Complex varieties

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Suppose now, our algebraically closed ground field is given a topology, making it into a topological field. The most interesting case is when k = $\mathbb{C}$. By equipping all varieties over $\mathbb{C}$ with its analytical topology(which is called the *strong topology* in our following statement), we notice that the following properties hold:

1. the strong topology is stronger than the Zaraski-topology i.e.,Zaraski closed(resp. open) sets are closed(resp. open) in strong topology.
2. all morphisms are strongly continuous.
3. the strong topology of a locally closed (i.e.,intersection of an open and a closed set) subvariety $X \subset Z$ is the induced topology on X.
4. the strong topology on $X \times Y$ is the product of strong topologies on X and Y.

Recall :

Theorem 1 (Noether's normalization, geometric form). *Suppose X is an affine variety (of dim n), then there exist a surjective morphism:*

$$\pi : X \longrightarrow \mathbb{A}^n,$$

and for every $P \in \mathbb{A}^n$, $\pi^{-1}(P)$ is a finite but nonempty set.(In the following statement, we will call such morphism a finite morphism)

Proof. (existence) Embed X into $\mathbb{A}^n$. Its coordinate ring is noted as $A := k[x_1, x_2, \dots, x_m]/I$. By Noether's normalization, there exists $y_1, y_2, \dots, y_n \in A$ which are algebraically independent, and, when the field contains infinite elements, are of the form of linear combination of $\overline{x}_1, \dots, \overline{x}_m$, making A a finite $k[y_1, y_2, \dots, y_n]$ algebra. Then the inclusion $\pi^* : k[y_1, y_2, \dots, y_n] \hookrightarrow A$ induce the morphism:

$$\pi : X \longrightarrow \mathbb{A}^n$$

(finite) As $\overline{x}_i$ is integral on $K := k[y_1, y_2, \dots, y_n]$,so we have

$$\overline{x}_i^N + f_N^i \overline{x}_i^{N-1} + \cdots + f_0^i = 0, \ f_j^i \in K$$

when $\{y_i\}$ is given, the solution is finite.

(nonempty) Let $P = (b_1, \dots, b_n)$, It suffice to show $V((y_1 - b_1, \dots, y_n - b_n)) \neq A$, applying nakayama's theorem on $K \subset A$, we know it is true. □

Lemma 2 (Hausdoff axiom). *X is a variety $\Rightarrow \Delta(X) := \{(x, x) | x \in X\}$ is closed.*

Remark 1. It sometimes becomes the definition of the variety, in other words, like the definition of manifolds, a (abstract) variety X is a locally affine topological space satisfying the Hausdoff axiom.

Proposition 3. *TFAE:*

1. *$\Delta(X)$ is closed.*
2. *for any morphism $f, g \colon Y \to X$, $\{y \in Y | f(y) = g(y)\}$ is closed.*
3. *for any morphism $f \colon Y \to X$, the graph of f is closed.(i.e., the image of $Y \xrightarrow{(f,id)} X \times Y$ is closed)*

The following definition and theorem is mentioned by HY Gao.

Definition 4 (Complete variety). X is a complete variety if for any variety Y, $X \times Y \to Y$ is a closed map.

Theorem 5. *Any projective variety is complete.*

Our work starts here. The first comparison of two topologies states that strong topology is not "too strong".

Theorem 6. *Let X be a irreducible variety, and U a nonempty open subvariety. Then U strongly dense in X.*

Proof. Since the statement is local, we can suppose X is affine with coordinate ring $A \subset k[x_1, \dots, x_m]$, By Noether's theorem, we can find $K := k[y_1, \dots, y_n]$ s.t, A is a finite algebra over K and there exist a finite surjective map

$$\pi : X \to \mathbb{A}^n.$$

Let $Z = X \setminus U$ then Z be Zaraski closed *i.e.*, $Z = V(f_1, \dots, f_n)$. Choose $h \in I(Z)$, since h is integral on K, it satisfies an equation $h^N + f_N h^{N-1} + \cdots + f = 0$, $f_i, f \in K$.When $x \in Z \Rightarrow h(x) = 0 \Rightarrow f(y) = 0$,therefore $\pi(Z) \subset V(f))$. Now fixed $x_0 \in X$, we want to construct a sequence $\{y^i\} \in \mathbb{A}^n \setminus \pi(Z)$ strongly converge to $\pi(x_0)$. In order to do this, take $y \notin V(f)$ and

$$\tilde{h} := f((1-t)y + t\pi(x_0)), \tilde{h}(t) \not\equiv 0.$$

Therefore $\tilde{h}(t)$ has finite zero points, so we can take $t^i \to 0$, and the corresponding $y^i := (1-t^i)y + t^i\pi(x_0) \notin \pi(Z)$ and $y^i \to \pi(x_0)$,. The problem now is to lift the converge from $\mathbb{A}^n$ into X. Let$\pi(\pi^{-1}(x_0)) = \{x_0.x^1, x^2, \dots, x^{n'}\}$, there exist a $g \in A$ s.t. $g(x_0) = 0$ and $g(x^i) \neq 0$ (T_4 property of Zaraski topology). Due to the integral dependency of $g, \exists\ A_1, \dots, A_N \in K$ satisfy:

$$F(x) := g^N + A_N g^{n-1} + \cdots + A_0 = 0.$$

The inclusions of

$$K \xrightarrow{i_1} K[g] = k[y_1, \dots, y_n, y_{n+1}]/F \xrightarrow{i_2} A$$

induce a morphism chain:

$$\begin{tikzcd}
X \arrow[r, hook, "\pi_1"] & V(F) \subset \mathbb{A}^{n+1} \arrow[r, hook, "\pi_2"] & \mathbb{A}^n \\
(x_1, x_2, \dots, x_m) \arrow[r, mapsto, "\pi_1"] & (y_1, \dots, y_n, g(x)) \arrow[r, mapsto, "\pi_2"] & (y_1, \dots, y_n)
\end{tikzcd}$$

We claim: π_1, π_2 is finite.
Now that, $F(x_0) = F((\pi(x_0), g(x_0)) = 0 \Rightarrow A_0(\pi(x_0)) = 0$, $A_0(y^i) \to 0$. On the other hand,

$$A_0(y^i) = \prod_{F(y^i, t^i_k) = 0} t^i_k.$$

Therefore we can find roots t^i of $F(y^i, t) = 0$ such that $(y^i, t^i) \to \pi_1(x_0)$.

Similarly, since $x_1, x_2, \dots, x_n$ integral on $K[g]$, $\exists\ a_{ij} \in K[g]$ such that

$$x_i^N + a_{in} x_i^{N-1} + \cdots + a_{i0} = 0$$

When Σ is a compact set $\subset V(F)$ in the strong topology, $\{a_{ij}(\Sigma)\}$ is bounded. So $\pi^{-1}(\Sigma)$ is closed and bounded strongly, thus compact strongly. Choose $\Sigma = \{(y^i, t^i)\}$. We claim that there exist a subsequence $\{z^i\} \in X$ strongly converge to x_0. If not, for compact set, there must be a subsequence $\{z^{i^k}\}$ converge to $x' \neq x$. then

$$\pi(x') = \lim_{i^k \to \infty} \pi(z^{i^k}) = \pi(x)$$

therefore x' must be some x^i, but

$$t^{i^k} = \lim_{i^k \to \infty} g(z^{i^k}) = 0$$

Contradiction! thus there exist a sequence in U converge to x_0 strongly. $\square$

$\square$

Corllary 7. *If $Z \subset X$ is locally closed, the strong closure and Zaraski closure actually the same.*

Proof: Z is open in $\bar{Z}^{zar}$,so strongly dense in $\bar{Z}^{zar}$. The strong topology of $\bar{Z}^{zar}$ is induced by that of X, therefore, $\bar{Z}^{str} = \bar{Z}^{zar}$. $\square$

Theorem 8. *Let X be a complex variety, then X in complete iff X compact strongly.*

Proof: ($\Longrightarrow$) Suppose X is strongly compact. Assume Y is affine (moreover,$\mathbb{A}^n$), $Z \subset X \times Y$ irreducible closed set, and $p_2 : X \times Y \to Y$ the projection. Since X compact, p_2 is a proper map strongly (the inverse image of compact set is compact), with locally compactness of Y=$\mathbb{A}^n$, P_2 is closed in the strong topology. And as Z irreducible, $p_2(Z)$ contains a Zaraski open set in $\overline{p_2(Z)}^{zar}$, by corllary1, we get $\overline{p_2(Z)}^{zar} = \overline{p_2(Z)}^{str} = p_2(Z)$.

($\Longleftarrow$) As we know, the projective variety over $\mathbb{C}$ is compact and complete. The following lemma shows us how complete varieties are related with projective varieties.

Lemma 9 (Chow)**.** *X is a complete variety over a algebraically closed field, then there exist a projective variety Y and a surjective, birational map:*

$$\pi : Y \to X$$

Proof:

STEP 1 (construct Y) Cover X by a colletion of open affine subsets $U_i \subset \mathbb{A}^n \subset \mathbb{P}^n$. Since $k[x_1, \ldots, k_n]$ is noetherian, the cover is finite. Take $\bar{U}_i$, the closure of U_i in $\mathbb{P}^n$, $U^* = \bigcap_{i=1}^n U_i$ an open set.Then consider the "multidiagonal" morphism:

$$U^* \xrightarrow{\Delta^n} U^* \times U^* \times \cdots \times U^* \hookrightarrow \bar{U}_1 \times \cdots \times \bar{U}_m$$

the closure of the image in $\bar{U}_1 \times \cdots \times \bar{U}_m$ is denoted by Y. Y is a projective thus complete variety. Now we want to construct the birational morphism π.

STEP 2 (construct π)Consider following morphism:

$$U^* \xrightarrow{\Delta} U^* \times U^* \xrightarrow{(i,\ \Delta^n)} X \times Y$$

the projection restricted on the closure of image(noted as $\tilde{Y}$) give us two commutative diagrams:

$$\begin{tikzcd} U^* \arrow[r, hook] \arrow[d, "id"'] & \tilde{Y} \arrow[d, "p"] \\ U^* \arrow[r, hook] & X \end{tikzcd} \qquad \begin{tikzcd} U^* \arrow[r, hook] \arrow[d, "id"'] & \tilde{Y} \arrow[d, "q"] \\ U^* \arrow[r, hook] & y \end{tikzcd}$$

as U^* is (Zaraski) open and dense in $\tilde{Y}$, p, q are birational. Moreover, as X,Y are both complete, $p(\tilde{Y}), q(\tilde{Y})$ is closed set contains U^*, it follows that p, q are surjection. Let $\pi = p \circ q^{-1}$, if q *is isomorphism*, the everything is proven. By following lemma, we can prove it easily;

STEP 3 (q isomorphism) first, let us look at this lemma.

Lemma 10. *Let S and T be varieties, with isomorphic open subset V, look at the morphism:*

$$V \xrightarrow{\Delta} V \times V \hookrightarrow S \times T.$$

If $\bar{V}$ is the closure of the image, then

$$\bar{V} \cap (S \times V) = \bar{V} \cap (V \times T) = (\Delta(V)).$$

Proof: It suffice to show that $\Delta(V)$ is already closed in $V \times T$ and in $S \times V$. But$\bar{V} \cap (S \times V)$ is just the graph of the inclusion morphism $V \to T$. Hence it is closed.

Back to our proof, we want to analyze the projection $X \times Y$ on $X \times \bar{U}_i$. Look at the diagram:

$$\begin{tikzcd} & U^* \times U^* \ldots U^* \arrow[r, hook] \arrow[d] & X \times \bar{U}_1 \ldots \bar{U}_n \arrow[d, "r_i"] \\ U^* \arrow[ur, "\Delta^n"] \arrow[r, "\Delta"] & U^* \times U^* \arrow[r, hook] & X \times \bar{U}_i \end{tikzcd}$$

We claim that $r_i(\tilde{Y}) = \overline{\Delta(U^*)}$ because r_i is a closed map. Therefore

$$\begin{aligned} r_i(\tilde{Y}) \cap (X \times U_i) &= r_i(\tilde{Y}) \cap (U_i \times \bar{U}_i) \\ &= \{(x, x) | x \in U_i\} \end{aligned}$$

Therefore

$$\tilde{Y} \cap (X \times \bar{U}_1 \times \cdots \times U_i \times \cdots \times \bar{U}_n) = \tilde{Y} \cap (U_i \times \bar{U}_1 \times \cdots \times \bar{U}_n)$$

Called these set $\tilde{Y}_i$. From the second form it follows that $\{\tilde{Y}_i\}$ is a open covering of $\tilde{Y}$. From the first form of the intersection, it follows that

$$\tilde{Y}_i \quad = \quad q^{-1}(Y_i)$$

if

$$Y_i \quad = \quad \tilde{Y} \cap (\bar{U}_1 \times \cdots \times U_i \times \cdots \times \bar{U}_n)$$

Since q is surjective, this implies that Y_i must be an open covering of Y, now define

$$\begin{aligned} &\sigma_i : Y_i \to \tilde{Y}_i \\ &\sigma_i(u_1, \ldots, u_n) = (u_i, u_1, \ldots, u_m)) \end{aligned}$$

Then σ_i is an inverse of q restrict to $\tilde{Y}_i$:

(a) $q(\sigma_i(u_1, \ldots, u_n)) = q(u_i, u_1, \ldots, u_n) = (u_1, \ldots, u_n)$

(b) by the lemma, all points $(v, u_1, \ldots, u_n)$ of $\tilde{Y}_i$ satisfy $v = u_i$, hence

$$\sigma_i(q(v, u_1, \ldots, u_n)) = \sigma_i(u_1, \ldots, u_n) = (v, u_1, \ldots, u_n)$$

By gluing σ_i together,we can see q is an isomorphism, and π is constructed. □

Complex and algebraic geometry

Algebraic varieties are locally defined as the common zero sets of polynomials and since polynomials over the complex numbers are holomorphic functions, algebraic varieties over C can be interpreted as analytic spaces. Similarly, regular morphisms between varieties are interpreted as holomorphic mappings between analytic spaces. Somewhat surprisingly, it is often possible to go the other way, to interpret analytic objects in an algebraic way.

Actually,(Reference: The Red Book §10)this is because the strong topology in X induces a strong structure sheaf $\Omega_x(U)$, which is called the "holomorphic function" on U, such that:

1. $O_x(U) \subset \Omega_x(u)$, i.e., All regular function is holomorphic.
2. All morphisms $f : X \to Y$ are "holomorphic", i.e., $g \in \Gamma(\Omega_y)$, $f^*g \in \Gamma(\Omega_X)$.
3. $\Omega_Z)$ on locally closed subvariety Z is induced by the restriction on $\Omega_x(U)$.

Ex Riemann sphere. it is easy to prove that the analytic functions from the Riemann sphere to itself are either the rational functions or the identically infinity function (an extension of Liouville's theorem). This fact shows that there is no essential difference between the view of $\hat{C}$ or $\mathbb{P}^1$

Here are some important result:

Theorem 11. *Compact riemann surfaces are projective variety.*

Ex The torus $\mathbb{C}/\Lambda$. Its function field (meromorphic function) over $\mathbb{C}$ are generated by Weierstrass function $\wp(z)$ and $\wp\prime(z)$. (Reference: GTM228 §7.5) There is a equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

Therefore the function field is isomorphic to $Q(\mathbb{C}[x, y, z]/_{y^2z-4x^2z+g_2xz^2+g_3z^3})$ which determines a elliptic curve.

Theorem 12 (Chow)**.** *Any closed analytic subspace of complex projective space is an algebraic subvariety.*

Theorem 13 (Lefschetz principle)**.** $(m \models ACF-_0) \Rightarrow (m \models Th((\mathbb{C}, 0, 1, +, \times)))$

Theorem 14. *Statement of GAGA*