MA1522 Linear Algebra for Computing

Lecture 1: Introduction

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Outline

Grading Policy

Three components:

- Online quizzes (20%): Test your understanding after you watch the video lecture. They have due dates and will be marked by Canvas.

- Assignments (30%) Three assignments 10% each. No late submissions.
  Assignment 1: 10—16 February 2025.
  Assignment 2: 10—16 March 2025.
  Assignment 3: 31 March—6 April 2025.

- Final 50%—9:00 am, 26 April (Saturday), 2025 (Closed book with one A4 helpsheet).

E-Hybrid

- Learning is done by watching online videos.
- Followed by doing online quizzes.
- Lectures on Mondays (repeated on Thursdays). More explanations, more examples and Q and A.
- Tutorials (from week 3 onwards). More interactions.

Contact

- Information can be found on Canvas.
- I can be reached by email: `matyangy@nus.edu.sg`.
- Challenges ahead (for both the lecturers and students).

Algebra vs. Geometry

...algebra is to the geometer what you might call the "Faustian Offer". As you know, Faust in Goethe's story was offered whatever he wanted by the devil in return for selling his soul. Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, and it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this powerful machine." Of course we like to have things both ways: we would probably cheat on the devil, pretend we are selling our soul, and not give it away...

Sir Michael Atiyah, Am Math. Monthly *[2001]*

Computer Graphics as an Example

- 3D objects in space, their rotation, translation, scaling and projection onto 2D screens, seem to have nothing to do with algebra.
- However, they are often represented and manipulated using matrices.
- Also in Computer Vision and Feature extractions.

Other Reasons

- Data in computer science, especially in areas like machine learning, graphics, and network analysis, is often represented using vectors and matrices.
- Machine learning models rely heavily on linear algebra for operations like matrix multiplication, eigenvalue decomposition, and singular value decomposition (SVD).
- In cryptography, linear algebra is used in encoding and decoding information.
- Graph Representations: Adjacency matrices and incidence matrices represent graphs.
- Google's PageRank algorithm is based on eigenvector computation.

Linear Systems

- Solving systems will be behind everything we do later.
- We will learn how to solve them.
- Also learn the structure of their solutions.

In Secondary School

- We learnt "substitution" and "Elimination" methods.
- For example, solve:

$$\begin{aligned} x + 2y &= 3 \qquad (1) \\ 4x + 5y &= 6 \qquad (2) \end{aligned}$$

- Using "Substitution": we change (1) into $x = 3 - 2y$ and substitute into (2):

$$4(3 - 2y) + 5y = 6. \qquad (3).$$

Thus, $-3y = -6$, so $y = 2$, and $x = 3 - 2(2) = -1$.

Using Elimination

- We perform "(2):= (2)$+(-4)\times$ (1)" and get:

$$-3y = -6.$$

- Observe that substitution and elimination have the same impact on equation (2).

- This leads to Gaussian Elimination.

Linear Systems in General Form

Definition

A *system of linear equations*, or a *linear system* consists of a finite number of linear equations. In general, a linear system with n variables and m equations in *standard form* is written as

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

The linear system is *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$, that is, all the linear equations are homogeneous.

Homogeneous equations play an important role when we discuss the "structure" of solutions.

Augmented Matrix

As the variables are like "place holders", we can use augmented matrix to express the system:

A linear system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

can be expressed uniquely as an augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

Elementary Row Operations

In the process of elimination, we need 3 types of operations, called elementary row operations.

1. Exchanging 2 rows, $R_i \leftrightarrow R_j$,
2. Adding a multiple of a row to another, $R_i + cR_j$, $c \in \mathbb{R}$,
3. Multiplying a row by a nonzero constant, aR_j, $a \neq 0$.

Remark

Performing elementary row operations to the augmented matrix of a linear system preserves the solutions.

Goal: Row-Echelon Form

Definition

In an *(augmented) matrix*, a ***zero row*** is a row with all entries 0. A row is called a ***nonzero row*** otherwise. The first nonzero entry from the left of a nonzero row is called a ***leading entry***.

An *(augmented) matrix* is in ***row-echelon form*** (REF) if

1. If zero rows exist, they are at the bottom of the matrix.
2. The leading entries are further to the right as we move down the rows.

An *augmented matrix* in REF has the form

$$\left(\begin{array}{cccccccc|c} * & & & & & & & & * \\ 0 & \cdots & 0 & * & & & & & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & * & & * \\ \vdots & & & & & & & \vdots & \vdots \\ 0 & \cdots & & & & & \cdots & 0 & 0 \end{array}\right).$$

Pivot Columns

Definition

In row-echelon form, a column containing a leading entry is called a *pivot column*. It is called a *non-pivot column* otherwise.

Reduced Row-Echelon Form

The *(augmented) matrix* is in reduced row-echelon form (RREF) if further

3. The leading entries are 1.
4. In each pivot column, all entries except the leading entry is 0.

An *augmented matrix* in RREF has the form

$$\left(\begin{array}{cccccccccc|c} 0 & \cdots & 1 & & * & 0 & & * & 0 & * & * \\ 0 & \cdots & 0 & \cdots & 0 & 1 & & * & 0 & * & * \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & * & * \\ 0 & \cdots & 0 & & & 0 & & & 0 & 0 & 0 \\ \vdots & & & & & \vdots & & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & & 0 & & \cdots & 0 & 0 & 0 \end{array}\right).$$

Can obtain the solution when augmented matrix is in REF (by back substitution), or RREF (read off directly).

Question

The augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(a) is in RREF, but not REF

(b) is in both RREF and REF

(c) is in REF, but not RREF

(d) is neither in RREF nor in REF

Example

Solve the system.

$$\begin{cases} x_1 & & & + & 3x_3 & = & 13 \\ -x_1 & - & x_2 & - & x_3 & = & -8 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 14 \\ 2x_1 & + & 3x_2 & & & = & 11 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 13 \\ -1 & -1 & -1 & -8 \\ 3 & 1 & 2 & 14 \\ 2 & 3 & 0 & 11 \end{array}\right) \xrightarrow[\substack{R_3-3R_1 \\ R_4-2R_1}]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 13 \\ 0 & -1 & 2 & 5 \\ 0 & 1 & -7 & -25 \\ 0 & 3 & -6 & -15 \end{array}\right)$$

$$\xrightarrow[R_4+3R_2]{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 13 \\ 0 & -1 & 2 & 5 \\ 0 & 0 & -5 & -20 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Can perform back substitution: $x_3 = \frac{-20}{-5} = 4$, $-x_2 + 2(4) = 5 \Rightarrow x_2 = 3$, $x_1 + 3(4) = 13 \Rightarrow x_1 = 1$.

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Example (continued)

If we continue the reduction to RREF:

$$\xrightarrow{-\frac{1}{5}R_3}\xrightarrow{-R_2}\left(\begin{array}{ccc|c} 1 & 0 & 3 & 13 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1-3R_3]{R_2+2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

we can read off the solution directly.

Example (continued)

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Solutions to Linear System

Definition

Given a linear system

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

we say that

$$x_1 = c_1,\ x_2 = c_2,\ \dots,\ x_n = c_n$$

is a ***solution*** to the linear system if the equations are simultaneously satisfied after making the substitution.

Three Possibilities for Solutions

- A linear system can be *inconsistent*, that is, have no solutions. For example,

$$\begin{cases} x & + & y & = & 0 \\ x & + & y & = & 2 \end{cases}$$

- It may have a unique solution. For example,

$$\begin{cases} x & - & y & = & 0 \\ x & + & y & = & 2 \end{cases}$$

- It can have infinitely many solutions. For example,

$$\begin{cases} 2x & + & 2y & = & 4 \\ x & + & y & = & 2 \end{cases}$$

Geometric Interpretation

There are no other possibilities, why?

`https://www.geogebra.org/calculator`

$$\begin{cases} x - y = 0 \\ x + y = 2 \end{cases}$$

`https://www.geogebra.org/3d`

$$\begin{cases} 3x - y + 2z = 0 \\ 3x + 3y - z = 1 \end{cases}$$

2 planes intersect at a line, infinitely many solutions with 1 parameter.

Challenge

Let **R** be a $n \times m$ matrix in reduced row-echelon form. Which of the following statements are true?

(a) The number of pivot columns of **R** is equal to the number of nonzero rows of **R**.

(b) The number of nonpivot columns of **R** is equal to the number of zero rows in **R**.

(c) The number of nonpivot columns of **R** is equal to the number of nonzero rows in **R**.

(d) The number of pivot columns of **R** is equal to the number of zero rows of **R**.