Lecture notes 5

Solutions to the wave equation

Last class we found that linearized general relativity can be written in this form:

$$\boxed{-\Box\bar{h}_{\mu\nu} = 16\pi T_{\mu\nu}}$$

When $T_{\mu\nu} = 0$, this is just the **wave equation**

$$\Box\bar{h}_{\mu\nu} = 0$$

Refresher on the wave equation

You might have seen the wave equation before. On a 1D string with shape $u(x, t)$, it takes the form

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

and describes waves moving with speed v. In practice, getting the boundary conditions right is also very important to get the solution right. A string with fixed ends and a bump initially moving to the right looks like this under the wave equation:

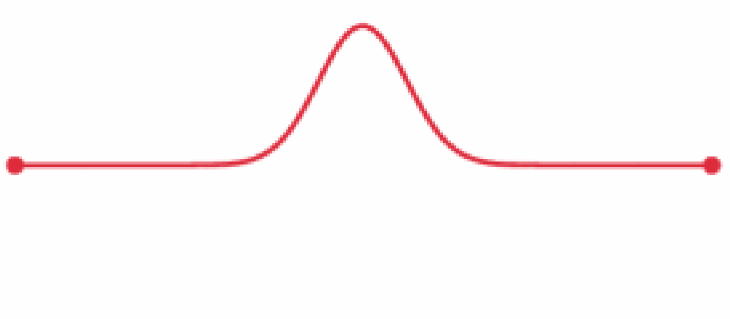


Figure 1: Wave_equation_1D_fixed_endpoints.gif

There are many ways to solve the wave equation, but we'll use the *plane wave decomposition.*

Let's assume that $u(x, t) = e^{-i\omega t} f(x)$, separating out the part that depends on x and the part that depends on t.

Plugging this in we get:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= v^2 \frac{\partial^2 u}{\partial x^2} \\
-\omega^2 e^{-i\omega t} f(x) &= v^2 e^{-i\omega t} \frac{\partial^2 f}{\partial x^2} \\
-\left(\frac{\omega}{v}\right)^2 f(x) &= \frac{\partial^2 f}{\partial x^2}
\end{aligned}$$

This has solutions $f(x) = Ae^{\pm ikx}$ with $k = \omega/v$. k is also called the wave number and is the inverse wavelength.

The combined solution then looks something like $u(x,t) = Ae^{i(\pm kx - \omega t)}$.

Gravitational plane waves

From our experience solving the 1D wave equation, let's start with

$$h_{\mu\nu} = \mathrm{Re}(A_{\mu\nu} e^{ik_\sigma x^\sigma})$$

Here k_σ is the wavenumber along each coordinate, and $A_{\mu\nu}$ is some unknown constant tensor.

Okay, let's try plugging this into our wave equation

$$\partial_\alpha x^\beta = \delta^\beta_\alpha$$

$$\begin{aligned}
\Box h_{\mu\nu} &= \partial^\alpha \partial_\alpha (A_{\mu\nu} e^{ik_\sigma x^\sigma}) \\
&= \partial^\alpha (ik_\sigma \delta^\sigma_\alpha)(A_{\mu\nu} e^{ik_\sigma x^\sigma}) \\
&= \eta^{\alpha\beta} \partial_\beta (ik_\sigma \delta^\sigma_\alpha)(A_{\mu\nu} e^{ik_\sigma x^\sigma}) \\
&= \eta^{\alpha\beta} (ik_\sigma \delta^\sigma_\beta)(ik_{\sigma'} \delta^{\sigma'}_\alpha)(A_{\mu\nu} e^{ik_\sigma x^\sigma}) \\
&= -k_\alpha k^\alpha (A_{\mu\nu} e^{ik_\sigma x^\sigma}) \\
&= 0
\end{aligned}$$

In general, $h_{\mu\nu} \neq 0$ (otherwise what's the point?), so that means that $k_\alpha k^\alpha = 0$.

Null vector! This means that k^α is something like $(1, 1, 0, 0)$. Then when we contract this vector with $\eta_{\alpha\beta}$, we get $-(k^0)^2 + (k^1)^2 = 0$. In relativity, null vectors are also called light-like vectors, because light always travels along null directions.

Gravitational waves must also travel at the speed of light then!

We still need to figure out the components of $A_{\mu\nu}$ and k^σ though. The equation we have works for any k^σ that is light-like.

We also assumed our gauge condition, that $\bar{h}_{\mu\nu}{}^{,\nu} = 0$. Applying this enforces that $A_{\mu\nu} k^\nu = 0$.

You might guess then that since $A_{\mu\nu}$ is symmetric and has 4 orthogonality constraints, then we're left with (10-4 = 6) components.

In practice, there is some residual gauge freedom we haven't used. Remember that any coordinate transformation of the form $x^\mu \to x^\mu + \xi^\mu$ with $\Box \xi^\mu = 0$ works. We can pick a more specific ξ^μ to pin down $A_{\mu\nu}$ more precisely and eliminate 4 more degrees of freedom. We won't go through the details here, but if interested see the argument in MTW Chapter 35.

After completely fixing the gauge freedom this way, we are in what's called the **transverse traceless gauge**. We are left with only two degrees of freedom in $A_{\mu\nu}$. We can write it:

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

when $k^\sigma = (\omega, 0, 0, \omega)$. ($z$-direction motion).

We will name these A_+ and $A_\times$ because they are gauge invariant.

$$A_+ = A_{11}$$
$$A_\times = A_{12}$$

These are also called the two **polarizations** of the wave, you'll see why shortly.

The plane gravitational wave metric

Okay, so what do these waves look like? Let's look at the metric built from a wave traveling in the z-direction

$$ds^2 = -\mathrm{d}t^2 + \mathrm{Re}(1+A_+e^{ik_\alpha x^\alpha})\mathrm{d}x^2 + 2\mathrm{Re}(A_\times e^{ik_\alpha x^\alpha})\mathrm{d}x\mathrm{d}y + \mathrm{Re}(1-A_+e^{ik_\alpha x^\alpha})\mathrm{d}y^2 + \mathrm{d}z^2$$

Consider a wave with $A_\times = 0$ (this is called a linearly polarized wave in the $+$ direction), then we have

$$A_\times = 0$$
$$\mathrm{Re}(A_+e^{ik_\alpha x^\alpha}) = A_+\cos(kz - \omega t)$$

$$ds^2 = -\mathrm{d}t^2 + (1 + A_+\cos(kz - \omega t))\mathrm{d}x^2 + (1 - A_+\cos(kz - \omega t))\mathrm{d}y^2 + \mathrm{d}z^2$$

If we call $kz - \omega t \equiv \Phi$, then we have:

Circularly polarized waves also exist, have $A_\times = A_+e^{i\pi/2}$.

A circularly (or elliptically) polarized wave traveling towards you looks like this:

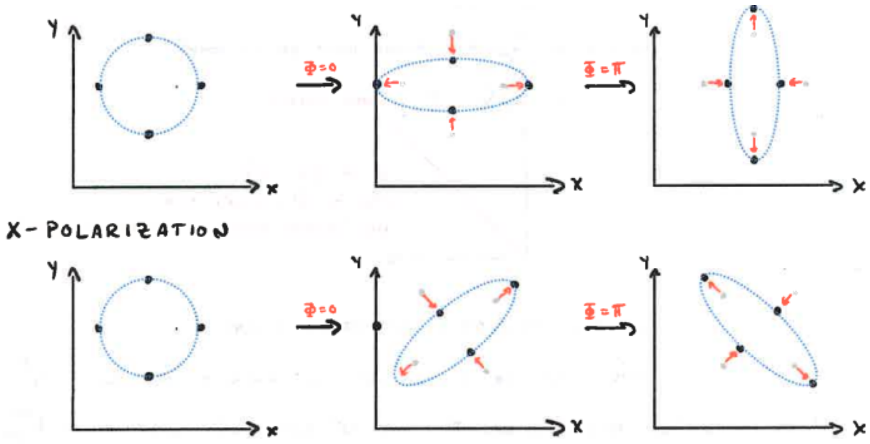


Figure 2: gw_polarizations.png

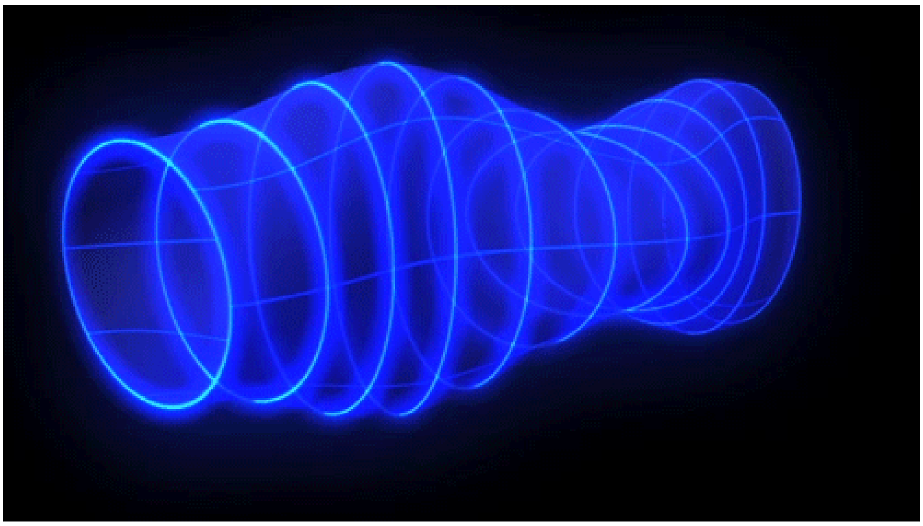
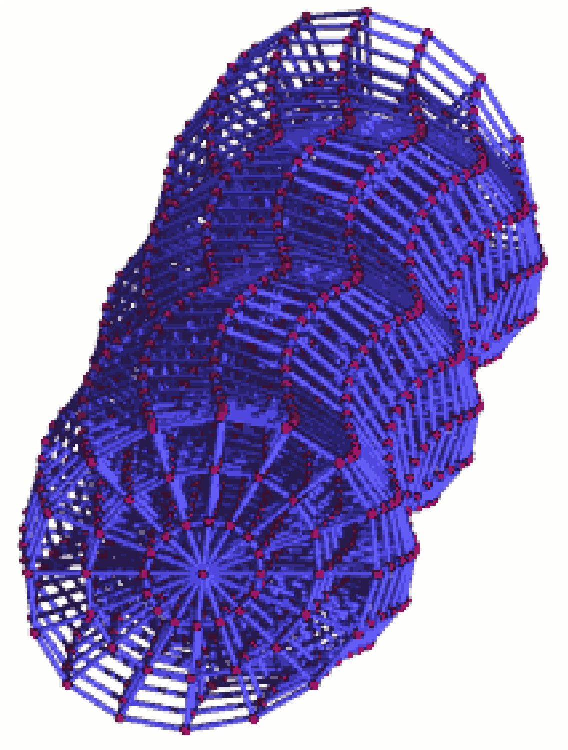


Figure 3: Gravitational_waves.gif



How is it to experience a gravitational wave?

So, what does all of this mean? If you were minding your own business floating around in flat space, what would a gravitational wave do to you?

An observer in *weak gravity* can always choose their coordinates to be very similar to ours! If a wave was passing by in the z-dir, and we moved x and y to make the wave +-polarized:

$$ds^2 = -\mathrm{d}t^2 + (1 + A_+ \cos(kz - \omega t))\mathrm{d}x^2 + (1 - A_+ \cos(kz - \omega t))\mathrm{d}y^2 + \mathrm{d}z^2$$

Distances along x-dir lengthen, while distances along y-dir shrink! Then y lengthens while x shrinks.

Time seems to pass normally and z-dir is unaffected.

Space is *actually changing* as these waves pass – you might even notice stress and strain in materials! Or lengthen the path covered by light.

Some good reading here: https://www.einstein-online.info/en/spotlight/gw_waves/

Next class

How big are these waves? Spoiler: very small

How do we make them?