NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 5**

1. For each of the following sets of vectors S,

   (i) Determine if S is linearly independent.

   (ii) If S is linearly dependent, express one of the vectors in S as a linear combination of the others.

   (a) $S = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} \right\}$.

   Solution: The set S is linearly dependent since it contains 4 vectors from $\mathbb{R}^3$.

$$\begin{pmatrix} 2 & 0 & 2 & 3 \\ -1 & 3 & 4 & 6 \\ 0 & 2 & 3 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{9}{2} \\ 0 & 1 & 0 & \frac{15}{2} \\ 0 & 0 & 1 & -3 \end{pmatrix}.$$

   Thus

$$\begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = \frac{9}{2} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \frac{15}{2} \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}.$$

   (b) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \right\}$.

   Solution: The set S is linearly independent since S has only two vectors which are not multiples of each other.

   (c) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

   Solution: Any set containing the zero vector is linearly dependent. Indeed we have

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}.$$

   (d) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$.

Solution: Solving $a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\1\end{pmatrix}+c\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$,

$$\left(\begin{array}{ccc|c}1&0&1&0\\0&1&2&0\\0&1&-1&0\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right).$$

So $a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\1\end{pmatrix}+c\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ has only the trivial solution and S is a linearly independent set.

2. Suppose $\mathbf{u},\mathbf{v},\mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine which of the sets S_1 to S_5 are linearly independent.

   (a) $S_1=\{\mathbf{u},\mathbf{v}\}$,

   Solution: Any subset of a linearly independent set is linearly independent. Indeed we have
   $$a\mathbf{u}+b\mathbf{v}=\mathbf{0}\quad\Leftrightarrow\quad a\mathbf{u}+b\mathbf{v}+0\mathbf{w}=\mathbf{0}.$$

   (b) $S_2=\{\mathbf{u}-\mathbf{v},\mathbf{v}-\mathbf{w},\mathbf{w}-\mathbf{u}\}$,

   Solution: Observe that $(\mathbf{u}-\mathbf{v})+(\mathbf{v}-\mathbf{w})+(\mathbf{w}-\mathbf{u})=\mathbf{0}$. So, S_2 is linearly dependent.

   (c) $S_3=\{\mathbf{u}-\mathbf{v},\mathbf{v}-\mathbf{w},\mathbf{u}+\mathbf{w}\}$,

   Solution: We have

   $$a(\mathbf{u}-\mathbf{v})+b(\mathbf{v}-\mathbf{w})+c(\mathbf{w}+\mathbf{u})=\mathbf{0}\quad\Leftrightarrow\quad(a+c)\mathbf{u}+(-a+b)\mathbf{v}+(-b+c)\mathbf{w}=\mathbf{0}.$$

   Since $\mathbf{u},\mathbf{v},\mathbf{w}$ are linearly independent, we have
   $$\begin{cases}a\phantom{{}+b}+c=0\\-a+b\phantom{{}+c}=0\\-b+c=0\end{cases}$$

   The system has only the trivial solution $a=0,b=0,c=0$. Thus S_3 is linearly independent.

   (d) $S_4=\{\mathbf{u},\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}+\mathbf{w}\}$.

   Solution: We have

   $$a\mathbf{u}+b(\mathbf{u}+\mathbf{v})+c(\mathbf{u}+\mathbf{v}+\mathbf{w})=\mathbf{0}\quad\Leftrightarrow\quad(a+b+c)\mathbf{u}+(b+c)\mathbf{v}+c\mathbf{w}=\mathbf{0}.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, we have $a+b+c = b+c = c = 0$. Solving for a, b, c gives the trivial solution $a = 0, b = 0, c = 0$. Thus S_4 is linearly independent.

(e) $S_5 = \{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$.

Solution: We have

$$(\mathbf{u}+\mathbf{v}) + (\mathbf{v}+\mathbf{w}) + (\mathbf{u}+\mathbf{w}) - \mathbf{2}(\mathbf{u}+\mathbf{v}+\mathbf{w}) = \mathbf{0}.$$

So, S_5 is linearly dependent.

3. For each of the following subspaces V, write down a basis for V.

(a) $V = \left\{ \begin{pmatrix} a+b \\ a+c \\ c+d \\ b+d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$.

Solution:

$$\begin{aligned} V &= \left\{ \begin{pmatrix} a+b \\ a+c \\ c+d \\ b+d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\} \\ &= \left\{ a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

This means that the fourth vector is redundant. Notice that the last vector linearly depend on the first 3. After throwing this vector out, one can check that

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for V.

(b) $V = \text{span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$.

Solution: Since the set contains 4 vectors in $\mathbb{R}^3$, it cannot be linearly independent. One can check that $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}\right\}$ is linearly independent. Hence it is a basis for $V = \mathbb{R}^3$.

(c) V is the solution space of the following homogeneous linear system

$$\begin{cases} a_1 \quad\quad\quad + a_3 + a_4 - a_5 = 0 \\ \quad\quad a_2 + a_3 + 2a_4 + a_5 = 0 \\ a_1 + a_2 + 2a_3 + a_4 - 2a_5 = 0 \end{cases}$$

Solution: Solving the homogeneous system:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

Thus the solution set is

$$V = \left\{ s\begin{pmatrix}-1\\-1\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\1\\0\\-1\\1\end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}.$$

It is easy to see that $\left\{\begin{pmatrix}-1\\-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\-1\\1\end{pmatrix}\right\}$ is linearly independent, and is thus a basis for V.

4. For what values of a will $\mathbf{u}_1 = \begin{pmatrix}a\\1\\-1\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-1\\a\\1\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}1\\-1\\a\end{pmatrix}$ form a basis for $\mathbb{R}^3$?

Solution:

$$\left(\begin{array}{ccc|c} a & -1 & 1 & 0 \\ 1 & a & -1 & 0 \\ -1 & 1 & a & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ a & -1 & 1 & 0 \\ -1 & 1 & a & 0 \end{array}\right) \xrightarrow[R_3 + R_1]{R_2 - aR_1}$$

$$\left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ 0 & -1-a^2 & 1+a & 0 \\ 0 & 1+a & a-1 & 0 \end{array}\right) \xrightarrow{-R_2} \left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ 0 & 1+a^2 & -1-a & 0 \\ 0 & 1+a & a-1 & 0 \end{array}\right) \xrightarrow{R_3 - \frac{1+a}{1+a^2}R_2}$$

$$\left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ 0 & 1+a^2 & -1-a & 0 \\ 0 & 0 & \frac{a(a^2+3)}{1+a^2} & 0 \end{array}\right)$$

So $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ will form a basis for $\mathbb{R}^3$ if and only if $a \neq 0$.

Alternatively, the determinant of $\begin{pmatrix} a & -1 & 1 \\ 1 & a & -1 \\ -1 & 1 & a \end{pmatrix}$ is $a(a^2+3)$, which is 0 if and only if $a = 0$.

5. Let U and V be subspaces of $\mathbb{R}^n$. We define the sum $U + V$ to be the set of vectors

$$\{\ \mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U \text{ and } \mathbf{v} \in V\ \}.$$

Suppose $U = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}\right\}$, $V = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}\right\}$.

Solution: Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$.

(a) Is $U \cup V$ a subspace of $\mathbb{R}^4$?

Solution: No. We will show that it is not closed under linear combinations. We have $\mathbf{u}_1 \in U \cup V$ and $\mathbf{v}_1 \in U \cup V$. The sum is

$$\mathbf{u}_1 + \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

We will check that this is neither in U nor in V, and thus it cannot be in the union $U \cup V$.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

shows that $\mathbf{u}_1 + \mathbf{v}_1 \notin U$, and

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

shows that $\mathbf{u}_1 + \mathbf{v}_1 \notin V$. So $\mathbf{u}_1 + \mathbf{v}_1 \notin U \cup V$.

(b) Show that $U+V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

Solution: A vector is in U if and only if it can be written as $\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2$, and a vector is in V if and only if it can be written as $\beta_1\mathbf{v}_1+\beta_2\mathbf{v}_2$. So a vector in $U+V$ if and only if it has the form $\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\beta_1\mathbf{v}_1+\beta_2\mathbf{v}_2$. This shows that $U+V=\text{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2\}$, and hence, $U+V$ is a subspace.

We will also give a general proof that for any subspaces U and V, $U+V$ is a subspace. Indeed clearly $\mathbf{0}\in U+V$. Suppose now $\mathbf{w}_1,\mathbf{w}_2\in U+V$. Then by definition of $U+V$, we can find $\mathbf{u}_1,\mathbf{u}_2\in U$ and $\mathbf{v}_1,\mathbf{v}_2\in V$ such that $\mathbf{w}_1=\mathbf{u}_1+\mathbf{v_1}$ and $\mathbf{w}_2=\mathbf{u}_2+\mathbf{v}_2$. Then for any $\alpha,\beta\in\mathbb{R}$,

$$\alpha\mathbf{w}_1+\beta\mathbf{w}_2=\alpha(\mathbf{u}_1+\mathbf{v_1})+\beta(\mathbf{u}_2+\mathbf{v}_2)=(\alpha\mathbf{u}_1+\beta\mathbf{u}_2)+(\alpha\mathbf{v}_1+\beta\mathbf{v}_2).$$

Since U and V are subspaces, $(\alpha\mathbf{u}_1+\beta\mathbf{u}_2)\in U$ and $(\alpha\mathbf{v}_1+\beta\mathbf{v}_2)\in V$. So $\alpha\mathbf{w}_1+\beta\mathbf{w}_2\in U+V$.
In general, if $U=\text{span}\{\mathbf{u}_1,...,\mathbf{u}_k\}$ and $V=\text{span}\{\mathbf{v}_1,...,\mathbf{v}_l\}$, then $U+V=\text{span}\{\mathbf{u}_1,...,\mathbf{u}_k,\mathbf{v}_1,...,\mathbf{v}_l\}$.

Since $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2\}$ is a spanning set, it suffices to find a linearly independent subset of it to form a basis.

$$\begin{pmatrix}1&1&1&1\\1&2&0&0\\1&2&1&2\\1&1&0&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0&-2\\0&1&0&1\\0&0&1&2\\0&0&0&0\end{pmatrix}.$$

This shows that $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1\}$ is a basis for $U+V$, and hence $\dim(U+V)=3$.

(c) Show that $U+V$ contains U and V. This shows that $U+V$ is a subspace containing $U\cup V$.

Solution: This is clear from (a) since $\text{span}\{\mathbf{u}_1,\mathbf{u}_2\}$ and $\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}$ are subsets of $\text{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2\}$. In fact, $U+V$ is the smallest subspace that contains $U\cup V$.

(d) What are the dimensions of U and V?

Solution:

$$\begin{pmatrix}1&1\\1&2\\1&2\\1&1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}.$$

Hence, $\dim(U)=2$.

$$\begin{pmatrix}1&1\\0&0\\1&2\\0&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}.$$

Hence, $\dim(V) = 2$.

(e) Show that $U \cap V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

Solution: A vector in $\mathbf{w} \in U \cap V$ must be able to be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, and as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In other words, we must be able to find $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\mathbf{w} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix},$$

in other words, we are solving the homogeneous linear system

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} - \beta_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \beta_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & -1 & -2 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So for any choice of $s \in \mathbb{R}$, $\alpha_1 = -2s$ and $\alpha_2 = s$, or $\beta_1 = -2s$ and $\beta_2 = s$ will work, that is, $\mathbf{w} = -s(2\mathbf{u}_1 - \mathbf{u}_2) = -s(2\mathbf{v}_1 - \mathbf{v}_2)$. Hence, $U \cap V = \operatorname{span}\{2\mathbf{u}_1 - \mathbf{u}_2\} = \operatorname{span}\{2\mathbf{v}_1 - \mathbf{v}_2\}$, and this shows that $U \cap V$ is a subspace, with $\dim(U \cap V) = 1$.

In fact, we can show in general that for any subspaces U and V in $\mathbb{R}^n$, $U \cap V$ is a subspace of $\mathbb{R}^n$. Indeed clearly $\mathbf{0} \in U \cap V$, so it is nonempty. Suppose now $\mathbf{w}_1, \mathbf{w}_2 \in U \cap V$ and $\alpha, \beta \in \mathbb{R}$. Then since U and V are subspace, $\alpha\mathbf{w}_1 + \beta\mathbf{w}_2$ belongs to U and V. Hence, it is in the intersection, $\alpha\mathbf{w}_1 + \beta\mathbf{w}_2 \in U \cap V$.

(f) Verify that $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$.

Solution: Indeed, $3 = 2 + 2 - 1$.

Extra problems

1. Let $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ be vectors in $\mathbb{R}^n$ and $\mathbf{P}$ a square matrix of order n.

   (a) Show that if $\mathbf{Pu_1}, \mathbf{Pu_2}, ..., \mathbf{Pu_k}$ are linearly independent, then $\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_k}$ are linearly independent.

   Solution: Note that

$$\begin{aligned} & c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0} \\ \Rightarrow\ & \mathbf{P}(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = \mathbf{P0} \\ \Rightarrow\ & c_1\mathbf{Pu}_1 + c_2\mathbf{Pu}_2 + \cdots + c_k\mathbf{Pu}_k = \mathbf{0}. \end{aligned}$$

   Since $\mathbf{Pu}_1, \mathbf{Pu}_2, ..., \mathbf{Pu}_k$ are linearly independent, we conclude that $c_1 = c_2 = \cdots = c_k = 0$. Thus, $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ are linearly independent.

   (b) Suppose $\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_k}$ are linearly independent.

      (i) Show that if $\mathbf{P}$ is invertible, then $\mathbf{Pu_1}, \mathbf{Pu_2}, ..., \mathbf{Pu_k}$ are linearly independent.

      Solution: Note that

$$\begin{aligned} & c_1\mathbf{Pu}_1 + c_2\mathbf{Pu}_2 + \cdots + c_k\mathbf{Pu}_k = \mathbf{0} \\ \Rightarrow\ & \mathbf{P}(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = \mathbf{0} \\ \Rightarrow\ & c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0} \end{aligned}$$

      where the last implication follows from the fact that $\mathbf{P}$ is invertible. Since $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ are linearly independent, we conclude that $c_1 = c_2 = \cdots = c_k = 0$. Thus, $\mathbf{Pu}_1, \mathbf{Pu}_2, ..., \mathbf{Pu}_k$ are linearly independent.

      (ii) If $\mathbf{P}$ is not invertible, are $\mathbf{Pu_1}, \mathbf{Pu_2}, ..., \mathbf{Pu_k}$ are linearly independent?

      Solution: No conclusion. For example, let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. It is clear that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

      If $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $\mathbf{Pu}_1 = \mathbf{u}_1$ and $\mathbf{Pu}_2 = \mathbf{u}_2$ are linearly independent.

      If $\mathbf{P} = \mathbf{0}_{3\times 3}$, then $\mathbf{Pu}_1 = \mathbf{Pu}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ are linearly dependent.

2. Prove the theorem that $V \subseteq \mathbb{R}^n$ is a subspace if and only if $V = \operatorname{span}\{\mathbf{u}_1, ..., \mathbf{u}_k\}$ for some $\mathbf{u}_i \in \mathbb{R}^n$, $i = 1, ..., k$.

Solution:

($\Leftarrow$) A spanning set contains the origin and is closed under linear combination.

($\Rightarrow$) Suppose $V = \{\mathbf{0}\}$. Then $V = \text{span}\{\mathbf{0}\}$. Hence, we may assume $V \neq \{\mathbf{0}\}$. Since V is a non-trivial subspace, we may pick a nonzero vector $\mathbf{u}_1 \neq \mathbf{0}$ in V. If $V = \text{span}\{\mathbf{u}_1\}$, we are done. Otherwise, pick a $\mathbf{u}_2 \in V \backslash \text{span}\{\mathbf{u}_1\}$. Now $\{\mathbf{u}_1, \mathbf{u}_2\}$ is necessarily linearly independent since $\mathbf{u}_2 \notin \text{span}\{\mathbf{u}_1\}$. Suppose the vectors $\mathbf{u}_1, ..., \mathbf{u}_i$ has been chosen such that they are linearly independent. If $\text{span}\{\mathbf{u}_1, ..., \mathbf{u}_i\} = V$, we are done. Otherwise, pick a $\mathbf{u}_{i+1} \in V \backslash \text{span}\{\mathbf{u}_1, ..., \mathbf{u}_i\}$. We claim that $\mathbf{u}_1, ..., \mathbf{u}_i, \mathbf{u}_{i+1}$ are linearly independent. Suppose

$$c_1\mathbf{u}_1 + \cdots + c_i\mathbf{u}_i + c_{i+1}\mathbf{u}_{i+1} = \mathbf{0}.$$

If $c_{i+1} \neq 0$, then

$$\frac{-c_1}{c_{i+1}}\mathbf{u}_1 + \cdots + \frac{-c_i}{c_{i+1}}\mathbf{u}_i = \mathbf{u}_{i+1},$$

a contradiction to $\mathbf{u}_{i+1} \in V \backslash \text{span}\{\mathbf{u}_1, ..., \mathbf{u}_i\}$. This mean that

$$c_1\mathbf{u}_1 + \cdots + c_i\mathbf{u}_i = \mathbf{0},$$

which shows that $c_1 = \cdots = c_i = 0$ too since $\mathbf{u}_1, ..., \mathbf{u}_i$ are linearly independent. This proves the claim. The process must stop since there can be at most n linearly independent vectors in $\mathbb{R}^n$. Hence there must be a k such that $\text{span}\{\mathbf{u}_1, ..., \mathbf{u}_k\} = V$.

Remark. The proof of ($\Rightarrow$) yields a useful result for a subspace V of $\mathbb{R}^n$. Let $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ be a set of linearly independent vectors in V. If S is not a basis of V, then we could add vectors into the set S so that

$$\{\mathbf{u}_1, ..., \mathbf{u}_k, \mathbf{u}_{k+1}, ..., \mathbf{u}_l\}$$

is a basis of V.

3. (a) Let U and V be subspaces of $\mathbb{R}^n$. Prove that $U \cup V$ is a subspace if and only if $U \subseteq V$ or $V \subseteq U$.

Solution:

($\Leftarrow$) Suppose $U \subseteq V$. Then $U \cup V = V$ is a subspace. The prove for $V \subseteq U$ is analogous.

($\Rightarrow$) Suppose $U \nsubseteq V$ nor $V \nsubseteq U$. We will show that $U \cup V$ is not a subspace. Since $U \nsubseteq V$, find a $\mathbf{u} \in U \backslash V$. Similarly, we can find a $\mathbf{v} \in V \backslash U$. It is clear that $\mathbf{u}, \mathbf{v} \in U \cup V$. We claim that $\mathbf{u} + \mathbf{v} \notin U \cup V$. For otherwise, then it must be either $\mathbf{u} + \mathbf{v} \in U$ or $\mathbf{u} + \mathbf{v} \in V$. Suppose $\mathbf{u} + \mathbf{v} \in U$, then since U is closed under linear combination,

$$\mathbf{v} = \mathbf{u} - (\mathbf{u} + \mathbf{v}) \in U, \text{ a contradiction to the choice of } \mathbf{v}.$$

Similarly, if $\mathbf{u} + \mathbf{v} \in V$, then

$$\mathbf{u} = \mathbf{v} - (\mathbf{u} + \mathbf{v}) \in V, \text{ a contradiction to the choice of } \mathbf{u}.$$

Hence $\mathbf{u} + \mathbf{v} \notin U \cup V$, which shows that $U \cup V$ cannot be a subspace.

(b) Let U and V be subspaces of $\mathbb{R}^n$. Show that $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$.

Solution: Note that in the remark to Question 2, we have shown that given a linearly independent subset $\{\mathbf{u}_1, ..., \mathbf{u}_i\}$ of a subspace V, we can always extend the set into a basis $\{\mathbf{u}_1, ..., \mathbf{u}_i, ..., \mathbf{u}_k\}$ for V.

Let $\{\mathbf{w}_1, ..., \mathbf{w}_r\}$ be a basis for $U \cap V$. Then $\dim(U \cap V) = r$. Note that $\{\mathbf{w}_1, ..., \mathbf{w}_r\}$ is a linearly independent subset of U. Thus we may extend it to be a basis $\{\mathbf{w}_1, ..., \mathbf{w}_r, \mathbf{u}_1, ..., \mathbf{u}_k\}$ of U. Then $\dim(U) = r + k$. Similarly, $\{\mathbf{w}_1, ..., \mathbf{w}_r\}$ is a linearly independent subset of V. So, we may extend it to be a basis $\{\mathbf{w}_1, ..., \mathbf{w}_r, \mathbf{v}_1, ..., \mathbf{v}_l\}$ of V. Then $\dim(V) = r + l$. From question 5b, we have

$$U + V = \text{span}\{\mathbf{w}_1, ..., \mathbf{w}_r, \mathbf{u}_1, ..., \mathbf{u}_k, \mathbf{v}_1, ..., \mathbf{v}_l\}.$$

We claim that $\{\mathbf{w}_1, ..., \mathbf{w}_r, \mathbf{u}_1, ..., \mathbf{u}_k, \mathbf{v}_1, ..., \mathbf{v}_l\}$ is linearly independent. Suppose

$$a_1\mathbf{w}_1 + \cdots + a_r\mathbf{w}_r + b_1\mathbf{u}_1 + \cdots + b_k\mathbf{u}_k + c_1\mathbf{v}_1 + \cdots + c_l\mathbf{v}_l = \mathbf{0}. \tag{1}$$

This is equivalent to

$$a_1\mathbf{w}_1 + \cdots + a_r\mathbf{w}_r + b_1\mathbf{u}_1 + \cdots + b_k\mathbf{u}_k = -(c_1\mathbf{v}_1 + \cdots + c_l\mathbf{v}_l).$$

The left hand side of the equation is a vector in U and the right side is a vector in V, and hence, $c_1\mathbf{v}_1 + \cdots + c_l\mathbf{v}_l \in U \cap V$. Hence, there exists $d_1, ..., d_r$ such that

$$c_1\mathbf{v}_1 + \cdots + c_l\mathbf{v}_l = d_1\mathbf{w}_1 + \cdots + d_r\mathbf{w}_r$$

which is equivalent to

$$c_1\mathbf{v}_1 + \cdots + c_l\mathbf{v}_l - d_1\mathbf{w}_1 - \cdots - d_r\mathbf{w}_r = \mathbf{0}.$$

Since $\{\mathbf{w}_1, ..., \mathbf{w}_r, \mathbf{v}_1, ..., \mathbf{v}_l\}$ is a basis of V, in particular, it is independent, we have $c_1 = \cdots = c_l = 0 = d_1 = \cdots = d_r$. Furthermore,

$$a_1\mathbf{w}_1 + \cdots + a_r\mathbf{w}_r + b_1\mathbf{u}_1 + \cdots + b_k\mathbf{u}_k = -(c_1\mathbf{v}_1 + \cdots + c_l\mathbf{v}_l) = \mathbf{0}.$$

Finally, since $\{\mathbf{w}_1, ..., \mathbf{w}_r, \mathbf{u}_1, ..., \mathbf{u}_k\}$ is a basis for U, we can conclude that $a_1 = \cdots = a_r = 0 = b_1 = \cdots = b_k$. This shows that equation 1 has only the trivial solution, and hence $\{\mathbf{w}_1, ..., \mathbf{w}_r, \mathbf{u}_1, ..., \mathbf{u}_k, \mathbf{v}_1, ..., \mathbf{v}_l\}$ is independent. Therefore

$$\dim(U + V) = r + k + l = (r + k) + (r + l) - r = \dim(U) + \dim(V) - \dim(U \cap V).$$

4. **(MATLAB)** In the last tutorial, we introduced the "binary n-space" $\mathbb{B}^n$, which is governed by the special addition rule: $1+1=0$. Our goal in this problem is to investigate how working with binary vectors can help us detect and correct errors in information transmission.

   When a computer transmits a piece of information (a binary string) to another device, there is always a possibility of an error—that is, the receiving device might receive an incorrect binary string, perhaps due to external interference or noise in the communication channel. One way that a computer might "protect" its message is by adding extra information to the binary string so that the receiving device can detect—and ideally, correct—any errors that may have occured in transmission.

   Consider the following scenario: your friend Annette wants to send you a message $\boldsymbol{u}$. For the sake of simplicity, let's assume that Annette's message contains four bits—that is, $\boldsymbol{u}$ is a vector in $\mathbb{B}^4$ (as opposed to a standard *byte*, which is a vector in $\mathbb{B}^8$). Annette, however, is afraid that a transmission error might send you the wrong message—say, by accidentally changing a 1 to a 0, or a 0 to a 1.

   1. Rather than just sending you the message $\boldsymbol{u}$, Annette instead sends you the 8-vector that results when each bit in $\boldsymbol{u}$ is repeated twice. You receive the vector

   $$\left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right),$$

   where divider bars have been used to split the string up into segments, each representing one bit in Annette's original message. Do you have enough information to decode Annette's original message?

   Solution: We can detect that an error was made in transmitting the fourth bit of Annette's original message. However, we do not have information to recover actual message—it could easily be either $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$.

   As the above situation suggests, a simple—perhaps naïve—error-correcting code would employ *repetition*: sending each bit repeatedly, with the hope that the recipient will be able to spot any errors. Observe, however, that this method significantly increases the required amount of data to be transmitted. In the above example, Annette needed to send out twice as much data—this may be problematic for longer messages, specially since bandwidth is expensive!

   2. Annette, who is running out of mobile data, attempts a different error-correcting code, invented by the 20th Century mathematician Richard Hamming. Recall that, taking the non-zero vectors in $\mathbb{B}^3$ as columns, we can create the *Hamming matrix*

   $$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

   From this, we formed the matrix $\boldsymbol{M}$ by taking the basis vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ for

$S = \{\boldsymbol{x} \in \mathbb{B}^7 \mid \boldsymbol{H}\boldsymbol{x} = \boldsymbol{0}\}$ (i.e., the basis for the null space of $\boldsymbol{H}$) as its columns:

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Explain why the set S is identical to the column space of $\boldsymbol{M}$. Hence, without explicitly calcuating any matrix products, explain why $\boldsymbol{H}(\boldsymbol{M}\boldsymbol{x}) = \boldsymbol{0}$ for all vectors $\boldsymbol{x} \in \mathbb{B}^4$.

Solution: The column space of $\boldsymbol{M}$ is simply the linear span of its columns, which is precisely the set $S = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$. Observe that for an arbitrary vector $\boldsymbol{x} \in \mathbb{B}^4$, we may write the matrix product $\boldsymbol{M}\boldsymbol{x}$ as

$$\boldsymbol{M}\boldsymbol{x} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 + x_4\boldsymbol{v}_4.$$

That is, the vector $\boldsymbol{M}\boldsymbol{x}$ is a linear combination of the columns of $\boldsymbol{M}$ and must hence lie in the column space of $\boldsymbol{M}$. Since the column space of $\boldsymbol{M}$ and the set S are identical, $\boldsymbol{M}\boldsymbol{x}$ must lie in $S = \{\boldsymbol{x} \in \mathbb{B}^7 \mid \boldsymbol{H}\boldsymbol{x} = \boldsymbol{0}\}$ as well—that is, $\boldsymbol{M}\boldsymbol{x}$ is a solution to the homogeneous linear system with coefficient matrix $\boldsymbol{H}$. Thus, $\boldsymbol{H}(\boldsymbol{M}\boldsymbol{x}) = \boldsymbol{0}$.

<u>An alternative method of checking $\boldsymbol{H}(\boldsymbol{M}\boldsymbol{x}) = \boldsymbol{0}$.</u> The columns $\mathbf{v}_1, \ldots, \mathbf{v}_4$ of $\mathbf{M}$ are in the nullspace of $\mathbf{H}$ so

$$\mathbf{H}\mathbf{v}_1 = \cdots = \mathbf{H}\mathbf{v}_4 = \mathbf{0}.$$

We compute

$$\mathbf{HM} = \mathbf{H} \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{H}\boldsymbol{v}_1 & \mathbf{H}\boldsymbol{v}_2 & \mathbf{H}\boldsymbol{v}_3 & \mathbf{H}\boldsymbol{v}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} = \mathbf{0}.$$

Hence

$$\mathbf{H}(\mathbf{M}\mathbf{x}) = (\mathbf{HM})\mathbf{x} = (\mathbf{0})\mathbf{x} = \mathbf{0}.$$

(b) Consider the vector $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{u}$, which is a vector in $\mathbb{B}^7$. The first three entries of $\boldsymbol{v}$ will later be used to detect errors. What are the last four entries of $\boldsymbol{v}$?

Solution: Observe that the last four rows of the matrix $\boldsymbol{M}$ form the identity matrix of order 4. We can partition the matrix $\boldsymbol{M}$ into two blocks—the first three rows forming the matrix $\boldsymbol{A}$, and the last four rows forming $\boldsymbol{I}_4$. Thus,

$$\boldsymbol{v} = \boldsymbol{M}\boldsymbol{u} = \left(\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \boldsymbol{u} = \left(\begin{array}{c} \uparrow \\ \boldsymbol{A}\boldsymbol{u} \\ \downarrow \\ \hline \uparrow \\ \boldsymbol{I}_4\boldsymbol{u} \\ \downarrow \end{array}\right) = \left(\begin{array}{c} * \\ * \\ * \\ \hline \uparrow \\ \boldsymbol{u} \\ \downarrow \end{array}\right),$$

and the last four entries of the vector $\boldsymbol{v}$ contains the original message $\boldsymbol{u}$.

(c) Instead of transmitting the vector $\boldsymbol{u}$, Annette's computer instead sends out $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{u}$. Assume that *at most* one error can occur during transmission.

- What vector will you receive if no errors occur during transmission?

  Solution: If no error occurs during transmission, then all the entries of the vector $\boldsymbol{v}$ will remain unchanged—thus, we will receive the vector $\boldsymbol{v}$.

- Let $\boldsymbol{e}_i$ denote the standard unit vector whose i-th entry is 1 and remaining entries are all 0's. Explain why you would receive a vector of the form $\boldsymbol{v}+\boldsymbol{e}_i$, for some $i \in \{1, \ldots, 7\}$, if an error has occured during transmission.

  Solution: If an error has occurred during transmission, then one component of the vector $\boldsymbol{v}$ would have been altered. That is, for one of the vector's components—say, the i-th component—a 0 would have been turned into a 1, or a 1 to a 0. Since $0 + 1 = 1$ and $1 + 1 = 0$, the error would be akin to adding a 1 to the i-th component of $\boldsymbol{v}$. Thus, we would receive $\boldsymbol{v} + \boldsymbol{e}_i$: the vector $\boldsymbol{v}$ whose i-th entry has been altered.

(d) Let $\boldsymbol{w}$ be the vector you receive on your device. Explain how calculating the matrix product $\boldsymbol{H}\boldsymbol{w}$ will allow you to detect and correct a potential transmission error. [*Hint:* First consider what the vector $\boldsymbol{w}$ would look like if no error has occured, then consider the possibility that an error has occured during transmission.]

Solution: There are two possibilities for $\boldsymbol{w}$, depending on whether or not an error occurs during transmission.

- If no error has occurred, the vector $\boldsymbol{w}$ we receive would simply be the vector $\boldsymbol{v}$ that Annette's computer sends out. Since $\boldsymbol{w} = \boldsymbol{v} = \boldsymbol{M}\boldsymbol{u}$, calculating the matrix product $\boldsymbol{H}\boldsymbol{w}$ will yield

  $$\boldsymbol{H}\boldsymbol{w} = \boldsymbol{H}\boldsymbol{v} = \boldsymbol{H}(\boldsymbol{M}\boldsymbol{u}) = \boldsymbol{0}.$$

  Since the last four entries of the vector $\boldsymbol{v}$ contains Annette's original message $\boldsymbol{u}$, we are easily able to decode Annette's message.

- If an error has occured, then the vector $\boldsymbol{w}$ we receive would be of the form $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{e}_i$. Calculating the matrix product $\boldsymbol{H}\boldsymbol{w}$ yields

  $$\boldsymbol{H}\boldsymbol{w} = \boldsymbol{H}(\boldsymbol{M}\boldsymbol{u} + \boldsymbol{e}_i) = \boldsymbol{H}(\boldsymbol{M}\boldsymbol{u}) + \boldsymbol{H}\boldsymbol{e}_i = \boldsymbol{0} + \boldsymbol{H}\boldsymbol{e}_i = \boldsymbol{H}\boldsymbol{e}_i.$$

In particular, the product $\boldsymbol{Hw} = \boldsymbol{He}_i$ is the i-th column of $\boldsymbol{H}$. Since all the columns of $\boldsymbol{H}$ are distinct, we can deduce the vector $\boldsymbol{e}_i$ by finding the corresponding column in $\boldsymbol{H}$. This tells us which component of the vector $\boldsymbol{v}$ contains an error.

In summary, if $\boldsymbol{Hw}$ is the zero vector, then no error has occurred; if $\boldsymbol{Hw}$ yields the i-th column of $\boldsymbol{H}$, then an error has occurred in transmitting the i-th component of $\boldsymbol{v}$.

(e) Your device receives the vector

$$\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Has an error been made during transmission? Do you have enough information to deduce Annette's original message?

Solution: We calculate $\boldsymbol{Hw}$ on MATLAB, performing our operations modulo 2:

```
>> H=[1 0 0 1 0 1 1; 0 1 0 1 1 0 1; 0 0 1 1 1 1 0];

>> w=[1 0 1 0 1 0 0]';

>> mod(H*w,2)
```

We find that $\boldsymbol{Hw} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Since $\boldsymbol{Hw} \neq \mathbf{0}$, an error has been made. Now $\boldsymbol{Hw}$ is the 7th column of the matrix $\boldsymbol{H}$, and we deduce that we received the vector $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{e}_7$, and an error has been made in transmitting the 7th component of $\boldsymbol{v}$. Thus, we recover the transmission sent from Annette's computer to be

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{e}_7 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Since the last four entries of $\boldsymbol{v}$ is the vector $\boldsymbol{u}$, we deduce that Annette's original message is $\boldsymbol{u} = (0, 1, 0, 1)^T$.