# Lecture notes 7

## Quick aside on units

We are using units with $G = c = 1$. What does that mean in practice? How do we calculate things?

Short answer: **length / time / mass / energy all become the same thing. Let's call it km**. We can always convert back to SI units by remembering how things relate to distance.

- time: multiply by $c$
- distance: convert to km
- mass: convert to (half of) Schwarzschild radius. multiply by $G/c^2$. $r_s = 2M = 2GM/c^2$
- energy: convert to mass first, $E = mc^2$

**Why are we allowed to do this?** Why not? Our physical laws uniquely relate distance, mass, and time. There's no real reason to use different units as far as the math goes. For human reasons, converting back to SI units is definitely more convenient though.

Example: this class is 1 hour, or $c \cdot 1\,\text{hour} \approx 10^9\,\text{km}$.

**Very useful units to remember**:

- the Schwarzschild radius of the Sun is about $3\,\text{km} \approx 2GM_\odot/c^2$.
- One parsec is about $200,000$ AU.
- One AU is about $150 \times 10^6\,\text{km}$.
- $c$ is about $300,000\,\text{km/s}$.

### How big are GWs?

For a circular binary,

$$h_+ = \frac{8M}{r}\Omega^2 R^2 \cos 2\Omega t$$
$$\Omega = \sqrt{\frac{M}{4R^3}}$$

$M \sim 2M_\odot$ $R \sim 1\,\text{AU}$ $r = 8\,\text{kpc}$

Okay, let's try to calculate the amplitude of $h_+$ using our unit tricks:

$$
\begin{aligned}
\frac{8M}{r}\frac{M}{4R^3}R^2 &= \frac{8M^2}{r}\frac{1}{4R} \\
&= 2\left(\frac{GM}{c^2}\right)^2\frac{1}{rR} \\
&\approx 2\frac{(3\,\mathrm{km})^2}{(8\,\mathrm{kpc})(1\,\mathrm{AU})} \\
&= 2\frac{(3\,\mathrm{km})^2}{[8\times 10^3\cdot(2\times 10^5)\cdot(150\times 10^6)\,\mathrm{km}][1\cdot(150\times 10^6)\,\mathrm{km}]} \\
&= 2\frac{9}{[16\cdot 150\times 10^{14}][150\times 10^6]} \\
&= \frac{1.8}{1.6 * 1.5^2 * 10^{24}} \\
&\approx 10^{-24}
\end{aligned}
$$

If the stars were not stars but black holes and much closer, say $R = 100\,\mathrm{km}$, then that's $(1.5\times 10^8)/10^2 \sim 10^6$ times closer and $h_+ \sim 10^{-18}$.

If they were way closer, let's say $r = 1\,\mathrm{pc}$ away, then that's another factor of $\sim 10^4$ bigger, and $h_+ \sim 10^{-14}$.

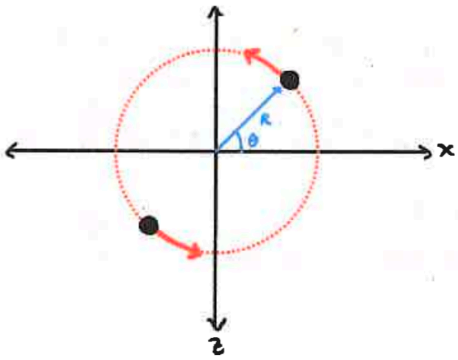
**even the biggest gravitational waves hitting us are *tiny***

For comparison, rocks start to break at strains of $\sim 10^{-4}$ (big earthquakes).

Steel has plastic deformation at $\sim 10^{-3}$, breaks around $\sim 10^{-1}$.

**Why pick km?** No reason really, other than solar-mass black holes are close to $\mathcal{O}(1)$ numbers in km. When we work with supermassive black holes, we will probably switch to AU to keep having $\mathcal{O}(1)$ numbers.

## Strain from an edge-on binary




$$\vec{x}_1 \to (0,\ R\cos\Omega t,\ 0,\ R\sin\Omega t)$$
$$\vec{x}_2 \to (0,\ -R\cos\Omega t,\ 0,\ -R\sin\Omega t)$$

$T_{\mu\nu} = T_{00}\delta_{\mu 0}\delta_{\nu 0}$

$T^{00}(t, x^i) = M\delta(x^2)\left[\delta(x^1 - R\cos\Omega t)\delta(x^3 - R\sin\Omega t) + \delta(x^1 + R\cos\Omega t)\delta(x^3 + R\sin\Omega t)\right]$

This is almost the same as before, except sources are moving in the $xz$-plane.

$$I_{ij} = \int T^{ij}x^i x^j d^3x$$

We get $I_{11}, I_{33} = MR^2(1 \pm \cos 2\Omega t)$ and $I_{13} = I_{31} = MR^2 \sin 2\Omega t$, with all of the $I_{i2} = I_{2i} = 0$.

Then our tensor becomes $t_r = t - r/c$

$$\bar{h}_{ij} = \frac{2}{r}\ddot{I}_{ij} = \frac{8M}{r}\Omega^2 R^2 \begin{pmatrix} \cos 2\Omega t_r & 0 & -\sin 2\Omega t_r \\ 0 & 0 & 0 \\ -\sin 2\Omega t_r & 0 & -\cos 2\Omega t_r \end{pmatrix}$$

For an observer in the $z$-dir, the GW part of the metric is only $h_{11}, h_{22}, h_{12}, h_{21}$ .

Then we have

$$h_+ = \frac{4M}{r}R^2\Omega^2 \cos 2\Omega t$$
$$h_\times = 0$$

Linearly-polarized GWs! We can also get $h_\times$ only on the line $x = y$.