1. Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ and $n \times p$ matrices respectively.

   (a) Suppose the homogeneous linear system $\mathbf{Bx} = \mathbf{0}$ has infinitely many solutions. How many solutions does the system $\mathbf{ABx} = \mathbf{0}$ have?

   (b) Suppose $\mathbf{Bx} = \mathbf{0}$ has only the trivial solution. Can we tell how many solutions are there for $\mathbf{ABx} = \mathbf{0}$.

1. (a) Infinitely many. Every solution of $Bx = 0$ is still a solution of $ABx = 0$.

(b) No. Let $B = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $Bx = 0$ $\exists!$ (trivial) solution.

But if $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $A_1 B x = 0$ has a unique solution

if $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $A_2 B x = 0$ has infinitely many solutions

2. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Find a 4×3 matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{I}_3$.

   Hint: Write $\mathbf{X} = (\mathbf{x}_1 \ \ \mathbf{x}_2 \ \ \mathbf{x}_3)$, where $\mathbf{x}_i$ is a 4×1 matrix, for $i = 1, 2, 3$.

   (b) Let $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Find a 3×4 matrix $\mathbf{Y}$ such that $\mathbf{YB} = \mathbf{I}_3$.

2. (a) Write $X = (x_1 \ x_2 \ x_3)$. then we have $Ax_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $Ax_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $Ax_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Solve them simultaneously

$$(A \ \ I_3) \xrightarrow{rref} \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

general solutions: $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ $x_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

$s_1, s_2, s_3 \in \mathbb{R}$

So one possible $X = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(b) Write $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, then $YB = I_3 \iff$ $y_1 B = (1 \ 0 \ 0)$
$y_2 B = (0 \ 1 \ 0)$
$y_3 B = (0 \ 0 \ 1)$

$\iff B^T y_1^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $B^T y_2^T = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $B^T y_3^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So $$(B^T \ \ I_3) \xrightarrow{rref} \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$ $\longrightarrow$ transpose

general solutions $y_1^T = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$ $y_2^T = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$ $y_3^T = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

$s_1, s_2, s_3 \in \mathbb{R}$.

one possible $Y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

Remark: We can also use the block matrix to understand it:

Let $X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$ $\quad A_1 = (A_1 \ \ A_2)$ where $A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad A_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Then $I_3 = AX = A_1 X_1 \Rightarrow X_1 = A_1^{-1}$, $\quad X = \begin{pmatrix} A_1^{-1} \\ 0 \end{pmatrix}$

3. (i) Reduce the following matrices $\mathbf{A}$ to its reduced row-echelon form $\mathbf{R}$.

(ii) For each of the elementary row operation, write the corresponding elementary matrix.

(iii) Write the matrices $\mathbf{A}$ in the form $\mathbf{E}_1\mathbf{E}_2\ldots\mathbf{E}_n\mathbf{R}$ where $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n$ are elementary matrices and $\mathbf{R}$ is the reduced row-echelon form of $\mathbf{A}$.

(a) $\mathbf{A} = \begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix}$. $\quad$ (b) $\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$. $\quad$ (c) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

3. (a)

$$A \xrightarrow[\tilde{E}_1]{R_2 + \frac{2}{5}R_1} \xrightarrow[\tilde{E}_2]{\frac{1}{5}R_1} \xrightarrow[\tilde{E}_3]{5R_2} \xrightarrow[\tilde{E}_4]{R_1 + \frac{2}{5}R_2} R = \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}$$

$$\tilde{E}_1 = \begin{pmatrix} 1 & 0 \\ \frac{2}{5} & 1 \end{pmatrix} \quad \tilde{E}_2 = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix} \quad \tilde{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \quad \tilde{E}_4 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{pmatrix}$$

$\tilde{E}_4 \tilde{E}_3 \tilde{E}_2 \tilde{E}_1 A = R \longrightarrow E_i = \tilde{E}_i^{-1}$, $i = 1, 2, 3, 4$

$$A = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}$$

(b)

$$A \xrightarrow[\begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix}]{R_2 + 2R_1} \xrightarrow[\begin{pmatrix} 1 & & \\ & 1 & \\ -4 & & 1 \end{pmatrix}]{R_3 - 4R_1} \xrightarrow[\begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix}]{R_3 + R_2} \xrightarrow[\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}]{-R_1} \xrightarrow[\begin{pmatrix} 1 & & \\ & \frac{1}{10} & \\ & & 1 \end{pmatrix}]{\frac{1}{10}R_2} \xrightarrow[\begin{pmatrix} 1 & 3 & \\ & 1 & \\ & & 1 \end{pmatrix}]{R_1 + 3R_2} R$$

$$A = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ 4 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 10 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{19}{10} \\ 0 & 1 & -\frac{7}{10} \\ 0 & 0 & 0 \end{pmatrix}$$

(c)

$$A \xrightarrow[\begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix}]{R_2 - 2R_1} \xrightarrow[\begin{pmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{pmatrix}]{R_3 - R_1} \xrightarrow[\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}]{R_2 \leftrightarrow R_3} \xrightarrow[\begin{pmatrix} 1 & & \\ & \frac{1}{3} & \\ & & 1 \end{pmatrix}]{\frac{1}{3}R_2} \xrightarrow[\begin{pmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{pmatrix}]{R_2 - R_3} \xrightarrow[\begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}]{R_1 + R_2} R$$

$$A = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} 1 & & \\ & 3 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Determine if the following matrices are invertible. If the matrix is invertible, find its inverse.

(a) $\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}$. $\quad$ (b) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

4. (a)

$$\left(\begin{array}{cc|c} -1 & 3 & \multirow{2}{*}{I_2} \\ 3 & -2 & \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|cc} 1 & 0 & 2/7 & 3/7 \\ 0 & 1 & 3/7 & 1/7 \end{array}\right)$$

So $\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 2/7 & 3/7 \\ 3/7 & 1/7 \end{pmatrix}$

(b)

$$\left(\begin{array}{ccc|c} -1 & 3 & -4 & \\ 2 & 4 & 1 & I_3 \\ -4 & 2 & -9 & \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$$

So not invertible.

5. Write down the conditions so that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ is invertible.

5. $$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \xrightarrow[R_3 - a^2 R_1]{R_2 - aR_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \xrightarrow{R_3-(a+b)R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

So we need that $a \neq b$, $b \neq c$, $c \neq a$.

Remark: This type of matrix is called the Vandermonde matrix.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_{n+1} \\ a_1^2 & a_2^2 & \cdots & a_{n+1}^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_{n+1}^n \end{pmatrix}$$ invertible iff all a_i's are distinct

6. (a) Suppose $\mathbf{A}$ is a square matrix such that $\mathbf{A}^2 = \mathbf{0}$. Show that $\mathbf{I} - \mathbf{A}$ is invertible, with inverse $\mathbf{I} + \mathbf{A}$.

(b) Suppose $\mathbf{A}^3 = \mathbf{0}$. Is $\mathbf{I} - \mathbf{A}$ invertible?

(c) A square matrix $\mathbf{A}$ is said to be *nilpotent* if there is a positive integer n such that $\mathbf{A}^n = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\mathbf{I} - \mathbf{A}$ is invertible.

6. (a) $(I-A)(I+A) = I^2 + I\cdot A - A\cdot I - A^2 = I + A - A - A^2 = I$.

(b) Yes. Polynomial $1 - x^3 = (1-x)(1+x+x^2)$

So $I = I - A^3 = (I-A)(I+A+A^2)$

(c) Similarly $1 - x^n = (1-x)\left(\sum_{i=0}^{n-1} x^i\right)$

implying $I = I - A^n = (I-A)(I + A + A^2 + \cdots + A^{n-1})$

Remark: Try to prove that this matrix is nilpotent with index $n-1$:

$$N_n := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 \\ & & & & 0 \end{pmatrix}_{n\times n}$$