MA1522: Linear Algebra for Computing

Chapter 1: Linear Systems

1.1 Introduction to Linear Systems

Introduction to Linear Systems

Consider the following

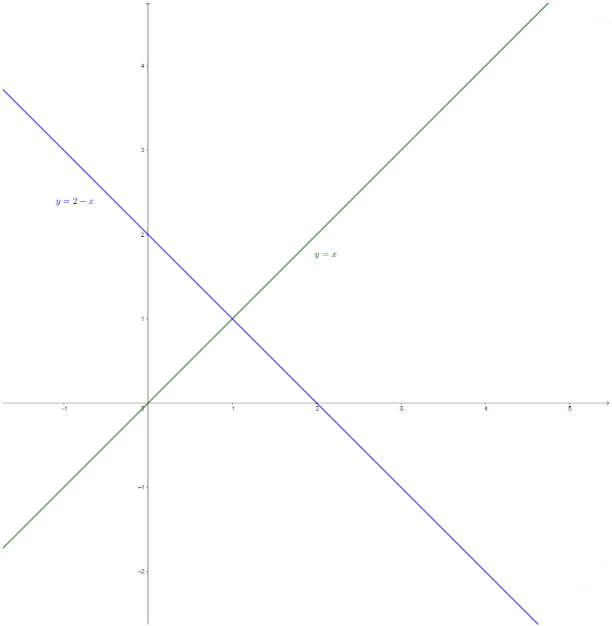
$$\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$$

which is (formerly and fondly) know as
simultaneous equations.

(Only) Solutions: $x = 1$, $y = 1$.

Geometrically, the solution is the intersection of 2 lines.

The above is an example of a *linear system*. In most applications, a linear system involves a large number of variables and equations. We will learn what they are and how to solve them in this chapter.



Definition of Linear Equations

Definition

A *linear* equation with n variables in *standard form* has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Here $a_1, a_2, ..., a_n$ are known constants, called the *coefficients*, b is called the *constant*, and $x_1, x_2, ..., x_n$ are *variables*.

The linear equation is *homogeneous* if $b = 0$, i.e.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0.$$

Examples

Which of the following equations are linear in x and y?

1. $2x + y = 3$
2. $x = 2$
3. $y = x \sin(\frac{\pi}{6})$
4. $x = ky$, for some $k \in \mathbb{R}$.
5. $xy = 3$
6. $y = x^2$
7. $y = \sin(x)$
8. $3\cos(x) + 4\sin(y) = 2$

Which of the linear equations are in standard form? Which of the linear equations are homogeneous?

Definition of Linear Systems

Definition

A *system of linear equations*, or a *linear system* consists of a finite number of linear equations. In general, a linear system with n variables and m equations in *standard form* is written as

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

The linear system is *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$, that is, all the linear equations are homogeneous.

Solutions to a Linear System

Definition

Given a linear system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

we say that

$$x_1 = c_1,\ x_2 = c_2,\ \ldots,\ x_n = c_n$$

is a *solution* to the linear system if the equations are simultaneously satisfied after making the substitution.

Example

$x = 1$, $y = 1$ is the (only) solution to the following linear system.

$$\left\{ \begin{array}{ccccc} x & - & y & = & 0 \\ x & + & y & = & 2 \end{array} \right.$$

- $x = 2, y = 2$ is a solution to the first equation, but not the second.
- $x = 2, y = 0$ is a solution to the second equation, but not the first.

Question

Give a solution to the following linear system.

$$\begin{cases} x & + & 2y & = & 5 \\ 2x & + & 4y & = & 10 \end{cases}$$

Some possible solutions are

- $x = 1, y = 2$
- $x = 3, y = 1$
- $x = 4, y = \frac{1}{2}$

In fact, we may choose y to be any real number, then $x = 5 - 2y$ will be a solution to the system. Alternatively, we may choose x to be any real number, then $y = \frac{5-x}{2}$.

General Solutions

Definition

The ***general solution*** to a linear system captures all possible solutions to the linear system.

When there are infinitely many solutions to a linear system, we can assign some variables as parameters, and express the other variables in terms of the parameters.

The general solution of the example above will be $x = 5 - 2s,\ y = s,\ s \in \mathbb{R}$. Here the symbol $s \in \mathbb{R}$ means that s can be any real number, which we call a parameter of the general solution. Alternatively, the general solution can be expressed as $y = \frac{5-t}{2},\ x = t,\ t \in \mathbb{R}$.

Caution

Do not use the variables as parameters. That is, do not write $x = 5 - 2x$, as it is begging the question.

Inconsistent Linear System

Consider now another linear system.

$$\begin{cases} x & + & y & = & 2 \\ x & - & y & = & 0 \\ 2x & + & y & = & 1 \end{cases}$$

Are you able to provide any solution to the system?

Observe that $x = 1 = y$ is the only solution to the first 2 equations, but not a solution to the third equations.

Hence, the system has not solution. In this case, we say that the system is <u>inconsistent</u>.

Inconsistent Linear Systems

Definition

A linear system is said to be ***inconsistent*** if it does not have any solutions. It is ***consistent*** otherwise, that is, a linear system is consistent if it has at least one solution.

We have seen examples of linear system that are inconsistent, consistent with a unique solution, or consistent with infinitely many solutions. Are there examples of linear systems that have more than 1 but only finitely many solutions?

Linear Systems with 2 Variables

Consider a linear system with 2 variables

$$\begin{cases} a_1 x & + & b_1 y & = & c_1 \\ a_2 x & + & b_2 y & = & c_2 \end{cases}$$

for some real numbers $a_i, b_i, c_i,\ i = 1, 2$.

Each equation is a line in the 2-dimensional graph . The solution of this system corresponds to a point of intersection of the lines, and there are three possibilities.

(i) The lines are parallel and distinct. System has no solution.

(ii) The lines intersect at only one point. System has a unique solution.

(iii) The lines coincide. System has infinitely many solutions.

Head over to the following link to visualize the linear system.
https://www.geogebra.org/m/ahvsz6j9

Example

Go to www.geogebra.org/classic and type in the following equations.

$$\begin{cases} 2x + 3y = 1 \\ x - y = 3 \end{cases}$$

(i) Is the system consistent?

(ii) How many solutions are there?

The system is consistent with only 1 solution, $x = 2, y = -1$.

Example

Refresh the page www.geogebra.org/classic and type in the following equations.

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 5 \end{cases}$$

(i) Is the system consistent?

(ii) How many solutions are there?

The system is inconsistent.

Example

Refresh the page www.geogebra.org/classic and type in the following equations.

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 3 \end{cases}$$

(i) Is the system consistent?

(ii) How many solutions are there?

The system is consistent with infinitely many solution. A general solution will be $x = 1 + 2s$, $y = s$, $s \in \mathbb{R}$.

Linear Systems with 3 Variables

Consider a linear system with 3 variables

$$\begin{cases} a_1x & + & b_1y & + & c_1z & = & d_1 \\ a_2x & + & b_2y & + & c_2z & = & d_2 \\ a_3x & + & b_3y & + & c_3z & = & d_3 \end{cases}$$

for some real numbers $a_i, b_i, c_i, d_i,\ i = 1, 2, 3$.

Each equation is a plane in the 3d-space. First note that 2 planes intersect along a line.

(i) The system has no solution if either 2 planes are parallel but distinct, or the planes intersects, but the lines of intersections do not intersect.

(ii) The system has a unique solution if all 3 planes intersects at a point.

(iii) The system has infinitely many solutions with 1 parameter in the general solution if the 3 planes intersects along a line.

(iv) The system has infinitely many solutions with 2 parameters in the general solution if the 3 planes coincide.

Head over to the following link to visualize the linear system.
https://www.geogebra.org/m/wxneuj7k

Example

Go to \color{blue}www.geogebra.org/3d?lang=en. Type in the following equations.

$$\begin{cases} x + y + z = 1 \\ x + y - z = 1 \\ x - y + z = 1 \end{cases}$$

(i) Is the system consistent?

(ii) If the system is consistent, is the solution unique?

(iii) If the solution is not unique, how many parameters are needed in the general solution?

The system is consistent with a unique solution $x = 1, y = z = 0$.

Example

Fresh the page `\color{blue}www.geogebra.org/3d?lang=en`. Type in the following equations.

$$\begin{cases} x+y+z=1 \\ x-y+z=1 \\ x+z=1 \end{cases}$$

(i) Is the system consistent?

(ii) If the system is consistent, is the solution unique?

(iii) If the solution is not unique, how many parameters are needed?

All 3 planes intersection along a line.

Example

Fresh the page www.geogebra.org/3d?lang=en. Type in the following equations.

$$\begin{cases} x+y+z=1 \\ x+y+z=2 \\ x+z=1 \end{cases}$$

(i) Is the system consistent?

(ii) If the system is consistent, is the solution unique?

(iii) If the solution is not unique, how many parameters are needed?

The system is inconsistent. Any 2 planes intersects along a line, but the lines of intersections do not intersect.

Discussions

1. Give an example of a linear system with 3 variables such that the general solution has 2 parameters.

2. Is it possible to have a linear system with 3 variables, 3 equations, with the general solution having 3 parameters?

3. In the next section we will see that it is true that a linear system will always only have either no solution, a unique solution, or infinitely many solutions.

1.2 Solving a Linear System and Row-Echelon Form

Example

Solve the following linear systems.

1. $$\left\{\begin{array}{rcrcrcr} x & & & & & = & 1 \\ & & y & & & = & 2 \\ & & & & z & = & 3 \end{array}\right. \quad \Rightarrow \quad \begin{array}{rcl} x & = & 1 \\ y & = & 2 \\ z & = & 3 \end{array}$$

2. $$\left\{\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ & & y & - & \frac{7}{2}z & = & -\frac{17}{2} \\ & & & & z & = & 3 \end{array}\right. \quad \Rightarrow \quad \begin{array}{rcl} x & = & 1 \\ y & = & 2 \\ z & = & 3 \end{array}$$

3. $$\left\{\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ 2x & + & 4y & - & 3z & = & 1 \\ 3x & + & 6y & - & 5z & = & 0 \end{array}\right. \quad \Rightarrow \quad \begin{array}{rcl} x & = & 1 \\ y & = & 2 \\ z & = & 3 \end{array}$$

Augmented Matrix

A linear system

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

can be expressed uniquely as an augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

That is, we first express the linear system in standard form, then put the coefficients in the entries of the left hand side of the bar in the augmented matrix, and the constant on the right hand side of the bar in the augmented matrix.

Remark

The augmented matrix is an example of a matrix, which we will discuss in further details in Chapter 2.

Example

Linear system:

$$\begin{cases} 3x + 2y - z = 1 \\ \phantom{3x + {}} 5y + z = 3 \\ x \phantom{{}+2y} + z = 2 \end{cases}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 0 & 5 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{array}\right)$$

Example

Linear system:

$$\begin{cases} 2x & - & 1 & = & 3y \\ 3 & - & 9y & = & 6x \end{cases}$$

Augmented matrix:
We must first write the system in standard form.

$$\begin{cases} 2x & - & 3y & = & 1 \\ 6x & + & 9y & = & 3 \end{cases}$$

Now we can write down the augmented matrix.

$$\left(\begin{array}{cc|c} 2 & -3 & 1 \\ 6 & 9 & 3 \end{array}\right)$$

Example

Linear system:

$$\left\{\begin{array}{rcrcrcrcr} x_1 & - & 4x_2 & + & ax_3 & - & 6x_4 & = & 2 \\ 3x_1 & + & 2x_2 & & & & & = & a \\ 6x_1 & + & 2x_2 & - & x_3 & + & (a-1)x_4 & = & -1 \end{array}\right.$$

for some $a \in \mathbb{R}$.

Augmented matrix:

Here a is a fixed real number. Hence, the a in the first and equation is the coefficient of x_3 and the constant, respectively. Similarly, the $a-1$ is the coefficient of x_4 in the third equation. Therefore, the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -4 & a & -6 & 2 \\ 3 & 2 & 0 & 0 & a \\ 6 & 2 & -1 & a-1 & -1 \end{array}\right)$$

which includes the terms involving a.

Row-Echelon Form

Definition

In an (augmented) matrix, a *zero row* is a row with all entries 0. A row is called a *nonzero row* otherwise. The first nonzero entry from the left of a nonzero row is called a *leading entry*.

An (augmented) matrix is in *row-echelon form* (REF) if

1. If zero rows exists, they are at the bottom of the matrix.
2. The leading entries are further to the right as we move down the rows.

An augmented matrix in REF has the form

$$\left(\begin{array}{cccccccc|c} * & & & & & & & & * \\ 0 & \cdots & 0 & * & & & & & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & * & & * \\ \vdots & & & & & & & \vdots & \vdots \\ 0 & \cdots & & & & & \cdots & 0 & 0 \end{array}\right).$$

Definition

In row-echelon form, a column containing a leading entry is called a *pivot column*. It is called a *non-pivot column* otherwise.

Reduced Row-Echelon Form

The (augmented) matrix is in reduced row-echelon form (RREF) if further

3. The leading entries are 1.
4. In each pivot column, all entries except the leading entry is 0.

An augmented matrix in RREF has the form

$$\left(\begin{array}{cccccccccc|c} 0 & \cdots & 1 & & * & 0 & & * & 0 & * & * \\ 0 & \cdots & 0 & \cdots & 0 & 1 & & * & 0 & * & * \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & * & * \\ 0 & \cdots & 0 & & & 0 & & & 0 & 0 & 0 \\ \vdots & & & & & \vdots & & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & & 0 & & \cdots & 0 & 0 & 0 \end{array}\right).$$

Remark

Note that a (an augmented) matrix in reduced row-echelon form is also in row-echelon form. The reduced row-echelon form is a special case of row-echelon form.

Examples

The following are examples of augmented matrices not in row-echelon form.

1. $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right)$ is not in REF since the leading entry in the third row is in the same column as the leading entry in the second row.

2. $\left(\begin{array}{ccc|c} 0 & 2 & 0 & -5 \\ 1 & 0 & 1 & 3 \end{array}\right)$ is not in REF as the leading entry in row 2 is on the left of the leading entry in row 1.

3. $\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right)$ is not in REF since the second row is a zero row while the third is a non zero row.

Examples

The following are examples of augmented matrices in row-echelon form but not in reduced row-echelon form.

1. $\left(\begin{array}{ccc|c} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{array}\right)$ is not in RREF as the leading entries in the first and third row are not 1.

2. $\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 1 \end{array}\right)$ is not in RREF as the third column is a pivot column, but the entry in the first row is a nonzero non leading entry.

3. $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 3 \end{array}\right)$ is not in RREF as the leading entry in the second row is not 1.

Question

Consider the following augmented matrix

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & 1 \\ 0 & g & h & i \end{array}\right)$$

for some real numbers $a, b, c, d, e, f, g, h, i$. Suppose the augmented matrix is in row-echelon form.

1. What are the possible values of g?
2. If $h = 0$, what are the possible values of i?
3. If the augmented matrix is in reduced row-echelon form, and $f = -1$, what are the possible values of e?

Solutions from REF and RREF

When the augmented matrix is in row-echelon or reduced row-echelon form, it is easy to extract the solutions of the linear system.

- If the augmented matrix is in row-echelon form, we perform back substitution to obtain the solutions.
- If the augmented matrix is in reduced row-echelon form, we will read off the solutions directly.

Example

1. The augmented matrix

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right)$$

is in reduced row-echelon form. it corresponds to the linear system

$$\left\{\begin{array}{ccccccccc} x_1 & - & x_2 & & & + & 3x_4 & = & -2 \\ & & & & x_3 & + & 2x_4 & = & 5 \end{array}\right.$$

First, observe that if we parameterize $x_4 = t$, then the second equation tells us that $x_3 = 5 - 2t$. If we parameterize $x_2 = s$, then the first equation tells us that $x_1 = -2 + s - 3t$. Hence, the general solution is

$$x_1 = -2 + s - 3t, \quad x_2 = s, \quad x_3 = 5 - 2t, \quad x_4 = t, \quad s, t \in \mathbb{R}.$$

Observe that we can obtain the general solution directly by assigning parameters to the variables in the non-pivot columns in the LHS of the augmented matrix, here $x_2 = s$ and $x_4 = t$. Then from row 1, we get $x_1 = -2 + s - 3t$, and from row 2, we get $x_3 = 5 - 2t$.

Example

2 Consider another example

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This augmented matrix is in reduced row-echelon form. The non-pivot columns in the LHS of the augmented matrix are the second and fourth. We let $x_3 = s$ and $x_4 = t$. Then from the first row, we have $x_1 = 2 - s$, and from the second, $x_2 = 3 - s + t$. Hence, a general solution is

$$x_1 = 2 - s, \quad x_2 = 3 - s + t, \quad x_3 = s, \quad x_4 = t, \quad s, t \in \mathbb{R}.$$

Example

3 Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 11 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

It is in row-echelon form only. So, we need to perform *back-substitution*. Since there are no non-pivot columns in the LHS of the augmented matrix, we do not need to assign any parameters. We start from the last row; it tells us that $x_3 = -1$. Then from the second row, we have $x_2 = 7 + x_3 = 7 - 1 = 6$. Finally, the first row tells us that $x_1 = 11 - x_2 + 2x_3 = 11 - 6 - 2 = 3$. So, the system has a unique solution

$$x_1 = 3, \quad x_2 = 6, \quad x_3 = -1.$$

Example

4 Consider the augmented matrix

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

Note that it is in row-echelon form only. So, we will perform back-substitution. The non-pivot columns in the LHS of the augmented matrix are columns 1 and 4. Hence, we let $x_1 = s$ and $x_4 = t$. From row 3, we get $x_5 = 4$. From row 2, we have $x_3 = -1 + x_4 = -1 + t$. Finally, row 1 tells us that $x_2 = 2 + x_5 = 2 + 4 = 6$. So, the general solution is

$$x_1 = s, \quad x_2 = 6, \quad x_3 = t - 1, \quad x_4 = t, \quad x_5 = 4, \quad s, t, \in \mathbb{R}.$$

Example

5 Consider the last example

$$\left(\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Observe that the third row tells us that $0x_1 + 0x_2 + 0x_3 = 1$. There exists no solution x_1, x_2, x_3 such that $0x_1 + 0x_2 + 0x_3 = 1$. Hence, we can conclude that the system is inconsistent. Observe that we get such an equation if and only if RHS of the augmented column is a pivot column. We will state it as a result.

The linear system is inconsistent if and only if the RHS (last column) of the augmented matrix in row-echelon form is a pivot column.

Remarks

1. It is easy to obtain the solutions when the augmented matrix is in row-echelon form (by performing back-substitution) or reduced row-echelon form (reading off the solutions directly).

2. The linear system is inconsistent if and only if the RHS (last column) of the augmented matrix in row-echelon form is a pivot column.

3. Assign parameters to the variables corresponding to the non-pivot columns in the LHS of the augmented matrix.

4. The number of parameters needed is equal to the number of non-pivot columns in the LHS of the augmented matrix.

5. We can convert/reduce the augmented matrix of a linear system to a row-echelon form or its reduced row-echelon form to find the solutions (if exists). This is achieved using elementary row operations.

Challenge

Let $\mathbf{R}$ be a $n \times m$ matrix in reduced row-echelon form. Which of the following statements are true?

1. The number of pivot columns of $\mathbf{R}$ is equal to the number of nonzero rows of $\mathbf{R}$.

2. The number of nonpivot columns of $\mathbf{R}$ is equal to the number of zero rows in $\mathbf{R}$.

For each statement that is false, what restrictions can we impose on $\mathbf{R}$ such that the statement is true?

1.3 Elementary Row Operations

Motivation

Consider the following linear system

$$\begin{cases} x & + & 2y & = & 3 \\ 4x & + & 5y & = & 6 \end{cases}$$

To solve it, we perform the following operations

$$\begin{array}{lcrl} (\text{equation 2}) - 4 \times (\text{equation 1}) & \to & -3y = -6 & (\text{equation 3}) \\ -\frac{1}{3}(\text{equation 3}) & \to & y = 2 & (\text{equation 4}) \end{array}$$

Substitute equation 4 into equation 1, or

$$(\text{equation 1}) - 2 \times (\text{equation 4}) \quad \to \quad x = -1 \quad (\text{equation 5})$$

we conclude that $x = -1, y = 2$ is the (unique) solution to the system.

Observe that in all the operations above, the solution(s) to the system is preserved. This motivated the introduction of *elementary row operations*, that is, a fundamental set of operations that do not alternative the solution set when performed on the augmented matrix of a linear system.

Elementary Row Operations

There are 3 types of elementary row operations.

1. Exchanging 2 rows, $R_i \leftrightarrow R_j$,
2. Adding a multiple of a row to another, $R_i + cR_j$, $c \in \mathbb{R}$,
3. Multiplying a row by a nonzero constant, aR_j, $a \neq 0$.

Remark

Performing elementary row operations to the augmented matrix of a linear system preserves the solutions.

Examples

1. Exchange row 2 and row 4

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4 & 3 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_4} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

2. Add 2 times row 3 to row 2

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4 & 3 \end{array}\right) \xrightarrow{R_2 + 2R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4 & 3 \end{array}\right)$$

3. Multiply row 2 by $1/2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4 & 3 \end{array}\right) \xrightarrow{\frac{1}{2} R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4 & 3 \end{array}\right)$$

Example

Consider the following linear system

$$\left\{\begin{array}{ccccc} x & + & y & = & 2 \\ x & - & y & = & 0 \end{array}\right., \quad \text{Augmented Matrix: } \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right).$$

$x = 1, y = 1$ is the only solution. It should be clear that if we swap the rows, the solution would remain unchanged.

Now if we take a multiple of row 2, say takes 3 times of row 2, we have

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{3R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & -3 & 0 \end{array}\right).$$

The solution is still $x = y = 1$. Finally, if we take a multiple of row 1 and add it to row 2, say -1 times of row 1 to row 2, we have

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right).$$

The solution is still $x = y = 1$.

Row Equivalent Matrices

Definition

Two (augmented) matrices are *row equivalent* if one can be obtained from the other by performing a set of elementary row operations.

Theorem

Two linear systems have the same solutions if their augmented matrices are row equivalent.

The proof of the theorem can be found in the appendix of chapter 2.

Remarks

Recall that when multiplying a row by a constant, aR_i, we require the multiple to be nonzero, $a \neq 0$.

Consider the following linear system.

$$\begin{array}{ccccc} x & + & y & = & 2 \\ x & - & y & = & 0 \end{array}$$

It has a unique solution $x = y = 1$. Now suppose we multiple the row two of the augmented matrix by 0,

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{0R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

we obtain a system with infinitely many solution; the general solution is $x = 2 - s$, $y = s$, $s \in \mathbb{R}$.

This demonstrates that if we multiply a row in the augmented matrix by 0, we may change the solution of the linear system.

Remarks

The order by which we perform the row operations matters (that is row operations do not commute).

Consider the following augmented matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

By taking 2 times of row two, then exchanging the row, we obtain

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{2R_1} \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 2 & 0 & 0 \end{array}\right).$$

However, if we first exchange the rows then take 2 times of row two, we have

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_1} \xrightarrow{2R_1} \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 1 & 0 & 0 \end{array}\right).$$

Remarks

Hence, when performing a few row operations on an augmented matrix, we have to write the row operations from left to right.

However, if the row operations commute, that is, in some cases where it does not matter which row operations we perform first, we may stack them together as such

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow[2R_1]{2R_2} \left(\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right)$$

Remarks

The notation $R_i + cR_j$ acts on R_i, but keeps R_j unchanged.

For example

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right),$$

whereas

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right).$$

Remarks

Here is another example. $R_1 + 2R_2$ is an elementary row operation,

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1+2R_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right),$$

which is different from the row operation $2R_2 + R_1$

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{2R_2+R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right).$$

In fact, $2R_2 + R_1$ is not an elementary row operation, but a combination of 2 elementary row operations, first perform $2R_2$, then perform $R_2 + R_1$;

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{2R_2} \xrightarrow{R_2+R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right).$$

Discussion

Consider the following elementary row operations.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R_2+3R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{??} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

What is the (unique) elementary row operation that when performed, would change the middle augmented matrix back to its original one?

Since the second row in the augmented matrix in the middle had an extra 3 times of row 1, to undo it, we need to remove the extra 3 times of row 1. That is, if we perform $R_2 - 3R_1$, we will get the original augmented matrix. This shows that the elementary row operation $R_i + cR_j$ can be undone, or "reversed" if we perform $R_i - cR_j$. This is true for all elementary row operations.

Reverse of Elementary Row Operations

Every elementary row operation has a *reverse elementary row operation*. The reverse of the row operations are given as such.

1. The reverse of exchanging 2 rows, $R_i \leftrightarrow R_j$, is itself.

2. The reverse of adding a multiple of a row to another, $R_i + cR_j$ is subtracting the multiple of that row, $R_i - cR_j$.

3. The reverse of multiplying a row by a nonzero constant, aR_j is the multiplication of the reciprocal of the constant, $\frac{1}{a}R_j$.

Readers should convinced themselves that the reverse of the elementary row operations are indeed the ones given above. In fact, this is the definitive property of an elementary row operation; all (linear) operations that can be reversed are composed (made up of a series) of elementary row operations.

Examples

1. The reverse of $R_2 \leftrightarrow R_4$ is itself,

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

2. The reverse of $R_2 + 3R_3$ is $R_2 - 3R_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{R_2 + 3R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 11 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{R_2 - 3R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

3. The reverse of $2R_3$ is $\frac{1}{2}R_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{2R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

Question

What is the reverse of the elementary row operation $R_2 - \frac{1}{2}R_1$?

Remark

- Since performing elementary row operations do not change the solution set of a linear system, and
- the solutions of a linear system can be obtained easily from the augmented matrix in row-echelon form,
- our aim is therefore to use elementary row operations to reduce the augmented matrix of a linear system until it is in row-echelon or even better, in reduced row-echelon form.
- This will be the discussion of the next section.

1.4 Row Reduction, Gaussian and Gauss-Jordan Elimination

Introduction

- Aim: to reduce an augmented matrix to a row-echelon, or reduced row-echelon form using elementary row operations.
- Gaussian or Gauss-Jordan elimination are algorithms that do so.
- These algorithm guarantees to reduce the augmented matrix, but might not be the most efficient.
- May have to modify the algorithms when the augmented matrix contains unknown coefficients.

Example

Solve the following linear system

$$\begin{cases} x & + & y & + & 2z & = & 4 \\ -x & + & 2y & - & z & = & 1 \\ 2x & & & + & 3z & = & -2 \end{cases}$$

The augmented matrix of the linear system is $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -1 & 2 & -1 & 1 \\ 2 & 0 & 3 & -2 \end{array}\right)$.

The aim is to reduce it to either in row-echelon, or reduced row-echelon form. Recall that in the row-echelon form, the leading entries are further to the right as we move down the rows. This would mean that in the first column, the entries in the second and third row must be 0. So, we use the first entry in the first row to eliminate the first entry in the second and third row.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -1 & 2 & -1 & 1 \\ 2 & 0 & 3 & -2 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & -2 & -1 & -10 \end{array}\right).$$

Note that since these 2 operations commute, we may stack them.

Example

Next, we need the leading entry in the third row to be in the third column.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & -2 & -1 & -10 \end{array}\right) \xrightarrow{R_3+\frac{2}{3}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -\frac{1}{3} & -\frac{20}{3} \end{array}\right).$$

The augmented matrix is now in row-echelon form, and one can perform back substitution to obtain the solution.

$$z = 20, \quad y = \frac{5-20}{3} = -5, \quad x = 4 - (-5) - 2(20) = -31.$$

Alternatively, we may continue to reduce the augmented matrix to its reduced row-echelon form. To proceed, we need the leading entry of the third row to be 1.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -\frac{1}{3} & -\frac{20}{3} \end{array}\right) \xrightarrow{-3R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 20 \end{array}\right).$$

Example

Next, we need the entries above the leading entry in the third row to be 0. And finally, we need the leading entry in the second row to be 1, and the entry above it to be 0.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 20 \end{array}\right) \xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 0 & -15 \\ 0 & 0 & 1 & 20 \end{array}\right) \xrightarrow{\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 20 \end{array}\right) \xrightarrow[R_1 - 2R_3]{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -31 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 20 \end{array}\right).$$

The augmented matrix is now in reduced row-echelon form and we arrive at the conclusion that the unique solution to the linear system is

$$x = -31, \quad y = -5, \quad z = 20,$$

which is consistent to the result obtained via back-substitution.

Example

In summary, these are the steps we performed to reduce the augmented matrix to its RREF.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -1 & 2 & -1 & 1 \\ 2 & 0 & 3 & -2 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & -2 & -1 & -10 \end{array}\right) \xrightarrow{R_3+\frac{2}{3}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -\frac{1}{3} & -\frac{20}{3} \end{array}\right) \xrightarrow{-3R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 20 \end{array}\right) \xrightarrow{R_2-R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 0 & -15 \\ 0 & 0 & 1 & 20 \end{array}\right) \xrightarrow{\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 20 \end{array}\right) \xrightarrow[R_1-2R_3]{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -31 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 20 \end{array}\right)$$

Gaussian Elimination

Step 1: Locate the leftmost column that does not consist entirely of zeros.

Step 2: Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

Example

(i) $$\left(\begin{array}{cccc|c} 0 & 3 & 9 & \cdots & * \\ 1 & 2 & -3 & \cdots & * \\ 4 & 1 & 0 & \cdots & * \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 2 & -3 & \cdots & * \\ 0 & 3 & 9 & \cdots & * \\ 4 & 3 & 0 & \cdots & * \end{array}\right)$$

(ii) $$\left(\begin{array}{cccc|c} 0 & 0 & 5 & \cdots & * \\ 0 & 1 & 2 & \cdots & * \\ 0 & -3 & 2 & \cdots & * \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 0 & 1 & 2 & \cdots & * \\ 0 & 0 & 5 & \cdots & * \\ 0 & -3 & 2 & \cdots & * \end{array}\right)$$

(iii) $$\left(\begin{array}{cccc|c} 1 & 0 & 2 & \cdots & * \\ 0 & 5 & -1 & \cdots & * \\ 0 & 4 & 1 & \cdots & * \end{array}\right)$$ no interchanging of rows needed.

Gaussian Elimination

Step 3: For each row below the top row, add a suitable multiple of the top row to it so that the entry below the leading entry of the top row becomes zero.

$$\text{(i)}\quad \left(\begin{array}{cccc|c} 1 & 2 & -3 & \cdots & * \\ 0 & 3 & 9 & \cdots & * \\ 4 & 1 & 0 & \cdots & * \end{array}\right) \xrightarrow{R_3-4R_1} \left(\begin{array}{cccc|c} 1 & 2 & -3 & \cdots & * \\ 0 & 3 & 9 & \cdots & * \\ 0 & -7 & 12 & \cdots & * \end{array}\right)$$

$$\text{(ii)}\quad \left(\begin{array}{cccc|c} 0 & 1 & 2 & \cdots & * \\ 0 & 0 & 5 & \cdots & * \\ 0 & -3 & 2 & \cdots & * \end{array}\right) \xrightarrow{R_3+3R_1} \left(\begin{array}{cccc|c} 0 & 1 & 2 & \cdots & * \\ 0 & 0 & 5 & \cdots & * \\ 0 & 0 & 8 & \cdots & * \end{array}\right)$$

$$\text{(iii)}\quad \left(\begin{array}{cccc|c} 1 & 0 & 2 & \cdots & * \\ 0 & 5 & -1 & \cdots & * \\ 0 & 4 & 1 & \cdots & * \end{array}\right) \quad \text{no change required.}$$

Gaussian Elimination

Step 4: Now cover the top row in the augmented matrix and begin again with Step 1 applied to the submatrix that remains. Continue this way until the entire matrix is in row-echelon form.

(i) $$\left(\begin{array}{cccc|c} 1 & 2 & -3 & \cdots & * \\ 0 & 3 & 9 & \cdots & * \\ 0 & -7 & 12 & \cdots & * \end{array}\right) \xrightarrow{R_3+\frac{7}{3}R_2} \left(\begin{array}{cccc|c} 1 & 2 & -3 & \cdots & * \\ 0 & 3 & 9 & \cdots & * \\ 0 & 0 & 33 & \cdots & * \end{array}\right)$$

(ii) $$\left(\begin{array}{cccc|c} 0 & 1 & 2 & \cdots & * \\ 0 & 0 & 5 & \cdots & * \\ 0 & 0 & 8 & \cdots & * \end{array}\right) \xrightarrow{R_3-\frac{8}{5}R_2} \left(\begin{array}{cccc|c} 0 & 1 & 2 & \cdots & * \\ 0 & 0 & 5 & \cdots & * \\ 0 & 0 & 0 & \cdots & * \end{array}\right)$$

(iii) $$\left(\begin{array}{cccc|c} 1 & 0 & 2 & \cdots & * \\ 0 & 5 & -1 & \cdots & * \\ 0 & 4 & 1 & \cdots & * \end{array}\right) \xrightarrow{R_3-\frac{4}{5}R_2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & \cdots & * \\ 0 & 5 & -1 & \cdots & * \\ 0 & 0 & 9/5 & \cdots & * \end{array}\right)$$

The result of step 1 to 4 reduces to (augmented) matrix to a row-echelon form. The process up to step 4 is called *Gaussian Elimination*.

Gauss-Jordan Elimination

Step 5: Multiply a suitable constant to each row so that all the leading entries become 1.

Step 6: Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading entries.

$$\left(\begin{array}{cccccccc|c}
1 & & * & 0 & & * & 0 & & * \\
0 & \cdots & 0 & 1 & & * & 0 & & * \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & & * \\
0 & & & 0 & & & 0 & 0 & 0 \\
\vdots & & & \vdots & & & \vdots & \vdots & \vdots \\
0 & \cdots & & 0 & & \cdots & 0 & 0 & 0
\end{array}\right).$$

If we continue from step 1 to 6, then the entire process is known as *Gauss-Jordan elimination*.

Question

Can we still perform Gaussian and/or Gauss-Jordan elimination if the coefficients contains unknown? For example,

$$\begin{cases} ax & + & y & - & z & = & 1 \\ x & + & y & + & 2z & = & 3 \\ x & + & (a-1)y & - & z & = & 0 \end{cases}$$

for some constant $a \in \mathbb{R}$?

Challenge

1. Is it possible to reduce an augmented matrix to 2 different row-echelon forms?

2. Is it possible to reduce an augmented matrix to 2 different reduced row-echelon form?

1.5 More on Linear Systems

Summary

1. Write the linear system in its standard form.
2. Form the augmented matrix of the linear system.
3. Reduce the augmented matrix to either a row-echelon form or reduced row echelon form. May use Gaussian/Gauss-Jordan elimination.
4. Decide if the system is consistent
   - If the last column is a pivot column, the system is inconsistent.
   - Otherwise, the system is consistent, assign the variables corresponding to the nonpivot columns to be parameters, s, t, $s, t \in \mathbb{R}$, etc.
5. If the system is in reduced row-echelon form, read off the solutions directly.
6. If the system is in row-echelon form only, do back substitution, starting from the lowest nonzero row.
7. Write down the (general) solution to the system.

More Examples

Solve the following linear system

$$\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & + & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}$$

$$\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right)
\xrightarrow{R_2-2R_1}\xrightarrow{R_4-2R_1}
\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right)$$

$$\xrightarrow{R_3+5R_2}\xrightarrow{R_4+4R_2}
\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 2
\end{array}\right)
\xrightarrow{R_3\leftrightarrow R_4}
\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 0 & 0 & 0 & 6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

The augmented matrix is now in row-echelon form. One may choose to use back-substitution to obtain the answer. Which variables should we assign as parameters?

More Examples

Let us continue to reduce the system to its reduced row-echelon form.

$$\xrightarrow{-R_2}\xrightarrow{(1/6)R_3}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2-3R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1+2R_2} \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_2 = r, x_4 = s, x_5 = t$. Then the general solution is

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = 1/3, \quad r, s, t \in \mathbb{R}.$$

Linear System with Unknowns in the Coefficient and Constants

Consider the following linear system

$$\begin{array}{ccccccc} x_1 & + & ax_2 & + & 2x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 1 \\ x_1 & & & + & ax_3 & = & 2 \end{array}$$

for some fixed real number $a \in \mathbb{R}$. Here we want to determine the value of a such that the system is

(a) inconsistent,

(b) consistent with a unique solution and find the unique solution, or

(c) consistent with infinitely many solutions and write down the general solution.

$$\left(\begin{array}{ccc|c} 1 & a & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & a & 2 \end{array}\right) \xrightarrow{R_2-R_1} \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & a & 2 & 0 \\ 0 & -a & -1 & 1 \\ 0 & -a & a-2 & 2 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & a & 2 & 0 \\ 0 & -a & -1 & 1 \\ 0 & 0 & a-1 & 1 \end{array}\right).$$

Observe that if we want to continue the Gauss-Jordan elimination, we would need to multiple row 2 by $-\frac{1}{a}$ and row 3 by $\frac{1}{a-1}$. But this is not well defined if $a = 0$ and $a = 1$, respectively. Hence, we need to consider if $a = 0, 1$.

Linear System with Unknowns in the Coefficient and Constants

If $a = 1$, the augmented matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

The system is inconsistent.

If $a = 0$, the augmented matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right) \xrightarrow{R_3 - R_2} \xrightarrow{R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system is consistent with infinitely many solution. The general solution is

$$x_1 = 2, \; x_2 = s, \; x_3 = -1, \quad s \in \mathbb{R}.$$

Linear System with Unknowns in the Coefficient and Constants

Finally, suppose $a \neq 0, 1$. By back substitution or reducing the augmented matrix to its reduced row-echelon form, we obtain the unique solution

$$x_1 = \frac{a-2}{a-1}, \quad x_2 = \frac{1}{1-a}, \quad x_3 = \frac{1}{a-1}.$$

The details are left as an exercise. Note that there are infinitely many values of a such that the system has a unique solution, and the solution depends on these values of a. Do not pick a particular value of a for the solution.

(a) The system has no solution if $a = 1$.

(b) The system has a unique solution when $a \neq 0, 1$. The unique solution is

$$x_1 = \frac{a-2}{a-1}, \quad x_2 = \frac{1}{1-a}, \quad x_3 = \frac{1}{a-1}.$$

(c) The system has infinitely many solutions when $a = 0$. The general solution is

$$x_1 = 2,\ x_2 = s,\ x_3 = -1, \quad s \in \mathbb{R}.$$

Another Example

Consider the following linear system

$$\left\{\begin{array}{ccccccc} ax & + & 2ay & - & z & = & 2a+2 \\ x & + & y & & & = & 1 \\ x & + & y & + & (a-1)z & = & 3 \end{array}\right.$$

for some fixed real number $a \in \mathbb{R}$.

The augmented matrix of the system is $\left(\begin{array}{ccc|c} a & 2a & -1 & 2a+2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & a-1 & 3 \end{array}\right)$. Here, if we follow Gaussian elimination strictly, one might consider the operation $R_2 - \frac{1}{a}R_1$. However, this is not well-defined if $a = 0$. Hence, instead of considering cases, we might consider doing a row swap.

$$\left(\begin{array}{ccc|c} a & 2a & -1 & 2a+2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & a-1 & 3 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ a & 2a & -1 & 2a+2 \\ 1 & 1 & a-1 & 3 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - aR_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & a & -1 & a+2 \\ 0 & 0 & a-1 & 2 \end{array}\right)$$

Another Example

We should consider whether $a = 0, 1$ or not.

(a) The system is inconsistent if $a = 1$.

(b) The system has a unique solution if $a \neq 0, 1$. The unique solution is

$$x = \frac{2}{1-a}, \quad y = \frac{a+1}{a-1}, \quad z = \frac{2}{a-1}.$$

(c) The system has infinitely many solutions with 1 parameter if $a = 0$. The general solution is

$$x = 1 - s, \quad y = s, \quad z = -2, \quad s \in \mathbb{R}.$$

The details are left as an exercise.

Hard Example

Consider the following linear system

$$\begin{cases} x_1 & & & + & 3x_3 & + & x_4 & = & 2 \\ 3x_1 & + & ax_2 & + & 9x_3 & & & = & 6 \\ 2x_1 & & & + & (a+6)x_3 & + & ax_4 & = & b+2 \\ 2x_1 & & & + & 6x_3 & + & bx_4 & = & b+2 \end{cases}$$

where a and b are some constants.

(a) Find the conditions on a and b such that the system has no solution.

(b) Find the conditions on a and b such that the system has a unique solution, and write down the unique solution.

(c) Find the conditions on a and b such that the system has infinitely many solutions with 1 parameter, and write down a general solution.

(d) Find the conditions on a and b such that the system has infinitely many solutions with 2 parameter, and write down a general solution.

Hard Example

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 3 & a & 9 & 0 & 6 \\ 2 & 0 & a+6 & a & b+2 \\ 2 & 0 & 6 & b & b+2 \end{array}\right) \xrightarrow[\substack{R_3-2R_1 \\ R_4-2R_1}]{R_2-3R_1} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & a & 0 & -3 & 0 \\ 0 & 0 & a & a-2 & b-2 \\ 0 & 0 & 0 & b-2 & b-2 \end{array}\right).$$

The cases to consider are $a = 0$ or not and $b = 2$ or not; a total of 4 cases.

Suppose $b = 2$. The augmented matrix becomes

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & a & 0 & -3 & 0 \\ 0 & 0 & a & a-2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

If $a = 0$ then columns 2 and 3 are non-pivots. Hence, the system has infinitely many solutions with 2 parameters.The general solution is

$$x_1 = 2 - 3t, \quad x_2 = s, \quad x_3 = t, \quad x_4 = 0, \quad s, t, \in \mathbb{R}.$$

If $a \neq 0$, then column 4 is non-pivot, and hence the system has infinitely many solutions with 1 parameters. The general solution is

$$x_1 = 2 + \frac{2a-6}{a}s, \quad x_2 = \frac{3}{a}s, \quad x_3 = \frac{(2-a)}{a}s, \quad x_4 = s, \quad s \in \mathbb{R}.$$

Hard Example

Suppose $b \neq 2$. We may proceed to reduce the augmented matrix.

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & a & 0 & -3 & 0 \\ 0 & 0 & a & a-2 & b-2 \\ 0 & 0 & 0 & b-2 & b-2 \end{array}\right) \xrightarrow{\frac{1}{b-2}R_4} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & a & 0 & -3 & 0 \\ 0 & 0 & a & a-2 & b-2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[R_2+3R_4]{R_3-(a-2)R_4} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & a & 0 & 0 & 3 \\ 0 & 0 & a & 0 & b-a \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

From row 2, we can see that the system is inconsistent if $a = 0$.

Otherwise, if $a \neq 0$, all 4 columns on the LHS will be pivot columns, and hence, the system has a unique solution

$$x_1 = \frac{4a - 3b}{a}, \quad x_2 = \frac{3}{a}, \quad x_3 = \frac{b-a}{a}, \quad x_4 = 1.$$

Hard Example

(a) The system has no solution if $a = 0$ and $b \neq 2$.

(b) The system has a unique solution if $a \neq 0$ and $b \neq 2$. The unique solution is

$$x_1 = \frac{4a - 3b}{a}, \quad x_2 = \frac{3}{a}, \quad x_3 = \frac{b - a}{a}, \quad x_4 = 1.$$

(c) The system has infinitely many solutions with 1 parameter if $a \neq 0$ and $b = 2$. The general solution is

$$x_1 = 2 + \frac{2a - 6}{a}s, \quad x_2 = \frac{3}{a}s, \quad x_3 = \frac{(2 - a)}{a}s, \quad x_4 = s, \quad s \in \mathbb{R}.$$

(d) The system has infinitely many solutions with 2 parameters if $a = 0$ and $b = 2$. The general solution is

$$x_1 = 2 - 3t, \quad x_2 = s, \quad x_3 = t, \quad x_4 = 0, \quad s, t, \in \mathbb{R}.$$

Constructing a Linear System from the General Solution

Find a linear system with 2 equations such that

$$x_1 = 1 - 2s + t, \quad x_2 = s, \quad x_3 = t, \quad x_4 = 0, \quad s, t \in \mathbb{R}$$

is the general solution.

We will substitute $x_2 = s$ and $x_3 = t$ into the first equation and get

$$x_1 = 1 - 2x_2 + x_3.$$

Include the equation $x_4 = 0$, and changing it to a standard form, we obtain the linear system

$$\left\{ \begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & & & = & 1 \\ & & & & & & x_4 & = & 0 \end{array} \right. .$$

The augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

which is in reduced row-echelon form.

Another Example

Construct an augmented matrix with 3 variables and 4 equations such that it has the following general solution

$$x = 3 - 5s, \quad y = 2 + 2s, \quad z = s, \quad s \in \mathbb{R}.$$

Substituting $z = s$, into the first 2 equations, we get

$$x = 3 - 5z \text{ and } y = 2 + 2z \quad \Rightarrow \begin{cases} x \phantom{{}+y} + 5z = 3 \\ \phantom{x+{}} y - 2z = 2 \end{cases}$$

However, we need 4 equations. Since we have used all the given information, the other 2 more equations should not contribute anymore information, that is, they should be derived from the first 2. We may consider taking multiples or adding the first 2 equations to obtain more equations. For example.

$$\begin{cases} x \phantom{{}+y} + 5z = 3 \\ \phantom{x+{}} y - 2z = 2 \\ x + y + 3z = 5 \\ \phantom{x+{}} 2y - 4z = 4 \end{cases} \text{ which corresponds to the augmented matrix } \left(\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -2 & 2 \\ 1 & 1 & 3 & 5 \\ 0 & 2 & -4 & 4 \end{array}\right)$$

Question

Construct an augmented matrix with 3 variables and 3 equations such that it has the following solution

$$x = 3, \quad y = 2, \quad z = 1.$$