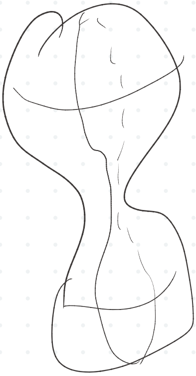


Constant power peanut

GW power:

$$|h_+|^2 + |h_\times|^2$$

<u>Old solution (face-on):</u>

$$h_+ = \frac{8M}{r} R^2 \Omega^2 \cos 2\Omega t$$

$$h_\times = -\frac{8M}{r} R^2 \Omega^2 \sin 2\Omega t$$

<u>New solution (edge on):</u>

$$h_+ = \frac{4M}{r} R^2 \Omega^2 \cos(2\Omega t)$$

$$h_\times = 0$$

$\Rightarrow$ Face-on power is 8x edge on!

How do orbits evolve when emitting GWs?

- GWs carry energy and angular momentum

$\Rightarrow$ orbits shrink!

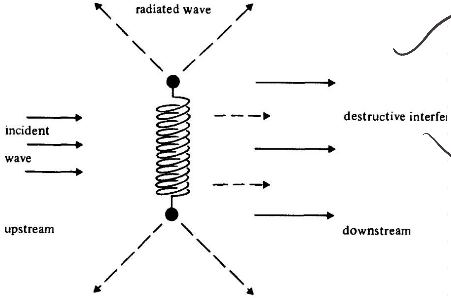
$$\boxed{\Omega^2 = \frac{GM_{tot}}{R^3}}$$

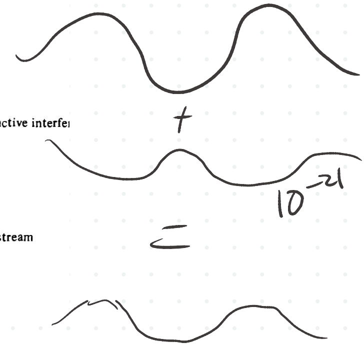
$$2\Omega\dot{\Omega} = -3\frac{GM}{R^4}\dot{R} \Rightarrow \dot{R} = -\frac{2}{3}R\frac{\dot{\Omega}}{\Omega}$$

One cool way to calculate energy loss:

arbitrarily, since we know from § 9.3 what waves they produce.

Schutz §9.4





Energy in spring is lost by GW, destructive interference down stream

$$\boxed{\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega \, \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle \, .}$$

This mostly comes from $\boxed{\frac{d}{dt} T^{00}}$, we want go into the details.

Power in GWs : $\langle P \rangle = \frac{1}{5} \langle \dddot{I}_{jk} \dddot{I}^{jk} \rangle$

$= -\langle \dot{E} \rangle$

$$h_{\ell m} = \frac{2}{r} \ddot{I}_{\ell m}(t-r)$$

$$\langle \dot{E} \rangle = -\frac{32}{5} \left(\frac{m_1 m_2}{m_1 + m_2} \right)^2 R^4 \Omega^6$$

$$= -\frac{32}{5} \frac{m_1^2 m_2^2}{R^5} (m_1 + m_2) \quad (1)$$

$$E = -\frac{m_1 m_2}{2R}$$

$$\frac{d}{dt}E \Rightarrow \dot{E} = \frac{m_1 m_2}{2R^2}\dot{R} \qquad (2)$$

$$(1) + (2) \Rightarrow \dot{R} = \frac{-64}{5}\frac{m_1 m_2 (m_1 + m_2)}{R^3}$$

Reduced mass: $\mu \equiv \dfrac{m_1 m_2}{m_1 + m_2}$

Total mass: $M \equiv m_1 + m_2$

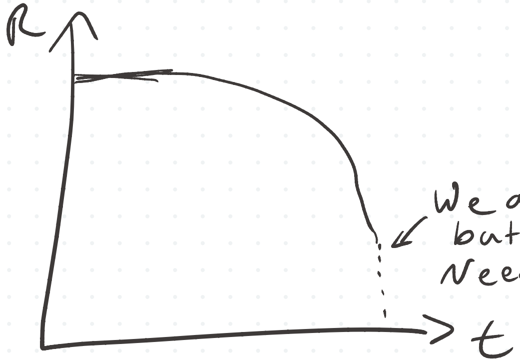
$$\dot{R} = -\frac{64}{5}\frac{\mu M^2}{R^3}$$

$$\int \cancel{dt}\, R^3 \cdot \frac{dR}{\cancel{dt}} = \int_t^{t_c} \frac{-64}{5}\mu M^2 \, dt$$

$$\int_R^0 R^3 \, dR = \frac{-64}{5}\mu M^2 (t_c - t)$$

$$-\frac{R^4}{4} = \frac{-64}{5}\mu M^2 (t_c - t)$$

$$\Rightarrow R(t) = \left(\frac{256}{5}\mu M^2\right)^{1/4} (t_c - t)^{1/4}$$



We assumed $v \ll c$, but $v \to c$ here. Need more terms.

"Post-Newtonian" expansion

Gravitational Wave frequency

$$\Omega_{orb} = \underbrace{2\pi f_{orb}}_{= \pi f_{GW}} = \frac{M^{1/2}}{R^{3/2}} \quad \text{(Kepler)}$$

$$\frac{d}{dt}\left(\pi f_{GW} = \frac{M^{1/2}}{R^{3/2}}\right)$$

$$\dot{R} = -\frac{64}{5}\frac{\mu M^2}{R^3}$$

$$\pi f_{GW}\,\dot{f}_{GW} = -\frac{3}{2}\frac{M^{1/2}}{R^{5/2}}\dot{R}$$

$$\dot{f}_{GW} = \frac{96}{5} M^{2/3} \mu\, \pi^{8/3} f_{GW}^{11/3}$$

$$\boxed{\text{Chirp Mass}: \mathcal{M} = M_c \equiv \mu^{3/5} M^{2/5}}$$

\mathcal{M}

$$\boxed{\dot{f}_{GW} = \frac{96}{5}\pi^{8/3} M_c^{5/3} f^{11/3}}$$

GW Amplitude:

increasing

$$A(t) \sim \frac{\mu M}{r\,R(t)} f^2(t)$$

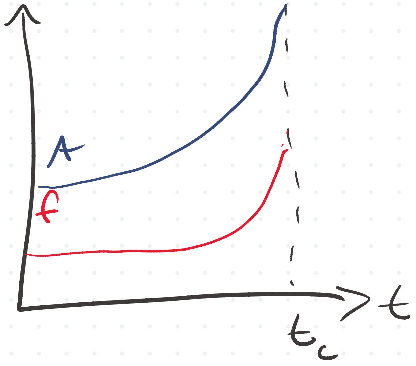
↳ shrinking

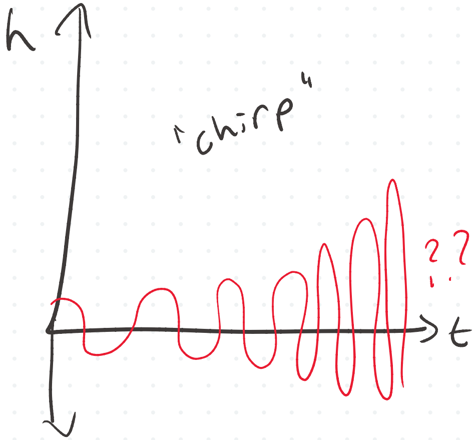
$$= 2\frac{M_c^{5/3}}{r}(\pi f_{GW})^{2/3}$$

$$R \sim (t_c - t)^{1/4}$$

$$f_{GW} \sim R^{-3/2} \sim (t_c - t)^{-3/8}$$

$$A \sim f^{2/3} \sim (t_c - t)^{-1/4}$$





So far we have assumed:

$v \ll c$

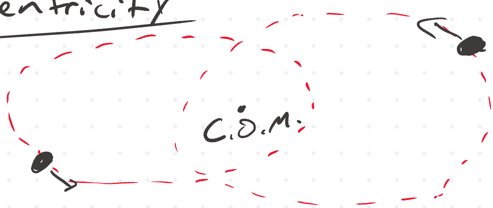
$R \gg M$

sources are points

Let's relax these:

- eccentricity
- $v \to c$, $a \to M$
- spin
- matter

Eccentricity



To get GWs:
- write mass distribution into T^{00}
- Calculate $\ddot{I}_{jk}$

$$R(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$$

a : semi major axis

e : [-1, 1] eccentricity

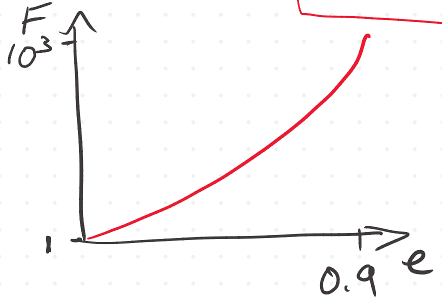
$e < 0 \Rightarrow$ hyperbolic / parabolic path
unbound! Not an orbit

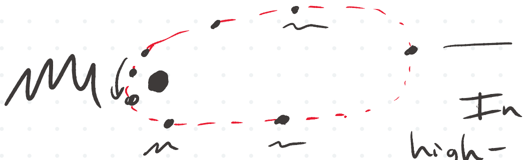
$e = 0 \Rightarrow$ circular 

$e = 1 \Rightarrow$ head-on collision 

We can derive:

$$\langle \dot{E} \rangle = \langle \dot{E} \rangle_{e=0} \underbrace{(1-e^2)^{-7/2}\left[1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 + \dots\right]}_{F(e)}$$





In practice, these high-eccentricity binaries only emit strongly at closest approach (peri apsis).

$$\dot{a} = \dot{a}_{GW} F(e)$$

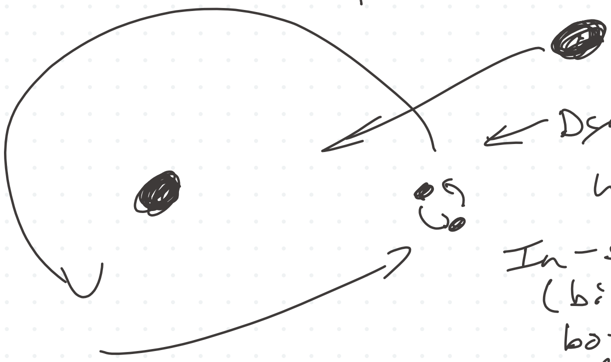
$$\dot{e} = F(e)/e \qquad \underbrace{(1-e^2)^{-7/2}\left[1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 + \dots\right]}_{F(e)}$$

$\dot{e} > \dot{a} \Rightarrow$ orbits circularize!

By the time of merger, we expect most binaries to be $e \approx 0$.

If not:

- started with a very large eccentricity
- formed with small a
- both



Dynamical formation? high ecc.?

In-situ formation (binary star, both become BHs, merge)?

When $e > 0$, the GWs are not just quadrupolar:

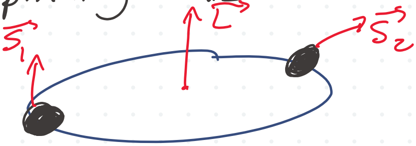
$$h_{em} \sim \underbrace{\int T_{00} x^l x^m d^3x}_{\text{quadrupole}} + \hat{k}_k \underbrace{\int T_{00} x^l x^m x^k d^3x}_{\text{octopole}} + \dots$$

$$h(t) = A e^{2\pi i f t}\left(1 + a_0 e^{2\pi(2f)it} + a_1 e^{2\pi(3f)it} + \dots\right)$$

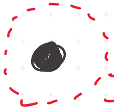
"higher harmonics"

Radiation pattern changes, use spin-weighted spherical harmonics.

Spinning BHs



Orbits: $\vec{S} = 0$



ISCO: innermost stable circular orbit

$r_{isco} = 3 r_s = 6M$

$|\vec{S}| > 0$

co-rotating ISCO: very close

counter-rotating ISCO: much farther away

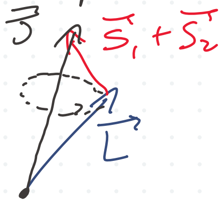
If max. spin, $r_{isco} = M$ (corotating) or $9M$ (counter rotating)

Penrose process: how to gain angular momentum from spinning BH.

$f_{max} > f_{max}^{s=0}$

$f_{max} < f_{max}^{s=0}$

Spin-orbit precession



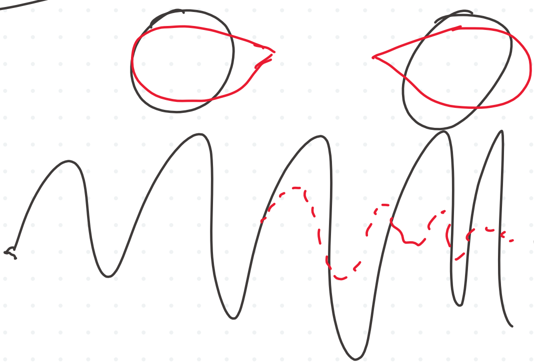
$\vec{J}$ stays close to constant, but $\vec{L}$ undergoes precession

As they inspiral, plane of orbit changes, amplitude of GWs varies.

$\vec{S} \parallel \vec{L} \Rightarrow$ merger frequency

$\vec{S} \perp \vec{L} \Rightarrow$ Precession

Matter



Tidal forces deform objects that aren't BHs.

Neutron stars, white dwarfs, etc.

← Tides absorb some orbital energy, smaller GWs.