Lecture notes 4

Solving Einstein's Equations

Remember, when we have

$$G_{\mu\nu} = 8\pi T_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$T_{\mu\nu}$ and $G_{\mu\nu}$ are symmetric! So in 4d, there are 10 Einstein's equations. In general these are coupled, second order in derivatives of $g_{\mu\nu}$, and non-linear!

Special relativity and the metric in flat spacetime

If you guys remember your special relativity, what all observers agree on is the *spacetime interval*, which we can normally write

$$\Delta s^2 = -(c\Delta t)^2 + \Delta \vec{x}^2$$

The equivalent flat space metric in GR then must be

$$\eta_{\mu\nu} \equiv \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Technically, everything we have been studying so far was for *Riemannian manifolds*, which have $\det g_{\mu\nu} > 0$. Spacetime is actually a *pseudo-Riemannian manifold*, with $\det g_{\mu\nu} < 0$. This has some mathematical consequences, but everything we've looked at so far in this class still works for either.

Vacuum solutions

Weak, static, spherically symmetric (outside of a star) If we use polar coordinates for spacetime (t, r, θ, ϕ), then we get:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & & & \\ & \left(1 + \frac{2M}{r}\right) & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$$

Alternatively this can be written:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Also the shorthand $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is very common.

This solution works well for our solar system or objects in orbit around the Earth. It matches Newtonian gravity, but also includes new predictions from GR! Explains well the precession of Mercury, light from stars bending while near the sun.

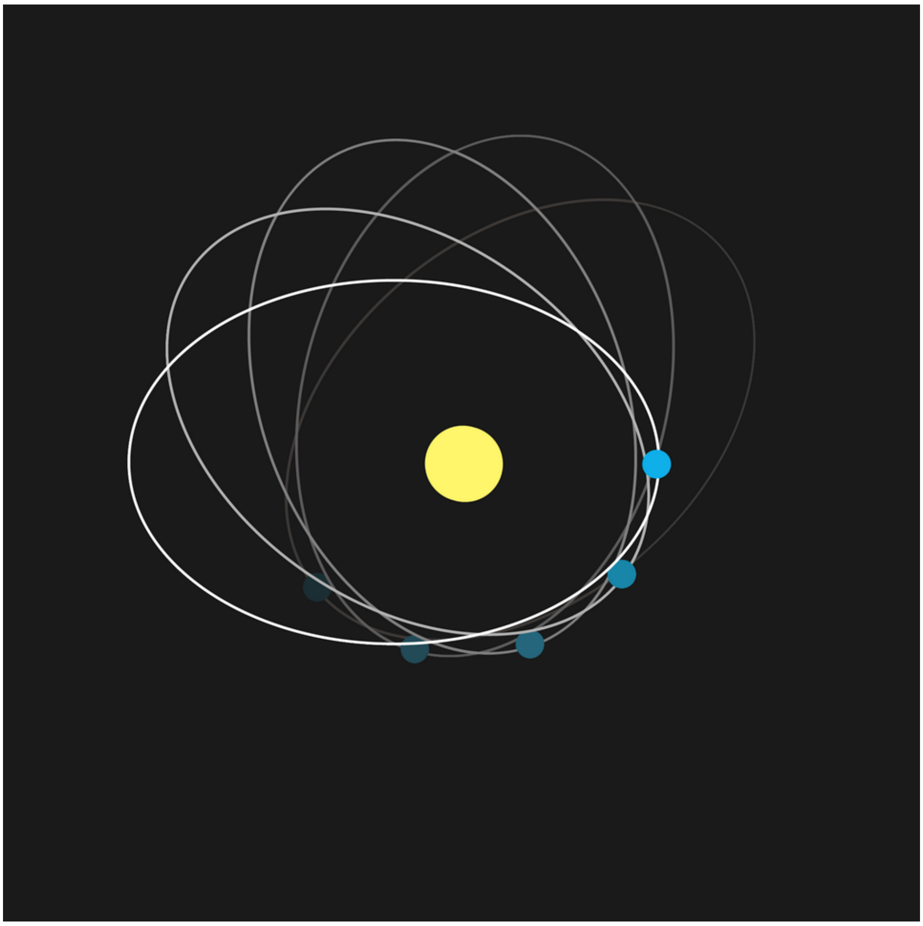

Figure 1: mercury_precession.png

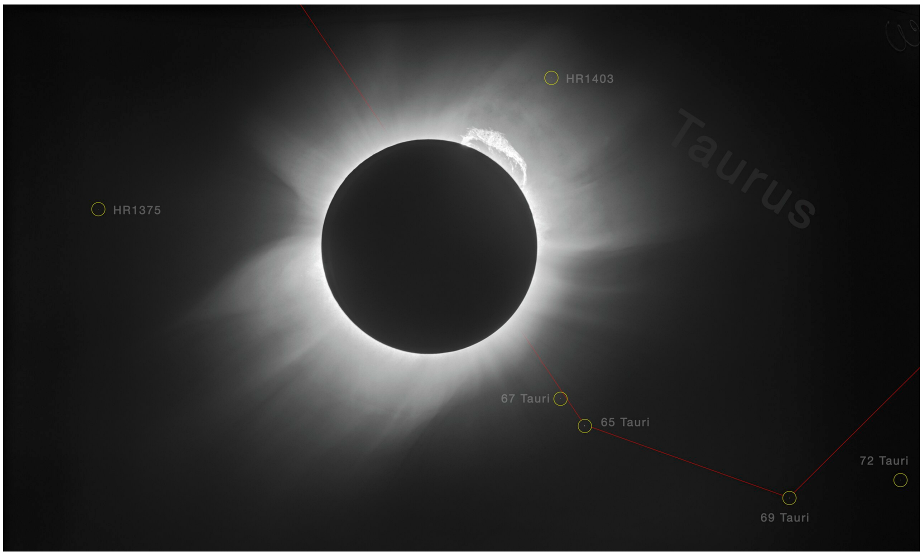

Figure 2: eddington_eclipse.png

Strong field version

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(\frac{1}{1 - 2M/r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Notice that, when M/r is small, we get the weak solution again:

$$\frac{1}{1 - 2M/r} = 1 + \frac{2M}{r} + \mathcal{O}\left((M/r)^2\right)$$

This strong-field solution for a point mass has a better-known name, it is the **Schwarzschild solution** and also describes non-rotating, uncharged black holes.

Weird things happen to our coordinates when $r = 2M$, or, restoring the constants, when $r = 2GM/c^2$. This is the black hole's event horizon.

> HW: calculate what the event horizon of a an object the mass of our sun would be.

Isotropic, homogeneous universe

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right)$$

This metric describes a universe full of a uniform mixture of matter, and is often used to describe the large-scale behavior of our universe in cosmology. It is usually called the **Friedmann-Robertson-Walker-Lemaitre (FLRW) metric**.

k sets the curvature of the universe, and could in principle be positive or negative. For our universe, it has been measured to be essentially 0.

Here $a(t)$ is the size of the universe over time. Even though this solution technically works without matter, once we add matter, that determines $a(t)$ and k.

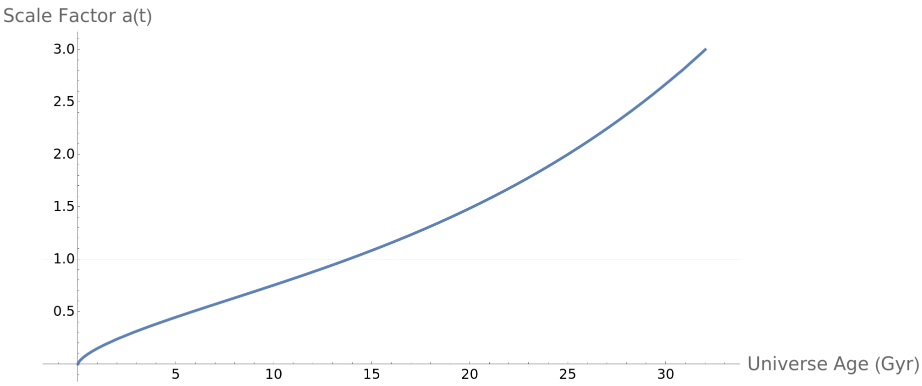

Figure 3: ./scale_factor.png

Gravitational waves

This is the moment we've been building up towards! Gravitational waves are another vacuum solution to GR. This means that they can keep traveling away from any sources. As we'll see, you need matter to actually source them.

So, let's try to solve Einstein's equations! We will assume that we have a vacuum solution, so $T_{\mu\nu} = 0$. Then that means $G_{\mu\nu}$ must also be 0. There are exact solutions for gravitational waves, but we will study an approximate solution for now, because it is easier.

Linearized gravity

To start, let's assume that gravitational waves must be an unknown *small approximation* around flat space.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

This approximation being "small" means $|h_{\mu\nu}| \ll 1$, or practically that **we'll take any terms that appear with h^2 and ignore them!**

Let's take this guess for $g_{\mu\nu}$ and use it to calculate the Einstein tensor $G_{\mu\nu}$, just keeping $h_{\mu\nu}$ as an unknown tensor.

Remember that $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

Raising / lowering We can raise and lower indices using only $\eta_{\mu\nu}$, the difference will always be $\mathcal{O}(h^2)$.

$$A^\mu = g^{\mu\nu} A_\nu \to A^\mu \sim \eta^{\mu\nu} A_\nu$$
$$h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$$

Christoffel symbols

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda}\right)$$

Okay, let's simplify this. We know that we can raise and lower with η, and that the derivatives of η are all 0. $\eta_{\mu\nu,\alpha} = 0$.

Using this info, we can simplify:

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}\eta^{\alpha\lambda}\left(h_{\lambda\nu,\mu} + h_{\mu\lambda,\nu} - h_{\mu\nu,\lambda}\right)\\ &= \frac{1}{2}\left(h^{\alpha}{}_{\nu,\mu} + h^{\alpha}{}_{\mu,\nu} - h_{\mu\nu}{}^{,\alpha}\right)\end{aligned}$$

Riemann tensor

$$R^{\rho}_{\sigma\mu\nu} = \Gamma^{\rho}_{\nu\sigma,\mu} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

In our case, we are dropping $\mathcal{O}(h^2)$ and $\Gamma \sim \mathcal{O}(h)$, so we should drop the Γ^2 terms. Let's actually try to skip some unnecessary algebra and go straight to the Ricci tensor

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

Ricci tensor Once we drop $\mathcal{O}(\Gamma^2)$, we have

$$\begin{aligned}R_{\mu\nu} &= \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu}\\ &= \frac{1}{2}(h^{\alpha}_{\nu,\mu\alpha} + h_{\mu}{}^{\alpha}{}_{,\nu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu} - h_{\mu}{}^{\alpha}{}_{,\alpha\nu} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu})\\ &= \frac{1}{2}(h^{\alpha}{}_{\nu,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu}).\end{aligned}$$

We can try to simplify more. Let's define

$$h^{\alpha}_{\alpha} = \mathrm{tr}(h_{\alpha\beta}) \equiv h$$

and the d'Alembertian operator:

$$\begin{aligned}{}_{,\alpha}^{,\alpha} &= -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\\ &= -\partial_t^2 + \nabla^2\\ &\equiv \Box\end{aligned}$$

Then we can rewrite the Ricci tensor as

$$R_{\mu\nu} = \frac{1}{2}(-\Box h_{\mu\nu} + h^{\alpha}{}_{\nu,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu} - h_{,\mu\nu})$$

Ricci scalar

$$
\begin{aligned}
R^\mu_\mu &= \frac{1}{2}(-\Box h^\mu{}_\mu + h^{\alpha\mu}{}_{,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha\mu} - h_{,\mu}{}^{\mu}) \\
&= \frac{1}{2}(-\Box h + h^{\alpha\mu}{}_{,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha\mu} - \Box h) \\
&= h^{\mu\alpha}{}_{,\mu\alpha} - \Box h
\end{aligned}
$$

Einstein Tensor

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \\
&= \frac{1}{2}(-\Box h_{\mu\nu} + h^\alpha{}_{\nu,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha}{}_\nu - h_{,\mu\nu} - h^{\alpha\beta}{}_{,\alpha\beta}\eta_{\mu\nu} + \eta_{\mu\nu}\Box h)
\end{aligned}
$$

Simplifying the solution

Someone smart (probably Einstein) at some point came up with the idea to define a special object called $\bar{h}_{\alpha\beta}$ that helps simplify the Einstein tensor in this case

$$
\begin{aligned}
\bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - \frac{1}{2}h\,\eta_{\mu\nu} \\
\bar{h} &= \bar{h}^\alpha_\alpha = -h
\end{aligned}
$$

$g^\alpha_\beta = \delta^\alpha_\beta$

Let's go through the terms one-by-one and rewrite them

$$
\begin{aligned}
-\Box h_{\mu\nu} &= -\Box\bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\bar{h} \\
h_\nu{}^\alpha{}_{,\mu\alpha} &= (\bar{h}^\alpha_\nu - \frac{1}{2}\bar{h}\eta^\alpha_\nu)_{,\mu\alpha} \\
&= \bar{h}_\nu{}^\alpha{}_{,\mu\alpha} - \frac{1}{2}\eta^\alpha_\nu\bar{h}_{,\mu\alpha} \\
&= \bar{h}_\nu{}^\alpha{}_{,\mu\alpha} - \frac{1}{2}\bar{h}_{,\mu\nu} \\
h_{\mu\alpha}{}^{,\alpha}{}_\nu &= \bar{h}_{\mu\alpha}{}^{,\alpha}{}_\nu - \frac{1}{2}\bar{h}_{,\mu\nu} \\
-h_{,\mu\nu} &= \bar{h}_{,\mu\nu} \\
-h^{\alpha\beta}{}_{,\alpha\beta}\eta_{\mu\nu} &= \left(-\bar{h}^{\alpha\beta}{}_{,\alpha\beta} + \frac{1}{2}\Box\bar{h}\right)\eta_{\mu\nu} \\
\eta_{\mu\nu}\Box h &= -\eta_{\mu\nu}\Box\bar{h}
\end{aligned}
$$

Whew! Okay, now let's rewrite the Einstein tensor with all of these terms in terms of $\bar{h}$.

A lot of terms end up cancelling! and we get the linearized field equations:

HW: try combining these yourself and see if you can get the same equation. It should be straightforward if you remember that $h_{\alpha\beta}$ is symmetric.

$$\boxed{-\Box\bar{h}_{\mu\nu} + \bar{h}_{\nu}{}^{\alpha}{}_{,\mu\alpha} + \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{\nu} - \bar{h}^{\alpha\beta}{}_{,\alpha\beta}\eta_{\mu\nu} = 16\pi T_{\mu\nu}}$$

First term: d'Alembertian operator

Other terms: preserve gauge invariance (more about this in a sec)

A nice gauge choice (Lorentz gauge)

Gauge symmetries in physics in general The idea of "gauge" comes from railroad cars, and is the distance between the rails. The exact distance you pick between the rails matters, but any width of rails will make a train run.

In general, "choosing a gauge" in physics means that our theory contains additional degrees of freedom compared to our physical system. Usually these extra degrees of freedom make the math of the theory much nicer, but can sometimes get in the way.

Example you might know: the electric field can be written as the gradient of a scalar field (the electric potential). But the exact height of the electric potential doesn't matter, only differences in electric potential give you an electric field. In other words, the ground in a circuit can be at 0V or at 100V, as long as the voltage differences are the same the physics will be identical. Choosing ground = 0V is a gauge choice.

Gauge choices in GR Gauge choices in general relativity are basically convenient coordinate transformations, that we can assume to work to cancel some terms in our equations. If we pick a certain gauge, our equations are still true, but they become not fully general in every coordinate system.

But choosing a gauge can simplify equations a lot, and not restrict our coordinate choices very much. In that sense, gauge choice is very useful!

We will try to choose coordinates so that $\bar{h}^{\mu\nu}_{,\nu} = 0$ (our linear perturbation around flat space is constant in some directions in our coordinates)

Consider a coordinate transformation of the form

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}$$
$$x^{\alpha} = x'^{\alpha} - \xi^{\alpha}$$

Then our Jacobian is

$$\frac{\partial x^{\beta}}{\partial x'^{\alpha}} = \delta^{\beta}_{\alpha} - \xi^{\beta}_{,\alpha}$$

And in the new coordinates:

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}_{,\alpha}$$

If we take a derivative and cancel out similar terms, we get that

$$\bar{h}'^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \Box\xi^{\mu}$$

If we want this to be zero, then we just need to choose $\Box\xi^{\mu} = \bar{h}^{\mu\nu}_{,\nu}$. This is a well known equation (inhomogenous wave equation), and a solution always exists for any RHS. This is called the **Lorentz Gauge**.

Putting in this gauge choice, our Einstein equations become

$$\boxed{-\Box\bar{h}_{\mu\nu} = 16\pi T_{\mu\nu}}$$

When $T_{\mu\nu} = 0$, this is just the wave equation!!

If only $T_{00} = \rho$ is nonzero (static mass), this equation actually reduces to the weak field solution we saw earlier!