# MA1522 Linear Algebra for Computing

## Lecture 12: Singular Value Decomposition


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# Outline

# Beginning Remarks

- SVD brings everything (eigenvalue, orthogonal basis, diagonalization, etc) together.
- It applies to arbitrary matrices.
- You should watch the Youtube video at the end of Dr. Teo's Lecture to get the geometric idea.

# Exercise in Section 6.5

1. Show that

$$\Sigma^T\Sigma = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mu_n \end{pmatrix},$$

   where $\mu_i$, $i = 1, \dots, n$, are the eigenvalues of $\mathbf{A}^T\mathbf{A}$. If we let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ of the associated eigenvectors of $\mathbf{A}^T\mathbf{A}$, then $\mathbf{A}^T\mathbf{A} = \mathbf{P}\Sigma^T\Sigma\mathbf{P}^T$, where $\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$.

2. Show that $\mathbf{A}\mathbf{v}_i \neq \mathbf{0}$ for all $i \leq r$ and $\mathbf{A}\mathbf{v}_i = \mathbf{0}$ for all $i > r$.

# Slides 79 and 80: Singular Values

Let $\mathbf{A}$ be an $m \times n$ matrix. Since $\mathbf{A}^T\mathbf{A}$ is an order $n$ symmetric matrix, it is orthogonally diagonalizable. Let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}^T\mathbf{A}$ with associated eigenvalues $\mu_i$, not necessarily distinct.

## Lemma

*The eigenvalue $\mu_i$ of $\mathbf{A}^T\mathbf{A}$ is nonnegative.*

Reordering if necessary, we may assume that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0.$$

The singular values of $\mathbf{A}$ are

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0,$$

where $\sigma_i = \sqrt{\mu_i}$, $i = 1, ..., n$. Let $r$ be the largest integer such that $1 \leq r \leq n$ and $\sigma_i > 0$ for all $i \leq r$, that is

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = 0 = \cdots = \sigma_m = 0.$$

# Slides 79 and 80: The Matrix $\Sigma$

Define the matrix $m \times n$ matrix $\Sigma$ to be

$$\Sigma = \begin{pmatrix} \mathbf{D} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{pmatrix}, \text{ where } \mathbf{D} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{pmatrix}.$$

# Answer to Exercise in Section 6.5 (part 1)

Since

$$\Sigma = \begin{pmatrix} \mathbf{D} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{pmatrix}, \Sigma^T = \begin{pmatrix} \mathbf{D} & \mathbf{0}_{r\times(m-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{0}_{(n-r)\times(m-r)} \end{pmatrix}.$$

Thus,

$$\Sigma^T\Sigma = \begin{pmatrix} \mathbf{D}^2 & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{0}_{(n-r)\times(n-r)} \end{pmatrix}.$$

Furthermore, the $i$-th entry in $\mathbf{D}^2$ is $\sigma_i^2 = \mu_i$, when $i \leq r$. The result follows.

# Answer to Exercise in Section 6.5 (part 2)

2. Show that $\mathbf{Av}_i \neq \mathbf{0}$ for all $i \leq r$ and $\mathbf{Av}_i = \mathbf{0}$ for all $i > r$.

The following more general **Fact** is useful: For any $m \times n$ matrix $\mathbf{A}$ and $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{Ax} = \mathbf{0}$ iff $\mathbf{A}^T\mathbf{Ax} = \mathbf{0}$, consequently,

$$\text{Null}(\mathbf{A}) = \text{Null}(\mathbf{A}^T\mathbf{A}).$$

Proof of the fact: ($\subseteq$): If $\mathbf{Ax} = \mathbf{0}$, then clearly, $\mathbf{A}^T\mathbf{Ax} = \mathbf{0}$.

($\supseteq$) On the other hand, if $\mathbf{A}^T\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}^T\mathbf{A}^T\mathbf{Ax} = 0$, i.e., $(\mathbf{Ax})^T(\mathbf{Ax}) = 0$, which says $\|\mathbf{Ax}\|^2 = 0$, so $\mathbf{Ax} = \mathbf{0}$.

Answer: For $i \leq r$, we have $\mu_i \neq 0$. Hence the associated eigenvector $\mathbf{v}_i \notin \text{Null}(\mathbf{A}^T\mathbf{A})$. By the fact, $\mathbf{Av}_i \neq \mathbf{0}$.

For $i > r$, $\mu_i = 0$. Hence, the associated eigenvector $\mathbf{v}_i \in \text{Null}(\mathbf{A}^T\mathbf{A})$. By the fact, $\mathbf{Av}_i = \mathbf{0}$.

# Question in Section 6.5

Let $\mathbf{A}$ be an $m \times n$ matrix. Prove the following statements.

1. $\text{rank}(\mathbf{A}) = n$ if and only if all the singular values of $\mathbf{A}$ are positive.
2. $\text{rank}(\mathbf{A}) = m$ if and only if all the singular values of $\mathbf{A}^T$ are positive.

Recall that the Rank-Nullity Theorem says: For an $m \times n$ matrix $\mathbf{A}$,

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n.$$

By the Fact above, we have $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T\mathbf{A})$.

# Answer to Question in Section 6.5 (part 1)

Let $\mathbf{A}$ be an $m \times n$ matrix. Prove the following statements.

1. $\text{rank}(\mathbf{A}) = n$ if and only if all the singular values of $\mathbf{A}$ are positive.

Proof: $(\Rightarrow)$ If $\text{rank}(\mathbf{A}) = n$ then $\text{rank}(\mathbf{A}^T\mathbf{A}) = n$. Therefore $\mathbf{D} = \mathbf{P}^T\mathbf{A}^T\mathbf{A}\mathbf{P}$ also has rank $n$. Thus, every eigenvalue $\mu_i > 0$. So every singular value $\sigma_i > 0$.

$(\Leftarrow)$ If all the singular values of $\mathbf{A}$ are positive, then the eigenvalues $\mu_i$ of $\mathbf{A}^T\mathbf{A}$ are positive. In other words, 0 is not an eigenvalue of $\mathbf{A}^T\mathbf{A}$. Hence $\text{nullity}(\mathbf{A}^T\mathbf{A}) = 0$. By the Fact, $\text{nullity}(\mathbf{A}) = 0$. By Rank-Nullity Theorem, $\text{rank}(\mathbf{A}) = n$.

# Question in Section 6.5 (part 2)

Let $\mathbf{A}$ be an $m \times n$ matrix. Prove the following statements.

2. $\text{rank}(\mathbf{A}) = m$ if and only if all the singular values of $\mathbf{A}^T$ are positive.

Answer: Since $\text{rank}(\mathbf{A}^T) = \text{rank}(\mathbf{A}) = m$, apply part 1, we have all the singular values of $\mathbf{A}^T$ are positive.

# Slide 83: Singular Value Decomposition

Suppose $\mathbf{A}$ is an $m \times n$ matrix. Let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}^T\mathbf{A}$. Let

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

be the nonzero singular values of $\mathbf{A}$. Define

$$\mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i, \quad i = 1, .., r.$$

Lemma

$\{\mathbf{u}_1, ..., \mathbf{u}_r\}$ *is an orthonormal basis for the column space of* $\mathbf{A}$, *and* $\text{rank}(\mathbf{A}) = r$.

# Slide 85: Singular Value Decomposition

Extend $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r\}$ to an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ for $\mathbb{R}^m$ (if $r \neq m$). Define

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{pmatrix},$$

it is an order $m$ orthogonal matrix. Define

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix},$$

then $\mathbf{V}$ is an order $n$ orthogonal matrix. Let $\Sigma$ be the matrix defined by the nonzero singular values $\sigma_1, \sigma_2, ..., \sigma_r$. Then

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T.$$

# Slide 86: Algorithm to Singular Value Decomposition

Let $\mathbf{A}$ be an $m \times n$ matrix with $\text{rank}(\mathbf{A}) = r$.

1. Find the eigenvalues of $\mathbf{A}^T\mathbf{A}$. Arrange the nonzero eigenvalues in descending order (counting multiplicity)

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r > 0 = \mu_{r+1} = \cdots = \mu_n,$$

and let $\sigma_i = \sqrt{\mu_i}$, $i = 1, ..., r$. Set

$$\Sigma = \begin{pmatrix} \mathbf{D} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{pmatrix}, \text{ where } \mathbf{D} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{pmatrix}.$$

2. Find an orthonormal basis for each eigenspace, and let $\mathbf{v}_i$ be the unit vector associated to $\mu_i$. Set

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}.$$

# Slide 86: Algorithm to Singular Value Decomposition (conti.)

3. Let $\mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$ for $i = 1, ..., r$. Extend $\{\mathbf{u}_1, ..., \mathbf{u}_r\}$ to an orthonormal basis $\{\mathbf{u}_1, ..., \mathbf{u}_r, \mathbf{u}_{r+1}, ..., \mathbf{u}_m\}$ of $\mathbb{R}^m$, that is, solve for $\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_r \end{pmatrix}^T \mathbf{x} = \mathbf{0}$ and find an orthonormal basis for the solution space. Let
$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{pmatrix}.$$

Then $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$.

# Example 1

Find a singular value decomposition of $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

1. $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

   $\det(x\mathbf{I} - \mathbf{A}^T\mathbf{A}) = \begin{vmatrix} x-2 & -1 \\ -1 & x-2 \end{vmatrix} = (x-3)(x-1)$. Then eigenvalues of $\mathbf{A}^T\mathbf{A}$ are $\mu_1 = 3$, $\mu_2 = 1$, and thus the singular values of $\mathbf{A}$ are $\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$. Hence,

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

# Example 1 (conti.)

2. Find and orthonormal basis for the eigenspace of $\mathbf{A}^T\mathbf{A}$,

$$3\mathbf{I} - \mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\mathbf{I} - \mathbf{A}^T\mathbf{A} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

So, $\mathbf{V} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

# Example 1 (conti.)

3. $\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}1/\sqrt{6}\\ 1/\sqrt{6}\\ 2/\sqrt{6}\end{pmatrix}$, and
$\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\\ 0\end{pmatrix}$. Next, find an vector orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$,

$$\begin{pmatrix}1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{pmatrix} \xrightarrow{RREF} \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$$

$$\Longrightarrow \qquad \mathbf{u}_3 = \begin{pmatrix}-1/\sqrt{3}\\ -1/\sqrt{3}\\ 1/\sqrt{3}\end{pmatrix}.$$

# Example 1 (conti)

Therefore, $\mathbf{U} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix}$. Hence,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

## Example 2

Find a singular value decomposition of $\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.

1. $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{pmatrix}$.

$$\det(x\mathbf{I} - \mathbf{A}^T\mathbf{A}) = \begin{vmatrix} x-10 & -2 & -6 \\ -2 & x-2 & 2 \\ -6 & 2 & x-10 \end{vmatrix} = (x-16)(x-6)x.$$

Then eigenvalues of $\mathbf{A}^T\mathbf{A}$ are $\mu_1 = 16$, $\mu_2 = 6$, $\mu_3 = 0$ and thus the singular values of $\mathbf{A}$ are $\sigma_1 = 4$, $\sigma_2 = \sqrt{6}$, $\sigma_3 = 0$. Hence,

$$\Sigma = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{pmatrix}.$$

# Example 2 (conti.)

2. Find and orthonormal basis for the eigenspace of $\mathbf{A}^T\mathbf{A}$,

$$16\mathbf{I} - \mathbf{A}^T\mathbf{A} = \begin{pmatrix} 6 & -2 & -6 \\ -2 & 14 & 2 \\ -6 & 2 & 6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

$$6\mathbf{I} - \mathbf{A}^T\mathbf{A} = \begin{pmatrix} -4 & -2 & -6 \\ -2 & 4 & 2 \\ -6 & 2 & -4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \quad \mathbf{v}_2 = \begin{pmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}.$$

# Example 2 (conti.)

$$0\mathbf{I} - \mathbf{A}^T\mathbf{A} = \begin{pmatrix} -10 & -2 & -6 \\ -2 & -2 & 2 \\ -6 & 2 & -10 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \quad \mathbf{v}_3 = \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}.$$

So, $\mathbf{V} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$.

# Example 2 (conti.)

3.

$$\begin{aligned}
\mathbf{u}_1 &= \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1 = \frac{1}{4}\begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}. \\
\mathbf{u}_2 &= \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.
\end{aligned}$$

Therefore, $\mathbf{U} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$. Hence,

$$\begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}.$$

# Question One in Section 7.1

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & -1 & -1 \\ 1 & -1 & 0 & 2 \end{pmatrix}$. Then $\mathbf{A}$ defines a linear transformation defined $T$ by matrix multiplication.

1. What are the domain and codomain of $T$?
2. Write down the formula of $T$.

# Slide 6: Linear Transformation

## Definition

A mapping (function) $T : \mathbb{R}^n \to \mathbb{R}^m$, is a *linear transformation* if for all vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, and scalars $\alpha, \beta$,

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\, T(\mathbf{u}) + \beta\, T(\mathbf{v}).$$

The Euclidean space $\mathbb{R}^n$ is called the *domain* of the mapping, and the Euclidean space $\mathbb{R}^m$ is called the *codomain* of the mapping.

# Answer to Question One in Section 7.1

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & -1 & -1 \\ 1 & -1 & 0 & 2 \end{pmatrix}$. Then $\mathbf{A}$ defines a linear transformation defined $T$ by matrix multiplication.

1. What are the domain and codomain of $T$?
2. Write down the formula of $T$.

Answer: The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$.

$$T\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \begin{pmatrix} x+z-w \\ 2x+y-z-w \\ x-y+2w \end{pmatrix}.$$

# Question two in Section 7.1

Is the mapping $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\mathbf{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \sqrt[3]{(x^3 + y^3)} \\ 0 \end{pmatrix},$$

a linear transformation?

Slide 12:
A mapping $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is not a linear transformation if any of the following statements hold.

(i) $\mathbf{T}$ does not map the zero vector to the zero vector, $\mathbf{T}(\mathbf{0}) \neq \mathbf{0}$.

(ii) There is a scalar $\alpha$ and a vector $\mathbf{u}$ in $\mathbb{R}^n$ such that $\mathbf{T}(\alpha\mathbf{u}) \neq \alpha\mathbf{T}(\mathbf{u})$.

(iii) There are vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ such that $\mathbf{T}(\mathbf{u} + \mathbf{v}) \neq \mathbf{T}(\mathbf{u}) + \mathbf{T}(\mathbf{v})$.

# Answer to Question two in Section 7.1

Q: Is the mapping $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\mathbf{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \sqrt[3]{(x^3 + y^3)} \\ 0 \end{pmatrix},$$

a linear transformation?

Answer: No, it is not a linear mapping.

$$\begin{aligned} \mathbf{T}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= \mathbf{T}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} \sqrt[3]{1^3 + 1^3} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt[3]{2} \\ 0 \end{pmatrix} \\ \neq \quad \begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \begin{pmatrix} \sqrt[3]{1} \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt[3]{1} \\ 0 \end{pmatrix} = \mathbf{T}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \mathbf{T}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right). \end{aligned}$$

# Challenge

Find a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$$

for all scalar $\alpha$ and vector $\mathbf{u}$ in $\mathbb{R}^n$, but is not a linear transformation.

Let $T$ be as in Question 2. We knew it is not a linear transformation. However,

$$T\left(\alpha \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \sqrt[3]{\alpha^3 x^3 + \alpha^3 y^3} \\ 0 \end{pmatrix} = \alpha T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

# Question three in Section 7.1

1. Is the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$,
$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
a linear transformation? If it is, find its standard matrix.
2. Is $T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ for some constants $a_1, a_2, ..., a_n$ a linear transformation? If it is, find its standard matrix.
3. What is the standard matrix of the following linear transformation $\mathbf{T}\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$?

# Slides 15, 16: Standard Matrix

## Theorem

*A mapping* $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear transformation if and only if there is a unique* $m \times n$ *matrix* $\mathbf{A}$ *such that*

$$T(\mathbf{u}) = \mathbf{Au} \quad \textit{for all vectors } \mathbf{u} \textit{ in } \mathbb{R}^n.$$

*The matrix* $\mathbf{A}$ *is given by*

$$\mathbf{A} = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{pmatrix},$$

*where* $E = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ *is the standard basis for* $\mathbb{R}^n$*. That is, the* $i$*-th column of* $\mathbf{A}$ *is* $T(\mathbf{e}_i)$*, for* $i = 1, ..., n$*.*

## Definition

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The unique $m \times n$ matrix $\mathbf{A}$ such that $T(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u}$ in $\mathbb{R}^n$ is called the *standard matrix* of $T$.

# Remarks

- Slide 4: Matrix multiplication is linear, for all vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and scalars $\alpha, \beta$,

$$\mathbf{A}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{A}\mathbf{u} + \beta\mathbf{A}\mathbf{v}.$$

- For Question 3, one can check the definition for each map, and apply them to the standard basis to get their matrices respectively.
- We just reformulate them as matrix multiplication, so that we can show their linearity and find their standard matrices in one shot.

# Answer to Question three in Section 7.1, part 1

1. Is the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$,
$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
a linear transformation? If it is, find its standard matrix.

Answer:
$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (x + 2y + 3z) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} x + 2y + 3z \\ 2x + 4y + 6z \\ 3x + 6y + 9z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$
Thus, it is a linear transformation with matrix
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

# Answer to Question three in Section 7.1, part 2

2. Is $T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ for some constants $a_1, a_2, \ldots, a_n$ a linear transformation? If it is, find its standard matrix.

Answer: Since

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

It is a linear transformation with matrix

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}.$$

# Answer to Question three in Section 7.1, part 3

3. What is the standard matrix of the following linear transformation $\mathbf{T}\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$?

Answer: The standard matrix is the identity matrix $\mathbf{I}_n$.