Review

Vector spaces (linear spaces) : (10 axioms , for addition / scalar multiplication well-defined)

classical examples :

think of . (forgetting other dimensions)

Euclidean n-space $\mathbb{R}^n$

Linear span of several vectors $\text{Span}\{u_1, \cdots, u_k\}$

justify if $v \in \text{Span}\{u_1, \cdots, u_k\}$

$\text{rref}\,(u_1 \; u_2 \; \cdots \; u_k \mid v)$

justify if $\mathbb{R}^n = \text{Span}\{u_1, \cdots, u_k\}$

① $\text{rref}\,(u_1 \; u_2 \; \cdots \; u_k)$

② if $k < n$ No

③ if $k = n$ $\det(u_1 \cdots u_k) \neq 0$

Solution sets to homogeneous linear systems

$\updownarrow$

" linear restrictions "

determine the relation between Spans

$\text{rref}\,(u_1 \cdots u_k \; v_1 \cdots v_l)$

$\text{rref}\,(v_1 \cdots v_l \; u_1 \cdots u_k)$

$\mathbb{R}^2$ straight lines crossing through the origin

$\mathbb{R}^3$ straight lines / planes crossing through the origin

$\vdots$

$\mathbb{R}^n$

1 coefficient $\longrightarrow$ lines $\longleftarrow$ $(n-1)$ restrictions

2 coefficients $\longrightarrow$ planes $\longleftarrow$ $(n-2)$ restrictions

solution sets to non-homogeneous system $Ax = b$ $\longrightarrow$ affine space $= u + V$

where u is an arbitrary sln to $Ax = b$, and $V = \{x \mid Ax = 0\}$

one particular solution u + general solution to the corre. homogeneous linear system

i.e. translation of hyperspaces along u

proof. "$\supseteq$" : for every vector $z = u + v \in u + V$. $Az = Au + Av = b$

"$\subseteq$" : if y is a solution to $Ax = b$.

then $A(y-u) = Ay - Au = 0$.

namely $y - u \in V$.

1. Let $A = \{ (1+t, 1+2t, 1+3t) \mid t \in \mathbb{R} \}$ be a subset in $\mathbb{R}^3$.

(a) Describe A geometrically.

(b) Show that $A = \{ (x,y,z) \mid x+y-z=1 \text{ and } x-2y+z=0 \}$.

(c) Write down a matrix equation $\mathbf{M}\mathbf{x} = \mathbf{b}$ where $\mathbf{M}$ is a 3×3 matrix and $\mathbf{b}$ is a 3×1 matrix such that its solution set is A.

1. (a) straight line connecting $(1,1,1)$, $(2,3,4)$

(b) $\boxed{1}$ solve $\begin{cases} x+y-z=1 \\ x-2y+z=0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}(2+s) \\ y = \frac{1}{3}(1+2s) \\ z = s \end{cases}$ $s \in \mathbb{R}$

$s = 1+3t$

$\boxed{2}$ $A \subseteq$ RHS, A & RHS linear spaces, $\dim A = \dim$ RHS

$\Rightarrow A =$ RHS

(c) $M = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

2. Let $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$.

(a) If possible, express each of the following vectors as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(i) $\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} -4 \\ 6 \\ -13 \\ 4 \end{pmatrix}$

(b) Is it possible to find 2 vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that they are not a multiple of each other, and both are not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

2. (a) i.e. solve for $u_1x_1 + u_2x_2 + u_3x_3 = v$ $\quad (u_1 \; u_2 \; u_3 \mid v) \xrightarrow{\text{rref}}$

$$\left(\begin{array}{ccc|cccc} 2 & 3 & -1 & 2 & 0 & 1 & -4 \\ 1 & -1 & 0 & 3 & 0 & 1 & 6 \\ 0 & 5 & 2 & -7 & 0 & 1 & -13 \\ 3 & 2 & 1 & 3 & 0 & 1 & 4 \end{array}\right) \to \left(\begin{array}{ccc|cccc} 1 & & & 2 & 0 & 0 & 3 \\ & 1 & & -1 & 0 & 0 & -3 \\ & & 1 & -1 & 0 & 0 & 1 \\ & & & 0 & 0 & 1 & 0 \end{array}\right)$$

alternatively

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & a_1 \\ 1 & -1 & 0 & a_2 \\ 0 & 5 & 2 & a_3 \\ 3 & 2 & 1 & a_4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & a_2 \\ 0 & 5 & -1 & a_1 - 2a_2 \\ 0 & 0 & 3 & -a_1 + 2a_2 + a_3 \\ 0 & 0 & 0 & a_1 + 7a_2 + 2a_3 - 3a_4 \end{array}\right)$$

(b) Yes $\quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix} = v_1 + u_1$

3. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x - y - z = 0 \right\}$ be a subset of $\mathbb{R}^3$.

(a) Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\}$. Show that $\operatorname{span}(S) = V$.

(b) Let $T = S \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Show that $\operatorname{span}(T) = \mathbb{R}^3$.

3. (a) $\operatorname{Span}(S) \subseteq V$, V linear space. $\dim(V) = 2 \Rightarrow V = \operatorname{Span} S$

(b) sufficient to prove $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ linearly independent

$\Leftrightarrow$ $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$ only has the trivial solution

$\Leftrightarrow$ rref $\Leftrightarrow$ $\det(a_1 \ a_2 \ a_3) \neq 0$ necessity & sufficiency

necessary & sufficient conditions

4. Which of the following sets S spans $\mathbb{R}^4$?

(i) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(ii) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(iii) $S = \left\{ \begin{pmatrix} 6 \\ 4 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -2 \\ -1 \end{pmatrix} \right\}$.

(iv) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

4. (ii) No. $\dim S \leq 3 < 4$

(i) $\det \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \neq 0$ ✓

(iii) $\operatorname{rref}(a_1 \ a_2 \ a_3 \ a_4 \ a_5) = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

a row of zeros in RREF $\Rightarrow$ No

(iv) $\operatorname{rref}(a_1 \ a_2 \ a_3 \ a_4 \ a_5) = \begin{pmatrix} 1 & & & & 3 \\ & 1 & & & 0 \\ & & 1 & & 5 \\ & & & 1 & -1 \end{pmatrix}$ Yes

5. Determine whether span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq$ span$\{\mathbf{v}_1, \mathbf{v}_2\}$ and/or span$\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq$ span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if

(a) $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(b) $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}$.

5. (a) $\det(u_1 \;\; u_2 \;\; u_3) = 0$

$$\operatorname{rref}(u_1 \;\; u_2 \;\; u_3 \;\; v_1 \;\; v_2) = \left(\begin{array}{ccc|cc} 1 & 0 & -\frac{9}{2} & 3 & 0 \\ 0 & 1 & -9 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\operatorname{rref}(v_1 \;\; v_2 \;\; u_1 \;\; u_2 \;\; u_3) = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & \frac{9}{10} \end{array}\right)$$

So $\operatorname{Span}\{v_1, v_2\} \not\subseteq \operatorname{Span}\{u_1, u_2, u_3\}$

& $\operatorname{Span}\{u_1, u_2, u_3\} \not\subseteq \operatorname{Span}\{v_1, v_2\}$

(b) $\det(u_1 \;\; u_2 \;\; u_3) = 0$

$\operatorname{rref}(u_1 \;\; u_2 \;\; u_3 \;\; v_1 \;\; v_2) =$

$\operatorname{rref}(v_1 \;\; v_2 \;\; u_1 \;\; u_2 \;\; u_3) =$

So $\operatorname{Span}\{u_1, u_2, u_3\} = \operatorname{Span}\{v_1, v_2\}$

6. Determine which of the following sets are subspaces. For those sets that are subspaces, express the set as a linear span. For those sets that are not, explain why.

(a) $S = \left\{ \begin{pmatrix} p \\ q \\ p \\ q \end{pmatrix} \middle| \; p, q \in \mathbb{R} \right\}$.

(b) $S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; a \geq b \text{ or } b \geq c \right\}$.

(c) $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \; 4x = 3y \text{ and } 2x = -3w \right\}$.

(d) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right\}$.

(e) $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \; w + x = y + z \right\}$.

(f) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; ab = cd \right\}$.

(g) S is the solution set of $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.

6. (a) $S = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ ✓

(b) No. $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot (-1) = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$

(c) $S = \text{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ ✓

(d) $\det = a - c - d = 0$ ✓

(e) ✓

(f) No. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \notin S$

(g) ✓