Lecture notes 6

Generation of gravitational waves

Gravitational waves in linear theory **must be small**. Remember, we assumed that we could neglect $\mathcal{O}(h^2)$ terms in Einsteins' equations.

A "large" gravitational wave would have $|h| \sim \mathcal{O}(1)$.

Solving the sourced wave equation

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

There are many ways to skin a cat, but we will solve this equation methodically with a good guess for the form of the solution (based on the book by Schutz). Another way is with the *Green's function* method (see MTW). There's nothing too crazy about this solution, if you've taken E&M you've seen worse!

Last class we took $T_{\mu\nu} = 0$, this time, let's take

$$T_{\mu\nu} = S_{\mu\nu}(x^i)e^{-i\omega t}$$

In practice, this is not a big restriction. If we can solve this case, then we can write any possible time dependence as a sum of different choices for ω (this is just Fourier analysis). Many sources are periodic too.

It will also make our solution easier if the region where $S_{\mu\nu} \neq 0$ is much smaller than one wavelength of the gravitational wave. This is equivalent to all the sources are moving slowly compared to the speed of light. $\lambda_{\mathrm{GW}} = 2\pi/\omega$. Sometimes this is called the **flow motion approximation**. In practice, this is true for all but the most powerful GW sources.

Let's look for a solution of the form

$$\bar{h}_{\mu\nu} = B_{\mu\nu}(x^i)e^{-i\omega t}$$

Then plugging that in,

$$\begin{aligned}
\Box h_{\mu\nu} &= -\partial_t^2 \bar{h}_{\mu\nu} + \partial_j \partial^j \bar{h}_{\mu\nu} &&= -16\pi T_{\mu\nu} \\
&= B_{\mu\nu}\omega^2 e^{-i\omega t} + \partial_j \partial^j B_{\mu\nu} e^{-i\omega t} &&= -16\pi S_{\mu\nu} e^{-i\omega t} \\
(\omega^2 &+ \partial_j \partial^j) B_{\mu\nu} &&= -16\pi S_{\mu\nu}
\end{aligned}$$

Let's try to use spherical coordinates with the source in a small region around $r = 0$. Then we can guess that $B_{\mu\nu}$ takes the form

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r} e^{i\omega r} + \frac{C_{\mu\nu}}{r} e^{-i\omega r}$$

This is the same as our solution to the wave equation on the string the other day, but we allow for both left and right solutions at the same time. In this case, they are in the $\pm r$-directions.

We can assume that we won't have any waves going in $-r$, so let's just take $C_{\mu\nu} = 0$.

Let's integrate our equation, again assuming that $S_{\mu\nu}$ is only nonzero in a small radius around the center ($r < \varepsilon$):

$$\begin{gathered}
\omega^2 \int B_{\mu\nu} \mathrm{d}^3 x + \int \nabla^2 B_{\mu\nu} \mathrm{d}^3 x = -16\pi \int S_{\mu\nu} \mathrm{d}^3 x \\
\omega^2 \frac{4}{3} \pi \varepsilon^3 \bar{B}_{\mu\nu} + \oint \vec{n}.\vec{\nabla} B_{\mu\nu} \mathrm{d}S = -16\pi J_{\mu\nu}
\end{gathered}$$

we defined $\bar{B}_{\mu\nu}$ to mean the average $B_{\mu\nu}$ over the source ball, and similarly defined the integral of $S_{\mu\nu}$ to just be $J_{\mu\nu}$.

The integral of the Laplacian, by Gauss's theorem, can be written as a surface integral of the gradient over a bounding surface (we'll take a sphere). Simplifying more:

$$\begin{aligned}
\oint \vec{n}.\vec{\nabla} B_{\mu\nu} \mathrm{d}S &= 4\pi\varepsilon^2 \left.\frac{\mathrm{d}B_{\mu\nu}}{\mathrm{d}r}\right|_{r=\varepsilon} \\
&= 4\pi\varepsilon^2 \left[-\frac{A_{\mu\nu}}{r^2} e^{i\omega r} + i\omega \frac{A_{\mu\nu}}{r} e^{i\omega r} \right] \\
&= 4\pi\varepsilon^2 \left[-\frac{A_{\mu\nu}}{\varepsilon^2} + i\omega \frac{A_{\mu\nu}}{\varepsilon} \right] e^{i\omega\varepsilon} \\
&\simeq 4\pi[-A_{\mu\nu} + i \cdot 0] \cdot 1 \\
&= -4\pi A_{\mu\nu}
\end{aligned}$$

where we used in the second to last line that $\varepsilon \ll 2\pi/\omega$ and that the equation must be real.

Now, taking the limit that $\varepsilon \to 0$,

$$\begin{gathered}
0 - 4\pi A_{\mu\nu} = -16\pi J_{\mu\nu} \\
A_{\mu\nu} = 4 J_{\mu\nu}
\end{gathered}$$

$$\bar{h}_{\mu\nu} = B_{\mu\nu} e^{-i\omega t}$$

$$\boxed{\bar{h}_{\mu\nu} = \frac{4 J_{\mu\nu}}{r} e^{i\omega(r-t)}}$$

These are the GW generated by a source, neglecting $1/r^2$ terms and $1/r$ terms at order ε.

The components of $J_{\mu\nu}$

We can further simplify with some tensor tricks. Remember that

$$J_{\mu\nu}e^{-i\omega t} = \int T_{\mu\nu}\mathrm{d}^3x$$

Let's raise the indices and take a time derivative of the $\mu 0$ component:

$$-i\omega J^{\mu 0}e^{-i\omega t} = \int T^{\mu 0}_{,0}\mathrm{d}^3x$$

In general $T^{\mu\nu}{}_{,\nu} = T^{\mu\nu}{}_{,\mu} = 0$ (this is the **conservation of mass and energy**), so

$$T^{\mu 0}{}_{,0} = -T^{\mu k}{}_{,k}$$

where k is spatial-only. Then

$$-i\omega J^{\mu 0}e^{-i\omega t} = -\int T^{\mu k}{}_{,k}\mathrm{d}^3x = \oint T^{\mu k}n_k \mathrm{d}S = 0$$

If $\omega \neq 0$, then $J^{\mu 0} = \bar{h}^{\mu 0} = 0$.

What about J_{ij}? We can try using the **virial theorem**

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int T^{00}x^l x^m \mathrm{d}^3x = \int T^{lm}\mathrm{d}^3x$$

See Schutz Chapter 9.3. For a slowly-moving source, $T^{00} \simeq \rho$ (total mass energy is just Newtonian mass density). Let's call $\int T^{00}x^l x^m \mathrm{d}^3x \equiv I^{lm}$ the **mass quadrupole moment**. Then our virial theorem becomes

$$\ddot{I}^{lm} = 2\int T^{lm}\mathrm{d}^3x$$

If our system permits us to take $I^{lm} = D^{lm}e^{-i\omega t}$, then $\ddot{I}^{lm} = -\omega^2 D^{lm}e^{-i\omega t}$.

Combining everything we know then, we have these componens for $h^{\mu\nu}$:

$$\bar{h}_{\mu\nu} = \frac{4}{r}J_{\mu\nu}e^{i\omega(r-t)} \tag{1}$$
$$\bar{h}_{0\nu} = 0 \tag{2}$$
$$\bar{h}_{lm} = \frac{4}{r}J_{lm}e^{-i\omega t}e^{i\omega r} \tag{3}$$
$$\bar{h}_{lm} = \frac{2}{r}\ddot{I}_{lm}e^{i\omega r} \tag{4}$$
$$\bar{h}_{lm} = -\frac{2}{r}\omega^2 D_{lm}e^{i\omega(r-t)} \tag{5}$$

Re-absorbing the last term gives us **the quadrupole formula**:

$$\boxed{\bar{h}_{lm} = \frac{2}{r}\ddot{I}_{lm}(t-r)}$$

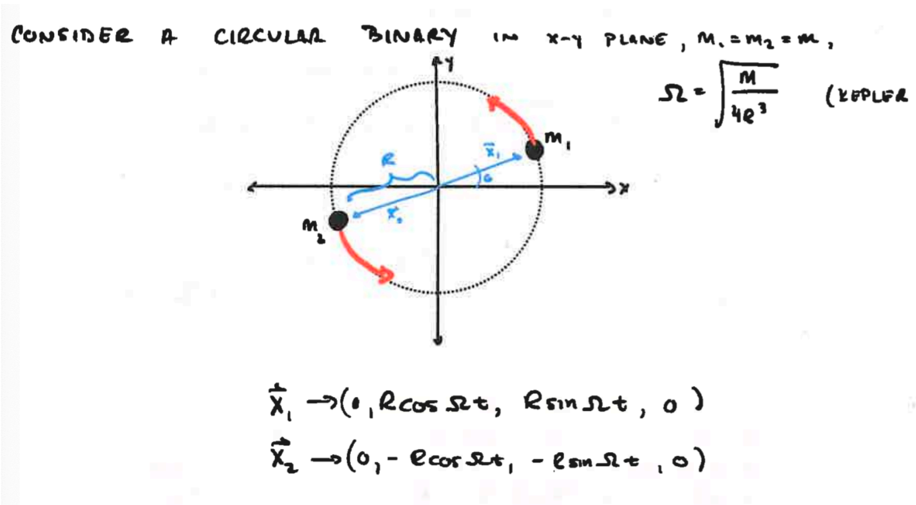

Figure 1: Screenshot 2026-01-29 at 01.10.33.png

An example source

Okay, so

$$T^{00}(t, x^i) = M\delta(x^3)\left[\delta(x^1 - R\cos\Omega t)\delta(x^2 - R\sin\Omega t) + \delta(x^1 + R\cos\Omega t)\delta(x^2 + R\sin\Omega t)\right]$$

Basically just setting up a product of 3 deltas at each mass position.

Then we need to calculate I_{ij}.

HW: try working out the integral

We get that

$$\begin{aligned} I_{11} &= MR^2(1 + \cos 2\Omega t) \\ I_{22} &= MR^2(1 - \cos 2\Omega t) \\ I_{12} &= MR^2 \sin 2\Omega t \\ I_{i3} &= 0 \end{aligned}$$

Then taking the time derivatives and plugging into our quadrupole formula

$$\bar{h}_{ij} = \frac{2}{r}\ddot{I}_{ij} = \frac{8M}{r}\Omega^2 R^2 \begin{pmatrix} -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ -\sin 2\Omega t & \cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This means that

$$\begin{aligned} h_+ &= \frac{8M}{r}\Omega^2 R^2 \cos 2\Omega t \\ h_\times &= -\frac{8M}{r}\Omega^2 R^2 \sin 2\Omega t \end{aligned}$$

Things to notice!

- GW frequency is twice the orbital frequency!
- GW amplitude scales with M and Ω.
- GW observations directly depend on distance (so if we know M, Ω, R, we can get r)
- GW from binaries are circularly polarized

Homework

We did this for an observer in the z axis (face-on). What if we were in the x, y plane (edge-on), what would we see? Find h_+ and $h_\times$, and compare to the face-on case.

Astrophysical GW sources

Anything with $\ddot{I} \neq 0$!

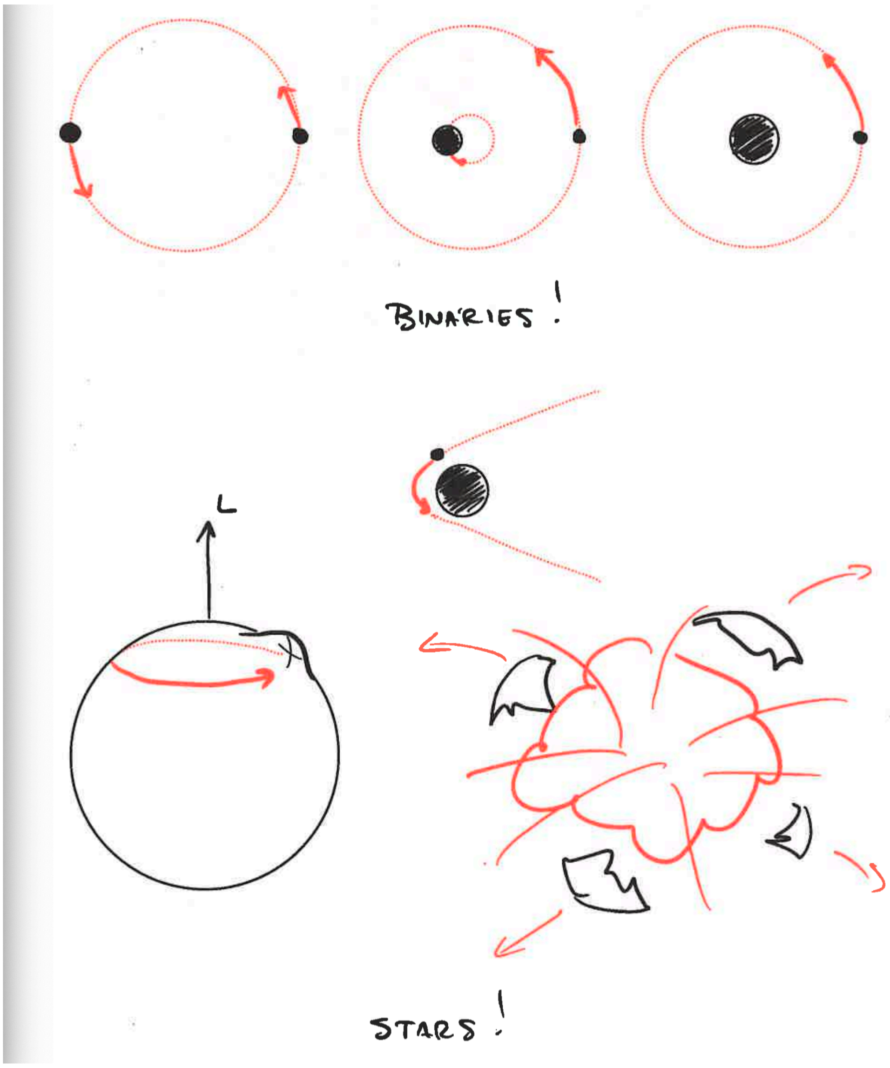

Figure 2: Screenshot 2026-01-29 at 01.27.30.png

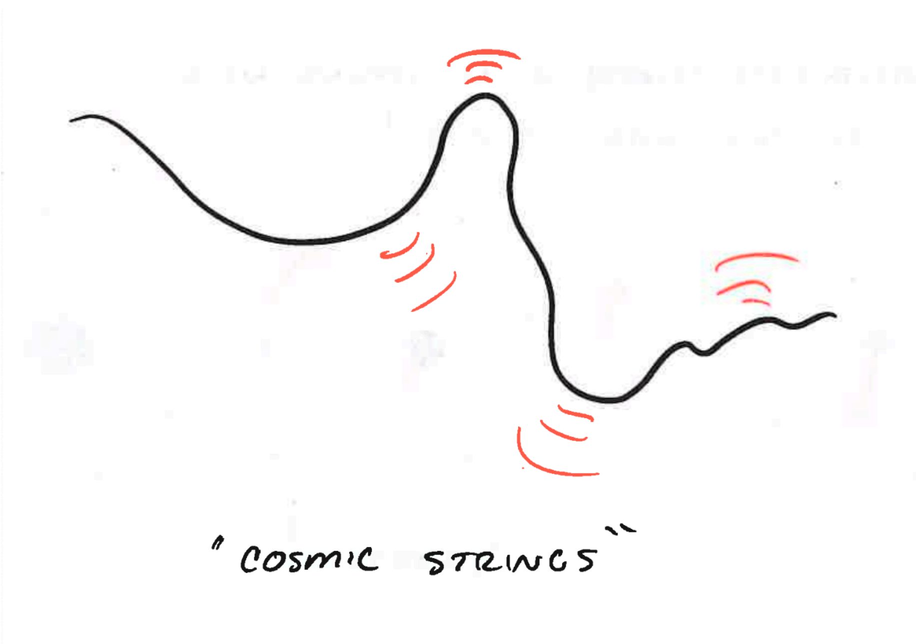

Figure 3: Screenshot 2026-01-29 at 01.27.46.png