MA1522 Linear Algebra for Computing

Lecture 7: Basis and Dimension

Yang Yue

Department of Mathematics
National University of Singapore

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Outline

Questions posed in Dr.Teo's Lectures

Challenges posed in Dr.Teo's Lectures

Question One in Section 3.6

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a set of vectors in $\mathbb{R}^n$ and V a subspace. Let $\mathbf{v}$ be a vector in V.

(i) Suppose there is a non-pivot column in the left side of the reduced row-echelon form of

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right).$$

What can you conclude?

(ii) Suppose

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right)$$

is inconsistent. What can you conclude?

Slide 118: Algorithm for Computing Relative Coordinate

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for V.

- Let $\mathbf{v}$ be a vector in V. To find $[\mathbf{v}]_S$, we must find the coefficients $c_1, c_2, ..., c_k$ such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

- Converting it to a matrix equation, we have

$$\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots \mathbf{u}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{v},$$

- which is equivalent to solving the linear system

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right).$$

Answer to Question One in Section 3.6, part 1

Q: Given $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$, a subspace V and $\mathbf{v} \in V$,

(i) suppose there is a non-pivot column in the left side of the reduced row-echelon form of

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right).$$

What can you conclude?

Answer: By assumption, the left side of the reduced row-echelon form of

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{0} \end{array} \right).$$

also has a non-pivot column. In other words, the linear system

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$$

has nontrivial solutions. Hence, we conclude that S is linearly dependent.

Answer to Question One in Section 3.6, part 2

Q: Given $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$, a subspace V and $\mathbf{v} \in V$,

(ii) suppose

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right)$$

is inconsistent. What can you conclude?

Answer: To be more precise, it is the linear system

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{v}$$

that is inconsistent.

Hence, we conclude that $\mathbf{v}$ is not in the span of
$S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.

Question Two in Section 3.6

Prove the following theorem:

Theorem

Let V be a subspace of $\mathbb{R}^n$ and B a basis for V.

(i) *For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u} = \mathbf{v}$ if and only if $[\mathbf{u}]_B = [\mathbf{v}]_B$.*

(ii) *For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$, and $c_1, c_2, \ldots, c_m \in \mathbb{R}$,*

$$[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m]_B = c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + \cdots + c_m[\mathbf{v}_m]_B.$$

Slide 114: Coordinates Relative to a Basis

Definition

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for V, a subspace of $\mathbb{R}^n$ and

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

be the unique expression of a vector $\mathbf{v}$ in V in terms of the basis S.

The vector in $\mathbb{R}^k$ defined by the coefficients of the linear combination is called the *coordinates of* $\mathbf{v}$ *relative to basis* S, and is denoted as

$$[\mathbf{v}]_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

Answer to Question Two in Section 3.6, part 1

Q: Let V be a subspace of $\mathbb{R}^n$ and B a basis for V. Prove that

(i) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u} = \mathbf{v}$ if and only if $[\mathbf{u}]_B = [\mathbf{v}]_B$.

One direction (the "only if") is by logic.

For the "if" direction, suppose that $B = \{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k\}$ and $[\mathbf{u}]_B = [\mathbf{v}]_B = (c_1, c_2, \ldots, c_k)^T$. Then

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k = \mathbf{v}.$$

Answer to Question Two in Section 3.6, part 2

Q: Let V be a subspace of $\mathbb{R}^n$ and B a basis for V. Prove that

(ii) For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$, and $c_1, c_2, \ldots, c_m \in \mathbb{R}$,

$$[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m]_B = c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + \cdots + c_m[\mathbf{v}_m]_B.$$

Answer: For typing reasons, let's only prove the case when $m = 2$.
Again, suppose that $B = \{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k\}$.
Let $[\mathbf{v}_1]_B == (a_1, a_2, \ldots, a_k)^T$ and $[\mathbf{v}_2]_B = (b_1, b_2, \ldots, b_k)^T$. Then
$\mathbf{v}_1 = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k$ and
$\mathbf{v}_2 = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \cdots + b_k\mathbf{w}_k$. Thus

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = (c_1 a_1 + c_2 b_1)\mathbf{w}_1 + \cdots + (c_1 a_k + c_2 b_k)\mathbf{w}_k,$$

in other words, $[c_1\mathbf{v}_1 + c_2\mathbf{v}_2]_B = (c_1 a_1 + c_2 b_1, \ldots, c_1 a_k + c_2 b_k)^T$,
which is the same as $c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B$.

Challenge in Section 3.6

Recall that the set of 2×2 matrices, $\mathbb{R}^{2\times 2}$, is a vector space. Show that the set

$$\left\{ \mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^{2\times 2}$.

Slide 100: Basis

Definition

Let V be a subspace of $\mathbb{R}^n$. A set $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_k\} \subseteq V$ is a ***basis*** for V if

(i) S spans V, $\text{span}(S) = V$, and

(ii) S is linearly independent.

Theorem

Suppose S is a basis for V. then every vector $\mathbf{v} \in V$ can be written as a linear combination of vectors in S uniquely.

Idea:

(i) $\text{span}(S) = V$ tells us that every vector $\mathbf{v} \in V$ can be written as a combination of vectors in S.

(ii) S is linearly independent tells us the uniqueness.

Answer to the Challenge in Section 3.6

Q: Recall that the set of 2×2 matrices, $\mathbb{R}^{2\times 2}$, is a vector space. Show that the set

$$\left\{ \mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^{2\times 2}$.

Answer: We check the conditions ("span" and "independence") one by one.

Span: Any element $\mathbf{A}$ in $\mathbb{R}^{2\times 2}$ is a matrix of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Clearly, $\mathbf{A} = a\mathbf{M}_1 + b\mathbf{M}_2 + c\mathbf{M}_3 + d\mathbf{M}_4$.

Answer to the Challenge in Section 3.6 (conti.)

Q: Recall that the set of 2×2 matrices, $\mathbb{R}^{2\times 2}$, is a vector space. Show that the set

$$\left\{ \mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^{2\times 2}$.

It remains to check the independence:
Suppose that

$$c_1\mathbf{M}_1 + c_2\mathbf{M}_2 + c_3\mathbf{M}_3 + c_4\mathbf{M}_4 = \mathbf{0}_2,$$

where $\mathbf{0}_2$ is the 2×2 zero matrix. But LHS is just $\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$. It follows that $c_1 = c_2 = c_3 = c_4 = 0$. The result follows.

Challenge in Section 3.7

Let V be a k-dimensional subspace of $\mathbb{R}^n$. Using the dimension of V (instead of proving using equivalent statements of invertibility), prove that a subset S in V containing k vectors, $|S| = k$, is linearly independent if and only if it spans V.

Remarks:

- To use invertibility here, one has to use the coordinate space $\mathbb{R}^k$, see slide 125. But we use this opportunity to make some revision.
- Dr. Teo actually proved this later in Slides 138 and 139, which I will remind you also.

Slide 110: Equivalent Statements for Invertibility

Theorem

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) $\mathbf{A}$ *is invertible.*

(ii) $\mathbf{A}^T$ *is invertible.*

(iii) *(left inverse) There is a matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{I}$.*

(iv) *(right inverse) There is a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$.*

(v) *The reduced row-echelon form of $\mathbf{A}$ is the identity matrix.*

(vi) $\mathbf{A}$ *can be expressed as a product of elementary matrices.*

(vii) *The homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.*

(viii) *For any $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.*

(ix) *The determinant of $\mathbf{A}$ is nonzero, $\det(\mathbf{A}) \neq 0$.*

(x) *The columns/rows of $\mathbf{A}$ are linearly independent.*

(xi) *The columns/rows of $\mathbf{A}$ spans $\mathbb{R}^n$.*

Slide 128: Dimension

Corollary

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *and* $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ *are bases for a subspace* $V \subseteq \mathbb{R}^n$*. Then* $k = m$*.*

Definition

Let V be a subspace of $\mathbb{R}^n$. The ***dimension*** of V, denoted by $\dim(V)$, is defined to be the number of vectors in any basis of V.

Slides 133 and 134

Spanning Set Theorem Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a subset of vectors in $\mathbb{R}^n$, and let $V = \text{span}(S)$. Suppose V is not the zero space, $V \neq \{\mathbf{0}\}$. Then there must be a subset of S that is a basis for V.

Linear Independence Theorem Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ a linearly independent subset of V, $S \subseteq V$. Then there must be a set T containing S, $S \subseteq T$ such that T is a basis for V.

Answer to the Challenge in Section 3.7

Q: Let V be a k-dimensional subspace of $\mathbb{R}^n$. Using the dimension of V (instead of proving using equivalent statements of invertibility), prove that a subset S in V containing k vectors, $|S| = k$, is linearly independent if and only if it spans V.

Answer: When $k = 0$, $S = \emptyset$, it is linearly independent and spans $\{\mathbf{0}\}$. The statement holds. Assume that $k \neq 0$.

($\Rightarrow$) Suppose that S is linearly independent. By Linear Independence Theorem, there is a basis $T \supseteq S$. However, $|T| = \dim V = k = |S|$. Thus $S = T$ and $\text{span}(S) = \text{span}(T) = V$.

($\Leftarrow$) Suppose that $\text{span}(S) = V$. By Spanning Set Theorem, there is a basis $U \subseteq S$. Again $|U| = k = \dim V = |S|$. Thus $S = T$ and S is linearly independent.

Slide 138: Equivalent ways to check for Basis

Theorem B1 Let V be a k-dimensional subspace of $\mathbb{R}^n$, $\dim(V) = k$. Suppose S is a linearly independent subset of V containing k vectors, $|S| = k$. Then S is a basis for V.

Theorem B2 Let V be a k-dimensional subspace of $\mathbb{R}^n$, $\dim(V) = k$. Suppose S is a set containing k vectors, $|S| = k$, such that $V \subseteq \text{span}(S)$. Then S is a basis for V.

Challenge in Section 3.8: Inverse of Transition Matrix

Theorem

Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from T to S. Then $\mathbf{P}^{-1}$ is the transition matrix from S to T.

Proof.

Exercise. Note that you cannot assume that $\mathbf{P}$ is invertible. $\square$

Slide 148: Transition Matrix

Definition

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for the subspace V. Define the *transition matrix* from T to S to be

$$\mathbf{P} = \begin{pmatrix} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \cdots & [\mathbf{v}_k]_S \end{pmatrix},$$

the matrix whose columns are the coordinates of the vectors in T relative to the basis S.

Theorem

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for the subspace V. Let $\mathbf{P}$ be the transition matrix from T to S. Then for any vector $\mathbf{w}$ in V,

$$[\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T.$$

Slide 150: Algorithm to find Transition Matrix

Let $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be basis for a subspace V in $\mathbb{R}^n$.

- To find $\mathbf{P}$, the transition matrix from T to S, we need to find $[\mathbf{v}_i]_S$ for $i = 1, 2, ..., k$.
- This is equivalent to solving $\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_i \end{array}\right)$ for $i = 1, 2, ..., k$.
- Since these linear systems have the same coefficient matrix, we can solve them simultaneously,

$$\left(\begin{array}{cccc|cccc} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{array}\right).$$

- Now since S is a basis, the system must have a unique solution, and the reduced row-echelon form of the augmented matrix above will be of the form

$$\left(\begin{array}{c|cccc} \mathbf{I}_k & [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \dots & [\mathbf{v}_k]_S \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0} & \mathbf{0} & & \mathbf{0} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{(n-k)\times k} \end{array}\right)$$

  where $\mathbf{P}$ is the transition matrix from T to S.

In summary,

$$\left(\begin{array}{c|c} \text{“}S\text{”} & \text{“}T\text{”} \end{array}\right) = \left(\begin{array}{cccc|cccc} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{array}\right) \xrightarrow{rref} \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{(n-k)\times k} \end{array}\right).$$

First Answer of Challenge in Section 3.8

Q: Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from T to S. Then $\mathbf{P}$ is invertible and $\mathbf{P}^{-1}$ is the transition matrix $\mathbf{Q}$ from S to T.

Answer: For a $\mathbf{v}$ in V, write

$$\begin{aligned}
\mathbf{v} &= c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix} [\mathbf{v}]_S \\
\mathbf{v} &= d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix} [\mathbf{v}]_T \\
&= \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix} \mathbf{Q}[\mathbf{v}]_S
\end{aligned}$$

Since this is true for all $\mathbf{v}$ in V, then

$$\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix} \mathbf{Q}.$$

First Answer of Challenge in Section 3.8 (conti.)

Analogous argument shows that

$$\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix} \mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix}.$$

Post multiplying both sides of the equation by $\mathbf{Q}$, we get

$$\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix} \mathbf{P}\mathbf{Q} = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix} \mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix}.$$

Since S is linearly independent, $\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix}$ has a left inverse (see chapter 4.2). Premultiplying both sides of the equation by a left inverse of $\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix}$, we get

$$\mathbf{P}\mathbf{Q} = \mathbf{I}.$$

Since $\mathbf{P}$ is a square matrix, we can conclude that it is invertible with inverse

$$\mathbf{P}^{-1} = \mathbf{Q}.$$

Alternative Answer to the Challenge in Section 3.8

Q: Suppose $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from T to S. Then $\mathbf{P}^{-1}$ is the transition matrix from S to T.

Answer: Let $\mathbf{Q}$ be the transition matrix from S to T. Then we have invertible matrices $\mathbf{E}$ and $\mathbf{F}$ such that

$$\mathbf{ES} = \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \quad \mathbf{ET} = \begin{pmatrix} \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix},$$

$$\mathbf{FT} = \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \quad \mathbf{FS} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix}.$$

Thus

$$\mathbf{T} = \mathbf{F}^{-1} \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \quad \mathbf{S} = \mathbf{E}^{-1} \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix}. \tag{1}$$

Answer to the Challenge in Section 3.8 (conti.)

Thus

$$
\begin{aligned}
& \begin{pmatrix} \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \begin{pmatrix} \mathbf{Q} \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \\
= \; & \mathbf{ETFS} \\
= \; & \mathbf{EF}^{-1} \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \mathbf{FE}^{-1} \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \quad \text{by (1)} \\
= \; & \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix}
\end{aligned}
$$

The last equality holds because

$$
\mathbf{F}^{-1} \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \mathbf{F} = \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} = \mathbf{E} \begin{pmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \mathbf{E}^{-1}.
$$

Thus $\mathbf{PQ} = \mathbf{I}_k$. The result follows.