MA1522: Linear Algebra for Computing

Tutorial 3

Revision

Properties of Inverse

Theorem (Cancellation law for matrices)

Let $\mathbf{A}$ *be an invertible matrix of order* n*.*

(i) *(Left cancellation) If* $\mathbf{B}$ *and* $\mathbf{C}$ *are* $n \times m$ *matrices with* $\mathbf{AB} = \mathbf{AC}$*, then* $\mathbf{B} = \mathbf{C}$*.*

(ii) *(Right cancellation) If* $\mathbf{B}$ *and* $\mathbf{C}$ *are* $m \times a$ *matrices with* $\mathbf{BA} = \mathbf{CA}$*, then* $\mathbf{B} = \mathbf{C}$*.*

Theorem (Properties of Inverse)

Let $\mathbf{A}$ *be an invertible matrix of order* n*.*

(i) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

(ii) *For any nonzero real number* $a \in \mathbb{R}$, $(a\mathbf{A})$ *is invertible with inverse* $(a\mathbf{A})^{-1} = \frac{1}{a}\mathbf{A}^{-1}$.

(iii) $\mathbf{A}^T$ *is invertible with inverse* $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

(iv) *If* $\mathbf{B}$ *is an invertible matrix of order* n*, then* $(\mathbf{AB})$ *is invertible with inverse* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

(v) *In general,* $(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1} \cdots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$*, if* $\mathbf{A}_i$ *is an invertible matrix for* $i = 1, \ldots, k$*.*

Inverse of Elementary Matrices

Every elementary matrices $\mathbf{E}$ are invertible. The inverse $\mathbf{E}^{-1}$ of an elementary matrix corresponding to the reverse of the corresponding row operation.

(i)

$$\mathbf{I}_n \xrightarrow{R_i+cR_j} \mathbf{E} \xrightarrow{R_i-cR_j} \mathbf{I}_n \quad \Rightarrow \quad \mathbf{E} : R_i + cR_j,\ \mathbf{E}^{-1} : R_i - cR_j.$$

(ii)

$$\mathbf{I}_n \xrightarrow{R_i \leftrightarrow R_j} \mathbf{E} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{I}_n \quad \Rightarrow \quad \mathbf{E} : R_i \leftrightarrow R_j,\ \mathbf{E}^{-1} : R_i \leftrightarrow R_j.$$

(iii)

$$\mathbf{I}_n \xrightarrow{cR_i} \mathbf{E} \xrightarrow{\frac{1}{c}R_i} \mathbf{I}_n \quad \Rightarrow \quad \mathbf{E} : cR_i,\ \mathbf{E}^{-1} : \frac{1}{c}R_i.$$

LU Decomposition

Suppose $\mathbf{A} \xrightarrow{r_1, r_2, \ldots, r_k} \mathbf{U}$, where each row operation r_l is of the form $R_i + cR_j$ for some $i > j$ and $\mathbf{U}$ is an row-echelon form of $\mathbf{A}$. Then

$$\mathbf{A} = \mathbf{L}\mathbf{U} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{pmatrix} \begin{pmatrix} * & & & & & \cdots & * \\ 0 & \cdots & 0 & * & & \cdots & * \\ \vdots & & & & & & \vdots \\ 0 & \cdots & & & & \cdots & * \end{pmatrix},$$

where

$$\mathbf{L} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1} \cdots \mathbf{E}_k^{-1},$$

and $\mathbf{E}_l$ is the elementary matrix corresponding to r_l.

To solve $\mathbf{LUx} = \mathbf{Ax} = \mathbf{b}$, solve $\mathbf{Ly} = \mathbf{b}$, and $\mathbf{Ux} = \mathbf{y}$.

Definition of Determinant

1. For $n = 1$, $\mathbf{A} = (a)$, $\det(\mathbf{A}) = a$.
2. For $n = 2$, $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(\mathbf{A}) = ad - bc$.
3. For $n = 3$, $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $\det(\mathbf{A}) = aei - afh - bdi + bfg + cdh - ceg$.
4. In general,

$$\det(\mathbf{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{k=1}^{n} a_{ik}A_{ik} \tag{1}$$

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{k=1}^{n} a_{kj}A_{kj} \tag{2}$$

This is called the cofactor expansion along $\left\{ \begin{array}{ccc} \text{row} & i & (1) \\ \text{column} & j & (2) \end{array} \right.$. Here $A_{ij} = (-1)^{i+j} det(\mathbf{M}_{ij})$ is called the (i,j)-cofactor, where $\mathbf{M}_{ij}$ is the (i,j)-matrix minor, the matrix obtained from $\mathbf{A}$ by deleting the i-th row and j-th column.

Properties of Determinant

1. $\det(\mathbf{A}^T) = \det(\mathbf{A})$.

2. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ for any square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order. More generally (by induction), $\det(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k) = \det(\mathbf{A}_1)\det(\mathbf{A}_2)\cdots\det(\mathbf{A}_k)$.

3. $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$ if $\mathbf{A}$ is invertible.

4. $\det(c\mathbf{A}) = c^n \det \mathbf{A}$, for any $c \in \mathbb{R}$ and square matrix $\mathbf{A}$ of order n.

5. $\det(diag(d_1, d_2, ..., d_n)) = d_1 d_2 \cdots d_n$.

6.

$\mathbf{A} \xrightarrow{R_i + aR_j} \mathbf{B}$	$\det(\mathbf{B}) = \det(\mathbf{A})$
$\mathbf{A} \xrightarrow{cR_i} \mathbf{B}$	$\det(\mathbf{B}) = c\det(\mathbf{A})$
$\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B}$	$\det(\mathbf{B}) = -\det(\mathbf{A})$

Adjoint

Let **A** be an order n square matrix. Define the adjoint of **A** to be

$$adj(\mathbf{A}) = (A_{ij})^T = \begin{pmatrix} \det(\mathbf{M}_{11}) & -det(\mathbf{M}_{21}) & \cdots & (-1)^{n+1}\det(\mathbf{M}_{n1}) \\ -\det(\mathbf{M}_{12}) & det(\mathbf{M}_{22}) & \cdots & (-1)^{n+2}\det(\mathbf{M}_{n2}) \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{1+n}\det(\mathbf{M}_{1n}) & (-1)^{2+n}det(\mathbf{M}_{2n}) & \cdots & \det(\mathbf{M}_{nn}) \end{pmatrix}.$$

Theorem (Adjoint formula)

$$\mathbf{A}adj(\mathbf{A}) = \det(\mathbf{A})\mathbf{I}.$$

Tutorial 3 Solutions

Question 1

Let $\mathbf{A}$ be the 4×4 matrix obtained from $\mathbf{I}$ by the following sequence of elementary row operations:

$$\mathbf{I} \xrightarrow{\frac{1}{2}R_2} \xrightarrow{R_1 - R_2} \xrightarrow{R_2 \leftrightarrow R_4} \xrightarrow{R_3 + 3R_1} \mathbf{A}.$$

Write $\mathbf{A}^{-1}$ as a product of four elementary matrices.

$$\mathbf{A} = \mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{I} = \mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1.$$

So,

$$\begin{aligned}
\mathbf{A}^{-1} &= (\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{E}_4^{-1} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Question 2(a)

Find an LU factorization for the matrix $\mathbf{A}$, and solve the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$.

$$\mathbf{A} \xrightarrow{R_2+3R_1,\ R_3-4R_1\ R_3+R_2} \mathbf{U} = \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{pmatrix},\ \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}.$$

Solve $\mathbf{Ly} = \mathbf{b}$. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right) \Rightarrow \mathbf{y} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$. Solve for $\mathbf{Ux} = \mathbf{y}$. $\left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & -3 & 4 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right) \Rightarrow \mathbf{x} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$.

Question 2(b)

Find an LU factorization for the matrix $\mathbf{A}$, and solve the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 17 \end{pmatrix}$.

$$\mathbf{A} \xrightarrow{R_2 - 3R_1,\ R_3 + \frac{1}{2}R_1,\ R_3 + 2R_2} \mathbf{U} = \begin{pmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix},\ \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{pmatrix}.$$

Solve $\mathbf{Ly} = \mathbf{b}$. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1/2 & -2 & 1 & 17 \end{array}\right) \Rightarrow \mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 17 \end{pmatrix}$. Solve for $\mathbf{Ux} = \mathbf{y}$. $\left(\begin{array}{cccc|c} 2 & -4 & 4 & -2 & 0 \\ 0 & 3 & -5 & 3 & 0 \\ 0 & 0 & 0 & 5 & 17 \end{array}\right) \Rightarrow$

$$= \begin{pmatrix} -17/5 + 4s/3 \\ -17/5 + 5s/3 \\ s \\ 17/5 \end{pmatrix}, \quad s \in \mathbb{R}..$$

Question 3(a)

Find an LU factorization of $\mathbf{A} = \begin{pmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{pmatrix}$.

```
>> A=[2 -6 6;-4 5 -7;3 5 -1;-6 4 -8;8 -3 9];
>> A(2,:)=A(2,:)+2*A(1,:);A(3,:)=A(3,:)-(3/2)*A(1,:);A(4,:)=A(4,:)+3*A(1,:);
A(5,:)=A(5,:)-4*A(1,:)
>> A(3,:)=A(3,:)+2*A(2,:);A(4,:)=A(4,:)-2*A(2,:);A(5,:)=A(5,:)+3*A(2,:)
```

$$\mathbf{U} = \begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Question 3(a)

To compute **L**.

```
A=eye(5);
>> A(2,:)=A(2,:)+2*A(1,:);A(3,:)=A(3,:)-(3/2)*A(1,:);A(4,:)=A(4,:)+3*A(1,:);
A(5,:)=A(5,:)-4*A(1,:);
A(3,:)=A(3,:)+2*A(2,:);A(4,:)=A(4,:)-2*A(2,:);A(5,:)=A(5,:)+3*A(2,:)
>> L=inv(A)
```

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{pmatrix}.$$

Question 3(b)

```
>> A=[2 -6 6;-4 5 -7;3 5 -1;-6 4 -8;8 -3 9];
>> [L U]=lu(sym(A))
L =

[  1,  0, 0, 0, 0]
[ -2,  1, 0, 0, 0]
[3/2, -2, 1, 0, 0]
[ -3,  2, 0, 1, 0]
[  4, -3, 0, 0, 1]

U =

[2, -6, 6]
[0, -7, 5]
[0,  0, 0]
[0,  0, 0]
[0,  0, 0]
```

Question 4

Let $\mathbf{A} = \begin{pmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{pmatrix}$. Compute the determinant of $\mathbf{A}$ and find all the values of x such that $\mathbf{A}$ is singular.

```
>> syms x
>> A=[-x 1 0;0 -x 1;2 -5 4-x];
>> det(A)
>> simplify(ans)
```

The matrix $\mathbf{A}$ is singular if and only if $\det \mathbf{A} = 0$ which is $x = 1$ or $x = 2$.

Question 5

Show that $\begin{vmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix} = (1+x^3)\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$.

```
>> syms a b c p q r u v w x;
>> A=[a+p*x b+q*x c+r*x;p+u*x q+v*x r+w*x;u+a*x v+b*x w+c*x]
>> A(2,:)=A(2,:)-x*A(3,:);A=simplify(A)
>> A(1,:)=A(1,:)-x*A(2,:); A=simplify(A)
```

$$\begin{pmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{pmatrix} \xrightarrow{R_2-xR_3} \begin{pmatrix} a+px & b+qx & c+rx \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{pmatrix} \xrightarrow{R_1-xR_2} \begin{pmatrix} a(1+x^3) & b(1+x^3) & c(1+x^3) \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{pmatrix}.$$

Assume $x \neq -1$,

```
>> A(1,:)=A(1,:)/(1+x^3);A=simplify(A)
>> A(2,:)=A(2,:)+x^2*A(1,:);A=simplify(A)
>> A(3,:)=A(3,:)-x*A(1,:)
```

Question 5

Challenge: Write $\begin{pmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{pmatrix}$ as a product of 2 order 3 matrices such that one of them is $\begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix}$.

Question 6

Let $\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. Compute

(a) $\det(3\mathbf{A}^T)$;

(b) $\det(3\mathbf{A}\mathbf{B}^{-1})$; and

(c) $\det((3\mathbf{B})^{-1})$.

$\det(\mathbf{A}) = -2$ and $\det(\mathbf{B}) = 3$.

(a) $\det(3\mathbf{A}^T) = 3^4 \det(\mathbf{A}^T) = 3^4 \det(\mathbf{A}) = -162$

(b) $\det(3\mathbf{A}\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}) \det(\mathbf{B}^{-1}) = 3^4 \det(\mathbf{A}) \dfrac{1}{\det(\mathbf{B})} = -54$

(c) $\det((3\mathbf{B})^{-1}) = \dfrac{1}{\det(3\mathbf{B})} = \dfrac{1}{3^4 \det(\mathbf{B})} = \dfrac{1}{3^5} = \dfrac{1}{243}$

Question 7

Use Cramer's rule to solve

$$\begin{cases} x + 5y + 3z = 1 \\ \quad\; 2y - 2z = 2 \\ \quad\;\; y + 3z = 0 \end{cases}$$

Cramer's rule: if $\mathbf{A}$ is invertible, unique solution to $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \dfrac{1}{\det(\boldsymbol{A})}\begin{pmatrix}\det(\boldsymbol{A}_1)\\ \det(\boldsymbol{A}_2)\\ \\ det(\boldsymbol{A}_3)\end{pmatrix}$, where $\boldsymbol{A}_i$ is the matrix constructed from $\boldsymbol{A}$ by replacing the i-th column with $\boldsymbol{b}$.

```
>> A=[1 5 3;0 2 -2;0 1 3]; b=[1;2;0]; A1=A;A1(:,1)=b;A2=A;A2(:,2)=b;A3=A;A3(:,3)=b;
>> A, A1, A2, A3
>> x=(1/det(A))*[det(A1);det(A2);det(A3)]
```

Question 8

Compute the adjoint of $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 6 \end{pmatrix}$, and use it to compute $\mathbf{A}^{-1}$.

$$adj(\mathbf{A}) = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 12 & 6 & -5 \\ 3 & 0 & -1 \\ -6 & -3 & 2 \end{pmatrix}.$$

Therefore

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A}) = \frac{-1}{3} \begin{pmatrix} 12 & 6 & -5 \\ 3 & 0 & -1 \\ -6 & -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & -2 & 5/3 \\ -1 & 0 & 1/3 \\ 2 & 1 & -2/3 \end{pmatrix}.$$

```
A=[1 -1 2;0 2 1;3 0 6]; adjoint(A)
```