MA1522 Linear Algebra for Computing

Lecture 10: Applications (of Orthogonality) and Eigenvalues

Yang Yue

Department of Mathematics
National University of Singapore

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Outline

Exercise One in Section 5.4

1. Prove that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_n$.

2. Prove that the diagonal entries of $\mathbf{R}$ are positive, $r_{ii} > 0$ for all $i = 1, ..., n$.

3. Prove that the upper triangular matrix
$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{pmatrix}$$ is invertible.

(Exercises are part of proofs that Dr. Teo skipped.)

Slide 57: QR Factorization

Theorem (QR Factorization)

Suppose $\mathbf{A}$ *is an* $m \times n$ *matrix with linearly independent columns. Then* $\mathbf{A}$ *can be written as*

$$\mathbf{A} = \mathbf{QR}$$

for some $m \times n$ *matrix* $\mathbf{Q}$ *such that* $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_n$ *and invertible upper triangular matrix* $\mathbf{R}$ *with positive diagonal entries.*

Definition

The decomposition given in the theorem above is called a *QR factorization* of $\mathbf{A}$.

Where do $\mathbf{Q}$ and $\mathbf{R}$ come from?

Suppose $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ whose columns are linearly independent. Applying the Gram-Schmidt process on the columns, we obtain a matrix $\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{pmatrix}$ whose columns are orthonormal.

By Gram-Schmidt, $\text{span}\{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_i\} = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_i\}$. Thus,

$$\begin{aligned} \mathbf{a}_i &= r_{1i}\mathbf{q}_1 + r_{2i}\mathbf{q}_2 + \cdots + r_{ii}\mathbf{q}_i + 0\mathbf{q}_{i+1} + \cdots + 0\mathbf{q}_n \\ &= \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_i & \cdots & \mathbf{q}_n \end{pmatrix} \begin{pmatrix} r_{1i} \\ \vdots \\ r_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned}$$

Where do **Q** and **R** come from? (conti.)

Putting things together,

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{pmatrix} \\
&= \mathbf{QR}
\end{aligned}
$$

for some $m \times n$ matrix **Q** with orthonormal columns, and an upper triangular $n \times n$ matrix **R**.

Algorithm to QR Factorization

Let $\mathbf{A}$ be an $m \times n$ matrix with linearly independent columns.

1. Perform Gram-Schmidt on the columns of $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ to obtain an orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n\}$.
2. Set $\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{pmatrix}$.
3. Compute $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$.

Note, item 3 gave us an alternative way to calculate $\mathbf{R}$.

Answer to Exercise One in Section 5.4 (part 1)

1. Prove that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_n$.

Proof. Since $\mathbf{Q}$ is an $m \times n$ matrix, $\mathbf{Q}^T$ is $n \times m$. Thus $\mathbf{Q}^T\mathbf{Q}$ is $n \times n$. By block multiplication,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{pmatrix} = \begin{pmatrix} \mathbf{q}_i^T\mathbf{q}_j \end{pmatrix}$$

Observe that $\mathbf{q}_i^T\mathbf{q}_j$ (as matrix multiplication) is equal to the dot product $\mathbf{q}_i \cdot \mathbf{q}_j$, and $\{\mathbf{q}_1, \cdots, \mathbf{q}_n\}$ is orthonormal, we then have

$$\begin{pmatrix} \mathbf{q}_i^T\mathbf{q}_j \end{pmatrix} = \mathbf{I}_n.$$

Answer to Exercise One in Section 5.4 (part 2)

2. Prove that the diagonal entries of $\mathbf{R}$ are positive, $r_{ii} > 0$ for all $i = 1, ..., n$.

Proof. By earlier slides, we have, for each $i \leq n$,

$$\mathbf{a}_i = r_{1i}\mathbf{q}_1 + r_{2i}\mathbf{q}_2 + \cdots + r_{ii}\mathbf{q}_i. \tag{1}$$

Using $\mathbf{q}_i$ to dot multiply both sides of (1), we have

$$r_{ii} = \mathbf{a}_i \cdot \mathbf{q}_i.$$

Now back to Gram-Schmidt, we have

$$\mathbf{v}_i = \mathbf{a}_i - \left(\frac{\mathbf{v}_1 \cdot \mathbf{a}_i}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{a}_i}{\|\mathbf{v}_2\|^2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{v}_{i-1} \cdot \mathbf{a}_i}{\|\mathbf{v}_{i-1}\|^2}\right)\mathbf{v}_{i-1}. \tag{2}$$

and $\mathbf{q}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$, using $\mathbf{q}_i$ to dot multiply both sides of (2), we have

$$\mathbf{a}_i \cdot \mathbf{q}_i = \mathbf{v}_i \cdot \mathbf{q}_i = \|\mathbf{v}_i\| > 0.$$

Consequently, $r_{ii} > 0$.

Answer to Exercise One in Section 5.4 (part 3)

3. Prove that the upper triangular matrix

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{pmatrix}$$

is invertible.

Proof. It follows from

$$\det(\mathbf{R}) = r_{11} \ldots r_{nn} > 0.$$

Exercise Two in Section 5.4

Use QR factorization to prove the following

Corollary

Suppose $\mathbf{A}$ *is an* $m \times n$ *matrix with linearly independent columns, i.e.* $\operatorname{rank}(\mathbf{A}) = n$. *Then* $\mathbf{A}^T\mathbf{A}$ *is invertible, and* $\mathbf{A}$ *has a left inverse; that is, there is a* $\mathbf{B}$ *such that*

$$\mathbf{BA} = \mathbf{I}_n.$$

Answer to Exercise Two in Section 5.4

Use QR factorization to prove that if $\mathbf{A}$ is an $m \times n$ matrix with linearly independent columns, then $\mathbf{A}^T\mathbf{A}$ is invertible, and $\mathbf{A}$ has a left inverse; that is, there is a $\mathbf{B}$ such that

$$\mathbf{BA} = \mathbf{I}_n.$$

Proof. Since $\mathbf{A}$ has independent columns, $\mathbf{A}$ has a QR-decomposition $\mathbf{A} = \mathbf{QR}$. Then

$$\mathbf{A}^T\mathbf{A} = \mathbf{R}^T\mathbf{Q}^T\mathbf{QR} = \mathbf{R}^T\mathbf{R},$$

because $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_n$ by Exercise above. Since $\mathbf{R}$ is invertible (by the same exercise), $\mathbf{R}^T$ is also invertible, and hence $\mathbf{A}^T\mathbf{A}$ is invertible. Let $\mathbf{P}$ be its inverse, then $\mathbf{B} = \mathbf{PA}^T$ is a left inverse of $\mathbf{A}$.

Question in Section 5.5

Suppose the system $\mathbf{Ax} = \mathbf{b}$ is consistent.

1. Suppose $\mathbf{u}$ is a solution to $\mathbf{Ax} = \mathbf{b}$. Is $\mathbf{u}$ a least square solution to $\mathbf{Ax} = \mathbf{b}$?

2. Suppose $\mathbf{u}$ is a least square solution to $\mathbf{Ax} = \mathbf{b}$. Is $\mathbf{u}$ a solution to $\mathbf{Ax} = \mathbf{b}$?

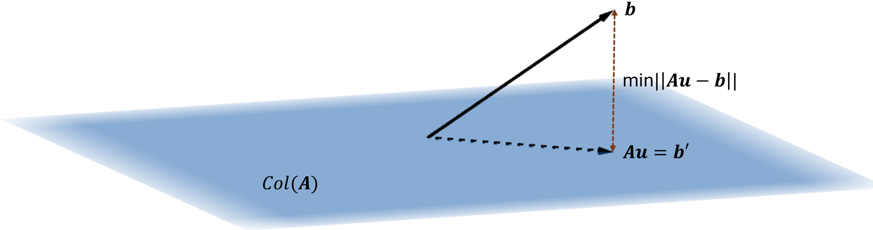
Slide 64: Least Square Approximation

Definition

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b}$ a vector in $\in \mathbb{R}^m$. A vector $\mathbf{u}$ in $\mathbb{R}^n$ is a *least square solution* of $\mathbf{Ax} = \mathbf{b}$ if for every vector $\mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{Au} - \mathbf{b}\| \leq \|\mathbf{Av} - \mathbf{b}\|.$$

Geometrically, by the best approximation theorem, the vector $\mathbf{b}' = \mathbf{Au}$ in $\mathrm{Col}(\mathbf{A})$ closest to $\mathbf{b}$ is the projection of $\mathbf{b}$ onto $\mathrm{Col}(\mathbf{A})$.



Answer to Question in Section 5.5

Suppose the system $\mathbf{Ax} = \mathbf{b}$ is consistent.

1. Suppose $\mathbf{u}$ is a solution to $\mathbf{Ax} = \mathbf{b}$. Is $\mathbf{u}$ a least square solution to $\mathbf{Ax} = \mathbf{b}$?

   Answer: Yes, because for every vector $\mathbf{v} \in \mathbb{R}^n$,

   $$\|\mathbf{Au} - \mathbf{b}\| = 0 \leq \|\mathbf{Av} - \mathbf{b}\|.$$

2. Suppose $\mathbf{u}$ is a least square solution to $\mathbf{Ax} = \mathbf{b}$. Is $\mathbf{u}$ a solution to $\mathbf{Ax} = \mathbf{b}$?

   Answer: Yes. By the assumption that $\mathbf{Ax} = \mathbf{b}$ is consistent, there is some $\mathbf{v}$ with $\mathbf{Av} = \mathbf{b}$. Since $\mathbf{u}$ is a least square solution,

   $$\|\mathbf{Au} - \mathbf{b}\| \leq \|\mathbf{Av} - \mathbf{b}\| = 0.$$

   Hence $\mathbf{Au} = \mathbf{b}$.

Exercise in Section 5.5

Suppose $\mathbf{A}$ is an $m \times n$ matrix with linearly independent columns, i.e. $\text{rank}(\mathbf{A}) = n$. QR factorize $\mathbf{A}$,

$$\mathbf{A} = \mathbf{QR}.$$

Show that the unique least square solution of $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{u} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}.$$

That is, suffice to solve for

$$\mathbf{Rx} = \mathbf{Q}^T\mathbf{b}.$$

This is easy to solve by hand since $\mathbf{R}$ is an upper triangular matrix (i.e. an REF).

Slide 65: Least Square Approximation

Theorem

Let $\mathbf{A}$ *be an* $m \times n$ *matrix and* $\mathbf{b}$ *a vector in* $\mathbb{R}^m$*. A vector* $\mathbf{u}$ *in* $\mathbb{R}^n$ *is a least square solution to* $\mathbf{Ax} = \mathbf{b}$ *if and only if* $\mathbf{Au}$ *is the projection of* $\mathbf{b}$ *onto the column space of* $\mathrm{Col}(\mathbf{A})$*.*

Theorem

Let $\mathbf{A}$ *be an* $m \times n$ *matrix and* $\mathbf{b}$ *a vector in* $\mathbb{R}^m$*. A vector* $\mathbf{u}$ *in* $\mathbb{R}^n$ *is a least square solution to* $\mathbf{Ax} = \mathbf{b}$ *if and only if* $\mathbf{u}$ *is a solution to* $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$*.*

Answer to Exercise in Section 5.5

Suppose $\mathbf{A}$ is an $m \times n$ matrix with linearly independent columns, and $\mathbf{A} = \mathbf{QR}$ is a QR factorization. Show that the unique least square solution of $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{u} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}.$$

Proof. By the Theorem on previous slide, the least square solution $\mathbf{u}$ is the solution to

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}.$$

Substituting $\mathbf{A} = \mathbf{QR}$, and using $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $\mathbf{R}$ and $\mathbf{R}^T$ are invertible, we have

$$\begin{aligned} \mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b} \quad &\Leftrightarrow \quad \mathbf{R}^T\mathbf{Q}^T\mathbf{QRu} = \mathbf{R}^T\mathbf{Q}^T\mathbf{b} \\ &\Leftrightarrow \quad \mathbf{R}^T\mathbf{Ru} = \mathbf{R}^T\mathbf{Q}^T\mathbf{b} \\ &\Leftrightarrow \quad \mathbf{u} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}. \end{aligned}$$

Questions in Section 6.1

1. Let $\mathbf{A}$ and $\mathbf{B}$ be row equivalent order n square matrices.
   (a) If λ is an eigenvalue of $\mathbf{A}$, is it an eigenvalue of $\mathbf{B}$?
   (b) If $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, is it an eigenvector of $\mathbf{B}$?

2. Can we compute the characteristic polynomial of a square matrix using row reduction instead of cofactor expansion?

Slide 6: Eigenvalues and Eigenvectors

Definition

Let $\mathbf{A}$ be a square matrix of order n. A real number λ is an *eigenvalue* of $\mathbf{A}$ if there is a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

In this case, the nonzero vector $\mathbf{v}$ is called an *eigenvector* associated to λ.

Remarks:

- Think of a matrix as an *action* on $\mathbb{R}^n$ (we will learn linear transformation later). Geometrically, eigenvectors are the vectors that are being scaled (stretch, dilate, or reflect) when acted upon by $\mathbf{A}$, and eigenvalues are the amount to scale the eigenvectors.

- We require the eigenvector to be nonzero, $\mathbf{v} \neq \mathbf{0}$, otherwise, the definition becomes uninteresting.

Slides 8 and 9: Characteristic Polynomial

Definition

Let $\mathbf{A}$ be a square matrix of order n, the *characteristic polynomial* of $\mathbf{A}$, denoted as $\text{char}(\mathbf{A})$, is the degree n polynomial

$$\det(x\mathbf{I} - \mathbf{A}).$$

Theorem

Let $\mathbf{A}$ be a square matrix of order n. λ is an eigenvalue of $\mathbf{A}$ if and only if λ is a root of the characteristic polynomial $\det(x\mathbf{I} - \mathbf{A})$.

Answer to Questions in Section 6.1 (part 1)

1. Let **A** and **B** be row equivalent order n square matrices.
   (a) If λ is an eigenvalue of **A**, is it an eigenvalue of **B**?
   (b) If **v** is an eigenvector of **A**, is it an eigenvector of **B**?

Answer: Both (a) and (b) are false. For example, let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix},$$

which are row equivalent. But

$$\det(x\mathbf{I} - \mathbf{A}) = (x-1)(x-4) \quad \det(x\mathbf{I} - \mathbf{B}) = (x+2)(x-2),$$

which shows that they don't share the same eigenvalues. Moreover $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of **A** associated with 1, but it is not an eigenvalue of **B**.

Answer to Questions in Section 6.1 (part 2)

2. Can we compute the characteristic polynomial of a square matrix using row reduction instead of cofactor expansion?

Answer: Yes, as long as the row operations are on the matrix $x\mathbf{I} - \mathbf{A}$ (not on $\mathbf{A}$).

Challenge one in Section 5.5

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b}$ a vector in $\mathbb{R}^m$. Prove that for any choice of least square solution $\mathbf{u}$, that is, for any solution $\mathbf{u}$ of $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$, the projection $\mathbf{A}\mathbf{u}$ is unique.

Recall that on Slide 65, we have

Theorem

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b}$ a vector in $\mathbb{R}^m$. A vector $\mathbf{u}$ in $\mathbb{R}^n$ is a least square solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{A}\mathbf{u}$ is the projection of $\mathbf{b}$ onto the column space of $\mathrm{Col}(\mathbf{A})$.

The Challenge is essentially the “only if” direction.

Answer to Challenge one in Section 5.5

Q: Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b}$ a vector in $\mathbb{R}^m$. Prove that for any choice of least square solution $\mathbf{u}$, that is, for any solution $\mathbf{u}$ of $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$, the projection $\mathbf{A}\mathbf{u}$ is unique.

Answer: Suppose that $\mathbf{A}^T\mathbf{A}\mathbf{u} = \mathbf{A}^T\mathbf{b}$. Then $\mathbf{A}^T(\mathbf{A}\mathbf{u} - \mathbf{b}) = \mathbf{0}$, in other words, $\mathbf{A}\mathbf{u} - \mathbf{b} \in \text{Null}(\mathbf{A}^T)$.

By Orthogonal to a Subspace algorithm (on Slide 16 in Chapter 5), $\mathbf{A}\mathbf{u} - \mathbf{b}$ is orthogonal to the column space of $\mathbf{A}$. Hence

$$\mathbf{b} = \mathbf{A}\mathbf{u} + (\mathbf{b} - \mathbf{A}\mathbf{u})$$

is a decomposition as in Orthogonal Projection Theorem. Thus $\mathbf{A}\mathbf{u}$, being the unique orthogonal projection of $\mathbf{b}$ to $\text{Col}(\mathbf{A})$, is unique.

Challenge two in Section 5.5

Let $V \subseteq \mathbb{R}^n$ be a subspace and suppose $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k\}$ is an orthonormal basis of V. Write

$$\mathbf{Q} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_k \end{pmatrix}.$$

Then for any $\mathbf{w} \in \mathbb{R}^n$, the projection of $\mathbf{w}$ onto V is

$$\mathbf{Q}\mathbf{Q}^T\mathbf{w}.$$

Answer to Challenge two in Section 5.5

Let $V \subseteq \mathbb{R}^n$ be a subspace and suppose $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k\}$ is an orthonormal basis of V. Write

$$\mathbf{Q} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_k \end{pmatrix}.$$

Then for any $\mathbf{w} \in \mathbb{R}^n$, the projection of $\mathbf{w}$ onto V is $\mathbf{Q}\mathbf{Q}^T\mathbf{w}$.

From Challenge one, we know that if $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$, then $\mathbf{A}\mathbf{u}$ is the projection of $\mathbf{b}$ onto $\mathrm{Col}(\mathbf{A})$.

In Challenge Two, $\mathbf{A}$ is $\mathbf{Q}$ and $\mathbf{b}$ is $\mathbf{w}$. It suffices to check that $\mathbf{u} = \mathbf{Q}^T\mathbf{w}$ satisfies

$$\mathbf{Q}^T\mathbf{Q}\mathbf{u} = \mathbf{Q}^T\mathbf{w},$$

which follows immediately from $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$.

Challenges in Section 6.1

Let $\mathbf{A}$ be an $n \times n$ matrix.

1. Show that the characteristic polynomial of $\mathbf{A}$ is equal to the characteristic polynomial of $\mathbf{A}^T$. Hence $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues.

2. Let λ be an eigenvalue of $\mathbf{A}$. Show that the geometric multiplicity of λ as an eigenvalue of $\mathbf{A}$ is equal to its geometric multiplicity as an eigenvalue of $\mathbf{A}^T$.

Slide 17: Eigenspace

Recall that eigenvectors of $\mathbf{A}$ associated to eigenvalue λ are nontrivials solution to

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}.$$

Since the system is homogeneous, the set of all solutions is a subspace. We will call it the eigenspace of $\mathbf{A}$ associated to eigenvalue λ.

Definition

Let $\mathbf{A}$ be an order n square matrix. The *eigenspace* associated to an eigenvalue λ of $\mathbf{A}$ is

$$E_\lambda = \{ \ \mathbf{v} \in \mathbb{R}^n \ \middle| \ \mathbf{A}\mathbf{v} = \lambda \mathbf{v} \ \} = \mathrm{Null}(\lambda \mathbf{I} - \mathbf{A}).$$

The *geometric multiplicity* of an eigenvalue λ is the dimension of its eigenspace,

$$\dim(E_\lambda) = \mathrm{nullity}(\lambda \mathbf{I} - \mathbf{A}).$$

Slide 37 in Chapter 4: Rank-Nullity Theorem

Theorem (Rank-Nullity Theorem)

Let $\mathbf{A}$ *be a* $m \times n$ *matrix. The sum of its rank and nullity is equal to the number of columns,*

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n.$$

Sketch of Proof: This follows from the fact that the nullity of $\mathbf{A}$ is equal to the number of non-pivot columns in its reduced row-echelon form, and that the rank of $\mathbf{A}$ is equal to the number of pivot columns of its reduced row-echelon form.

Answer to Challenges in Section 6.1 (part 1)

Let $\mathbf{A}$ be an $n \times n$ matrix.

1. Show that the characteristic polynomial of $\mathbf{A}$ is equal to the characteristic polynomial of $\mathbf{A}^T$. Hence $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues.

Proof. The following fact is useful for both parts: For a square matrix $\mathbf{A}$

$$(\lambda \mathbf{I} - \mathbf{A})^T = (\lambda \mathbf{I})^T - \mathbf{A}^T = \lambda \mathbf{I} - \mathbf{A}^T.$$

For part 1, we have

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det(\lambda \mathbf{I} - \mathbf{A})^T = \det(\lambda \mathbf{I} - \mathbf{A}^T),$$

where we used the fact in the last equality.

Answer to Challenges in Section 6.1 (part 2)

Let $\mathbf{A}$ be an $n \times n$ matrix.

2. Let λ be an eigenvalue of $\mathbf{A}$. Show that the geometric multiplicity of λ as an eigenvalue of $\mathbf{A}$ is equal to its geometric multiplicity as an eigenvalue of $\mathbf{A}^T$.

Proof. For part 2, we have

$$\begin{aligned}\text{nullity}(\lambda\mathbf{I} - \mathbf{A}) &= n - \text{rank}(\lambda\mathbf{I} - \mathbf{A}) \\ &= n - \text{rank}(\lambda\mathbf{I} - \mathbf{A})^T \\ &= n - \text{rank}(\lambda\mathbf{I} - \mathbf{A}^T) \\ &= \text{nullity}(\lambda\mathbf{I} - \mathbf{A}^T).\end{aligned}$$