Review

Assume u, v are column vectors in $\mathbb{R}^n$

$$\underbrace{u \cdot v}_{\text{inner product}} = u_1 v_1 + \cdots + u_n v_n = u^T v = v^T u$$

view as matrices

$$\|u\| = \sqrt{u \cdot u} \qquad \theta = \arccos\left(\frac{u \cdot v}{\|u\|\|v\|}\right)$$

Orthogonal $\quad u \cdot v = 0$

$\sim$ to a set T $\quad u \cdot n = 0, \quad \forall\, n \in T$

If $T = V$ is a subspace with $V = \mathrm{Span}\{v_1, \cdots, v_k\}$, then

u is orthogonal to V iff $u \cdot v_i = 0 \quad \forall\ 1 \le i \le k$

$$\| \qquad v_i^T u$$

iff $\begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{pmatrix} u = 0$

iff $\underbrace{(v_1 \ v_2 \cdots v_k)}_{A}{}^T u = 0$

Orthogonal complement of a subspace V

$$V^\perp := \{\text{all vectors orthogonal to } V\}$$

$$= \{ w \in \mathbb{R}^n \mid w \cdot v = 0 \quad \forall\, v \in V\}$$

$$(\mathrm{Col}(A))^\perp = V^\perp = \mathrm{Null}(A^T)$$

$$\left(\Leftrightarrow \ \mathrm{Null}(A) = (\mathrm{Col}(A^T))^\perp = (\mathrm{Row}(A))^\perp \right)$$

proof: $\quad \mathrm{Col}(A) = \mathrm{Span}\{v_1, \cdots, v_k\}$.

so $v \in (\mathrm{Col}(A))^\perp$ iff $v_i^T v = v_i \cdot v = 0, \quad \forall\ i = 1, \cdots, k$

iff $\begin{pmatrix} v_1^T \\ \vdots \\ v_k^T \end{pmatrix} v = 0 \quad$ i.e. $A^T v = 0$

$(\mathrm{Col}(A))^\perp = \mathrm{Null}(A^T)$

iff $v \in \mathrm{Null}(A^T)$

$$\mathrm{Null}(A) = \mathrm{Null}((A^T)^T) = (\mathrm{Col}(A^T))^\perp = (\mathrm{Row}(A))^\perp \qquad \square$$

Consider a set $S = \{v_1, \cdots, v_k\} \subseteq \mathbb{R}^n$

Orthogonal $\quad v_i \cdot v_j = 0$ if $i \neq j$

Orthonormal $\quad v_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ $\quad$ Kronecker delta function

$\boxed{\text{Orthonormal} \Rightarrow \text{L. I.}}$

Let S be an orthonormal basis of a subspace V of $\mathbb{R}^n$.

Aim: compute $[u]_S$ for every $u \in V$.

Suppose $u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$

then
$$
\begin{aligned}
u \cdot v_i &= (c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) \cdot v_i \\
&= c_1 \delta_{1i} + c_2 \delta_{2i} + \cdots + c_k \delta_{ki} \\
&= c_i
\end{aligned}
$$

$$\Rightarrow \quad [u]_S = (u \cdot v_1, \; u \cdot v_2, \cdots, \; u \cdot v_k)^T$$

If we only assume S is an orthogonal basis, then

$$[u]_S = \left(\frac{u \cdot v_1}{v_1 \cdot v_1}, \frac{u \cdot v_2}{v_2 \cdot v_2}, \cdots, \frac{u \cdot v_k}{v_k \cdot v_k} \right)^T.$$

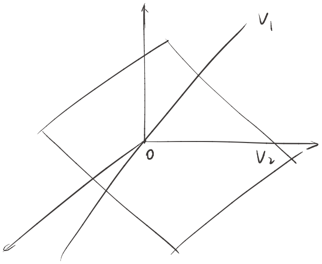
How to determine S is orthogonal / orthonormal ?

Define $Q = (u_1 \; u_2 \cdots u_k) \in M_{n \times k}$

$$Q^T Q = \begin{pmatrix} u_1^T u_1 & u_1^T u_2 & \cdots & u_1^T u_k \\ \vdots & \vdots & & \vdots \\ u_k^T u_1 & u_k^T u_2 & \cdots & u_k^T u_k \end{pmatrix} = \begin{pmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \cdots & u_1 \cdot u_k \\ \vdots & \vdots & & \vdots \\ u_k \cdot u_1 & u_k \cdot u_2 & \cdots & u_k \cdot u_k \end{pmatrix} \in M_{k \times k}$$

$$Q^T Q \left\langle \begin{array}{ccc} \text{diagonal} & \longleftrightarrow & \text{orthogonal} \\ \text{identity} & \longleftrightarrow & \text{orthonormal} \end{array} \right\rangle S$$

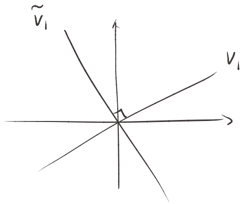
Geometric explanation of linear systems (Cont.)



$\mathbb{R}^3$

$V_1^{\perp} = V_2 \qquad V_2^{\perp} = V_1$

$$\underbrace{\dim V_1 = 1 \qquad \dim V_2 = 2}_{\substack{\text{sum} \\ \| \\ 3}}$$

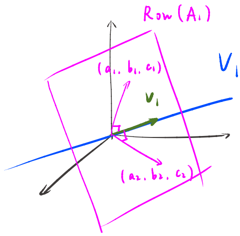


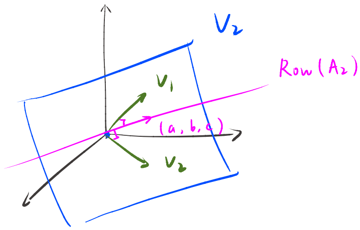
$\mathbb{R}^2$

$V_1^{\perp} = \tilde{V}_1, \qquad \tilde{V}_1^{\perp} = V_1$

$$\underbrace{\dim V_1 = 1, \qquad \dim \tilde{V}_1 = 1}_{\substack{\text{sum} \\ \| \\ 2}}$$

$\longrightarrow$ more general version see Remarks in Ex 4.





$$V_1 = \left\{ (x, y, z) \;\middle|\; \begin{matrix} a_1 x + b_1 y + c_1 z = 0 \\ a_2 x + b_2 y + c_2 z = 0 \end{matrix} \right\}$$

$$V_2 = \{ (x, y, z) \mid ax + by + cz = 0 \}$$

$$A_1 = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

$$A_2 = (a \;\; b \;\; c)$$

$Null(A_1) = V_1$

$Row(A_1) = V_1^{\perp}$

$Col(A_1) = \mathbb{R}^2$

$Null(A_2) = V_2$

$Row(A_2) = V_2^{\perp}$

$Col(A_2) = \mathbb{R}^1$.

1. (a) Let $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ be a linear equation. Express this linear system as $\mathbf{a} \cdot \mathbf{x} = b$ for some (column) vectors $\mathbf{a}$ and $\mathbf{x}$.

1. (a)

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

(b) Find the solution set of the linear system

$$\begin{array}{rcrcrcrl} x_1 & + & 3x_2 & - & 2x_3 & & & = 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & = 0 \\ & & & + & 5x_3 & + & 10x_4 & = 0 \end{array}$$

(b) Let $A = \begin{pmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 0 & 0 & 5 & 10 \end{pmatrix}$ $\quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ $\quad Av = 0$

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 3 & 0 & 4 \\ & & 1 & 2 \\ & & & \end{pmatrix}$$

The solution set is $\left\{ s\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \end{pmatrix},\ s, t \in \mathbb{R} \right\}$

(c) Find a nonzero vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{a}_1 \cdot \mathbf{v} = 0$, $\mathbf{a}_2 \cdot \mathbf{v} = 0$, and $\mathbf{a}_3 \cdot \mathbf{v} = 0$, where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 6 \\ -5 \\ -2 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 10 \end{pmatrix}.$$

(c) $A = \begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \end{pmatrix}$ $\qquad Av = 0$

May choose $s = 1$ & $t = 0$ in (b). $\quad v = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Remark :

This exercise demonstrates the fact that if $\mathbf{A}$ is a $m \times n$ matrix, then the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ consist of all the vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of $\mathbf{A}$.

Indeed, write $A = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}$, then

$$Ax = 0 \iff \underbrace{r_i x}_{= r_i^T \cdot x} = 0 \quad \forall\ 1 \le i \le m$$

2. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set. Suppose

$$\mathbf{x} = \mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3 \quad \text{and} \quad \mathbf{y} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Determine the value for each of the following

(a) $\mathbf{x} \cdot \mathbf{y}$.
(b) $||\mathbf{x}||$ and $||\mathbf{y}||$.
(c) The angle θ between $\mathbf{x}$ and $\mathbf{y}$.

2. (a) $v_i \cdot v_j = \delta_{ij}$

$\Rightarrow \quad x \cdot y = 6$

(b) $x \cdot x = 9 \qquad y \cdot y = 14$

$\Rightarrow \quad \|x\| = 3 \qquad \|y\| = \sqrt{14}$

(c) $\cos\theta = \dfrac{x \cdot y}{\|x\|\|y\|} = \dfrac{2}{\sqrt{14}} \qquad \theta = \arccos\dfrac{2}{\sqrt{14}} = 57.69°$

3. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{V} = (\mathbf{v}_1 \quad \mathbf{v}_2)$.

(a) Compute $\mathbf{v}_1 \cdot \mathbf{v}_1$, $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_2 \cdot \mathbf{v}_1$ and $\mathbf{v}_2 \cdot \mathbf{v}_2$.
(b) Compute $\mathbf{V}^T\mathbf{V}$. What does the entries of $\mathbf{V}^T\mathbf{V}$ represent?

3. (a) $v_1 \cdot v_1 = 6 \qquad v_1 \cdot v_2 = v_2 \cdot v_1 = 0 \qquad v_2 \cdot v_2 = 2$

(b) $V = (v_1 \quad v_2) \qquad V^T = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$

$$V^T V = \begin{pmatrix} v_1^T v_1 & v_1^T v_2 \\ v_2^T v_1 & v_2^T v_2 \end{pmatrix} = \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

4. Let W be a subspace of $\mathbb{R}^n$. The *orthogonal complement* of W, denoted as $W^\perp$, is defined to be

$$W^\perp := \{\ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\ \}.$$

Let $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}$, and $\mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, and $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

(a) Show that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.
(b) Show that S is orthogonal.

4. (a) $A := (w_1 \quad w_2 \quad w_3)$

M1. $\text{rref}(A)$ $\qquad$ M2. $A^T A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(b) $w_1 \cdot w_2 = w_1 \cdot w_3 = w_3 \cdot w_2 = 0$

(c) Show that $W^\perp$ is a subspace of $\mathbb{R}^5$ by showing that it is a span of a set. What is the dimension? (**Hint**: See Question 1.)

(d) Obtain an orthonormal set T by normalizing $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

(c) $v \in W^\perp \iff w_i \cdot v = 0$ for all $i = 1, 2, 3$

$\iff v$ solves the linear system $\begin{pmatrix} w_1^T \\ w_2^T \\ w_3^T \end{pmatrix} x = 0 \quad (*)$

i.e. $A^T x = 0$

Hence $\dim(W^\perp) = 5 - \operatorname{rank}(A^T) = 2$

$W^\perp = \operatorname{Span}\{(2, -1, -2, 1, 0)^T, (1, -2, -3, 0, 4)^T\}$.

(d) $v_i = \dfrac{w_i}{\|w_i\|}$. $v_1 = \frac{1}{\sqrt{5}} w_1 \quad v_2 = \frac{1}{\sqrt{10}} w_2 \quad v_3 = \frac{1}{2} w_3 \quad T = \{v_1, v_2, v_3\}$

(e) Let $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. Find the projection of $\mathbf{v}$ onto W.

(f) Let $\mathbf{v}_W$ be the projection of $\mathbf{v}$ onto W. Show that $\mathbf{v} - \mathbf{v}_W$ is in $W^\perp$.

(e) $v \cdot v_1 = \frac{3}{\sqrt{5}} \quad v \cdot v_2 = -\frac{1}{\sqrt{10}} \quad v \cdot v_3 = 1$

$\Rightarrow v_W = \frac{3}{\sqrt{5}} v_1 - \frac{1}{\sqrt{10}} v_2 + v_3 = \frac{1}{10}(10, -1, 12, 3, 6)^T$

(f) $v - v_W = \frac{1}{10}(10, 1, -2, 7, -16)^T$ solves the linear system $(*)$

$\Rightarrow v - v_W \in W^\perp$

Remark:

This exercise demonstrated the fact that every vector $\mathbf{v}$ in $\mathbb{R}^5$ can be written as $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_W^\perp$, for some $\mathbf{v}_W$ in W and $\mathbf{v}_W^\perp$ in $W^\perp$. In other words, $W + W^\perp = \mathbb{R}^5$.

This property holds for any $\mathbb{R}^n$ & its subspace W.

Characterize $W^\perp$:

① $W + W^\perp = \mathbb{R}^n$

② $W \cap W^\perp = \{0\}$

$\longrightarrow$ We say $\mathbb{R}^n = W \oplus W^\perp$ ← direct sum

pf of ②: Fix any $w_0 \in W \cap W^\perp$. Then $\forall\, w \in W$, $w_0 \cdot w = 0$.

Take $w = w_0 \Rightarrow \|w_0\| = 0 \quad w_0 = 0$.

③ $n = \dim W + \dim W^\perp$

④ $(W^\perp)^\perp = W$

pf of ④: Fix any $w \in W$. Then $\forall\, v \in W^\perp$, $u \cdot v = 0$, i.e. $u \in (W^\perp)^\perp$.

Moreover, $W \oplus W^\perp = \mathbb{R}^n = W^\perp \oplus (W^\perp)^\perp$

$\dim W + \dim W^\perp = n = \dim W^\perp + \dim (W^\perp)^\perp$

$\Rightarrow W = (W^\perp)^\perp$.

5. Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{u_4} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Check that S is an orthogonal basis for $\mathbb{R}^4$.

(b) Is it possible to find a nonzero vector $\mathbf{w}$ in $\mathbb{R}^4$ such that $S \cup \{\mathbf{w}\}$ is an orthogonal set?

5. (a) $A := (u_1 \quad u_2 \quad u_3 \quad u_4)$ $\qquad A^T A = \mathrm{diag}(10, 4, 4, 10)$

(b) No. Otherwise $\#(\underbrace{S \cup \{w\}}_{\text{linearly independent}}) = 5 > \dim \mathbb{R}^4$

(c) Obtain an orthonormal set T by normalizing $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$.

(c) $$T = \Big\{ \underbrace{\tfrac{1}{\sqrt{10}} u_1}_{= u_1'}, \underbrace{\tfrac{1}{2} u_2}_{= u_2'}, \underbrace{\tfrac{1}{2} u_3}_{= u_3'}, \underbrace{\tfrac{1}{\sqrt{10}} u_4}_{= u_4'} \Big\}$$

(d) Let $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$. Find $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

(d) Assume $v = c_1 u_1 + \cdots + c_4 u_4$

Then $v \cdot u_i = (c_1 u_1 + \cdots + c_4 u_4) \cdot u_i = c_i\, u_i \cdot u_i$

$$\Rightarrow \quad c_i = \frac{v \cdot u_i}{u_i \cdot u_i} \qquad [v]_S = \left(\frac{v \cdot u_1}{u_1 \cdot u_1}, \cdots, \frac{v \cdot u_4}{u_4 \cdot u_4} \right)^T$$

$$= \left(\frac{3}{10}, \frac{1}{2}, -1, \frac{9}{10} \right)^T$$

$$(u_1' \quad u_2' \quad u_3' \quad u_4') = (u_1 \quad u_2 \quad u_3 \quad u_4) \underbrace{\begin{pmatrix} \frac{1}{\sqrt{10}} & & & \\ & \frac{1}{2} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{\sqrt{10}} \end{pmatrix}}_{= P}$$

P is the transition matrix from T to S.

$$[v]_T = P^{-1} [v]_S = \left(\frac{3}{\sqrt{10}}, 1, -2, \frac{9}{\sqrt{10}} \right)^T$$

(e) Suppose $\mathbf{w}$ is a vector in $\mathbb{R}^4$ such that $[\mathbf{w}]_S = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Find $[\mathbf{w}]_T$.

(e) $[w]_T = P^{-1} [w]_S = (\sqrt{10}, 4, 2, \sqrt{10})^T$.

Revisit Tutorial 2 Exercise 2.

$$A = \begin{pmatrix} 1 & 1 & & 1 \\ & 1 & 1 & \\ & & 1 & 1 \end{pmatrix}_{3\times 4} \qquad \text{Find } X \in M_{4\times 3} \text{ s.t. } AX = I_3.$$

Answer : $X = (x_1 \ x_2 \ x_3)$ with

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s_1 v \qquad s_1, s_2, s_3 \in \mathbb{R}$$

$$x_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s_2 v \qquad v = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ fixed}$$

$$x_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s_3 v$$

Why v fixed ?

Write $A = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$, then $AX = I_3 \iff r_i \cdot x_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$i, j \in \{1, 2, 3\}$

Are there any vectors orthogonal to all r_1, r_2, r_3 ?

Yes ! Let $V = \operatorname{Span}\{r_1, r_2, r_3\}$ then $\dim V = 3$

$\implies \dim V^\perp = 4 - \dim V = 1$.

i.e. $V^\perp = \operatorname{Span}\{u\}$ with some $u \neq 0$.

One can check : $u = \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \lambda v$ (for some $\lambda \neq 0$)

or $\underline{V^\perp = \operatorname{Span}\{v\}}$!

So : if $\begin{cases} Ax_1 = e_1 \\ Ax_2 = e_2 \\ Ax_3 = e_3 \end{cases}$, then $\begin{cases} A(x_1 + s_1 v) = e_1 \\ A(x_2 + s_2 v) = e_2 \\ A(x_3 + s_3 v) = e_3 \end{cases}$ $\forall s_1, s_2, s_3 \in \mathbb{R}$

$X = (x_1 \ x_2 \ x_3)$ is a solution to $AX = I_3$

$\tilde{X} = (x_1 + s_1 v \quad x_2 + s v \quad x_3 + s v)$ is also a solution to $A\tilde{X} = I_3$