Separable kernel $K(x,z)=\sum_{i=1}^{N} g_i(x)h_i(z)$ for Fredholm equations $y(x)=f(x)+\lambda\int_a^b K(x,z)y(x)\mathrm{d}x$.

$y(x)=f(x)+\lambda\int_a^b\left(\sum_{j=1}^{N} g_j(x)h_j(z)\right)y(z)\mathrm{d}z=f(x)+\lambda\sum_{j=1}^{N} g_j(x)\int_a^b h_j(z)y(z)\mathrm{d}z=f(x)+\lambda\sum_{j=1}^{N} c_j g_j(x)$.

$c_i=\int_a^b h_i(z)y(z)\mathrm{d}z$, $f_i=\int_a^b h_i(z)f(z)\mathrm{d}z$, $K_{ij}=\int_a^b h_i(z)g(z)\mathrm{d}z \Longrightarrow \int_a^b h_i(x)y(x)\mathrm{d}x=\int_a^b h_i(x)f(x)\mathrm{d}x+\lambda\sum_{j=1}^{N} c_j\int_a^b h_i(x)g_j(x)\mathrm{d}x$

$c_i=f_i+\lambda\sum_{j=1}^{N} K_{ij}c_j \Rightarrow \mathbf{c}=\mathbf{f}+\lambda\mathbf{K}\mathbf{c}\Rightarrow\mathbf{c}=(\mathbf{I}-\lambda\mathbf{K})^{-1}\mathbf{f}$. For homogeneous equation, $\mathbf{c}=\lambda\mathbf{K}\mathbf{c}$, $\mathbf{K}\mathbf{c}=\lambda^{-1}\mathbf{c}$.

$$\boxed{\text{Procedure: Separable kernel} \Rightarrow \text{Construct } \mathbf{K}\text{, compute } \mathbf{f} \text{ and } (\mathbf{I}-\lambda\mathbf{K})^{-1}\Rightarrow \text{Get } \mathbf{c}\Rightarrow y(x)=f(x)+\lambda\sum_{j=1}^{N} c_j g_j(x).}$$

Example. $y(x)=\lambda\int_0^{\pi/2}\cos(x-z)y(z)\mathrm{d}z$. Separable kernel $\cos(x-z)=\cos x\cos z+\sin x\sin z$. $g_1=h_1=\cos x$, $g_2=h_2=\sin x$.

Construct $\mathbf{K}=\frac{1}{4}\begin{pmatrix}\pi & 2\\ 2 & \pi\end{pmatrix}$. The eigenvalues for $\mathbf{K}$ are $\frac{\pi\pm 2}{4}$ with eigenvectors $\mathbf{c}=\begin{pmatrix}1\\ \pm 1\end{pmatrix}$. The equation would have solutions for $\lambda=\frac{4}{\pi\pm 2}$.

The eigenfunctions for $\mathscr{K}$ are $\sum^{N}_{j=1} c_j g_j(x)$, which are $\frac{4}{\pi-2}(\cos x-\sin x)=\frac{4}{\pi-2}\cos\left(x+\frac{\pi}{4}\right)$ and $\frac{4}{\pi+2}\sin\left(x+\frac{\pi}{4}\right)$ (normalised).

Example. $y(x)=1-x+\frac{2}{\pi}\int_0^{\pi/2}\cos(x-z)y(z)\mathrm{d}z$. $\mathbf{K}=\frac{1}{4}\begin{pmatrix}\pi & 2\\ 2 & \pi\end{pmatrix}$, $(\mathbf{I}-\lambda\mathbf{K})^{-1}=\frac{2\pi}{\pi^2-4}\begin{pmatrix}\pi & 2\\ 2 & \pi\end{pmatrix}$, $\mathbf{f}=\begin{pmatrix}2-\pi/2\\ 0\end{pmatrix}\Rightarrow\mathbf{c}=\frac{\pi(4-\pi)}{\pi^2-4}\begin{pmatrix}\pi\\ 2\end{pmatrix}$.

$\Rightarrow y(x)=f(x)+\lambda\sum_{j=1}^{N} c_j g_j(x)=1-x+\frac{2(4-\pi)}{\pi^2-4}(\pi\cos x+2\sin x)$.

Consider $y(x)=f(x)+\lambda\mathscr{K}y(x)$, if $\lambda=\lambda_n$, where λ_n are eigenvalues of $\mathscr{K}$ and $\frac{1}{\lambda_n}$ are eigenvalues of $\mathbf{K}$. Then $\mathbf{I}-\lambda\mathbf{K}$ is not invertible.

$y=f+\lambda_1\mathscr{K}y$. Let $y=c_1u_1(x)+c_2u_2(x)$, where u_1,u_2 are eigenfunctions of $\mathscr{K}$. $c_1u_1+c_2u_2=f+\lambda_1\left(\frac{1}{\lambda_1}c_1u_1+\frac{1}{\lambda_2}c_2u_2\right)=f+c_1u_1+\frac{\lambda_1}{\lambda_2}c_2u_2\Rightarrow c_2u_2=f+\frac{\lambda_1}{\lambda_2}c_2u_2\Rightarrow$ There are solutions if f is orthogonal to u_1, the eigenfunction corresponds to λ_1.

Source-resolvable functions. Consider the eigenvalue equation $\phi_n(x)=\lambda_n\mathscr{K}\phi_n(x)$. If $f(x)=\int_a^b K(x,z)\rho(z)\mathrm{d}z$, then f can be expanded as $f=\sum_{n=1}^{N} f_n\phi_n(x)$. In this $N-$dimensional space, $K(x,z)=\sum_n\frac{\phi_n(x)\phi_n^*(z)}{\lambda_n}$. In the full space, $f(x)=\sum_n f_n\phi_n(x)+u(x)$, $y(x)=\sum_n y_n\phi_n(x)+v(x)$, where u,v are remainders and are orthogonal to ϕ_n. The expansion gives:

$\sum_n y_n\phi_n(x)+v(x)=\sum_n f_n\phi_n(x)+u(x)+\lambda\int_a^b\sum_n\frac{\phi_n(x)\phi_n^*(z)}{\lambda_n}\left(\sum_m y_m\phi_m(z)+v(z)\right)\mathrm{d}z=\sum_n f_n\phi_n(x)+u(x)+\lambda\sum_{n,m}\frac{\phi_n(x)}{\lambda_n}y_m\delta_{nm}$.

$\Rightarrow y_n=f_n+\frac{\lambda}{\lambda_n}y_n$ and $u=v\Rightarrow y_n=\frac{\lambda_n}{\lambda_n-\lambda}f_n=\left(1+\frac{\lambda}{\lambda_n-\lambda}\right)f_n\Rightarrow y(x)=f(x)+\sum_n\frac{\lambda}{\lambda_n-\lambda}\langle\phi_n|f\rangle\phi_n(x)$

$$\boxed{y(x)=f(x)+\lambda\int_a^b\sum_n\frac{\phi_n(x)\phi_n^*(z)}{\lambda_n-\lambda}f(z)\mathrm{d}z\equiv f(x)+\lambda\int_a^b R(x,z;\lambda)f(z)\mathrm{d}z.}$$

Example. $y(x)=f(x)+\lambda\int_0^1 xzy(z)\mathrm{d}z$. $G(x,z)=xz=g(x)h(z)$. The eigenfunction is $\phi_1=x$, $\int_0^1 xz\cdot z\mathrm{d}z=\frac{1}{3}x\Rightarrow\lambda_1=3$. Normalisation $\int_0^1|\phi_1|^2\mathrm{d}x=1\Rightarrow\phi_1=\sqrt{3}x$. The resolvent kernel $R(x,z;\lambda)=\frac{\phi_1(x)\phi_1^*(z)}{\lambda_1-\lambda}=\frac{3xz}{3-\lambda}$. For $\lambda\neq 3$, $y(x)=f(x)+\lambda\int_0^1\frac{3xz}{3-\lambda}f(z)\mathrm{d}z$.

Rearrange the resolvent equation gives $f(x)=y(x)-\lambda\int_a^b R(x,z;\lambda)f(z)\mathrm{d}z\Rightarrow f$ with source y and resolvent kernel $-R(x,z;\lambda)$.

Perturbation theory. Perturbation expansion (Fredholm). $y=f+\lambda\mathscr{K}y\Rightarrow y^{(0)}+\lambda y^{(1)}+\lambda^2y^{(2)}+\cdots=f+\lambda\mathscr{K}\left(y^{(0)}+\lambda y^{(1)}+\lambda^2y^{(2)}+\cdots\right)$

$\Rightarrow y^{(0)}=f, y^{(1)}=\mathscr{K}y^{(0)}, y^{(2)}=\mathscr{K}y^{(1)}\Rightarrow y^{(n)}=\mathscr{K}y^{(n-1)}=\mathscr{K}^nf\Rightarrow y=\sum_{n=0}^{\infty}(\lambda\mathscr{K})^nf$. $\mathscr{K}f=\int_a^b K(x,z)f(z)\mathrm{d}z$. $\mathscr{K}^2f=\mathscr{K}(\mathscr{K}f)=\int_a^b K(x,z')\left[\int_a^b K(z',z)f(z)\mathrm{d}z\right]\mathrm{d}z'=\int_a^b\left[\int_a^b K(x,z')K(z',z)\mathrm{d}z'\right]f(z)\mathrm{d}z\equiv\int_a^b K_2(x,z)f(z)\mathrm{d}z\Rightarrow\mathscr{K}^nf(x)=\int_a^b K_n(x,z)f(z)\mathrm{d}z$.

$$\boxed{y(x)=f(x)+\sum_{n=1}^{\infty}\lambda^n\int_a^b K_n(x,z)f(z)\mathrm{d}z=f(x)+\lambda\int_a^b K(x,z)f(z)\mathrm{d}z+\lambda^2\int_a^b K_2(x,z)f(z)\mathrm{d}z+\cdots,\ K_2(x,z)=\int_a^b K(x,z')K(z',z)\mathrm{d}z'.}$$

Example. $y(x)=x+\lambda\int_0^1 xzy(z)\mathrm{d}z$. $K(x,z)=xz$. $K_2(x,z)=\int_0^1 K(x,s)K(s,z)\mathrm{d}s=\int_0^1 xssz\mathrm{d}s=xz\int_0^1 s^2\mathrm{d}s=\frac{1}{3}xz$. $K_3(x,z)=\frac{1}{3^2}xz$.

$y(x)=x+\lambda\int_0^1 xzz\mathrm{d}z+\lambda^2\int_0^1\frac{1}{3}xzz\mathrm{d}z+\cdots=x+\sum_{n=1}^{\infty}\lambda\left(\frac{\lambda}{3}\right)^{n-1}\cdot\frac{1}{3}x=x\left(1+\sum_{n=1}^{\infty}\left(\frac{\lambda}{3}\right)^n\right)=\frac{x}{1-\frac{\lambda}{3}}$. $|\lambda|^2\int_a^b\int_a^b|K(x,z)|^2\,\mathrm{d}x\,\mathrm{d}z<1\Rightarrow\lambda<3$.

Perturbation theory for Volterra equations. $K_2(x,z)=\int_z^x K(x,z')K(z',z)\mathrm{d}z'$.

Example. $y(x)=f(x)+\lambda\int_0^x e^{x-z}y(z)\mathrm{d}z$. $K_2(x,z)=\int_z^x \mathrm{e}^{x-z'}\mathrm{e}^{z'-z}\,\mathrm{d}z'=(x-z)\mathrm{e}^{x-z}$, $K_3(x,z)=\int_z^x \mathrm{e}^{x-z'}\left(z'-z\right)\mathrm{e}^{z'-z}\,\mathrm{d}z'=\frac{1}{2}(x-z)^2\mathrm{e}^{x-z}$
$K_n(x,z)=\frac{(x-z)^{n-1}}{(n-1)!}\mathrm{e}^{x-z}$. $R(x,z;\lambda)=\sum_{n=1}^{\infty}\lambda^{n-1}K_n(x,z)=\mathrm{e}^{(1+\lambda)(x-z)}$. $y(x)=f(x)+\lambda\int_0^x R(x,z;\lambda)f(z)\mathrm{d}z$.

Initial value b.c. $\mathscr{L}=-p_0v''-p_1v'+qv$. $y(a)=0$, $y'(a)=0$. $G(x,z)=\Theta(x-z)A(z)\left[y_1(z)y_2(x)-y_2(z)y_1(x)\right]$.

Example. $\mathscr{L}y(t)\equiv\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}+\omega^2y(t)$ for $t\geq 0$ with $y(0)=0$ and $y'(0)=0$. $y_1(x)=\sin\omega x$, $y_2(x)=\cos\omega x$. $W=\begin{vmatrix}\sin\omega x & \cos\omega x\\ \omega\cos\omega x & -\omega\sin\omega x\end{vmatrix}=$
$-\omega$. $p=-1$. $A=-\frac{1}{pW}=-\frac{1}{\omega}$. $G(t,t')=-\Theta(t-t')\frac{1}{\omega}(\sin\omega t'\cos\omega t-\sin\omega t\cos\omega t')=\Theta(t-t')\frac{\sin\left[\omega(t-t')\right]}{\omega}$. $y=\int_{-\infty}^t\frac{\sin\left[\omega(t-t')\right]}{\omega}f(t')\mathrm{d}t'$.

Example. $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-b\frac{\mathrm{d}x}{\mathrm{d}t}+f(t)$. $mx''+bx'=0$ has general solution $x_1(t)=1$, $x_2(t)=e^{-\gamma t}$, where $\gamma=\frac{b}{m}$. $W=\begin{vmatrix}1 & e^{-\gamma t}\\ 0 & -\gamma e^{-\gamma t}\end{vmatrix}=-\gamma e^{-\gamma t}$.
Continuity gives $G(t,t')=\Theta(t-t')A(t')\left(x_1(t')x_2(t)-x_2(t')x_1(t)\right)$. $\int_{t'-\varepsilon}^{t'+\varepsilon}mG''(t,t')\mathrm{d}t=mG'(t,t')\Big|_{t'-\varepsilon}^{t'+\varepsilon}=mA(t')\left(x_1(t')x_2'(t')-x_2(t')x_1'(t)'\right)$
$=mA(t')W=1\Rightarrow A(t')=\frac{1}{mW}=-\frac{1}{b}e^{-\gamma t'}$. $G(t,t')=-\Theta(t-t')\frac{1}{b}e^{-\gamma t'}\left(e^{-\gamma t}-e^{-\gamma t'}\right)=\Theta(t-t')\left(1-e^{-\gamma(t-t')}\right)/b$.
If $f(t)=-mg$. $x(t)=-\frac{mg}{b}\int_0^t\left(1-e^{-\gamma(t-t')}\right)\mathrm{d}t'=-\frac{mg}{b}\left[t'-\frac{1}{\gamma}e^{-\gamma(t-t')}\right]_0^t=-\frac{mg}{b}\left(t-\frac{1}{\gamma}\left(1-e^{-\gamma t}\right)\right)$.

Eigenfunction expansion. $G(x,z)=\sum_n\frac{\phi_n(x)\phi_n^*(z)}{\lambda_n}$.

Example. $\mathscr{L}u(x)=-i\frac{\mathrm{d}}{\mathrm{d}x}u(x)=ku(x)$ defined for $x\in[0,1]$ with $u(1)=Cu(0)$, $C=e^{-i\alpha}$.
The eigenfunctions are $u_n=e^{ik_nx}$, $u(1)=e^{ik_n}=e^{-i\alpha}\Rightarrow k_n=\alpha+2\pi n\Rightarrow G(x,y)=\sum_{-\infty}^{\infty}\frac{e^{i(\alpha+2\pi n)(x-y)}}{(\alpha+2\pi n)^2}$. The homogeneous solutions are $u(x)=c$. $G(x,y)=\begin{cases}c_1(y), & x<y\\ c_2(y), & x>y\end{cases}$. $c_2(y)=e^{-i\alpha}c_1(y)$. G is discontinuous at $x=y$. $1=\int_{y-\varepsilon}^{y+\varepsilon}-i\frac{\mathrm{d}}{\mathrm{d}x}G(x,y)\mathrm{d}x=-iG(x,y)\Big|_{y-\varepsilon}^{y+\varepsilon}=$
$-i(c_2-c_1)\Rightarrow c_1=\frac{i}{C-1}, c_2=\frac{iC}{C-1}$.

First order ODE. $\mathscr{L}_ty(t)=f(t)$, $\mathscr{L}_tG(t,t')=\delta(t-t')\Rightarrow G(t,t')=\Theta(t-t')A(t')y_1(t)$.

Example. $\mathscr{L}y\equiv\dot y-\kappa y$, $y_1=e^{-\kappa t}$. $\int_{t'-\varepsilon}^{t'+\varepsilon}\mathscr{L}_tG(t,t')=A(t')y_1(t')=1\Rightarrow A(t')=e^{\kappa t'}\Rightarrow G(t,t')=\Theta(t-t')e^{-\kappa(t-t')}$.

Differential equation in time and space. $\tilde G(\mathbf{K},\tau)=\int e^{-i\mathbf{K}\cdot\mathbf{R}}G(\mathbf{R},\tau)\mathrm{d}^n\mathbf{R}$. $G(\mathbf{R},\tau)=\frac{1}{(2\pi)^n}\int e^{i\mathbf{K}\cdot\mathbf{R}}\tilde G(\mathbf{K},\tau)\mathrm{d}^n\mathbf{K}$.
$\mathscr{L}_{\mathbf{r},t}\equiv\mathscr{L}_\tau-\nabla^2\Rightarrow\mathscr{L}_{\mathbf{r},t}G(\mathbf{R},t)=\frac{1}{(2\pi)^n}\int e^{i\mathbf{K}\cdot\mathbf{R}}\left(\mathscr{L}_\tau+K^2\right)\tilde G(\mathbf{K},\tau)\mathrm{d}^n\mathbf{K}=\delta(\tau)\frac{1}{(2\pi)^n}\int e^{i\mathbf{K}\cdot\mathbf{R}}\mathrm{d}^n\mathbf{K}\Rightarrow\left(\mathscr{L}_\tau+K^2\right)\tilde G(\mathbf{K},\tau)=\delta(\tau)$.

Example. TDKG equation $\frac{\partial^2\Psi(\mathbf{r},t)}{\partial t^2}-\nabla^2\Psi(\mathbf{r},t)+m^2\Psi(\mathbf{r},t)=\Phi(\mathbf{r},t)$. F.T. $\Rightarrow\frac{\partial^2\tilde G(\mathbf{K},t)}{\partial t^2}+K^2\tilde G+m^2\tilde G=\delta(\tau)\Rightarrow\frac{\partial^2\tilde G}{\partial t^2}+\omega_k^2\tilde G=\delta(\tau)$.
$\tilde G(\tau)=\Theta(\tau)\frac{\sin\omega_k\tau}{\omega_k}\Rightarrow G(\mathbf{r},\mathbf{r}',t,t')=\Theta(t-t')\frac{1}{(2\pi)^3}\int e^{i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')}\frac{\sin\left[\omega_k(t-t')\right]}{\omega_k}\mathrm{d}^n\mathbf{K}$.

Oscillatory source for wave equation. Wave equation $-\nabla^2\psi+\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=\rho(\mathbf{r},t)$ and $\rho(\mathbf{r},t)=\rho(\mathbf{r})e^{-i\omega t}$. Then $\psi(\mathbf{r},t)=\psi(\mathbf{r})e^{-i\omega t}$.
$\Rightarrow$ Helmholtz equation $-\nabla^2\psi(\mathbf{r})-k_0^2\psi(\mathbf{r})=\rho(\mathbf{r})$. Solution to the wave eq. $\psi(\mathbf{r},t)=\int\mathrm{d}^3r'\int_{-\infty}^tG_W(\mathbf{r}-\mathbf{r}',t-t')\rho(\mathbf{r}',t)\mathrm{d}t$. Solution to the Helmholtz eq. $\psi(\mathbf{r})=\int\mathrm{d}^3r'G_H(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r})\Rightarrow\int\mathrm{d}^3r'\int_{-\infty}^tG_W(\mathbf{r}-\mathbf{r}',t-t')\rho(\mathbf{r})e^{-i\omega t'}\mathrm{d}t'=e^{-i\omega t}\int\mathrm{d}^3r'G_H(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r})\Rightarrow$
$G_H(\mathbf{R})e^{-i\omega t}=\int_{-\infty}^tG_W(\mathbf{R},t-t')e^{-i\omega t'}\mathrm{d}t'$.

Euler's equation. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$ or $F-y'\frac{\partial F}{\partial y'}=\mathrm{const}$ if $F=F(y,y')$.

Rayleigh-Ritz variational technique. $-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)y'(x)\right)+q(x)y(x)=\lambda\rho(x)y(x)$. $I[y]=\int_a^bp(x)y'(x)^2+q(x)y(x)^2\,\mathrm{d}x$, $J[y]=\int_a^b\rho(x)y(x)^2\,\mathrm{d}x$.
$\lambda[y]=\frac{I[y]}{J[y]}$ has a minimum.

Example. $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\lambda xy(x)$ for $x\in[0,1]$ with $y(1)=0$. Trial function $u=1-x^2$. Then $I[y]=\int_0^1x(u')^2\mathrm{d}x$. $J[y]=\int_0^1xu^2\mathrm{d}x$.

$$u_1(x) = x^{n-1}, \quad u_2(x) = x^{n-1} - x^{-n-1}$$

$$W = \begin{vmatrix} x^{n-1} & x^{n-1} - x^{-n-1} \\ (n-1)x^{n-2} & (n-1)x^{n-2} + (n+1)x^{-n-2} \end{vmatrix} = (n-1)x^{2n-3} + (n+1)x^{-3} - (n-1)x^{2n-3} + (n-1)x^{-3} = 2nx^{-3}$$

$$G(x,z) = \begin{cases} A(z)u_2(z)u_1(x) \\ A(z)u_1(z)u_2(x) \end{cases} \quad , \quad 1 = \lim_{\varepsilon\to 0}\int_{z-\varepsilon}^{z+\varepsilon} x^2 G''(x,z)\mathrm{d}x = x^2 G'(x,z)\Big|_{z-\varepsilon}^{z+\varepsilon} = -z^2 A(z)W(z) \Rightarrow A(z) = -\frac{z}{2n}$$

$$G(x,z) = \begin{cases} -\frac{1}{4}\left(z^{n-1} - z^{-n-1}\right) \end{cases}$$