Review

rref $\Longrightarrow$ LU - factorization (lower unit L, upper U)

only permit operations $R_i + cR_j$ $i > j$

$\longrightarrow$ Not every matrix has an LU - factorization

but every matrix does, if admiting some

permutation matrix P. i.e. $PA = LU$

product of e.m. corres. to row swaps

Determinant $\longrightarrow$ cofactor expanding (or Laplace expansion)

(i,j) - cofactor $A_{ij} = (-1)^{i+j} \det(M_{ij})$

(i,j) matrix minor of A

elementary row operations

$R_i + aR_j \rightsquigarrow \times 1$

$cR_i \rightsquigarrow \times c$

$R_i \leftrightarrow R_j \rightsquigarrow \times(-1)$

$\det(cA) = c^n \det A$

trick: adding a row & column

$\det(AB) = \det A \det B$ $\det(A) = \det(A^T)$

$(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det(M_{ji})$

A invertible $\Longleftrightarrow$ $\det A \neq 0$ $\Longleftrightarrow$ $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$

$\Longleftrightarrow$ $Ax = b$ $\exists!$ solution, which is (Cramer's rule)

$x_i = \dfrac{\det(A_i)}{\det(A)}$ A_i formed by replacing the i^{th} column by b

$(AB)^{-1} = B^{-1}A^{-1}$

Find the inverse matrix
- adjoint matrix
- $\operatorname{rref}(A \;\; I) = (I \;\; A^{-1})$

Remarks :

- Why $\mathrm{rref}(A \;\; I) = (I \;\; A^{-1})$?

(From tutorial 1 & 2) if A invertible. $A = E_1 E_2 \cdots E_n$ (i.e. $R = I_n$)

$$(A \;\; I) = (E_1 \cdots E_n \;\; I) \xrightarrow[\text{corre. to } E_1^{-1}]{\text{elementary op.}} (E_2 \cdots E_n \;\; E_1^{-1}) \xrightarrow{\cdots E_2^{-1}} (E_3 \cdots E_n \;\; E_2^{-1} E_1^{-1})$$

$$\xrightarrow{n \text{ steps}} \cdots \longrightarrow (I_n \;\; \underbrace{E_n^{-1} \cdots E_2^{-1} E_1^{-1}}_{\substack{\| \\ (E_1 \cdots E_n)^{-1} = A^{-1}}})$$

- How & Why block matrices ?

$$A_{m\times n} B_{n \times p} \qquad m \Bigg\{ \overbrace{\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix}}^{n} \begin{array}{l} n_1' \\ n_2' \\ \vdots \\ n_{s_2}' \end{array} \overbrace{\begin{pmatrix} & & \\ & B & \\ & & \end{pmatrix}}^{p} \Bigg\} n$$

with the columns of A grouped as $n_1 \;\; n_2 \;\; \cdots \;\; n_{s_1}$

$$n = n_1 + \cdots + n_{s_1} = n_1' + \cdots + n_{s_2}'$$

$$= \begin{pmatrix} A_{11} & \cdots & A_{1s_1} \\ \vdots & & \vdots \\ A_{r1} & \cdots & A_{rs_1} \end{pmatrix} \begin{pmatrix} B_{11} & \cdots & B_{1t} \\ \vdots & & \vdots \\ B_{s_2 1} & \cdots & B_{s_2 t} \end{pmatrix}$$

$$\overset{s_1 = s_2 = s}{=} \begin{pmatrix} \sum_{1\le i\le s} A_{1i} B_{i1} & \cdots & \sum_{1\le i \le s} A_{1i} B_{it} \\ \vdots & & \vdots \\ \sum_{1\le i\le s} A_{ri} B_{i1} & \cdots & \sum_{1\le i\le s} A_{ri} B_{it} \end{pmatrix} \longleftarrow$$

to make sense of every submatrix need $n_i = n_i'$, $i = 1, 2, \cdots, s$

$\rightsquigarrow$ partitions meet

Block matrices allow us to do some "formal computations" :

e.g. $\det \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} = \det(A_{11}) \det(A_{22})$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \xrightarrow[\text{e.o. } R_2 - A_{21} A_{11}^{-1} R_1]{\text{if } A_{11} \text{ invertible}} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \overset{\text{if } A_{11} \text{ invertible}}{=\!=\!=} \det(A_{11}) \det(A_{22} - A_{21} A_{11}^{-1} A_{12})$$

Jordan canonical form $\rightsquigarrow$ $\mathrm{diag}(J_1, \cdots, J_s)$

- Unanswered questions in Tutorial 2 :

Q2. $AX = I_3$. $X = (X_1 \;\; X_2 \;\; X_3) \longrightarrow$ solution

$$X_1 = \mathring{X}_1 + s_1 \underline{v}$$
$$X_2 = \mathring{X}_2 + s_2 \underline{v}$$
$$X_3 = \mathring{X}_3 + s_3 \underline{v}$$

same vector Why ?

I will answer after Tutorial 7.

1. Let $\mathbf{A}$ be the 4×4 matrix obtained from $\mathbf{I}$ by the following sequence of elementary row operations:

$$\mathbf{I} \xrightarrow{\frac{1}{2}R_2} \xrightarrow{R_1 - R_2} \xrightarrow{R_2 \leftrightarrow R_4} \xrightarrow{R_3 + 3R_1} \mathbf{A}.$$

Write $\mathbf{A}^{-1}$ as a product of four elementary matrices.

1.

$$I \underset{E_1}{\xrightarrow{\frac{1}{2}R_2}} \underset{E_2}{\xrightarrow{R_1 - R_2}} \underset{E_3}{\xrightarrow{R_2 \leftrightarrow R_1}} \underset{E_4}{\xrightarrow{R_3 + 3R_1}} A$$

$$A = E_4 E_3 E_2 E_1 \longrightarrow A^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$E_1^{-1} = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ -3 & & 1 & \\ & & & 1 \end{pmatrix}$$

2. Find an LU factorization for the matrices $\mathbf{A}$, and solve the equation $\mathbf{Ax} = \mathbf{b}$.

(a) $\mathbf{A} = \begin{pmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$.

(b) $\mathbf{A} = \begin{pmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 17 \end{pmatrix}$.

2. (a)

$$A \underset{E_1}{\xrightarrow{R_2 + 3R_1}} \underset{E_2}{\xrightarrow{R_3 - 4R_1}} \underset{E_3}{\xrightarrow{R_3 + R_2}} U = \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L = E_3 E_2 E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}$$

So $\quad Ax = b \iff \begin{cases} Ux = y \\ Ly = b \end{cases}$

solve $Ly = b \Rightarrow y = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \Rightarrow$ solve $Ux = y \Rightarrow x = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

(b)

$$A \underset{E_1}{\xrightarrow{R_2 - 3R_1}} \underset{E_2}{\xrightarrow{R_3 + \frac{1}{2}R_1}} \underset{E_3}{\xrightarrow{R_3 + 2R_2}} U = \begin{pmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$$L = E_3 E_2 E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix}$$

solve $Ly = b \Rightarrow y = \begin{pmatrix} 0 \\ 0 \\ 17 \end{pmatrix} \Rightarrow$ solve $Ux = y \Rightarrow x = \begin{pmatrix} -\frac{17}{5} + \frac{4}{3}s \\ -\frac{17}{5} + \frac{5}{3}s \\ s \\ \frac{17}{5} \end{pmatrix} \quad s \in \mathbb{R}$

3. Let $\mathbf{A} = \begin{pmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{pmatrix}$.

(a) Find an LU factorization of $\mathbf{A}$.

(b) We can find a LU factorization in MATLAB using the command `lu`. Enter the following codes.

```
>> A=[2 -6 6;-4 5 -7;3 5 -1;-6 4 -8;8 -3 9];

>> [L U]=lu(sym(A)).
```

Compare the results with the answer in (a).

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$$L = \begin{pmatrix} 1 & & & & \\ -2 & 1 & & & \\ \frac{3}{2} & -2 & 1 & & \\ -3 & 2 & 0 & 1 & \\ 4 & -3 & 0 & 0 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & -6 & 6 \\ & -7 & 5 \\ & & \\ & & \\ & & \end{pmatrix}$$

4. Let $\mathbf{A} = \begin{pmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{pmatrix}$. Compute the determinant of $\mathbf{A}$ and find all the values of x such that $\mathbf{A}$ is singular.

4、

$$\det A = -x \cdot \begin{vmatrix} -x & 1 \\ -5 & 4-x \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 2 & 4-x \end{vmatrix}$$

$$= x^2(4-x) - 5x + 2$$

$$= -(x-1)^2(x-2)$$

A singular when $x = 1$ or $x = 2$

Question 4

Let $\mathbf{A} = \begin{pmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{pmatrix}$. Compute the determinant of $\mathbf{A}$ and find all the values of x such that $\mathbf{A}$ is singular.

```
>> syms x
>> A=[-x 1 0;0 -x 1;2 -5 4-x];
>> det(A)
>> simplify(ans)
```

The matrix $\mathbf{A}$ is singular if and only if $\det \mathbf{A} = 0$ which is $x = 1$ or $x = 2$.

5. Show that $\begin{vmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix} = (1+x^3)\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$.

5. Claim:

$$\det\begin{pmatrix} a_{11}+a_{11}' & \cdots & a_{1n}+a_{n1} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

cofactor expansion along 1^{st} row

$$= \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \det\begin{pmatrix} a_{11}' & \cdots & a_{1n}' \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Use this claim

$$\Rightarrow \quad LHS = \begin{vmatrix} a & b & c \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix} + x\begin{vmatrix} p & q & r \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix}$$

$$= RHS + x^3\, RHS$$

Question 5

Show that $\begin{vmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix} = (1+x^3)\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$.

```
>> syms a b c p q r u v w x;
>> A=[a+p*x b+q*x c+r*x;p+u*x q+v*x r+w*x;u+a*x v+b*x w+c*x]
>> A(2,:)=A(2,:)-x*A(3,:);A=simplify(A)
>> A(1,:)=A(1,:)-x*A(2,:); A=simplify(A)
```

$$\begin{pmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{pmatrix} \xrightarrow{R_2-xR_3} \begin{pmatrix} a+px & b+qx & c+rx \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{pmatrix} \xrightarrow{R_1-xR_2} \begin{pmatrix} a(1+x^3) & b(1+x^3) & c(1+x^3) \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{pmatrix}.$$

Assume $x \neq -1$,

```
>> A(1,:)=A(1,:)/(1+x^3);A=simplify(A)
>> A(2,:)=A(2,:)+x^2*A(1,:);A=simplify(A)
>> A(3,:)=A(3,:)-x*A(1,:)
```

6. Let $\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. Compute

(a) $\det(3\mathbf{A}^T)$;

(b) $\det(3\mathbf{A}\mathbf{B}^{-1})$; and

(c) $\det((3\mathbf{B})^{-1})$.

6. $\det A = -2 \qquad \det B = 3$

(a) -162

(b) -54

(c) $\frac{1}{243}$

7. Use Cramer's rule to solve

$$\begin{cases} x + 5y + 3z = 1 \\ 2y - 2z = 2 \\ y + 3z = 0 \end{cases}$$

7、

Question 7

Use Cramer's rule to solve

$$\begin{cases} x + 5y + 3z = 1 \\ 2y - 2z = 2 \\ y + 3z = 0 \end{cases}$$

Cramer's rule: if $\mathbf{A}$ is invertible, unique solution to $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A}_1) \\ \det(\boldsymbol{A}_2) \\ \\ det(\boldsymbol{A}_3) \end{pmatrix}$, where $\boldsymbol{A}_i$ is the matrix constructed from $\boldsymbol{A}$ by replacing the i-th column with $\boldsymbol{b}$.

```
>> A=[1 5 3;0 2 -2;0 1 3]; b=[1;2;0]; A1=A;A1(:,1)=b;A2=A;A2(:,2)=b;A3=A;A3(:,3)=b;
>> A, A1, A2, A3
>> x=(1/det(A))*[det(A1);det(A2);det(A3)]
```

8. Compute the adjoint of $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 6 \end{pmatrix}$, and use it to compute $\mathbf{A}^{-1}$.

8.

$$\operatorname{adj} A = \begin{pmatrix} 12 & 6 & -5 \\ 3 & 0 & 1 \\ -6 & -3 & -2 \end{pmatrix}$$

$$\det A = -3$$

$$A^{-1} = \begin{pmatrix} -4 & -2 & \frac{5}{3} \\ -1 & 0 & \frac{1}{3} \\ 2 & 1 & -\frac{2}{3} \end{pmatrix}$$

adjoint (A)

Another possible approach to Q5 :

trick : adding a row & column

$$\begin{vmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix} = \begin{vmatrix} 1 & a & b & c \\ 0 & a+px & b+qx & c+rx \\ 0 & p+ux & q+vx & r+wx \\ 0 & u+ax & v+bx & w+cx \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & b & c \\ -1 & px & qx & rx \\ 1/x & ux & vx & wx \\ -\frac{1}{x^2} & ax & bx & cx \end{vmatrix}$$

take out the factor x

$$= x^3 \begin{vmatrix} 1 & a & b & c \\ -\frac{1}{x} & p & q & r \\ \frac{1}{x^2} & u & v & w \\ -\frac{1}{x^3} & a & b & c \end{vmatrix}$$

$$= x^3 \begin{vmatrix} 1+\frac{1}{x^3} & 0 & 0 & 0 \\ -\frac{1}{x} & p & q & r \\ \frac{1}{x^2} & u & v & w \\ -\frac{1}{x^3} & a & b & c \end{vmatrix}$$

$$= (x^3+1) \begin{vmatrix} p & q & r \\ u & v & w \\ a & b & c \end{vmatrix}$$