MA1522: Linear Algebra for Computing

Tutorial 8

Revision

Orthogonal Projection

Orthogonal to a Subspace

A vector $\mathbf{n} \in \mathbb{R}^n$ is orthogonal to a subspace V if for every $\mathbf{v} \in V$, $\mathbf{n} \cdot \mathbf{v} = 0$. Denote it as $\mathbf{n} \perp V$.

Theorem

Let $V \subseteq \mathbb{R}^n$ be a subspace and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a spanning set for V, span$(S) = V$. Then $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to V if and only if $\mathbf{w} \cdot \mathbf{u}_i = 0$ for all $i = 1, ..., k$.

Orthogonal Projection

Let $V \subseteq \mathbb{R}^n$ be a subspace. Every vector $\mathbf{w} \in \mathbb{R}^n$ can be decomposed uniquely as

$$\mathbf{w} = \mathbf{w}_p + \mathbf{w}_n,$$

where $\mathbf{w}_n$ is orthogonal to V and

$$\mathbf{w}_p = \left(\frac{\mathbf{w} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 + \left(\frac{\mathbf{w} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right)\mathbf{u}_2 + \cdots + \left(\frac{\mathbf{w} \cdot \mathbf{u}_k}{\|\mathbf{u}_k\|^2}\right)\mathbf{u}_k$$

is a vector in V, called the orthogonal projection (or just projection) of $\mathbf{w}$ onto V. Here, $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is an orthogonal basis for V.

Gram-Schmidt Process

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a linearly independent set. Let

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}_2}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}_3}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_3}{\|\mathbf{v}_2\|^2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}_k}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_k}{\|\mathbf{v}_2\|^2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{u}_k}{\|\mathbf{v}_{k-1}\|^2}\right)\mathbf{v}_{k-1}.
\end{aligned}
$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is an orthogonal set (of nonzero vectors), and hence,

$$\left\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, ..., \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|}\right\}$$

is an orthonormal set such that $\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_k\} = \text{span}\{\mathbf{u}_k, ..., \mathbf{u}_k\}$.

Orthogonal Matrices

A square matrix $\mathbf{A}$ of order n is an orthogonal matrix if $\mathbf{A}^T = \mathbf{A}^{-1}$, equivalently, $\mathbf{A}^T\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^T$.

Theorem

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) $\mathbf{A}$ *is orthogonal.*

(ii) *The columns of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$.*

(iii) *The rows of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$.*

QR Factorization

Theorem (QR Factorization)

Suppose $\mathbf{A}$ *is a* $m \times n$ *matrix with linearly independent columns. Then* $\mathbf{A}$ *can be written as*

$$\mathbf{A} = \mathbf{QR}$$

for some $m \times n$ *matrix* $\mathbf{Q}$ *such that* $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_n$ *and invertible upper triangular matrix* $\mathbf{R}$ *with positive diagonal entries.*

This is called a QR factorization of $\mathbf{A}$.

Algorithm to QR Factorization

1. Perform Gram-Schmidt on the columns of $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ to obtain an orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n\}$.

2. Set $\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{pmatrix}$.

3. $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$. Change $\mathbf{q}_j$ to $-\mathbf{q}_j$ if necessary to ensure that $\mathbf{R}$ has positive diagonal entries.

Least Square Approximation

Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. A vector $\mathbf{u} \in \mathbb{R}^n$ is a least square solution to $\mathbf{Ax} = \mathbf{b}$ if for every vector $\mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{Au} - \mathbf{b}\| \leq \|\mathbf{Av} - \mathbf{b}\|.$$

Theorem

Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. $\mathbf{u} \in \mathbb{R}^n$ is a least square solution to $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{Au}$ is the projection of $\mathbf{b}$ onto the column space of $\mathrm{Col}(\mathbf{A})$.

Theorem

Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. A vector $\mathbf{u} \in \mathbb{R}^n$ is a least square solution to $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{u}$ is a solution to $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.

Theorem (Finding projection using Least Square Approximation)

Let $V \subseteq \mathbb{R}^n$ be a subspace and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for V. Then the orthogonal projection of a vector $\mathbf{w} \in \mathbb{R}^n$ onto V is $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{w}$, where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$.

Tutorial 8 Solutions

Question 1(a)

Apply Gram-Schmidt Process to convert $\left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\2\\0\\1\end{pmatrix} \right\}$ into an orthonormal basis for $\mathbb{R}^4$.

```
>> u1=[1;1;1;1];u2=[1;-1;1;0];u3=[1;1;-1;-1];u4=[1;2;0;1];
v1=u1, v2=u2-(u2'*v1)/(v1'*v1)*v1, v3=u3-(u3'*v1)/(v1'*v1)*v1-(u3'*v2)/(v2'*v2)*v2,
v4=u4-(u4'*v1)/(v1'*v1)*v1-(u4'*v2)/(v2'*v2)*v2-(u4'*v3)/(v3'*v3)*v3
>> v2=[3;-5;3;-1];v3=[7;3;-4;-6];v4=[1;-1;-2;2]; V=[v1 v2 v3 v4]; V'*V
```

Orthonormal basis

$$\left\{ \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \frac{1}{2\sqrt{11}}\begin{pmatrix}3\\-5\\3\\-1\end{pmatrix}, \frac{1}{\sqrt{110}}\begin{pmatrix}7\\3\\-4\\-6\end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix}1\\-1\\-2\\2\end{pmatrix} \right\}.$$

Question 1(b)

Apply Gram-Schmidt Process to convert $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ into an orthonormal set. Is the set obtained an orthonormal basis? Why?

```
>> u1=[1;2;2;1];u2=[1;2;1;0];u3=[1;0;1;0];u4=[1;0;2;1];
v1=u1, v2=u2-(u2'*v1)/(v1'*v1)*v1, v3=u3-(u3'*v1)/(v1'*v1)*v1-(u3'*v2)/(v2'*v2)*v2,
v4=u4-(u4'*v1)/(v1'*v1)*v1-(u4'*v2)/(v2'*v2)*v2-(u4'*v3)/(v3'*v3)*v3
>> v2=[3;6;-4;-7];v3=[4;-3;2;-2]; V=[v1 v2 v3]; V'*V
```

The orthonormal set obtained is

$$\left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{110}} \begin{pmatrix} 3 \\ 6 \\ -4 \\ -7 \end{pmatrix}, \frac{1}{\sqrt{33}} \begin{pmatrix} 4 \\ -3 \\ 2 \\ -2 \end{pmatrix} \right\}.$$

It is not a basis since it only contains 3 vectors. The vector $\mathbf{v}_4 = 0$ means that $\mathbf{u}_4$ minus its projection onto $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the zero vector. Hence $\mathbf{u}_4$ is contained in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Question 2(a)

Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ -1 \\ 1 \end{pmatrix}$. Is the linear system $\mathbf{Ax} = \mathbf{b}$ consistent?

```
>> A=[0 1 1 0;1 -1 1 -1;1 0 1 0;1 1 1 1];b=[6;3;-1;1]; rref([A b])
```

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 6 \\ 1 & -1 & 1 & -1 & 3 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

So the linear system $\mathbf{Ax} = \mathbf{b}$ is inconsistent.

Question 2(b)

Find a least squares solution to the system. Is the solution unique?

```
>> rref([A'*A A'*b])
```

$$\left(\ \mathbf{A}^T\mathbf{A} \mid \mathbf{A}^T\mathbf{b}\ \right) = \left(\begin{array}{cccc|c} 3 & 0 & 3 & 0 & 3 \\ 0 & 3 & 1 & 2 & 4 \\ 3 & 1 & 4 & 0 & 9 \\ 0 & 2 & 0 & 2 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -6 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

General solution $x_1 = -6 - s,\ x_2 = -1 - s,\ x_3 = 7 + s,\ x_4 = s,\ s \in \mathbb{R}.$. Choose $s = 0$, have $\mathbf{v} = \begin{pmatrix} -6 \\ -1 \\ 7 \\ 0 \end{pmatrix}$.

The lest square solutions are not unique.

The least square solutions are not unique follows from the fact that since $\mathbf{A}$ is not invertible (shown in (a)), $\mathbf{A}^T$ is not invertible too, and hence $\mathbf{A}^T\mathbf{A}$ is not invertible.

Question 2(c)

Use your answer in (b), compute the projection of **b** onto the column space of **A**. Is the solution unique?

Take any least square solution **v** found in (b), the projection is **Av**. The projection is unique; for any choice of s,

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -6-s \\ -1-s \\ 7+s \\ s \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Question 3

A line

$$p(x) = a_1 x + a_0$$

is said to be the *least squares approximating line* for a given a set of data points (x_1, y_1), (x_2, y_2), ..., (x_m, y_m) if the sum

$$S = [y_1 - p(x_1)]^2 + [y_2 - p(x_2)]^2 + \cdots + [y_m - p(x_m)]^2$$

is minimized. Writing

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \ \text{and } p(\mathbf{x}) = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_m) \end{pmatrix} = \begin{pmatrix} a_1 x_1 + a_0 \\ a_1 x_2 + a_0 \\ \vdots \\ a_1 x_m + a_0 \end{pmatrix}$$

the problem is now rephrased as finding a_0, a_1 such that

$$S = ||\mathbf{y} - p(\mathbf{x})||^2$$

is minimized.

Question 3

Observe that if we let

$$\mathbf{N} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix},$$

then $\mathbf{Na} = p(\mathbf{x})$. And so our aim is to find $\mathbf{a}$ that minimizes $||\mathbf{y} - \mathbf{Na}||^2$.

Question 3(a)

It is known the equation representing the dependency of the resistance of a cylindrically shaped conductor (a wire) at $20^o C$ is given by

$$R = \rho \frac{L}{A},$$

where R is the resistance measured in Ohms Ω, L is the length of the material in meters m, A is the cross-sectional area of the material in meter squared m^2, and ρ is the resistivity of the material in Ohm meters Ωm. A student wants to measure the resistivity of a certain material. Keeping the cross-sectional area constant at $0.002 m^2$, he connected the power sources along the material at varies length and measured the resistance and obtained the following data.

L	0.01	0.012	0.015	0.02
R	2.75×10^{-4}	3.31×10^{-4}	3.92×10^{-4}	4.95×10^{-4}

It is known that the Ohm meter might not be calibrated. Taking that into account, the student wants to find a linear graph $R = \frac{\rho}{0.002} L + R_0$ from the data obtained to compute the resistivity of the material. Relabeling, we let $R = y$, $\frac{\rho}{0.002} = a_1$ and $R_0 = a_0$. Is it possible to find a graph $y = a_1 x + a_0$ satisfying the points?

Question 3(a)

```
>> L=[0.01;0.012;0.015;0.02];y=[2.75;3.31;3.92;4.95];N=[[1;1;1;1] L]; rref([N y])
```

This linear system is inconsistent. Hence, no such graph exists.

Question 3(b)

Find the least square approximating line for the data points and hence find the resistivity of the material. Would this material make a good wire?

`>> rref([N'*N N'*y])` or `>> inv(N'*N)*N'*y` (why?)

So the least square approximating line is $y = 0.0216x + 0.0001$. So $\frac{\rho}{0.002} = 0.0216\Omega$, and hence $\rho = 4.32 \times 10^{-5}\Omega m$. It would not make a good wire, the resistivity of metals is in the $10^{-8}\Omega m$ range.

Question 4

Suppose the equation governing the relation between data pairs is not known. We may want to then find a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

of degree n, $n \leq m-1$, that best approximates the data pairs (x_1, y_1), (x_2, y_2), ..., (x_m, y_m). A *least square approximating polynomial* of degree n is such that

$$||\mathbf{y} - p(\mathbf{x})||^2$$

is minimized. If we write

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \ \mathbf{N} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^n \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix},$$

then $p(\mathbf{x}) = \mathbf{Na}$, and the task is to find $\mathbf{a}$ such that $||\mathbf{y} - \mathbf{Na}||^2$ is minimized. Observe that $\mathbf{N}$ is a matrix minor of the Vandermonde matrix. If at least $n+1$ of the x-values $x_1, x_2, ..., x_m$ are distinct, the columns of $\mathbf{N}$ are linearly independent, and thus $\mathbf{a}$ is uniquely determined by

$$\mathbf{a} = (\mathbf{N}^T\mathbf{N})^{-1}\mathbf{N}^T\mathbf{y}.$$

Question 4

We shall now find a quartic polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

that is a least square approximating polynomial for the following data points

x	4	4.5	5	5.5	6	6.5	7	8	8.5
y	0.8651	0.4828	2.590	-4.389	-7.858	3.103	7.456	0.0965	4.326

Enter the data points.

```
>> x=[4 4.5 5 5.5 6 6.5 7 8 8.5]';
>> y=[0.8651 0.4828 2.590 -4.389 -7.858 3.103 7.456 0.0965 4.326]';
```

Next, we will generate the 10×10 Vandermonde matrix.

```
>> N=fliplr(vander(x));
```

We only want the matrix minor up to the 4-th power, that is, up to the the 5-th column,

```
>> N=N(:,1:5);
```

Use this to find the least square approximating polynomial of degree 4.

```
>> a=inv(N'*N)*N'*y
```

$$-0.2720x^4 + 6.1528x^3 - 49.7013x^2 + 169.2099x - 204.0716.$$

Question 5(a)

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Find a QR factorization of $\mathbf{A}$.

```
>> a1=[1;1;1;0];a2=[1;1;1;1];a3=[0;0;1;1];
q1=a1;q2=a2-(a2'*q1)/(q1'*q1)*q1, q3=a3-(a3'*q1)/(q1'*q1)*q1-(a3'*q2)/(q2'*q2)*q2,
>> q3=q3*3
>> Q=[q1 q2 q3]; Q'*Q
```

So, let

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ 0 & 1 & 0 \end{pmatrix}. \text{ Then } \mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{pmatrix} \sqrt{3} & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}.$$

Question 5(b)

Use your answer in (a) to find the least square solution to $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Write $\mathbf{A} = \mathbf{QR}$. Then $\mathbf{A}^T\mathbf{A} = \mathbf{R}^T\mathbf{Q}^T\mathbf{QR} = \mathbf{R}^T\mathbf{R}$, and $\mathbf{A}^T\mathbf{b} = \mathbf{R}^T\mathbf{Q}^T\mathbf{b}$. Hence, solving for $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ is equivalent to solving for $\mathbf{R}^T\mathbf{Rx} = \mathbf{R}^T\mathbf{Q}^T\mathbf{b}$, which is equivalent to solving for $\mathbf{Rx} = \mathbf{Q}^T\mathbf{b}$, since $\mathbf{R}$ is invertible (and hence, so is $\mathbf{R}^T$).

$$\mathbf{Q}^T\mathbf{b} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{3} \\ 0 \\ -2/\sqrt{6} \end{pmatrix} \qquad \left(\begin{array}{ccc|c} \sqrt{3} & \sqrt{3} & 1/\sqrt{3} & 2/\sqrt{3} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \sqrt{2}/\sqrt{3} & -2/\sqrt{6} \end{array}\right)$$

$\Rightarrow \mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is the least square solution to $\mathbf{Ax} = \mathbf{b}$.