MA1522: Linear Algebra for Computing

Tutorial 4

Revision

Linear Span

The span of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k \in \mathbb{R}^n$ is

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \left\{ \; c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \;\middle|\; c_1, c_2, ..., c_k \in \mathbb{R} \; \right\}.$$

It is the set of all linear combinations of the vectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$.

Theorem

1. $\mathbf{v} \in$ *span*$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \Leftrightarrow \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}\mathbf{x} = \mathbf{v}$ *is consistent.* $\Leftrightarrow \left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array}\right)$ *is consistent.*
2. *span*$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \mathbb{R}^n \Leftrightarrow$ *the reduced row-echelon form* $\mathbf{R}$ *of* $\mathbf{A}$ *has no zero rows.*

Linear system: $\left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \right. \quad \leftrightarrow \quad$ Matrix Equation: $\mathbf{Ax} = \mathbf{b}$

Vector equation: $x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \leftrightarrow \quad x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$

where $\mathbf{a}_i$ is the i-th column of $\mathbf{A}$.

Set relations between spans

Theorem (Properties of Linear Spans)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$. *Then*

(i) *(Contains the origin)* $\mathbf{0} \in span(S)$, *and*

(ii) *(Closed under addition) for any* $\mathbf{u}, \mathbf{v} \in span(S)$, $\mathbf{u} + \mathbf{v} \in span(S)$.

(iii) *(Closed under scalar multiplication) for any* $\mathbf{v} \in span(S)$ *and real number* $\alpha \in \mathbb{R}$, $\alpha\mathbf{v} \in span(S)$.

(iv) *For any* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m \in span(S)$ *and real numbers* $c_1, c_2, ..., c_m \in \mathbb{R}$, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \in span(S)$. *That is,* $span\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\} \subseteq span(S)$.

Theorem

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *and* $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$, *both subsets of* $\mathbb{R}^n$. *Then* $span(T) \subseteq span(S)$ *if and only if* $\mathbf{v}_i \in span(S)$ *for every* $i = 1, ..., m$.

To check if $\text{span}(S) = \text{span}(T)$, we check that

- $\text{span}(S) \subseteq \text{span}(T)$, that is,

$$\left(\text{“}T\text{''} \mid \text{“}S\text{''} \right) = \left(\begin{array}{cccc|c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{array} \right) \text{ is consistent, and}$$

- $\text{span}(T) \subseteq \text{span}(S)$, that is,

$$\left(\text{“}S\text{''} \mid \text{“}T\text{''} \right) = \left(\begin{array}{cccc|c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \end{array} \right) \text{ is consistent.}$$

Subspaces

A subset $V \subseteq \mathbb{R}^n$ is a subspace if

(i) (Contains the origin) $\mathbf{0} \in V$, and

(ii) (Closed under linear combination) for any $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{R}$,

$$\alpha\mathbf{u} + \beta\mathbf{v} \in V.$$

Theorem (Equivalent Definition for Subspaces)

A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if is a linear span, $V = span(S)$, for some finite set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.

Solution Set to Linear Systems

Solution set to a linear system can be expressed implicitly or explicitly

- Implicit form: $\left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \mathbf{Ax} = \mathbf{b} \text{ or } \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \right\}.$

- Explicit form:
$$\{ \; \mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots s_k\mathbf{v}_k \;\middle|\; s_1, s_2, \ldots, s_k \in \mathbb{R} \; \},$$
where $\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots s_k\mathbf{v}_k, \; s_1, s_2, \ldots, s_k \in \mathbb{R}$ is the general solution to $\mathbf{Ax} = \mathbf{b}$.

Solution Space of Homogeneous System

Theorem

The solution set $V = \{ \ \mathbf{u} \ | \ \mathbf{Au} = \mathbf{b} \ \}$ *to a linear system* $\mathbf{Ax} = \mathbf{b}$ *is a subspace if and only if* $\mathbf{b} = \mathbf{0}$, *that is, the system is homogeneous.*

Let

$$s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, ..., s_k \in \mathbb{R}$$

be the general solution to the homogeneous system $\mathbf{Ax} = \mathbf{0}$. Then the solution set of $\mathbf{Ax} = \mathbf{0}$ is spanned by $\mathbf{u}_1, ..., \mathbf{u}_k$; that is,

$$\{\text{Solutions to } \mathbf{Ax} = \mathbf{0}\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$$

Solution Set to Non-homogeneous System

Theorem

The solution set $W = \{ \mathbf{w} \mid \mathbf{A}\mathbf{w} = \mathbf{b} \}$ *of a linear system* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *is given by* $\mathbf{u} + V$*, where* $V = \{ \mathbf{v} \mid \mathbf{A}\mathbf{v} = \mathbf{0} \}$ *is the solution space to the associated homogeneous system and* $\mathbf{u}$ *is a particular solution,* $\mathbf{A}\mathbf{u} = \mathbf{b}$*.*

If

$$\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k, s_1, s_2, ..., s_k \in \mathbb{R}$$

is a general solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$,

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k, s_1, s_2, ..., s_k \in \mathbb{R}$$

is a general solution to $\mathbf{A}\mathbf{x} = \mathbf{0}$.

Tutorial 4 Solutions

Question 1(a)

Let $A = \{\ (1+t, 1+2t, 1+3t) \mid t \in \mathbb{R}\ \}$ be a subset in $\mathbb{R}^3$. Describe A geometrically.

https://www.geogebra.org/calculator/uc7pfr7a

A is a line joining the points $(1, 1, 1)$ and $(2, 3, 4)$.

Question 1(b)

Show that $A = \{ \ (x, y, z) \mid x + y - z = 1 \text{ and } x - 2y + z = 0 \ \}$.

https://www.geogebra.org/calculator/uc7pfr7a

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 0 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & -2/3 & 1/3 \end{array} \right).$$

General solution: $\begin{pmatrix} \frac{2}{3} + \frac{1}{3}s \\ \frac{1}{3} + \frac{2}{3}s \\ s \end{pmatrix}$, $s \in \mathbb{R}$. Let $s = 1 + 3t$, we get $\begin{pmatrix} 1 + t \\ 1 + 2t \\ 1 + 3t \end{pmatrix}$, $t \in \mathbb{R}$, which is the set in (a).

Question 1(c)

Write down a matrix equation $\mathbf{Mx} = \mathbf{b}$ where $\mathbf{M}$ is a 3×3 matrix and $\mathbf{b}$ is a 3×1 matrix such that its solution set is A.

From (b), we obtain the first 2 rows of $\mathbf{M}$ and $\mathbf{b}$. Since we have exhausted all information, the last row of $\mathbf{M}$ and $\mathbf{b}$ must give us no useful information. Hence, we may let it be the zero row; i.e. $\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Question 2(a)

Let $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$. If possible, express each of the following vectors as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. (i) $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix}$ (ii) $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (iii) $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (iv) $\mathbf{v}_4 = \begin{pmatrix} -4 \\ 6 \\ -13 \\ 4 \end{pmatrix}$.

Solve for $\left(\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \mid \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \mathbf{v}_4 \right)$.

$$\left(\begin{array}{ccc|cccc} 2 & 3 & -1 & 2 & 0 & 1 & -4 \\ 1 & -1 & 0 & 3 & 0 & 1 & 6 \\ 0 & 5 & 2 & -7 & 0 & 1 & -13 \\ 3 & 2 & 1 & 3 & 0 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Question 2(a)

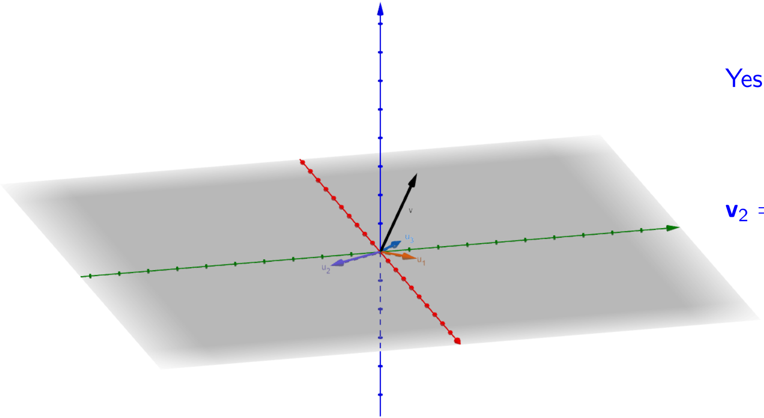
Alternatively,

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & x_1 \\ 1 & -1 & 0 & x_2 \\ 0 & 5 & 2 & x_3 \\ 3 & 2 & 1 & x_4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & x_2 \\ 0 & 5 & -1 & x_1 - 2x_2 \\ 0 & 0 & 3 & -x_1 + 2x_2 + x_3 \\ 0 & 0 & 0 & x_1 + 7x_2 + 2x_3 - 3x_4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2x_1+11x_2+x_3}{15} \\ 0 & 1 & 0 & \frac{2x_1-4x_2+x_3}{15} \\ 0 & 0 & 1 & \frac{-x_1+2x_2+x_3}{3} \\ 0 & 0 & 0 & 0 \end{array}\right),$$

assuming $x_1 + 7x_2 + 2x_3 - 3x_4 = 0$. So a vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ if and only if it satisfies $x_1 + 7x_2 + 2x_3 - 3x_4 = 0$. If that is true, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$, where $a = \frac{2x_1+11x_2+x_3}{15}$, $b = \frac{2x_1-4x_2+x_3}{15}$, $c = \frac{-x_1+2x_2+x_3}{3}$.

Question 2(b)

Is it possible to find 2 vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that they are not a multiple of each other, and both are not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?



Yes, for example, take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and

$$\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix}.$$

Question 3(a)

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x - y - z = 0 \right\}$ be a subset of $\mathbb{R}^3$. Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\}$. Show that $\text{span}(S) = V$.

General solution to $x - y - z = 0$ is $x = s + t$, $y = s$, $z = t$ where $s, t \in \mathbb{R}$. Hence, $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

- Since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ satisfy the equation $x - y - z = 0$, they are in V and hence $\text{span}(S) \subseteq V$.
- Check $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq \text{span}(S)$: $\left(\begin{array}{cc|cc} 1 & 5 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \end{array} \right)$.

So, $\text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\} = V$.

Question 3(b)

Let $T = S \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Show that $\text{span}(T) = \mathbb{R}^3$.

Consider the row-echelon form of the matrix:

$$\begin{pmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}.$$

Since there are no zero rows in $\mathbf{R}$, we conclude that T spans $\mathbb{R}^3$.

Question 4(i)

Does $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ spans $\mathbb{R}^4$?

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So S spans $\mathbb{R}^4$.

Question 4(ii)

Does $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans $\mathbb{R}^4$?

3 vectors cannot span $\mathbb{R}^4$.

Question 4(iii)

Does $S = \left\{ \begin{pmatrix} 6 \\ 4 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -2 \\ -1 \end{pmatrix} \right\}$ spans $\mathbb{R}^4$?

$$\begin{pmatrix} 6 & 2 & 3 & 5 & 0 \\ 4 & 0 & 2 & 6 & 4 \\ -2 & 0 & -1 & -3 & -2 \\ 4 & 1 & 2 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{R}$$

Since there is a row of zeros in $\mathbf{R}$, S does not span $\mathbb{R}^4$.

Question 4(iv)

Does $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$ spans $\mathbb{R}^4$?

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} = \mathbf{R}.$$

So S spans $\mathbb{R}^4$.

Question 5(a)

Determine whether $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and/or $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

```
>> u1=[2;-2;0];u2=[-1;1;-1];u3=[0;0;9];v1=[1;-1;-5];v2=[0;1;1];
>> rref([u1 u2 u3 v1 v2])
```

$$\left(\begin{array}{ccc|cc} 2 & -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & -1 & 1 \\ 0 & -1 & 9 & -5 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & -\frac{9}{2} & 3 & 0 \\ 0 & 1 & -9 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

```
>> rref([v1 v2 u1 u2 u3])
```

$$\left(\begin{array}{cc|ccc} 1 & 0 & 2 & -1 & 0 \\ -1 & 1 & -2 & 1 & 0 \\ -5 & 1 & 0 & -1 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & \frac{9}{10} \end{array}\right).$$

So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \not\subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \not\subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Question 5(b)

Determine whether $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and/or $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}.$$

```
>> u1=[1;6;4];u2=[2;4;-1];u3=[-1;2;5];v1=[1;-2;-5];v2=[0;8;9];
>> rref([u1 u2 u3 v1 v2])
>> rref([v1 v2 u1 u2 u3])
```

So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. We conclude that the two linear spans are equal.

Question 6(a)

Determine if $S = \left\{ \begin{pmatrix} p \\ q \\ p \\ q \end{pmatrix} \;\middle|\; p, q \in \mathbb{R} \right\}$ is a subspace. If it is, express the set as a linear span. If not, explain why.

$S = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

Question 6(b)

Determine if $S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; a \geq b \text{ or } b \geq c \right\}$ is a subspace. If it is, express the set as a linear span. If not, explain why.

S is not a linear span (thus not a subspace) since $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is in S but $(-1)\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is not.

Question 6(c)

Determine if $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \; 4x = 3y \text{ and } 2x = -3w \right\}$ is a subspace. If it is, express the set as a linear span. If not, explain why.

$$S = \left\{ \begin{pmatrix} x \\ \frac{4x}{3} \\ z \\ -\frac{2x}{3} \end{pmatrix} \middle| \; x, z \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ \frac{4}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Question 6(d)

Determine if $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right\}$ is a subspace. If it is, express the set as a linear span. If not, explain why.

```
>> syms a b c d; det([1 0 1 0;0 1 0 0;1 0 0 1;a b c d])
```

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = a - c - d.$$

So the set S can be rewritten as

$$S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| a - c - d = 0 \right\} = \left\{ \begin{pmatrix} s+t \\ u \\ s \\ t \end{pmatrix} \middle| s, t, u \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Question 6(e)

Determine if $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \ w + x = y + z \right\}$ is a subspace. If it is, express the set as a linear span. If not, explain why.

S can be rewritten as $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \ w + x - y - z = 0 \right\}$. Solving the equation $w + x - y - z = 0$, we have

$$S = \left\{ \begin{pmatrix} -s + t + u \\ s \\ t \\ u \end{pmatrix} \middle| \ s, t, u \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Question 6(f)

Determine if $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \ ab = cd \right\}$ is a subspace. If it is, express the set as a linear span. If not, explain why.

S is not a linear span (thus not a subspace) since $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ are vectors in S but $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ is not.

Question 6(g)

S is the solution set of $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.

```
>> rref([2 2 -1 0 1;-1 -1 2 -3 1;0 0 1 1 1;1 1 -2 0 -1]).
```

$$\left(\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set of $\mathbf{Ax} = \mathbf{0}$ is $\left\{ \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$. So, $S = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.