MA1522 Linear Algebra for Computing

Lecture 8: Row/Column Spaces, Rank and Nullity

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Outline

Selected Revision of Section 4.1

- Section 4.1 talks about Row and Column Spaces and Null Spaces.
- We take a brief revision, with emphasize on how to find basis in each of the spaces.
- Along the way, we will answer the questions and challenges posed by Dr. Teo.

Slide 3: Definition of Column and Row Space

Let $\mathbf{A}$ be an $m \times n$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The ***row space*** of $\mathbf{A}$, is the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$,

$$\text{Row}(\mathbf{A}) = \text{span}\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}, \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}, \ldots, \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\}.$$

The ***column space*** of $\mathbf{A}$, is the subspace of $\mathbb{R}^m$ spanned by the columns of $\mathbf{A}$,

$$\text{Col}(\mathbf{A}) = \text{span}\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}.$$

Slide 4: Finding A Basis for Row Space

Algorithm of Finding a basis for Row($\mathbf{A}$).

1. Reduce $\mathbf{A}$ into its reduced row echelon form $\mathbf{R}$.
2. The nonzero rows of $\mathbf{R}$ form a basis of Row($\mathbf{A}$).

Remarks:

- It is based on the important **Fact** (see Slide 9): Row operations preserve row space.
- In fact, it suffices to reduce to the row echelon form and take the nonzero rows.
- Note: The basis are from the rows of $\mathbf{R}$, not from $\mathbf{A}$. We will come back to this point later.

Examples from Slide 11

1. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

   So $\left\{ \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix} \right\}$ is a basis for Row($\mathbf{A}$).

2. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 7 \\ -1 & 7 & -19 \\ 1 & 9 & -13 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

   So $\left\{ \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -3 \end{pmatrix} \right\}$ is a basis for Row($\mathbf{A}$).

Challenge one from Section 4.1

Example continued:

3. $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

So $\left\{ \begin{pmatrix} 1 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \right\}$ is a basis for Row($\mathbf{A}$).

Challenge: However, in this case, we could have taken the original rows of $\mathbf{A}$

$$\left\{ \begin{pmatrix} 1 & 1 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \right\}$$

as a basis for Row($\mathbf{A}$) too. Why?

Answer to Challenge one from Section 4.1

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Q: We could have taken the original rows of $\mathbf{A}$

$$\left\{ \mathbf{u}_1 = \begin{pmatrix} 1 & 1 & 2 & -1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \right\}$$

as a basis for Row($\mathbf{A}$) too. Why?

Answer: From the RREF, we know that $\dim(\text{Row}(\mathbf{A})) = 3$. Clearly $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \text{Row}(\mathbf{A})$. By Slide 138 of Chap 3, we know that they are a basis of Row($\mathbf{A}$).

Slide 16: Finding A Basis for Column Space

Algorithm of Finding a basis for Col($\mathbf{A}$).

1. Reduce $\mathbf{A}$ into its reduced row echelon form $\mathbf{R}$.
2. The columns of $\mathbf{A}$ corresponding to the pivot columns in $\mathbf{R}$ form a basis for Col($\mathbf{A}$).

Remarks:

- It is based on the important **Fact** (see Slide 15): Row operations preserve linear relations between columns.

- In fact, it suffices to reduce to the row echelon form and do step 2.

- Back to row space of $\mathbf{A}$, if we apply the above algorithm for $\mathbf{A}^T$, we will get a basis for Row($\mathbf{A}$) consisting only row vectors of $\mathbf{A}$.

Example (on Slide 17)

Q:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \\ 0 & 0 & 1 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Which columns of $\mathbf{A}$ form a basis for $\mathrm{Col}(\mathbf{A})$?

Answer: Let $\mathbf{v}_i$ ($i = 1, \ldots, 5$) denote the column vectors of $\mathbf{A}$. Since the first and third columns of $\mathbf{R}$ are the pivot columns, $\{\mathbf{v}_1, \mathbf{v}_3\}$ forms a basis for $\mathrm{Col}(\mathbf{A})$.

Illustrating Preservation of Linear Relations

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \\ 0 & 0 & 1 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We also use this example to illustrate "Row operations preserve linear relations between columns".

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{5}{6} \\ -\frac{1}{6} \\ 0 \\ 0 \end{pmatrix} = \frac{1}{6}\left(5\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{6}\left(5\begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \end{pmatrix}\right), \quad \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix} = \frac{1}{3}\left(\begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \end{pmatrix}\right)$$

Challenge Two from Section 4.1

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \\ 0 & 0 & 1 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Q: We could take any 2 columns of $\mathbf{A}$ except columns 1 and 2, to be a basis for $\mathrm{Col}(\mathbf{A})$. Why?

Answer: We know that $\dim(\mathrm{Col}(\mathbf{A})) = 2$. It suffices to check that any such 2 columns are linearly independent. For example, take $\mathbf{v}_2$ and $\mathbf{v}_4$.

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{\text{by above}} \mathbf{R} = \begin{pmatrix} 1/2 & 5/6 \\ 0 & -1/6 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which shows that $\mathbf{v}_2$ and $\mathbf{v}_4$ are linearly independent. Other pairs are similar.

Question One from Section 4.1

Which of the following statements is/are true?

1. Suppose $\mathbf{A}$ is a 3×3 matrix whose reduced row-echelon form is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for the column space of $\mathbf{A}$.

2. Suppose $\mathbf{A}$ is a 4×3 matrix whose reduced row-echelon form is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we can conclude that the first 2 rows of $\mathbf{A}$ are linearly independent.

These two questions are related to (or trying to extend) the facts:

- Row operations preserve row space.
- Row operations preserve linear relations between columns.

Remarks (Slide 20)

1. Row operations do not preserve column space. Consider

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\mathrm{Col}(\mathbf{A}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \neq \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

2. Row operations do no preserve linear relations between the rows. Consider

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

row 2 of $\mathbf{A} = 2\times$ row 1 of $\mathbf{A}$, row 2 of $\mathbf{B} = 0\times$ row 1 of $\mathbf{B}$.

Answer to Question One in Section 4.1, part 1

Q: Suppose $\mathbf{A}$ is a 3×3 matrix whose reduced row-echelon form is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for the column space of $\mathbf{A}$.

Note: Here, unlike the algorithm, the columns are picked from the RREF, instead of $\mathbf{A}$.

Answer: False. For example, let

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But $(1, 0, 0)^T$ is not even in the column space of $\mathbf{A}$.

Answer to Question One in Section 4.1, part 2

Q: Suppose **A** is a 4×3 matrix whose reduced row-echelon form is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we can conclude that the first 2 rows of **A** are linearly independent.

Note: Here, clearly the first two rows of the RREF being linearly independent, does it also hold for rows of **A**?

Answer: False. For example, let

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first two rows of **A** is clearly linearly dependent.

Slide 23: Nullspace

Definition

The *nullspace* of an $m \times n$ matrix $\mathbf{A}$ is the solution space to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ with coefficient matrix $\mathbf{A}$. It is denoted as

$$\mathrm{Null}(\mathbf{A}) = \left\{ \ \mathbf{v} \in \mathbb{R}^n \ \middle| \ \mathbf{A}\mathbf{v} = \mathbf{0} \ \right\}.$$

The *nullity* of $\mathbf{A}$ is the dimension of the nullspace of $\mathbf{A}$, denoted as

$$\mathrm{nullity}(\mathbf{A}) = \dim(\mathrm{Null}(\mathbf{A})).$$

Algorithms for finding a basis of nullspace: Same as finding a basis for solution space.

Question Two in Section 4.1

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \\ 0 & 0 & 1 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1. Find a basis for the nullspace of $\mathbf{A}$.
2. What is the nullity of $\mathbf{A}$?

From the RREF, we immediately get the general solution:

$$x_1 = -\frac{1}{2}r - \frac{5}{6}s - \frac{1}{3}t, x_2 = r, x_3 = \frac{1}{6}s - \frac{1}{3}t, x_4 = s, x_5 = t,$$

where $r, s, t \in \mathbb{R}$.

Answer to Question Two in Section 4.1

(continue from previous slide) Answer: The general solution of $\mathbf{Ax} = \mathbf{0}$ is

$$\mathbf{x} = r \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{5}{6} \\ 0 \\ \frac{1}{6} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The vectors above form a basis and the Nullity of $\mathbf{A}$ is 3.

Slide 27: Rank

Let $\mathbf{A}$ be an $m \times n$ matrix.

Definition

Define the *rank* of $\mathbf{A}$ to be the dimension of its column or row space

$$\operatorname{rank}(\mathbf{A}) = \dim(\operatorname{Col}(\mathbf{A})) = \dim(\operatorname{Row}(\mathbf{A})).$$

Justification: Let $\mathbf{R}$ be the reduced row-echelon form of $\mathbf{A}$.

$$\begin{aligned}\dim(\operatorname{Col}(\mathbf{A})) &= \#\text{ of pivot columns in RREF of } \mathbf{A},\\ &= \#\text{ of leading entries in RREF of } \mathbf{A},\\ &= \#\text{ of nonzero rows in RREF of } \mathbf{A} = \dim(\operatorname{Row}(\mathbf{A})).\end{aligned}$$

Challenge One of Section 4.2

Prove the following theorem.

Theorem

The linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ *is consistent if and only if the rank of* $\mathbf{A}$ *is equal to the rank of the augmented matrix* $\begin{pmatrix} \mathbf{A} \mid \mathbf{b} \end{pmatrix}$,

$$\operatorname{rank}(\mathbf{A}) = \operatorname{rank}((\mathbf{A} \mid \mathbf{b})).$$

Preparation: Suppose that $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the column vectors of $\mathbf{A}$. Let $\mathbf{x} = (c_1, \dots, c_n)^T$. Then the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be expressed as:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{b}.$$

Answer to Challenge One Section 4.2, the "only if" direction

Q: The linear system $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if

$$\text{rank}(\mathbf{A}) = \text{rank}((\mathbf{A} \mid \mathbf{b})).$$

Answer: ($\Rightarrow$) Suppose that the linear system $\mathbf{Ax} = \mathbf{b}$ is consistent. By previous slide, there are $c_1, \ldots, c_n$ such that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{b},$$

that is, $\mathbf{b} \in \text{Col}(\mathbf{A})$. Thus,

$$\text{rank}(\mathbf{A}) = \dim(\text{Col}(\mathbf{A})) = \dim(\text{Col}(\mathbf{A} \mid \mathbf{b})) = \text{rank}((\mathbf{A} \mid \mathbf{b})).$$

Answer to Challenge One Section 4.2, the "if" direction

Q: The linear system $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if

$$\mathrm{rank}(\mathbf{A}) = \mathrm{rank}((\mathbf{A} \mid \mathbf{b})).$$

Answer: ($\Leftarrow$) Suppose that $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}((\mathbf{A} \mid \mathbf{b}))$, that is,

$$\dim(\mathrm{Col}(\mathbf{A})) = \dim(\mathrm{Col}(\mathbf{A} \mid \mathbf{b})).$$

Let W be a set of basis of $\mathrm{Col}(\mathbf{A})$, then W is linearly independent and also $W \subseteq \mathrm{Col}(\mathbf{A} \mid \mathbf{b})$. Since $|W| = \dim(\mathrm{Col}(\mathbf{A} \mid \mathbf{b}))$, we have $\mathrm{span}(W) = \mathrm{Col}(\mathbf{A} \mid \mathbf{b})$. Therefore, $\mathbf{b} \in \mathrm{span}(W)$, hence also in the span of column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbf{A}$, i.e., there are $c_1, \ldots, c_n$, such that,

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{b},$$

that is, $(c_1, \ldots, c_n)^T$ is a solution of $\mathbf{Ax} = \mathbf{b}$.

Question in Section 4.2

Show that if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B})$.

Answer: Since elementary row operations does not change the row space, we have $\text{Row}(\mathbf{A}) = \text{Row}(\mathbf{B})$. Hence,

$$\text{rank}(\mathbf{A}) = \dim(\text{Row}(\mathbf{A})) = \dim(\text{Row}(\mathbf{B})) = \text{rank}(\mathbf{B}).$$

Challenge Two in Section 4.2

Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of the same size. Prove that

$$\text{rank}(\mathbf{A}+\mathbf{B}) \le \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}).$$

Answer: Let $\mathbf{A} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ where $\mathbf{u}_i$ are the column vectors of $\mathbf{A}$, and $\mathbf{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ where $\mathbf{v}_j$ are the column vectors of $\mathbf{B}$. Form the matrix $\mathbf{C} = (\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}_1, \ldots, \mathbf{v}_n)$. Then we have

- $\text{Col}(\mathbf{A}+\mathbf{B}) \subseteq \text{Col}(\mathbf{C})$, because every column vector in $\mathbf{A}+\mathbf{B}$ is $\mathbf{u}_i + \mathbf{v}_i$ for some i. thus $\text{rank}(\mathbf{A}+\mathbf{B}) \le \text{rank}(\mathbf{C})$.

- On the other hand, let $X \subseteq \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $Y \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases of $\text{Col}(\mathbf{A})$ and $\text{Col}(\mathbf{B})$ respectively. We have $\text{rank}(\mathbf{A}) = |X|$ and $\text{rank}(\mathbf{B}) = |Y|$. Since $\text{Col}(\mathbf{C}) \subseteq \text{span}(X \cup Y)$,

$$\begin{aligned} & \text{rank}(\mathbf{C}) = \dim(\text{Col}(\mathbf{C})) \le \dim(\text{span}(X \cup Y)) \\ \le \quad & |X| + |Y| = \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}). \end{aligned}$$

Challenge Three in Section 4.2

Let $\mathbf{A}$ be a $m \times n$ matrix such that $\text{rank}(\mathbf{A}) = m$. Suppose $m > n$. By the equivalent statements of full rank equals number of columns, $(\mathbf{A}^T\mathbf{A})$ invertible and $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ is a left inverse of $\mathbf{A}$.

Now consider the system $\mathbf{Ax} = \mathbf{b}$ for some vector $\mathbf{b}$ in $\mathbb{R}^m$. Premultiplying the left inverse above on both sides of the equation, we get

$$\mathbf{x} = ((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{Ax} = ((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{b},$$

that is, $((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{b}$ is a solution to $\mathbf{Ax} = \mathbf{b}$. But this is true for every $\mathbf{b}$, which by the equivalent statements of full rank equals number of rows, means that the rank of $\mathbf{A}$ is equal to m, the number of row. This is a contradiction to $m > n$.

What is the mistake in the argument above?

Side Remark: Matrices of the form $\mathbf{A}^T\mathbf{A}$ will appear in sections on Least Square Problems.

Slide 46: Full Rank Equals Number of Columns

Theorem

Suppose $\mathbf{A}$ *is an* $m \times n$ *matrix. The following statements are equivalent.*

(i) $\mathbf{A}$ *is full rank, where the rank is equal to the number of columns,* $\text{rank}(\mathbf{A}) = n$.

(ii) *The rows of* $\mathbf{A}$ *spans* $\mathbb{R}^n$, $Row(\mathbf{A}) = \mathbb{R}^n$.

(iii) *The columns of* $\mathbf{A}$ *are linearly independent.*

(iv) *The homogeneous system* $\mathbf{A}\mathbf{x} = \mathbf{0}$ *has only the trivial solution, that is,* $\text{Null}(\mathbf{A}) = \{\mathbf{0}\}$.

(v) $\mathbf{A}^T\mathbf{A}$ *is an invertible matrix of order* n.

(vi) $\mathbf{A}$ *has a left inverse.*

Note: In this case, $m \geq n$.

Slide 49: Full Rank Equals Number of Rows

Theorem

Suppose $\mathbf{A}$ *is an* $m \times n$ *matrix. The following statements are equivalent.*

(i) $\mathbf{A}$ *is full rank, where the rank is equal to the number of rows,* $\text{rank}(\mathbf{A}) = m$.

(ii) *The columns of* $\mathbf{A}$ *spans* $\mathbb{R}^m$, $Col(\mathbf{A}) = \mathbb{R}^m$.

(iii) *The rows of* $\mathbf{A}$ *are linearly independent.*

(iv) *The linear system* $\mathbf{Ax} = \mathbf{b}$ *is consistent for every* $\mathbf{b} \in \mathbb{R}^m$.

(v) $\mathbf{AA}^T$ *is an invertible matrix of order* m.

(vi) $\mathbf{A}$ *has a right inverse.*

Note: In this case, $m \leq n$.

Answer to the Challenge Three in Section 4.2

Q: After apply Full Rank Theorems (both the column form and the row form), some contradiction occurred. What is the mistake?

Answer: When $m > n$, one can only use the Theorem about full rank equals number of columns, the theorem for rows would not apply.