MA1522: Linear Algebra for Computing

Tutorial 5

Revision

Linearly Independent

A set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly independent if

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0},$$

has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.

$\Leftrightarrow$ the homogeneous linear system $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ has only the trivial solution.

Theorem

$S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ *linearly independent* $\Leftrightarrow$ *RREF of* $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ *has no non-pivot columns.*

Idea

1. A set is linearly independence if and only if there are no redundancy when taking the span, i.e. no subset of $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ can span $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.
2. A set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly independent if and only if the linear combination $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ is unique.

Special Cases

1. $\{\mathbf{0}\}$, where $\mathbf{0} \in \mathbb{R}^n$ is the zero vector is always linearly dependent.
2. If $\mathbf{v} \neq \mathbf{0}$, then $\{\mathbf{v}\} \in \mathbb{R}^n$ is linearly independent.
3. $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if one is a scalar multiple of the other, $\alpha\mathbf{v}_1 = \mathbf{v}_2$ or $\mathbf{v}_1 = \beta\mathbf{v}_2$.
4. The empty set $\{\} = \varnothing$ is linearly independent.
5. Any subset of $\mathbb{R}^n$ containing more than n vectors must be linearly dependent.
6. If $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent, then for any $\mathbf{u} \in \mathbb{R}^n$, $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ is linearly dependent.
7. $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ linearly independent and $\mathbf{u} \notin \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \Rightarrow \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ linearly independent.
8. Subset of linearly independent set is linearly independent.
9. A set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ containing n vectors in $\mathbb{R}^n$ is linearly independent if and only if it spans $\mathbb{R}^n$.

Basis

Let $V \subseteq \mathbb{R}^n$ be a subspace. A set $S = \{\mathbf{u}_1, \cdots \mathbf{u}_k\} \subseteq V$ is a basis for V if

(i) $\text{span}(S) = V$, and

(ii) S is linearly independent.

- Basis to the solution space: Let $V = \{\ \mathbf{u} \mid \mathbf{Au} = \mathbf{0}\ \}$, and $s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, \quad s_1, s_2, ..., s_k \in \mathbb{R}$ the general solution to the homogeneous system $\mathbf{Ax} = \mathbf{0}$. Then $\{\mathbf{u}_1, \cdots \mathbf{u}_k\}$ is a basis for V.
- A subset $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$ if and only if $k = n$ and $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$ is an invertible matrix.

Equivalent Statements for Invertibility

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) $\mathbf{A}$ is invertible.

(ii) $\mathbf{A}$ has a left inverse.

(iii) $\mathbf{A}$ has a right inverse.

(iv) The reduced row-echelon form of $\mathbf{A}$ is the identity matrix.

(v) $\mathbf{A}$ can be expressed as a product of elementary matrices.

(vi) The homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

(vii) For any $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

(viii) The determinant of $\mathbf{A}$ is nonzero, $\det(\mathbf{A}) \neq 0$.

(ix) The columns/rows of $\mathbf{A}$ are linearly independent.

(x) The columns/rows of $\mathbf{A}$ spans $\mathbb{R}^n$.

Dimension

Theorem

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *and* $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ *are bases for a subspace* $V \subseteq \mathbb{R}^n$*. Then* $k = m$*.*

The dimension of a subspace $V \subseteq \mathbb{R}^n$ is the number of vectors in any basis, denoted as $\dim(V)$.

Theorem

The dimension of a solution space $V = \{ \ \mathbf{u} \mid \mathbf{Au} = \mathbf{0} \ \}$ *is the number of non-pivot columns in the RREF of* $\mathbf{A}$.

Theorem

Let V *be a* k*-dimensional subspace. Then*

(i) *any subset of* V *containing more than* k *vectors must be linearly dependent;*

(ii) *any subset of* V *containing less than* k *vectors cannot span* V*.*

Basis

Theorem (Spanning set theorem)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ be a subset of vectors in $\mathbb{R}^n$, and let $V = \text{span}(S)$. Then there must be a subset of S that is a basis for V.

The basis S' of V that is a basis subset of S has $\dim(V)$ vectors, i.e. need to remove $m - \dim(V)$ vectors from S to obtain a basis S'.

Theorem (Linear independence theorem)

Let $T = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ a linearly independent subset of V, $T \subseteq V$. Then there must be a set T' containing T, $T \subseteq T'$ such that T' is a basis for V.

The basis T' has $\dim(V)$ vectors. Need to add $\dim(V) - m$ more independent vectors to extend T to be a basis for V.

Tutorial 5 Solutions

Question 1(a)

$$S = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} \right\}.$$

(i) Determine if S is linearly independent.

(ii) If S is linearly dependent, express one of the vectors in S as a linear combination of the others.

The set S is linearly dependent since it contains 4 vectors from $\mathbb{R}^3$.

$$\begin{pmatrix} 2 & 0 & 2 & 3 \\ -1 & 3 & 4 & 6 \\ 0 & 2 & 3 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{9}{2} \\ 0 & 1 & 0 & \frac{15}{2} \\ 0 & 0 & 1 & -3 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = \frac{9}{2} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \frac{15}{2} \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}.$$

Question 1(b)

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \right\}.$$

(i) Determine if S is linearly independent.

(ii) If S is linearly dependent, express one of the vectors in S as a linear combination of the others.

The set S is linearly independent since S has only two vectors which are not multiples of each other.

Question 1(c)

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

(i) Determine if S is linearly independent.

(ii) If S is linearly dependent, express one of the vectors in S as a linear combination of the others.

Any set containing the zero vector is linearly dependent. Indeed we have

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}.$$

Question 1(d)

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

(i) Determine if S is linearly independent.

(ii) If S is linearly dependent, express one of the vectors in S as a linear combination of the others.

```
>> rref([1 0 1;0 1 2;0 1 -1])
```

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

So $a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has only the trivial solution and S is a linearly independent set.

Question 2(a)

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine if $S_1 = \{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

Any subset of a linearly independent set is linearly independent.

Question 2(b)

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine if $S_2 = \{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$ is linearly independent.

Observe that $(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w}) + (\mathbf{w} - \mathbf{u}) = \mathbf{0}$. So, S_2 is linearly dependent.

Question 2(c)

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine if $S_3 = \{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{w}\}$ is linearly independent.

We have

$$a(\mathbf{u} - \mathbf{v}) + b(\mathbf{v} - \mathbf{w}) + c(\mathbf{w} + \mathbf{u}) = \mathbf{0} \quad \Leftrightarrow \quad (a + c)\mathbf{u} + (-a + b)\mathbf{v} + (-b + c)\mathbf{w} = \mathbf{0}.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, we have

$$\left\{ \begin{array}{rcrcrcl} a & & & + & c & = & 0 \\ -a & + & b & & & = & 0 \\ & - & b & + & c & = & 0 \end{array} \right.$$

The system has only the trivial solution $a = 0, b = 0, c = 0$. Thus S_3 is linearly independent.

Question 2(d)

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine if $S_4 = \{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is linearly independent.

$$a\mathbf{u} + b(\mathbf{u}+\mathbf{v}) + c(\mathbf{u}+\mathbf{v}+\mathbf{w}) = \mathbf{0} \quad \Leftrightarrow \quad (a+b+c)\mathbf{u} + (b+c)\mathbf{v} + c\mathbf{w} = \mathbf{0}.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, we have $a+b+c = b+c = c = 0$. Solving for a, b, c gives the trivial solution $a = 0, b = 0, c = 0$. Thus S_4 is linearly independent.

Question 2(e)

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine if $S_5 = \{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is linearly independent.

$$(\mathbf{u}+\mathbf{v}) + (\mathbf{v}+\mathbf{w}) + (\mathbf{u}+\mathbf{w}) - 2(\mathbf{u}+\mathbf{v}+\mathbf{w}) = \mathbf{0}.$$

So, S_5 is linearly dependent.

Question 3(a)

Find a basis for $V = \left\{ \begin{pmatrix} a+b \\ a+c \\ c+d \\ b+d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$.

$$\begin{pmatrix} a+b \\ a+c \\ c+d \\ b+d \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow V = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{RRED} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

So, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for V.

Question 3(b)

Find a basis for $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

The set contains 4 vectors in $\mathbb{R}^3$. Subset of any 3 vectors will form a basis for V, that is, $V = \mathbb{R}^3$.

```
>> V=[1 -1 0 1;0 2 3 -1;-1 3 0 1];
a=[1:4];
for i = 1:4
V(:,setdiff(a,i))
rref(V(:,setdiff(a,i)))
end
```

Question 3(c)

Find a basis for V, the solution space of the following homogeneous linear system

$$\left\{\begin{array}{ccccccccccc} a_1 & & & + & a_3 & + & a_4 & - & a_5 & = & 0 \\ & & a_2 & + & a_3 & + & 2a_4 & + & a_5 & = & 0 \\ a_1 & + & a_2 & + & 2a_3 & + & a_4 & - & 2a_5 & = & 0 \end{array}\right.$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

So, $\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}\right\}$ is a basis for V.

Question 4

For what values of a will $\mathbf{u}_1 = \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ a \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

$\left\{ \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ a \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$ if and only if $\begin{pmatrix} a & -1 & 1 \\ 1 & a & -1 \\ -1 & 1 & a \end{pmatrix}$ is invertible, if and only if its determinant is nonzero.

```
>> syms a; A=[a -1 1;1 a -1;-1 1 a];
>> simplify(det(A))
```

The set is a basis if and only if $a \neq 0$.

Question 5(a)

Suppose $U = \text{span}\left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \right\}$, $V = \text{span}\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} \right\}$. Define the sum $U + V = \{ \mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U \text{ and } \mathbf{v} \in V \}$. Is $U \cup V$ a subspace of $\mathbb{R}^4$?

No. $\mathbf{u}_1, \mathbf{v}_1 \in U \cup V$ but we will show that $\mathbf{w} = \mathbf{u}_1 + \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ is not, that is, $\mathbf{w}$ is neither in U nor V.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

```
>> u1=[1;1;1;1];u2=[1;2;2;1];v1=[1;0;1;0];v2=[1;0;2;-1];
```

Question 5(b)

Show that $U + V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

Any vector in $U + V$ can be written as

$$\mathbf{u} + \mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2,$$

that is, $U + V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$, and hence, $U + V$ is a subspace.Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set, it suffices to find a linearly independent subset of it to form a basis.

```
>> rref([u1 u2 v1 v2])
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\}$ is a basis for $U + V$, and hence $\dim(U + V) = 3$.

Question 5(c)

Show that $U + V$ contains U and V. This shows that $U + V$ is a subspace containing $U \cup V$.

Given any $\mathbf{u} \in U$, let $\mathbf{v} = \mathbf{0} \in V$, and so $\mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u} + \mathbf{v} \in U + V$. This shows that $U \subseteq U + V$. Similarly, given any $\mathbf{v} \in V$, let $\mathbf{u} = \mathbf{0} \in U$, and so $\mathbf{v} = \mathbf{0} + \mathbf{v} = \mathbf{u} + \mathbf{v} \in U + V$. Since $U \subseteq U + V$ and $V \subseteq U + V$, $U \cup V \subseteq U + V$.

Alternatively, since $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ are subsets of $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$. In fact, $U + V$ is the smallest subspace that contains $U \cup V$.

Question 5(d)

What are the dimensions of U and V?

It is clear that $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent sets. Hence, $\dim(U) = \dim(V) = 2$.

Question 5(e)

Show that $U \cap V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

A vector in $\mathbf{w} \in U \cap V$ must be able to be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, and as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In other words, we must be able to find $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $\mathbf{w} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2$, in other words, we are solving the homogeneous linear system $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 - \beta_1\mathbf{v}_1 - \beta_2\mathbf{v}_2 = \mathbf{0}$.

```
>> rref([u1 u2 -v1 -v2])
```

So for any choice of $s \in \mathbb{R}$, $\alpha_1 = -2s$ and $\alpha_2 = s$, or $\beta_1 = -2s$ and $\beta_2 = s$ will work, that is, $\mathbf{w} = -s(2\mathbf{u}_1 - \mathbf{u}_2) = -s(2\mathbf{v}_1 - \mathbf{v}_2)$. Hence, $U \cap V = \text{span}\{2\mathbf{u}_1 - \mathbf{u}_2\} = \text{span}\{2\mathbf{v}_1 - \mathbf{v}_2\}$, and this shows that $U \cap V$ is a subspace, with $\dim(U \cap V) = 1$.

Question 5(f)

Verify that $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$.

Indeed, $3 = 2 + 2 - 1$.