NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 1**

1. (a) Find a linear equation in the variables x and y that has a general solution $x = 1+2t$ and $y = t$ where t is an arbitrary parameter.

   Solution: One such linear equation is $x - 2y = 1$.

   (b) Show that $x = t$ and $y = \frac{1}{2}t - \frac{1}{2}$, where t is an arbitrary parameter, is also a general solution for the equation constructed in part (a).

   Solution: If we set $x = t$, then $y = \frac{1}{2}(t-1)$.

2. Find a linear equation in the variables x, y, and z that has a general solution

$$\left\{\begin{array}{lcl} x & = & 3 - 4s + t \\ y & = & s \\ z & = & t \end{array}\right. \qquad s, t \in \mathbb{R}\ .$$

Solution: Replacing $y = s$ and $z = t$ into the first equation, get $x + 4y - z = 3$.

3. Solve the following linear systems.

   (a)

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & 4x_3 & = & 3 \\ 2x_1 & + & 3x_2 & + & 3x_3 & = & 15 \\ 5x_1 & - & 3x_2 & + & x_3 & = & 14 \end{array}\right.$$

Solution:

$$\left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 2 & 3 & 3 & 15 \\ 5 & -3 & 1 & 14 \end{array}\right) \xrightarrow{R_3-\frac{5}{3}R_1, R_2-\frac{2}{3}R_1} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 0 & \frac{5}{3} & \frac{17}{3} & 13 \\ 0 & -\frac{19}{3} & \frac{23}{3} & 9 \end{array}\right) \xrightarrow{R_3+\frac{19}{5}R_1} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 0 & \frac{5}{3} & \frac{17}{3} & 13 \\ 0 & 0 & \frac{146}{5} & \frac{292}{5} \end{array}\right)$$

$$\xrightarrow{\frac{5}{146}R_3, 3R_2} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 0 & 5 & 17 & 39 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_2+17R_3} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{\frac{1}{5}R_2} \xrightarrow{R_1-2R_2} \xrightarrow{R_1+4R_2}$$

$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 9 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

System has a unique solution $x_1 = 3, x_2 = 1, x_3 = 2$.

(b)

$$\left\{\begin{array}{rcrcrcrcr} a & + & b & - & c & - & 2d & = & 0 \\ 2a & + & b & - & c & + & d & = & -2 \\ -a & + & b & - & 3c & + & d & = & 4 \end{array}\right.$$

Solution: The reduced row-echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & -\frac{19}{2} & 2 \\ 0 & 0 & 1 & -\frac{9}{2} & 0 \end{array}\right).$$

So a general solution to the system is $a = -2 - 3s, b = 2 + \frac{19s}{2}, c = \frac{9s}{2}, d = s, s \in \mathbb{R}$.

(c)

$$\left\{\begin{array}{rcrcrcr} x & - & 4y & + & 2z & = & -2 \\ x & + & 2y & - & 2z & = & -3 \\ x & - & y & & & = & 4 \end{array}\right.$$

Solution: The linear system is inconsistent. Its reduced row-echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

4. Determine the values of a and b so that the linear system

$$\left\{\begin{array}{rcrcrcr} ax & & & + & bz & = & 2 \\ ax & + & ay & + & 4z & = & 4 \\ & & ay & + & 2z & = & b \end{array}\right.$$

(a) has no solution;

(b) has only one solution;

(c) has infinitely many solutions and a general solution has one arbitrary parameter;

(d) has infinitely many solutions and a general solution has two arbitrary parameters.

Solution:

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) \to \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right).$$

Case 1: $b \neq 2$.

$$\to \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} a & 0 & 0 & 2-b \\ 0 & a & 0 & b-2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

If $a = 0$, system is inconsistent. If $a \neq 0$, system has a unique solution.
Case 2: $b = 2$.

$$\to \left(\begin{array}{ccc|c} a & 0 & 2 & 2 \\ 0 & a & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

If $a = 0$, then system has infinitely many solutions with two parameter. If $a \neq 0$, then system has infinitely many solution with one parameter.

(a) $a = 0$ and $b \neq 2$;

(b) $a \neq 0$ and $b \neq 2$;

(c) $a \neq 0$ and $b = 2$;

(d) $a = 0$ and $b = 2$.

5. (a) Does an inconsistent linear system with more unknowns than equations exist?

Solution: Yes, for example

$$\begin{cases} x + y + z = 0 \\ x + y + z = 1 \end{cases}$$

(b) Does a linear system which has only one solution, but more equations than unknowns, exist?

Solution: Yes, for example

$$\begin{cases} x + y = 0 \\ x - y = 0 \\ 2x + 4y = 0 \end{cases}$$

(c) Does a linear system which has only one solution, but more unknowns than equations, exists?

Solution: No. A linear system with more unknowns than equations will either have no solution or infinitely many solutions.

(d) Does a linear system which has infinitely many solutions, but more equations than unknowns, exists?

Solution: Yes, for example

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \\ 3x + 3y = 3 \end{cases}$$

6. Solve the following system of non-linear equations:

$$\begin{array}{rcrcrcl} x^2 & - & y^2 & + & 2z^2 & = & 6 \\ 2x^2 & + & 2y^2 & - & 5z^2 & = & 3 \\ 2x^2 & + & 5y^2 & + & z^2 & = & 9 \end{array}$$

Solution: Let $X = x^2$, $Y = y^2$, $Z = z^2$, then the system of non-linear equations above is converted to a system of linear equations

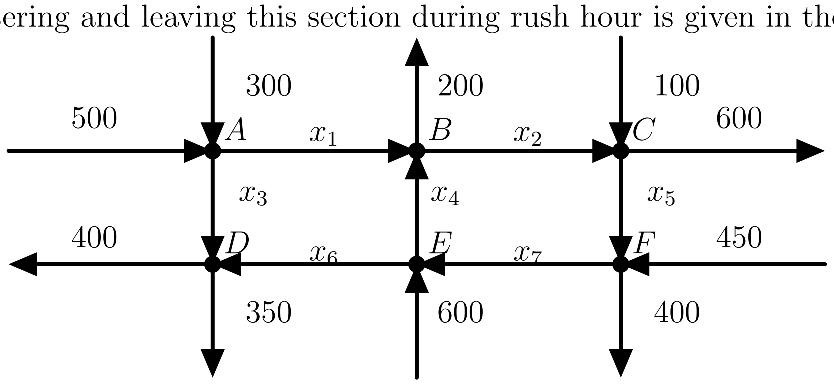
$$\begin{array}{rcrcrcl} X & - & Y & + & 2Z & = & 6 \\ 2X & + & 2Y & - & 5Z & = & 3 \\ 2X & + & 5Y & + & Z & = & 9 \end{array}$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 2 & 2 & -5 & 3 \\ 2 & 5 & 1 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

The solution is $x^2 = X = 4$, $y^2 = Y = 0$, $z^2 = Z = 1$, and hence, $x = \pm 2$, $y = 0$, $z = \pm 1$ are solutions to the system of non-linear equations.

A network of one-way streets of a downtown section can be represented by the diagram below, with traffic flowing in the direction indicated. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram.



(a) Do we have enough information to find the traffic volumes x_1, x_2, x_3, x_4, x_5, x_6, and x_7?

Solution: No, there are 6 equations (junctions), but 7 unknowns.

$$\begin{array}{rcrcrcrcrcrcrcl} x_1 & & & + & x_3 & & & & & & & & & = & 800 \\ x_1 & - & x_2 & & & + & x_4 & & & & & & & = & 200 \\ & & x_2 & & & & & - & x_5 & & & & & = & 500 \\ & & & & x_3 & & & & & + & x_6 & & & = & 750 \\ & & & & & - & x_4 & & & - & x_6 & + & x_7 & = & -600 \\ & & & & & & & & x_5 & & & - & x_7 & = & -50 \end{array}$$

In fact, the RREF is $\left(\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 450 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$, which shows that we need 2 parameters.

The general solution is $x_1 = 50 + s$, $x_2 = 450 + t$, $x_3 = 750 - s$, $x_4 = 600 - s + t$, $x_5 = t - 50$, $x_6 = s$, $x_7 = t$.

(b) Suppose $x_6 = 50$ and $x_7 = 100$. What is x_1, x_2, x_3, x_4, and x_5?

Solution: $x_1 = 100$, $x_2 = 550$, $x_3 = 700$, $x_4 = 650$, $x_5 = 50$.

(c) Can the road between junction A and B be closed for construction while still keeping the traffic flowing in the same directions on the other streets? Explain.

Solution: No, for in that case, $x_6 = -50$, a contradiction.

Extra problems

1. The following is the reduced row-echelon form of the augmented matrix of a linear system:
$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & k \end{array}\right),$$
where $a, b, c, d, e, f, g, h, k$ are constants. Suppose the solution set of this system is represented by a line that passes through the origin and the point $(1, 1, 1)$. Find the values of $a, b, c, d, e, f, g, h, k$. Justify your answer.

   Solution: As the line passes through the origin and the point $(1, 1, 1)$, we have $d = g = h = k = 0$ and $a + b + c = 0$, $e + f = 0$. Since the row-echelon form must have two nonzero rows (so that the solutions form a line in the xyz-space), $e \neq 0$. Finally, the augmented matrix is in reduced row-echelon form, we get $a = e = 1, b = 0$ and hence $f = c = -1$.

2. Determine which of the following statements are true. Justify your answer.

   (a) A homogeneous system can have a non-trivial solution.

   Solution: True. For example,
$$\begin{cases} x \phantom{{}+y} - z = 0 \\ \phantom{x+{}} y - z = 0 \end{cases}$$
The point $(1, 1, 1)$ is a solution.

   (b) A non-homogeneous system can have a trivial solution.

   Solution: False. Given a system of linear equations
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$
substituting $x_1 = x_2 = \cdots = x_n = 0$, we have $b_1 = b_2 = \cdots = b_m = 0$, and hence the system is homogeneous.

   (c) If a homogeneous system has the trivial solution, then it cannot have a non-trivial solution.

   Solution: False. Use the example in (a).

(d) If a homogeneous system has a non-trivial solution, then it cannot have a trivial solution.

Solution: False. Every homogeneous system has the trivial solution.

(e) If a homogeneous system has a unique solution, then the solution has to be trivial.

Solution: True, since the trivial solution is always a solution.

(f) If a homogeneous system has the trivial solution, then the solution has to be unique.

Solution: False. Use the example in (a).

(g) If a homogeneous system has a non-trivial solution, then there are infinitely many solutions to the system.

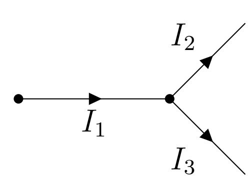
Solution: True. Any multiple of an non-trivial solution to a homogeneous system is solution too. More precisely, if (a, b, c) is a non-trivial solution, then (ka, kb, kc) for any real number k is also a solution.

3. **(Application)** Electrical networks provides information about power sources, such as batteries, and devices powered by these sources, such as light bulbs or motors. A power source 'forces' a current of electrons to flow through the network, where it encounters various resistors, each of which requires that a certain amount of force be applied in order for the current to flow through it.

The fundamental law of electricity is Ohm's law, which states exactly how much force E is needed to drive a current I through a resistor with resistance R. Ohm's law states $E = IR$, in other words, force = current × resistance. Here, force is measured in volts, resistance in ohms and current in amperes.

The following two laws (discovery due to Kirchhoff), govern electrical networks. The first is a 'conservation of flow' law at each node; the second is a 'balancing of votage' law around each loop.

(Kirchoff's Current Law (KCL)) At each node, the sum of the currents flowing into any node is equal to the sum of the currents flowing out of that node. For example, in the diagram below, by KCL, we have $I_1 = I_2 + I_3$.



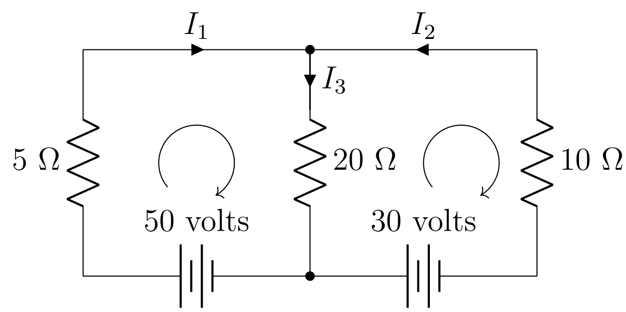
(Kirchoff's Voltage Law (KVL)) In one traversal of any closed loop, the sum of the voltage rises equals to the sum of the voltage drops.

In circuits with multiple loops and batteries there is usually no way to tell in advance which way the currents are flowing, so the usual procedure in circuit analysis is to assign *arbitrary* directions to the current flows in the branches and let the mathematical computations determine whether the assignments are correct. In addition to assigning directions to the current flows, Kirchoff's Voltage Law requires a direction of travel for each closed loop. The choice is arbitrary, but for the sake of consistency we will always take this direction to be *clockwise*. We will also make the following conventions:

- A voltage drop occurs at a resistor if the direction assigned to the current through the resistor is the same as the direction assigned in the loop, and a voltage rise occurs at a resistor if the direction assigned to the current through the resistor is the opposite to that assigned in the loop.
- A voltage rise occurs at a battery if the direction assigned to the loop is from $-$ to $+$ through the battery, and a voltage drop occurs at a battery if the direction assigned to the loop is from $+$ to $-$ through the battery.

If we follow these conventions when calculating currents, then those currents whose directions were assigned correctly will have positive values and those whose direction were assigned incorrectly will have negative values.

For each of the following circuits, use KCL and KVL to write down a linear system with equations involving variables $I_1, I_2, \ldots$. Solve the linear system by Gaussian Elimination.

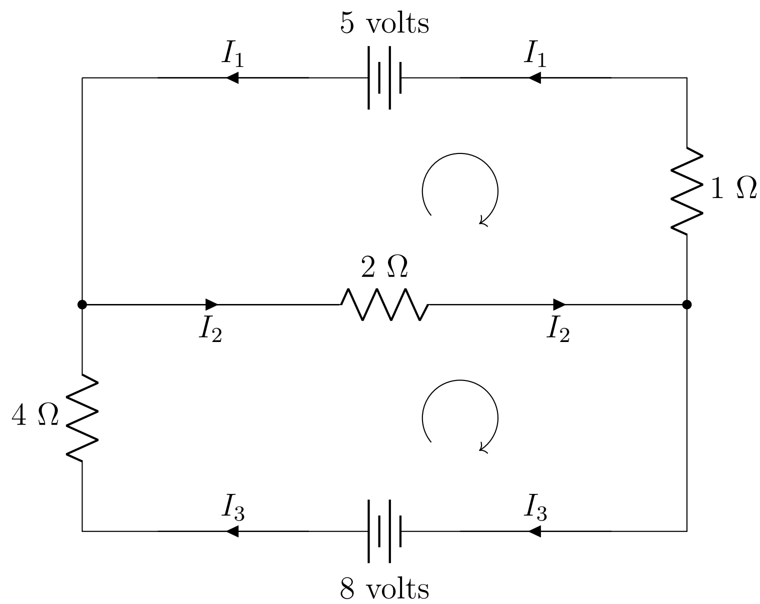


(a)

Solution: The linear system is

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 5I_1 + 20I_3 = 50 \\ -10I_2 - 20I_3 = 30 \end{cases}$$

Solving by Gaussian elimination, we have $I_1 = 6$, $I_2 = -5$ and $I_3 = 1$. Hence the actual direction of the current I_2 is in the opposite direction from what is shown in the figure.



(b)

Solution: The linear system is

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + 2I_2 = 5 \\ 2I_2 + 4I_3 = 8 \end{cases}$$

Solving by Gaussian elimination, we have $I_1 = 1, I_2 = 2, I_3 = 1$. So the actual direction of all the currents are in the from direction as shown in the figure.