

Lecture notes 3

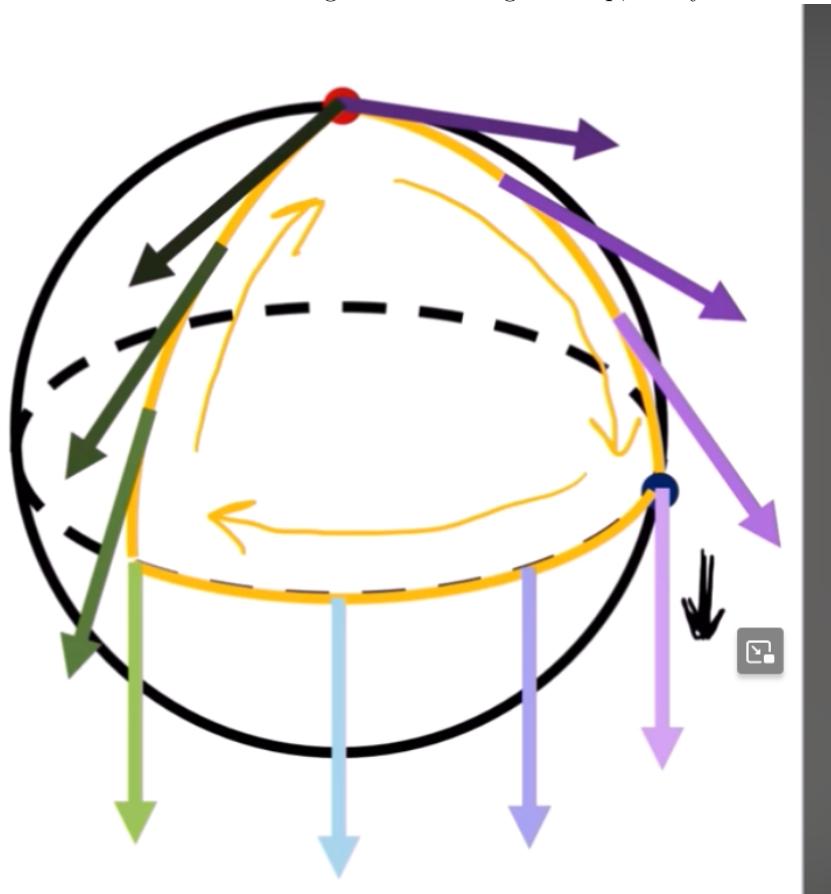
The Riemann tensor, Ricci tensor, and Ricci scalar

If you think about it for long enough, the Γ_{jk}^i really give us a lot of information about the *curvature* of our space. They tell us how much our derivatives differ from flat space derivatives.

One object we can use to measure this curvature in terms of the Γ_{jk}^i is called the **Riemann tensor**

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda$$

This formula looks very complicated (and it is), but it is conceptually not that bad. It measures how much a vector changes when moving in a loop, in any direc-



tion.

About the Riemann tensor

The Riemann tensor is the most complete description of the curvature of a manifold. Flat space in Cartesian coordinates has $R_{\sigma\mu\nu}^\rho = 0$. If we use different coordinates or are in curved space, then some of the Riemann tensor components will be nonzero.

Ricci tensor and scalar Often we work with simplified versions of the Riemann tensor. One is called the **Ricci tensor**

$$R_{\sigma\nu} \equiv R_{\sigma\rho\nu}^{\rho}$$

We contracted the first and third indices to average some parts of the Riemann tensor.

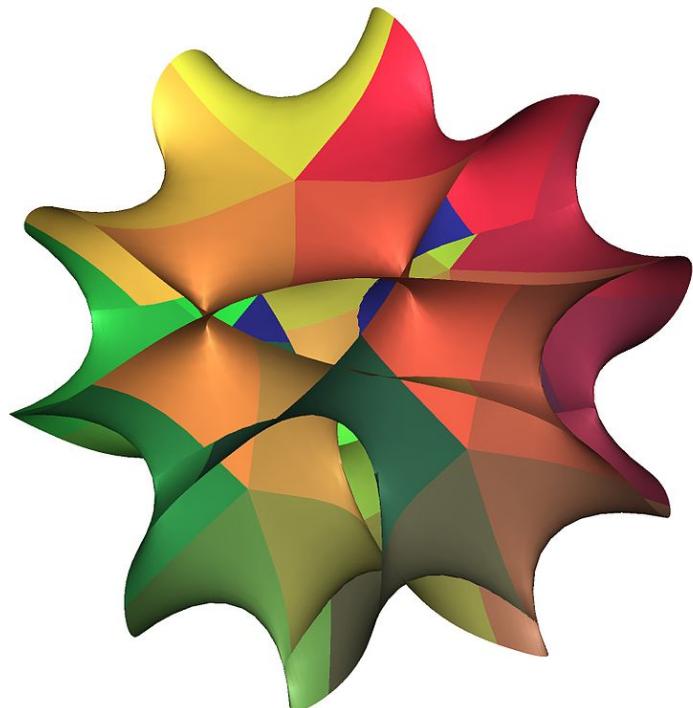
Sometimes we average even more components, and study the **Ricci scalar**, the trace of the Ricci tensor

$$R \equiv R_{\alpha}^{\alpha}$$

Normally, **what we call a flat space has $R = 0$** . The coordinate system cannot change the Ricci scalar because it is a scalar!

The Ricci scalar closely corresponds to our description of Gaussian curvature the other day: the surface of a cylinder would have $R = 0$, the surface of a saddle $R < 0$, and the surface of a sphere $R > 0$.

$R = 0$ includes *many more spaces than you would think of as being flat*. To avoid confusion, people sometimes call them **Ricci-flat**, like this Calabi-Yau manifold (we're looking at a 2D slice of the full 6D manifold):



Einstein Tensor

This combination of curvature enters into the actual Einstein equations, and is directly related to the matter content of spacetime:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

This has a ton of derivatives of the metric!! Real solutions in GR are very hard to find! ### Einstein equations

Exact Solutions to Einstein's Field Equations Second Edition

HANS STEPHANI
DIETRICH KRAMER
MALCOLM MACCALLUM
CORNELIUS HOENSELAERS
EDUARD HERLT

CAMBRIDGE MONOGRAPHS
ON MATHEMATICAL PHYSICS

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$T_{\mu\nu}$ is called the **stress-energy tensor** and is related to the mass and energy content of spacetime. Both $G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric, rank-2 tensors. This gives us 10 (coupled, non-linear) partial differential equations in 4 dimensions.

Normally in GR, we will label coordinates t, x, y, z or similar, with coordinate 0 being time. Greek letter indices μ, ν, \dots will go from 0 ... 3 while Latin letter indices i, j, k, \dots go from 1 ... 3 (spatial directions only).

Ideal recipe for a solution of Einstein's equations: 1. Put in your matter and energy into $T_{\mu\nu}$ 2. Solve (with difficulty!) for the metric $g_{\mu\nu}$ 3. ... Profit!

(Cynical) recipe for an exact solution in GR: 1. Pick a metric $g_{\mu\nu}$ 2. Take derivatives to build $R_{\beta\gamma\delta}^\alpha$ and $G_{\mu\nu}$ 3. Invent matter so that $T_{\mu\nu} = G_{\mu\nu}/(8\pi)$ (doesn't need to be possible) 4. ... Profit!

Gravitational waves, at least as we will study them, are actually only an approximate solution to GR. Many approximation methods for higher accuracy solutions, we'll study some of them.

Stress-energy tensor

Components: - T_{00} : energy density - T_{i0} : energy flux - T_{ii} : pressure - T_{ij} : shear

Continue our example with tensor calculus

In polar coordinates, let's compute the Christoffel symbols.

Answers:

$$\Gamma_{\theta\theta}^r = -r\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = 1/r$$

Riemann tensor is all zero!

Geodesics

“Straightest possible paths” in curved space.

$$\ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = 0$$

Geodesics are the paths which parallel transport a vector without it rotating, even on a curved surface. They're also the shortest paths between any two points.

Geodesics on: - sphere: great circles. - plane: straight lines - saddle: hyperbolae