NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 11**

1. (i) Determine whether the following are linear transformations.
   (ii) Write down the standard matrix for each other the linear transformations.
   (iii) Find a basis for the range for each of the linear transformations.
   (iv) Find a basis for the kernel for each of the linear transformations.

   (a) $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   Solution: $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\begin{pmatrix} 1 \\ -1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. The standard matrix is $\mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The range of T_1 is the column space of $\mathbf{A}_1$. Since the columns of $\mathbf{A}_1$ are linearly independent, a basis for the column space is $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$. Alternatively, one might observe that $\mathbf{A}_1$ is invertible, and hence the range is the whole $\mathbb{R}^2$, and one might choose the standard basis to be the basis of the range of T_1. The kernel of T_1 is the null space of $\mathbf{A}_1$, which is the zero space. The basis is the empty set.

   (b) $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2^x \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   Solution: The function T_2 is not a linear transformation. Indeed

   $$T_2\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 8 \\ 0 \end{pmatrix} \quad \text{and} \quad 3T_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 3\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$$

   Hence $T_2\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) \neq 3T_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$.

   (c) $T_3 \colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   Solution: $T_3 : \mathbb{R}^2 \to \mathbb{R}^3$, $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

   The standard matrix is $\mathbf{A}_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. A basis for the range of T_3 is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$, and a basis for the kernel is $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

(d) $T_4\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_4\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y-x \\ y-z \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

Solution: T_4 is not a linear transformation because $T_4\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(e) $T_5\colon \mathbb{R}^5 \to \mathbb{R}$ such that $T_5\left(\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}\right) = x_3 + 2x_4 - x_5$ for $\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^5$.

Solution: We see that

$$T_5\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = 0x_1 + 0x_2 + x_3 + 2x_4 - x_5 = (0\ 0\ 1\ 2\ \ -1)\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}.$$

Hence the standard matrix is $\mathbf{A}_5 = \begin{pmatrix} 0 & 0 & 1 & 2 & -1 \end{pmatrix}$. The range is $\mathbb{R}$, and a basis is $\{1\}$. A basis for the kernel is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

(f) $T_6 : \mathbb{R}^n \to \mathbb{R}$ such that $T_6(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$.

Solution: We have

$$\begin{aligned} T_6(2\mathbf{e}_1) &= (2\mathbf{e}_1) \cdot (2\mathbf{e}_1) = 4 \text{ and} \\ 2T_6(\mathbf{e}_1) &= 2(\mathbf{e}_1 \cdot \mathbf{e}_1) = 2. \end{aligned}$$

Thus $T_6(2\mathbf{e}_1) \neq 2T_6(\mathbf{e}_1)$. The function T_6 is not a linear transformation.

2. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $G : \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformations such that

$$F\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 - 3x_3 \\ 5x_2 - x_3 \end{pmatrix} \text{ and } G\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix},$$

and let $\mathbf{A}_F$ and $\mathbf{B}_G$ be the standard matrix of F and G, respectively.

(a) Find $\mathbf{A}_F$ and $\mathbf{B}_G$.

Solution:

$$\begin{aligned}
F\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) &= \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 - 3x_3 \\ 5x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 0 & 5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\
G\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) &= \begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\end{aligned}$$

$$\mathbf{A}_F = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 0 & 5 & -1 \end{pmatrix}, \quad \mathbf{B}_G = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

(b) Define

$$(F+G)(\mathbf{x}) := F(\mathbf{x}) + G(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathbb{R}^3.$$

Is $(F+G)$ a linear transformation? If it is, find its standard matrix.

Solution: For any $\mathbf{x} \in \mathbb{R}^3$,

$$(F+G)(\mathbf{x}) := F(\mathbf{x}) + G(\mathbf{x}) = \mathbf{A}_F\mathbf{x} + \mathbf{B}_G\mathbf{x} = (\mathbf{A}_F + \mathbf{B}_G)(\mathbf{x}).$$

Therefore $(F+G)$ is a linear transformation and the standard matrix is $(\mathbf{A}_F + \mathbf{B}_G)$.

(c) Write down the formula for $F(G(\mathbf{x}))$ and find its standard matrix.

Solution:

$$\begin{aligned}
F(G(x)) &= F\begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} (x_3 - x_1) - 2(x_2 + 5x_1) \\ (x_3 - x_1) + (x_2 + 5x_1) - 3(x_1 + x_2 + x_3) \\ 5(x_2 + 5x_1) - (x_1 + x_2 + x_3) \end{pmatrix} \\
&= \begin{pmatrix} -11x_1 - 2x_2 + x_3 \\ x_1 - 2x_2 - 2x_3 \\ 24x_1 + 4x_2 - x_3 \end{pmatrix} = \begin{pmatrix} -11 & -2 & 1 \\ 1 & -2 & -2 \\ 24 & 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{aligned}$$

The standard matrix is

$$\begin{pmatrix} -11 & -2 & 1 \\ 1 & -2 & -2 \\ 24 & 4 & -1 \end{pmatrix},$$

which is the product $\mathbf{A}_F\mathbf{B}_G$,

$$\begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 0 & 5 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -11 & -2 & 1 \\ 1 & -2 & -2 \\ 24 & 4 & -1 \end{pmatrix}.$$

(d) Find a linear transformation $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$H(G(\mathbf{x})) = \mathbf{x}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

Solution: From the observation in part (b), if $\mathbf{C}_H$ is the standard matrix $\mathbf{H}$, then we must have

$$\mathbf{C}_H \mathbf{B}_G = \mathbf{I}_3,$$

where $\mathbf{I}_3$ is the identity matrix. This means that $\mathbf{C}_H$ is the inverse of $\mathbf{B_G}$, which is

$$\mathbf{C_H} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{5}{3} & -\frac{2}{3} & \frac{5}{3} \\ \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}.$$

Hence,

$$H\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{5}{3} & -\frac{2}{3} & \frac{5}{3} \\ \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x_1 + x_2 - x_3 \\ -5x_1 - 2x_2 + 5x_3 \\ 4x_1 + x_2 - x_3 \end{pmatrix}$$

3. For each of the following linear transformations, (i) determine whether there is enough information for us to find the formula of T; and (ii) find the formula and the standard matrix for T if possible.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \; T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 4 \end{pmatrix}, \text{ and } T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 6 \end{pmatrix}.$$

Solution: There is enough information. The standard matrix is

$$\left(T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \; T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \; T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \right) = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \\ 0 & -1 & 1 \\ 1 & 4 & 6 \end{pmatrix},$$

that is, $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + 2y \\ 3x + 2y + 4z \\ -y + z \\ x + 4y + 6z \end{pmatrix}$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \; T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \text{ and } T\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Solution: There is enough information.

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\0\end{pmatrix}, \text{ and } \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix}\right).$$

Hence, the standard matrix is

$$\left(T\begin{pmatrix}1\\0\end{pmatrix} \quad T\begin{pmatrix}0\\1\end{pmatrix}\right) = \left(\tfrac{1}{2}T\begin{pmatrix}2\\0\end{pmatrix} \quad \tfrac{1}{2}\left(T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}1\\-1\end{pmatrix}\right)\right) = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}.$$

Thus $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x-y\\x+y\end{pmatrix}$.

(c) $T : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}\right) = -1,\ T\left(\begin{pmatrix}0\\1\\-1\end{pmatrix}\right) = 1 \text{ and } T\left(\begin{pmatrix}-1\\0\\1\end{pmatrix}\right) = 0.$$

Solution: There is not enough information. For example, $T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = y$ and $T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = -x-z$ are different linear transformations that satisfy the given information.

This is because the vectors $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ belongs to the subspace defined by the equation $x+y+z=0$. In particular the three vectors do not span the domain $\mathbb{R}^3$. Hence there are not enough information to define the transformation T.

4. For each of the following linear transformations T, determine its rank and nullity, and whether it is one-to-one, and/or onto.

(a) $T\colon \mathbb{R}^4 \to \mathbb{R}^6$ such that the rank is 4.

Solution: $\text{nullity}(T) = 4 - \text{rank}(T) = 4 - 4 = 0$. T is one-to-one, but not unto.

(b) $T\colon \mathbb{R}^6 \to \mathbb{R}^4$ such that the nullity is 2.

Solution: $\text{rank}(T) = 6 - 2 = 4$. T is unto, but not one-to-one.

(c) $T : \mathbb{R}^4 \to \mathbb{R}^6$ such that the reduce row-echelon form of its standard matrix has 3 nonzero rows.

Solution: Since the rref of the standard matrix has 3 nonzero rows, $\text{rank}(T) = 3$. So, $\text{nullity}(T) = 3 - 4 = 1$. T is neither one-to-one nor unto.

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that T is one-to-one.

Solution: Since T is one-to-one, $\text{nullity}(T) = 0$ and hence $\text{rank}(T) = 3 - 0 = 3$, which means that T is unto too.

Extra problems

1. (a) Show that if λ is an eigenvalue of a stochastic matrix $\mathbf{P}$, then $|\lambda| \leq 1$.
   Hint: Pick an eigenvector $\mathbf{v}$ of $\mathbf{P}^T$ associated with λ. Let $k \in \{1, 2, \ldots, n\}$ be a coordinate of $\mathbf{v}$ with the maximum absolute value, $|v_k| \geq |v_i|$ for all $i = 1, \ldots, n$. Consider the k-th coordinate of the equation $\mathbf{P}^T\mathbf{v} = \lambda\mathbf{v}^T$.

   Solution: Let $\mathbf{v}$ and v_k be chosen according to the hint. Taking the absolute value of the k-th coordinate of the equation $\mathbf{P}^T\mathbf{v} = \lambda\mathbf{v}^T$, we have

$$\begin{aligned} |\lambda v_k| &= |p_{1k}v_1 + p_{2k}v_2 + \cdots + p_{nk}v_n| \\ &\leq p_{1k}|v_1| + p_{2k}|v_2| + \cdots + p_{nk}|v_n| \\ &\leq p_{1k}|v_k| + p_{2k}|v_k| + \cdots + p_{nk}|v_k| \\ &\leq (p_{1k} + p_{2k} + \cdots + p_{nk})|v_k| \\ &= |v_k|, \end{aligned}$$

   which shows that $|\lambda||v_k| \leq |v_k|$. The second line follow from the fact that $p_{ij} \geq 0$ for all $i, j = 1, \ldots, n$. Since $\mathbf{v}$ is an eigenvector, necessarily $v_k \neq 0$. Hence, $|\lambda| \leq 1$.

   (b) Let $\mathbf{P}$ be a stochastic matrix. For any vector $\mathbf{v}$, define $\mathbf{v}^{(k)} = \mathbf{P}^k\mathbf{v}$. Show that if $\mathbf{v}$ is an eigenvector of $\mathbf{P}$ that is not associated to eigenvalue 1, then $\mathbf{v}^{(k)} \to 0$ as $k \to \infty$.

   Solution: Let λ be the eigenvalue associated to $\mathbf{v}$. Since $\lambda \neq 1$, by (a), $|\lambda| < 1$. Note that since $|\lambda| < 1$, then $\lambda^k \to 0$ as $k \to \infty$. So, $\mathbf{v}^{(k)} = \mathbf{P}^k\mathbf{v} = \lambda^k\mathbf{v} \to 0\mathbf{v} = \mathbf{0}$.

2. Let $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$ and $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be some vectors in $\mathbb{R}^m$. Prove that there is a unique transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{u}_i) = \mathbf{v}_i$ for $i = 1, ..., n$.

   Solution: Given any $\mathbf{u} \in \mathbb{R}^n$, since $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$, write $\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$. Define

$$T(\mathbf{u}) = c_1T(\mathbf{u}_1) + \cdots + c_nT(\mathbf{u}_n) = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

   Since $c_1, ..., c_n$ are unique, T is well-defined on $\mathbb{R}^n$.

   We will next show that T is a linear transformation. Given vectors $\mathbf{u}$, $\mathbf{w}$ in $\mathbb{R}^n$ and scalars $\alpha, \beta \in \mathbb{R}$, write $\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$ and $\mathbf{w} = d_1\mathbf{u}_1 + \cdots + d_n\mathbf{u}_n$. Then by definition of T,

$$\begin{aligned} T(\alpha\mathbf{u} + \beta\mathbf{v}) &= T((\alpha c_1 + \beta d_1)\mathbf{u}_1 + \cdots + (\alpha c_n + \beta d_n)\mathbf{u}_n) \\ &= (\alpha c_1 + \beta d_1)\mathbf{v}_1 + \cdots + (\alpha c_n + \beta d_n)\mathbf{v}_n \\ &= \alpha(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) + \beta(d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n). \end{aligned}$$

   On the other hand,

$$\begin{aligned} \alpha T(\mathbf{u}) + \beta T(\mathbf{w}) &= \alpha T(c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n) + \beta T(d_1\mathbf{u}_1 + \cdots + d_n\mathbf{u}_n) \\ &= \alpha(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) + \beta(d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n), \end{aligned}$$

which is equal to $T(\alpha\mathbf{u} + \beta\mathbf{v})$. This shows that T is linear.

Finally, we prove the uniqueness of T. Suppose L is another linear transformation such that $L(\mathbf{u}_i) = \mathbf{v}_i$ for all $i = 1, \ldots, n$. Then for any $\mathbf{u} \in \mathbb{R}^n$, we may write $\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$. By linearity,

$$\begin{aligned} L(\mathbf{u}) &= L(c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n) \\ &= c_1 L(\mathbf{u}_1) + \cdots + c_n L(\mathbf{u}_n) \\ &= c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \\ &= T(\mathbf{u}). \end{aligned}$$

This shows that T and L are the same linear transformation.

3. Prove that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective (one-to-one and onto) linear transformation if and only if there is a basis $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ such that the standard matrix of T is a transition matrix from S to E, where E is the standard basis for $\mathbb{R}^n$.

   Solution: Let $\mathbf{A} = \begin{pmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{pmatrix}$ be the standard matrix of T. First note that T is one-to-one, that is, $\text{nullity}(T) = 0$, if and only if $\text{rank}(T) = n$, that is, T is onto too. Hence, if T is either one-to-one or unto, then T is bijective. This is equivalent to $\mathbf{A}$ being invertible, since $\text{rank}(T) = \text{rank}(\mathbf{A})$. This would mean that $\{\mathbf{A}\mathbf{e}_1 = \mathbf{u}_1, \ldots, \mathbf{A}\mathbf{e}_n = \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$. Furthermore, since $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix}$, $\mathbf{A}$ is the transition matrix from S to E. Conversely, suppose $\mathbf{A}$ is a transition matrix. Then it is invertible, and hence, T is bijective.