MA1522 ASSIGNMENT 3

- Do NOT upload this assignment problem set to any website.
- The assignment carries a total number of 30 marks. The marks for each question or part are as indicated.
- You do not need to show the steps of the row reductions.

(1) [1 mark for each question] In each of the following, assume $\mathbf{A}$ is an $n \times n$ matrix over real numbers.

Which of the following statements are true? Which are false? You do not need to justify your answers.

Summarized Answer: $\boxed{\text{F, T, F, F, T, F, F, T, F, T}}$.

(a) The row and the column spaces of $\mathbf{A}$ are always the same.

Answer: $\boxed{\text{False}}$. For example, let $n = 2$ and $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then the row space of $\mathbf{A}$ is the x-axis whereas the column space of $\mathbf{A}$ is the y-axis.

(b) If $\mathbf{B}$ is an $m \times n$ matrix, then $\text{rank}(\mathbf{B}) \le \min\{m, n\}$.

Answer: $\boxed{\text{True}}$. The rank r is the dimension of the row space, so $r \le m$. It is also the dimension of the column space, so $r \le n$.

(c) If $\mathbf{B}$ is a 2025×5 matrix and the nullity of $\mathbf{B}$ is 2, then the rank of $\mathbf{B}$ is 2023.

Answer: $\boxed{\text{False}}$. By Dimension Theorem, the rank of $\mathbf{B}$ = $n - \text{nullity}(\mathbf{B}) = 5 - 2 = 3$.

(d) If S and T are subspaces of $\mathbb{R}^n$, then so is $S \cup T$.

Answer: $\boxed{\text{False}}$. For example, let S and T be the x-axis and y-axis of $\mathbb{R}^2$ respectively. Then S and T are subspaces, but $S \cup T$ is not.

(e) If S and T are subspaces of $\mathbb{R}^n$, then so is $S \cap T$.

Answer: $\boxed{\text{True}}$. One can verify that $S \cap T$ satisfies the three conditions of being a subspace.

(f) Let $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ be three nonzero vectors in $\mathbb{R}^n$. If $\mathbf{x} \perp \mathbf{y}$ and $\mathbf{y} \perp \mathbf{z}$, then $\mathbf{x} \perp \mathbf{z}$.

Answer: $\boxed{\text{False}}$. For example, one can take $\mathbf{x} = \mathbf{z} = (1, 0)^T$ and $\mathbf{y} = (01)^T$ in $\mathbb{R}^2$. Then $\mathbf{x} \perp \mathbf{y}$ and $\mathbf{y} \perp \mathbf{z}$, but $\mathbf{x}$ can't be perpendicular to itself.

(g) If $\mathbf{Q}$ is an orthogonal matrix, then $\det(\mathbf{Q}) = 1$.

Answer: $\boxed{\text{False}}$. For example, $\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a orthogonal matrix, however, $\det(\mathbf{Q}) = -1$.

(h) If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthonormal set of vectors in $\mathbb{R}^n$, and

$$\mathbf{U} = (\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k)$$

is the matrix whose j-th column is the vector $\mathbf{u}_j$ for $1 \le j \le k$, then $\mathbf{U}^T\mathbf{U} = \mathbf{I}_k$.

Answer: $\boxed{\text{True}}$. The (i, j) entry in $\mathbf{U}^T\mathbf{U}$ is $\mathbf{u}_i^T\mathbf{u}_j$ as matrix multiplication, which is equal to $\mathbf{u}_i \cdot \mathbf{u}_j$, which is 0 when $i \ne j$; and is 1 when $i = j$.

(i) If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthonormal set of vectors in $\mathbb{R}^n$, and

$$\mathbf{U} = (\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k)$$

is the matrix whose j-th column is the vector $\mathbf{u}_j$ for $1 \le j \le k$, then $\mathbf{U}\mathbf{U}^T = \mathbf{I}_n$.

Answer: $\boxed{\text{False}}$. For example, let $n = 2$, $k = 1$ and

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{U}.$$

Then

$$\mathbf{U}\mathbf{U}^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(j) If $\mathbf{A}^2 = \mathbf{A}$ and λ is an eigenvalue of $\mathbf{A}$, then λ must be either 0 or 1.

Answer: $\boxed{\text{True}}$. Suppose $\mathbf{v}$ is an eigenvector associated with λ. Then $\mathbf{v} \ne \mathbf{0}$ and $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, and $\mathbf{A}^2\mathbf{v} = \lambda^2\mathbf{v}$. Since $\mathbf{A}^2 = \mathbf{A}$, we have

$$\mathbf{0} = \mathbf{A}^2\mathbf{v} - \mathbf{A}\mathbf{v} = (\lambda^2 - \lambda)\mathbf{v}.$$

Since $\mathbf{v} \ne \mathbf{0}$, $\lambda^2 - \lambda = 0$. Hence $\lambda = 0$ or 1.

(2) [1 mark for each question] Let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{pmatrix}.$$

(i) Find a basis for the row space of $\mathbf{A}$.

(ii) Find a basis for the column space of $\mathbf{A}$.

(iii) Find a basis for the nullspace of $\mathbf{A}$.

Answer: By row operations

$$\mathbf{A} \longrightarrow \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & 4 & 3 \end{pmatrix} \left(\longrightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{13}{20} \\ 0 & 1 & 0 & \frac{21}{20} \\ 0 & 0 & 1 & \frac{3}{4} \end{pmatrix} \right)$$

We then follow the algorithms from the lectures to get the basis.

(i) One can take the row vectors from the RREF as a basis of the row space, namely,

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{13}{20} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{21}{20} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \right\}}.$$

One can also take the row vectors from the above row echelon form. Since the rank is three, one can also take the original three row vectors from $\mathbf{A}$ too.

(ii) One can go back to the matrix $\mathbf{A}$ to pick the columns corresponding to the pivot columns in the RREF as a basis for the column space. Namely,

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \right\}}.$$

One can take any three independent vectors from the column space as a basis, so alternative solution exists.

(iii) Let $x_4 = t$ as a parameter, we get the general solution of $\mathbf{Ax} = \mathbf{0}$: $(\frac{13}{20}t, -\frac{21}{20}t, -\frac{3}{4}t, t)^T$ where $t \in \mathbb{R}$. Thus

$$\boxed{\left\{ \begin{pmatrix} \frac{13}{20} \\ -\frac{21}{20} \\ -\frac{3}{4} \\ 1 \end{pmatrix} \right\}}$$

is a basis of the nullspace.

(3) [2 marks] Let $\mathbf{A}$ be a 4×4 matrix with reduced row echelon form given by

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If the first two columns of $\mathbf{A}$ are

$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 5 \\ 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ -3 \\ 7 \\ -1 \end{pmatrix},$$

determine the remaining two columns of $\mathbf{A}$.

Answer: Let $\mathbf{u}_i$ and $\mathbf{v}_i$ denote the column vectors in $\mathbf{U}$ and $\mathbf{A}$ respectively. From $\mathbf{U}$, we observe that $\mathbf{u}_3 = 2\mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{u}_4 = \mathbf{u}_1 + 4\mathbf{u}_2$. Since row operations preserve linear relations of the columns, we have

$$\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2 = \boxed{\begin{pmatrix} -2 \\ 7 \\ 11 \\ 1 \end{pmatrix}},$$

and

$$\mathbf{v}_4 = \mathbf{v}_1 + 4\mathbf{v}_2 = \boxed{\begin{pmatrix} 13 \\ -7 \\ 30 \\ -3 \end{pmatrix}}.$$

(4) [2 marks] Let S be the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ -2 \end{pmatrix}.$$

Find a basis for $S^{\perp}$.

Answer: Let $\mathbf{z} = (a, b, c, d)^T$ be an arbitrary element in $S^{\perp}$. Then

$$\mathbf{z} \cdot \mathbf{x}_1 = 0 = \mathbf{z} \cdot \mathbf{x}_2,$$

which shows that $S^{\perp}$ is the nullspace of

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -2 \end{pmatrix}.$$

Since $\mathbf{A}$ is already in RREF, we get the general solutions as

$$\mathbf{z} = s(2, -3, 1, 0)^T + t(-1, 2, 0, 1)^T$$

where $s, t \in \mathbb{R}$ are parameters. Thus, a basis of $S^{\perp}$ can be taken as

$$\boxed{\left\{\begin{pmatrix}2\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\\0\\1\end{pmatrix}\right\}}.$$

(5) [2 marks] Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be an orthonormal basis for a subspace V of $\mathbb{R}^n$. If $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ is a vector with the properties $\|\mathbf{x}\| = 5$, $\mathbf{u}_1 \cdot \mathbf{x} = 4$ and $\mathbf{x} \perp \mathbf{u}_2$. Find all possible values of c_1, c_2 and c_3.

Answer: From the assumption, we have

$$\begin{aligned} 25 &= \|\mathbf{x}\|^2 = c_1^2 + c_2^2 + c_3^2 \\ 4 &= \mathbf{u}_1 \cdot \mathbf{x} = c_1 \\ 0 &= \mathbf{u}_2 \cdot \mathbf{x} = c_2. \end{aligned}$$

One can easily get that $\boxed{c_1 = 4, c_2 = 0 \text{ and } c_3 = \pm 3}$.

(6) Let

$$\mathbf{A} = \begin{pmatrix}1 & 1 & 0\\1 & 1 & 0\\1 & 1 & 1\\0 & 1 & 1\end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

(i) [2 marks] Find a QR factorization of $\mathbf{A}$.

Answer: To save space, we identify $\mathbf{v}$ with its transpose $\mathbf{v}^T$. Let $\boldsymbol{u}_1 = (1,1,1,0)$, $\boldsymbol{u}_2 = (1,1,1,1)$, $\boldsymbol{u}_3 = (0,0,1,1)$. Apply Gram-Schmidt process, we get:

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{u}_1 = (1,1,1,0), \\ \boldsymbol{v}_2 &= \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 = (0,0,0,1), \\ \boldsymbol{v}_3 &= \boldsymbol{u}_3 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, 0\right). \end{aligned}$$

Normalize, we have

$$\begin{aligned} \boldsymbol{q}_1 &= \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{\sqrt{3}}(1,1,1,0), \\ \boldsymbol{q}_2 &= \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = (0,0,0,1) \\ \boldsymbol{q}_3 &= \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|} = \frac{1}{\sqrt{6}}(-1,-1,2,0). \end{aligned}$$

Thus,

$$\boxed{\boldsymbol{Q} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ 0 & 1 & 0 \end{pmatrix}}.$$

By the algorithm from the lecture notes,

$$\boxed{\boldsymbol{R} = \boldsymbol{Q}^T\boldsymbol{A} = \begin{pmatrix} \sqrt{3} & \sqrt{3} & \frac{\sqrt{3}}{3} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{\sqrt{6}}{3} \end{pmatrix}}.$$

Alternatively, one can also use

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{q}_1 \cdot \boldsymbol{u}_1 & \boldsymbol{q}_1 \cdot \boldsymbol{u}_2 & \boldsymbol{q}_1 \cdot \boldsymbol{u}_3 \\ 0 & \boldsymbol{q}_2 \cdot \boldsymbol{u}_2 & \boldsymbol{q}_2 \cdot \boldsymbol{u}_3 \\ 0 & 0 & \boldsymbol{q}_3 \cdot \boldsymbol{u}_3 \end{pmatrix}$$

to get the same result.

(ii) [2 marks] Use the result in (i) to find the least squares solution to $\mathbf{Ax} = \mathbf{b}$.

Answer: From the lecture notes, we know that the least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is given by $\boldsymbol{Rx} = \boldsymbol{Q}^T\boldsymbol{b}$.
Now $\boldsymbol{Q}^T\boldsymbol{b} = (\frac{\sqrt{3}}{3}, 0, -\frac{\sqrt{6}}{6})^T$. We form the augmented matrix:

$$(\boldsymbol{R} \mid \boldsymbol{Q}^T\boldsymbol{b}) = \left(\begin{array}{ccc|c} \sqrt{3} & \sqrt{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} \end{array}\right).$$

Using backward substitution, we get

$$\begin{aligned} \frac{\sqrt{6}}{3}x_3 = -\frac{\sqrt{6}}{6} &\Rightarrow x_3 = -\frac{1}{2}, \\ x_2 + (-\frac{1}{2}) = 0 &\Rightarrow x_2 = \frac{1}{2}, \\ \sqrt{3}x_1 + \sqrt{3} \cdot \frac{1}{2} + \frac{\sqrt{3}}{3}(-\frac{1}{2}) = \frac{\sqrt{3}}{3} &\Rightarrow x_1 = 0. \end{aligned}$$

Then the least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is given by

$$\boxed{\boldsymbol{x} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}}.$$

(iii) [2 marks] Determine the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$.

Answer: By Lecture notes, we know that the projection is $\mathbf{A}\boldsymbol{x}$ where $\boldsymbol{x}$ is the least square solution we found in (ii). Thus,

$$\boxed{\boldsymbol{p} = \boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}}.$$

(7) [5 marks] Find the eigenvalues and their associated eigenspaces respectively for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}.$$

Two marks for the correct eigenvalues. One mark each for the correct eigenspaces.

Answer: The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} \det(x\mathbf{I} - \mathbf{A}) &= \begin{vmatrix} x-1 & -2 & -1 \\ 0 & x-3 & -1 \\ 0 & -5 & x+1 \end{vmatrix} \\ &= (x-1)(x-4)(x+2) \end{aligned}$$

Thus, the eigenvalues are: $\boxed{\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = -2}$.

For $\lambda_1 = 1$, its eigenspace is the nullspace of $\mathbf{I} - \mathbf{A}$, thus

$$\boxed{E_1 = \{(s, 0, 0)^T : s \in \mathbb{R}\}}.$$

For $\lambda_2 = 4$, its eigenspace is the nullspace of $\mathbf{4I} - \mathbf{A}$, thus

$$\boxed{E_4 = \{(s, s, s)^T : s \in \mathbb{R}\}}.$$

For $\lambda_3 = -2$, its eigenspace is the nullspace of $-\mathbf{2I} - \mathbf{A}$, thus

$$\boxed{E_{-2} = \{(-\frac{1}{5}s, -\frac{1}{5}s, s)^T : s \in \mathbb{R}\}}.$$

END OF ASSIGNMENT THREE