NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 7**

1. (a) Let $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ be a linear equation. Express this linear system as $\mathbf{a} \cdot \mathbf{x} = b$ for some (column) vectors $\mathbf{a}$ and $\mathbf{x}$.

   (b) Find the solution set of the linear system

$$\begin{array}{rcrcrcrl} x_1 & + & 3x_2 & - & 2x_3 & & & = 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & = 0 \\ & & & + & 5x_3 & + & 10x_4 & = 0 \end{array}$$

   (c) Find a nonzero vector $\mathbf{v} \in \mathbb{R}^4$ such that $\mathbf{a}_1 \cdot \mathbf{v} = 0$, $\mathbf{a}_2 \cdot \mathbf{v} = 0$, and $\mathbf{a}_3 \cdot \mathbf{v} = 0$, where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 6 \\ -5 \\ -2 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 10 \end{pmatrix}.$$

   This exercise demonstrates the fact that if $\mathbf{A}$ is a $m \times n$ matrix, then the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ consist of all the vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of $\mathbf{A}$.

2. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set. Suppose

$$\mathbf{x} = \mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3 \quad \text{and} \quad \mathbf{y} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

   Determine the value for each of the following

   (a) $\mathbf{x} \cdot \mathbf{y}$.

   (b) $||\mathbf{x}||$ and $||\mathbf{y}||$.

   (c) The angle θ between $\mathbf{x}$ and $\mathbf{y}$.

3. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$.

   (a) Compute $\mathbf{v}_1 \cdot \mathbf{v}_1$, $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_2 \cdot \mathbf{v}_1$ and $\mathbf{v}_2 \cdot \mathbf{v}_2$.

   (b) Compute $\mathbf{V}^T\mathbf{V}$. What does the entries of $\mathbf{V}^T\mathbf{V}$ represent?

4. Let W be a subspace of $\mathbb{R}^n$. The *orthogonal complement* of W, denoted as $W^\perp$, is defined to be

$$W^\perp := \{ \ \mathbf{v} \in \mathbb{R}^n \ \big| \ \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W \ \}.$$

   Let $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}$, and $\mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, and $W = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

(a) Show that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

(b) Show that S is orthogonal.

(c) Show that $W^\perp$ is a subspace of $\mathbb{R}^5$ by showing that it is a span of a set. What is the dimension? (**Hint**: See Question 1.)

(d) Obtain an orthonormal set T by normalizing $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

(e) Let $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. Find the projection of $\mathbf{v}$ onto W.

(f) Let $\mathbf{v}_W$ be the projection of $\mathbf{v}$ onto W. Show that $\mathbf{v} - \mathbf{v}_W$ is in $W^\perp$.

This exercise demonstrated the fact that every vector $\mathbf{v}$ in $\mathbb{R}^5$ can be written as $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_W^\perp$, for some $\mathbf{v}_W$ in W and $\mathbf{v}_W^\perp$ in $W^\perp$. In other words, $W + W^\perp = \mathbb{R}^5$.

5. Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \ \mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \ \mathbf{u_3} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{u_4} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Check that S is an orthogonal basis for $\mathbb{R}^4$.

(b) Is it possible to find a nonzero vector $\mathbf{w}$ in $\mathbb{R}^4$ such that $S \cup \{\mathbf{w}\}$ is an orthogonal set?

(c) Obtain an orthonormal set T by normalizing $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$.

(d) Let $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$. Find $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

(e) Suppose $\mathbf{w}$ is a vector in $\mathbb{R}^4$ such that $[\mathbf{w}]_S = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Find $[\mathbf{w}]_T$.

Extra problems

1. Let $\mathbf{A}$ be an $m \times n$ matrix.

   (a) Show that the nullspace of $\mathbf{A}$ is equal to the nullspace of $\mathbf{A}^T\mathbf{A}$.

   (b) Show that $\text{nullity}(\mathbf{A}) = \text{nullity}(\mathbf{A}^T\mathbf{A})$ and $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T\mathbf{A})$.

   (c) Is it true that $\text{nullity}(\mathbf{A}) = \text{nullity}(\mathbf{A}\mathbf{A}^T)$? Justify your answer.

   (d) Is it true that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}^T)$? Justify your answer.

2. Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices of the same size. Show that

$$\text{rank}(\mathbf{A} + \mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}).$$

3. (a) Let W be a subspace of $\mathbb{R}^n$. Prove that the orthogonal complement of the orthogonal complement of W is W, i.e.

$$(W^\perp)^\perp = W.$$

   (b) Show that for any matrix $\mathbf{A}$, the column space of $\mathbf{A}$ is the orthogonal complement of the nullspace of $\mathbf{A}^T$,

$$\text{Col}(\mathbf{A})^\perp = \text{Null}(\mathbf{A}^T),$$

   or equivalently, the row space of $\mathbf{A}$ is the orthogonal complement of the nullspace of $\mathbf{A}$,

$$\text{Row}(\mathbf{A})^\perp = \text{Null}(\mathbf{A}).$$