MA1522: Linear Algebra for Computing

Chapter 4: Subspaces Associated to a Matrix

4.1 Column Space, Row Space, and Nullspace

Column and Row Space

Definition

Let $\mathbf{A}$ be an $m \times n$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The *row space* of $\mathbf{A}$, is the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$,

$$\mathrm{Row}(\mathbf{A}) = \mathrm{span}\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}, \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}, \dots, \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\}$$

The *column space* of $\mathbf{A}$, is the subspace of $\mathbb{R}^m$ spanned by the columns of $\mathbf{A}$,

$$\mathrm{Col}(\mathbf{A}) = \mathrm{span}\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}$$

Question

Let $\mathbf{A}$ be an $m \times n$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The row space is a subspace of

(a) $\mathbb{R}^n$

(b) $\mathbb{R}^m$

The column space is a subspace of

(a) $\mathbb{R}^n$

(b) $\mathbb{R}^m$

Example

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$.

Columns space of $\mathbf{A}$:

$$\text{Col}(\mathbf{A}) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Row space of $\mathbf{A}$:

$$\begin{aligned} \text{Row}(\mathbf{A}) &= \text{span}\{\begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 2 \end{pmatrix}\} \\ &= \text{span}\{\begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix}\}. \end{aligned}$$

Remark: Can use column vectors to represent vectors in row space.

Question

Let $\mathbf{A} = \begin{pmatrix} 5 & 2 & -1 & -1 \\ 1 & -1 & 4 & -3 \\ 8 & 3 & -1 & -2 \\ 9 & 3 & 0 & -3 \end{pmatrix}$. Which of the following statements are true?

(i) $\left\{ \begin{pmatrix} 5 \\ 1 \\ 8 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -2 \\ -3 \end{pmatrix} \right\}$ is a basis for the column space of $\mathbf{A}$.

(ii) $\left\{ \begin{pmatrix} 5 \\ 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 8 \\ 3 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 9 \\ 3 \\ 0 \\ -3 \end{pmatrix} \right\}$ is a basis for the row space of $\mathbf{A}$.

Example

Suppose

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 3 & 0 \\ 1 & -2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} = \mathbf{B}.$$

Then it is clear that $\text{Row}(\mathbf{A}) = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \text{Row}(\mathbf{B})$ since $\mathbf{b}_1 = \mathbf{a}_1$, $\mathbf{b}_2 = \mathbf{a}_3$, and $\mathbf{b}_3 = \mathbf{a}_2$.

Suppose

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix} \xrightarrow{-R_3} \begin{pmatrix} 1 & 2 & 2 & -1 \\ 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} = \mathbf{B}.$$

Then it is clear that $\text{Row}(\mathbf{A}) = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \text{Row}(\mathbf{B})$ since $\mathbf{b}_1 = \mathbf{a}_1$, $\mathbf{b}_2 = \mathbf{a}_2$, and $\mathbf{b}_3 = -\mathbf{a}_3$.

Example

Suppose

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & -4 & -2 & 4 \\ 0 & 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} = \mathbf{B}.$$

Then $\mathbf{b}_2 = \mathbf{a}_2 - \mathbf{a}_1$ tells us that $\mathbf{b}_2 \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Since $\mathbf{b}_1 = \mathbf{a}_1$ and $\mathbf{b}_3 = \mathbf{a}_3$, we can conclude that $\text{Row}(\mathbf{B}) = \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \subseteq \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = \text{Row}(\mathbf{A})$.

But $\mathbf{a}_2 = \mathbf{b}_2 + \mathbf{b}_1$ also tells us that $\mathbf{a}_2 \in \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. So, $\text{Row}(\mathbf{A}) = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subseteq \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \text{Row}(\mathbf{B})$.

Hence, we have equality $\text{Row}(\mathbf{A}) = \text{Row}(\mathbf{B})$.

This discussion shows that row operations do not change the row space.

Row Operations Preserves Row Space

Theorem (Row operations preserve row space)

Suppose $\mathbf{A}$ *and* $\mathbf{B}$ *are row equivalent matrices. Then the row space of* $\mathbf{A}$ *is equal to the row space of* $\mathbf{B}$, $Row(\mathbf{A}) = Row(\mathbf{B})$.

Sketch of proof.

(i) Suppose $\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B}$. Then the rows of $\mathbf{B}$ are the rows of $\mathbf{A}$ rearranged, which will not change the span.

(ii) Suppose $\mathbf{A} \xrightarrow{cR_i} \mathbf{B}$, $c \neq 0$. Then $\text{Row}(\mathbf{B}) \subseteq \text{Row}(\mathbf{A})$. But $\mathbf{B} \xrightarrow{\frac{1}{c}R_i} \mathbf{A}$ shows that $\text{Row}(\mathbf{A}) \subseteq \text{Row}(\mathbf{B})$ too.

(iii) Suppose $\mathbf{A} \xrightarrow{R_i + aR_j} \mathbf{B}$. Then $\text{Row}(\mathbf{B}) \subseteq \text{Row}(\mathbf{A})$. But $\mathbf{B} \xrightarrow{R_i - aR_j} \mathbf{A}$ shows that $\text{Row}(\mathbf{A}) \subseteq \text{Row}(\mathbf{B})$ too.

$\square$

Finding Basis for Row Space

Theorem

If a matrix $\mathbf{R}$ *is in reduced row-echelon form, then the nonzero rows of* $\mathbf{R}$ *form a basis for its row space.*

Sketch of proof.

Write $\mathbf{R} = \begin{pmatrix} 0 & \cdots & 1 & \cdots & * & 0 & \cdots & * & 0 & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & * & 0 & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & & & & & \vdots & & & \vdots & \vdots \end{pmatrix}$. By definition, the 1 in the leading entry in each nonzero row is only nonzero entry in that coordinate among the rows. This shows that each nonzero row cannot be a linear combination of the other rows. Hence, the rows of $\mathbf{R}$ are linearly independent. It is clear that the nonzero rows spans the row space of $\mathbf{R}$. $\square$

Theorem

For any matrix $\mathbf{A}$*, the nonzero rows of the reduced row-echelon form of* $\mathbf{A}$ *form a basis for for the row space of* $\mathbf{A}$*.*

Proof.

Follows from the fact that $\mathbf{A}$ is row equivalent to its reduced row-echelon form $\mathbf{R}$, and that the nonzero rows of $\mathbf{R}$ form a basis for Row($\mathbf{R}$). $\square$

Examples

1. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

   So $\left\{ \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix} \right\}$ is a basis for Row($\mathbf{A}$).

2. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 7 \\ -1 & 7 & -19 \\ 1 & 9 & -13 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

   So $\left\{ \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -3 \end{pmatrix} \right\}$ is a basis for Row($\mathbf{A}$).

Challenge

3. $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

   So $\left\{ \begin{pmatrix} 1 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \right\}$ is a basis for Row($\mathbf{A}$).

   However, in this case, we could have taken the original rows of $\mathbf{A}$

$$\left\{ \begin{pmatrix} 1 & 1 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \right\}$$

   as a basis for Row($\mathbf{A}$) too. Why?

Discussion

Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$. Claim: the 4th column is a linear combination of the first 3 columns. (It should be clear that the columns are linearly dependent since there are 4 columns and these are vectors in $\mathbb{R}^3$.)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

We might add a bar between the 3rd and 4th columns to emphasize that we are solving a linear system, but that would not affect the computation/reduction. From the reduced row-echelon form, we conclude that

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

But this is exactly the linear relations between the columns of $\mathbf{R}$,

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Discussion

In fact, the linear relations between $\mathbf{A}$ and its reduced row-echelon form are preserved for any columns. Consider

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \\ \begin{pmatrix} 2 & 8 & 2 & 4 & 2 \\ 1 & 6 & 0 & 4 & 3 \\ -1 & -4 & -1 & -2 & -1 \end{pmatrix} \end{array} \xrightarrow{RREF} \begin{array}{c} \begin{array}{ccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 3 & -2 & -3 \\ 0 & 1 & -1/2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = \mathbf{R}.$$

Then

$$\begin{aligned} \mathbf{r}_3 &= 3\mathbf{r}_1 - \tfrac{1}{2}\mathbf{r}_2 & \longleftrightarrow \quad \mathbf{a}_3 &= 3\mathbf{a}_1 - \tfrac{1}{2}\mathbf{a}_2 \\ \mathbf{r}_4 &= -2\mathbf{r}_1 + \mathbf{r}_2 & \longleftrightarrow \quad \mathbf{a}_4 &= -2\mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{r}_5 &= -3\mathbf{r}_1 + \mathbf{r}_2 & \longleftrightarrow \quad \mathbf{a}_5 &= -3\mathbf{a}_1 + \mathbf{a}_2 \end{aligned}$$

Also, $\{\mathbf{r}_1, \mathbf{r}_2\}$ is linearly independent, and so is $\{\mathbf{a}_1, \mathbf{a}_2\}$.

Row Operations Preserves Linear Relations Between Columns

Theorem (Row operations preserve linear relations between columns)

Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$ be row equivalent $m \times n$ matrices, where $\mathbf{a}_i$ and $\mathbf{b}_i$ is the i-th column of $\mathbf{A}$ and $\mathbf{B}$, respectively, for $i = 1, ..., n$. Then for any coefficients $c_1, c_2, ..., c_n$,

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

if and only if

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}.$$

Sketch of the proof.

Since $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, $\mathbf{A} = \mathbf{PB}$ for some invertible order m matrix $\mathbf{P}$. By block multiplication,

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} = \mathbf{A} = \mathbf{PB} = \begin{pmatrix} \mathbf{Pb}_1 & \mathbf{Pb}_2 & \cdots & \mathbf{Pb}_n \end{pmatrix} \quad \Rightarrow \quad \mathbf{a}_i = \mathbf{Pb}_i, \ i = 1, ..., n.$$

So, if $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$,

$$\mathbf{0} = \mathbf{P0} = \mathbf{P}(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n) = c_1\mathbf{Pb}_1 + c_2\mathbf{Pb}_2 + \cdots + c_n\mathbf{Pb}_n = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n.$$

Use $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}$ to prove the converse. $\square$

Finding basis for Column space

Theorem

Suppose $\mathbf{R}$ *is the reduced row-echelon form of a matrix* $\mathbf{A}$*. Then the columns of* $\mathbf{A}$ *corresponding to the pivot columns in* $\mathbf{R}$ *form a basis for the column space of* $\mathbf{A}$*.*

Proof.

First observe that the pivot columns in $\mathbf{R}$ are linearly independent since they are just the vectors in the standard basis. Also, the non-pivot columns of $\mathbf{R}$ linearly dependent on the pivot columns. Hence, the columns of $\mathbf{A}$ that corresponds to the pivot columns of $\mathbf{R}$ are linearly independent and are sufficient to span $\mathrm{Col}(\mathbf{A})$, and thus form a basis. $\square$

Question

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \\ 0 & 0 & 1 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Which columns of $\mathbf{A}$ form a basis for Col($\mathbf{A}$)?

Challenge

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \\ 0 & 0 & 1 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 5/6 \\ -1/6 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{6}\left(5\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \begin{pmatrix} 1/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right),$$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{6}\left(5\begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \end{pmatrix}\right), \quad \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix} = \frac{1}{3}\left(\begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \end{pmatrix}\right).$$

Since the first and third columns of **R** are the pivot columns, the first an third columns of **A** form a basis for Col(**A**). However, in this case, we could take any 2 columns of **A** except columns 1 and 2, to be a basis for the column space of **A**. Why?

Question

Which of the follow statements is/are true?

1. Suppose $\mathbf{A}$ is a 3×3 matrix whose reduced row-echelon form is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for the column space of $\mathbf{A}$.

2. Suppose $\mathbf{A}$ is a 4×3 matrix whose reduced row-echelon form is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we can conclude that the first 2 rows of $\mathbf{A}$ are linearly independent.

Remarks

1. Row operations do not preserve column space. Consider

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\mathrm{Col}(\mathbf{A}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \neq \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

2. Row operations do no preserve linear relations between the rows. Consider

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$\text{row 2 of } \mathbf{A} = 2 \times \text{ row 1 of } \mathbf{A}, \quad \text{row 2 of } \mathbf{B} = 0 \times \text{ row 1 of } \mathbf{B}$$

Question

Is the vector $\mathbf{v} = \begin{pmatrix} 6 \\ 7 \\ -3 \end{pmatrix}$ in the column space of $\mathbf{A} = \begin{pmatrix} 2 & 8 & 2 & 4 \\ 1 & 6 & 0 & 4 \\ -1 & -4 & -1 & -2 \end{pmatrix}$?

$\mathbf{v}$ in the column space of $\mathbf{A}$ if and only if there exists coefficients c_1, c_2, c_3, c_4 such that

$$c_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 8 \\ 6 \\ -4 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c_4 \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ -3 \end{pmatrix},$$

which is equivalent to solving the system

$$\begin{pmatrix} 2 & 8 & 2 & 4 \\ 1 & 6 & 0 & 4 \\ -1 & -4 & -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ -3 \end{pmatrix}.$$

This is equivalent to asking if the system $\mathbf{A}\mathbf{x} = \mathbf{v}$ is consistent.

Column Space and Consistency of Linear System

Let $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$.

- Then a vector $\mathbf{v}$ is in the column space of $\mathbf{A}$, $\mathbf{v} \in \mathrm{Col}(\mathbf{A})$, if and only if we can find a $\mathbf{u} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ such that

$$\mathbf{Au} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{v}.$$

- This is equivalent to the system $\mathbf{Ax} = \mathbf{v}$ being consistent. This is also equivalent to $\mathbf{v} = \mathbf{Au}$ for some $\mathbf{u}$ in $\mathbb{R}^k$.
- Hence, the column space can be characterized either by the set of vectors $\mathbf{v}$ such that $\mathbf{Ax} = \mathbf{v}$ is consistent, or the set of vectors $\mathbf{v}$ such that $\mathbf{v} = \mathbf{Au}$ for some $\mathbf{u}$,

$$\mathrm{Col}(\mathbf{A}) = \left\{ \ \mathbf{v} = \mathbf{Au} \ \middle| \ \mathbf{u} \in \mathbb{R}^k \ \right\} = \left\{ \ \mathbf{v} \ \middle| \ \mathbf{Ax} = \mathbf{v} \text{ is consistent } \right\}.$$

Nullspace

Definition

The *nullspace* of a $m \times n$ matrix $\mathbf{A}$ is the solution space to the homogeneous system $\mathbf{Ax} = \mathbf{0}$ with coefficient matrix $\mathbf{A}$. It is denoted as

$$\mathrm{Null}(\mathbf{A}) = \left\{\ \mathbf{v} \in \mathbb{R}^n \ \middle|\ \mathbf{Av} = \mathbf{0}\ \right\}.$$

The *nullity* of $\mathbf{A}$ is the dimension of the nullspace of $\mathbf{A}$, denoted as

$$\mathrm{nullity}(\mathbf{A}) = \dim(\mathrm{Null}(\mathbf{A})).$$

Question

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \\ 0 & 0 & 1 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1. Find a basis for the nullspace of $\mathbf{A}$.

2. What is the nullity of $\mathbf{A}$?

4.2 Rank

Question

For any matrix $\mathbf{A}$, the dimension of the column space of $\mathbf{A}$ is equal to the dimension of the row space of $\mathbf{A}$. True or false?

Rank

Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{R}$ its reduced row-echelon form.

$$\begin{aligned} \dim(\mathrm{Col}(\mathbf{A})) &= \#\text{ of pivot columns in RREF of } \mathbf{A}, \\ &= \#\text{ of leading entries in RREF of } \mathbf{A}, \\ &= \#\text{ of nonzero rows in RREF of } \mathbf{A} = \dim(\mathrm{Row}(\mathbf{A})) \end{aligned}$$

Definition

Define the *rank* of $\mathbf{A}$ to be the dimension of its column or row space

$$\mathrm{rank}(\mathbf{A}) = \dim(\mathrm{Col}(\mathbf{A})) = \dim(\mathrm{Row}(\mathbf{A})).$$

Exercise

Prove that the rank is invariant under transpose,

$$\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^T).$$

Examples

1. $\text{rank}(\mathbf{A}) = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

2. $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. So $\text{rank}(\mathbf{A}) = 3$.

3. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ -1 & 7 & 5 \\ 1 & 9 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. So $\text{rank}(\mathbf{A}) = 3$.

4. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 7 \\ -1 & 7 & -19 \\ 1 & 9 & -13 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So $\text{rank}(\mathbf{A}) = 2$.

Challenge: Rank and Consistency of Linear Systems

Prove the following theorem.

Theorem

The linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of $\mathbf{A}$ is equal to the rank of the augmented matrix $(\mathbf{A} \mid \mathbf{b})$,

$$\mathrm{rank}(\mathbf{A}) = \mathrm{rank}((\mathbf{A} \mid \mathbf{b})).$$

Properties Rank

Let $\mathbf{A} = \begin{pmatrix} 4 & 3 & 5 \\ 5 & -1 & 5 \\ -1 & 0 & 0 \\ 5 & 2 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \\ 3 & 3 \end{pmatrix}$. Then $\text{rank}(\mathbf{A}) = 3$ and $\text{rank}(\mathbf{B}) = 2$. Now,

$$\mathbf{AB} = \begin{pmatrix} 30 & 14 \\ 29 & 9 \\ -3 & 1 \\ 32 & 12 \end{pmatrix},$$

and $\text{rank}(\mathbf{AB}) = 2$.

Here $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{A})$ and $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{B})$.

Properties of Rank

Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{pmatrix}$. Check that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 2$. Next,

$$\mathbf{AB} = \begin{pmatrix} 8 & -1 & 6 \\ 1 & 0 & 1 \end{pmatrix}$$

and $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{A})$ and $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{B})$.

Properties of Rank

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 3 & -2 & 1 & -1 \\ 3 & -2 & 1 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -4 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. Check that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 2$. Now

$$\mathbf{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{rank}(\mathbf{AB}) = 0.$$

Here $\text{rank}(\mathbf{AB}) < \text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})$.

Properties of Rank

Lemma

Let $\mathbf{A}$ *be a* $m \times n$ *matrix and* $\mathbf{B}$ *a* $n \times p$ *matrix. The column space of the product* $\mathbf{AB}$ *is a subspace of the column space of* $\mathbf{A}$,

$$Col(\mathbf{AB}) \subseteq Col(\mathbf{A}).$$

Sketch of proof.

Write $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots \mathbf{b}_p \end{pmatrix}$. Then

$$\mathbf{AB} = \mathbf{A} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots \mathbf{b}_p \end{pmatrix} = \begin{pmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots \mathbf{Ab}_p \end{pmatrix}.$$

Recall that $\mathbf{Au} \in \text{Col}(\mathbf{A})$ for all $\mathbf{u}$, and hence, $\mathbf{Ab}_i \in \text{Col}(\mathbf{A})$ for all $i = 1, .., p$. Therefore

$$\text{Col}(\mathbf{AB}) = \text{span}\{\mathbf{Ab}_1, \mathbf{Ab}_2, ..., \mathbf{Ab}_p\} \subseteq \text{Col}(\mathbf{A}).$$

$\square$

Properties of Rank

Theorem

Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{B}$ a $n \times p$ matrix. Then

$$\operatorname{rank}(\mathbf{AB}) \leq \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}.$$

Proof.

By the previous lemma,

$$\operatorname{rank}(\mathbf{AB}) = \dim(\operatorname{Col}(\mathbf{AB})) \leq \dim(\operatorname{Col}(\mathbf{A})) = \operatorname{rank}(\mathbf{A}).$$

Next, using the previous lemma and the above derivation on $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, we have

$$\operatorname{rank}(\mathbf{AB}) = \operatorname{rank}((\mathbf{AB})^T) = \operatorname{rank}(\mathbf{B}^T\mathbf{A}^T) \leq \operatorname{rank}(\mathbf{B}^T) = \operatorname{rank}(\mathbf{B}).$$

Hence,

$$\operatorname{rank}(\mathbf{AB}) \leq \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}.$$

$\square$

Question

Show that if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B})$.

Rank-Nullity Theorem

Theorem (Rank-Nullity Theorem)

Let $\mathbf{A}$ *be a* $m \times n$ *matrix. The sum of its rank and nullity is equal to the number of columns,*

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n.$$

Sketch of Proof.

This follows from the fact that the nullity of $\mathbf{A}$ is equal to the number of non-pivot columns in its reduced row-echelon form, and that the rank of $\mathbf{A}$ is equal to the number of pivot columns of its reduced row-echelon form. $\square$

Examples

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 2 + 2 = 4 =$ number of columns of $\mathbf{A}$. Indeed, $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}$ is a basis of the column space, and $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}$ is a basis for the nullspace.

Observe that the column space is a subspace of $\mathbb{R}^3$ but the subspace is a subspace of $\mathbb{R}^4$.

Summary of the Subspaces Associated to a Matrix

Let $\mathbf{A}$ be a $m \times n$ matrix.

Subspace	**Subspace of**	**Basis**	**Dimension**
$\text{Col}(\mathbf{A})$	$\mathbb{R}^m$	Columns of $\mathbf{A}$ corresponding to pivot columns in RREF	$\text{rank}(\mathbf{A}) =$ no. of pivot columns in RREF
$\text{Row}(\mathbf{A})$	$\mathbb{R}^n$	Nonzero rows of RREF	$\text{rank}(\mathbf{A}) =$ no. of nonzero rows in RREF
$\text{Null}(\mathbf{A})$	$\mathbb{R}^n$	Vectors in general solution to $\mathbf{Ax} = \mathbf{0}$	$\text{nullity}(\mathbf{A}) =$ no. of nonpivot columns in RREF

Challenge

Let **A** and **B** be matrices of the same size. Prove that

$$\operatorname{rank}(\mathbf{A}+\mathbf{B}) \leq \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}).$$

Full rank

Let $\mathbf{A}$ be a $m \times n$ matrix.

$$\begin{aligned} \text{rank}(\mathbf{A}) &= \# \text{ of pivot columns in RREF} \leq \text{ no. of columns} = n \\ \text{rank}(\mathbf{A}) &= \# \text{ of nonzero rows in RREF} \leq \text{ no. of rows} = m \end{aligned}$$

So, the rank of $\mathbf{A}$ is no greater than the number of rows or columns, whichever is smaller,

$$\text{rank}(\mathbf{A}) \leq \min\{m, n\}.$$

The maximum rank a matrix can attain is when it is equal to either the number of rows or columns, whichever is smaller.

Definition

A $m \times n$ matrix $\mathbf{A}$ is said to be of *full rank* if its rank is equal to either the number of rows or columns,

$$\text{rank}(\mathbf{A}) = \min\{m, n\}.$$

Example

1. $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ $\text{rank}(\mathbf{A}) = 3 =$ number of rows, so $\mathbf{A}$ is full rank.
2. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ -1 & 7 & 5 \\ 1 & 9 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{A}) = 3 =$ number of columns, so $\mathbf{A}$ is full rank.
3. $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{A}) = 2$ which is strictly smaller than the number rows and number of columns. So $\mathbf{A}$ is not of full rank.
4. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{A}) = 2$ which is strictly smaller than the number rows and number of columns. So $\mathbf{A}$ is not of full rank.

Discussion

What happens when a $m \times n$ matrix $\mathbf{A}$ is full rank? Let us consider cases.

Consider the case when $\mathbf{A}$ is a square matrix $m = n$. Then $\mathbf{A}$ is full rank if and only if its is invertible. The proof is left as an exercise. We will include this in the list of equivalent statements for invertibility.

Equivalent Statements for Invertibility

Theorem

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) $\mathbf{A}$ *is invertible.*

(ii) $\mathbf{A}^T$ *is invertible.*

(iii) *(left inverse) There is a matrix* $\mathbf{B}$ *such that* $\mathbf{BA} = \mathbf{I}$.

(iv) *(right inverse) There is a matrix* $\mathbf{B}$ *such that* $\mathbf{AB} = \mathbf{I}$.

(v) *The reduced row-echelon form of* $\mathbf{A}$ *is the identity matrix.*

(vi) $\mathbf{A}$ *can be expressed as a product of elementary matrices.*

(vii) *The homogeneous system* $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution.*

(viii) *For any* $\mathbf{b}$, *the system* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

(ix) *The determinant of* $\mathbf{A}$ *is nonzero,* $\det(\mathbf{A}) \neq 0$.

(x) *The columns/rows of* $\mathbf{A}$ *are linearly independent.*

(xi) *The columns/rows of* $\mathbf{A}$ *spans* $\mathbb{R}^n$.

(xii) $\text{rank}(\mathbf{A}) = n$ *(*$\mathbf{A}$ *has full rank).*

(xiii) $\text{nullity}(\mathbf{A}) = 0$.

Discussion

Let $\mathbf{A}$ be a $m \times n$ matrix. Suppose $\mathbf{A}$ is not a square matrix $m \neq n$, and $\mathbf{A}$ is full rank.
Then either $\text{rank}(\mathbf{A}) = n < m$, $\text{rank}(\mathbf{A}) = m < n$. In either cases, some of the equivalent statements of invertibility will still be true of $\mathbf{A}$.

Lemma

Let $\mathbf{A}$ be a $m \times n$ matrix. Then the nullspace of $\mathbf{A}$ is equal to the nullspace of $\mathbf{A}^T\mathbf{A}$,

$$\text{Null}(\mathbf{A}) = \text{Null}(\mathbf{A}^T\mathbf{A}).$$

Proof.

Suppose $\mathbf{u}$ is in the nullspace of $\mathbf{A}$, $\mathbf{Au} = \mathbf{0}$. Then premultiplying by $\mathbf{A}^T$, $\mathbf{A}^T\mathbf{Au} = \mathbf{0}$ too. This shows that $\mathbf{u}$ is in the nullspace of $\mathbf{A}^T\mathbf{A}$. This proves $\text{Null}(\mathbf{A}) \subseteq \text{Null}(\mathbf{A}^T\mathbf{A})$.

Conversely, suppose $\mathbf{u}$ is in the nullspace of $\mathbf{A}^T\mathbf{A}$, $\mathbf{A}^T\mathbf{Au} = \mathbf{0}$. Premultiplying both sides by $\mathbf{u}^T$, and noting that $\mathbf{u}^T\mathbf{A}^T\mathbf{Au} = (\mathbf{Au}) \cdot (\mathbf{Au})$,

$$(\mathbf{Au}) \cdot (\mathbf{Au}) = \mathbf{u}^T\mathbf{A}^T\mathbf{Au} = \mathbf{u}^T(\mathbf{0}) = 0.$$

Hence, $\mathbf{Au} = \mathbf{0}$ too, that is, $\mathbf{u}$ is in the nullspace of $\mathbf{A}$. This proves that $\text{Null}(\mathbf{A}^T\mathbf{A}) \subseteq \text{Null}(\mathbf{A})$ too. $\square$

Full Rank Equals Number of Columns

Theorem

Suppose $\mathbf{A}$ *is a* $m \times n$ *matrix. The following statements are equivalent.*

(i) $\mathbf{A}$ *is full rank, where the rank is equal to the number of columns,* $\text{rank}(\mathbf{A}) = n$.

(ii) *The rows of* $\mathbf{A}$ *spans* $\mathbb{R}^n$, *Row*$(\mathbf{A}) = \mathbb{R}^n$.

(iii) *The columns of* $\mathbf{A}$ *are linearly independent.*

(iv) *The homogeneous system* $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution, that is,* $\text{Null}(\mathbf{A}) = \{\mathbf{0}\}$.

(v) $\mathbf{A}^T\mathbf{A}$ *is an invertible matrix of order* n.

(vi) $\mathbf{A}$ *has a left inverse.*

Proof.

We will only prove the equivalence of the last 3 statements, the rest are left as an exercise. Hint: One might try to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Observe that if $\text{rank}(\mathbf{A}) = n$, then the reduced row-echelon form is of the form

$$\mathbf{R} = \begin{pmatrix} \mathbf{I}_n \\ \mathbf{0}_{(m-n)\times n} \end{pmatrix}.$$

Full Rank Equals Number of Columns

Continue of Proof.

(iv) $\Rightarrow$ (v): Suppose the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution. By the lemma, the homogeneous system $\mathbf{A}^T\mathbf{Ax} = \mathbf{0}$ has only the trivial solution too. But since $\mathbf{A}^T\mathbf{A}$ is a square matrix, by the equivalent statements of invertibility, $\mathbf{A}^T\mathbf{A}$ is invertible.

(v) $\Rightarrow$ (vi): Suppose $\mathbf{A}^T\mathbf{A}$ is invertible. Then

$$\mathbf{I} = (\mathbf{A}^T\mathbf{A})^{-1}(\mathbf{A}^T\mathbf{A}) = ((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{A},$$

which shows that $((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)$ is a left inverse of $\mathbf{A}$.

(vi) $\Rightarrow$ (i): Let $\mathbf{B}$ be a left inverse of $\mathbf{A}$, $\mathbf{BA} = \mathbf{I}_n$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Then by the properties of rank,

$$n = \text{rank}(\mathbf{I}) = \text{rank}(\mathbf{BA}) \leq \text{rank}(\mathbf{A}).$$

But since $\text{rank}(\mathbf{A}) \leq n$, equality holds. $\square$

Example

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Its reduce row-echelon form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

- So, $\mathbf{A}$ is full rank, where the rank is equal to the number of columns.
- Check that $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ is invertbile, with inverse $(\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{7}\begin{pmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{pmatrix}$.
- Check that

$$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \frac{1}{7}\begin{pmatrix} 3 & 3 & -4 & 1 \\ -4 & 3 & 3 & 1 \\ 3 & -4 & 3 & 1 \end{pmatrix}$$

  is a left inverse of $\mathbf{A}$.

Full Rank Equals Number of Rows

Theorem

Suppose $\mathbf{A}$ *is a* $m \times n$ *matrix. The following statements are equivalent.*

(i) $\mathbf{A}$ *is full rank, where the rank is equal to the number of rows,* $\text{rank}(\mathbf{A}) = m$.

(ii) *The columns of* $\mathbf{A}$ *spans* $\mathbb{R}^m$, $Col(\mathbf{A}) = \mathbb{R}^m$.

(iii) *The rows of* $\mathbf{A}$ *are linearly independent.*

(iv) *The linear system* $\mathbf{Ax} = \mathbf{b}$ *is consistent for every* $\mathbf{b} \in \mathbb{R}^m$.

(v) $\mathbf{AA}^T$ *is an invertible matrix of order* m.

(vi) $\mathbf{A}$ *has a right inverse.*

Full Rank Equals Number of Rows

If $\text{rank}(\mathbf{A}) = m$, then the reduced row-echelon form is of the form

$$\begin{pmatrix} 0 & \cdots & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots \\ 0 & \cdots & 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 & \ldots \\ 0 & \cdots & 0 & \ldots & 0 & \ldots & 1 & \ldots & 0 & \ldots \\ \vdots & & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots \end{pmatrix}.$$

The proof follows from the previous theorem by replacing $\mathbf{A}$ with $\mathbf{A}^T$. For statement (iv), use rank-nullity theorem. The details are left to the readers.

Example

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$. The reduced row-echelon form of $\mathbf{A}$ is $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

- So, $\mathbf{A}$ is full rank, where the rank is equal to the number of rows.
- Check that $\mathbf{A}\mathbf{A}^T = \begin{pmatrix} 7 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ is invertbile, with inverse $\mathbf{A}\mathbf{A}^T)^{-1} = \frac{1}{12}\begin{pmatrix} 5 & -8 & 1 \\ -8 & 20 & -4 \\ 1 & -4 & 5 \end{pmatrix}$.
- Check that

$$\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1} = \frac{1}{6}\begin{pmatrix} 3 & -6 & 3 \\ -1 & 4 & 1 \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{pmatrix}$$

is a right inverse of $\mathbf{A}$.

Challenge

Let $\mathbf{A}$ be a $m \times n$ matrix such that $\text{rank}(\mathbf{A}) = m$. Suppose $m > n$. By the equivalent statements of full rank equals number of columns, $(\mathbf{A}^T\mathbf{A})$ invertible and $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$. is a left inverse of $\mathbf{A}$.

Now consider the system $\mathbf{Ax} = \mathbf{b}$ for some vector $\mathbf{b}$ in $\mathbb{R}^m$. Premultiplying the left inverse above on both sides of the equation, we get

$$\mathbf{x} = ((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{Ax} = ((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{b},$$

that is, $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{b}$ is a solution to $\mathbf{Ax} = \mathbf{b}$. But this is true for every $\mathbf{b}$, which by the equivalent statements of full rank equals number or rows, means that the rank of $\mathbf{A}$ is equal to m, the number of row. This is a contradiction to $m > n$.

What is the mistake in the argument above?