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- Each tutorial session will typically include

  - revision
  - solution to exercises

  \+ some possible extensions beyond the exercises

- For the revision part, I will try to give you some intuition about

  - why we consider this
  - why we compute that

  ...

- My tutorial notes are available on Canvas.

- Contact me via email

  Any questions or comments are welcome!

Introduction & Revision

Linear Algebra $\longrightarrow$ linear system

linear space

linear operator

$\vdots$

Why linear ?

linear $\xrightarrow{\text{base}}$ nonlinear

e.g. Taylor expansion

Linear system

1×1 $\quad ax = b \quad \xrightarrow{\text{generalization}} \quad m \times n \quad Ax = b$

$a, x, b \in \mathbb{R}$

$$\begin{cases} a_{11} x_1 + \cdots + a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n = b_m \end{cases}$$

solve : if a^{-1} exists $(a \neq 0)$

$x = a^{-1} a x = a^{-1} b$

if a^{-1} doesn't exist $(a = 0)$

$b = 0 \Rightarrow$ infinitely many

$b \neq 0 \Rightarrow$ no solutions

if A^{-1} exists

$x = A^{-1} A x = A^{-1} b$

if A^{-1} doesn't exist

infinitely many / no solutions

depending on REF / RREF

" essentially equivalent

simplest form in solving linear systems "

(LU - decomposition)

REF / RREF

$A = E_1 E_2 \cdots E_n R$

E_i : elementary matrix

(corre. to ele. operations)

Why equivalent ? $\longrightarrow$ E_i invertible.

$Ax = b \iff Rx = E_n^{-1} \cdots E_1^{-1} A x = E_n^{-1} \cdots E_1^{-1} b$

Why simple ? $\longrightarrow$ R has many zero-entries $\quad \begin{pmatrix} & * \\ 0 & \end{pmatrix}$

more zero-entries $\Rightarrow$ simpler

$$\operatorname{diag}(\lambda_1, \cdots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad O_n = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$$

1. (a) Find a linear equation in the variables x and y that has a general solution $x = 1+2t$ and $y = t$ where t is an arbitrary parameter.

(b) Show that $x = t$ and $y = \frac{1}{2}t - \frac{1}{2}$, where t is an arbitrary parameter, is also a general solution for the equation constructed in part (a).

1. (a) Eliminate the parameter t. $\Longrightarrow$ $x - 2y = 1$

denotes for a straight line (in $\mathbb{R}^2$)

so other possible answers will be $c \cdot (x - 2y = 1)$ $c \neq 0$

(b) Check: ① it is a solution

② any solution can be written into this form "general"

2. Find a linear equation in the variables x, y, and z that has a general solution

$$\begin{cases} x = 3 - 4s + t \\ y = s \\ z = t \end{cases} \qquad s, t \in \mathbb{R}\ .$$

2. Similarly as 1(a), eliminating s, t, $c \cdot (x + 4y - z = 3)$ $c \neq 0$

3. Solve the following linear systems.

(a)

$$\begin{cases} 3x_1 + 2x_2 - 4x_3 = 3 \\ 2x_1 + 3x_2 + 3x_3 = 15 \\ 5x_1 - 3x_2 + x_3 = 14 \end{cases}$$

(3a)

$$\left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 2 & 3 & 3 & 15 \\ 5 & -3 & 1 & 14 \end{array}\right) \xrightarrow{\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & 2/3 & -4/3 & 1 \\ 2 & 3 & 3 & 15 \\ 5 & -3 & 1 & 14 \end{array}\right)$$

$$\Big\downarrow \; R_2 - 2R_1 \;\&\; R_3 - 5R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2/3 & -4/3 & 1 \\ 0 & 1 & 17/5 & 39/5 \\ 0 & -19/3 & 23/3 & 9 \end{array}\right) \xleftarrow{\frac{3}{5}R_2} \left(\begin{array}{ccc|c} 1 & 2/3 & -4/3 & 1 \\ 0 & 5/3 & 17/3 & 13 \\ 0 & -19/3 & 23/3 & 9 \end{array}\right)$$

$$\Big\downarrow \; R_3 + \tfrac{19}{3}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2/3 & -4/3 & 1 \\ 0 & 1 & 17/5 & 39/5 \\ 0 & 0 & 438/15 & 876/15 \end{array}\right) \xrightarrow{\frac{15}{438} R_3} \left(\begin{array}{ccc|c} 1 & 2/3 & -4/3 & 1 \\ 0 & 1 & 17/5 & 39/5 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \text{REF}$$

$$\Big\downarrow \; R_2 - 17/5\, R_3 \;\; \& \;\; R_1 + 4/3\, R_3$$

$$\text{RREF} \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xleftarrow{R_1 - 2/3\, R_2} \left(\begin{array}{ccc|c} 1 & 2/3 & 0 & 11/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$\Rightarrow$ solution $x_1 = 3$ $\quad x_2 = 1$ $\quad x_3 = 2$

CHECK: plugging the solution back to equations

(b)

$$\begin{cases} a + b - c - 2d = 0 \\ 2a + b - c + d = -2 \\ -a + b - 3c + d = 4 \end{cases}$$

(3b)

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 2 & 1 & -1 & 1 & -2 \\ -1 & 1 & -3 & 1 & 4 \end{array}\right) \xrightarrow{R_2 - 2R_1 \;\&\; R_3 + R_1} \left(\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & -1 & 1 & 5 & -2 \\ 0 & 2 & -4 & -1 & 4 \end{array}\right)$$

$$\Big\downarrow \; R_3 + 2R_2$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{13}{2} & 0 \\ 0 & -1 & 0 & \frac{19}{2} & 2 \\ 0 & 0 & -2 & 9 & 0 \end{array}\right) \xleftarrow{R_1 - \frac{1}{2} R_3 \;\&\; R_2 + \frac{1}{2} R_3} \left(\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & -1 & 1 & 5 & 2 \\ 0 & 0 & -2 & 9 & 0 \end{array}\right) \quad \text{REF}$$

$$\Big\downarrow \; R_1 + R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & -1 & 0 & \frac{19}{2} & 2 \\ 0 & 0 & -2 & 9 & 0 \end{array}\right) \xrightarrow{-1 \cdot R_2 \;\&\; -\frac{1}{2} \cdot R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -\frac{19}{2} & -2 \\ 0 & 0 & 1 & -\frac{9}{2} & 0 \end{array}\right) \quad \text{RREF}$$

$\uparrow$ assign parameter s

$\Rightarrow$ solution

$d = s$

$c = \frac{9}{2} s$

$b = -2 + \frac{19}{2} s$

$a = 2 - 3s$

(c)

$$\begin{cases} x - 4y + 2z = -2 \\ x + 2y - 2z = -3 \\ x - y = 4 \end{cases}$$

(3c)

$$\left(\begin{array}{ccc|c} 1 & -4 & 2 & -2 \\ 1 & 2 & -2 & -3 \\ 1 & -1 & 0 & 4 \end{array}\right) \xrightarrow[\& R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & -4 & 2 & -2 \\ 0 & 6 & -4 & -1 \\ 0 & 3 & -2 & 6 \end{array}\right)$$

$$\xrightarrow{R_3 - \frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & -4 & 2 & -2 \\ 0 & 6 & -4 & -1 \\ 0 & 0 & 0 & 13/2 \end{array}\right) \text{ REF}$$

$$\xrightarrow{\frac{1}{6}R_2} \left(\begin{array}{ccc|c} 1 & -4 & 2 & -2 \\ 0 & 1 & -2/3 & -1/6 \\ 0 & 0 & 0 & 13/2 \end{array}\right)$$

Inconsistent! No solutions

Remark: What we're doing to the matrix is "dig-holes"

→ make the "zero"s in coefficient matrix as many as possible.

This trick will also useful in matrix theory.

4. Determine the values of a and b so that the linear system

$$\begin{cases} ax + bz = 2 \\ ax + ay + 4z = 4 \\ ay + 2z = b \end{cases}$$

(a) has no solution;

(b) has only one solution;

(c) has infinitely many solutions and a general solution has one arbitrary parameter;

(d) has infinitely many solutions and a general solution has two arbitrary parameters.

4.

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right)$$

Case 1: $b = 2$

$$\left(\begin{array}{ccc|c} a & 0 & 2 & 2 \\ 0 & a & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

subcase 1.1: $a = 0$ $\xrightarrow[\& \frac{1}{2} \cdot R_1]{R_2 - R_1}$ $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ RREF

solution: $x_3 = 1$ $x_1 = s$ $x_2 = t$ $s, t \in \mathbb{R}$

subcase 1.2 : $a \neq 0$ $\xrightarrow[\& \frac{1}{a} \cdot R_2]{\frac{1}{a} \cdot R_1}$ $\left(\begin{array}{ccc|c} 1 & 0 & 2/a & 2/a \\ 0 & 1 & 2/a & 2/a \\ 0 & 0 & 0 & 0 \end{array}\right)$ RREF

solution : $x_3 = s$ $x_2 = \frac{2}{a}(1-s)$ $x_1 = \frac{2}{a}(1-s)$ $s \in \mathbb{R}$

Case 2 : $b \neq 2$ $\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right) \xrightarrow{\frac{1}{b-2} \cdot R_3} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$

$\downarrow$ $R_1 - b \cdot R_3$, $R_2 + (b-4) \cdot R_3$

$\left(\begin{array}{ccc|c} a & 0 & 0 & 2-b \\ 0 & a & 0 & b-2 \\ 0 & 0 & 1 & 1 \end{array}\right)$

subcase 2.1 : $a \neq 0$

$\xrightarrow[\& \frac{1}{a} \cdot R_2]{\frac{1}{a} \cdot R_1}$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2-b}{a} \\ 0 & 1 & 0 & \frac{b-2}{a} \\ 0 & 0 & 1 & 1 \end{array}\right)$

solution : $x_1 = \frac{2-b}{a}$ $x_2 = \frac{b-2}{a}$ $x_3 = 1$

subcase 2.2 : $a = 0$ $\left(\begin{array}{ccc|c} \boxed{0 \quad 0 \quad 0} & & & \boxed{2-b} \\ 0 & 0 & 0 & b-2 \\ 0 & 0 & 1 & 1 \end{array}\right)$

note that $b-2 \neq 0$ zero non-zero no solution

In conclusion, (a) $a=0$ and $b \neq 2$

(b) $a \neq 0$ and $b \neq 2$

(c) $a \neq 0$ and $b = 2$

(d) $a = 0$ and $b = 2$

5. (a) Does an inconsistent linear system with more unknowns than equations exist?

5. (a) Yes. Inconsistent $\longrightarrow$ no solution.

Sufficient to assign "contradictory" equations

for example $\begin{cases} x + y + z = 0 \\ x + y + z = 1 \end{cases}$

(b) Does a linear system which has only one solution, but more equations than unknowns, exist?

(c) Does a linear system which has only one solution, but more unknowns than equations, exists?

(d) Does a linear system which has infinitely many solutions, but more equations than unknowns, exists?

(b) Yes. Have a solution but # unknowns < # equations (the number of)

⇝ some of the equations are useless

for example $\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$ useless

(c) No # unknowns > # equations ⇒ if have a solution, then never be unique

(d) Yes. Similarly as (b), firstly construct a linear system with infinitely many solutions, then add "useless" equations.

for example $\begin{cases} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 = 4 \\ 3x_1 + 3x_2 = 6 \end{cases}$

6. Solve the following system of non-linear equations:

$$\begin{array}{rcrcrcr} x^2 & - & y^2 & + & 2z^2 & = & 6 \\ 2x^2 & + & 2y^2 & - & 5z^2 & = & 3 \\ 2x^2 & + & 5y^2 & + & z^2 & = & 9 \end{array}$$

6. View x^2, y^2, z^2 as unknowns. Let $u = x^2$, $v = y^2$, $w = z^2$

Now become the linear system of u, v, w. It is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 2 & 2 & -5 & 3 \\ 2 & 5 & 1 & 9 \end{array}\right) \xrightarrow[\& R_3 - 2R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -3 & -3 \end{array}\right)$$

$$\Big\downarrow R_3 - \tfrac{7}{4} R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 1 & -\frac{9}{4} & -\frac{9}{4} \\ 0 & 0 & 1 & 1 \end{array}\right) \xleftarrow[\frac{4}{51} \cdot R_3]{\frac{1}{4} \cdot R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & \frac{51}{4} & \frac{51}{4} \end{array}\right) \quad \text{REF}$$

$$\Big\downarrow \begin{array}{l} R_1 - 2R_3 \\ \& \ R_2 + \frac{9}{4} R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$ RREF

solution $u = 4 \quad v = 0 \quad w = 1$

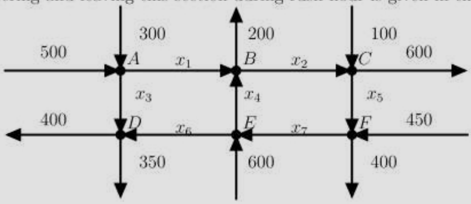
i.e. $x^2 = 4 \quad y^2 = 0 \quad z^2 = 1$

or equivalently $|x| = 2 \quad y = 0 \quad |z| = 1$

or equivalently $(x, y, z) = (2, 0, 1)$ or $(2, 0, -1)$

or $(-2, 0, 1)$ or $(-2, 0, -1)$

A network of one-way streets of a downtown section can be represented by the diagram below, with traffic flowing in the direction indicated. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram.



(a) Do we have enough information to find the traffic volumes x_1, x_2, x_3, x_4, x_5, x_6, and x_7?

(b) Suppose $x_6 = 50$ and $x_7 = 100$. What is x_1, x_2, x_3, x_4, and x_5?

(c) Can the road between junction A and B be closed for construction while still keeping the traffic flowing in the same directions on the other streets? Explain.

We have the following linear system :

A : $x_1 + x_3 = 300 + 500$

B : $x_1 + x_4 = x_2 + 200$

C : $x_2 + 100 = x_5 + 600$

D : $x_3 + x_6 = 400 + 350$

E : $x_4 + x_6 = x_7 + 600$

F : $x_5 + 450 = x_7 + 400$

(a) # unknowns > # equations

$\longrightarrow$ the solution is not unique.

(b) Matrix

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 800 \\ 1 & -1 & 0 & 1 & 0 & 200 \\ 0 & 1 & 0 & 0 & -1 & 500 \\ 0 & 0 & 1 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & 650 \\ 0 & 0 & 0 & 0 & 1 & 50 \end{array}\right)$$

$\Big\downarrow$ RREF

Solution

$x_1 = 100$

$x_2 = 550$

$x_3 = 700$

$x_4 = 650$

$x_5 = 50$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & 0 & 0 & 0 & 550 \\ 0 & 0 & 1 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & 650 \\ 0 & 0 & 0 & 0 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(c) the road connecting A & B closed $\Rightarrow$ $x_1 = 0$

Then the matrix of the linear system (with unknowns $x_2, \cdots, x_7$)

$$\left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 800 \\ -1 & 0 & 1 & 0 & 0 & 0 & 200 \\ 1 & 0 & 0 & -1 & 0 & 0 & 500 \\ 0 & 1 & 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 1 & 0 & 1 & -1 & 600 \\ 0 & 0 & 0 & -1 & 0 & 1 & 50 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 450 \\ 0 & 1 & 0 & 0 & 0 & 0 & 800 \\ 0 & 0 & 1 & 0 & 0 & -1 & 650 \\ 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 1 & 0 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

negative

$\longrightarrow$ contradiction