Lecture notes 1 SPA 689 GW

First half of this class: we're going to learn just enough general relativity to study gravitational waves. Not a lot!

General relativity: gravity is not a force! Mass causes spacetime to curve, and then objects move along straight lines in curved space (*geodesics*).

matter tells spacetime how to curve, and spacetime tells matter how to move

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

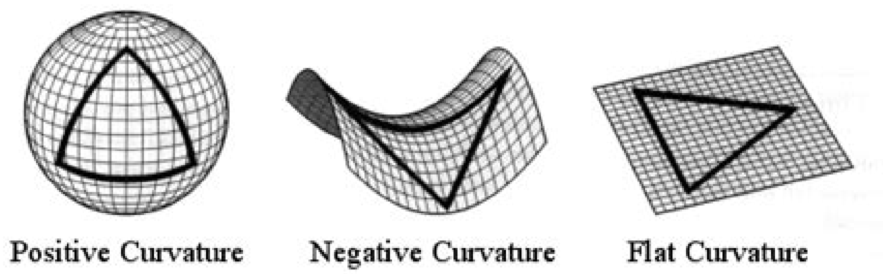
$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

Conceptual differential geometry

differential geometry = calculus in curved spaces

What is *curved space*? One easy way to think about it (in 2D)

- Flat space: triangles have angles sum up to 180 degrees
- Positive curvature: triangles have > 180 degrees
- Negative curvature: triangles have < 180 degrees



Alternative definitions:

- Flat space: parallel lines stay parallel
- Positive curvature: parallel lines eventually converge
- Negative curvature: parallel lines diverge

Intuitively, **negatively curved spaces feel "bigger"**.

Example spaces:

- Flat: a plane, a cylinder
- Positive: a sphere
- Negative: a saddle

Gaussian curvature

One easy way to try to measure the curvature of a surface: measure the 'princpal curvature' directions (directions of most and least curvature)

- concave up: -
- concave down: +

Figure 1: negative_curvature_crochet.jpg

- flat: 0

Take the product. - Saddle: $+ \cdot - \rightarrow -$ - Cylinder: $+ \cdot 0 \rightarrow 0$

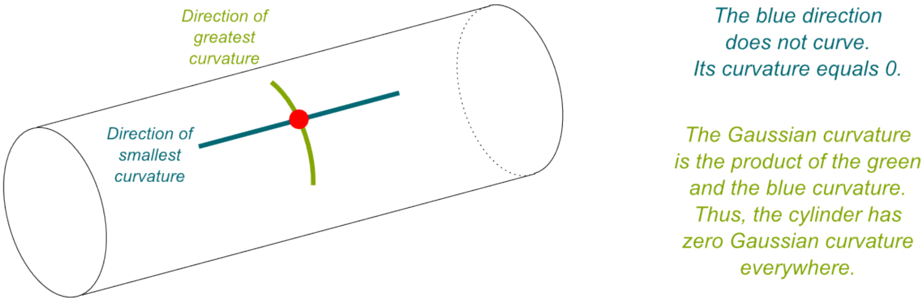
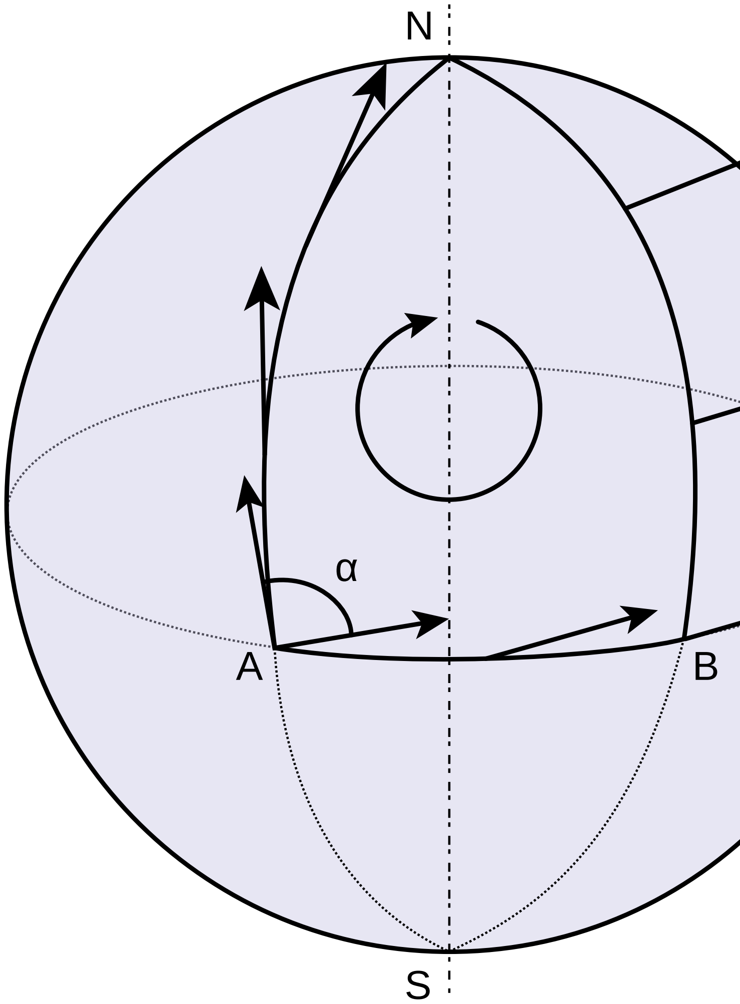


Figure 2: Curvature-of-Cylinder.png

Vectors

We need to be very careful about how vectors work in curved spaces!

Flat space: vectors always point the same direction, move them around however you want! Curved space: vectors can rotate depending on the path they take!!



We move vectors with **parallel transport**.

We brought the vector back to A, but it points at a different angle! The rotation amount is directly related to the curvature of the space.

Basic mathematics of differential geometry

We are going to describe the basics of differential geometry by slowly introducing **Penrose tensor index notation**, we will be more precise about this later.

Spaces (manifolds)

We can do differential geometry on any space that is **smooth** (we can take as many derivatives as we want), and can be covered in smooth coordinates everywhere (not necessarily the same ones). Technically this kind of space is called a **differentiable manifold**.

Our coordinates don't have to be good everywhere, as long as some set of coordinates work at every point. (This situation happens often in GR). **Most**

of the tricky math of differential geometry is related to coordinate transformations.

Important fact: spaces like this are **locally flat**! You only notice the curvature at big distances.

Scalars

Functions that just give a number at each point in the space are called **scalars**. These are just functions that you're used to!

In Penrose tensor notation, scalars have no free indices. *Scalars are the same in every coordinate system, they're the best.*

Vectors (contravariant vectors)

In a certain set of coordinates, we will write a **vector** with an upper index: $a^\mu = (a^0, a^1, a^2, ...)$ When we change coordinates, vectors will transform using the **Jacobian** of the space. Old coordinates: x^μ, new coordinates x'^μ.

$$a^\mu \longrightarrow \frac{\partial x'^\nu}{\partial x^\mu} a^\mu = a'^\nu$$

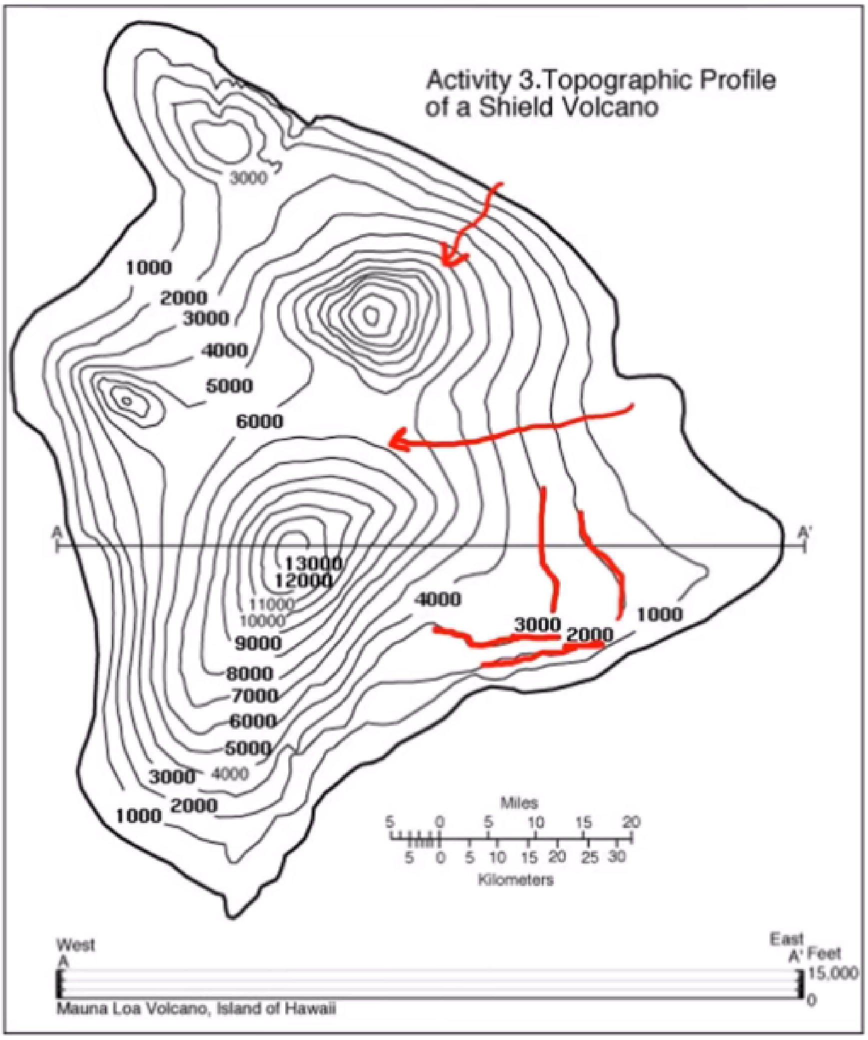
This is the same as in normal calculus!

Intuition: divide coordinates by 2, for vector to stay same length, vector components must grow by 2.

Covectors (one-forms, or covariant vectors)

Some objects transform the opposite way as vectors under coordinate transforms! We will call these **covectors** or **one-forms**.

Example: the rate of change of a scalar function. If our coordinates shrink by 2, the derivative must also shrink by 2!



When we combine a vector and a covector, we always get a scalar (e.g. height up on contour map). They transform in opposite ways and leave the scalar invariant.

In Penrose tensor notation, we write covectors with a lower index: $a_\mu = (a_0, a_1, a_2, ...)$, and they transform with the **inverse Jacobian**:

$$a_\mu \longrightarrow \frac{\partial x^\mu}{\partial x'^\nu} a_\mu = a_{'\nu}$$

Tensors

Tensors act like combinations of vectors and covectors. They can have any number of indices, up or down. They are often described by their **tensor rank**, which just counts how many indices act like vectors and how many act like covectors.

For example, $R^\alpha_{\beta\gamma\delta}$ is a rank (1,3) tensor. When you apply a coordinate transformation to a tensor, you just apply the vector or covector transformation rule to each index.

The metric

The metric is a special (0,2) tensor, often written $g_{\mu\nu}$. The metric encodes distances in our manifold within a certain coordinate system. You can figure out the metric by studying the basis vectors of your coordinate system, or by transforming a metric you know using the co-vector transformation rules

$$g'_{\mu\nu} = \frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu} g_{ij}$$

In Cartesian coordinates, the metric is just $\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 & ... \\ 0 & 1 & 0 & ... \\ & & \ddots & \\ 0 & 0 & ... & 1 \end{pmatrix}$. The inverse metric $g^{\mu\nu}$ can be figured out a similar way, or is just the matrix inverse of the metric.

The metric can be used to relate vectors to co-vectors and vice-versa (lowering and raising the index)!

$$a_\mu = g_{\mu\nu} a^\nu a^\mu = g^{\mu\nu} a_\nu$$

It can also be used to calculate lengths! $||x^\mu||^2 = g_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu$

Penrose tensor notation

The notation we have been using so far to write these tensors is called **Penrose index notation**. (There are many other ways to write them as well that you might see in books).

Here are the basic rules:

- When you have a repeated index, one of them must be up and the other down. (Sometimes this is also called a **contracted index or dummy index**). Then this acts like a sum over that index:

  $$a_i a^i = a^0 a_0 + a^1 a_1 + ...$$

  - We are allowed to rename dummy indices without changing the answer. For example $a_i a^i = a_j a^j = a^0 a_0 + a^1 a_1 + ...$
  - The number of **free indices** (non-contracted) tells you the rank of the tensor! For example, R^i_{aib} is a rank (0,2) tensor, i is a dummy index and the free indices are a and b.
  - When you add two terms, they must have the same rank and free indices! So expressions like this are allowed:
    * $a_i a^i + 1$ (the first term has 1 dummy index and 0 free indices, and the second term also has 0 free indices because it's a scalar)
    * $x_\mu x_\nu + b_{\nu\mu\gamma} c^\gamma$ (both terms are rank (0,2) and have the same free indices (in a different order, but that's ok)).
    * $z^a(v_a + b_a)$ overall this is a scalar, inside the parentheses is rank (0,1), contracted with rank (1,0) outside the parentheses.

Tensor notation practice homework

Check out the following tensor expressions. Note that **some of these make no sense**!

1. a_μ
2. $a_i b_j g^{ij}$
3. $a_i b_j c_k + d^i e^j f^k$
4. $1 + x^\mu$
5. $x_\mu x_\nu + g_{\mu\nu}$
6. $c^\mu (a^i a_i b_\mu + g_{\mu\nu} c^\nu) + 1$
7. $x^i x^i$
8. $\dot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k$
9. $g_{ab} + x^c$
10. $a_i + b_{ij} c^j + d_{ik} e^k$

Which ones of these are *invalid*? AKA they break the rules of Penrose tensor notation?

click for answer

3, 4, 7, 9

Write down the rank of each valid expression. Example: $a_i b_j$ is rank (0,2) and Γ^i_{jk} is rank (1,2).

click for answer

1. (0,1)
2. (0,0)
3. invalid
4. invalid
5. (0,2)
6. (0,0)
7. invalid
8. (1,0)
9. invalid
10. (0,1)

Write down the length of the vector a^μ in terms of the metric $g_{\mu\nu}$:

click for answer

$\sqrt{a_\mu g^{\mu\nu} a_\nu}$

If the metric can be written in 2D polar coordinates r, θ as $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, what is the distance between the points $x^\mu = (1, 0.1)$ and $y^\mu = (1, 1.0)$?