NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 4**

1. Let $A = \{ (1+t, 1+2t, 1+3t) \mid t \in \mathbb{R} \}$ be a subset in $\mathbb{R}^3$.

   (a) Describe A geometrically.

   Solution: A is a line joining the points $(1,1,1)$ and $(2,3,4)$.

   (b) Show that $A = \{ (x,y,z) \mid x+y-z=1 \text{ and } x-2y+z=0 \}$.

   Solution: Let $B = \{(x,y,z) \mid x+y-z=1 \text{ and } x-2y+z=0\}$. Since $x+y-z=1$ and $x-2y+z=0$ are two non-parallel planes, B is the line of intersection of the two planes. To show that $A=B$, it suffices to show that the line A lies on both planes. This is true because $(1+t)+(1+2t)-(1+3t)=1$ and $(1+t)-2(1+2t)+(1+3t)=0$ for all $t \in \mathbb{R}$.

   (c) Write down a matrix equation $\mathbf{Mx} = \mathbf{b}$ where $\mathbf{M}$ is a 3×3 matrix and $\mathbf{b}$ is a 3×1 matrix such that its solution set is A.

   Solution: For example $\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

2. Let $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$.

   (a) If possible, express each of the following vectors as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

   (i) $\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} -4 \\ 6 \\ -13 \\ 4 \end{pmatrix}$

   Solution:

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & x_1 \\ 1 & -1 & 0 & x_2 \\ 0 & 5 & 2 & x_3 \\ 3 & 2 & 1 & x_4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & x_2 \\ 0 & 5 & -1 & x_1 - 2x_2 \\ 0 & 0 & 3 & -x_1 + 2x_2 + x_3 \\ 0 & 0 & 0 & x_1 + 7x_2 + 2x_3 - 3x_4 \end{array}\right)$$

   Suppose $x_1 + 7x_2 + 2x_3 - 3x_4 = 0$. We may proceed

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2x_1+11x_2+x_3}{15} \\ 0 & 1 & 0 & \frac{2x_1-4x_2+x_3}{15} \\ 0 & 0 & 1 & \frac{-x_1+2x_2+x_3}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So a vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ if and only if it satisfies $x_1 + 7x_2 + 2x_3 - 3x_4 = 0$. If that is true, then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$$

where

$$a = \frac{2x_1 + 11x_2 + x_3}{15}, \qquad b = \frac{2x_1 - 4x_2 + x_3}{15}, \qquad c = \frac{-x_1 + 2x_2 + x_3}{3}$$

(i) $2 + 7(3) + 2(-7) - 3(3) = 0$. It is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, $\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix} = 2\mathbf{u}_1 - \mathbf{u}_2 - \mathbf{u}_3$.

(ii) This is clearly a linear combination with $a = b = c = 0$.

(iii) $1 + 7 + 2 - 3 = 7$. It is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(iv) $-4+7(6)+2(-13)-3(4) = 0$. It is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, $\begin{pmatrix} -4 \\ 6 \\ -13 \\ 4 \end{pmatrix} = 3\mathbf{u}_1 - 3\mathbf{u}_2 + \mathbf{u}_3$.

Alternatively, we can solve all 4 simultaneous,

$$\left(\begin{array}{ccc|cccc} 2 & 3 & -1 & 2 & 0 & 1 & -4 \\ 1 & -1 & 0 & 3 & 0 & 1 & 6 \\ 0 & 5 & 2 & -7 & 0 & 1 & -13 \\ 3 & 2 & 1 & 3 & 0 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

(b) Is it possible to find 2 vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that they are not a multiple of each other, and both are not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

Solution: Yes, for example, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

3. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x - y - z = 0 \right\}$ be a subset of $\mathbb{R}^3$.

(a) Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\}$. Show that $\text{span}(S) = V$.

Solution: Since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ satisfy the equation $x-y-z=0$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \in V$ and hence $\text{span}(S) \subseteq V$.

Note that a general solution of $x - y - z = 0$ is $x = s + t$, $y = s$, $z = t$ where $s, t \in \mathbb{R}$. Hence, $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. So, to check $V \subseteq \text{span}(S)$,

$$\left(\begin{array}{cc|cc} 1 & 5 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Since the system is consistent, it shows that $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq \text{span}(S)$.

So $V \subseteq \text{span}(S)$.

We have shown that $\text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\} = V$.

(b) Let $T = S \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Show that $\text{span}(T) = \mathbb{R}^3$.

Solution: Consider the row-echelon form of the matrix:

$$\begin{pmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}.$$

Since there are no zero rows in $\mathbf{R}$, we conclude that T spans $\mathbb{R}^3$.

4. Which of the following sets S spans $\mathbb{R}^4$?

(i) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(ii) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(iii) $S = \left\{ \begin{pmatrix} 6 \\ 4 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -2 \\ -1 \end{pmatrix} \right\}$.

(iv) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

Solution:

(i)

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So S spans $\mathbb{R}^4$.

(iii)

$$\begin{pmatrix} 6 & 2 & 3 & 5 & 0 \\ 4 & 0 & 2 & 6 & 4 \\ -2 & 0 & -1 & -3 & -2 \\ 4 & 1 & 2 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{R}$$

Since there is a row of zeros in $\mathbf{R}$, S does not span $\mathbb{R}^4$.

(iv)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} = \mathbf{R}.$$

So S spans $\mathbb{R}^4$.

5. Determine whether $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and/or $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if

(a) $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(b) $\mathbf{u}_1 = \begin{pmatrix}1\\6\\4\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}2\\4\\-1\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-1\\2\\5\end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix}1\\-2\\-5\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\8\\9\end{pmatrix}$.

Solution:

(a)

$$\left(\begin{array}{ccc|cc} 2 & -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & -1 & 1 \\ 0 & -1 & 9 & -5 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & -\frac{9}{2} & 3 & 0 \\ 0 & 1 & -9 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \not\subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

$$\left(\begin{array}{cc|ccc} 1 & 0 & 2 & -1 & 0 \\ -1 & 1 & -2 & 1 & 0 \\ -5 & 1 & 0 & -1 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & \frac{9}{10} \end{array}\right).$$

So $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \not\subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b)

$$\left(\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

$$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. We conclude that the two linear spans are equal.

6. Determine which of the following sets are subspaces. For those sets that are subspaces, express the set as a linear span. For those sets that are not, explain why.

(a) $S = \left\{ \begin{pmatrix}p\\q\\p\\q\end{pmatrix} \middle| \; p, q \in \mathbb{R} \right\}$.

Solution: $S = \text{span}\left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \right\}$.

(b) $S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \ a \geq b \text{ or } b \geq c \right\}$.

Solution: S is not a linear span (thus not a subspace) since $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is in S but $(-1)\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is not.

(c) $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \ 4x = 3y \text{ and } 2x = -3w \right\}$.

Solution: $S = \left\{ \begin{pmatrix} x \\ \frac{4x}{3} \\ z \\ -\frac{2x}{3} \end{pmatrix} \middle| \ x, z \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ \frac{4}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(d) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \ \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right\}$.

Solution:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = a - c - d.$$

So the set S can be rewritten as

$$\begin{aligned} S &= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \ a - c - d = 0 \right\} = \left\{ \begin{pmatrix} s+t \\ u \\ s \\ t \end{pmatrix} \middle| \ s, t, u \in \mathbb{R} \right\} \\ &= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

(e) $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \ w + x = y + z \right\}$.

Solution: S can be rewritten as $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \, w + x - y - z = 0 \right\}$. Solving the equation $w + x - y - z = 0$, we have

$$S = \left\{ \begin{pmatrix} -s+t+u \\ s \\ t \\ u \end{pmatrix} \middle| \, s, t, u \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(f) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \, ab = cd \right\}$.

Solution: S is not a linear span (thus not a subspace) since $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ are vectors in S but $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ is not.

(g) S is the solution set of $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.

Solution: Solving $\mathbf{Ax} = \mathbf{0}$, we have

$$\left(\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \end{array} \right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

So an arbitrary vector in the solution set of $\mathbf{Ax} = \mathbf{0}$ is

$$\left\{ \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\}.$$

So we can rewrite S as

$$\operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Extra problems

1. (a) Suppose $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$. Show that $\det(\mathbf{A}) = \det(\mathbf{D})$.

   Solution: $\det(\mathbf{A}) = \det(\mathbf{PDP}^{-1}) = \det(\mathbf{P})\det(\mathbf{D})\det(\mathbf{P}^{-1}) = \det(\mathbf{P})\det(\mathbf{P})^{-1}\det(\mathbf{D})$ $= \det(\mathbf{D})$. The second equality follows from commutativity of multiplication of real numbers, and that $\det(\mathbf{P}^{-1}) = \det(\mathbf{P})^{-1}$.

   (b) Suppose $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$ and $\mathbf{D}$ is a diagonal matrix. Show that $\mathbf{A}$ is invertible if and only if all the diagonal entries of $\mathbf{D}$ is nonzero.

   Solution: From (a), we have $\det(\mathbf{A}) = \det(\mathbf{D}) = d_{11}d_{22}\cdots d_{nn}$, where d_{ii} is the i-th diagonal entry of $\mathbf{D}$. Thus $\det(\mathbf{A})$ is nonzero if any only if $d_{ii} \neq 0$ for all $i = 1, ..., n$.

   (c) Recall that a square matrix $\mathbf{A}$ is nilpotent if there is a positive integer k such that $\mathbf{A}^k = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\det(\mathbf{A}) = 0$.

   Solution: $0 = \det(\mathbf{A}^k) = \det(\mathbf{A})^k \Rightarrow \det(\mathbf{A}) = 0$ since $\det(\mathbf{A})$ is a real number.

   (d) A square matrix is an *orthogonal* matrix if $\mathbf{A}^T = \mathbf{A}^{-1}$. Show that if $\mathbf{A}$ is orthogonal, then $\det(\mathbf{A}) = \pm 1$.

   Solution: Follows from $1 = \det(\mathbf{A}^{-1}\mathbf{A}) = \det(\mathbf{A}^T)\det(\mathbf{A}) = \det(\mathbf{A})^2$, since $\det(\mathbf{A}^T) = \det(\mathbf{A})$. Alternatively, $\det(\mathbf{A})^{-1} = \det(\mathbf{A}^{-1}) = \det(\mathbf{A}^T) = \det(\mathbf{A})$ tells us that $\det(\mathbf{A}) = \pm 1$.

2. (a) Show that the solution set to any homogeneous linear system

   $$V = \left\{ \ \mathbf{v} \in \mathbb{R}^n \ \middle| \ \mathbf{Av} = \mathbf{0} \ \right\}$$

   is a subspace.

   Solution: We will show that the solution set $V = \left\{ \ \mathbf{v} \in \mathbb{R}^n \ \middle| \ \mathbf{Av} = \mathbf{0} \ \right\}$ is nonempty and is closed under linear combinations. It is obviously nonempty, since it contains the trivial solution $\mathbf{A0} = \mathbf{0}$. Suppose $\mathbf{u}, \mathbf{v} \in V$. Then for any $\alpha, \beta \in \mathbb{R}$, $\mathbf{A}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{Au} + \beta\mathbf{Av} = \alpha\mathbf{0} + \beta\mathbf{0} = \mathbf{0}$. So $(\alpha\mathbf{u} + \beta\mathbf{v}) \in V$. Hence, V is a subspace of $\mathbb{R}^n$.

   (b) Let $V = \left\{ \ \mathbf{v} \in \mathbb{R}^n \ \middle| \ \mathbf{Av} = \mathbf{0} \ \right\}$. Show that if $\mathbf{Ax} = \mathbf{b}$ is consistent, then the solutions set is

   $$\mathbf{u}_p + V = \left\{ \ \mathbf{u}_p + \mathbf{v} \ \middle| \ \mathbf{v} \in V \ \right\},$$

   where $\mathbf{u}_p$ is a particular solution of $\mathbf{Ax} = \mathbf{b}$. (cf. Tutorial 1 Question 1)

Solution: Suppose $\mathbf{Ax} = \mathbf{b}$ is consistent. Let $\mathbf{u}_p$ be a particular solution, and let $S = \{ \mathbf{u} \in \mathbb{R}^n \mid \mathbf{Au} = \mathbf{b} \}$ be the solution set.
Suppose $\mathbf{u}$ is a vector in $\mathbf{u}_p + V$, that is, $\mathbf{u} = \mathbf{u}_p + \mathbf{v}$, for some $\mathbf{v} \in V$. Note that $\mathbf{Av} = \mathbf{0}$. Then $\mathbf{A}(\mathbf{u}_p + \mathbf{v}) = \mathbf{Au}_p + \mathbf{Av} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. So, $\mathbf{u} = \mathbf{u}_p + \mathbf{v}$ is in S. This shows that $\mathbf{u}_p + V \subseteq S$.
Now suppose $\mathbf{u}$ is a vector in S, that is, $\mathbf{Au} = \mathbf{b}$. Let $\mathbf{v} = \mathbf{u} - \mathbf{u}_p$. Then $\mathbf{Av} = \mathbf{A}(\mathbf{u} - \mathbf{u}_p) = \mathbf{Au} - \mathbf{Au}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Hence, $\mathbf{v} \in V$. So, $\mathbf{u} = \mathbf{u}_p + (\mathbf{u} - \mathbf{u}_p) = \mathbf{u}_p + \mathbf{v}$ is in $\mathbf{u}_p + V$. This show that $S \subseteq \mathbf{u}_p + V$ too, and hence, they are equal.

A subset of $\mathbb{R}^n$ is called an *affine space* if it is of the form $\{ \mathbf{u} + \mathbf{v} \mid \mathbf{v} \in V \}$ for some subspace $V \subseteq \mathbb{R}^n$. Geometrically, an affine space is a subset of $\mathbb{R}^n$ that is parallel to a subspace. This exercise shows that the solution set to the linear system $\mathbf{Ax} = \mathbf{b}$ is an affine space $\{ \mathbf{u}_p + \mathbf{v} \mid \mathbf{v} \in V \}$, where V is the solutions to homogeneous linear system $\mathbf{Ax} = \mathbf{0}$, and $\mathbf{u}_p$ is any particular solution.

3. Determine which of the following statements are true. Justify your answer.

   (a) If S_1 and S_2 are two subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$.

   Solution: False. For example, consider $S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$. Then $\text{span}(S_1) = \mathbb{R}^2 = \text{span}(S_1)$

   $$\text{span}(S_1 \cap S_2) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \neq \mathbb{R}^2 = \text{span}(S_1) \cap \text{span}(S_2).$$

   (b) If S_1 and S_2 are two subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) \cup \text{span}(S_2)$.

   Solution: False. For example, consider $S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ and $S_2 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Then $\text{span}(S_1 \cup S_2) = \mathbb{R}^2$, while $\text{span}(S_1) \cup \text{span}(S_2)$ is the union of 2 lines, the x and the y-axis in $\mathbb{R}^2$.

4. In computers, information is stored and processed in the form of strings of binary digits, 0 and 1. For this exercise, we will work in the "world" of binary digits

   $$\mathbb{B} = \{0, 1\}.$$

   Addition in $\mathbb{B}$ works just as it does in $\mathbb{R}$, save for one special rule:

   $$1 + 1 = 0.$$

   We can similarly perform scalar multiplication in $\mathbb{B}$—however, note that in our "binary world", we only have two possible scalars: 0 and 1 (as opposed to any real number).

   Remark. The special rule for binary addition is equivalent to performing our standard operations **modulo 2**. That is, in our "binary world," we evaluate a sum according to its remainder when divided by 2: if the remainder is 0 (i.e., when a number is even), then it corresponds to the binary digit 0, and if the remainder is 1 (i.e., when a number is odd), then it corresponds to the binary digit 1.

1. Using the rules on the basic operations in $\mathbb{B}$, complete the addition and multiplication tables below.

$+$	0	1
0		
1		

$\times$	0	1
0		
1		

Solution:

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

2. Recall that we created the Euclidean space $\mathbb{R}^n$ by taking the set of all n-vectors with real components (i.e., with components in $\mathbb{R}$). We can create the set $\mathbb{B}^n$ in a similar fashion, by taking the set of all n-vectors whose components are binary digits, 0 or 1. Observe, then, that the basic properties of addition and scalar multiplication in $\mathbb{R}^n$ directly apply to $\mathbb{B}^n$, as long as we remember that $1 + 1 = 0$ and the only scalars we are allowed to multiply by are 0 and 1.

(a) Consider the Euclidean 3-space $\mathbb{R}^3$, which has infinitely many vectors. How many vectors does $\mathbb{B}^3$ have?

Solution: We can list out the vectors in $\mathbb{B}^3$, noting that the elements of each vector can only either be a 0 or a 1.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We find that $\mathbb{B}^3$ only contains eight vectors.

(b) A *byte*—the fundamental unit of data used by many computers—is a string of 8 binary digits. Observe that we can treat each byte as a vector in $\mathbb{B}^8$. How many distinct bytes exist; that is, how many vectors are there in $\mathbb{B}^8$? How does this compare to Euclidean 8-space $\mathbb{R}^8$?

Solution: For an arbitrary string of 8 binary digits, we have two choices for each of the vector's entries: either 0 or 1. There are thus a total of $2^8 = 256$ different ways we can create a byte, and the set $\mathbb{B}^8$ contains 256 vectors, as opposed to the infinitely many vectors in $\mathbb{R}^8$.

(c) The Euclidean n-space $\mathbb{R}^n$ has infinitely many vectors. More generally, how many vectors are there in $\mathbb{B}^n$?

Solution: The set $\mathbb{B}^n$ has 2^n vectors.

For the purposes of this exercise, you may assume that $\mathbb{B}^n$ has all the properties of a subspace—that is, $\mathbb{B}^n$ is closed under addition and scalar multiplication. (Try to prove this yourself!)

3. To get a sense of how vectors work in $\mathbb{B}^n$, we take a simple example. Let's begin by working in $\mathbb{B}^3$—the set of all 3-vectors whose components are binary digits.

   (a) Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ be the set of standard unit vectors in $\mathbb{R}^3$. Show that S forms a basis for $\mathbb{B}^3$.

   Solution: To show that S is a basis for $\mathbb{B}^3$, we need to show that $\operatorname{span}(S) = \mathbb{B}^3$ and that S is linearly independent. To show that S spans $\mathbb{B}^3$, consider the vector equation

   $$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ where } x, y, z \in \mathbb{B}.$$

   Simplifying the left-hand side of the equation, we have

   $$(c_1, c_2, c_3) = (x, y, z).$$

   Note that we are taking the scalars c_1, c_2, c_3 from $\mathbb{B} = \{0, 1\}$ as well; thus, S spans $\mathbb{B}$. In the case when $x = y = z = 0$, we require that $c_1 = c_2 = c_3 = 0$. Thus, S must be linearly independent as well.