MA1522 Linear Algebra for Computing

Lecture 6: Span and Linear Dependence/Independence

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Outline

Question one in Section 3.3

Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(i) Is $\mathbf{v}$ in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$?

(ii) If it is, write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$,

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

(iii) Are the coefficients c_1, c_2, c_3 unique?

Find a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that is not in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Slide 24: Linear Combinations

Definition

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. A *linear combination* of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

for some $c_1, c_2, \ldots, c_k \in \mathbb{R}$. The scalars $c_1, c_2, \ldots, c_k$ are called *coefficients*.

Slide 26: Linear Span

Definition

Let $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. The *span* of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ is the subset of $\mathbb{R}^n$ containing all the linear combinations of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$,

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \left\{ \; c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \;\middle|\; c_1, c_2, ..., c_k \in \mathbb{R} \; \right\}.$$

We also define the span of the empty set $\operatorname{span}\emptyset = \{\mathbf{0}\}$.

That is every vector $\mathbf{v}$ in the set $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$,

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

for some scalars $c_1, c_2, ..., c_k$.

Slide 31: Algorithm to Check for Linear Combination

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

- Form the $n \times k$ matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ whose columns are the vectors in S.
- Then a vector $\mathbf{v}$ in $\mathbb{R}^n$ is in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ if and only if the system $\mathbf{Ax} = \mathbf{v}$ is consistent.
- If the system is consistent, then the solutions to the system are the possible coefficients of the linear combination. That is, if $\mathbf{u} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ is a solution to $\mathbf{Ax} = \mathbf{v}$, then

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

Explicitly, $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ if and only if $\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right)$ is consistent.

Answer to Question one in Section 3.3, part 1

Q: Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Is $\mathbf{v}$ in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$?

Answer: By the algorithm on Slide 31, we form the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system is consistent. Hence, $\mathbf{v}$ is in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Answer to Question one in Section 3.3, part 1 (conti.)

Q: (following (i))

(ii) Write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$,

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

(iii) Are the coefficients c_1, c_2, c_3 unique?

Answer: We have had the REF form of the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system has solutions $c_1 = 1 - 2s, c_2 = 1 - s$ and $c_3 = s$, where $s \in \mathbb{R}$. We can let $s = 0$, and $c_1 = c_2 = 1$ and $c_3 = 0$. Namely

$$\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2.$$

The answer is not unique, for example, we let $s = 1$ and $c_1 = -1, c_2 = 0$ and $c_3 = 1$,

$$\mathbf{v} = -\mathbf{u}_1 + \mathbf{u}_3.$$

Answer to Question one in Section 3.2, part 2

Q: Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Find a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that is not in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Answer: We can take $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and form the augmented matrices

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

The system is inconsistent. Thus $\mathbf{v}$ is not in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Question Two in Section 3.3

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of k vectors in $\mathbb{R}^n$.

1. Show that if $k < n$ then $\text{span}(S) \neq \mathbb{R}^n$.

2. If $k > n$, can we make any conclusion?

Slide 37: Algorithm to check if span$(S) = \mathbb{R}^n$.

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

- Form the $n \times k$ matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ whose columns are the vectors in S.
- Then span$(S) = \mathbb{R}^n$ if and only if the system $\mathbf{Ax} = \mathbf{v}$ is consistent for all $\mathbf{v}$.
- This is equivalent to the reduced row-echelon form of $\mathbf{A}$ having no zero rows.

Explicitly, span$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \mathbb{R}^n$ if and only if the reduced row-echelon form of $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ has no zero rows.

Answer to Question two in Section 3.2, part 1

Q: Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of k vectors in $\mathbb{R}^n$. Show that if $k < n$ then $\text{span}(S) \neq \mathbb{R}^n$.

Answer: By slide 37, we study the reduced row-echelon form of $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$. Since $k < n$, the number of pivotal columns can be at most k, in other words, there are at most k many nonzero rows. Thus, there must be zero rows, because $n > k$. Therefore, we conclude $\text{span}(S) \neq \mathbb{R}^n$.

Answer to Question two in Section 3.2, part 2

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of k vectors in $\mathbb{R}^n$. If $k > n$, can we make any conclusion?

Answer: We cannot make any conclusion. For example, let $n = 2$, and $k = 3$.

If we take $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then span$(S) = \mathbb{R}^2$ (because $\mathbf{u}_1$ and $\mathbf{u}_2$ already span $\mathbb{R}^2$).

On the other hand, if we take $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\mathbf{u}_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Then span$(S) \neq \mathbb{R}^2$ (because the RREF has a zero row).

Question one in Section 3.4

1. Show that the set containing the zero vector $\{\mathbf{0}\}$ is a subspace.

2. Construct a set V such that it satisfies condition (i) and (ii) but not (iii); that is, V contains the origin and is closed under scalar multiplication, but not closed under addition.

3. Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ a subset of V, $S \subseteq V$. Show that the span of S is contained in V, $\text{span}(S) \subseteq V$.

Slide 58: Subspace

Definition

A subset V of $\mathbb{R}^n$ is a *subspace* if it satisfies the following properties.

(i) V contains the zero vector $\mathbf{0} \in V$.

(ii) V is closed under scalar multiplication. For any vector $\mathbf{v}$ in V and scalar α, the vector $\alpha\mathbf{v}$ is in V.

(iii) V is closed under addition. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.

Remark

(1) Property (i) can be replaced with property (i'): V is nonempty.

(2) Properties (ii) and (iii) is equivalent to property (ii'): For any $\mathbf{u}, \mathbf{v}$ in V, and scalars α, β, the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is in V.

Remarks

- Algebraic terminology: Let U be a set, $f: U \to U$ a function/operation on U. A subset X of U is *closed under f*, if for any $x \in X$, $f(x) \in X$.
- Subspace is an instance of "substructure" in algebra.
- It turns out that for a subset V of the Euclidean space $\mathbb{R}^n$ to satisfy all 10 axioms of being a vector space, suffice for it to satisfies only 3 of them.
- (This is because all other axioms are in "universal" form. If a "universal" property holds for a big set, then it holds for all its subsets.)

Answer to Question one in Section 3.4, part 1

Q: Show that the set containing the zero vector $Z = \{\mathbf{0}\}$ is a subspace.

Answer: We check the three properties:

(i) V contains the zero vector $\mathbf{0} \in V$. Clearly, $\mathbf{0} \in Z$.

(ii) V is closed under scalar multiplication. For any vector $\mathbf{v}$ in V and scalar α, the vector $\alpha\mathbf{v}$ is in V.
Only $\mathbf{0}$ is in Z, and for any scalar α, $\alpha\mathbf{0} = \mathbf{0} \in Z$.

(iii) V is closed under addition. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.
Again, we only have one sum $\mathbf{0} + \mathbf{0} = \mathbf{0} \in Z$.

Answer to Question one in Section 3.4, part 2

Q: Construct a set V such that it satisfies condition (i) and (ii) but not (iii); that is, V contains the origin and is closed under scalar multiplication, but not closed under addition.

Answer: Consider the space $\mathbb{R}^2$ and V to be the two axis. That is,

$$V = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}.$$

Then (i) and (ii) are satisfied, but (iii) failed because $(1, 0) + (0, 1) = (1, 1) \notin V$.

Answer to Question one in Section 3.4, part 3

Q: Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ a subset of V, $S \subseteq V$. Show that the span of S is contained in V, $\text{span}(S) \subseteq V$.

Answer: Recall that the elements $\mathbf{w}$ in $\text{span}(S)$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$, i.e., of the form

$$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k$$

for some $c_1, \ldots c_k \in \mathbb{R}$.

Given $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k \in V$, since V is a subspace, V is closed under scalar multiplication, so $c_1\mathbf{u}_1, \ldots, c_k\mathbf{u}_k$ are in V. V is also closed under addition, because it is a subspace, we have

$$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k \in V.$$

Question two in Section 3.4

Is $\mathbb{R}^2 \subseteq \mathbb{R}^3$?

Answer: No, because every element is $\mathbb{R}^2$ has two coordinates, whereas $\mathbb{R}^3$ has three.

That said, $\mathbb{R}^2$ can be "embedded" into $\mathbb{R}^3$ by $(x, y) \mapsto (x, y, 0)$. In other words, if we identify (x, y) with $(x, y, 0)$, then $\mathbb{R}^2$ can be viewed as a subspace of $\mathbb{R}^3$.

Question one in Section 3.5

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. Let

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_1 + \mathbf{u}_2, \\
\mathbf{v}_3 &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3.
\end{aligned}$$

Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent too.

Slide 82: Linearly Independent

Definition

A set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is *linearly independent* if the only coefficients $c_1, c_2, ..., c_k$ satisfying the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0},$$

are $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, we say that the set is *linearly dependent*.

Remarks on Linear Independence

- In symbols, $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly independent if for all $c_1, \dots, c_k \in \mathbb{R}$,

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0} \Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

- $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent if for some $c_1, \dots, c_k \in \mathbb{R}$, not all equal to 0 such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

- (It is a good exercise in logic to show the second statement is indeed a negation of the first.)

Answer to Question one in Section 3.5

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. Let

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1, \\ \mathbf{v}_2 &= \mathbf{u}_1 + \mathbf{u}_2, \\ \mathbf{v}_3 &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3. \end{aligned}$$

Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent too.

Answer: Suppose that for some $c_1, c_2, c_3 \in \mathbb{R}$ with

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

We must show that $c_1 = c_2 = c_3 = 0$.

Answer to Question one in Section 3.5 (conti.)

Substitute the u vectors, we have

$$c_1\mathbf{u}_1 + c_2(\mathbf{u}_1 + \mathbf{u}_2) + c_3(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) = \mathbf{0}.$$

Namely,

$$(c_1 + c_2 + c_3)\mathbf{u}_1 + (c_2 + c_3)\mathbf{u}_2 + c_3\mathbf{u_3} = \mathbf{0}.$$

Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, we have

$$c_1 + c_2 + c_3 = c_2 + c_3 = c_3 = 0.$$

Hence $c_1 = c_2 = c_3 = 0$.

Question two in Section 3.5

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in of $\mathbb{R}^n$. Show that if $k > n$, then S is linearly dependent.

Answer: We need to show that for some scalars $c_1, \ldots, c_k \in \mathbb{R}$, not all zero, $c_1\mathbf{u_1} + \cdots + c_k\mathbf{u}_k = \mathbf{0}$. In other words, the linear system with c_i as unknowns has nonzero solutions.

Form the augmented matrix, which is $n \times k$. Since $k > n$, it has infinitely many solutions (with at least $k - n$ parameters). Thus S is linearly dependent.

Challenge in Section 3.4

Prove that if a subset V of $\mathbb{R}^n$ satisfies the 3 criteria of a subspace, then it satisfies all 10 axioms of a vector space.

Answer: By (ii) and (iii), V also equipped with the same addition and scalar multiplication.

Axioms 1,2,5,6,7,8 are in universal form, thus they also hold in V.

Axiom 3 holds by (i).

Axiom 4 holds because $-\mathbf{u} = (-1)\mathbf{u} \in V$.

Definition of Abstract Vector Spaces

A set V equipped with addition and scalar multiplication is said to be a *vector space* over $\mathbb{R}$ if it satisfies the following axioms.

1. (Commutative) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. (Associative) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
3. (Zero vector) There is a vector $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$ in V.
4. (Negative) For any vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. (Distribution) For any scalar a in $\mathbb{R}$ and vectors $\mathbf{u}, \mathbf{v}$ in V, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
6. (Distribution) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
7. (Associativity of scalar multiplication) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $a(b\mathbf{u}) = (ab)\mathbf{u}$.
8. For any vector $\mathbf{u}$ in V, $1\mathbf{u} = \mathbf{u}$.

Challenge in Section 3.3

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. Referring to the properties of a spanning set or otherwise, show that the set $V = \text{span}(S)$ is a (abstract) vector space. That is, it satisfies the 10 axioms of the definition of vector spaces.

Answer: By Challenge in Section 3.4, it suffices to show that V satisfies all three conditions of subspaces. Details skipped.