MA1522 Linear Algebra for Computing

Lecture 3: Matrices

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Outline

Some Highlights

Questions posed in Dr.Teo's Lectures

Challenges posed in Dr.Teo's Lectures

Matrix Multiplication

$$\mathbf{AB} = (a_{ij})_{m\times p}(b_{ij})_{p\times n} = (\sum_{k=1}^{p} a_{ik}b_{kj})_{m\times n}.$$

For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}.$$

Cautions! (I)

Matrix multiplication is not commutative,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Two nonzero matrices can have product equal to zero:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(in algebra terms, they are called zero divisors.)

Cautions! (II)

In general, we do NOT have cancellation laws.

Below is an example of the failure of (left) cancellation law, (that is, from $\mathbf{AB} = \mathbf{AC}$ and $\mathbf{A} \neq \mathbf{0}$, we may not conclude that $\mathbf{B} = \mathbf{C}$):

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix},$$

but $\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$.

However, when $\mathbf{A}$ is invertible, the cancellation laws hold.

Slides 70 and 75 of Ch.2

Definition

An $n \times n$ square matrix $\mathbf{A}$ is *invertible* if there exists a matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I}_n = \mathbf{BA}.$$

A matrix is said to be *non-invertible* otherwise.

Theorem

Let $\mathbf{A}$ be an invertible matrix of order n.

(i) *(Left cancellation) If $\mathbf{B}$ and $\mathbf{C}$ are $n \times m$ matrices with $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.*

(ii) *(Right cancellation) If $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$ matrices with $\mathbf{BA} = \mathbf{CA}$, then $\mathbf{B} = \mathbf{C}$.*

Theorem (Equivalent statements of invertibility)

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) $\mathbf{A}$ *is invertible.*

(ii) $\mathbf{A}^T$ *is invertible.*

(iii) *(left inverse) There is a matrix* $\mathbf{B}$ *such that* $\mathbf{BA} = \mathbf{I}$.

(iv) *(right inverse) There is a matrix* $\mathbf{B}$ *such that* $\mathbf{AB} = \mathbf{I}$.

(v) *The reduced row-echelon form of* $\mathbf{A}$ *is the identity matrix.*

(vi) $\mathbf{A}$ *can be expressed as a product of elementary matrices.*

(vii) *The homogeneous system* $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution.*

(viii) *For any* $\mathbf{b}$, *the system* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

Questions in Section 2.1.

Q1: Which of the following statements are true?

(i) Every square matrix is row equivalent to an upper triangular matrix.

(ii) Every square matrix is row equivalent to a strictly upper triangular matrix.

Q2: Is it true that a symmetric upper triangular matrix is a zero matrix? If not, what type of matrix is it?

Key concepts involved: "upper triangular", "strictly upper triangular" and "symmetric" matrices.

Review of some Concepts

On Slide 14 of Ch.2: (For square matrices) $\mathbf{A} = (a_{ij})$ is upper triangular, if $\mathbf{A} = (a_{ij})$, $a_{ij} = 0$ for all $i > j$; $\mathbf{A}$ is strictly upper triangular, if $a_{ij} = 0$ for all $i \geq j$. For example

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

are upper triangular and strict upper triangular matrices, respectively.

On Slide 16 of Ch.2: A matrix $\mathbf{A} = (a_{ij})_n$ is symmetric if $a_{ij} = a_{ji}$.

For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

is symmetric.

Answer to Q1(i) in Section 2.1.

T or F?

(i) Every square matrix is row equivalent to an upper triangular matrix.

Recall: Two matrices **B** and **C** are *row equivalent* if one can obtain **C** from **B** by a series of elementary row operations.

Answer: True. We know that every matrix is row equivalent to one in REF. It suffices to show that matrices in REF are upper triangular.

Let $i > j$. We show that $a_{ij} = 0$ by contradiction. Suppose that, for the sake of contradiction, $a_{ij} \neq 0$. Then for all $i' \leq i$, the row $R_{i'}$ is nonzero (because in REF, zero rows are in the bottom). Thus, the leading entry in each $R_{i'}$ must be in some column $C_{j'}$ for some $j' \leq j$. Since the leading entries are in different columns, this violates the Pigeonhole Principle.

Answer to Q1(ii) in Section 2.1.

T or F?

(ii) Every square matrix is row equivalent to a strictly upper triangular matrix.

Answer: False. For example, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not row equivalent to any strictly upper triangular matrix.

The reason is that each 2×2 strictly upper triangular matrix is of the form: $B = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. If A is row equivalent to B, then the first column of A must be 0.

Question Two in Section 2.1

Q2: Is it true that a symmetric upper triangular matrix is a zero matrix? If not, what type of matrix is it?

Answer: No, it is not true. For example, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a nonzero symmetric upper triangular matrix.

Any such matrix $A = (a_{ij})$ must be diagonal. Reason: If $i > j$, then $a_{ij} = 0$ because A is upper triangular. If $i < j$, then $a_{ij} = a_{ji} = 0$ because A is symmetric.

Questions in Section 2.2

Show that for diagonal matrices **A** and **B**,

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{A}\mathbf{B} + \mathbf{B}^2.$$

Suffice to prove that $\mathbf{AB} = \mathbf{BA}$ for diagonal matrices **A** and **B**.

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}a_{11} & 0 & \cdots & 0 \\ 0 & b_{22}a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}a_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Further Question

Q: How about only one of $\mathbf{A}$ or $\mathbf{B}$ is diagonal?

For example, take $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ to be diagonal and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Then

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} &= \begin{pmatrix} 2 & 4 \\ 9 & 12 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} &= \begin{pmatrix} 2 & 6 \\ 6 & 12 \end{pmatrix}, \end{aligned}$$

which shows that $\mathbf{AB} \neq \mathbf{BA}$.

Question in Sec 2.3.

Q: Which of the following statements is/are true?

(i) If the homogeneous system has a unique solution, it must be the trivial solution.

(ii) If the homogeneous system has the trivial solution, it must be the unique solution.

Key concepts: Homogeneous system and trivial solution.

Review of some Concepts (I)

On Slide 45 of Ch.2: a linear system is homogeneous if it has the following corresponding matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{0},$$

for some $m \times n$ matrix $\mathbf{A}$, a variable n-vector $\mathbf{x}$, and the $m \times 1$ zero matrix (or zero m-vector) $\mathbf{0} = \mathbf{0}_{m\times 1}$.

In standard form, a homogeneous system looks like:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0. \end{cases}$$

Review of some Concepts (II)

On Slides 45 and 46 of Ch.2:

Theorem

A homogeneous linear system is always consistent.

(Because $x_1 = \cdots = x_n = 0$ is always a solution.)

Definition

The zero solution is called the *trivial solution*. If $\mathbf{x} \neq \mathbf{0}$ is a nonzero solution to the homogeneous system, it is called a *nontrivial solution*.

Question in Sec 2.3.

Q: Which of the following statements is/are true?

(i) If the homogeneous system has a unique solution, it must be the trivial solution.

(ii) If the homogeneous system has the trivial solution, it must be the unique solution.

Answer: (i) is true. By Theorem, homogeneous system always has zero solution. If the solution is unique, it has to be the zero solution, i.e., trivial solution.

(ii) False, for example,

$$\begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0. \end{cases}$$

has trivial solution, but also has solutions $x_1 = -2t, x_2 = t$ where $t \in \mathbb{R}$.

Question one in Section 2.4.

Q1: Verify that the homogeneous system

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has only the trivial solution.

Answer: We perform Gaussian elimination (we dropped the last column in the augmented matrix):

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \\ & \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}. \end{aligned}$$

The statement follows.

Question two in Sec 2.4.

Q2: Statement (iv) of the theorem on properties of inverse says that the product of invertible matrices is invertible. Is the converse true? That is, if the product of two square matrices $\mathbf{AB}$ is invertible, can we conclude that both $\mathbf{A}$ and $\mathbf{B}$ are invertible? Why?

Is the inverse of an invertible symmetric matrix symmetric?

Review of Slide 87 of Ch. 2

Recall:

Theorem

Let $\mathbf{A}$ *be an invertible matrix of order* n*.*

(i) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

(ii) *For any nonzero real number* $a \in \mathbb{R}$, $(a\mathbf{A})$ *is invertible with inverse* $(a\mathbf{A})^{-1} = \frac{1}{a}\mathbf{A}^{-1}$.

(iii) $\mathbf{A}^T$ *is invertible with inverse* $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

(iv) *If* $\mathbf{B}$ *is an invertible matrix of order* n*, then* $(\mathbf{AB})$ *is invertible with inverse* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Q2 in Sec 2.4., part 1

Q2: If the product of two square matrices $\mathbf{AB}$ is invertible, can we conclude that both $\mathbf{A}$ and $\mathbf{B}$ are invertible? Why?

Answer: Since $\mathbf{AB}$ is invertible, there is a matrix $\mathbf{C}$ such that

$$\mathbf{C}(\mathbf{AB}) = \mathbf{I} = (\mathbf{AB})\mathbf{C}.$$

By associativity, we get $(\mathbf{CA})\mathbf{B} = \mathbf{I} = \mathbf{A}(\mathbf{BC})$. By Equivalent statements of invertibility on Slide 117, both $\mathbf{A}$ and $\mathbf{B}$ are invertible.

Q2 in Sec 2.4., part 2

Q2: Is the inverse of an invertible symmetric matrix symmetric?

Concept involved (Slide 34): Let $\mathbf{A} = (a_{ij})$ be a $m \times n$ matrix. The *transpose* of $\mathbf{A}$, denoted as $\mathbf{A}^T$, is the $n \times m$ matrix whose (i, j)-entry is the (j, i)-entry of $\mathbf{A}$, $\mathbf{A}^T = (b_{ij})_{n\times m}$, $b_{ij} = a_{ji}$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \qquad \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}_{n\times m}$$

Slide 36: Properties of Transpose

Theorem

(i) $(\mathbf{A}^T)^T = \mathbf{A}$.

(ii) $(c\mathbf{A})^T = c\mathbf{A}^T$.

(iii) $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(iv) $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$.

A square matrix $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^T$.

Q2 in Sec 2.4., part 2

Q2: Is the inverse of an invertible symmetric matrix symmetric?

Answer: Yes. Let $\mathbf{A}$ be an invertible symmetric matrix. Then by (iii) in Slide 87, we have

$$(\mathbf{A}^{-1})^{\mathbf{T}} = (\mathbf{A}^{\mathbf{T}})^{-1} = \mathbf{A}^{-1}.$$

Hence $\mathbf{A}^{-1}$ is symmetric.

Challenge in Sec 2.1., part 1

Q: Is it true that if $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the same order, then so is $\mathbf{AB}$?

Answer: False. For example,

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

Challenge in Section 2.3

Let $\mathbf{v}$ be a particular solution to a non-homogeneous system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$.

Show that

$$\mathbf{v} + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, ..., s_k \in \mathbb{R}$$

is the general solution to $\mathbf{Ax} = \mathbf{b}$ if and only if

$$s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, ..., s_k \in \mathbb{R}$$

is the general solution to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$.

Remarks

The geometric picture of solutions tell us: When a linear system has infinitely many solutions, the solutions are not random points, rather, they have certain *structure*.

This structure is better described using homogeneous equations.

In Chapters 3 and 4, we will learn that the solutions of homogeneous systems form a *vector space*, and the number of parameters will the its dimension.

Finally, the solutions for nonhomogeneous systems and for homogeneous systems are linked via Challenge in Section 2.3.

Challenge in Section 2.3, "if" part

Let $\mathbf{v}$ be a particular solution to a non-homogeneous system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$. Show that if

$$s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, ..., s_k \in \mathbb{R}$$

is the general solution to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$, then

$$\mathbf{v} + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, ..., s_k \in \mathbb{R}$$

is the general solution to $\mathbf{Ax} = \mathbf{b}$.

Proof. Let $\mathbf{z}$ be a solution of $\mathbf{Ax} = \mathbf{b}$. Then $\mathbf{A}(\mathbf{z} - \mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$. In other words, $\mathbf{z} - \mathbf{v}$ is a solution to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$. By assumption,

$$\mathbf{z} - \mathbf{v} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k,$$

hence

$$\mathbf{z} = \mathbf{v} + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k$$

where $s_1, s_2, ..., s_k \in \mathbb{R}$.

Challenge in Section 2.3, "only if" part

Let $\mathbf{v}$ be a particular solution to a non-homogeneous system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$. Show that if

$$\mathbf{v} + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, \ldots, s_k \in \mathbb{R}$$

is the general solution to $\mathbf{Ax} = \mathbf{b}$, then

$$s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, \ldots, s_k \in \mathbb{R}$$

is the general solution to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$.

Proof. Let $\mathbf{y}$ be a solution of $\mathbf{Ax} = \mathbf{0}$. Then $\mathbf{A}(\mathbf{y} + \mathbf{v}) = \mathbf{0} + \mathbf{b} = \mathbf{b}$. In other words, $\mathbf{y} + \mathbf{v}$ is a solution to the linear system $\mathbf{Ax} = \mathbf{b}$. By assumption,

$$\mathbf{y} + \mathbf{v} = \mathbf{v} + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k,$$

hence

$$\mathbf{y} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k$$

where $s_1, s_2, \ldots, s_k \in \mathbb{R}$.