MA1522: Linear Algebra for Computing

Tutorial 2

Revision

Scalar Multiplication and Matrix Addition

- Scalar multiplication: $c\mathbf{A} = c(a_{ij}) = (ca_{ij})$.
- Matrix addition: $\mathbf{A} + \mathbf{B} = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$.

Theorem (Properties of matrix addition and scalar multiplication)

For matrices $\mathbf{A} = (a_{ij})_{m\times n}$, $\mathbf{B} = (b_{ij})_{m\times n}$, $\mathbf{C} = (c_{ij})_{m\times n}$, and real numbers $a, b \in \mathbb{R}$,

(i) *(Commutative)* $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$,

(ii) *(Associative)* $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$,

(iii) *(Additive identity)* $\mathbf{0}_{m\times n} + \mathbf{A} = \mathbf{A}$,

(iv) *(Additive inverse)* $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}_{m\times n}$,

(v) *(Distributive law)* $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$,

(vi) *(Scalar addition)* $(a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$,

(vii) *(Associative)* $(ab)\mathbf{A} = a(b\mathbf{A})$,

(viii) *If $a\mathbf{A} = \mathbf{0}_{m\times n}$, then either $a = 0$ or $\mathbf{A} = \mathbf{0}$.*

Matrix Multiplication

$\mathbf{AB} = (a_{ij})_{m\times p}(b_{ij})_{p\times n} = (\sum_{k=1}^{p} a_{ik}b_{kj})_{m\times n}$

Theorem (Properties of matrix multiplication)

(i) *(Associative)* $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

(ii) *(Left distributive law)* $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$.

(iii) *(Right distributive law)* $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$.

(iv) *(Commute with scalar multiplication)* $c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B})$.

(v) *(Multiplicative identity)* $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$.

(vi) *(Zero divisor) There exists* $\mathbf{A} \neq \mathbf{0}_{m\times p}$ *and* $\mathbf{B} \neq \mathbf{0}_{p\times n}$ *such that* $\mathbf{AB} = \mathbf{0}_{m\times n}$.

(vii) *(Zero matrix)* $\mathbf{A}\mathbf{0}_{n\times p} = \mathbf{0}_{m\times p}$ *and* $\mathbf{0}_{p\times m}\mathbf{A} = \mathbf{0}_{p\times n}$

Homogeneous Linear System

- Homogeneous linear system: $\mathbf{Ax} = \mathbf{0}$.
- The trivial solution $\mathbf{x} = \mathbf{0}$ is always a solution.
- If there is a solution $\mathbf{x} \neq \mathbf{0}$, then the homogeneous linear system admits nontrivial solutions.
- Homogeneous system has infinitely many solutions if and only if it has a nontrivial solution.

Transpose

$\mathbf{A} = (a_{ij})_{m\times n}$, $\mathbf{A}^T = (b_{ij})_{n\times m}$, $b_{ij} = a_{ji}$.

Theorem (Properties of transpose)

(i) $(\mathbf{A}^T)^T = \mathbf{A}$.

(ii) $(c\mathbf{A})^T = c\mathbf{A}^T$.

(iii) $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(iv) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Block Multiplication

let $\mathbf{b}_i$ be the j-th column of $\mathbf{B}$. Then the j-th column of the product $\mathbf{AB}$ is $\mathbf{Ab}_j$,

$$\mathbf{AB} = \mathbf{A}\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_n \end{pmatrix}.$$

Also, if $\mathbf{a}_i$ is the i-th row of $\mathbf{A}$, then the i-row of the product $\mathbf{AB}$ is $\mathbf{a}_i\mathbf{B}$,

$$\mathbf{AB} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} \mathbf{B} = \begin{pmatrix} \mathbf{a}_1\mathbf{B} \\ \mathbf{a}_2\mathbf{B} \\ \vdots \\ \mathbf{a}_m\mathbf{B} \end{pmatrix}.$$

Combining Augmented Matrices

In general: p linear systems with the same coefficient matrix $\mathbf{A} = (a_{ij})_{m\times n}$, for $k = 1, \ldots, p$,

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_{1k} \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_{2k} \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_{mk} \end{array}\right.$$

Combined augmented matrix:

$$\left(\begin{array}{cccc|c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} & & b_{1p} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & b_{22} & & b_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} & b_{m2} & & b_{mp} \end{array}\right)$$

Elementary Matrices

A square matrix of order n $\mathbf{E}$ is called an elementary matrix if it can be obtained from the identity matrix $\mathbf{I}_n$ by performing a single elementary row operation

$$\mathbf{I}_n \xrightarrow{r} \mathbf{E},$$

where r is an elementary row operation.

Let $\mathbf{A}$ be an $n \times m$ matrix and let $\mathbf{E}$ be the elementary matrix corresponding to the elementary row operation r. Then the product $\mathbf{EA}$ is the resultant of the row operation r on $\mathbf{A}$,

$$\mathbf{A} \xrightarrow{r} \mathbf{EA}.$$

Here, the order of the elementary matrix is determined by the number of rows of the matrix $\mathbf{A}$.

Row Equivalent Matrices

Suppose the matrix $\mathbf{B}$ is obtained from $\mathbf{A}$ by performing row operations $r_1, r_2, ..., r_k$,

$$\mathbf{A} \xrightarrow{r_1} \xrightarrow{r_2} \cdots \xrightarrow{r_k} \mathbf{B}.$$

Let $\mathbf{E}_1, \mathbf{E}_2, ..., \mathbf{E}_k$ be the corresponding elementary matrices. Then

$$\mathbf{B} = \mathbf{E}_k \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A}.$$

That is, if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then there exists elementary matrices $\mathbf{E}_1, \mathbf{E}_2, ..., \mathbf{E}_k$ such that the above equation holds.

Inverse of a Matrix

A square matrix $\mathbf{A}$ of order n is invertible if there exists a square matrix $\mathbf{B}$ of order n such that

$$\mathbf{AB} = \mathbf{I}_n = \mathbf{BA}.$$

- Only square matrices are invertible.
- $\mathbf{A}$ is invertible $\Leftrightarrow$ there is a $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$ $\Leftrightarrow$ there is a $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{I}$.
- For any square matrix $\mathbf{B}$, $\mathbf{BA} = \mathbf{I}_n \Leftrightarrow \mathbf{AB} = \mathbf{I}_n$.
- If $\mathbf{B}$ and $\mathbf{C}$ are both inverses of a square matrix $\mathbf{A}$, then $\mathbf{B} = \mathbf{C}$.

Denote the (unique) inverse of $\mathbf{A}$ as $\mathbf{A}^{-1}$

Algorithm to Determine Invertibility and Finding Inverse

Let $\mathbf{A}$ be a square matrix of order n.

Step 1: Form a new $n \times 2n$ matrix $\left(\begin{array}{c|c} \mathbf{A} & \mathbf{I}_n \end{array} \right)$.

Step 2: Reduce the matrix $\left(\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array} \right) \longrightarrow \left(\begin{array}{c|c} \mathbf{R} & \mathbf{B} \end{array} \right)$ to its REF or RREF.

Step 3: If RREF $\mathbf{R} \neq \mathbf{I}$ or REF has a zero row, then $\mathbf{A}$ is not invertible. If RREF $\mathbf{R} = \mathbf{I}$ or REF has no zero row, $\mathbf{A}$ is invertible with inverse $\mathbf{A}^{-1} = \mathbf{B}$.

Equivalent Statements for Invertibility

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) $\mathbf{A}$ is invertible.

(ii) (left inverse) There is a matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{I}$.

(iii) (right inverse) There is a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$.

(iv) The reduced row-echelon form of $\mathbf{A}$ is the identity matrix.

(v) $\mathbf{A}$ can be expressed as a product of elementary matrices.

(vi) The homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

(vii) For any $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

Tutorial 2 Solutions

Question 1(a)

Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ and $n \times p$ matrices respectively. Suppose the homogeneous linear system $\mathbf{Bx} = \mathbf{0}$ has infinitely many solutions. How many solutions does the system $\mathbf{ABx} = \mathbf{0}$ have?

Claim: Any solution of $\mathbf{Bx} = \mathbf{0}$ is a solution to $\mathbf{ABx} = \mathbf{0}$, that is,

$$\{\text{all solutions to } \mathbf{Bx} = \mathbf{0}\} \subseteq \{\text{all solutions to } \mathbf{ABx} = \mathbf{0}\}.$$

Suppose $\mathbf{u}$ is a solution to $\mathbf{Bx} = \mathbf{0}$, that is, $\mathbf{Bu} = \mathbf{0}$. Premultiplying both sides of $\mathbf{Bu} = \mathbf{0}$ by $\mathbf{A}$, we have $\mathbf{ABu} = \mathbf{0}$, which shows that $\mathbf{u}$ is also a solution to $\mathbf{ABx} = \mathbf{0}$.

This shows that the set of solutions to $\mathbf{ABx} = \mathbf{0}$ contains the set of solutions to $\mathbf{Bx} = \mathbf{0}$, which is an infinite set. Hence, $\{\text{all solutions to } \mathbf{ABx} = \mathbf{0}\}$ is an infinite set too.

Question 1(b)

Suppose $\mathbf{Bx} = \mathbf{0}$ has only the trivial solution. Can we tell how many solutions are there for $\mathbf{ABx} = \mathbf{0}$.

No, for example, let $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{Bx} = \mathbf{0}$ has only the trivial solution. Now consider two cases

(i) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{ABx} = \mathbf{0}$ has only the trivial solution.

(ii) $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $\mathbf{ABx} = \mathbf{0}$ has infinitely many solutions.

Question 2(a)

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Find a 4×3 matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{I}_3$.

By block multiplication,

$$\mathbf{AX} = \mathbf{I} = \mathbf{A}\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{Ax}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Ax}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{Ax}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

General solution: $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad s_1, s_2, s_3 \in \mathbb{R}.$

```
A=[1 1 0 1;0 1 1 0;0 0 1 1], rref([A eye(3)])
```

Question 2(b)

Let $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Find a 3×4 matrix $\mathbf{Y}$ such that $\mathbf{YB} = \mathbf{I}_3$.

Solve $\mathbf{B}^T\mathbf{Y}^T = (\mathbf{YB})^T = \mathbf{I}_3^T = \mathbf{I}_3$ instead. Then by part (a), we may let $\mathbf{Y}^T = \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_1 & \mathbf{y}_3 \end{pmatrix}$.

$$\left(\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 & 1/2 \end{array}\right)$$

General solution:

$$\mathbf{y}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}, \ \mathbf{y}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}, \ \mathbf{y}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}, \quad s_1, s_2, s_3 \in \mathbb{R}.$$

```
B=[1 0 1;1 1 0;0 1 1;0 0 1]; rref([B' eye(3)])
```

Question 3(a)

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix}.$$

(i) Reduce the following matrices $\mathbf{A}$ to its reduced row-echelon form $\mathbf{R}$.

(ii) For each of the elementary row operation, write the corresponding elementary matrix.

(iii) Write the matrices $\mathbf{A}$ in the form $\mathbf{E}_1\mathbf{E}_2 \ldots \mathbf{E}_n\mathbf{R}$ where $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n$ are elementary matrices and $\mathbf{R}$ is the reduced row-echelon form of $\mathbf{A}$.

Question 3(a)

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(i) $\mathbf{A} \xrightarrow{r_1: R_2 + \frac{2}{5}R_1} \xrightarrow{r_2: \frac{1}{5}R_1} \xrightarrow{r_3: 5R_2} \xrightarrow{r_4: R_1 + \frac{2}{5}R_2} \mathbf{R} = \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}.$

Question 3(a)

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix}.$$

(i) Reduce the following matrices $\mathbf{A}$ to its reduced row-echelon form $\mathbf{R}$.

(ii) For each of the elementary row operation, write the corresponding elementary matrix.

(iii) Write the matrices $\mathbf{A}$ in the form $\mathbf{E}_1\mathbf{E}_2\ldots\mathbf{E}_n\mathbf{R}$ where $\mathbf{E}_1, \mathbf{E_2}, \ldots, \mathbf{E}_n$ are elementary matrices and $\mathbf{R}$ is the reduced row-echelon form of $\mathbf{A}$.

(i) $\mathbf{A} \xrightarrow{r_1: R_2+\frac{2}{5}R_1} \xrightarrow{r_2: \frac{1}{5}R_1} \xrightarrow{r_3: 5R_2} \xrightarrow{r_4: R_1+\frac{2}{5}R_2} \mathbf{R} = \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}.$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ \frac{2}{5} & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}.$$

Question 3(b)

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}.$$

(i) $\mathbf{A} \xrightarrow{r_1: R_2+2R_1} \xrightarrow{r_2: R_3-4R_1} \xrightarrow{r_3: R_3+R_2} \xrightarrow{r_4: -R_1} \xrightarrow{r_5: \frac{1}{10}R_2} \xrightarrow{r_6: R_1+3R_2} \mathbf{R}$

Question 3(b)

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}.$$

(i) $\mathbf{A} \xrightarrow{r_1: R_2+2R_1} \xrightarrow{r_2: R_3-4R_1} \xrightarrow{r_3: R_3+R_2} \xrightarrow{r_4: -R_1} \xrightarrow{r_5: \frac{1}{10}R_2} \xrightarrow{r_6: R_1+3R_2} \mathbf{R}$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$\mathbf{E}_6 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{19}{10} \\ 0 & 1 & -\frac{7}{10} \\ 0 & 0 & 0 \end{pmatrix}.$$

Question 3(c)

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

(i) $\mathbf{A} \xrightarrow{r_1: R_2 - 2R_1} \xrightarrow{r_2: R_3 - R_1} \xrightarrow{r_3: R_2 \leftrightarrow R_3} \xrightarrow{r_4: \frac{1}{3}R_2} \xrightarrow{r_5: R_2 - R_3} \xrightarrow{r_6: R_1 + R_2} \mathbf{R}$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$,

$\mathbf{E}_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 4(a)

Determine if the matrix $\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}$ is invertible. If it is invertible, find its inverse.

$$\left(\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_2+3R_1,\ -R_1,\ \frac{1}{7}R_2,\ R_1+3R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & \frac{3}{7} \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} \end{array}\right).$$

Hence the matrix is invertible and its inverse is $\frac{1}{7}\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$.

Alternatively, may use

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Question 4(b)

Determine if the matrix $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$ is invertible. If it is invertible, find its inverse.

$$\left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2+2R_1,\ R_3-4R_1,\ R_3+R_2} \left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right).$$

The matrix is not invertible.

Question 5

Write down the conditions so that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ is invertible.

- `syms a b c; A=[1 1 1;a b c;a^2 b^2 c^2];`
- `A(2,:)=A(2,:)-a*A(1,:); A(3,:)=A(3,:)-a^2*A(1,:)`
- `A(3,:)=A(3,:)-(b+a)*A(2,:)`
- `A=simplify(A)`

Alternatively, may use `det(A)`.

Question 5

Write down the conditions so that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ is invertible.

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \xrightarrow[R_3-a^2R_1]{R_2-aR_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \xrightarrow{R_3-(b+a)R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

So we need $c \neq a$ and $c \neq b$ for the last row to be nonzero. Suppose so, we proceed,

$$\xrightarrow{\frac{1}{(c-a)(b-a)}R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1-R_3]{R_2-(c-a)R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $b \neq a$, then it is clear that the matrix can be reduced to the identity matrix. Thus the conditions are $a \neq b$, $b \neq c$, $c \neq a$, that is, they are distinct points.

Question 6(a)

Suppose $\mathbf{A}$ is a square matrix such that $\mathbf{A}^2 = \mathbf{0}$. Show that $\mathbf{I} - \mathbf{A}$ is invertible, with inverse $\mathbf{I} + \mathbf{A}$.

To show that $\mathbf{I} - \mathbf{A}$, suffice to check that it has a left inverse. Indeed,

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A}) = \mathbf{I}^2 - \mathbf{A}^2 = \mathbf{I}.$$

Question 6(b)

Suppose $\mathbf{A}^3 = \mathbf{0}$. Is $\mathbf{I} - \mathbf{A}$ invertible?

Substituting $\mathbf{A}$ into the polynomial identity $(1 - x)(1 + x + x^2) = 1 - x^3$, we get

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2) = \mathbf{I} - \mathbf{A}^3 = \mathbf{I}.$$

Question 6(c)

A square matrix $\mathbf{A}$ is said to be *nilpotent* if there is a positive integer n such that $\mathbf{A}^n = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\mathbf{I} - \mathbf{A}$ is invertible.

Substituting $\mathbf{A}$ into the polynomial identity $(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1-x^n$, we get

$$(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}+\cdots+\mathbf{A}^{n-1}) = \mathbf{I}-\mathbf{A}^n = \mathbf{I}.$$

Hence the inverse matrix of $\mathbf{I}-\mathbf{A}$ is $(\mathbf{I}+\mathbf{A}+\cdots+\mathbf{A}^{n-1})$.

Remark: The inverse could be derived from the formula for the sum of a geometric progression,

$$\sum_{k=1}^{n} x^{k-1} = \frac{1-x^n}{1-x},$$

which is equivalent to $(1-x)\sum_{k=1}^{n} x^{k-1} = 1-x^n$.

Extra: Show that every strictly upper or lower triangular matrix is nilpotent.