MA1522 Linear Algebra for Computing

Lecture 4: Invertible Matrices and LU Decomposition

Yang Yue

Department of Mathematics
National University of Singapore

3 February, 2025

Outline

Questions posed in Dr.Teo's Lectures

Further Questions (if time permits)

Question in Section 2.4

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

By row operations, we have:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right).$$

Is $\mathbf{A}$ invertible? If it is, what is its inverse?

Note this question was posed after Slide 83: If $\mathbf{A}$ is invertible, then we can find its inverse by

$$\left(\mathbf{A} \mid \mathbf{I}\right) \xrightarrow{RREF} \left(\mathbf{I} \mid \mathbf{A}^{-1}\right).$$

Slide 115: Algorithm for Finding Inverse

Below is an algorithm to testing if a matrix is invertible, and finding its inverse if it is invertible.

Let $\mathbf{A}$ be an $n \times n$ matrix.

Step 1: Form the $n \times 2n$ (augmented) matrix $\left(\begin{array}{c|c} \mathbf{A} & \mathbf{I}_n \end{array} \right)$.

Step 2: Reduce the matrix $\left(\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array} \right) \longrightarrow \left(\begin{array}{c|c} \mathbf{R} & \mathbf{B} \end{array} \right)$ to its RREF (or REF).

Step 3: If RREF $\mathbf{R} \neq \mathbf{I}$ (or REF has a zero row), then $\mathbf{A}$ is not invertible. If RREF $\mathbf{R} = \mathbf{I}$ (or REF has no zero row), $\mathbf{A}$ is invertible with inverse $\mathbf{A}^{-1} = \mathbf{B}$.

Answer to Question in Section 2.4.

Q:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right).$$

Is $\mathbf{A}$ invertible? If it is, what is its inverse?

By Slide 115, we conclude that $\mathbf{A}$ is not invertible.

Question in Section 2.5

Find the inverse of this elementary matrix

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Slide 93: Elementary Matrices

Definition

A square matrix $\mathbf{E}$ of order n is called an *elementary matrix* if it can be obtained from the identity matrix $\mathbf{I}_n$ by performing a single elementary row operation

$$\mathbf{I}_n \xrightarrow{r} \mathbf{E},$$

where r is an elementary row operation. The elementary row operation is said to be the row operation corresponding to the elementary matrix.

Slide 94: Elementary Matrices and Elementary Row Operations

Let $\mathbf{A}$ be an $n \times m$ matrix and let $\mathbf{E}$ be the $n \times n$ elementary matrix corresponding to the elementary row operation r. Then the product $\mathbf{EA}$ is the resultant of performing the row operation r on $\mathbf{A}$,

$$\mathbf{A} \xrightarrow{r} \mathbf{EA}.$$

That is, performing elementary row operations is equivalent to premultiplying by the corresponding elementary matrix.

For example, $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & -1 \\ 2 & 1 & 4 & 2 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 4 & 5 & -1 \\ 2 & 1 & 4 & 2 \end{pmatrix}$

corresponds to $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & -1 \\ 2 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 4 & 5 & -1 \\ 2 & 1 & 4 & 2 \end{pmatrix}$.

Slide 101: Inverse of Elementary Matrices

Theorem

Every elementary matrix $\mathbf{E}$ *is invertible. The inverse* $\mathbf{E}^{-1}$ *is the elementary matrix corresponding to the reverse of the row operation corresponding to* $\mathbf{E}$.

(i)

$$\mathbf{I}_n \xrightarrow{R_i + cR_j} \mathbf{E} \xrightarrow{R_i - cR_j} \mathbf{I}_n \quad \Rightarrow \quad \mathbf{E} : R_i + cR_j,\ \mathbf{E}^{-1} : R_i - cR_j.$$

(ii)

$$\mathbf{I}_n \xrightarrow{R_i \leftrightarrow R_j} \mathbf{E} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{I}_n \quad \Rightarrow \quad \mathbf{E} : R_i \leftrightarrow R_j,\ \mathbf{E}^{-1} : R_i \leftrightarrow R_j.$$

(iii)

$$\mathbf{I}_n \xrightarrow{cR_i} \mathbf{E} \xrightarrow{\frac{1}{c}R_i} \mathbf{I}_n \quad \Rightarrow \quad \mathbf{E} : cR_i,\ \mathbf{E}^{-1} : \frac{1}{c}R_i.$$

Answer to Question in Section 2.5

Find the inverse of this elementary matrix

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Answer: By slide 101, the inverse is

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Question in Section 2.6.

Let **A** and **B** be $n \times m$ matrices. Show that **A** and **B** are row equivalent if and only if $\mathbf{B} = \mathbf{PA}$ for some invertible $n \times n$ matrix **P**.

Recall:

- Two matrices are *row equivalent* if we can obtain one matrix from the other by performing a series elementary row operations.
- Performing elementary row operations is equivalent to premultiplying by the corresponding elementary matrix.
- Every elementary matrix **E** is invertible. Product of invertible matrices is invertible.
- **A** is invertible iff **A** can be expressed as a product of elementary matrices.

Theorem (Equivalent statements of invertibility)

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

(i) $\mathbf{A}$ *is invertible.*

(ii) $\mathbf{A}^T$ *is invertible.*

(iii) *(left inverse) There is a matrix* $\mathbf{B}$ *such that* $\mathbf{BA} = \mathbf{I}$.

(iv) *(right inverse) There is a matrix* $\mathbf{B}$ *such that* $\mathbf{AB} = \mathbf{I}$.

(v) *The reduced row-echelon form of* $\mathbf{A}$ *is the identity matrix.*

(vi) $\mathbf{A}$ *can be expressed as a product of elementary matrices.*

(vii) *The homogeneous system* $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution.*

(viii) *For any* $\mathbf{b}$, *the system* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

Answer to Question in Section 2.6.

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times m$ matrices. Show that $\mathbf{A}$ and $\mathbf{B}$ are row equivalent if and only if $\mathbf{B} = \mathbf{PA}$ for some invertible $n \times n$ matrix $\mathbf{P}$.

Proof of the "if" direction: Suppose that $\mathbf{B} = \mathbf{PA}$ for some invertible $n \times n$ matrix $\mathbf{P}$. Then $\mathbf{P}$ is a product of some elementary matrices, say

$$\mathbf{P} = \mathbf{E_m E_{m-1} \cdots E_1}.$$

By the correspondence between elementary row operations and premultiplication of elementary matrices, $\mathbf{B}$ is obtained from $\mathbf{A}$ by performing the sequence of row operations corresponding to $\mathbf{E_1}, \mathbf{E_2}, \ldots, \mathbf{E_m}$ one by one. Thus, $\mathbf{A}$ and $\mathbf{B}$ are row equivalent.

Answer to Question in Section 2.6.

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times m$ matrices. Show that $\mathbf{A}$ and $\mathbf{B}$ are row equivalent if and only if $\mathbf{B} = \mathbf{PA}$ for some invertible $n \times n$ matrix $\mathbf{P}$.

Proof of the "only if" direction: Suppose that $\mathbf{A}$ and $\mathbf{B}$ are row equivalent and $\mathbf{B}$ is obtained from $\mathbf{A}$ by performing the sequence of row operations $r_1, r_2, \ldots, r_m$. By the correspondence between elementary row operations and premultiplication of elementary matrices, $\mathbf{B} = \mathbf{E_m} \cdots \mathbf{E_2 E_1 A}$, where $\mathbf{E_k}$ is the elementary matrix corresponding to the row operation r_k. Note that each $\mathbf{E_k}$ is invertible and their product $\mathbf{E_m} \cdots \mathbf{E_2 E_1}$ is also an invertible $n \times n$ matrix $\mathbf{P}$. Thus, $\mathbf{B} = \mathbf{PA}$.

Question One in Section 2.7

1. What if we use other row operations that are not of the type $R_i + cR_j$ for some $i > j$ and real number c? Is $\mathbf{L}$ still a unit lower triangular matrix?

2. Is it possible to reduce any matrix $\mathbf{A}$ to a row-echelon form with only the type of row operations mentioned above?

Question Two in Section 2.7

Let $\mathbf{A} = \mathbf{LU}$ be an LU factorization of $\mathbf{A}$.

1. Show that the system $\mathbf{Ly} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$.

2. Is every matrix LU factorizable? If not, provide a counter-example.

Slide 123: LU Factorization

Definition

A square matrix $\mathbf{L}$ is a *unit lower triangular* matrix if $\mathbf{L}$ is a lower triangular matrix with 1 in the diagonal entries.

An *LU factorization* of an $m \times n$ matrix $\mathbf{A}$ is the decomposition

$$\mathbf{A} = \mathbf{LU},$$

where $\mathbf{L}$ is a unit lower triangular matrix, and $\mathbf{U}$ is a row-echelon form of $\mathbf{A}$.

If such LU factorization exists for $\mathbf{A}$, we say that $\mathbf{A}$ is *LU factorizable*.

Slide 125: Algorithm to LU Factorization

Suppose $\mathbf{A} \xrightarrow{r_1, r_2, \ldots, r_k} \mathbf{U}$, where each row operation r_l is of the form $R_i + cR_j$ for some $i > j$ and real number c, and $\mathbf{U}$ is a row-echelon form of $\mathbf{A}$. Let $\mathbf{E}_i$ be the elementary matrix corresponding for r_i, for $i = 1, 2, ..., k$. Then

$$\mathbf{E}_k \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{U} \quad \Rightarrow \quad \mathbf{A} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \cdots \mathbf{E}_k^{-1} \mathbf{U} = \mathbf{LU},$$

where $\mathbf{L} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \cdots \mathbf{E}_k^{-1}$. Then

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{pmatrix} \begin{pmatrix} * & & & & & \cdots & * \\ 0 & \cdots & 0 & * & & \cdots & * \\ \vdots & & & & & & \vdots \\ 0 & \cdots & & & & \cdots & * \end{pmatrix}$$

is an LU factorization of $\mathbf{A}$.
In this case, we could obtain $\mathbf{L}$ quickly without computing $\mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \cdots \mathbf{E}_k^{-1}$. For each row operation $r_l = R_i + c_l R_j$ for some $i > j$ and real number c_l, we will put $-c_l$ in the (i, j)-entry of $\mathbf{L}$.

Question One in Section 2.7, part 1

Q: What if we use other row operations that are not of the type $R_i + cR_j$ for some $i > j$ and real number c? Is **L** still a unit lower triangular matrix?

Answer: The other types of row operations are:

- Exchanging two rows: $R_i \leftrightarrow R_j$.
- Multiply a nonzero constant a to a row: aR_i.
- It is of the type $R_i + cR_j$ but for some $i < j$.

We check one by one that **L** is no longer a unit lower triangular matrix, except the trivial cases when $a = 1$ and when $c = 0$.

Question One in Section 2.7, part 2

Q: Is it possible to reduce any matrix $\mathbf{A}$ to a row-echelon form with only the type of row operations mentioned above?

Answer: No. For example, let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Observe that the only row operations of the type mentioned above is $R_2 + cR_1$, and the resulting matrix is of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix}$$

which is not in REF.

Question Two in Section 2.7, part 1

Q: Let $\mathbf{A} = \mathbf{LU}$ be an LU factorization of $\mathbf{A}$. Show that the system $\mathbf{Ly} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$.

Answer: Observe that any unit lower triangular matrix is row equivalent to the identity matrix. Thus $\mathbf{L}$ is invertible.

(If you have reached section 2.10, you can use that $\det(\mathbf{L}) = 1 \neq 0$.)

Thus, by slide 117 item (viii), the system $\mathbf{Ly} = \mathbf{b}$ has a unique solution, which is $\mathbf{y} = \mathbf{L}^{-1}\mathbf{b}$.

Question Two in Section 2.7, part 2

Q: Is every matrix LU factorizable? If not, provide a counter-example.

Answer: No, we can use the same example $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as before.

If
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & c \\ 0 & d \end{pmatrix},$$
then we have $b = 0$ and $ab = 1$, which is impossible.

Extra Question 1

Suppose the $n \times n$ matrix $\mathbf{A}$ is not invertible. Let $\mathbf{b}$ be a $n \times 1$ vector. Which of the following statements is true?

(a) $\mathbf{Ax} = \mathbf{b}$ must have a unique solution.

(b) $\mathbf{Ax} = \mathbf{b}$ may have a unique solution.

(c) $\mathbf{Ax} = \mathbf{b}$ must have infinitely many solutions.

(d) $\mathbf{Ax} = \mathbf{b}$ must be inconsistent.

(e) None of the above statements are true.

You should read the word “must” as “for all $\mathbf{b}$, and “may” as “for some $\mathbf{b}$.

Extra Question 1, part (a)

Q: Suppose the $n \times n$ matrix $\mathbf{A}$ is not invertible. Let $\mathbf{b}$ be a $n \times 1$ vector. Which of the following statements is true?

(a) $\mathbf{Ax} = \mathbf{b}$ must have a unique solution.

Recall (viii) of the theorem on Slide 117 says: $\mathbf{A}$ is invertible iff for any $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

Answer: Since $\mathbf{A}$ is not invertible, (a) is false.

Extra Question 1, part (b)

Q: Suppose the $n \times n$ matrix $\mathbf{A}$ is not invertible. Let $\mathbf{b}$ be a $n \times 1$ vector. Which of the following statements is true?

(b) $\mathbf{Ax} = \mathbf{b}$ may have a unique solution.

Answer: Since the square matrix $\mathbf{A}$ is not invertible, its row echelon form must have zero rows. When we follow the same reduction to the augmented matrix $(\mathbf{A}|\mathbf{b})$, we will see the bottom of the rows like $(0, 0, \ldots, 0|c_i)$. If one of such c_i is nonzero, then $\mathbf{Ax} = \mathbf{b}$ has no solution; otherwise, it has infinitely many solutions. In any case, $\mathbf{Ax} = \mathbf{b}$ has no unique solution. The statement is false.

Note: This argument gives us a fact similar to (viii) of the theorem on Slide 117, namely: For square matrix $\mathbf{A}$,
(viii)$'$ $\mathbf{A}$ is invertible iff for some $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

Extra Question 1, parts (c), (d) and (e)

Suppose the $n \times n$ matrix $\mathbf{A}$ is not invertible. Let $\mathbf{b}$ be a $n \times 1$ vector. Which of the following statements is true?

(c) $\mathbf{Ax} = \mathbf{b}$ must have infinitely many solutions.

(d) $\mathbf{Ax} = \mathbf{b}$ must be inconsistent.

(e) None of the above statements are true.

Answer: Part (c) and (d) are false, as showed by the following examples, respectively:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus only statement (e) is true.

Extra Challenge

Suppose $\mathbf{A}_1$, $\mathbf{A}_2$,...,$\mathbf{A}_k$ are $n \times n$ matrices such that

$$\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k\mathbf{x} = \mathbf{0}$$

has nontrivial solutions. What can you conclude about the homogeneous system

$$\mathbf{A}_k \cdots \mathbf{A}_2\mathbf{A}_1\mathbf{x} = \mathbf{0}?$$

Recall (vii) of the theorem on Slide 117 says: $\mathbf{A}$ is invertible iff the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Also recall that (in Lecture 3) For square matrices $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{AB}$ is invertible, then both $\mathbf{A}$ and $\mathbf{B}$ are invertible.

Solution of Extra Challenge

Suppose $\mathbf{A}_1$, $\mathbf{A}_2$,...,$\mathbf{A}_k$ are $n \times n$ matrices such that

$$\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k\mathbf{x} = \mathbf{0}$$

has nontrivial solutions. What can you conclude about the homogeneous system

$$\mathbf{A}_k \cdots \mathbf{A}_2\mathbf{A}_1\mathbf{x} = \mathbf{0}?$$

Answer: We conclude that $\mathbf{A}_k \cdots \mathbf{A}_2\mathbf{A}_1\mathbf{x} = \mathbf{0}$ must also have nontrivial solutions.

If not, $\mathbf{A}_k \cdots \mathbf{A}_2\mathbf{A}_1$ is invertible. Hence every $\mathbf{A}_i$ is invertible. Thus $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k$ would be invertible. By (vii) again,

$$\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k\mathbf{x} = \mathbf{0}$$

has only trivial solutions. Contradiction.

Extra Question 2

Let $\mathbf{A} = \mathbf{LU}$ be a LU factorization of $\mathbf{A}$. Which of the following statements are true?

(a) If $\mathbf{A}$ is $n \times n$, then $\mathbf{A}$ is invertible.

(b) If $\mathbf{A}$ is $m \times n$, then $m \geq n$.

(c) $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b}$.

Extra Question 2, part (a)

Q: Let $\mathbf{A} = \mathbf{L}\mathbf{U}$ be a LU factorization of $\mathbf{A}$. T or F?

(a) If $\mathbf{A}$ is $n \times n$, then $\mathbf{A}$ is invertible.

Note that we have argued $\mathbf{L}$, being unit lower triangular, must be invertible.

We only know that $\mathbf{U}$ is in REF, which may or may not be invertible.

Answer: False. For example, $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ is not invertible, but has LU factorization

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Extra Question 2, part (b)

Q: Let $\mathbf{A} = \mathbf{LU}$ be a LU factorization of $\mathbf{A}$. T or F?

(b) If $\mathbf{A}$ is $m \times n$, then $m \geq n$.

Answer: False. For example,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Extra Question 2, part (c)

Let $\mathbf{A} = \mathbf{LU}$ be a LU factorization of $\mathbf{A}$. T or F?

(c) $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b}$.

Answer: False. For example, let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\begin{cases} x + y = 1 \\ 2x + 2y = 1 \\ 3x + 3y = 1 \end{cases}$$

is inconsistent, but

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$