NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 9**

1. A father wishes to distribute an amount of money among his three sons Jack, Jim, and John. He wish to distribute such that the following conditions are all satisfied.

   (i) The amount Jack receives plus twice the amount Jim receives is \$300.

   (ii) The amount Jim receives plus the amount John receives is \$300.

   (iii) Jack receives \$300 more than twice of what John receives.

   (a) Is it possible for the following conditions to all be satisfied?

   Solution: Let x, y, z be the amount of money that Jack, Jim, and John receives, respectively. The conditions are

   $$\begin{cases} x + 2y = 300 \\ y + z = 300 \\ x - 2z = 300 \end{cases}$$

   This system is inconsistent. So, there are no solution to the distribution problem.

   (b) If it is not possible, find a least square solution. (Make sure that your least square solution is feasible. For example, one cannot give a negative amount of money to anybody.)

   Solution: The least square solutions to the system in (a) is

   $$x = 200 + 2t, \quad y = 100 - t, \quad z = t, \quad t \in \mathbb{R}.$$

   However, to make sure that x, y, z are all non-negative, we need to have $0 \leq t \leq 100$.

2. (a) Suppose $\mathbf{A}$ is a $m \times n$ matrix where $m > n$. Let $\mathbf{A} = \mathbf{QR}$ be a QR factorization of $\mathbf{A}$. Explain how you might use this to write

   $$\mathbf{A} = \mathbf{Q}'\mathbf{R}',$$

   where $\mathbf{Q}'$ is an $m \times m$ orthogonal matrix, and $\mathbf{R}'$ a $m \times n$ matrix with $m - n$ zero rows at the bottom. This is known as the *full* QR factorization of $\mathbf{A}$.

   Solution: Write $\mathbf{Q} = \begin{pmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_n \end{pmatrix}$. Note that $T = \{\mathbf{w}_1, ..., \mathbf{w}_n\}$ is an orthonormal set in $\mathbb{R}^m$. Extend T to an orthonormal basis $\{\mathbf{w}_1, ..., \mathbf{w}_n, \mathbf{w}_{n+1}, ..., \mathbf{w}_m\}$ for $\mathbb{R}^m$. Define

   $$\mathbf{Q}' = \begin{pmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_n & \mathbf{w}_{n+1} & \cdots & \mathbf{w}_m \end{pmatrix}.$$

   Then $\mathbf{Q}'$ is a $m \times m$ orthogonal matrix. Define

   $$\mathbf{R}' = \begin{pmatrix} \mathbf{R} \\ \mathbf{0}_{(m-n)\times n} \end{pmatrix}.$$

(b) In MATLAB, enter the following.

```
>> A=sym([1 1 0;1 1 0;1 1 1;0 1 1])

>> [Q R]=qr(A)
```

What is $\mathbf{Q}$ and $\mathbf{R}$? Compare this with the answer in tutorial 8 question 5(a).

Solution: $\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} \sqrt{3} & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & 0 \end{pmatrix}$. That is, the code `qr` returns the full QR factorization of $\mathbf{A}$.

(c) Explain how you might use the command `qr` in MATLAB to find a QR factorization of a $m \times n$ matrix $\mathbf{A}$?

Solution: Let $\mathbf{A} = \mathbf{Q}'\mathbf{R}'$ be the full QR factorization computed in MATLAB. Then let $\mathbf{Q}$ be the first n columns of $\mathbf{Q}'$, and $\mathbf{R}$ be the first n (nonzero) rows of $\mathbf{R}'$.

3. (Cayley-Hamilton theorem)
Consider
$$p(\mathbf{X}) = \mathbf{X}^3 - 4\mathbf{X}^2 - \mathbf{X} + 4\mathbf{I}.$$

(a) Compute $p(\mathbf{X})$ for $\mathbf{X} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

Solution: $p(\mathbf{X}) = \mathbf{0}$ the order 3 zero square matrix.

(b) Find the characteristic polynomial of $\mathbf{X}$.

Solution: $\begin{vmatrix} x-1 & -1 & -2 \\ -1 & x-2 & -1 \\ -2 & -1 & x-1 \end{vmatrix} = x^3 - 4x^2 - x + 4 = p(x)$.

(c) Show that $\mathbf{X}$ invertible. Express the inverse of $\mathbf{X}$ as a function of $\mathbf{X}$.

Solution: Since $\mathbf{X}^3 - 4\mathbf{X}^2 - \mathbf{X} + 4\mathbf{I} = \mathbf{0}$, we have $\mathbf{X}(\mathbf{X}^2 - 4\mathbf{X} - \mathbf{I}) = -4\mathbf{I}$, and hence, $\mathbf{X}^{-1} = -\frac{1}{4}(\mathbf{X}^2 - 4\mathbf{X} - \mathbf{I})$.

This question demonstrated the *Cayley-Hamilton theorem*, which states that if $p(x)$ is the characteristic polynomial of $\mathbf{X}$, then $p(\mathbf{X}) = 0$. This also show that if 0 is not an eigenvalue of $\mathbf{X}$, then the constant term of the characteristic polynomial $p(x)$ is nonzero, and we can use that to compute the inverse of $\mathbf{X}$.

4. For each of the following matrices $\mathbf{A}$, determine if $\mathbf{A}$ is diagonalizable. If $\mathbf{A}$ is diagonalizable, find an invertible $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and determine $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

   (a) $\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$.

   Solution: $\begin{vmatrix} x-1 & 3 & -3 \\ -3 & x+5 & -3 \\ -6 & 6 & x-4 \end{vmatrix} = x^3 - 12x - 16 = (x+2)^2(x-4)$. The eigenvalues are $\lambda = -2$ and $\lambda = 4$ with multiplicities $r_{-2} = 2, r_4 = 1$. Then $\mathbf{A}$ is diagonalizable if and only if the geometric multiplicity of eigenvalue $\lambda = -2$ is 2, $\dim(E_{-2}) = 2 = r_{-2}$.

   $$-2\mathbf{I} - \mathbf{A} = \begin{pmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } E_{-2}.$$

   This shows that $\dim(E_{-2}) = 2 = r_{-2}$, and hence, $\mathbf{A}$ is diagonalizable.

   $$4\mathbf{I} - \mathbf{A} = \begin{pmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\} \text{ is a basis for } E_4.$$

   Hence, we may let $\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix} 9 & 8 & 6 & 3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

   Solution: Since $\mathbf{A}$ is a triangular matrix, the eigenvalues are the diagonals $\lambda = -1$,

$\lambda = 2$, $\lambda = 3$, $\lambda = 9$, each with multiplicity 1. Hence, $\mathbf{A}$ is diagonalizable.

$$9\mathbf{I} - \mathbf{A} \xrightarrow{RREF} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } E_9,$$

$$-\mathbf{I} - \mathbf{A} \xrightarrow{RREF} \begin{pmatrix} 1 & 4/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} -4 \\ 5 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } E_{-1},$$

$$2\mathbf{I} - \mathbf{A} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } E_2,$$

$$3\mathbf{I} - \mathbf{A} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & -5/6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 5 \\ -6 \\ 0 \\ 6 \end{pmatrix} \right\} \text{ is a basis for } E_3.$$

So, let $\mathbf{P} = \begin{pmatrix} 1 & -4 & -2 & 5 \\ 0 & 5 & 1 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$ and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

Solution: This is a lower triangular matrix, the eigenvalue is 1, with algebraic multiplicity 3.

$$\mathbf{I} - \mathbf{A} = \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which shows that the geometric multiplicity of eigenvalue 1 is 1. Hence, $\mathbf{A}$ is not diagonalizable.

(d) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Solution: The characteristic polynomial of $\mathbf{A}$ is $(x-1)^2(x+1)^2$, that is, the eigenvalues are $\lambda = \pm 1$, with multiplicity $r_1 = r_{-1} = 2$.

$$-\mathbf{I} - \mathbf{A} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } E_{-1},$$

$$\mathbf{I} - \mathbf{A} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } E_{1}.$$

Hence, $\mathbf{A}$ is diagonalizable, with $\mathbf{P} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and

$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(e) $\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ -4 & 2 & 3 \end{pmatrix}$.

Solution: The characteristic polynomial of $\mathbf{A}$ is

$$x^3 - 3x^2 + 4x - 2 = (x-1)(x^2 - 2x + 2).$$

$(x^2 - 2x + 2)$ cannot be factorized into linear factor since its discriminant is -4. Since the characteristic polynomial does not factor into linear factors, $\mathbf{A}$ is not diagonalizable.

5. (a) Show that λ is an eigenvalue of $\mathbf{A}$ if and only if it is an eigenvalue of $\mathbf{A}^T$.

Solution:

$$\det(x\mathbf{I} - \mathbf{A}) = \det((x\mathbf{I} - \mathbf{A})^T) = \det((x\mathbf{I})^T - \mathbf{A}^T) = \det(x\mathbf{I} - \mathbf{A}^T).$$

Hence the roots of $\det(x\mathbf{I} - \mathbf{A})$ are exactly the roots of $\det(x\mathbf{I} - \mathbf{A}^T)$.

(b) Suppose $\mathbf{A}$ is diagonalizable. Is $\mathbf{A}^T$ diagonalizable? Justify your answer.

Solution: Yes. Write $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Then

$$\mathbf{A}^T = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})^T = (\mathbf{P}^{-1})^T\mathbf{D}^T\mathbf{P}^T = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1},$$

where the last equality follows from the fact that $\mathbf{D}$ is diagonal and thus symmetric, and letting $\mathbf{Q} = (\mathbf{P}^{-1})^T$ (recall that $(\mathbf{P}^{-1})^T = (\mathbf{P}^T)^{-1}$). That is, $\mathbf{Q} = (\mathbf{P}^{-1})^T$ diagonalizes $\mathbf{A}^T$.

(c) Suppose $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ associated to eigenvalue λ. Show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^k$ associated to eigenvalue λ^k for any positive integer k.

Solution: By definition, we have $\mathbf{Av} = \lambda\mathbf{v}$. Then

$$\mathbf{A}^k\mathbf{v} = \mathbf{A}^{k-1}\mathbf{A}\mathbf{v} = \lambda\mathbf{A}^{k-2}\mathbf{A}\mathbf{v} = \lambda^2\mathbf{A}^{k-3}\mathbf{A}\mathbf{v} = \cdots = \lambda^{k-1}\mathbf{A}\mathbf{v} = \lambda^k\mathbf{v}.$$

Since $\mathbf{v} \neq \mathbf{0}$, $\mathbf{v}$ is a witness to λ^k being an eigenvalue of $\mathbf{A}^k$.

(d) If $\mathbf{A}$ is invertible, show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^k$ associated to eigenvalue λ^k for any negative integer k.

Solution: Suppose $k = -1$. First note that since $\mathbf{A}$ is invertible, $k \neq 0$. Then $\mathbf{Av} = \lambda\mathbf{v} \iff \lambda^{-1}\mathbf{v} = \mathbf{A}^{-1}\mathbf{v}$. Hence, λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$. The rest of the argument follows from (c).

(e) A square matrix is said to be *nilpotent* if there is a positive integer k such that $\mathbf{A}^k = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then 0 is the only eigenvalue.

Solution: Let λ be an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ be an eigenvector associated to λ. By (c), $\mathbf{0} = \mathbf{A}^k\mathbf{v} = \lambda^k\mathbf{v}$. Since $\mathbf{v} \neq \mathbf{0}$, necessarily $\lambda^k = 0$, and hence $\lambda = 0$.

(f) Let $\mathbf{A}$ be a $n \times n$ matrix with one eigenvalue λ with algebraic multiplicity n. Show that $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ is a scalar matrix, $\mathbf{A} = \lambda\mathbf{I}$.

Solution: Suppose $\mathbf{A}$ is diagonalizable, say $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$. Now since λ is the only eigenvalue with multiplicity n, necessarily $\mathbf{D} = \lambda\mathbf{I}$. Hence,

$$\mathbf{A} = \mathbf{P}\lambda\mathbf{I}\mathbf{P}^{-1} = \lambda\mathbf{P}\mathbf{I}\mathbf{P}^{-1} = \lambda\mathbf{I},$$

that is, $\mathbf{A} = \lambda\mathbf{I}$ is a scalar matrix. It is clear that a scalar matrix is diagonalizable.

(g) Show that the only diagonalizable nilpotent matrix is the zero matrix.

Solution: Let $\mathbf{A}$ be a nilpontent matrix. Then 0 is the only eigenvalue. If $\mathbf{A}$ is diagonalizable, then $\mathbf{A} = \mathbf{P}\,\mathrm{diag}(0, 0, ..., 0)\mathbf{P}^{-1} = \mathbf{0}$ for some invertible matrix P.

Extra problems

1. Let $\mathbf{A}$ be an orthogonal matrix of order n and let $\mathbf{u}, \mathbf{v}$ be any two vectors in $\mathbb{R}^n$. Show that

   (a) $||\mathbf{u}|| = ||\mathbf{Au}||$;

   Solution: Given any vectors $\mathbf{u}, \mathbf{v}$,

   $$(\mathbf{Au}) \cdot (\mathbf{Av}) = \mathbf{u}^T\mathbf{A}^T\mathbf{Av} = \mathbf{u}^T\mathbf{v} = \mathbf{u} \cdot \mathbf{v},$$

   where the second equality follows from the fact that $\mathbf{A}$ is orthogonal, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

   Hence,

   $$\|\mathbf{Au}\| = \sqrt{(\mathbf{Au}) \cdot (\mathbf{Au})} = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \|\mathbf{u}\|.$$

   (b) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{Au}, \mathbf{Av})$; and

   Solution: By (a),

   $$d(\mathbf{Au}, \mathbf{Av}) = \|\mathbf{Au} - \mathbf{Av}\| = \|\mathbf{A}(\mathbf{u} - \mathbf{v})\| = \|\mathbf{u} - \mathbf{v}\| = d(\mathbf{u}, \mathbf{v}).$$

   (c) the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to the angle between $\mathbf{Au}$ and $\mathbf{Av}$.

   Solution: By part (a),

   $$\frac{\mathbf{Au} \cdot \mathbf{Av}}{\|\mathbf{Au}\|\|(\mathbf{Av})\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|(\mathbf{v})\|}.$$

   Since cos is one-to-one on $[0, \pi]$,

   $$\cos^{-1}\left(\frac{\mathbf{Au} \cdot \mathbf{Av}}{\|\mathbf{Au}\|\|(\mathbf{Av})\|}\right) = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|(\mathbf{v})\|}\right),$$

   that is, the angle between $\mathbf{Au}$ and $\mathbf{Av}$ is equal to the angle between $\mathbf{u}$ and $\mathbf{v}$.

2. Let $\mathbf{A}$ be an orthogonal matrix of order n. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ and define $T = \{\mathbf{Au_1}, \mathbf{Au_2}, ..., \mathbf{Au_k}\}$.

   (a) If S is orthogonal, show that T is orthogonal.

   Solution: Given any vectors $\mathbf{u}, \mathbf{v}$,

   $$(\mathbf{Au}) \cdot (\mathbf{Av}) = \mathbf{u}^T\mathbf{A}^T\mathbf{Av} = \mathbf{u}^T\mathbf{v} = \mathbf{u} \cdot \mathbf{v},$$

   where the second equality follows from the fact that $\mathbf{A}$ is orthogonal, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. Hence, $(\mathbf{Au}_i) \cdot (\mathbf{Au}_j) = \mathbf{u}_i \cdot \mathbf{u}_j = 0$ for all $i \neq j$.

(b) If S is orthonormal, is T orthonormal?

Solution: Yes. Follows from the argument in (a) that

$$(\mathbf{Au}_i) \cdot (\mathbf{Au}_j) = \mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

3. (a) Suppose $\mathbf{A}$ and $\mathbf{B}$ are similar matrices, that is, $\mathbf{A} = \mathbf{PBP}^{-1}$ for some invertible matrix $\mathbf{P}$. Then the characteristic polynomial of $\mathbf{A}$ and $\mathbf{B}$ are equal.

Solution: Since $\mathbf{A} = \mathbf{PBP}^{-1}$, we have

$$\begin{aligned} \det(x\mathbf{I} - \mathbf{A}) &= \det(\mathbf{P})\det(\mathbf{P})^{-1}\det(x\mathbf{I} - \mathbf{A}) \\ &= \det(\mathbf{P})\det(x\mathbf{I} - \mathbf{A})\det(\mathbf{P}^{-1}) \\ &= \det(\mathbf{P}(x\mathbf{I} - \mathbf{A})\mathbf{P}^{-1}) \\ &= \det(x\mathbf{PIP}^{-1} - \mathbf{PAP}^{-1}) \\ &= \det(x\mathbf{I} - \mathbf{B}). \end{aligned}$$

(b) Suppose the characteristic polynomial of $\mathbf{A}$ and $\mathbf{B}$ are equal. Can we conclude that $\mathbf{A}$ and $\mathbf{B}$ are similar? Justify your answer.

Solution: No. For example $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has the same characteristic polynomial $(x-1)^2$, but for any invertible matrix $\mathbf{P}$,

$$\mathbf{PIP}^{-1} = \mathbf{I} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(c) Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices with the same determinant. Is it true that their characteristic polynomials are equal? Justify your answer.

Solution: No. For example, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ have the same determinant 0, but $\det(x\mathbf{I} - \mathbf{A}) = (x-1)x$ and $\det(x\mathbf{I} - \mathbf{B}) = (x-2)x$.

4. (a) Let $\mathbf{A}$ be a 2×2 matrix. Prove that the characteristic polynomial of $\mathbf{A}$ is

$$x^2 - tr(\mathbf{A})x + \det(\mathbf{A}),$$

where $tr(\mathbf{A})$ is the sum of the diagonal entries of $\mathbf{A}$.

Solution: Write $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have $tr(\mathbf{A}) = a + d$ and $\det(\mathbf{A}) = ad - bc$. The characteristic polynomial is

$$\begin{aligned}
\det(x\mathbf{I} - \mathbf{A}) &= \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = (x-a)(x-d) - bc \\
&= x^2 - (a+d)x + ad - bc \\
&= x^2 - tr(\mathbf{A})x + \det(\mathbf{A}).
\end{aligned}$$

(b) Let $\mathbf{A}$ be a $n \times n$ matrix and $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Prove that $a_0 = (-1)^n \det(\mathbf{A})$.

Solution:

$$a_0 = p(0) = \det(0\mathbf{I} - \mathbf{A}) = \det(-\mathbf{A}) = (-1)^n \det(\mathbf{A}).$$