MA1522 Linear Algebra for Computing

Lecture 2: Gaussian Elimination

Yang Yue

Department of Mathematics
National University of Singapore

20 January, 2025

Outline

Summary to Solving Linear Systems

1. Write the linear system in its standard form.
2. Form the augmented matrix of the linear system.
3. Reduce the augmented matrix to either a row-echelon form or reduced row echelon form. May use Gaussian/Gauss-Jordan elimination.
4. Decide if the system is consistent
   - If the last column is a pivot column, the system is inconsistent.
   - Otherwise, the system is consistent, assign the variables corresponding to the nonpivot columns to be parameters, s, t, $s, t \in \mathbb{R}$, etc.
5. If the system is in reduced row-echelon form, read off the solutions directly.
6. If the system is in row-echelon form only, do back substitution, starting from the lowest nonzero row.
7. Write down the (general) solution to the system.

Consistency of Linear system and Number of Parameters

In row-echelon form

(i) No solution: a row of zero before the bar (coefficient matrix) and a non zero number after the bar $\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & \neq 0 \end{array}\right)$.

(ii) Unique solution: all columns of coefficient matrix are pivot columns
$\left(\begin{array}{cccc|c} \neq 0 & * & \cdots & * & * \\ 0 & \neq 0 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \neq 0 & * \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right)$. Not possible if $\#$ variables $>$ $\#$ equations.

(iii) Infinitely many solutions: when there is a non-pivot column in the augmented matrix before the bar $\left(\begin{array}{ccccccccc|c} \neq 0 & \cdots & * & * & & * & * & * & * \\ 0 & \cdots & 0 & \neq 0 & & * & * & * & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \neq 0 & * & * \\ 0 & & & 0 & & & 0 & 0 & 0 \\ \vdots & & & \vdots & & & \vdots & \vdots & \vdots \end{array}\right)$. In this case, number of parameters $=$ number of nonpivot columns before the bar.

Question in Section 1.1.

1. Give an example of a linear system with 3 variables such that the general solution has 2 parameters.
2. Is it possible to have a linear system with 3 variables, 3 equations, with the general solution having 3 parameters?

Key concept involved: "parameters in solutions". See Slides 10 and 34 in Chapter 1.

Remarks (Slide 39 in Chapter 1)

1. It is easy to obtain the solutions when the augmented matrix is in row-echelon form (by performing back-substitution) or reduced row-echelon form (reading off the solutions directly).

2. The linear system is inconsistent if and only if the RHS (last column) of the augmented matrix in row-echelon form is a pivot column.

3. Assign parameters to the variables corresponding to the non-pivot columns in the LHS of the augmented matrix.

4. The number of parameters needed is equal to the number of non-pivot columns in the LHS of the augmented matrix.

5. We can convert/reduce the augmented matrix of a linear system to a row-echelon form or its reduced row-echelon form to find the solutions (if exists). This is achieved using elementary row operations.

Answer to Question in Section 1.1.

1. To have a linear system with 3 variables such that the general solution has 2 parameters, we need two non-pivot columns in the LHS of the augmented matrix.

   Thus we may take $x + 2y + 3z = 4$, which has augmented matrix:
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \end{array}\right).$$

2. To have general solution with 3 parameters, we need three non-pivot columns in the LHS of the augmented matrix.

   Thus we have the (degenerated) case $0x + 0y + 0z = 0$, which has augmented matrix:
$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Question in Section 1.2

Consider the following augmented matrix

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & 1 \\ 0 & g & h & i \end{array}\right)$$

for some real numbers $a, b, c, d, e, f, g, h, i$. Suppose the augmented matrix is in row-echelon form.

1. What are the possible values of g?
2. If $h = 0$, what are the possible values of i?
3. If the augmented matrix is in reduced row-echelon form, and $f = -1$, what are the possible values of e?

Key Concept: Row-Echelon Form

In Slide 28 in Chapter 1, we have

Definition

An (augmented) matrix is in *row-echelon form* (REF) if

1. If zero rows exists, they are at the bottom of the matrix.
2. The leading entries are further to the right as we move down the rows.

An augmented matrix in REF has the form

$$\left(\begin{array}{cccccccc|c} * & & & & & & & & * \\ 0 & \cdots & 0 & * & & & & & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & * & & * \\ \vdots & & & & & & & \vdots & \vdots \\ 0 & \cdots & & & & & \cdots & 0 & 0 \end{array}\right).$$

Answer to Question in Sec 1.2., part 1

Q: If

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & 1 \\ 0 & g & h & i \end{array}\right)$$

is in row-echelon form, what are the possible values of g?

Answer: Since the first entry of row two is 0, its leading entry can be the second one (that is, when $e \neq 0$) or the third one (that is, when $e = 0$ and $f \neq 0$), or the fourth one (that is, when $e = f = 0$).

By condition 2 in the definition, the leading entry of the third row must not be the second one. Therefore $g = 0$.

Answer to Question in Sec 1.2., part 2

Q: If

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & 1 \\ 0 & g & h & i \end{array}\right)$$

is in row-echelon form, and $h = 0$, what are the possible values of i?

Answer: In the answer of part 1, we saw the leading entry of the second row can be its second (i.e., e), third (i.e. f) or fourth (when $e = f = 0$) entry.

In the first two cases, i can be any real number; whereas in the last case, that is, when $e = f = 0$, i must be 0.

Key Concept: Reduced Row-Echelon Form

In Slide 29 of Chapter 1, we have:

The (augmented) matrix is in reduced row-echelon form (RREF) if further

3. The leading entries are 1.
4. In each pivot column, all entries except the leading entry is 0.

An augmented matrix in RREF has the form

$$\left(\begin{array}{cccccccccc|c}
0 & \cdots & 1 & & * & 0 & & * & 0 & * & * \\
0 & \cdots & 0 & \cdots & 0 & 1 & & * & 0 & * & * \\
0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & * & * \\
0 & \cdots & 0 & & & 0 & & & 0 & 0 & 0 \\
\vdots & & & & & \vdots & & & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & & 0 & & \cdots & 0 & 0 & 0
\end{array}\right).$$

Answer to Question in Sec 1.2., part 3

Q: If

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & 1 \\ 0 & g & h & i \end{array}\right)$$

is in reduced row-echelon form, and $f = -1$, what are the possible values of e?

Answer: By condition 3 in the definition, $f = -1$ cannot be the leading entry of the second row. Therefore e must be its leading entry. By condition 3 again, $e = 1$.

Further Question in Sec 1.2., part 3

On Monday's lecture, some student asked: If

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & 1 \\ 0 & g & h & i \end{array}\right)$$

is in reduced row-echelon form, and $f = -1$, what are the possible values of h and i?

Answer: By part 1, $g = 0$. Now, h cannot be a leading entry, because of condition 4 (since $f = -1$ and h are in the same column). Thus, $h = 0$. By the same reason (using 1 and i are in the same column), $i = 0$.

Question in Section 1.3

What is the reverse of the elementary row operation $R_2 - \frac{1}{2}R_1$?

Recall in Slide 43 in Chapter 1, we have:
There are 3 types of elementary row operations.

1. Exchanging 2 rows, $R_i \leftrightarrow R_j$,

2. Adding a multiple of a row to another, $R_i + cR_j$, $c \in \mathbb{R}$,

3. Multiplying a row by a nonzero constant, aR_j, $a \neq 0$.

Note that we never write $cR_j + R_i$, in other words, the multiplier c must apply to the second "summand" R_j, and it is the first "summand" R_i that gets changed.

Slide 53 in Chapter 1

Every elementary row operation has a *reverse elementary row operation*. The reverse of the row operations are given as such.

1. The reverse of exchanging 2 rows, $R_i \leftrightarrow R_j$, is itself.

2. The reverse of adding a multiple of a row to another, $R_i + cR_j$ is subtracting the multiple of that row, $R_i - cR_j$.

3. The reverse of multiplying a row by a nonzero constant, aR_j is the multiplication of the reciprocal of the constant, $\frac{1}{a}R_j$.

Answer of Question in Section 1.3

The reverse of $R_2 - \frac{1}{2}R_1$ is $R_2 + \frac{1}{2}R_1$.

For example,

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{R_2 - \frac{1}{2}R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\xrightarrow{R_2 + \frac{1}{2}R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

Question in Section 1.5

Construct an augmented matrix with 3 variables and 3 equations such that it has the following solution

$$x = 3, \quad y = 2, \quad z = 1.$$

Answer: Just take

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Challenge in Section 1.2

Let $\mathbf{R}$ be a $n \times m$ matrix in reduced row-echelon form. Which of the following statements are true?

1. The number of pivot columns of $\mathbf{R}$ is equal to the number of nonzero rows of $\mathbf{R}$.

2. The number of nonpivot columns of $\mathbf{R}$ is equal to the number of zero rows in $\mathbf{R}$.

For each statement that is false, what restrictions can we impose on $\mathbf{R}$ such that the statement is true?

Key Concepts involved

Recall that on Slide 28 of Chapter 1, we have defined

- "nonzero rows"
- "leading entries"
- "pivot columns".

Observe that each nonzero row, say R_i contains a unique leading entry which belongs to a unique column C_j for some j.

We give an exceedingly mathematical proof below.

Answer to Challenge in Section 1.2, part 1

Let N and P denote the sets of nonzero rows and pivot columns in $\mathbf{R}$, respectively.

Define $f: N \to P$ by mapping nonzero row R_i to the column that contains its leading entry. By the observation in previous slide, f is a function (i.e., the output is unique for each input) and the domain of f is N.

f is one-to-one: If $R_{i_1}, R_{i_2} \in N$ and $i_1 \neq i_2$, say $i_1 < i_2$, then by condition 2 in the definition of REF, their leading entries are in different columns, thus $f(R_{i_1}) \neq f(R_{i_2})$.

f is onto: For any C_j in P, C_j must contain a leading entry of some nonzero column, say R_i, thus $f(R_i) = C_j$.

Since we have a one-one correspondence between N and P, they have the same number of elements. In other words, statement 1 is true.

Answer to Challenge in Section 1.2, part 2

How about the statement

2. The number of nonpivot columns of **R** is equal to the number of zero rows in **R**.

Let us use $\#P$ and $\#N$ denote the number of pivot columns and nonzero rows in **R**, respectively. By part 1, we knew $\#P = \#N$

Then the number of nonpivot columns is $m - \#P$ and the number of zero rows is $n - \#N$. Thus if $m \neq n$, these two numbers are not equal. For example, in

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we have two nonpivot columns but only one zero rows. Statement 2 is false.

When **R** is a square matrix, that is, $m = n$, by the discussion above, Statement 2 becomes true again.

Question in Section 1.4

Can we still perform Gaussian and/or Gauss-Jordan elimination if the coefficients contains unknown? For example,

$$\begin{cases} ax + y - z = 1 \\ x + y + 2z = 3 \\ x + (a-1)y - z = 0 \end{cases}$$

for some constant $a \in \mathbb{R}$?

A short answer is Yes. We can split into two cases: Case 1, $a = 0$ and Case 2, $a \neq 0$.

Alternatively, we may swap Row one with row 2 (or row 3) and continue, but sooner or later we have to do a case study.

All alternatives are tedious.

Question in Section 1.4 (Method 1)

Given augmented matrix

$$\left(\begin{array}{ccc|c} a & 1 & -1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & (a-1) & -1 & 0 \end{array}\right),$$

We split into two cases. Case 1, $a = 0$, the matrix becomes

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & -1 & -1 & 0 \end{array}\right).$$

One can do Gaussian elimination as usual.

Question in Section 1.4 (Case $a \neq 0$)

We do Gaussian elimination

$$\left(\begin{array}{ccc|c} a & 1 & -1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & (a-1) & -1 & 0 \end{array}\right) \xrightarrow[R_3 - \frac{1}{a}R_1]{R_2 - \frac{1}{a}R_1} \left(\begin{array}{ccc|c} a & 1 & -1 & 1 \\ 0 & \frac{a-1}{a} & \frac{2a+1}{a} & \frac{3a-1}{a} \\ 0 & \frac{a^2-a-1}{a} & \frac{-a+1}{a} & -\frac{1}{a} \end{array}\right)$$

We have to split into two cases again. Case 2.1, $a = 1$, the matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & -1 & 0 & -1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 3 & 2 \end{array}\right)$$

One can get a solution.

Question in Section 1.4 (Case $a \neq 0, 1$)

We do Gaussian elimination

$$\left(\begin{array}{ccc|c} a & 1 & -1 & 1 \\ 0 & \frac{a-1}{a} & \frac{2a+1}{a} & \frac{3a-1}{a} \\ 0 & \frac{a^2-a-1}{a} & \frac{-a+1}{a} & -\frac{1}{a} \end{array}\right)$$

$$\xrightarrow[aR_3]{\frac{a}{a-1}R_2} \left(\begin{array}{ccc|c} a & 1 & -1 & 1 \\ 0 & 1 & \frac{2a+1}{a-1} & \frac{3a-1}{a-1} \\ 0 & a^2-a-1 & -a+1 & -1 \end{array}\right)$$

$$\xrightarrow{R_3-(a^2-a-1)R_2} \left(\begin{array}{ccc|c} a & 1 & -1 & 1 \\ 0 & 1 & \frac{2a+1}{a-1} & \frac{3a-1}{a-1} \\ 0 & 0 & * & * \end{array}\right)$$

etc. (skipped)

Question in Section 1.4 (Method 2)

Alternatively, we can try to swap the rows first, for example, we swap R_1 and R_2.

$$\left(\begin{array}{ccc|c} a & 1 & -1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & (a-1) & -1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ a & 1 & -1 & 1 \\ 1 & (a-1) & -1 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 - R_1]{R_2 - aR_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1-a & -2a-1 & -3a+1 \\ 0 & a-2 & -3 & -3 \end{array}\right)$$

$$\xrightarrow{R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -1 & -2a-4 & -3a-2 \\ 0 & a-2 & -3 & -3 \end{array}\right)$$

$$\xrightarrow{R_3 + (a-2)R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -1 & -2a-4 & -3a-2 \\ 0 & 0 & -2(a^2-4) & -3a^2-8a-7 \end{array}\right)$$

We then split into three cases again (skipped).

Question in Section 1.4 (Method 2)

If we swap row one and row three, we may get

$$\left(\begin{array}{ccc|c} a & 1 & -1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & (a-1) & -1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & (a-1) & -1 & 0 \\ 1 & 1 & 2 & 3 \\ a & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 - aR_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & (a-1) & -1 & 0 \\ 0 & 2-a & 3 & 3 \\ 0 & -a^2+a+1 & a-1 & 1 \end{array}\right)$$

We have to split into two cases again (skipped).

The second alternative is clearer.

Challenge in Section 1.4

1. Is it possible to reduce an augmented matrix to 2 different row-echelon forms?
2. Is it possible to reduce an augmented matrix to 2 different reduced row-echelon form?

Part 1 is true and it does not challenge us too much. For example, let

$$A = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & 2 \end{array} \right) \text{ and } B = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right).$$

Both are in REF, and B is one more reduction away from A.

Challenge in Section 1.4, part 2

The second statement is false. Dr. Teo had included a proof using matrices in the appendix of Chapter 2.

Since we are in Chapter 1, it would be desirable to have a proof using only linear systems. Luckily, Kuttler provides us such a proof.

We need two Lemmas.

Challenge in Section 1.4, part 2 (conti.)

Lemma 1 Two linear systems of equations corresponding to two equivalent augmented matrices have exactly the same solutions.

Proof. Just use the remark on Slide 43 of Chapter 1, which says: performing elementary row operations to the augmented matrix of a linear system preserves the solutions.

Lemma 2 Let A and B be two augmented matrices in reduced row-echelon form for two homogeneous systems of m equations in n variables, such that the two systems have exactly the same solutions, then $A = B$.

Proof skipped. Interested reader can read Section 1.4 in Kuttler, and do pay attention to where the four conditions in the definition of RREF are used.

Proof of Statement 2 (assuming Lemmas 1 and 2)

Finally, we can show that every matrix A is equivalent to a unique matrix in reduced row-echelon form.

Proof. Let A be an $m \times n$ matrix and let B and C be equivalent to A matrices in reduced row-echelon form. We show that $B = C$. Let A^+ be the matrix A augmented with a new rightmost column consisting entirely of zeros. Similarly, augment matrices B and C to obtain B^+ and C^+ by adding a rightmost zero column respectively. Note that B^+ and C^+ are matrices in RREF which are obtained from A^+ by respectively applying the same sequence of elementary row operations which were used to obtain B and C from A.

Next, these matrices can all be considered as augmented matrices of homogeneous linear systems in the variables $x_1, x_2, \ldots, x_n$. Since they are row-equivalent, Lemma 1 says all three homogeneous linear systems have exactly the same solutions. By Lemma 2, we conclude that $B^+ = C^+$. Hence $B = C$.