MA1522: Linear Algebra for Computing

Tutorial 6

Revision

Equivalent ways to check for basis

- $U \subseteq V$, then $\dim(U) \leq \dim(V)$ with equality $\Leftrightarrow U = V$.
- $V \subseteq \mathbb{R}^n$ subspace with $\dim(V) = k$. $S \subseteq V$, linearly independent subset containing k vectors, $|S| = k \Rightarrow S$ is a basis for V.
- $V \subseteq \mathbb{R}^n$ subspace with $\dim(V) = k$. S contains k vectors, $|S| = k$ and $V \subseteq \text{span}(S) \Rightarrow S$ is a basis for V.

Definition	**(B1)**	**(B2)**
(1) $\text{span}(S) = V$ (2) S is L.I.	(1) $\|S\| = \dim(V)$ (2) $S \subseteq V$ and S is L.I.	(1) $V \subseteq \text{span}(S)$ (2) $\|S\| = \dim(V)$

Coordinates Relative to a Basis

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a basis for a subspace $V \subseteq \mathbb{R}^n$. Then any $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

for some unique $c_1, c_2, \ldots, c_k$. There is a unique correspondence

$$\mathbb{R}^k \xleftrightarrow{\text{via } S} V \subseteq \mathbb{R}^n, \qquad [\mathbf{v}]_S \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \leftrightarrow c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

$[\mathbf{v}]_S$ is the coordinates of $\mathbf{v}$ relative to the basis S; solve $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{v}.$

Transition Matrix

Let $V \subseteq \mathbb{R}^n$ be a subspace and S, T be 2 bases for V. The transition matrix from T to S is

$$\mathbf{P} = \begin{pmatrix} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \cdots & [\mathbf{v}_k]_S \end{pmatrix} \quad \Leftrightarrow \quad [\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T \text{ for all } \mathbf{w} \in V.$$

Algorithm to find **P**.

$$\left(\text{“}S\text{”} \;\middle|\; \text{“}T\text{”} \right) = \left(\begin{array}{cccc|cccc} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{array} \right) \xrightarrow{rref} \left(\begin{array}{c|cccc} \mathbf{I}_k & [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \cdots & [\mathbf{v}_k]_S \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0} & \mathbf{0} & & \mathbf{0} \end{array} \right)$$
$$= \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{(n-k)\times k} \end{array} \right),$$

If **P** is the transition matrix from T to S, then $\mathbf{P}^{-1}$ is the transition matrix from S to T.

Column and Row Space

Let $\mathbf{A}$ be an $m \times n$ matrix,

- Column space of $\mathbf{A}$ is the subspace of $\mathbb{R}^m$ spanned by the columns of $\mathbf{A}$.
- Row space of $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$.
- $\mathbf{v} \in \mathrm{Col}(\mathbf{A}) \Leftrightarrow \mathbf{A}\mathbf{x} = \mathbf{v}$ is consistent,

$$\mathrm{Col}(\mathbf{A}) = \left\{ \ \mathbf{A}\mathbf{u} \ \middle| \ \mathbf{u} \in \mathbb{R}^k \ \right\} = \{ \ \mathbf{v} \ | \ \mathbf{A}\mathbf{v} \text{ is consistent } \ .$$

Column and Row Space

- (Row operations preserve row space) Suppose $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices. Then $\text{Row}(\mathbf{A}) = \text{Row}(\mathbf{B})$.
- For any matrix $\mathbf{A}$, the nonzero rows of the reduced row-echelon form of $\mathbf{A}$ form a basis for $\text{Row}(\mathbf{A})$.
- (Row operations preserve linear relations between columns) Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$ be row equivalent $m \times n$ matrices, where $\mathbf{a}_i$ and $\mathbf{b}_i$ are the i-th column of $\mathbf{A}$ and $\mathbf{B}$, respectively, for $i = 1, ..., n$. Then for any $c_1, c_2, ..., c_n \in \mathbb{R}^n$, if

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

then (for the same coefficients $c_1, c_2, ..., c_n \in \mathbb{R}$),

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}.$$

- Suppose $\mathbf{R}$ is the reduced row-echelon form of a matrix $\mathbf{A}$. Then the columns of $\mathbf{A}$ corresponding to the pivot columns in $\mathbf{R}$ form a basis for the column space of $\mathbf{A}$.

Rank

Let $\mathbf{A}$ be a $n \times m$ matrix. Define the rank of $\mathbf{A}$ to be $\text{rank}(\mathbf{A}) = \dim(\text{Col}(\mathbf{A})) = \dim(\text{Row}(\mathbf{A}))$.

1. $\text{rank}(\mathbf{A}) =$ number of pivot columns in RREF.
2. $\text{rank}(\mathbf{A}) =$ number of nonzero rows in RREF.
3. $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$
4. $\text{rank}(\mathbf{A}) \leq \min\{m, n\}$.

$\mathbf{A}$ is said to be full rank if $\text{rank}(\mathbf{A}) = m$ or $\text{rank}(\mathbf{A}) = n$.

Nullspace and Nullity

The nullspace of a $m \times n$ matrix $\mathbf{A}$ is the solution space to the homogeneous system $\mathbf{Ax} = \mathbf{0}$ with coefficient matrix $\mathbf{A}$. It is denoted as

$$\text{Null}(\mathbf{A}) = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; \mathbf{Av} = \mathbf{0} \right\}.$$

The nullity of $\mathbf{A}$ is the dimension of the nullspace of $\mathbf{A}$, denoted as

$$\text{nullity}(\mathbf{A}) = \dim(\text{Null}(\mathbf{A})) = \text{ number of non-pivot columns in RREF of } \mathbf{A}.$$

Summary

Let $\mathbf{A}$ be a $m \times n$ matrix.

Subspace	**Subspace of**	**Basis**	**Dimension**
$\text{Col}(\mathbf{A})$	$\mathbb{R}^m$	Columns of $\mathbf{A}$ corresponding to pivot columns in RREF	$\text{rank}(\mathbf{A})$
$\text{Row}(\mathbf{A})$	$\mathbb{R}^n$	Nonzero rows of RREF	$\text{rank}(\mathbf{A})$
$\text{Null}(\mathbf{A})$	$\mathbb{R}^n$	Vectors in general solution to $\mathbf{Ax} = \mathbf{0}$	$\text{nullity}(A)$

Tutorial 6 Solutions

Question 1(a)

Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$. Show that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ forms a basis for $\mathbb{R}^3$.

Recall that $\{\mathbf{u}_1, ..., \mathbf{u}_n\} \in \mathbb{R}^n$ is a basis if and only if $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix}$ is an invertible matrix, and that $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.

```
>> S=[1 0 0;2 2 -1;-1 1 3]; det(S)
```

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ -1 & 1 & 3 \end{vmatrix} = 7 \neq 0.$$

Thus $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathbb{R}^3$.

Question 1(b)

Suppose $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find the coordinate vector of $\mathbf{w}$ relative to S.

```
>> w=[1;1;1]; rref([S w])
```

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 2 & -1 & 1 \\ -1 & 1 & 3 & 1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/7 \\ 0 & 0 & 1 & 5/7 \end{array}\right) \quad \Rightarrow \quad [\mathbf{w}]_S = \begin{pmatrix} 1 \\ -1/7 \\ 5/7 \end{pmatrix}.$$

Question 1(c)

Let $T = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be another basis for $\mathbb{R}^3$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$. Find the transition matrix from T to S.

```
>> T=[1 -1 2;5 3 2;4 7 4]; A=rref([S T]);P=A(:,[4:6])
```

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 2 & 2 & -1 & 5 & 3 & 2 \\ -1 & 1 & 3 & 4 & 7 & 4 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right)$$

So, the transition matrix $\mathbf{P}$ from T to S is $\mathbf{P} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Question 1(d)

Find the transition matrix from S to T.

```
>> Q=inv(P)
>> B=rref([T S]); Q=B(:,[4:6])
```

$$\mathbf{Q} = \mathbf{P}^{-1} = \begin{pmatrix} 3/4 & 1/2 & -3/4 \\ -1/2 & 0 & 1/2 \\ -1/8 & -1/4 & 5/8 \end{pmatrix}.$$

Question 1(e)

Use the vector **w** in Part (b). Find the coordinate vector of **w** relative to T.

```
>> Q*[1;-1/7;5/7]
```

$$[\mathbf{w}]_T = \mathbf{Q}[\mathbf{w}]_S = \begin{pmatrix} 3/4 & 1/2 & -3/4 \\ -1/2 & 0 & 1/2 \\ -1/8 & -1/4 & 5/8 \end{pmatrix} \begin{pmatrix} 1 \\ -1/7 \\ 5/7 \end{pmatrix} = \begin{pmatrix} 1/7 \\ -1/7 \\ 5/14 \end{pmatrix}.$$

Question 2(a)

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for a subspace V. Define $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3, \ \mathbf{v}_2 = \mathbf{u}_2 + \mathbf{u}_3 \text{ and } \mathbf{v}_3 = \mathbf{u}_2 - \mathbf{u}_3.$$

Show that T is a basis for V.

Use (B1): Clearly $T \subseteq V$, and $|T| = 3 = \dim(V)$. Finally, show that T is linearly independent.

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= c_1(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) + c_2(\mathbf{u}_2 + \mathbf{u}_3) + c_3(\mathbf{u}_2 - \mathbf{u}_3) \\ &= c_1\mathbf{u}_1 + (c_1 + c_2 + c_3)\mathbf{u}_2 + (c_1 + c_2 - c_3)\mathbf{u}_3 \end{aligned}$$

Then since S is linearly independent, $\Rightarrow \left\{ \begin{array}{ccccccc} c_1 & & & & & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \\ c_1 & + & c_2 & - & c_3 & = & 0 \end{array} \right.$ which has only the trivial solution $c_1 = c_2 = c_3 = 0$. Hence, T is linearly independent. Thus, T is a basis.

Question 2(b)

Find the transition matrix from S to T.

Observe that by construction,

$$[\mathbf{v}_1]_S = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad [\mathbf{v}_2]_S = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad [\mathbf{v}_3]_S = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Hence, the transition matrix $\mathbf{P}$ from T to S is

$$\mathbf{P} = \begin{pmatrix} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & [\mathbf{v}_3]_S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Hence, the transition matrix $\mathbf{Q}$ from S to T is

$$\mathbf{Q} = \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{pmatrix}.$$

Question 3(a)

Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Is $\mathbf{b}$ in the column space of $\mathbf{A}$? If it is, express it as a linear combination of the columns of $\mathbf{A}$.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Thus $\mathbf{b}$ is not a linear combination of the columns of $\mathbf{A}$.

Question 3(b)

Let $\mathbf{A} = \begin{pmatrix} 1 & 9 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = (5, 1, -1)$. Is $\mathbf{b}$ in the row space of $\mathbf{A}$? If it is, express it as a linear combination of the rows of $\mathbf{A}$.

Note that $\mathbf{b}$ is in the row space of $\mathbf{A}$ if and only if $\mathbf{b}^T$ is in the column space of $\mathbf{A}^T$. Hence we are solving for

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

```
>> A=[1 9 1;-1 3 1;1 1 1]; b=[5 1 -1]; rref([A' b'])
```

We get $\mathbf{b} = (5, 1, -1) = (1, 9, 1) - 3(-1, 3, 1) + (1, 1, 1)$.

Question 3(c)

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}$. Is the row space and column space of $\mathbf{A}$ the whole $\mathbb{R}^4$?

$\mathbf{A}$ is invertible if and only if either the columns or the rows of $\mathbf{A}$ form a basis for $\mathbb{R}^4$.

```
>> A=[1 2 0 1;0 1 2 1;1 2 1 3;0 1 2 2]; rref(A), det(A)
```

Question 4(a)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 1 & -4 & -1 & -9 \\ -1 & 0 & -3 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

(i) Find a basis for the row space of $\mathbf{A}$.

(ii) Find a basis for the column space of $\mathbf{A}$.

(iii) Find a basis for the nullspace of $\mathbf{A}$.

(iv) Hence determine rank($\mathbf{A}$), nullity($\mathbf{A}$) and verify the dimension theorem for matrices.

(v) Is $\mathbf{A}$ full rank?

```
>> A=[1 2 5 3;1 -4 -1 -9;-1 0 -3 1;2 1 7 0;0 1 1 2];
rref(A).
```

(i) A basis for the row space is $\{(1,0,3,-1),(0,1,1,2)\}$.

(ii) A basis for the column space is $\{(1,1,-1,2,0)^T,(2,-4,0,1,1,)^T\}$.

(iii) A basis for the nullspace is $\{(-3,-1,1,0)^T,(1,-2,0,1)^T\}$.

(iv) $\text{rank}(\mathbf{A}) = 2$, $\text{nullity}(\mathbf{A}) = 2$. Since $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 2 + 2 = 4$ which is the number of columns of $\mathbf{A}$, we have verified the dimension theorem for matrices.

(v) $\text{rank}(\mathbf{A}) = 2 < \min\{4,5\}$. $\mathbf{A}$ is not full rank.

Question 4(b)

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{pmatrix}.$$

(i) Find a basis for the row space of $\mathbf{A}$.

(ii) Find a basis for the column space of $\mathbf{A}$.

(iii) Find a basis for the nullspace of $\mathbf{A}$.

(iv) Hence determine rank($\mathbf{A}$), nullity($\mathbf{A}$) and verify the dimension theorem for matrices.

(v) Is $\mathbf{A}$ full rank?

```
>> A=[1 3 7;2 1 8;3 -5 -1;2 -2 2;1 1 5]; rref(A).
```

(i) A basis for the row space is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

(ii) A basis for the column space is $\{(1,2,3,2,1)^T,(3,1,-5,-2,1)^T,(7,8,-1,2,5)^T\}$.

(iii) The basis for the nullspace is the empty set.

(iv) $\text{rank}(\mathbf{A}) = 3$, $\text{nullity}(\mathbf{A}) = 0$. Since $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 3 + 0 = 3$ which is the number of columns of $\mathbf{A}$, we have verified the dimension theorem for matrices.

(v) $\text{rank}(\mathbf{A}) = 3 = \min\{3,5\}$. $\mathbf{A}$ is full rank.

Question 5(a)

Let W be a subspace of $\mathbb{R}^5$ spanned by the following vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 \begin{pmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 5 \\ 15 \\ 10 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} 2 \\ 1 \\ 15 \\ 8 \\ 6 \end{pmatrix}.$$

Find a basis for W.

```
>> U=[1 2 0 2;-2 -5 5 1;0 -3 15 15;0 -2 10 8;3 6 0 6];
>> rref(U)
>> rref(U')
```

Question 5(b)

What is $\dim(W)$?

From (a), $\dim(W) = 3$

Question 5(c)

Extend the basis W found in (a) to a basis for $\mathbb{R}^5$.

Find $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ such that it is not in the span of $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$.

```
>> syms x1 x2 x3 x4 x5; R=[U [x1;x2;x3;x4;x5]]
>> R(2,:)=R(2,:)+2*R(1,:);R(5,:)=R(5,:)-3*R(1,:);
R(3,:)=R(3,:)-3*R(2,:);R(4,:)=R(4,:)-2*R(2,:)
```

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ is not in the span if and only if $\begin{cases} -6x_1 - 3x_2 + x_3 = 0 \\ -3x_1 + x_5 = 0 \end{cases}$ May choose $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

Question 6

Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 5 \\ 12 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \\ 4 \end{pmatrix} \right\}$ and $V = \mathrm{span}(S)$. Find a subset $S' \subseteq S$ such that S' forms a basis for V.

```
>> S=[1 2 -1 0 3;0 -1 3 1 -1;1 0 5 2 1;3 1 12 5 4]; rref(S)
>> S1=S(:,[1 2])
```

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -1 & 3 & 1 & -1 \\ 1 & 0 & 5 & 2 & 1 \\ 3 & 1 & 12 & 5 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 5 & 2 & 1 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $S' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.