Review

Eigenvalues & Eigenvectors

λ & $v \neq 0$ $\qquad Av = \lambda v$

square matrix $A \in M_{n\times n}$

i.e. $v \in \text{Null}(\lambda I - A) =: E_\lambda$ eigenspace $\longrightarrow$ geometric multiplicity $g_\lambda = \dim(E_\lambda)$

Characteristic polynomial

$\text{char}(A) = \det(xI - A) = p_A(x)$ polynomial of x

write $p(x) = (x-\lambda_1)^{a_{\lambda_1}}(x-\lambda_2)^{a_{\lambda_2}} \cdots (x-\lambda_k)^{a_{\lambda_k}}$ $\longrightarrow$ algebraic multiplicity a_λ

$= x^n - \underbrace{a_{n-1}}_{\text{tr}(A)} x^{n-1} + a_{n-2}x^{n-2} - \cdots + (-1)^n \underbrace{a_0}_{\det(A)}$ $\qquad a_{\lambda_1} + \cdots + a_{\lambda_k} = n$

alternate

Thm: $\{\text{eigenvalues of } A\} = \{\text{roots of char}(A)\}$

Thm: A is invertible iff 0 is not an eigenvalue of A.

Similar matrices A, B, if $A = PBP^{-1}$ for some invertible matrix P

Diagonalizable if A is similar to a diagonal matrix D, i.e. $A = PDP^{-1}$

Let $P = (u_1 \ u_2 \ \cdots \ u_n)$ $\qquad D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$A = PDP^{-1}$ or $P^{-1}AP = D \iff Au_i = \lambda_i u_i, \ \forall i = 1, \cdots, n$

u_i: eigenvector (nonzero) $\qquad \lambda_i$: eigenvalue

Thm: A is diagonalizable iff A has n L.I. eigenvectors

very strong condition

Not all matrices are diagonalizable.

Which matrices can be diagonalized?

$\rightarrow$ Find P & D

* Eigenvectors from distinct eigenvalues are ~~orthogonal, hence~~ L.I.

$$P = (\underbrace{u_1^{(\lambda_1)} \cdots u_{a_{\lambda_1}}^{(\lambda_1)}}_{\text{e.v. from } \lambda_1} \ \underbrace{u_1^{(\lambda_2)} \cdots u_{a_{\lambda_2}}^{(\lambda_2)}}_{\text{e.v. from } \lambda_2} \cdots \underbrace{u_1^{(\lambda_k)} \cdots u_{a_{\lambda_k}}^{(\lambda_k)}}_{\text{e.v. from } \lambda_k})$$

$$D = \operatorname{diag}(\underbrace{\lambda_1, \cdots, \lambda_1}_{a_{\lambda_1} \text{ times}}, \underbrace{\lambda_2, \cdots, \lambda_2}_{a_{\lambda_2} \text{ times}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{a_{\lambda_k} \text{ times}})$$

$\Longrightarrow$ Want to find a basis $S_{\lambda_i} = \{ u_1^{(\lambda_i)}, \cdots, u_{g_{\lambda_i}}^{(\lambda_i)} \}$ for E_{λ_i}

$$\boxed{g_{\lambda_i} = \dim(E_{\lambda_i}) = \# S_{\lambda_i} = a_{\lambda_i} \quad \forall\, i = 1, \cdots, k. \quad \text{iff} \quad A \text{ is diagonalizable.}}$$

(One can prove $1 \le g_{\lambda_i} \le a_{\lambda_i}$ for arbitrary matrices A.)

Cor. If A has n distinct eigenvalues, then A is diagonalizable.

Return to the matrix similarity. $A = PBP^{-1}$

A & B share almost all properties that we have learnt so far:

rank, nullity, trace, determinant, eigenvalues,

algebraic multiplicities, geometric multiplicites, index of nilpotence

$\rightsquigarrow$ all are numbers that are used to characterize matrices

Motivation of diagonalization & matrix similarity (will revisit after Tutorial 11)

Solving linear systems $Ax = b$

Recall: if A is a diagonal matrix, then

$$\begin{cases} a_{11} x_1 = b_1 \\ a_{22} x_2 = b_2 \\ \vdots \\ a_{nn} x_n = b_n \end{cases} \Longrightarrow \begin{cases} x_1 = \dfrac{b_1}{a_{11}} \\ \vdots \\ x_n = \dfrac{b_n}{a_{nn}} \end{cases}$$

For general matrix A, view $v \mapsto Av$ as a linear transformation from $\mathbb{R}^n$ to itself

View x as the coordinate of some vector x relative to the standard basis E $\{e_1, \cdots, e_n\}$

P as the transition matrix of this standard basis to another basis F

$[x]_E = x$, $[b]_E = b$ $\Longrightarrow$ $[x]_F = Px \overset{!!}{=} y$ $\quad [b]_F = Pb \overset{!!}{=} c$

$Ax = b \rightsquigarrow AP^{-1}y = P^{-1}c$

$\rightsquigarrow \underbrace{(PAP^{-1})}^{B} y = c$

1. A father wishes to distribute an amount of money among his three sons Jack, Jim, and John. He wish to distribute such that the following conditions are all satisfied.

x_1 x_2 x_3

(i) The amount Jack receives plus twice the amount Jim receives is \$300.

(ii) The amount Jim receives plus the amount John receives is \$300.

(iii) Jack receives \$300 more than twice of what John receives.

(a) Is it possible for the following conditions to all be satisfied?

(b) If it is not possible, find a least square solution. (Make sure that your least square solution is feasible. For example, one cannot give a negative amount of money to anybody.)

1.
$$\begin{cases} x_1 + 2x_2 = 300 \\ x_2 + x_3 = 300 \\ x_1 - 2x_3 = 300 \end{cases} \qquad \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}}_{b}$$

(a) No. Inconsistent

(b) Solve for $A^TAx = A^Tb \Rightarrow \begin{cases} x_1 = 200 + 2s \\ x_2 = 100 - s \\ x_3 = s \end{cases} \quad s \in \mathbb{R}$

$x_1, x_2, x_3 > 0 \iff 0 \le s \le 100$

2. (a) Suppose **A** is a $m \times n$ matrix where $m > n$. Let $\mathbf{A} = \mathbf{QR}$ be a QR factorization of **A**. Explain how you might use this to write

$$\mathbf{A} = \mathbf{Q}'\mathbf{R}',$$

where $\mathbf{Q}'$ is an $m \times m$ orthogonal matrix, and $\mathbf{R}'$ a $m \times n$ matrix with $m - n$ zero rows at the bottom. This is known as the *full* QR factorization of **A**.

(b) In MATLAB, enter the following.

```
>> A=sym([1 1 0;1 1 0;1 1 1;0 1 1])
```

```
>> [Q R]=qr(A)
```

What is **Q** and **R**? Compare this with the answer in tutorial 8 question 5(a).

(c) Explain how you might use the command **qr** in MATLAB to find a QR factorization of a $m \times n$ matrix **A**?

2. (a) $A = QR \qquad Q \in M_{m\times n} \qquad R \in M_{n\times n}$

$Q = (q_1 \cdots q_n)$ orthonormal column vectors in $\mathbb{R}^m$ $(m > n)$

extend this set to be an orthonormal basis $\{q_1, \cdots, q_n, \tilde{q}_{n+1}, \cdots, \tilde{q}_m\}$ of $\mathbb{R}^m$

$Q' := (q_1 \cdots q_n \ \tilde{q}_{n+1} \cdots \tilde{q}_m) = (\underset{\uparrow m\times n}{Q} \ \ \underset{\uparrow m\times(m-n)}{\tilde{Q}}) \in M_{m\times m}$ orthonormal.

$R' := \begin{pmatrix} R \leftarrow n\times n \\ 0 \leftarrow (m-n)\times n \end{pmatrix} \in M_{m\times n}$ Then $A = QR = QR + \tilde{Q}\cdot 0 = Q'R'$.

(b) The code qr returns the full QR-factorization of A.

$$Q' = \begin{pmatrix} 1/\sqrt{3} & 0 & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad R' = \begin{pmatrix} \sqrt{3} & \sqrt{3} & 1/\sqrt{3} \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{6}/3 \\ 0 & 0 & 0 \end{pmatrix}$$

Q R

(c) Q = first n columns of Q' ; R = first n rows of R'

3. (Cayley-Hamilton theorem)
Consider

$$p(\mathbf{X}) = \mathbf{X}^3 - 4\mathbf{X}^2 - \mathbf{X} + 4\mathbf{I}.$$

(a) Compute $p(\mathbf{X})$ for $\mathbf{X} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

(b) Find the characteristic polynomial of $\mathbf{X}$.

(c) Show that $\mathbf{X}$ invertible. Express the inverse of $\mathbf{X}$ as a function of $\mathbf{X}$.

3. (a) $p(X) = 0$

(b) $f(\lambda) = \det(\lambda I - X) = \lambda^3 - 4\lambda^2 - \lambda + 4 = p(\lambda)$

(c) $0 = p(X) = X^3 - 4X^2 - X + 4I$

$\Rightarrow I = -\frac{1}{4} X(X^2 - 4X - 1)$, i.e. $X^{-1} = -\frac{1}{4}(X^2 - 4X - 1)$

Remark :

This question demonstrated the *Cayley-Hamilton theorem*, which states that if $p(x)$ is the characteristic polynomial of $\mathbf{X}$, then $p(\mathbf{X}) = 0$. This also show that if 0 is not an eigenvalue of $\mathbf{X}$, then the constant term of the characteristic polynomial $p(x)$ is nonzero, and we can use that to compute the inverse of $\mathbf{X}$.

The characteristic polynomial $\underset{a_n x^n + \cdots + a_1 x + a_0}{\overset{\parallel}{p_A}}$ is an annihilating polynomial for A.

① A^n can be written as linear combinations of $\{I, A, \cdots, A^{n-1}\}$;

② Since the roots of p_A are eigenvalues of A,

A is invertible $\Longleftrightarrow$ A has no zero eigenvalue

$\Longleftrightarrow$ $a_0 = p_A(0) \neq 0$

$\Longleftrightarrow$ A^{-1} can be written as linear combinations of $\{I, A, \cdots, A^{n-1}\}$

① & ② $\Rightarrow$ For invertible matrices $A \in M_{n\times n}$, A^k ($\forall k \in \mathbb{Z}$) can be written as linear combinations of $\{I, A, \cdots, A^{n-1}\}$

4. For each of the following matrices $\mathbf{A}$, determine if $\mathbf{A}$ is diagonalizable. If $\mathbf{A}$ is diagonalizable, find an invertible $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and determine $\mathbf{P}^{-1}\mathbf{AP}$.

(a) $\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$.

4. (a) $p(x) = (x+2)^2(x-4)$ eigenvalues $\lambda_1 = -2$ $\lambda_2 = 4$

algebraic multiplicity $a_{-2} = 2$ $a_4 = 1$

$-2I - A = \begin{pmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{pmatrix}$ $\dim(E_{-2}) = 3 - \text{rank}(-2I - A) = 2 = g_{-2}$

geometric multiplicity

$4I - A = \begin{pmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{pmatrix}$ $\dim(E_4) = 3 - \text{rank}(4I - A) = 1 = g_4$

$a_{-2} = g_{-2}$ & $a_4 = g_4 \Rightarrow A$ is diagonalizable

Find P. Solve for $(-2I - A)v = 0 \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ basis of E_{-2}

$(4I - A)v = 0 \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ basis of E_4

$P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $D = \begin{pmatrix} -2 & & \\ & -2 & \\ & & 4 \end{pmatrix}$ $A = PDP^{-1}$

(b) $\mathbf{A} = \begin{pmatrix} 9 & 8 & 6 & 3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(b) upper triangular matrix $\Rightarrow \lambda_1 = 9$ $\lambda_2 = -1$ $\lambda_3 = 2$ $\lambda_4 = 3$

$a_9 = a_{-1} = a_2 = a_3 = 1 \Rightarrow$ diagonalizable.

$(9I - A)v = 0 \Rightarrow E_9 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$(-I - A)v = 0 \Rightarrow E_{-1} = \text{Span}\left\{ \begin{pmatrix} -4 \\ 5 \\ 0 \\ 0 \end{pmatrix} \right\}$

$(2I - A)v = 0 \Rightarrow E_2 = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

$(3I - A)v = 0 \Rightarrow E_3 = \text{Span}\left\{ \begin{pmatrix} 5 \\ -6 \\ 0 \\ 6 \end{pmatrix} \right\}$

$P = \begin{pmatrix} 1 & -4 & -2 & 5 \\ 0 & 5 & 1 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$ $D = \begin{pmatrix} 9 & & & \\ & -1 & & \\ & & 2 & \\ & & & 3 \end{pmatrix}$ $A = PDP^{-1}$

(c) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

(c) only has one eigenvalue $\lambda = 1$ with $a_1 = 3$

$g_1 = \dim(E_1) = 3 - \text{rank}(I - A) = 1 < a_1 = 3$

A is not diagonalizable

(d) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(d) $p(x) = (x-1)^2(x+1)^2$ $\quad \lambda_1 = 1 \quad \lambda_2 = -1$

$a_1 = 2 \quad a_{-1} = 2$

$(-I - A)v = 0 \Rightarrow E_{-1} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\quad g_{-1} = \dim(E_{-1}) = 2 = a_{-1}$

$(I - A)v = 0 \Rightarrow E_1 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\quad g_1 = \dim(E_1) = 2 = a_1$

$A = PDP^{-1}$ is diagonalizable with $P = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

(e) $\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ -4 & 2 & 3 \end{pmatrix}$.

(e) $p(x) = (x-1)\underbrace{(x^2 - 2x + 2)}_{\text{discriminant } = -4}$

$\longrightarrow$ does not factor into linear factors, hence A is not diagonalizable.

Remark :

Examples shown in (c) are typical non-diagonalizable matrices.

Jordan blocks $\quad J_{\lambda,m} := \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \\ & & 1 & \lambda \end{pmatrix}_{m \times m}$ $\quad$ Jordan matrix $\quad J = \begin{pmatrix} J_{\lambda_1,m_1} & & \\ & \ddots & \\ & & J_{\lambda_k,m_k} \end{pmatrix}$

$\cup$

Diagonal matrices can be written as $\text{diag}(J_{\lambda_1,1}, J_{\lambda_2,1}, \cdots, J_{\lambda_n,1})$

Even though not all matrices are diagonalizable,

they must be similar to a Jordan matrix (in $\mathbb{C}$),

which is unique up to a permutation of diagonal blocks

Jordan canonical form $\quad$ (simplest & unique identification)

5. (a) Show that λ is an eigenvalue of $\mathbf{A}$ if and only if it is an eigenvalue of $\mathbf{A}^T$.

(b) Suppose $\mathbf{A}$ is diagonalizable. Is $\mathbf{A}^T$ diagonalizable? Justify your answer.

(c) Suppose $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ associated to eigenvalue λ. Show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^k$ associated to eigenvalue λ^k for any positive integer k.

(d) If $\mathbf{A}$ is invertible, show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^k$ associated to eigenvalue λ^k for any negative integer k.

5. (a) $\det(\lambda I - A) = \det(\lambda I - A^T)$ same roots

(b) Yes. If $A = PDP^{-1}$, then $A^T = (PDP^{-1})^T = (P^T)^{-1} D P^T$

(c) $Av = \lambda v \Rightarrow A^k v = A^{k-1}(\lambda v) = \lambda A^{k-1} v = \cdots = \lambda^k v$

(d) A is inveritble $\Rightarrow \lambda \neq 0$.

$Av = \lambda v \Leftrightarrow v = \lambda A^{-1} v \Leftrightarrow A^{-1} v = \lambda^{-1} v$

$A^k v = (A^{-1})^{-k} v = (\lambda^{-1})^{-k} v = \lambda^k v$

(e) A square matrix is said to be *nilpotent* if there is a positive integer k such that $\mathbf{A}^k = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then 0 is the only eigenvalue.

(f) Let $\mathbf{A}$ be a $n \times n$ matrix with one eigenvalue λ with algebraic multiplicity n. Show that $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ is a scalar matrix, $\mathbf{A} = \lambda\mathbf{I}$.

(g) Show that the only diagonalizable nilpotent matrix is the zero matrix.

(e) Fix an arbitrary eigenvalue λ of A, then

$\exists\, v \neq 0$ s.t. $Av = \lambda v$, thus $0 = A^k v = \lambda^k v$, $\lambda = 0$

(f) A is diagonalizable $\Leftrightarrow$ $A = PDP^{-1}$, $D = \lambda I_n$

$$PDP^{-1} = P(\lambda I_n)P^{-1} = \lambda I_n$$

(g) $A = PDP^{-1}$ with $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

$$\Rightarrow 0 = A^k = \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{k \text{ times}} = PD^kP^{-1}$$

$$\Rightarrow 0 = D^k = \mathrm{diag}(\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k)$$

$$\Rightarrow \lambda_1 = \cdots = \lambda_n = 0, \quad D = 0. \quad A = PDP^{-1} = 0.$$

Rmk: The more distinct eigenvalues a matrix has, the more likely it is to be diagonalizable.