Review

Vector spaces (or linear) $V \ni v$ **Vectors**

Subspaces
- V itself ; $\{0\}$ trivial subspace
- $\mathrm{Span}\{u_1, \cdots, u_k\}$
- solution set to homogeneous linear system

$v \in \mathrm{Span}\{u_1, \cdots, u_k\}$ iff $\exists\, \lambda_1, \cdots, \lambda_k \in \mathbb{R}$ s.t. $v = \lambda_1 u_1 + \cdots + \lambda_k u_k$

Determine whether $v \in \mathrm{Span}\{u_1, \cdots, u_k\}$:

solve the linear system $\begin{pmatrix} u_1 & \cdots & u_k \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = v$ $\quad$ $\mathrm{rref}\,(u_1 \cdots u_n\ v)$

$\longrightarrow$ Determine whether $\mathrm{Span}\{v_1, \cdots, v_l\} \subseteq \mathrm{Span}\{u_1, \cdots, u_k\}$

$\longrightarrow$ Determine whether $\mathbb{R}^n = \mathrm{Span}\{u_1, \cdots, u_k\}$

solve the above linear system for every v (or, if $k = n$,

compute $\det(u_1 \cdots u_n)$ $\begin{cases} = 0 & \text{No} \\ \neq 0 & \text{Yes} \end{cases}$)

$\lambda_1 u_1 + \cdots + \lambda_k u_k = 0$ solve for non-trivial $(\lambda_1, \cdots, \lambda_n) \neq (0, \cdots 0)$

$\Updownarrow$

$\begin{pmatrix} u_1 & \cdots & u_k \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = 0$

- ✓ **Linear dependence**
- X **Linear independence**
  i.e. only the trivial solution

Basis ① $\mathrm{Span}\, S = V$ ② S is linearly independent

Dimension $\dim V = \# S$ where S is a basis of V.

the maximum permitted number of linearly independent vectors in V

Remark : Based on a set $S \subseteq V$, how to construct a basis of V ?

Approach : consider span S .

if $V = \operatorname{Span} S$, then $\# S \geq \dim V$; if "=" holds then S is a basis ;

if ">" holds, then S linearly dependent $\xRightarrow[\text{theorem}]{\text{Spanning set}}$ $\exists$ a subset of S that is a basis of V ;

if $\operatorname{Span} S \subsetneq V$, let $S' \subseteq S$ linearly independent s.t. $\operatorname{Span} S' = \operatorname{Span} S$, then $\# S' < \dim V$ $\xRightarrow[\text{theorem}]{\text{linear independence}}$ $\exists\ T \supseteq S'$ s.t. T is a basis of V.

$\boxed{\text{Proposition : } U, V \text{ linear spaces. Then } U = V \text{ iff } \begin{array}{l} U \subseteq V \\ \dim U = \dim V \end{array}}$

$\longrightarrow$ Determine whether $\mathbb{R}^n = \operatorname{Span}\{u_1, \cdots, u_k\}$

if $k < n$, No ;

if $k \geq n$, "=" iff $\dim \operatorname{Span}\{u_1, \cdots, u_k\} = n$ iff $\exists\ \{v_1, \cdots, v_n\} \overset{\text{subset}}{\subseteq} \{u_1, \cdots, u_k\}$ s.t. $\det(v_1 \cdots v_n) \neq 0$

Equivalent conditions for basis :

S is a basis of V i.e. $\operatorname{span}(S) = V$ & S is L. I.

iff $\dim V = \# S$ & $S \subseteq V$ is L. I.

iff $V \subseteq \operatorname{Span}(S)$ & $\# S = \dim V$

(a) any set containing zero-vector is linearly dependent

(b) any subset of $\mathbb{R}^n$ containing more than n vectors must be linearly dependent

(c) subset of linearly independent set is linear independent

(d) a set containing n vectors in $\mathbb{R}^n$ is linearly independent

iff it spans $\mathbb{R}^n$ iff it forms a basis of $\mathbb{R}^n$ iff invertible matrix

(e) If $V = \{u : Au = 0\} = \{ s_1 u_1 + \cdots + s_k u_k ,\ s_1, \cdots, s_k \in \mathbb{R}\}$,

then $V = \operatorname{Span}\{u_1, \cdots, u_k\}$; if $\{u_1, \cdots, u_k\}$ linearly independent, then basis ;

in that case, $\dim V = k = \#$ non-pivot columns in the $\operatorname{rref}(A)$

1. For each of the following sets of vectors S,

   (i) Determine if S is linearly independent.

   (ii) If S is linearly dependent, express one of the vectors in S as a linear combination of the others.

(a) $S = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} \right\}.$

(b) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \right\}.$

(c) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$

(d) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}.$

1. (a) $\mathrm{rref}\left(\cdots \right) = \begin{pmatrix} 1 & & & 9/2 \\ & 1 & & 15/2 \\ & & 1 & -3 \end{pmatrix}$ No. $u_4 = \frac{9}{2} u_1 + \frac{15}{2} u_2 - 3 u_3$

(b) Yes. Not multiples of each other

(c) No. $u_3 = 0 \cdot u_1 + 0 \cdot u_2$

(d) $\mathrm{rref} = I_3$ or $\det \neq 0$ Yes.

2. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine which of the sets S_1 to S_5 are linearly independent.

   (a) $S_1 = \{\mathbf{u}, \mathbf{v}\}$,
   (b) $S_2 = \{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$,
   (c) $S_3 = \{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{w}\}$,
   (d) $S_4 = \{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$.
   (e) $S_5 = \{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{w}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$.

2. (a) ✓

(b) ✗ $(u - v) + (v - w) + (w - u) = 0$

(c) $x_1(u - v) + x_2(v - w) + x_3(u + w) = 0 \Rightarrow$ only trivial solution. ✓

(d) ✓

(e) ✗ $(u + v) + (v + w) + (w + u) - 2(u + v + w) = 0$

3. For each of the following subspaces V, write down a basis for V.

(a) $V = \left\{ \begin{pmatrix} a+b \\ a+c \\ c+d \\ b+d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}.$

3. (a) $\mathrm{Span} \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\text{basis}}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \xrightarrow{\mathrm{rref}} \begin{pmatrix} 1 & & & -1 \\ & 1 & & 1 \\ & & 1 & 1 \\ & & & \end{pmatrix}$

(b) $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(b) $V = \mathbb{R}^3$ $\{e_1, e_2, e_3\}$

(c) V is the solution space of the following homogeneous linear system

$$\begin{cases} a_1 + a_3 + a_4 - a_5 = 0 \\ a_2 + a_3 + 2a_4 + a_5 = 0 \\ a_1 + a_2 + 2a_3 + a_4 - 2a_5 = 0 \end{cases}$$

(c) $\text{rref} = \begin{pmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

Basis $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

Question 3(b)

Find a basis for $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

The set contains 4 vectors in $\mathbb{R}^3$. Subset of any 3 vectors will form a basis for V, that is, $V = \mathbb{R}^3$.

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>> V=[1 -1 0 1;0 2 3 -1;-1 3 0 1];
a=[1:4];
for i = 1:4
V(:,setdiff(a,i))
rref(V(:,setdiff(a,i)))
end
```

4. For what values of a will $\mathbf{u}_1 = \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -1 \\ a \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

4. 3 vectors in $\mathbb{R}^3$, hence iff they are linearly independent,

iff $\det(u_1\ u_2\ u_3) \neq 0$ iff $a \neq 0$

$$\det(u_1\ u_2\ u_3) = a^3 - 1 + 1 + a + a + a$$

$$= a(a^2 + 3)$$

simplify(det(A))

5. Let U and V be subspaces of $\mathbb{R}^n$. We define the sum $U+V$ to be the set of vectors

$$\{\, \mathbf{u}+\mathbf{v} \mid \mathbf{u}\in U \text{ and } \mathbf{v}\in V \,\}.$$

Suppose $U = \operatorname{span}\left\{ \overset{u_1}{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}, \overset{u_2}{\begin{pmatrix}1\\2\\2\\1\end{pmatrix}} \right\}$, $V = \operatorname{span}\left\{ \overset{v_1}{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}1\\0\\2\\-1\end{pmatrix}} \right\}$.

(a) Is $U \cup V$ a subspace of $\mathbb{R}^4$?

(b) Show that $U+V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

(c) Show that $U+V$ contains U and V. This shows that $U+V$ is a subspace containing $U \cup V$.

(d) What are the dimensions of U and V?

(e) Show that $U \cap V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

(f) Verify that $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$.

5. (a) No. $u_1 + v_1 \notin U \cup V$

(b) $U + V = \operatorname{Span}\{u_1, u_2, v_1, v_2\}$

$$\operatorname{rref}(u_1 \ u_2 \ v_1 \ v_2) = \begin{pmatrix} 1 & & & -2 \\ & 1 & & 1 \\ & & 1 & 2 \\ & & & \end{pmatrix}$$

$\{u_1 \ u_2 \ v_1\}$ basis. $\dim(U+V) = 3$

(c) ✓

(d) $\dim(U) = \dim(V) = 2$

(e) $\forall\, w \in U \cap V,\ \exists\ w = \alpha_1 u_1 + \alpha_2 u_2 = \beta_1 v_1 + \beta_2 v_2$

$\operatorname{rref}(u_1 \ u_2 \ -v_1 \ -v_2)$

$\Rightarrow \alpha_1 = -2s \quad \alpha_2 = s \quad \beta_1 = -2s \quad \beta_2 = s$

$\Rightarrow w = s(u_2 - 2u_1) = s(v_2 - 2v_1)$

Therefore, $U \cap V = \operatorname{Span}\{2v_1 - v_2\}$ $\quad \dim(U \cap V) = 1$.

(f) ✓

Remark: (f) holds for any linear spaces U, V.

It is generalization of $|A \cup B| = |A| + |B| - |A \cap B|$. $\quad A, B$ sets

Here, $U + V$ is the minimal linear space containing $U \cup V$.

in other words, $U + V = \operatorname{Span}\{U \cup V\}$