# MA1522: Linear Algebra for Computing

Tutorial 11

# Revision

# Linear Transformation

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$,

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}).$$

$T$ is a linear transformation $\Leftrightarrow$ there is a $m \times n$ matrix $\mathbf{A}$ such that $T(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

$$\mathbf{A} = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{pmatrix}$$

is called the standard matrix of $T$.

For any $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \in \mathbb{R}^n$ and $c_1, c_2, ..., c_k \in \mathbb{R}$,

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_k T(\mathbf{v}_k).$$

To show $T$ is not a linear transformation, either

- $T(\mathbf{0}) \neq \mathbf{0}$;
- there exists $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$;
- there exists $\mathbf{u} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $T(\alpha\mathbf{u}) \neq \alpha T(\mathbf{u})$.

# Range, Kernel, Rank, and Nullity

The range of $T$ is $R(T) = \{\ T(\mathbf{v}) = \mathbf{A}\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^n\ \} = \mathrm{Col}(\mathbf{A})$. $\Rightarrow$ Rank of $T$ is the rank of $\mathbf{A}$,

$$\mathrm{rank}(T) = \dim(R(T)) = \dim(\mathrm{Col}(\mathbf{A})) = \mathrm{rank}(\mathbf{A}).$$

The kernel of $T$ is $\ker(T) = \{\ \mathbf{u} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{u} = T(\mathbf{u}) = \mathbf{0}\ \} = \mathrm{Null}(\mathbf{A})$. $\Rightarrow$ Nullity of $T$ is the nullity of $\mathbf{A}$,

$$\mathrm{nullity}(T) = \dim(\ker(T)) = \mathrm{nullity}(\mathbf{A}).$$

### Theorem (Rank-Nullity Theorem)

*Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then*

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = n.$$

# One-to-one and Onto

$T$ is one-to-one (or injective) if for any $\mathbf{b} \in \mathbb{R}^m$, $T(\mathbf{x}) = \mathbf{b}$ has at most one solution $\mathbf{x}$ (or $T(\mathbf{u}) = T(\mathbf{v}) \Rightarrow \mathbf{u} = \mathbf{v}$). $\Leftrightarrow$ $\ker(T) = \{\mathbf{0}\}$ or $\text{nullity}(T) = 0$.

$T$ is onto (or surjective) if for any $\mathbf{b} \in \mathbb{R}^m$, there is an $\mathbf{u} \in \mathbb{R}^n$ such that $T(\mathbf{u}) = \mathbf{b}$, i.e. $T(\mathbf{x}) = \mathbf{b}$ is consistent. $\Leftrightarrow$ $R(T) = \mathbb{R}^m$, or $\text{rank}(T) = m$.

If $T$ is one-to-one or onto, then the standard matrix $\mathbf{A}$ is a full rank matrix.

If $m = n$, then $T$ is one-to-one $\Leftrightarrow$ $T$ is onto $\Leftrightarrow$ $\mathbf{A}$ is invertible.

# Full Rank Equals Number of Columns

**Theorem (Full rank equals to number of columns)**

*Suppose* $\mathbf{A}$ *is a* $m \times n$ *matrix. The following statements are equivalent.*

(i) $\mathbf{A}$ *is full rank, where the rank is equal to the number of columns,* $\text{rank}(\mathbf{A}) = n$.

(ii) *The rows of* $\mathbf{A}$ *spans* $\mathbb{R}^n$, $Row(\mathbf{A}) = \mathbb{R}^n$.

(iii) *The columns of* $\mathbf{A}$ *are linearly independent.*

(iv) *The homogeneous system* $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution, that is,* $\text{Null}(\mathbf{A}) = \{\mathbf{0}\}$.

(v) $\mathbf{A}^T\mathbf{A}$ *is an invertible matrix of order* $n$.

(vi) $\mathbf{A}$ *has a left inverse.*

(vii) $\mathbf{A}$ *has a QR factoization.*

(viii) *For any* $\mathbf{b} \in \mathbb{R}^m$, *the least square solutions to* $\mathbf{Ax} = \mathbf{b}$ *is unique.*

(ix) *The transformation* $T_{\boldsymbol{A}}$ *represented by* $\boldsymbol{A}$ *is injective.*

# Full Rank Equals Number of Rows

**Theorem**

*Suppose* $\mathbf{A}$ *is a* $m \times n$ *matrix. The following statements are equivalent.*

(i) $\mathbf{A}$ *is full rank, where the rank is equal to the number of rows,* $\text{rank}(\mathbf{A}) = m$.

(ii) *The columns of* $\mathbf{A}$ *spans* $\mathbb{R}^m$, $Col(\mathbf{A}) = \mathbb{R}^m$.

(iii) *The rows of* $\mathbf{A}$ *are linearly independent.*

(iv) *The linear system* $\mathbf{Ax} = \mathbf{b}$ *is consistent for every* $\mathbf{b} \in \mathbb{R}^m$.

(v) $\mathbf{AA}^T$ *is an invertible matrix of order* $m$.

(vi) $\mathbf{A}$ *has a right inverse.*

(vii) *The transformation* $T_{\boldsymbol{A}}$ *represented by* $\boldsymbol{A}$ *is surjective.*

# Equivalent Statements for Invertibility

## Theorem (Equivalent Statements for Invertibility)

*Let $\boldsymbol{A}$ be a square matrix of order $n$. The following statements are equivalent.*

(i) $\boldsymbol{A}$ is *invertible*.

(ii) $\boldsymbol{A}^T$ is *invertible*.

(iii) (*left inverse*) *There is a matrix* $\boldsymbol{B}$ *such that* $\boldsymbol{BA} = \boldsymbol{I}$.

(iv) (*right inverse*) *There is a matrix* $\boldsymbol{B}$ *such that* $\boldsymbol{AB} = \boldsymbol{I}$.

(v) *The reduced row-echelon form of* $\boldsymbol{A}$ *is the identity matrix.*

(vi) $\boldsymbol{A}$ *can be expressed as a product of elementary matrices.*

(vii) *The homogeneous system* $\boldsymbol{Ax} = \boldsymbol{0}$ *has only the trivial solution.*

(viii) *For any* $\boldsymbol{b}$, *the system* $\boldsymbol{Ax} = \boldsymbol{b}$ *has a unique solution.*

(ix) *The determinant of* $\boldsymbol{A}$ *is nonzero*, $\det(\boldsymbol{A}) \neq 0$.

(x) *The columns/rows of* $\boldsymbol{A}$ *are linearly independent.*

(xi) *The columns/rows of* $\boldsymbol{A}$ *spans* $\mathbb{R}^n$.

(xii) $\text{rank}(\boldsymbol{A}) = n$ ($\boldsymbol{A}$ *has full rank*).

(xiii) $\text{nullity}(\boldsymbol{A}) = 0$.

(xiv) $0$ *is not an eigenvalue of* $\boldsymbol{A}$.

(xv) *The transformation* $T_{\boldsymbol{A}}$ *represented by* $\boldsymbol{A}$ *is injective.*

(xvi) *The transformation* $T_{\boldsymbol{A}}$ *represented by* $\boldsymbol{A}$ *is surjective.*

(xvii) *The transformation* $T_{\boldsymbol{A}}$ *represented by* $\boldsymbol{A}$ *is bijective.*

# Finding Standard Matrix of a Linear Transformation

## Definition

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$. The *representation of $T$ with respect to basis $S$*, denoted as $[T]_S$, is defined to be the $m \times n$ matrix

$$[T]_S = \begin{pmatrix} T(\boldsymbol{u}_1) & T(\boldsymbol{u}_2) & \cdots & T(\boldsymbol{u}_n) \end{pmatrix}.$$

## Theorem

*Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$. Then for any vector $\boldsymbol{v}$ in $\mathbb{R}^n$,*

$$T(\boldsymbol{v}) = [T]_S[\boldsymbol{v}]_S,$$

*that is, the image $T(\boldsymbol{v})$ is the product of the representation of $T$ with respect to basis $S$ with the coordinates $\boldsymbol{v}$ with respect to basis $S$. Moreover, if $\boldsymbol{P}$ is the transition matrix from the standard basis $E$ of $\mathbb{R}^n$ to basis $S$, then the standard matrix $\boldsymbol{A}$ of $T$ is given by*

$$\boldsymbol{A} = [T]_S\boldsymbol{P} = \begin{pmatrix} T(\boldsymbol{u}_1) & T(\boldsymbol{u}_2) & \cdots & T(\boldsymbol{u}_n) \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}^{-1}.$$

# Tutorial 11 Solutions

# Question 1(a)

$T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(i) Determine whether the following are linear transformations.

(ii) Write down the standard matrix for each other the linear transformations.

(iii) Find a basis for the range for each of the linear transformations.

(iv) Find a basis for the kernel for each of the linear transformations.

$$T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\begin{pmatrix} 1 \\ -1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \Rightarrow \mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Either observe that the columns of $\mathbf{A}_1$ are linearly independent, or that $\mathbf{A}$ is invertible, we can conclude that $R(T_1) = \mathbb{R}^2$ (hence, $\text{rank}(T_1) = 2$) and $\ker(T_1) = \{\mathbf{0}\}$ (hence, $\text{nullity}(T_1) = 0$). May let $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ or $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ be the basis for the range of $T_1$, and $\{\}$ the empty set be the basis for the kernel of $T_1$

# Question 1(b)

$T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2^x \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(i) Determine whether the following are linear transformations.

(ii) Write down the standard matrix for each other the linear transformations.

(iii) Find a basis for the range for each of the linear transformations.

(iv) Find a basis for the kernel for each of the linear transformations.

$T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2^0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So, $T_2$ is not a linear transformation.

# Question 1(c)

$T_3 \colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(i) Determine whether the following are linear transformations.

(ii) Write down the standard matrix for each other the linear transformations.

(iii) Find a basis for the range for each of the linear transformations.

(iv) Find a basis for the kernel for each of the linear transformations.

$T_3 : \mathbb{R}^2 \to \mathbb{R}^3$, $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. $\Rightarrow \mathbf{A}_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. A basis for the range of $T_3$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$, and a basis for the kernel is $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

# Question 1(d)

$T_4 \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_4\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y-x \\ y-z \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

(i) Determine whether the following are linear transformations.

(ii) Write down the standard matrix for each other the linear transformations.

(iii) Find a basis for the range for each of the linear transformations.

(iv) Find a basis for the kernel for each of the linear transformations.

$T_4$ is not a linear transformation because $T_4\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

# Question 1(e)

$T_5 : \mathbb{R}^5 \to \mathbb{R}$ such that $T_5\left(\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}\right) = x_3 + 2x_4 - x_5$ for $\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^5$.

(i) Determine whether the following are linear transformations.

(ii) Write down the standard matrix for each other the linear transformations.

(iii) Find a basis for the range for each of the linear transformations.

(iv) Find a basis for the kernel for each of the linear transformations.

$T_5 \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = 0x_1 + 0x_2 + x_3 + 2x_4 - x_5 = (0\ 0\ 1\ 2\ -1) \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}$. $\Rightarrow \mathbf{A}_5 = \begin{pmatrix} 0 & 0 & 1 & 2 & -1 \end{pmatrix}$. The range is $\mathbb{R}$, and a basis is $\{1\}$. A basis for the kernel is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

# Question 1(f)

$T_6 : \mathbb{R}^n \to \mathbb{R}$ such that $T_6(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$.

(i) Determine whether the following are linear transformations.

(ii) Write down the standard matrix for each other the linear transformations.

(iii) Find a basis for the range for each of the linear transformations.

(iv) Find a basis for the kernel for each of the linear transformations.

$$T_6\left(2\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\right) = \begin{pmatrix}2\\0\\\vdots\\0\end{pmatrix} \cdot \begin{pmatrix}2\\0\\\vdots\\0\end{pmatrix} = 4 \neq 2 = 2\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = 2T_6\left(\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\right).$$

The function $T_6$ is not a linear transformation.

# Question 2(a)

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $G : \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformations such that

$$F\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 - 3x_3 \\ 5x_2 - x_3 \end{pmatrix} \text{ and } G\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix},$$

and let $\mathbf{A}_F$ and $\mathbf{B}_G$ be the standard matrix of $F$ and $G$, respectively. Find $\mathbf{A}_F$ and $\mathbf{B}_G$.

$$F\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 - 3x_3 \\ 5x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 0 & 5 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \Rightarrow \quad \mathbf{A}_F = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 0 & 5 & -1 \end{pmatrix}$$

$$G\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Rightarrow \quad \mathbf{B}_G = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

# Question 2(b)

Define

$$(F + G)(\mathbf{x}) := F(\mathbf{x}) + G(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathbb{R}^3.$$

Is $(F + G)$ a linear transformation? If it is, find its standard matrix.

For any $\mathbf{x} \in \mathbb{R}^3$,

$$(F + G)(\mathbf{x}) := F(\mathbf{x}) + G(\mathbf{x}) = \mathbf{A}_F\mathbf{x} + \mathbf{B}_G\mathbf{x} = (\mathbf{A}_F + \mathbf{B}_G)(\mathbf{x}).$$

Therefore $(F + G)$ is a linear transformation and the standard matrix is $(\mathbf{A}_F + \mathbf{B}_G)$.

# Question 2(c)

Write down the formula for $F(G(\mathbf{x}))$ and find its standard matrix.

$$\begin{aligned} F(G(\mathbf{x})) &= F\begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} (x_3 - x_1) - 2(x_2 + 5x_1) \\ (x_3 - x_1) + (x_2 + 5x_1) - 3(x_1 + x_2 + x_3) \\ 5(x_2 + 5x_1) - (x_1 + x_2 + x_3) \end{pmatrix} = \begin{pmatrix} -11x_1 - 2x_2 + x_3 \\ x_1 - 2x_2 - 2x_3 \\ 24x_1 + 4x_2 - x_3 \end{pmatrix} \\ &= \begin{pmatrix} -11 & -2 & 1 \\ 1 & -2 & -2 \\ 24 & 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \text{the standard matrix is } \begin{pmatrix} -11 & -2 & 1 \\ 1 & -2 & -2 \\ 24 & 4 & -1 \end{pmatrix}, \end{aligned}$$

which is the product $\mathbf{A}_F\mathbf{B}_G$, $\begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 0 & 5 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -11 & -2 & 1 \\ 1 & -2 & -2 \\ 24 & 4 & -1 \end{pmatrix}$.

More generally, if $S : \mathbb{R}^n \to \mathbb{R}^k$ and $T : \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations with standard matrices $\mathbf{A}$ and $\mathbf{B}$ respectively, then for all $\mathbf{x} \in \mathbb{R}^n$,

$$T(S(\mathbf{x})) = T(\mathbf{A}\mathbf{x}) = \mathbf{B}\mathbf{A}\mathbf{x}.$$

This is called the composition of $S$ and $T$, and is denoted as $T \circ S$.

# Question 2(d)

Find a linear transformation $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$H(G(\mathbf{x})) = \mathbf{x}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

By (c), we are finding a matrix $\mathbf{C}_H$ such that $\mathbf{C}_H \mathbf{B}_G \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, which is true if and only if $\mathbf{C}_H \mathbf{B}_G = \mathbf{I}_3$, that is $\mathbf{C}_H = \mathbf{B}_G^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{5}{3} & -\frac{2}{3} & \frac{5}{3} \\ \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$. Hence,

$$H\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{5}{3} & -\frac{2}{3} & \frac{5}{3} \\ \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x_1 + x_2 - x_3 \\ -5x_1 - 2x_2 + 5x_3 \\ 4x_1 + x_2 - x_3 \end{pmatrix}.$$

# Question 3

For each of the following linear transformations, (i) determine whether there is enough information for us to find the formula of $T$; and (ii) find the formula and the standard matrix for $T$ if possible.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$, and we are given the image of $T$ on $S$, that is, we are given $T(\mathbf{u}_1), T(\mathbf{u}_2), ..., T(\mathbf{u}_n)$. Then

$$\begin{pmatrix} T(\mathbf{u}_1) & T(\mathbf{u}_2) & \cdots & T(\mathbf{u}_n) \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{u}_1 & \mathbf{A}\mathbf{u}_2 & \cdots & \mathbf{A}\mathbf{u}_n \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix},$$

and hence,

$$\mathbf{A} = \begin{pmatrix} T(\mathbf{u}_1) & T(\mathbf{u}_2) & \cdots & T(\mathbf{u}_n) \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}^{-1}.$$

That is, the standard matrix of $T$ can be obtained by multiplying the matrix whose columns are formed by the image of $T$ on the basis, to the inverse of the matrix whose columns are formed by the basis.

# Question 3(a)

$T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\3\\0\\1\end{pmatrix}, \; T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\2\\-1\\4\end{pmatrix}, \text{ and } T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\4\\1\\6\end{pmatrix}.$$

In this case we can directly construct the standard matrix.

$$\mathbf{A} = \left( T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) \quad T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) \quad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) \right) = \begin{pmatrix}1 & 2 & 0\\3 & 2 & 4\\0 & -1 & 1\\1 & 4 & 6\end{pmatrix},$$

that is, $T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}x+2y\\3x+2y+4z\\-y+z\\x+4y+6z\end{pmatrix}.$

# Question 3(b)

$T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\0\end{pmatrix}, \quad T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\2\end{pmatrix}, \text{ and } T\left(\begin{pmatrix}2\\0\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix}.$$

First observe that $\left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$. Hence,

$$\mathbf{A} = \left(T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) \quad T\left(\begin{pmatrix}1\\1\end{pmatrix}\right)\right)\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1/2 & -1/2\\1/2 & 1/2\end{pmatrix} = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}.$$

Thus $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x-y\\x+y\end{pmatrix}$.

# Question 3(c)

$T : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right) = -1,\ T\left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) = 1 \text{ and } T\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) = 0.$$

Observe that $\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$ is not a basis since it is linearly dependent. Hence we do not have enough information to reconstruct $\mathbf{A}$. For example, $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = y$ and $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = -x - z$ are different linear transformations that satisfy the given information.

# Question 4(a)

Consider the transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^6$ such that the rank is 4. Determine its rank and nullity, and whether it is one-to-one, and/or onto.

By rank-nullity theorem, $4 = \text{rank}(T) + \text{nullity}(T) = 4 + \text{nullity}(T)$. So the nullity is 0. Since $\text{rank}(T) = 4 < 6 = \dim(\mathbb{R}^6)$, $T$ is not onto; and since $\text{nullity}(T) = 0$, $T$ is one-to-one.

# Question 4(b)

$T : \mathbb{R}^6 \to \mathbb{R}^4$ such that the nullity is 2.

By rank-nullity theorem, $6 = \text{rank}(T) + \text{nullity}(T) = \text{rank}(T) + 2 \Rightarrow \text{rank}(T) = 4 = \dim(\mathbb{R}^4)$. So $T$ is onto, but not one-to-one.

# Question 4(c)

$T : \mathbb{R}^4 \to \mathbb{R}^6$ such that the reduce row-echelon form of its standard matrix has 3 nonzero rows.

Since the rref of the standard matrix has 3 nonzero rows, rank$(T) = 3$. So, nullity$(T) = 3 - 4 = 1$. $T$ is neither one-to-one nor unto.

# Question 4(d)

$T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T$ is one-to-one.

Since $T$ is one-to-one, nullity$(T) = 0$ and hence rank$(T) = 3 - 0 = 3$, which means that $T$ is unto too. Alternatively, observe that the standard matrix $\mathbf{A}$ is a 3 by 3 square matrix and since the nullity is 0, $\mathbf{A}$ is invertible, and hence, $T$ is onto too.