MA1522: Linear Algebra for Computing

Tutorial 1

Introduction to MATLAB

Basic Operations

- App layout: Command Window, Script, Live Script, Workspace.
- Addition $+$, Subtraction $-$, Multiplication $*$, Division $\backslash$, Power $\hat{}$. To add a few entries, use sum, sum. sum([1,2,3,4,5]).
- Storing values, $a = 123$. Semicolon hides output $a = 123$;.
- Double click on stored values in workspace to view it.
- To clear all stored values, clear. To clear a certain stored value, clear a. To clear the command window, clc.
- May use % to add comments (seldom needed in our context).

Basic Functions

- To display in decimal places, use format long (for 16 decimal digits) and format short (for 4 decimal digits). Use format rat for fractions (Note that irrational numbers will be displayed as fractions too).
- Square root, sqrt.
- Trigonometric functions: sin(x), cos(x), tan(x), cot(x), sec(x), csc(x).
- Inverse trigonometric functions: asin(x), acos(x), atan(x), acot(x).
- The unit for trigonometric functions is in radian.
- Exponential and logarithmic functions: exp(x), log(x) (base e), log10(x) (base 10).

Solving Basic Algebraic Functions

- Declare an algebraic expression, syms.
- Solve algebraic expression using solve, solve(x^2 + x - 1 == 0, x).
- solve(a*x^2 + b*x + c == 0, x)
- May solve multivariable, syms x y;, [Sx,Sy] = solve([3*x+y==10, x+y==20], [x,y]).

Vector and Matrix

- Use square brackets to define vectors. For numbers in the same row, use space or comma. For a different row, use semi-colon.
- Row vector, v=[1 2 3 4 5] or v=[1,2,3,4,5]. Column vector, v=[1;2;3;4;5].
- Transpose, A' (more accurately, this is the Hermitian transpose, it is the conjugate transpose of complex matrices).
- Matrix multiplication, A*B.
- Powers are only defined for square matrices, A^n.
- To find determinant, det(A). To find inverse of square matrix, inv(A).
- Identity matrix of order n, eye(n). Zero matrix of size m by n, zeros(m,n). Diagonal matrix, diag(1,2,3,4,5).

Row Operations

- (i,j)-entry of A, A(i,j). i-th row of A, A(i,:). j-th column of A, A(:,j).
- Row swap $R_i \leftrightarrow R_j$, A([i,j],:)=A([j,i],:).
- A multiple of a row cR_i, A(i,:)=c*A(i,:).
- Add a multiple of a row to another $R_i + aR_j$, A(i,:)=A(i,:)+a*A(j,:).
- To find RREF, rref(A).

Linear System

- To solve the augmented matrix $(\mathbf{A} \mid \mathbf{b})$, rref([A b]).
  - Does not work if the system has unknown coefficients or constants.
- To obtain a particular solution, linsolve(A,b) or A\b.
  - Only if system is consistent.
  - Does not provide a general solution.
  - If system is inconsistent, it will return a least square solution instead.

Tutorial 1 Solutions

Question 1

(a) Find a linear equation in the variables x and y that has a general solution $x = 1 + 2t$ and $y = t$ where t is an arbitrary parameter.

Substitute $y = t$ into $x = 1 + 2t$,

$$x = 1 + 2y \quad \Rightarrow \quad x - 2y = 1.$$

All possible solutions are multiple of $x - 2y = -1$. To visualize, `https://www.geogebra.org/calculator`, type `(1+2t,t)`, and `x-2y=1`

(b) Show that $x = t$ and $y = \frac{1}{2}t - \frac{1}{2}$, where t is an arbitrary parameter, is also a general solution for the equation constructed in part (a).

Let $x = t$ in $x - 2y = 1$, get $t - 2y = 1 \Rightarrow 2y = t - 1 \Rightarrow t = \frac{1}{2}(t-1)$. It is a general solution: In (a), $y = s$, $x = 1 + 2s$, let $y = s = \frac{1}{2}(t-1)$, then $x = 1 + 2 \times \frac{1}{2}(t-1) = t$.
In the same GeoGebra graph above, key in `(t,t/2-1/2)`.

Question 2

Find a linear equation in the variables x, y, and z that has a general solution

$$\begin{cases} x = 3 - 4s + t \\ y = s \\ z = t \end{cases} \qquad s, t \in \mathbb{R}\ .$$

As in question 1, substitute $y = s$ and $z = t$ into $x = 3 - 4s + t$,

$$x = 3 - 4y + z \quad \Rightarrow \quad x + 4y - z = 3.$$

Similarly, all possible answers are multiple of $x + 4y - z = 3$.

Question 3(a)

Solve the linear system

$$\begin{cases} 3x_1 & + & 2x_2 & - & 4x_3 & = & 3 \\ 2x_1 & + & 3x_2 & + & 3x_3 & = & 15 \\ 5x_1 & - & 3x_2 & + & x_3 & = & 14 \end{cases}$$

- `R=[3 2 -4 3;2 3 3 15;5 -3 1 14]`
- `R(2,:)=R(2,:)-2*R(1,:)/3;R(3,:)=R(3,:)-5*R(1,:)/3`
- `R(3,:)=R(3,:)+19*R(2,:)/5, R(3,:)=5*R(3,:)/146`
- `R(2,:)=R(2,:)-17*R(3,:)/3; R(1,:)=R(1,:)+4*R(3,:)`
- `R(2,:)=3*R(2,:)/5; R(1,:)=R(1,:)-2*R(2,:); R(1,:)=R(1,:)/3`

Unique solution $x = 3$, $y = 1$, $z = 2$.

What does the $*$ means? Type `R(1,3)`, `R(2,3)`. This is due to rounding error.

Question 3(b)

Solve the linear system

$$\left\{\begin{array}{rcrcrcrcr} a & + & b & - & c & - & 2d & = & 0 \\ 2a & + & b & - & c & + & d & = & -2 \\ -a & + & b & - & 3c & + & d & = & 4 \end{array}\right.$$

Question didn't ask to show elementary row operations.

- `R=[1 1 -1 -2 0;2 1 -1 1 -2;-1 1 -3 1 4], rref(R)`
- RREF

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & -\frac{19}{2} & 2 \\ 0 & 0 & 1 & -\frac{9}{2} & 0 \end{array}\right).$$

- Let $d = s$ be the parameter. Then $a = -2 - 3s$, $b = 2 + \frac{19}{2}s$, $c = \frac{9}{2}s$.

General solution

$$a = -2 - 3s, \quad b = 2 + \frac{19s}{2}, \quad c = \frac{9s}{2}, \quad d = s, \quad s \in \mathbb{R}.$$

Question 3(c)

Solve the linear system

$$\begin{cases} x - 4y + 2z = -2 \\ x + 2y - 2z = -3 \\ x - y = 4 \end{cases}$$

- `R=[1 -4 2 -2;1 2 -2 -3;1 -1 0 4], rref(R)`

System is inconsistent.

Question 4

Determine the values of a and b so that the linear system

$$\begin{cases} ax & & + & bz & = & 2 \\ ax & + & ay & + & 4z & = & 4 \\ & & ay & + & 2z & = & b \end{cases}$$

(a) has no solution;

(b) has only one solution;

(c) has infinitely many solutions and a general solution has one arbitrary parameter;

(d) has infinitely many solutions and a general solution has two arbitrary parameters.

Question 4

- `syms a b; R=[a 0 b 2;a a 4 4;0 a 2 b]`
- `R(2,:)=R(2,:)-R(1,:), R(3,:)=R(3,:)-R(2,:)`

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right)$$

- Cases to consider: $a = 0$, $b = 2$ or not.
- `b=2; a=0; rref(eval(R))`. System has infinitely many solutions with 2 parameters when $b = 2, a = 0$. General solution: $x = s$, $y = t$, $z = 1$, $s, t \in \mathbb{R}$.
- `syms a; rref(eval(R))`. System has infinitely many solutions with 1 parameter when $a \neq 0$, $b = 2$. General solution: $x = y = \frac{2}{a}(1 - s)$, $z = s$, $s \in \mathbb{R}$.
- `syms b; a=0; rref(eval(R))`. System is inconsistent when $b \neq 2$, $a = 0$.
- `syms a b; rref(eval(R))`. System has a unique solution when $a \neq 0$, $b \neq 2$. Unique solution: $x = \frac{2-b}{a}$, $y = \frac{b-2}{a}$, $z = 1$.

Question 4

In summary

(a) $a = 0$ and $b \neq 2$: No solution;

(b) $a \neq 0$ and $b \neq 2$: Unique solution;

(c) $a \neq 0$ and $b = 2$: Infinitely many solution with one free variable;

(d) $a = 0$ and $b = 2$: Infinitely many solution with two free variables.

Question 5(a) and (b)

(a) Does an inconsistent linear system with more unknowns than equations exist?

Yes. Idea: $0 = 1$. Let something be equal to 0 and 1 at the same time.

$$\begin{cases} x + y + z = 0 \\ x + y + z = 1 \end{cases}$$

(b) Does a linear system which has only one solution, but more equations than unknowns, exist?

Yes. Create a linear system with a unique solution.

$$\begin{cases} x + y = 0 \\ x - y = 0 \end{cases}$$

Using the existing equations to create more equations by adding a multiple of a row to another, or multiplying by a nonzero scalar. For example: 3rd equation: $2x - 2y = 0$.

Question 5(c) and (d)

(c) Does a linear system which has only one solution, but more unknowns than equations, exists?

No. If variables > equations, there must be nonpivot columns in the LHS of augmented matrix. So either system has no solution or infinitely many.

(d) Does a linear system which has infinitely many solutions, but more equations than unknowns, exists?

Yes. Idea: create 1 equation 2 variables, then duplicate the equation by taking multiples.

$$\begin{cases} x & + & y & = & 1 \\ 2x & + & 2y & = & 2 \\ 3x & + & 3y & = & 3 \end{cases}$$

Question 6

Solve the following system of non-linear equations:

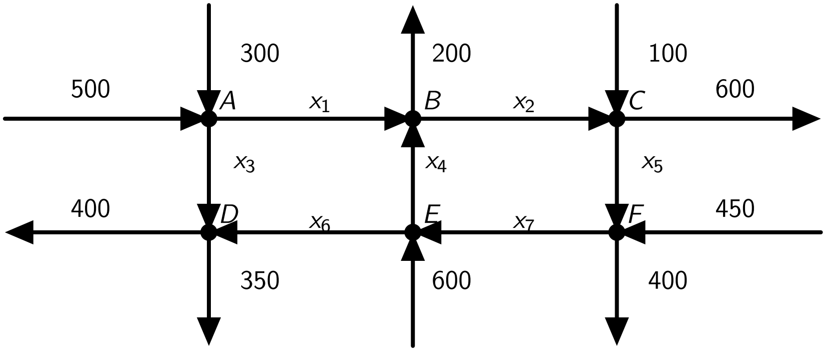
$$\begin{array}{ccccccc} x^2 & - & y^2 & + & 2z^2 & = & 6 \\ 2x^2 & + & 2y^2 & - & 5z^2 & = & 3 \\ 2x^2 & + & 5y^2 & + & z^2 & = & 9 \end{array}$$

Linearize the system. Let $X = x^2$, $Y = y^2$, $Z = z^2$, then the system of non-linear equations above is converted to a system of linear equations

$$\begin{array}{ccccccc} X & - & Y & + & 2Z & = & 6 \\ 2X & + & 2Y & - & 5Z & = & 3 \\ 2X & + & 5Y & + & Z & = & 9 \end{array}$$

`rref([1 -1 2 6;2 2 -5 3;2 5 1 9])`. Unique solution: $x^2 = X = 4$, $y^2 = Y = 0$, $z^2 = Z = 1$. Hence, $x = \pm 2$, $y = 0$, $z = \pm 1$ are solutions to the system of non-linear equations.

Question 7



$$
\begin{aligned}
A &: \quad 500 + 300 = x_1 + x_3 \\
B &: \quad x_1 + x_4 = x_2 + 200 \\
C &: \quad 100 + x_2 = x_5 + 600 \\
D &: \quad x_3 + x_6 = 400 + 350 \\
E &: \quad 600 + x_7 = x_4 + x_6 \\
F &: \quad 450 + x_5 = x_7 + 400
\end{aligned}
$$

Question 7(a)

Do we have enough information to find the traffic volumes x_1, x_2, x_3, x_4, x_5, x_6, and x_7?

No, since there are 7 variables and 6 equations.

$$\begin{array}{rrrrrrrl}
x_1 & & + \; x_3 & & & & & = 800 \\
x_1 & - \; x_2 & & + \; x_4 & & & & = 200 \\
& x_2 & & & - \; x_5 & & & = 500 \\
& & x_3 & & & + \; x_6 & & = 750 \\
& & & - \; x_4 & & - \; x_6 & + \; x_7 & = -600 \\
& & & & x_5 & & - \; x_7 & = -50
\end{array}$$

In fact, the RREF is $\left(\begin{array}{ccccccc|c} 71 & 0 & 0 & 0 & 0 & -1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 450 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$, which shows that we need 2 parameters.

Question 7(b)

Suppose $x_6 = 50$ and $x_7 = 100$. What is x_1, x_2, x_3, x_4, and x_5?

The general solution: $x_1 = 50 + s$, $x_2 = 450 + t$, $x_3 = 750 - s$, $x_4 = 600 - s + t$, $x_5 = t - 50$, $x_6 = s$, $x_7 = t$.

Substitute $s = 50$, $t = 100$, then $x_1 = 100$, $x_2 = 550$, $x_3 = 700$, $x_4 = 650$, $x_5 = 50$.

Question 7(c)

Can the road between junction A and B be closed for construction while still keeping the traffic flowing in the same directions on the other streets? Explain.

From general solution, $x_1 = 50 + s$. If the road is closed, $0 = x_1 = 50 + x \Rightarrow x_6 = s = -50$, a contradiction since we cannot have negative number of cars. Or we might interpret it as the traffic along DE is flowing in the opposition direction, which goes against the requirement of keeping the traffic flowing in the same directions.