathbf{u}) = (ab)\mathbf{u}$.

   Let $a, b \in \mathbb{R}$ and $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a vector in P_n. Then

$$\begin{aligned} & a(bf(x)) \\ = \; & a(ba_n x^n + \cdots + ba_1 x + ba_0) \\ = \; & aba_n x^n + \cdots + aba_1 x + aba_0 \\ = \; & (ab)(a_n x^n + \cdots + a_1 x + a_0) \\ = \; & (ab)f(x). \end{aligned}$$

8. For any vector $\mathbf{u}$ in V, $1\mathbf{u} = \mathbf{u}$. (Skipped)