MA1522 Linear Algebra for Computing

Lecture 9: Orthogonality

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17 March, 2025

Outline

General Remarks

- Linear Algebra is good at solving systems of equations. Can they do anything else? like geometry?
- Sure they can!
- Using dot product, one can talk about angles, distances, etc., even beyond 3D space.
- Below we focus on Orthogonality.

Question in Section 5.1

(i) Can an orthogonal set contain the zero vector $\mathbf{0}$?

(ii) Can an orthonormal set contain the zero vector $\mathbf{0}$?

Slide 8: Orthogonal and Orthonormal Sets

Definition

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ of vectors is *orthogonal* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for every $i \neq j$, that is, vectors in S are pairwise orthogonal.

The set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is *orthonormal* if for all $i, j = 1, ..., k$,

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

That is, S is orthogonal, and all the vectors are unit vectors.

Answer to Question in Section 5.1

(i) Can an orthogonal set contain the zero vector $\mathbf{0}$?

(ii) Can an orthonormal set contain the zero vector $\mathbf{0}$?

Answer: (i) Yes, for example, $\{\mathbf{0}, \mathbf{v}\}$ for any nonzero vector $\mathbf{v}$ is an orthogonal set.

(ii) No, because the length of the zero vector is 0, not 1.

Question One in Section 5.2

Note that this only works if $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is an orthogonal or orthonormal basis. Example, $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Then

$$\left(\frac{\mathbf{w} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} \right) \mathbf{u}_1 + \left(\frac{\mathbf{w} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2} \right) \mathbf{u}_2 = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} \neq \mathbf{w}.$$

Why is this so?

Slide 28: Coordinates Relative to an Orthogonal Basis

Theorem

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be an orthogonal basis for a subspace V of $\mathbb{R}^n$. Then for any $\mathbf{v} \in V$,

$$\mathbf{v} = \left(\frac{\mathbf{v}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 + \left(\frac{\mathbf{v}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right)\mathbf{u}_2 + \cdots + \left(\frac{\mathbf{v}\cdot\mathbf{u}_k}{\|\mathbf{u}_k\|^2}\right)\mathbf{u}_k$$

If further S is an orthonormal basis, then

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{u}_2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_k)\,\mathbf{u}_k.$$

that is S orthogonal, $[\mathbf{v}]_S = \begin{pmatrix} \frac{\mathbf{v}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2} \\ \frac{\mathbf{v}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2} \\ \vdots \\ \frac{\mathbf{v}\cdot\mathbf{u}_k}{\|\mathbf{u}_k\|^2} \end{pmatrix}$, *$S$ orthonormal,* $[\mathbf{v}]_S = \begin{pmatrix} \mathbf{v}\cdot\mathbf{u}_1 \\ \mathbf{v}\cdot\mathbf{u}_2 \\ \vdots \\ \mathbf{v}\cdot\mathbf{u}_k \end{pmatrix}$.

Answer to Question One in Section 5.2

Note that this only works if $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is an orthogonal or orthonormal basis. Example, $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Then

$$\left(\frac{\mathbf{w} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right) \mathbf{u}_1 + \left(\frac{\mathbf{w} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right) \mathbf{u}_2 = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} \neq \mathbf{w}.$$

Why is this so?

Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1$, they are not orthogonal. The Theorem on Slide 28 does not apply.

Question Two in Section 5.2

Let $\mathbf{A} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & \sqrt{2}/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & -\sqrt{2}/2 \end{pmatrix}$. Then

$$\begin{aligned} \mathbf{A}^T\mathbf{A} &= \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & \sqrt{2}/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & -\sqrt{2}/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Is $\mathbf{A}$ an orthogonal matrix?

Slide 35: Orthogonal Matrices

Definition

An $n \times n$ square matrix $\mathbf{A}$ is *orthogonal* if $\mathbf{A}^T = \mathbf{A}^{-1}$, equivalently, $\mathbf{A}^T\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^T$.

Theorem

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) *$\mathbf{A}$ is an orthogonal matrix.*

(ii) *The columns of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$.*

(iii) *The rows of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$.*

Answer to Question two in section 5.2: No, it is not an orthogonal matrix, because it is not a square matrix.

Challenge in Section 5.1

Let $\mathbf{A}$ be an $m \times n$ matrix. Show that the nullspace of $\mathbf{A}$ is the orthogonal complement of the row space of $\mathbf{A}$,

$$\text{Row}(\mathbf{A})^{\perp} = \text{Null}(\mathbf{A}).$$

Recall on Slides 18 and 14:

Definition

Let V be a subspace of $\mathbb{R}^n$. The *orthogonal complement* of V is the set of all vectors that are orthogonal to V, and is denoted as

$$V^{\perp} = \left\{ \ \mathbf{w} \in \mathbb{R}^n \ \middle| \ \mathbf{w} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \text{ in } V \ \right\}.$$

Theorem

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a spanning set for V, span$(S) = V$. Then a vector $\mathbf{w}$ is orthogonal to V if and only if $\mathbf{w} \cdot \mathbf{u}_i = 0$ for all $i = 1, ..., k$.

Slide 16: Algorithm to check for Orthogonal to a Subspace

Theorem

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a spanning set for V. Then $\mathbf{w}$ is orthogonal to V if and only if $\mathbf{w}$ is in the nullspace of $\mathbf{A}^T$, where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$;

$$\mathbf{w} \perp V \quad \Leftrightarrow \quad \mathbf{w} \in \mathrm{Null}(\mathbf{A}^T)$$

[Sketch of Proof] By previous theorem, $\mathbf{w} \perp V$ if and only if $\mathbf{u}_i^T\mathbf{w} = \mathbf{u}_i \cdot \mathbf{w} = 0$ for all $i = 1, 2, ..., k$. By block multiplication, this is equivalent to

$$\mathbf{A}^T\mathbf{w} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}^T \mathbf{w} = \begin{pmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_k^T \end{pmatrix} \mathbf{w} = \begin{pmatrix} \mathbf{u}_1^T\mathbf{w} \\ \mathbf{u}_2^T\mathbf{w} \\ \vdots \\ \mathbf{u}_k^T\mathbf{w} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Answer to Challenge in Section 5.1

Q: Let $\mathbf{A}$ be an $m \times n$ matrix. Show that the nullspace of $\mathbf{A}$ is the orthogonal complement of the row space of $\mathbf{A}$,

$$\mathrm{Row}(\mathbf{A})^{\perp} = \mathrm{Null}(\mathbf{A}).$$

Answer: By the theorem on Slide 16, we have

$$\mathrm{Null}(\mathbf{A}^T) = [\mathrm{Col}(\mathbf{A})]^{\perp}.$$

Thus,

$$\mathrm{Null}(\mathbf{A}) = \mathrm{Null}((\mathbf{A}^T)^T) = [\mathrm{Col}(\mathbf{A}^T)]^{\perp} = \mathrm{Row}(\mathbf{A})^{\perp}.$$

Challenge in Section 5.2

Let V be a subspace of $\mathbb{R}^n$ and S an orthonormal basis of V. Show that for any $\mathbf{u}, \mathbf{v} \in V$,

1. $\mathbf{u} \cdot \mathbf{v} = [\mathbf{u}]_S \cdot [\mathbf{v}]_S$.
2. $\|\mathbf{u} - \mathbf{v}\| = \|[\mathbf{u}]_S - [\mathbf{v}]_S\|$.

Answer: Let $S = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$, $\mathbf{u} = a_1\mathbf{w}_1 + \cdots + a_k\mathbf{w}_k$, and $\mathbf{v} = b_1\mathbf{w}_1 + \cdots + b_k\mathbf{w}_k$. Then we have

$$\mathbf{w}_i \cdot \mathbf{w}_i = 1 \text{ and } \mathbf{w}_i \cdot \mathbf{w}_j = 0 \text{ if } i \neq j,$$

and

$$[\mathbf{u}]_S = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}, \quad [\mathbf{v}]_S = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix}.$$

Answer to Challenge in Section 5.2 (conti.)

(Continue from previous Slide) Then we have

$$
\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k) \cdot (b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \cdots + b_k\mathbf{w}_k) \\
&= a_1 b_1(\mathbf{w}_1 \cdot \mathbf{w}_1) + a_1 b_2(\mathbf{w}_1 \cdot \mathbf{w}_2) + \cdots + a_1 b_k(\mathbf{w}_1 \cdot \mathbf{w}_k) \\
&+ \cdots\cdots\cdots \\
&+ a_k b_1(\mathbf{w}_k \cdot \mathbf{w}_1) + a_k b_2(\mathbf{w}_k \cdot \mathbf{w}_2) + \cdots + a_k b_k(\mathbf{w}_k \cdot \mathbf{w}_k) \\
&= a_1 b_1 + \cdots + a_k b_k \\
&= [\mathbf{u}]_S \cdot [\mathbf{v}]_S,
\end{aligned}
$$

which establishes the first statement.

Answer to Challenge in Section 5.2 (part 2)

Next, we show that $\|\mathbf{u}-\mathbf{v}\|^2 = \|[\mathbf{u}]_S - [\mathbf{v}]_S\|^2$. Since both sides are nonnegative, we can take square root and obtain the result. By part 1, we have

$$\begin{aligned}
\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) \\
&= (\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\mathbf{v}) - (\mathbf{v}\cdot\mathbf{u}) + (\mathbf{v}\cdot\mathbf{v}) \\
&= ([\mathbf{u}]_S \cdot [\mathbf{u}]_S) - ([\mathbf{u}]_S \cdot [\mathbf{v}]_S) - ([\mathbf{v}]_S \cdot [\mathbf{u}]_S) + ([\mathbf{v}]_S \cdot [\mathbf{v}]_S) \\
&= ([\mathbf{u}]_S - [\mathbf{v}]_S) \cdot ([\mathbf{u}]_S - [\mathbf{v}]_S) \\
&= \|[\mathbf{u}]_S - [\mathbf{v}]_S\|^2,
\end{aligned}$$

which establishes the second statement.

Challenge in Section 5.3

Theorem (Orthogonal projection theorem)

Let V be a subspace of $\mathbb{R}^n$. Every vector $\mathbf{w}$ in $\mathbb{R}^n$ can be decomposed uniquely *as a sum*

$$\mathbf{w} = \mathbf{w}_p + \mathbf{w}_n,$$

where $\mathbf{w}_n$ is orthogonal to V and $\mathbf{w}_p$ is a vector in V, that is, $\mathbf{w}_n \perp V$, $\mathbf{w}_p \in V$.

Challenge: Prove it!

Slide 44: Orthogonal Projection

Theorem (Orthogonal projection theorem)

Let V be a subspace of $\mathbb{R}^n$. Every vector $\mathbf{w}$ in $\mathbb{R}^n$ can be decomposed uniquely as

$$\mathbf{w} = \mathbf{w}_p + \mathbf{w}_n,$$

where $\mathbf{w}_n \perp V$, $\mathbf{w}_p \in V$. Moreover, if $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is an orthogonal basis for V, then

$$\mathbf{w}_p = \frac{\mathbf{w} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{w} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{w} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k}\mathbf{u}_k.$$

We call the vector $\mathbf{w}_p$ as the *orthogonal projection* (or just *projection*) of $\mathbf{w}$ onto the subspace V.

Answer to Challenge in Section 5.3

Prove the orthogonal projection theorem: Let V be a subspace of $\mathbb{R}^n$. Every vector $\mathbf{w}$ in $\mathbb{R}^n$ can be decomposed uniquely as a sum

$$\mathbf{w} = \mathbf{w}_p + \mathbf{w}_n,$$

where $\mathbf{w}_n \perp V$, $\mathbf{w}_p \in V$.

Proof: Let's follow the "Moreover" part to find $\mathbf{w}_p$: Fix an orthonormal basis $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ for V. Let

$$\mathbf{w}_p = (\mathbf{w} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{w} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{w} \cdot \mathbf{u}_k)\mathbf{u}_k,$$

which is in V. Next let $\mathbf{w}_n = \mathbf{w} - \mathbf{w}_p$. Since

$$\begin{aligned} \mathbf{w}_n \cdot \mathbf{u}_i &= (\mathbf{w} - \mathbf{w}_p) \cdot \mathbf{u}_i \\ &= \mathbf{w} \cdot \mathbf{u}_i - ((\mathbf{w} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{w} \cdot \mathbf{u}_k)\mathbf{u}_k) \cdot \mathbf{u}_i \\ &= \mathbf{w} \cdot \mathbf{u}_i - \mathbf{w} \cdot \mathbf{u}_i \\ &= 0 \end{aligned}$$

which says $\mathbf{w}_n \perp V$. Clearly $\mathbf{w} = \mathbf{w}_n + \mathbf{w}_p$.

Answer to Challenge in Section 5.3 (Uniqueness Part)

Suppose that

$$\mathbf{w} = \mathbf{w}_p + \mathbf{w}_n = \mathbf{z}_p + \mathbf{z}_n$$

where $\mathbf{w}_n, \mathbf{z}_n \in V^\perp$ and $\mathbf{w}_p, \mathbf{z}_p \in V$. Then

$$\mathbf{w}_p - \mathbf{z}_p = \mathbf{z}_n - \mathbf{w}_n \in V^\perp \cap V.$$

But any vector $\mathbf{x} \in V^\perp \cap V$ must be zero, because $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = 0$. Thus

$$\mathbf{w}_p - \mathbf{z}_p = \mathbf{z}_n - \mathbf{w}_n = \mathbf{0}.$$

Therefore, $\mathbf{w}_p = \mathbf{z}_p$ and $\mathbf{w}_n = \mathbf{z}_n$. We got the uniqueness.

Nonzero Orthogonal Vectors are Linearly Independent

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Proof: Suppose that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}.$$

If follows that

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n) \cdot \mathbf{u}_i = \mathbf{0} \cdot \mathbf{u}_i = 0.$$

Then

$$c_i\|\mathbf{u}_i\|^2 = 0.$$

Since $\mathbf{u}_i \neq \mathbf{0}$, we have $c_i = 0$ for all $i \leq n$. Hence they are linearly independent.

Exercise about Orthogonal Matrices

Suppose $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices. Which of the following are also orthogonal matrices?

(a) $\mathbf{AB}$

(b) $\mathbf{AB}^T$

(c) $\mathbf{A}^T\mathbf{B}$

(d) $\mathbf{A}+\mathbf{B}$

Answer: All of them except (d). $\mathbf{A}$ is orthogonal if and only if $\mathbf{A}^T$ is. Also, product of orthogonal matrices is an orthogonal matrix, because

$$(\mathbf{AB})^T\mathbf{AB} = \mathbf{B}^T\mathbf{A}^T\mathbf{AB} = \mathbf{B}^T\mathbf{B} = \mathbf{I}.$$

For (d), consider $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not an orthogonal matrix.

Slide 51: Gram-Schmidt Process

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a linearly independent set. Let

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}_2}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}_3}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_3}{\|\mathbf{v}_2\|^2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}_k}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_k}{\|\mathbf{v}_2\|^2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{u}_k}{\|\mathbf{v}_{k-1}\|^2}\right)\mathbf{v}_{k-1}.
\end{aligned}
$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is an orthogonal set (of nonzero vectors), and hence,

$$\left\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, ..., \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|}\right\}$$

is an orthonormal set such that $\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_k\} = \text{span}\{\mathbf{u}_k, ..., \mathbf{u}_k\}$.

An Example of Gram-Schmidt

Find an orthonormal basis for the column space of $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3).$$

We follow the algorithm.

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \|\mathbf{v}_1\| = 2.$$

Normalize,

$$\mathbf{w}_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}^T.$$

Calculating $\mathbf{v}_2$

$$\frac{\mathbf{v}_1 \cdot \mathbf{u}_2}{\|\mathbf{v}_1\|^2} = \frac{6}{4} = \frac{3}{2}.$$

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}_2}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 \\
&= \begin{pmatrix} -1 & 4 & 4 & 1 \end{pmatrix}^T - \frac{3}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T \\
&= \begin{pmatrix} -\frac{5}{2} & \frac{5}{2} & \frac{5}{2} & -\frac{5}{2} \end{pmatrix}^T, \quad \|\mathbf{v}_2\| = 5.
\end{aligned}$$

Normalize,

$$\mathbf{w}_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}^T.$$

Calculating $\mathbf{v}_3$

Next,

$$\frac{\mathbf{v}_1 \cdot \mathbf{u}_3}{\|\mathbf{v}_1\|^2} = 1 \quad \frac{\mathbf{v}_2 \cdot \mathbf{u}_3}{\|\mathbf{v}_2\|^2} = \frac{-10}{25} = -\frac{2}{5}.$$

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}_3}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_3}{\|\mathbf{v}_2\|^2}\right)\mathbf{v}_2 \\
&= \begin{pmatrix} 4 & -2 & 2 & 0 \end{pmatrix}^T - 1\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} - (-\frac{2}{5})\begin{pmatrix} -\frac{5}{2} & \frac{5}{2} & \frac{5}{2} & -\frac{5}{2} \end{pmatrix}^T \\
&= \begin{pmatrix} 2 & -2 & 2 & -2 \end{pmatrix}^T, \quad \|\mathbf{v}_3\| = 4
\end{aligned}$$

Normalize,

$$\mathbf{w}_3 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}^T.$$

In short, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis, whereas $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal one.