NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 7**

1. (a) Let $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ be a linear equation. Express this linear system as $\mathbf{a} \cdot \mathbf{x} = b$ for some (column) vectors $\mathbf{a}$ and $\mathbf{x}$.

Solution: $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

(b) Find the solution set of the linear system

$$\begin{array}{rcrcrcrl} x_1 & + & 3x_2 & - & 2x_3 & & & = 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & = 0 \\ & & & + & 5x_3 & + & 10x_4 & = 0 \end{array}$$

Solution: The RREF of the matrix coefficient is

$$\begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the solution set is

$$\left\{ s\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}.$$

(c) Find a nonzero vector $\mathbf{v} \in \mathbb{R}^4$ such that $\mathbf{a}_1 \cdot \mathbf{v} = 0$, $\mathbf{a}_2 \cdot \mathbf{v} = 0$, and $\mathbf{a}_3 \cdot \mathbf{v} = 0$, where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 6 \\ -5 \\ -2 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 10 \end{pmatrix}.$$

Solution: Write $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$. Then $\mathbf{a}_1 \cdot \mathbf{v} = 0$, $\mathbf{a}_2 \cdot \mathbf{v} = 0$, and $\mathbf{a}_3 \cdot \mathbf{v} = 0$ is equivalent to solving the following linear system

$$\left\{ \begin{array}{rcrcrcrl} v_1 & + & 3v_2 & - & 2v_3 & & & = 0 \\ 2v_1 & + & 6v_2 & - & 5v_3 & - & 2v_4 & = 0 \\ & & & + & 5v_3 & + & 10v_4 & = 0 \end{array} \right.$$

From (b), we may choose $s = 1$ and $t = 0$, that is, $\mathbf{v} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

This exercise demonstrates the fact that if $\mathbf{A}$ is a $m \times n$ matrix, then the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ consist of all the vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of $\mathbf{A}$.

2. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set. Suppose

$$\mathbf{x} = \mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3 \quad \text{and} \quad \mathbf{y} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Determine the value for each of the following

(a) $\mathbf{x} \cdot \mathbf{y}$.

Solution: Note that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. Furthermore, since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set, $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= (\mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3) \cdot (2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3) \\ &= 2(\mathbf{v}_1 \cdot \mathbf{v}_1) + 6(\mathbf{v}_2 \cdot \mathbf{v}_2) - 2(\mathbf{v}_3 \cdot \mathbf{v}_3) \\ &= 2 + 6 - 2 = 6. \end{aligned}$$

(b) $||\mathbf{x}||$ and $||\mathbf{y}||$.

Solution:

$$\begin{aligned} ||\mathbf{x}|| &= \sqrt{\mathbf{x} \cdot \mathbf{x}} \\ &= \sqrt{(\mathbf{v}_1 \cdot \mathbf{v}_1) + 4(\mathbf{v}_2 \cdot \mathbf{v}_2) + 4(\mathbf{v}_3 \cdot \mathbf{v}_3)} \\ &= \sqrt{1 + 4 + 4} = 3 \\ ||\mathbf{y}|| &= \sqrt{\mathbf{y} \cdot \mathbf{y}} \\ &= \sqrt{4(\mathbf{v}_1 \cdot \mathbf{v}_1) + 9(\mathbf{v}_2 \cdot \mathbf{v}_2) + (\mathbf{v}_3 \cdot \mathbf{v}_3)} \\ &= \sqrt{4 + 9 + 1} = \sqrt{14} \end{aligned}$$

(c) The angle θ between $\mathbf{x}$ and $\mathbf{y}$.

Solution: We note that $0 \leq \theta \leq 180°$.

$$\cos(\theta) = \frac{6}{3\sqrt{14}} \Rightarrow \theta = \cos^{-1} \frac{2}{\sqrt{14}} = 57.69°.$$

3. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$.

(a) Compute $\mathbf{v}_1 \cdot \mathbf{v}_1$, $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_2 \cdot \mathbf{v}_1$ and $\mathbf{v}_2 \cdot \mathbf{v}_2$.

Solution:

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_1 &= 1+4+1 = 6, & \mathbf{v}_1 \cdot \mathbf{v}_2 &= 1+0-1 = 0, \\ \mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{v}_1 \cdot \mathbf{v}_2 &= 0, & \mathbf{v}_2 \cdot \mathbf{v}_2 &= 1+0+1 = 2. \end{aligned}$$

(b) Compute $\mathbf{V}^T\mathbf{V}$. What does the entries of $\mathbf{V}^T\mathbf{V}$ represent?

Solution: $\mathbf{V}^T\mathbf{V} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$.

Since $\mathbf{V} = (\mathbf{v}_1\ \mathbf{v}_2)$, $\mathbf{V}^T = \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{pmatrix}$. Hence

$$\mathbf{V}^T\mathbf{V} = \begin{pmatrix} \mathbf{v}_1^T\mathbf{v}_1 & \mathbf{v}_1^T\mathbf{v}_2 \\ \mathbf{v}_2^T\mathbf{v}_1 & \mathbf{v}_2^T\mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{pmatrix}.$$

The (i,j)-entry of $\mathbf{V}^T\mathbf{V}$ is $\mathbf{v}_i \cdot \mathbf{v}_j$.

4. Let W be a subspace of $\mathbb{R}^n$. The *orthogonal complement* of W, denoted as $W^\perp$, is defined to be

$$W^\perp := \{\ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\ \}.$$

Let $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}$, and $\mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, and $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

(a) Show that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

Solution: Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & -1 \\ 1 & 0 & 0 \end{pmatrix}$. We compute

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

which shows that S is an orthogonal set of nonzero vectors.
An orthogonal set of nonzero vectors is linearly independent.
Therefore S is linear independent.

(b) Show that S is orthogonal.

Solution: Shown in (a).

(c) Show that $W^{\perp}$ is a subspace of $\mathbb{R}^5$ by showing that it is a span of a set. What is the dimension? (**Hint**: See Question 1.)

Solution: By Question 1, $W^{\perp}$ is the nullspace of $\mathbf{A}^T$. The fact that $W^{\perp}$ is a nullspace of some matrix proves that it is a subspace. Now we compute the nullspace.

$$\mathbf{A}^T \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 & -1/4 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 2 & 3/4 \end{pmatrix}$$

and the nullspace of $\mathbf{A}^T$ is spanned by $\left\{ \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -3 \\ 0 \\ 4 \end{pmatrix} \right\}$. This shows that $W^{\perp}$ is a subspace of $\mathbb{R}^5$ of dimension 2.

(d) Obtain an orthonormal set T by normalizing $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

Solution: From (b) $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal set.
From (a), $||\mathbf{w}_1||^2 = 5$, $||\mathbf{w}_2||^2 = 10$, and $||\mathbf{w}_3||^2 = 4$. Therefore

$$T = \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

(e) Let $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. Find the projection of $\mathbf{v}$ onto W.

Solution: The projection is

$$\frac{\mathbf{v} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{v} \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 + \frac{\mathbf{v} \cdot \mathbf{w}_3}{\mathbf{w}_3 \cdot \mathbf{w}_3}\mathbf{w}_3 = \frac{1}{10} \begin{pmatrix} 10 \\ -1 \\ 12 \\ 3 \\ 6 \end{pmatrix}.$$

(f) Let $\mathbf{v}_W$ be the projection of $\mathbf{v}$ onto W. Show that $\mathbf{v} - \mathbf{v}_W$ is in $W^\perp$.

Solution:

$$\mathbf{v} - \mathbf{v}_W = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{10}\begin{pmatrix} 10 \\ -1 \\ 12 \\ 3 \\ 6 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 10 \\ 1 \\ -2 \\ 7 \\ -16 \end{pmatrix}$$

$$\mathbf{A}^T(\mathbf{v} - \mathbf{v}_W) = \frac{1}{10}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 10 \\ 1 \\ -2 \\ 7 \\ -16 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This shows that $(\mathbf{v} - \mathbf{v}_W)$ is in the nullspace of $\mathbf{A}^T$, which is $W^\perp$.

This exercise demonstrated the fact that every vector $\mathbf{v}$ in $\mathbb{R}^5$ can be written as $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_W^\perp$, for some $\mathbf{v}_W$ in W and $\mathbf{v}_W^\perp$ in $W^\perp$. In other words, $W + W^\perp = \mathbb{R}^5$.

5. Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{u_4} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Check that S is an orthogonal basis for $\mathbb{R}^4$.

Solution: Let $\mathbf{U} = \begin{pmatrix} 1 & 1 & -1 & -2 \\ 2 & 1 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & -1 & 2 \end{pmatrix}$. Then

$$\mathbf{U}^T\mathbf{U} = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$

Hence, S is an orthogonal set. Since it is an orthogonal set of nonzero vectors, it is linearly independent. Moreover it contains 4 vectors so it is a basis of $\mathbb{R}^4$. Alternatively, since the product $\mathbf{U}^T\mathbf{U}$ is invertible, $\mathbf{U}$ is invertible. Hence the columns form a basis for $\mathbb{R}^4$.

(b) Is it possible to find a nonzero vector $\mathbf{w}$ in $\mathbb{R}^4$ such that $S \cup \{\mathbf{w}\}$ is an orthogonal set?

Solution: No. This is because if $\mathbf{w}$ were to exist, then $S \cup \{\mathbf{w}\}$ would be a linearly independent set in $\mathbb{R}^4$ containing 5 vectors. This is a contradiction.

Alternatively, from Tutorial 4 Question 6, $\mathbf{w}$ must be in the nullspace of U. However, U is invertible so its nullspace is the trivial subspace and $\mathbf{w} = \mathbf{0}$. This shows that there can be no nonzero vector that is orthogonal to the set S.

(c) Obtain an orthonormal set T by normalizing $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$.

Solution: From (a) we know that $||\mathbf{u}_1||^2 = 10$, $||\mathbf{u}_2||^2 = 4$, $||\mathbf{u}_3||^2 = 4$, and $||\mathbf{u}_4||^2 = 10$. Therefore

$$T = \left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

(d) Let $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$. Find $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

Solution: We have

$$\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{v} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 + \frac{\mathbf{v} \cdot \mathbf{u}_4}{\mathbf{u}_4 \cdot \mathbf{u}_4}\mathbf{u}_4,$$

which means that

$$[\mathbf{v}]_S = \begin{pmatrix} \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \\ \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \\ \frac{\mathbf{v} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} \\ \frac{\mathbf{v} \cdot \mathbf{u}_4}{\mathbf{u}_4 \cdot \mathbf{u}_4} \end{pmatrix} = \begin{pmatrix} 3/10 \\ 1/2 \\ -1 \\ 9/10 \end{pmatrix}.$$

Let

$$\mathbf{u}_1' = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \mathbf{u}_2' = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{u}_3' = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{u}_4' = \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

Then

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1')\mathbf{u}_1' + (\mathbf{v} \cdot \mathbf{u}_2')\mathbf{u}_2' + (\mathbf{v} \cdot \mathbf{u}_3')\mathbf{u}_3' + (\mathbf{v} \cdot \mathbf{u}_4')\mathbf{u}_4',$$

which means that

$$[\mathbf{v}]_T = \begin{pmatrix} \mathbf{v} \cdot \mathbf{u}_1' \\ \mathbf{v} \cdot \mathbf{u}_2' \\ \mathbf{v} \cdot \mathbf{u}_3' \\ \mathbf{v} \cdot \mathbf{u}_4' \end{pmatrix} = \begin{pmatrix} 3/\sqrt{10} \\ 1 \\ -2 \\ 9/\sqrt{10} \end{pmatrix}.$$

(e) Suppose $\mathbf{w}$ is a vector in $\mathbb{R}^4$ such that $[\mathbf{w}]_S = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Find $[\mathbf{w}]_T$.

Solution: Note that $\mathbf{u}'_i = \frac{\mathbf{u}_i}{||\mathbf{u}_i||}$, and so

$$\begin{aligned}
\mathbf{w} &= \frac{\mathbf{w} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{w} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{w} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 + \frac{\mathbf{w} \cdot \mathbf{u}_4}{\mathbf{u}_4 \cdot \mathbf{u}_4}\mathbf{u}_4 \\
&= \left(\frac{\mathbf{w} \cdot \mathbf{u}_1}{||\mathbf{u}_1||}\right)\frac{\mathbf{u}_1}{||\mathbf{u}_1||} + \left(\frac{\mathbf{w} \cdot \mathbf{u}_2}{||\mathbf{u}_2||}\right)\frac{\mathbf{u}_2}{||\mathbf{u}_2||} + \left(\frac{\mathbf{w} \cdot \mathbf{u}_3}{||\mathbf{u}_3||}\right)\frac{\mathbf{u}_3}{||\mathbf{u}_3||} + \left(\frac{\mathbf{w} \cdot \mathbf{u}_4}{||\mathbf{u}_4||}\right)\frac{\mathbf{u}_4}{||\mathbf{u}_4||} \\
&= \left(\frac{\mathbf{w} \cdot \mathbf{u}_1}{||\mathbf{u}_1||}\right)\mathbf{u}'_1 + \left(\frac{\mathbf{w} \cdot \mathbf{u}_2}{||\mathbf{u}_2||}\right)\mathbf{u}'_2 + \left(\frac{\mathbf{w} \cdot \mathbf{u}_3}{||\mathbf{u}_3||}\right)\mathbf{u}'_3 + \left(\frac{\mathbf{w} \cdot \mathbf{u}_4}{||\mathbf{u}_4||}\right)\mathbf{u}'_4
\end{aligned}$$

Let $[\mathbf{w}]_S(i)$ and $[\mathbf{w}]_T(i)$ denote the i-th coordinate of $[\mathbf{w}]_S$ and $[\mathbf{w}]_T$, respectively. Then, we have

$$[\mathbf{w}]_S(i) = \frac{\mathbf{w} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} = \frac{\mathbf{w} \cdot \mathbf{u}_i}{||\mathbf{u}_i||^2} = \frac{1}{||\mathbf{u}_i||}\frac{\mathbf{w} \cdot \mathbf{u}_i}{||\mathbf{u}_i||} = \frac{1}{||\mathbf{u}_i||}[\mathbf{w}]_T(i).$$

And so $[\mathbf{w}]_T = \begin{pmatrix} \sqrt{10} \\ 4 \\ 2 \\ \sqrt{10} \end{pmatrix}$.

Extra problems

1. Let $\mathbf{A}$ be an $m \times n$ matrix.

   (a) Show that the nullspace of $\mathbf{A}$ is equal to the nullspace of $\mathbf{A}^T\mathbf{A}$.

   Solution: We will prove the equality by showing that $\text{Null}(\mathbf{A}) \subseteq \text{Null}(\mathbf{A}^T\mathbf{A})$ and $\text{Null}(\mathbf{A}^T\mathbf{A}) \subseteq \text{Null}(\mathbf{A})$.

   Suppose $\mathbf{u}$ is in the nullspace of $\mathbf{A}$, that is, $\mathbf{Au} = \mathbf{0}$. By premultiplying both sides by $\mathbf{A}^T$, we get
   $$\mathbf{A}^T\mathbf{A} = \mathbf{A}^T\mathbf{0} = \mathbf{0}.$$
   Hence $\mathbf{u}$ is in the nullspace of $\mathbf{A}^T\mathbf{A}$ too. This shows that $\text{Null}(\mathbf{A}) \subseteq \text{Null}(\mathbf{A}^T\mathbf{A})$.

   Conversely, suppose $\mathbf{v}$ is in the nullspace of $\mathbf{A}^T\mathbf{A}$, that is, $\mathbf{A}^T\mathbf{Av} = \mathbf{0}$. Write $\mathbf{Av} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$. By premultiplying both sides by $\mathbf{v}^T$, we get
   $$\begin{aligned} 0 &= \mathbf{v}^T\mathbf{0} = \mathbf{v}^T\mathbf{A}^T\mathbf{Av} = (\mathbf{Av})^T(\mathbf{Av}) \\ &= b_1^2 + b_2^2 + \cdots + b_m^2. \end{aligned}$$
   This implies $b_i = 0$ for $i = 1, 2, ..., m$, and hence, $\mathbf{Av} = \mathbf{0}$, that is, $\mathbf{v}$ is in the nullspace of $\mathbf{A}$. This shows that $\text{Null}(\mathbf{A}^T\mathbf{A}) \subseteq \text{Null}(\mathbf{A})$.

   (b) Show that $\text{nullity}(\mathbf{A}) = \text{nullity}(\mathbf{A}^T\mathbf{A})$ and $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T\mathbf{A})$.

   Solution: By (a),
   $$\text{nullity}(\mathbf{A}) = \dim(\text{Null}(\mathbf{A})) = \dim(\text{Null}(\mathbf{A}^T\mathbf{A})) = \text{nullity}(\mathbf{A}^T\mathbf{A})$$
   Now $\mathbf{A}^T\mathbf{A}$ is a $n \times n$ matrix. By the dimension theorem,
   $$\text{rank}(\mathbf{A}^T\mathbf{A}) = n - \text{nullity}(\mathbf{A}^T\mathbf{A}) = n - \text{nullity}(\mathbf{A}) = \text{rank}(\mathbf{A}).$$

   (c) Is it true that $\text{nullity}(\mathbf{A}) = \text{nullity}(\mathbf{AA}^T)$? Justify your answer.

   Solution: No. For example, let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

   (d) Is it true that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{AA}^T)$? Justify your answer.

   Solution: Yes. Replace $\mathbf{A}$ with $\mathbf{A}^T$ in (b), we have
   $$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) = \text{rank}((\mathbf{A}^T)^T\mathbf{A}^T) = \text{rank}(\mathbf{AA}^T).$$

2. Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices of the same size. Show that

$$\operatorname{rank}(\mathbf{A}+\mathbf{B}) \leq \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}).$$

Solution: Write $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$, where $\mathbf{a}_i$ and $\mathbf{b}_i$ are the i-th column of $\mathbf{A}$ and $\mathbf{B}$, respectively, for $i = 1, \ldots, n$. Then

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} \mathbf{a}_1+\mathbf{b}_1 & \mathbf{a}_2+\mathbf{b}_2 & \cdots & \mathbf{a}_n+\mathbf{b}_n \end{pmatrix}.$$

Now for each $i = 1, 2, ..., n$, $\mathbf{a}_i + \mathbf{b}_i \in \operatorname{Col}(\mathbf{A}) + \operatorname{Col}(\mathbf{B})$ since $\operatorname{Col}(\mathbf{A}) + \operatorname{Col}(\mathbf{B}) = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n, \mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$. Hence,

$$\begin{aligned} \operatorname{Col}(\mathbf{A}+\mathbf{B}) &= \operatorname{span}\{\mathbf{a}_1+\mathbf{b}_1, \mathbf{a}_2+\mathbf{b}_2, ..., \mathbf{a}_n+\mathbf{b}_n\} \\ &\subseteq \operatorname{Col}(\mathbf{A}) + \operatorname{Col}(\mathbf{B}) \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{rank}(\mathbf{A}+\mathbf{B}) &= \dim(\operatorname{Col}(\mathbf{A}+\mathbf{B})) \\ &\leq \dim(\operatorname{Col}(\mathbf{A})) + \dim(\operatorname{Col}(\mathbf{B})) - \dim(\operatorname{Col}(\mathbf{A}) \cap \operatorname{Col}(\mathbf{B})) \\ &\leq \dim(\operatorname{Col}(\mathbf{A})) + \dim(\operatorname{Col}(\mathbf{B})) \\ &= \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}). \end{aligned}$$

3. (a) Let W be a subspace of $\mathbb{R}^n$. Prove that the orthogonal complement of the orthogonal complement of W is W, i.e.

$$(W^{\perp})^{\perp} = W.$$

Solution: Let $\mathbf{w}$ be a vector in W. Then by definition of the orthogonal complement $W^{\perp}$, every vector $\mathbf{v} \in W^{\perp}$ is orthogonal to $\mathbf{w}$, $\mathbf{w} \cdot \mathbf{v} = 0$. Hence, $\mathbf{w} \perp W^{\perp}$. This shows that $\mathbf{w}$ is in the orthogonal complement of $W^{\perp}$, that is, $W \subseteq (W^{\perp})^{\perp}$.

Now suppose $\mathbf{w}$ is a vector in $(W^{\perp})^{\perp}$. Write

$$\mathbf{w} = \mathbf{w}_p + \mathbf{w}_n,$$

where $\mathbf{w}_p$ is the projection of $\mathbf{w}$ onto W and $\mathbf{w}_n \perp W$; that is $\mathbf{w}_n = \mathbf{w} - \mathbf{w}_p \in W^{\perp}$. By assume, $\mathbf{w}$ is orthogonal to $W^{\perp}$, that is, $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in W^{\perp}$. In particular, $\mathbf{w} \cdot \mathbf{w}_n = 0$. So,

$$0 = \mathbf{w}_n \cdot (\mathbf{w}_p + \mathbf{w}_n) = \mathbf{w}_n \cdot \mathbf{w}_p + \mathbf{w}_n \cdot \mathbf{w}_n = \mathbf{w}_n \cdot \mathbf{w}_n,$$

where the last equality follows from the fact that $\mathbf{w}_n \cdot \mathbf{w}_p = 0$. This shows that $\mathbf{w}_n = \mathbf{0}$, and hence $\mathbf{w} = \mathbf{w}_p$, that is, $\mathbf{w} \in W$. This shows that $(W^{\perp})^{\perp} \subseteq W$.

Alternative solution to the second half of the proof.
Suppose $\mathbf{w}$ is a vector in $(W^{\perp})^{\perp}$. We write

$$\mathbf{w} = \mathbf{w}_p + \mathbf{w}_n,$$

where $\mathbf{w}_p$ is the projection of $\mathbf{w}$ onto W and $\mathbf{w}_n \perp W$. Now $\mathbf{w}_p$ lies in $W \subseteq (W^{\perp})^{\perp}$. Since $(W^{\perp})^{\perp}$ is a subspace, $\mathbf{w}_n = \mathbf{w} - \mathbf{w}_p \in (W^{\perp})^{\perp}$. Hence $\mathbf{w}_n \in W^{\perp} \cap (W^{\perp})^{\perp}$. The intersection $W^{\perp} \cap (W^{\perp})^{\perp}$ is the zero subspace so $\mathbf{w}_n = \mathbf{0}$ and $\mathbf{w} = \mathbf{w}_p \in W$. This shows that $(W^{\perp})^{\perp} \subseteq W$.

(b) Show that for any matrix $\mathbf{A}$, the column space of $\mathbf{A}$ is the orthogonal complement of the nullspace of $\mathbf{A}^T$,

$$\text{Col}(\mathbf{A})^{\perp} = \text{Null}(\mathbf{A}^T),$$

or equivalently, the row space of $\mathbf{A}$ is the orthogonal complement of the nullspace of $\mathbf{A}$,

$$\text{Row}(\mathbf{A})^{\perp} = \text{Null}(\mathbf{A}).$$

Solution: Write $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$, where $\mathbf{a}_i$ is the i-th column of $\mathbf{A}$, for $i = 1, ..., n$. Then $\text{Col}(\mathbf{A}) = \text{span}\{\mathbf{a}_1, ..., \mathbf{a}_n\}$. Hence, the result follows from

$$\begin{aligned}
\mathbf{w} \in \text{Col}(\mathbf{A})^{\perp} &\Leftrightarrow \mathbf{a}_i \cdot \mathbf{w} = 0 \quad \text{for all } i = 1, ..., n \\
&\Leftrightarrow \mathbf{a}_i^T \mathbf{w} = 0 \quad \text{for all } i = 1, ..., n \\
&\Leftrightarrow \begin{pmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{pmatrix} \mathbf{w} = \mathbf{0} \quad \Leftrightarrow \quad \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}^T \mathbf{w} = \mathbf{0} \\
&\Leftrightarrow \mathbf{A}^T \mathbf{w} = \mathbf{0} \\
&\Leftrightarrow \mathbf{w} \in \text{Null}(\mathbf{A}^T).
\end{aligned}$$