Lecture notes 2

Office hours will be T 1-2pm in CRH 2104 or Zoom

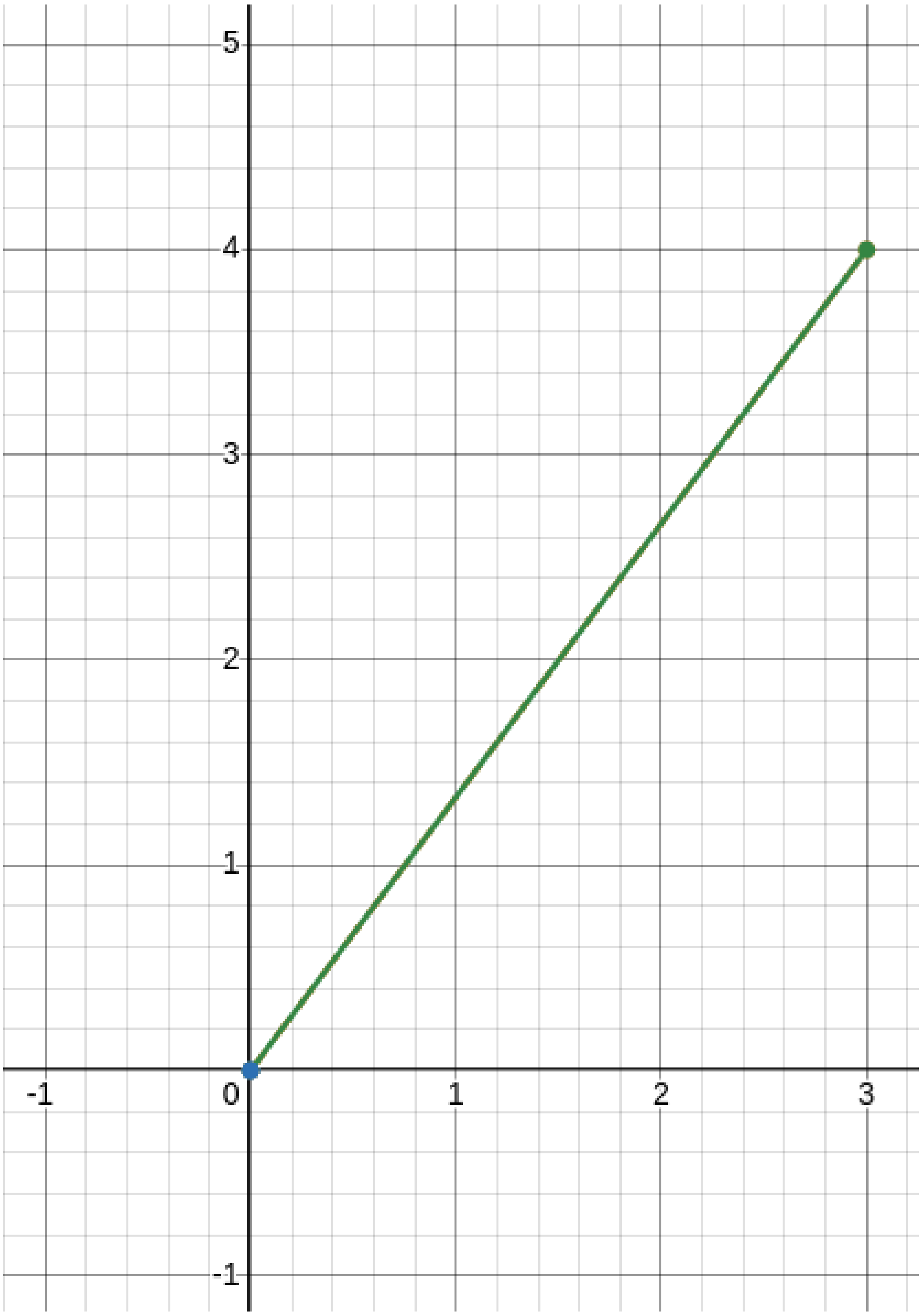
You can always email me if you have questions and can't make that time!

HW1 questions / review

Let's apply what we know so far to an example

The Cartesian plane in 2D: Our metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



For example, let's say we have a vector $a^i = (3, 4)$. The length of this vector from the origin is just $\sqrt{a^i a^j g_{ij}} = \sqrt{9 + 16} = 5$. Formally, we should be more careful: the metric gives us the small change in distance as we change in coordinates, and over long distances, we should integrate this up. So according to the metric $ds^2 = dx^2 + dy^2$. We want to integrate up along the length of this line, with

$\Delta y/\Delta x = 4/3$. We then would use the arc length integral

$$\int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^3 \sqrt{1 + \left(\frac{4}{3}\right)^2} dx = 3 * \sqrt{1 + 16/9} = 5.$$

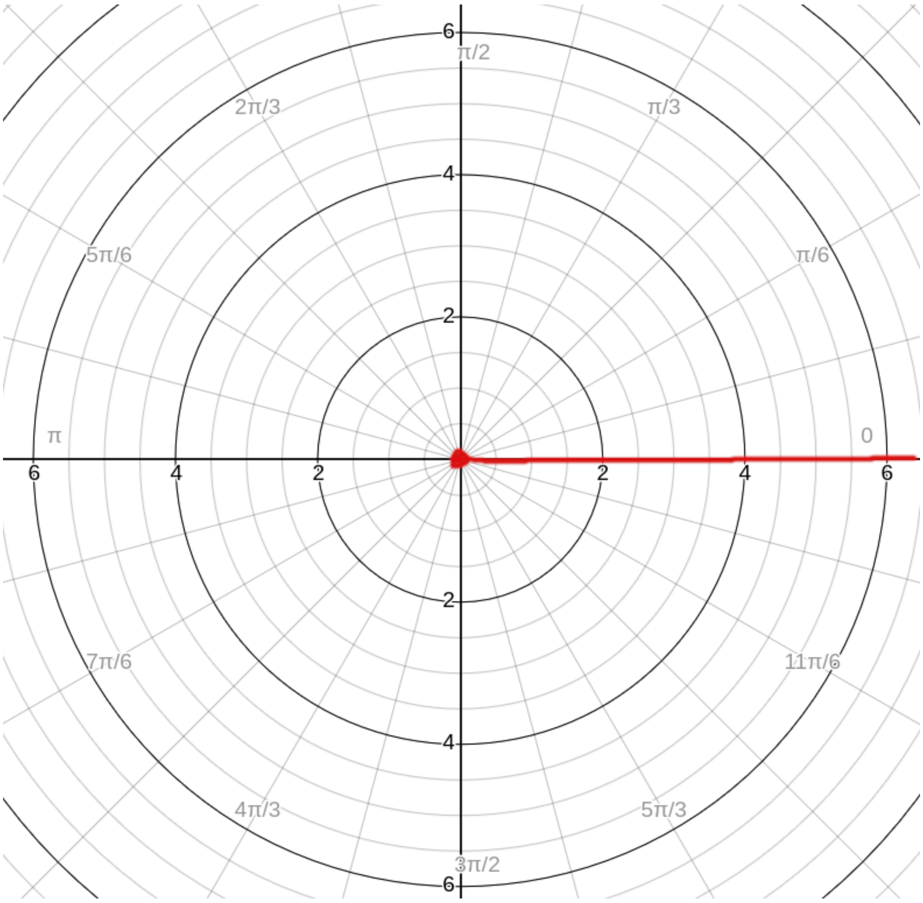
This gives the same answer in this case because the metric is independent of our coordinate system. But this procedure would work even if it did.

Let's try looking at things in polar coordinates! We'll use

$$x = r\cos\theta$$
$$y = r\sin\theta$$

, where $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$.

One small problem! These coordinates have some singular points: - when $r = 0$, θ can be anything and describe the same point - when θ crosses 2π, it wraps back to 0. This creates a jump in the coordinate (discontinous derivative)



Remember, this isn't a problem as long as we can cover these areas with other coordinates. For now let's say that we have another coordinate system with $\theta' \in [-\pi, \pi)$, this one will only have derivative issues on the $-x$ axis. And around the center point, we will convert back to Cartesian coordinates.

Small HW problem

> Using latitude and longitude on the sphere: do the coordinates cover the whole sphere? Are there any singular points? (Hint, yes). Come up with alternative coordinates that cover all of the problem areas

Okay, now let's try to compute the metric in our original polar coordinate system r, θ. Since we started with Cartesian coordinates, we can use how the metric transforms as a rank-2 tensor:

$$g'_{\mu\nu} = \frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu} g_{ij}.$$

We just need to compute the Jacobian of our tranformation:

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Then, the matrix version of our transform equation would be:

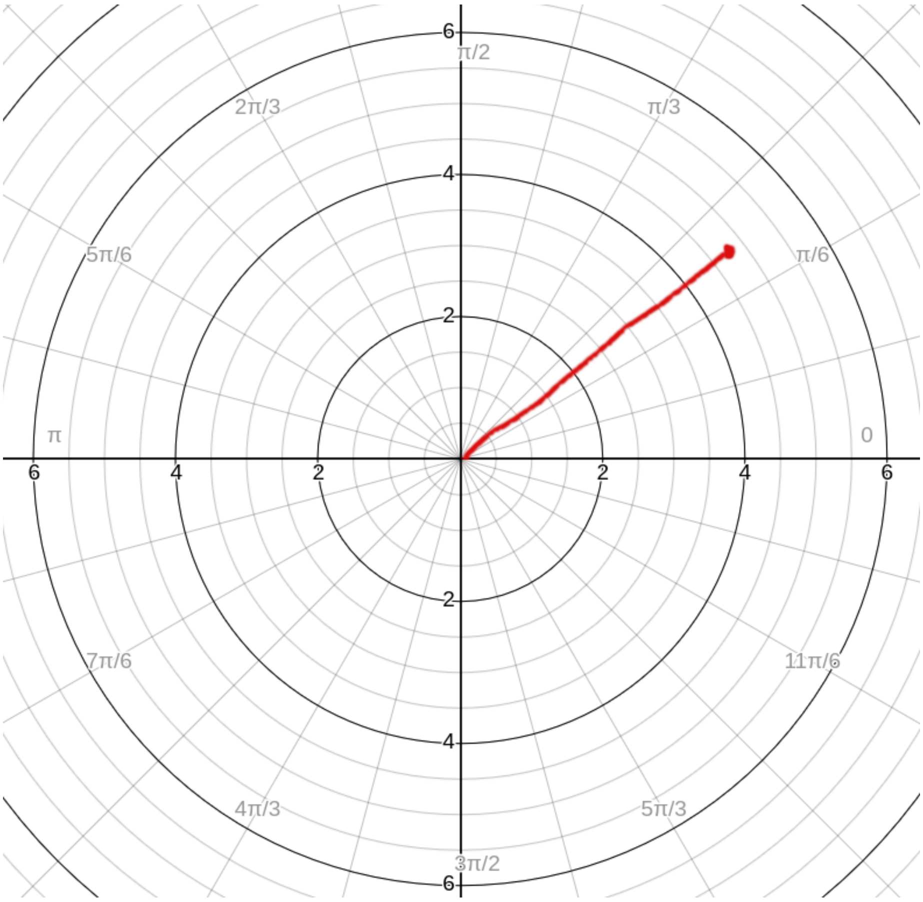
$$\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu} g_{ij} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^T$$

Note that the order matters with matrices, but not with index notation. Because the j index is on the right of g_{ij}, I needed to move the matrix $\frac{\partial x^j}{\partial x'^\nu}$ to the right side. I also needed to swap the order of the indices, to have ${}^j{}_\nu$ instead of ${}_\nu{}^j$. Swapping indices is the equivalent of a matrix transpose. Overall, it's much easier to avoid mistakes if you don't try to convert index notation into matrices, and just do the sums directly.

After doing the matrix multiplications and using $\cos^2\theta + \sin^2\theta = 1$, we have

$$g'_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

If we compare our other point at Cartesian coordinates $(3, 4)$, it's now at polar coordinates of $(5, \arctan 4/3)$, and has length 5. We can calculate its length, but we need be careful. Let's take a line from the origin to radius 5 along the constant-θ line.



this line has $\Delta\theta = 0$, and $\Delta r = 5$. Our total distance then, is just 5. Formally, the metric distance element is $ds^2 = dr^2 + r^2 d\theta^2$, and $d\theta = 0$ along this path. Then $\int ds = \int_0^5 dr = 5$.

Tensor calculus continued

If you are familiar with vector calculus from other classes (divergence, curl, gradient, etc), then tensor calculus should make sense to you.

Tensor calculus is basically the generalization of vector calculus to **any number of dimensions** and **any coordinate system**.

It's actually possible to rewrite Maxwell's equations in a very compact form using differential geometry (we won't get into this though).

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0$$
$$\nabla_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

(Anti-)symmetric tensors

One very common notational shortcut is to write symmetric or antisymmetric combinations of tensor indices.

Once you get used to these, they can be very useful!

$$A_{(ab)} = \frac{A_{ab} + A_{ba}}{2}$$
$$A_{(abc)} = \frac{A_{abc} + A_{acb} + A_{bac} + A_{bca} + A_{cba} + A_{cab}}{3!}$$

The anti-symmetric version is also common to see

$$A_{[ab]} = \frac{A_{ab} - A_{ba}}{2}$$
$$A_{[abc]} = \frac{A_{abc} - A_{acb} + A_{bca} - A_{bac} + A_{cab} - A_{cba}}{3!}$$

Here, basically:

- even number of index swaps: $+$
- odd number of index swaps: $-$

Maybe I'll make a cheat sheet and give it to y'all…

symmetric and antisymmetric parts Every rank-2 tensor can be split into its symmetric and antisymmetric parts. $A_{ab} = A_{(ab)} + A_{[ab]}$

Ordinary derivatives

Derivatives of tensor components work exactly as you would expect!

$$\frac{\partial}{\partial x}(x, y, z) = (1, 0, 0)$$

We also commonly use the gradient (partial derivative with respect to each component):

$$\frac{\partial}{\partial x^\mu} f$$

Taking a derivative with respect to a vector makes a co-vector! There are also several shortcut notations for derivatives that we will use

$$\frac{\partial}{\partial x^\mu} f = \partial_\mu f = f_{,\mu}$$

You can take these derivatives of any tensor, not just scalars!

$$A^a_{bc,d}$$

Covariant derivatives

The idea of the covariant derivative: **tensors transform with coordinate changes, so their derivatives have to too**. The ordinary partial derivative of a tensor is basis-dependent.

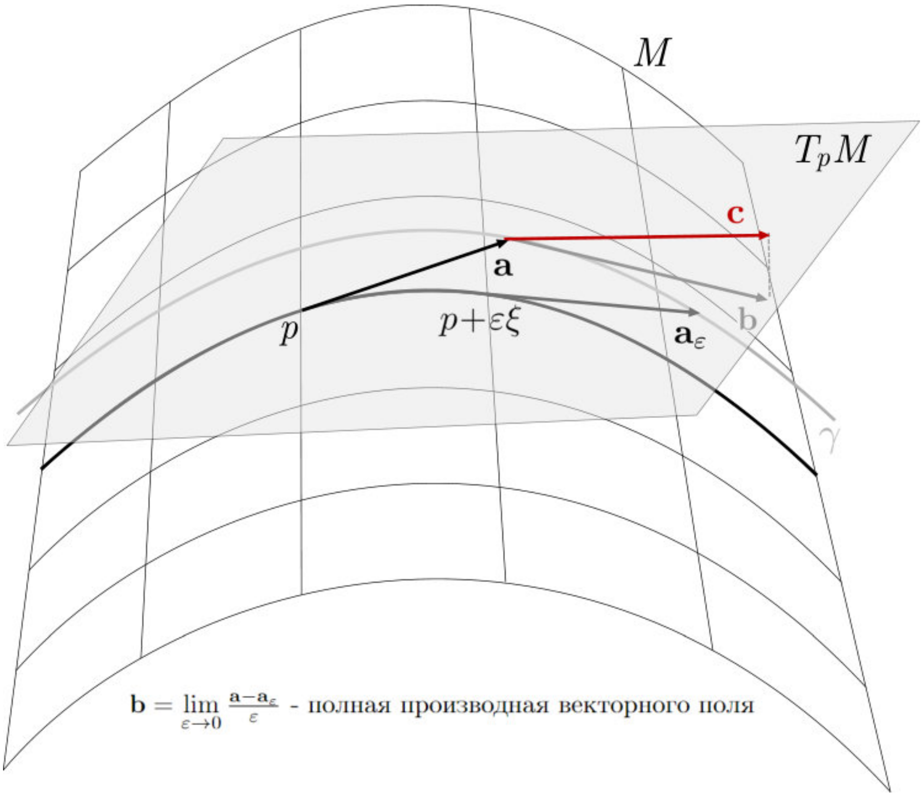


Figure 1: cov_deriv.jpg

The covariant derivative of a vector includes an extra term, correcting for how the vector transforms under coordinate changes:

$$\nabla_\mu v^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha_{\beta\mu} v^\beta$$
$$\nabla_\mu v_\alpha = \partial_\mu v_\alpha - \Gamma^\beta_{\alpha\mu} v_\beta$$

This Γ is a special tensor built from our metric that corrects for the transformation of our coordinates, see below.

For higher-rank tensors, there exist similar formulas. Basically each upper index gets a $+\Gamma$, each lower index gets a $-\Gamma$.

Often, we will use a shortcut for covariant derivatives:

$$\nabla_\mu v^\alpha \equiv v^\alpha_{;\mu}$$

So, semicolons for covariant derivatives, commas for regular derivatives.

For scalars, the covariant derivative is equal to the ordinary derivative

$$f_{;\mu} = f_{,\mu}$$

The Christoffel symbols Γ^i_{jk}

These measure how much we need to adjust our derivatives in order to make them covariant. We can derive these by requiring that the covariant derivative of the metric vanishes $\nabla_l g_{mk} = 0$. Sometimes these are also called the **Levi-Civita connection**.

We won't work through the math, but we get:

$$\Gamma_{mkl} = \frac{1}{2}\left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right)$$

Or often we'll work with the version with the first index raised:

$$\Gamma^i_{kl} = g^{im}\Gamma_{mkl}$$

Note: these feel like a gravitational force in GR.

> **Short HW problem** Use the formulas above to show that the last two indices of Γ are symmetric. $\Gamma^i_{jk} = \Gamma^i_{(jk)}$. Remember that the metric is symmetric too. How many independent elements are there of Γ^i_{jk} in dimension d?