Review

Dimension : the maximum permitted number of linearly independent vectors

S is a basis of V i.e. $\mathrm{span}(S) = V$ & S is L.I.

$\iff \dim V = \# S$ & $S \subseteq V$ is L.I.

$\iff V \subseteq \mathrm{Span}(S)$ & $\# S = \dim V$

* For linear spaces U, V, if $U \subseteq V$ & $\dim U = \dim V$, then $U = V$.

Assume S & T are 2 bases of V, then

$S = \{u_1, \cdots, u_k\}$
$T = \{v_1, \cdots, v_k\}$

$\forall\, v \in V$, coordinates of v relative to the basis S

$$[v]_S := (c_1 \cdots c_k)^T \quad \text{where} \quad v = c_1 u_1 + \cdots + c_k u_k$$

Transition matrix from T to S

$$P := ([v_1]_S \ \cdots \ [v_k]_S) \iff [v]_S = P[v]_T \quad \forall\, v \in V.$$

$\rightsquigarrow$ Transition matrix from S to T is P^{-1}.

Subspaces associated with a matrix $A \in M_{m \times n}$

Column space $\mathrm{Col}(A) \subseteq \mathbb{R}^m$

basis : columns of A corresponding to the pivot columns of $\mathrm{rref}(A)$

Row space $\mathrm{Row}(A) \subseteq \mathbb{R}^n$

basis : nonzero rows of $\mathrm{rref}(A)$

$\to$ $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A))$

Nullspace $\mathrm{Null}(A) \subseteq \mathbb{R}^n$ $\to$ $\mathrm{nullity}(A) = \dim(\mathrm{Null}(A))$

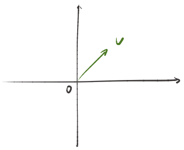
The Dimension Theorem : $n = \mathrm{rank}(A) + \mathrm{nullity}(A)$

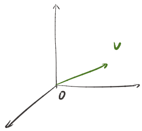
matrices $\longleftrightarrow$ linear transformations

Revisit : geometric explanations of linear systems

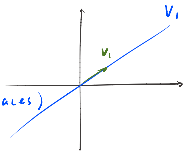
$\mathbb{R}^2$ $\mathbb{R}^3$ $\cdots$ $\mathbb{R}^n$ $(n>3)$

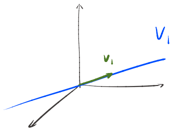
vectors

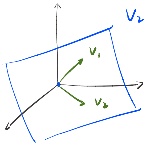




subspaces
(or vector spaces)







$V_1 = \{(x,y) \mid ax + by = 0\}$

$V_1 = \left\{(x,y,z) \;\middle|\; \begin{matrix} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{matrix}\right\}$

$V_2 = \{(x,y,z) \mid ax + by + cz = 0\}$

homogeneous linear systems $\quad Ax = 0$

$\mathrm{null}(A_1) = V_1$

$\mathrm{rank}\, A_1 = \#$ equations $= 1$

$\dim V_1 = 1$.

$\Longleftrightarrow$ V_1 has a basis $\{v_1\}$

$1 + 1 = 2$

$\mathrm{null}(A_1) = V_1$

$\mathrm{rank}\, A_1 = \#$ eq's $= 2$

$\dim V_1 = 1$

$\Longleftrightarrow$ $\cdots$ $\{v_1\}$

$2 + 1 = 3$

$\mathrm{null}(A_2) = V_2$

$\mathrm{rank}\, A_2 = 1$

$\dim V_2 = 2$

$\Longleftrightarrow$ V_2 has a basis $\{v_1, v_2\}$

$1 + 2 = 3$

$\longrightarrow$ $\boxed{\text{the dimension thm}}$

$A \in M_{m\times n}$ $\quad$ $\mathrm{rank}\, A$ = the number of "essential equations" in the linear system $Ax = 0$. $\quad$ i.e. independent equations

namely, the remaining equations $(m - \mathrm{rank}\, A)$ can be derived from those "essential equations" using only linear operations.

There are still other interesting questions :

1. relation between (a, b) and V_1 ?

2. $\text{Null}(A_1) = V_1$. What are $\text{Col}(A)$ & $\text{Row}(A)$?

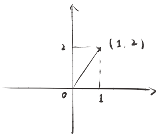
3. Conversely, $\text{null}(A)$ is a subspace of $\mathbb{R}^n$ and can be characterised as above. How about $\text{null}(A, b) := \{x : Ax = b\}$?

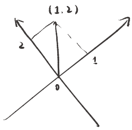
$\longrightarrow$ if $\text{null}(A, b) \neq \phi$, we have discussed in Tutorial 4 (affine space)

But what if $\text{null}(A, b) = \phi$?

4. Every time writing $V_1 = \{(x, y) \mid ax + by = 0\}$ we assume $\forall v \in \mathbb{R}^n$ $v = (x, y)$.

But





not the same vector

$\longrightarrow$ How ?

1. (a) Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$. Show that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ forms a basis for $\mathbb{R}^3$.

1. (a) $\det(u_1 \ \ u_2 \ \ u_3) = 7 \neq 0$

(b) Suppose $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find the coordinate vector of $\mathbf{w}$ relative to S.

(b) $w = x_1 u_1 + x_2 u_2 + x_3 u_3$

$$\operatorname{rref}(u_1 \ \ u_2 \ \ u_3 \mid w) = \left(\begin{array}{ccc|c} 1 & & & 1 \\ & 1 & & -\frac{1}{7} \\ & & 1 & \frac{5}{7} \end{array}\right) \quad [w]_S$$

format rat

(c) Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be another basis for $\mathbb{R}^3$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$. Find the transition matrix from T to S.

(c)
$$\operatorname{rref}(u_1 \ \ u_2 \ \ u_3 \mid v_1 \ \ v_2 \ \ v_3) = \left(\begin{array}{ccc|ccc} 1 & & & 1 & -1 & 2 \\ & 1 & & 2 & 3 & 0 \\ & & 1 & 1 & 1 & 2 \end{array}\right)$$

P transition matrix from T to S

(d) Find the transition matrix from S to T.

(e) Use the vector $\mathbf{w}$ in Part (b). Find the coordinate vector of $\mathbf{w}$ relative to T.

(d) $\operatorname{rref}(v_1 \ \ v_2 \ \ v_3 \mid u_1 \ \ u_2 \ \ u_3) = (I \quad Q)$ transition matrix from T to S

or directly $Q = P^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \end{pmatrix}$

(e) $(u_1 \ \ u_2 \ \ u_3)[w]_S = w = (v_1 \ \ v_2 \ \ v_3)[w]_T = (u_1 \ \ u_2 \ \ u_3) P [w]_T$

$\Leftrightarrow [w]_T = P^{-1}[w]_S = Q[w]_S = \begin{pmatrix} \frac{1}{7} \\ -\frac{1}{7} \\ \frac{5}{14} \end{pmatrix}$

2. Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for a subspace V. Define $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3, \ \mathbf{v}_2 = \mathbf{u}_2 + \mathbf{u}_3 \text{ and } \mathbf{v}_3 = \mathbf{u}_2 - \mathbf{u}_3.$$

(a) Show that T is a basis for V.

(b) Find the transition matrix from S to T.

2. (a) M1 $\quad x_1 v_1 + x_2 v_2 + x_3 v_3 = 0 \iff x_1 u_1 + (x_1 + x_2 + x_3) u_2 + (x_1 + x_2 - x_3) u_3 = 0$

T is a basis $\iff v_1, v_2, v_3$ linearly independent

$\iff \begin{cases} x_1 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$ has only the trivial solution ✓

M2 $\quad T$ is a basis $\iff \underbrace{(v_1 \ v_2 \ v_3)}_{\parallel}$ is invertible

$$(u_1 \ u_2 \ u_3) \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}}_{P}$$

$\iff P$ is invertible ✓

(b) The transition matrix is exactly $P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

3. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Is $\mathbf{b}$ in the column space of $\mathbf{A}$?

If it is, express it as a linear combination of the columns of $\mathbf{A}$.

3. (a) Denote $\quad A = (u_1 \ u_2 \ u_3)$

Want to solve $\quad u_1 x_1 + u_2 x_2 + u_3 x_3 = b$.

$$\text{rref}(A \ b) = \left(\begin{array}{ccc|c} 1 & & & \\ & 1 & -1 & \\ & & & 1 \end{array}\right)$$

No.

(b) Let $\mathbf{A} = \begin{pmatrix} 1 & 9 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = (5, 1, -1)$. Is $\mathbf{b}$ in the row space of $\mathbf{A}$? If it is, express it as a linear combination of the rows of $\mathbf{A}$.

(b) Denote $\quad A = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ $\quad$ Want to solve $\quad v_1^T x_1 + v_2^T x_2 + v_3^T x_3 = b^T$.

$$\text{rref}(A^T \ b^T) = \left(\begin{array}{ccc|c} 1 & & & 1 \\ & 1 & & -3 \\ & & 1 & 1 \end{array}\right)$$

Yes.

$b = v_1 - 3v_2 + v_3$

(c) Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}$. Is the row space and column space of $\mathbf{A}$ the whole $\mathbb{R}^4$?

(c) $\det(A) = 1 \neq 0$

$\Rightarrow$ A is full rank (both column rank & row rank)

$\Rightarrow$ row space = column space = $\mathbb{R}^4$.

4. For each of the following matrices $\mathbf{A}$,

$A = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} = (c_1 \cdots c_n)$

(i) Find a basis for the row space of $\mathbf{A}$.

(ii) Find a basis for the column space of $\mathbf{A}$.

(iii) Find a basis for the nullspace of $\mathbf{A}$.

(iv) Hence determine rank($\mathbf{A}$), nullity($\mathbf{A}$) and verify the dimension theorem for matrices.

(v) Is $\mathbf{A}$ full rank?

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 1 & -4 & -1 & -9 \\ -1 & 0 & -3 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{pmatrix}$.

4. (a) rref(A) rref(A^T)

(i) $\{r_3, r_5\}$ (ii) $\{c_1, c_2\}$

(iii) $\{(-3, -1, 1, 0)^T, (1, -2, 0, 1)^T\}$

(iv) rank(A) = 2 nullity(A) = 2

(v) No.

(b) rref(A) rref(A^T)

(i) $\{e_1, e_2, e_3\}$ (ii) $\{c_1, c_2, c_3\}$

(iii) $\emptyset$

(iv) rank(A) = 3 nullity(A) = 0

(v) Yes.

5. Let W be a subspace of $\mathbb{R}^5$ spanned by the following vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 5 \\ 15 \\ 10 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} 2 \\ 1 \\ 15 \\ 8 \\ 6 \end{pmatrix}.$$

(a) Find a basis for W.

(b) What is $\dim(W)$?

(c) Extend the basis W found in (a) to a basis for $\mathbb{R}^5$.

5. (a) $\operatorname{rref}(u_1 \ u_2 \ u_3 \ u_4)$ $\qquad \{u_1, u_2, u_4\}$

(b) $\dim(W) = 3$

(c) Find a vector that is not a linear combination of $\{u_1, u_2, u_3, u_4\}$

Let $w = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$

$$\operatorname{ref}(u_1 \ u_2 \ u_3 \ u_4 \mid w) = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & a_1 \\ & -1 & 5 & 5 & a_2 + 2a_1 \\ & & & & a_3 - 3a_2 - 6a_1 \\ & & & -2 & a_4 - 2a_2 - 4a_1 \\ & & & & a_5 - 3a_1 \end{array}\right)$$

$w \notin \operatorname{Span}\{u_1, u_2, u_3, u_4\} \iff \begin{cases} a_3 - 3a_2 - 6a_1 \neq 0 \\ a_5 - 3a_1 \neq 0 \end{cases}$ $\qquad$ May choose e_3, e_5.

6. Let $S = \left\{ \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \\ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, & \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, & \begin{pmatrix} -1 \\ 3 \\ 5 \\ 12 \end{pmatrix}, & \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}, & \begin{pmatrix} 3 \\ -1 \\ 1 \\ 4 \end{pmatrix} \end{array} \right\}$ and $V = \operatorname{span}(S)$. Find a subset $S' \subseteq S$ such that S' forms a basis for V.

6. $\operatorname{rref}(u_1 \ u_2 \ u_3 \ u_4 \ u_5) = \begin{pmatrix} 1 & & 5 & 2 & 1 \\ & 1 & -3 & -1 & 1 \\ & & & & \\ & & & & \end{pmatrix}$

$\Longrightarrow \ S' = \{u_1, u_2\}$.

More rigorously, rref $\Rightarrow u_3, u_4, u_5 \in \operatorname{Span}\{u_1, u_2\}$

Let $S' = \{u_1, u_2\}$, then $V = \operatorname{Span} S = \operatorname{Span} S'$

In addition, $\{u_1, u_2\}$ L.I., since $\operatorname{rref}(u_1 \ u_2)$ does not have non-pivot columns.

Hence S' is a basis of V.