MA1522: Linear Algebra for Computing

Tutorial 10

Revision

Powers of Diagonalizable Matrices

Suppose $\mathbf{A}$ is diagonalizable. Then $\mathbf{A}^k = \mathbf{P} \begin{pmatrix} \mu_1^k & 0 & \cdots & 0 \\ 0 & \mu_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n^k \end{pmatrix} \mathbf{P}^{-1}$. Moreover, if $\mathbf{A}$ is invertible, then the identity above holds for any integer $k \in \mathbb{Z}$.

Stochastic Matrices

A matrix **A** is a stochastic matrix if the sums of the entries in each column is 1, $\sum_{i=1}^{n} a_{ij} = 1$ for all $j = 1, \ldots, n$.

A Markov chain is a sequence of probability vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots,$ together with a stochatic matrix **P** such that

$$\mathbf{x}_1 = \mathbf{P}\mathbf{x}_0, \quad \mathbf{x}_2 = \mathbf{P}\mathbf{x}_1, \quad \ldots \quad , \quad \mathbf{x}_k = \mathbf{P}\mathbf{x}_{k-1}, \ldots$$

A steady-state vector, or equilibrium vector for a stochastic matrix **P** is a probability vector that is an eigenvector associated to eigenvalue 1.

The limit of any Markov chain is an equilibrium vector.

Singular Value Decomposition

Every matrix $m \times n$ matrix $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{U\Sigma V}^T$, where $\mathbf{U}$ is an order m orthogonal matrix, $\mathbf{V}$ an order n orthogonal matrix, and the matrix $\mathbf{\Sigma}$ has the form

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{D} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{pmatrix},$$

for some diagonal matrix $\mathbf{D}$ of order r, where $r \leq \min\{m, n\}$.

The singular values of $\mathbf{A}$, $\sigma_i = \sqrt{\mu_i}$, i=1,...,n, are the square root of the eigenvalues of $\mathbf{A}^T\mathbf{A}$.

The diagonal entries of the diagonal matrix $\mathbf{D}$ are the nonzero singular values of $\mathbf{A}$, arranged in decreasing order.

$\mathbf{V}$ is an orthogonal matrix that orthogonally diagonalize $\mathbf{A}^T\mathbf{A}$.

$\mathbf{U}$ is an orthogonal matrix whose first r columns are obtained from the columns of $\mathbf{V}$ via $\mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$, and $n - r$ columns are obtained via extending $\{\mathbf{u}_1, ..., \mathbf{u}_r\}$ to an orthonormal basis for $\mathbb{R}^n$.

Algorithm to Singular Value Decomposition

Let $\mathbf{A}$ be a $m \times n$ matrix with rank$(\mathbf{A}) = r$.

1. Find the eigenvalues of $\mathbf{A}^T\mathbf{A}$. Arrange the nonzero eigenvalues in descending order (counting multiplicity)

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r > 0 = \mu_{r+1} = \cdots = \mu_n,$$

   and let $\sigma_i = \sqrt{\mu_i}$, $i = 1, ..., r$. Set

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{D} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{pmatrix}, \text{ where } \mathbf{D} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{pmatrix}.$$

2. Find an orthogonal basis for each eigenspace, and let $\mathbf{v}_i$ be the unit vector associated to μ_i. Set

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}.$$

3. Let $\mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$ for $i = 1, ..., r$. Extend $\{\mathbf{u}_1, ..., \mathbf{u}_r\}$ to an orthonormal basis $\{\mathbf{u}_1, ..., \mathbf{u}_r, \mathbf{u}_{r+1}, ..., \mathbf{u}_m\}$ of $\mathbb{R}^m$, that is, solve for $\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_r \end{pmatrix}^T \mathbf{x} = \mathbf{0}$ and find an orthonormal basis for the solution space. Let

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$$

Tutorial 10 Solutions

Question 1(a)

A population of ants is put into a maze with 3 compartments labeled a, b, and c. If the ant is in compartment a, an hour later, there is a 20% chance it will go to compartment b, and a 40% change it will go to compartment c. If it is in compartment b, an hour later, there is a 10% chance it will go to compartment a, and a 30% chance it will go to compartment c. If it is in compartment c, an hour later, there is a 50% chance it will go to compartment a, and a 20% chance it will go to compartment b. Suppose 100 ants has been placed in compartment a. Find the transition probability matrix **A**. Show that it is a stochastic matrix.

Question 1(a)

A population of ants is put into a maze with 3 compartments labeled a, b, and c. If the ant is in compartment a, an hour later, there is a 20% chance it will go to compartment b, and a 40% change it will go to compartment c. If it is in compartment b, an hour later, there is a 10% chance it will go to compartment a, and a 30% chance it will go to compartment c. If it is in compartment c, an hour later, there is a 50% chance it will go to compartment a, and a 20% chance it will go to compartment b. Suppose 100 ants has been placed in compartment a. Find the transition probability matrix **A**. Show that it is a stochastic matrix.

$\begin{pmatrix} 0.4 & 0.1 & 0.5 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$. In fact, it is a doubly stochastic matrix, that is, the sum of the rows are also equal to 1.

Question 1(b)

By diagonalizing **A**, find the number of ants in each compartment after 3 hours.

```
A=[0.4 0.1 0.5;0.2 0.6 0.2;0.4 0.3 0.3]; syms x; solve(det(x*eye(3)-A))
```

The eigenvalues are $\lambda = -0.1, 0.4, 1$.

`rref(-0.1*eye(3)-A)` or `null(sym(-0.1*eye(3)-A))`
`rref(0.4*eye(3)-A)` or `null(sym(0.4*eye(3)-A))`
`rref(eye(3)-A)` or `null(sym(eye(3)-A))`

Hence $\mathbf{A} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0.4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$. Then

$$\mathbf{x}_3 = \mathbf{A}^3\mathbf{x}_0 = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.1^3 & 0 \\ 0 & 0 & 0.4^3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 35 \\ 31.2 \\ 33.8 \end{pmatrix}.$$

Question 1(c)

We can use MATLAB to diagonalize the matrix **A**. Type

```
>> [P D]=eig(sym(A))
```

The matrix **P** will be an invertible matrix, and **D** will be a diagonal matrix. Compare the answer with what you have obtained in (b).

The same answer is (b).

Question 1(d)

In the long run (assuming no ants died), where will the majority of the ants be?

$$\mathbf{A}^k = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1^k & 0 & 0 \\ 0 & (-0.1)^k & 0 \\ 0 & 0 & (0.4)^k \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \longrightarrow$$

$$\begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}.$$ So in the long run,

$$\begin{aligned} \mathbf{x}_\infty &= \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 33.33 \\ 33.33 \\ 33.33 \end{pmatrix}. \end{aligned}$$

Question 1(e)

Suppose initially the numbers of ants in compartments a, b and c are α, β, and γ respectively. What is the population distribution in the long run (assuming no ants died)?

$$\begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{3} \begin{pmatrix} \alpha+\beta+\gamma \\ \alpha+\beta+\gamma \\ \alpha+\beta+\gamma \end{pmatrix}.$$

This is always an equilibrium vector if $\alpha + \beta + \gamma \neq 0$.

Remark: This question demonstrate that the limit of a Markov chain is always an equilibrium vector.
Caution. It does not assume that Markov chain will converge.

Question 2

By diagonalizing $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, find a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$.

```
>> A=[1 0 3;0 4 0;0 0 4];
>> rref(eye(3)-A)
>> rref(4*eye(3)-A)
```

$$\mathbf{A} = \mathbf{PDP}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}.$$

Consider any of the 8 choices of $\mathbf{C} = \begin{pmatrix} \pm 2 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\mathbf{C}^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $\mathbf{B} = \mathbf{PCP}^{-1}$, then

$$\mathbf{B}^2 = \mathbf{PC}^2\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \mathbf{A}.$$

Question 3(a)

Find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

```
>> A=[3 1;1 3]; syms x; solve(det(x*eye(2)-A))
>> rref(2*eye(2)-A)
>> rref(4*eye(2)-A)
```

$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

Question 3(b)

Find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$.

```
>> A=[2 2 -2;2 -1 4;-2 4 -1]; syms x; solve(det(x*eye(3)-A))
>> rref(-6*eye(3)-A)
>> rref(3*eye(3)-A)
```

$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{pmatrix}$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Question 4(a)

Find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$, and compute $\mathbf{P}^T\mathbf{AP}$.

```
>> A=[1 -2 0 0;-2 1 0 0;0 0 1 -2;0 0 -2 1]; syms x; solve(det(x*eye(4)-A))
>> rref(-eye(4)-A)
>> rref(3*eye(4)-A)
```

$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$, then $\mathbf{P}^T\mathbf{AP} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

Question 4(b)

We will use MATLAB to orthogonally diagonalize $\mathbf{A}$. Type

```
>> A=[1 -2 0 0;-2 1 0 0;0 0 1 -2;0 0 -2 1];
>> [P D]=eig(A);
>> sym(P), sym(D)
```

Compare the result with your answer in (a).

The code results an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalize $\mathbf{A}$, and the diagonal matrix $\mathbf{D} = \mathbf{P}^T\mathbf{AP}$.

Question 5(a)

Find the SVD of $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}$.

`>> A=[3 2;2 3;2 -2]; B=A'*A; syms x; solve(det(x*eye(2)-B))` $\Rightarrow$ The singular values are $\sqrt{25} = 5 \geq \sqrt{9} = 3$.

`>> rref(25*eye(2)-B)` $\Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \Rightarrow \mathbf{u}_1 = \frac{1}{5}\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

`>> rref(9*eye(2)-B)` $\Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \Rightarrow \mathbf{u}_2 = \frac{1}{3}\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 1/(3\sqrt{2}) \\ -1/(3\sqrt{2}) \\ 4/(3\sqrt{2}) \end{pmatrix}$.

Question 5(a)

To extend $\{\mathbf{u}_1, \mathbf{u}_2\}$ to an orthonormal basis for $\mathbb{R}^3$, need to find a (unit) vector $\mathbf{u}_3$ orthogonal to $\mathbf{u}_1, \mathbf{u}_2$. Use tutorial 7 question 1,

`>> rref([1/sqrt(2) 1/sqrt(2) 0;1/(3/sqrt(2)) -1/(3/sqrt(2)) 4/(3/sqrt(2))])` $\Rightarrow \mathbf{u}_3 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$.

$$\mathbf{A} = \begin{pmatrix} 1/\sqrt{2} & \sqrt{2}/6 & -2/3 \\ 1/\sqrt{2} & -\sqrt{2}/6 & 2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

Question 5(b)

Find the SVD of $\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

The matrix is the transpose of the matrix in Part (a). Instead of computing the SVD from scratch, we use (a) to help us get the answer.
Suppose $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, then $\mathbf{A}^T = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T)^T = \mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T$, and note that $\mathbf{V}$ and $\mathbf{U}^T$ are orthogonal matrices too. So,

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{2}/6 & -\sqrt{2}/6 & 2\sqrt{2}/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix}.$$

Question 5(c)

Find the SVD of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

This is a symmetric matrix, and hence orthogonally diagonalizable. Write $\mathbf{A} = \mathbf{PDP}^T$. Observe that $\mathbf{A}^T\mathbf{A} = \mathbf{PD}^2\mathbf{P}^T$, and so the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are the squared of the eigenvalues of $\mathbf{A}$. This means that the singular value of $\mathbf{A}$ are the (absolute value of the) eigenvalues of $\mathbf{A}$. So, up to reordering (in descending order), we may let $\mathbf{\Sigma} = \mathbf{D}$. Since $\mathbf{P}$ and $\mathbf{P}^T$ are orthogonal matrices, up to rearranging the columns (so that the columns corresponds to the reordering of the eigenvalues in $\mathbf{D}$), we may let $\mathbf{P} = \mathbf{U} = \mathbf{V}$. That is

$$\mathbf{A} = \mathbf{PDP}^T = \mathbf{U\Sigma V}^T.$$

```
>> A=[1 0 1;0 1 1;1 1 2]; [P D]=eig(sym(A)).
```

$$\mathbf{P} = \mathbf{U} = \mathbf{V} = \begin{pmatrix} 1/\sqrt{6} & -1\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Question 6(a)

Find a SVD of $\mathbf{A} = \begin{pmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{pmatrix}$.

```
>> A=[-18 13 -4 4;2 19 -4 12;-14 11 -12 8;-2 21 4 8]; syms x; solve(det(x*eye(4)-A'*A))
>> rref(1600*eye(4)-A'*A), v1=[-1;2;-1/2;1]; v1=v1/sqrt(v1'*v1), u1=(1/40)*A*v1 ⇒
```

$\mathbf{v}_1 = \begin{pmatrix} -2/5 \\ 4/5 \\ -1/5 \\ 2/5 \end{pmatrix}$ and $\mathbf{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$.

`>> rref(400*eye(4)-A'*A), v2=[4;2;2;1]; v2=v2/sqrt(v2'*v2), u2=(1/20)*A*v2` ⇒ $\mathbf{v}_2 = \begin{pmatrix} 4/5 \\ 2/5 \\ 2/5 \\ 1/5 \end{pmatrix}$ and

$\mathbf{u}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$.

Question 6(a)

`>> rref(100*eye(4)-A'*A), v3=[1;-1/2;-2;1]; v3=v3/sqrt(v3'*v3), u3=(1/10)*A*v3` $\Rightarrow \mathbf{v}_3 = \begin{pmatrix} 2/5 \\ -1/5 \\ -4/5 \\ 2/5 \end{pmatrix}$

and $\mathbf{u}_3 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$.

`>> rref(A'*A), v4=[-1/4;-1/2;1/2;1]; v4=v4/sqrt(v4'*v4)` $\Rightarrow \mathbf{v}_4 = \begin{pmatrix} -1/5 \\ -2/5 \\ 2/5 \\ 4/5 \end{pmatrix}$. Now solve for $\mathbf{u}_4$.

`>> rref([u1';u2';u3']), u4=[-1;-1;1;1]; u4=u4/sqrt(u4'*u4)` $\Rightarrow \mathbf{u}_4 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$.

Question 6(a)

So, $\mathbf{U} = \begin{pmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix}$, $\boldsymbol{\Sigma} = \begin{pmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $\mathbf{V} = \begin{pmatrix} -2/5 & 4/5 & 2/5 & -1/5 \\ 4/5 & 2/5 & -1/5 & -2/5 \\ -1/5 & 2/5 & -4/5 & 2/5 \\ 2/5 & 1/5 & 2/5 & 4/5 \end{pmatrix}$.

Question 6(b)

In MATLAB, type

```
>> [U S V]=svd(A)
```

Compare the result with your answer in (a).

Up to a sign ($\pm$), they are equal.