Rob 501 - Mathematics for Robotics
Recitation #10

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1 Set Theory

1. In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$,
   - The distance from vector $x \in \mathcal{X}$ to vector $y \in \mathcal{X}$ is defined as $\mathrm{d}(x, y) := \|x - y\|$.
   - The distance from a vector $x \in \mathcal{X}$ to a set $S \subset \mathcal{X}$ is defined as $\mathrm{d}(x, S) := \inf_{y \in S} \|x - y\|$.

2. In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$, let $P \subset \mathcal{X}$ be a subset.
   - A point $x \in P$ is an interior point of P if $\exists\, \epsilon > 0$ such that $B_\epsilon(x) \subset P$.
     The interior of P, denoted as $\mathring{P}$, is the set of all the interior points of P.
   - A point $x \in \mathcal{X}$ is a closure point of P if $\forall\, \epsilon > 0$, $B_\epsilon(x) \bigcap P \neq \emptyset$.
     The closure of P, denoted as $\overline{P}$, is the set of all the closure points of P.
   - P is open if $P = \mathring{P}$.
   - P is closed if $P = \overline{P}$.

   Remark:
   - The interior of P is the largest open set contained in P.
   - The closure of P is the smallest closed set containing P.
   - P is open $\iff P = \{x \in \mathcal{X} \,|\, \exists\, \epsilon > 0 : B_\epsilon(x) \subset P\}$
     $\iff P = \{x \in \mathcal{X} \,|\, \exists\, \epsilon > 0 : B_\epsilon(x) \bigcap (\sim P) = \emptyset\}$
     $\iff P = \{x \in \mathcal{X} \,|\, \mathrm{d}(x, \sim P) > 0\}$
   - P is closed $\iff P = \{x \in \mathcal{X} \,|\, \forall\, \epsilon > 0,\, B_\epsilon(x) \bigcap P \neq \emptyset\}$
     $\iff P = \{x \in \mathcal{X} \,|\, \forall\, \epsilon > 0,\, \exists\, y \in P : \, ||x - y|| < \epsilon\}$
     $\iff P = \{x \in \mathcal{X} \,|\, \mathrm{d}(x, P) = 0\}$

3. **Proposition:** In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$,
   - A finite intersection of open sets is open.
   - A finite union of closed sets is closed.
   - An arbitrary union of open sets is open.
   - An arbitrary intersection of closed sets is closed.

   Remark:
   An infinite intersection of open sets can be either open or closed, or neither.
   An infinite union of closed sets can also be either open or closed, or neither.

4. Ex: In the following examples, $\mathbb{I}$ denote the set of irrational numbers.

   (a) In $(\mathbb{R}, \mathbb{R}, |\cdot|)$, $\mathrm{d}(\sqrt{2}, 1) = ?$ $\mathrm{d}(\sqrt{2}, \mathbb{Q}) = ?$ If given $x \in \mathbb{I}$, $\mathrm{d}(x, \mathbb{Q}) = ?$
   $\mathrm{d}(\sqrt{2}, 1) = \sqrt{2} - 1$

   Notice the fact that $\forall\, \epsilon > 0,\ \exists\, m, n \in \mathbb{Z},\ m$ and n are co-prime : $|\sqrt{2} - \dfrac{m}{n}| < \epsilon$.
   In words, $\sqrt{2}$ can be approximated arbitrarily close by a rational number.
   Thus, $\mathrm{d}(\sqrt{2}, \mathbb{Q}) = \inf\limits_{y \in \mathbb{Q}} |\sqrt{2} - y| = \inf\limits_{\epsilon > 0} \epsilon = 0$.

   This can be extended to any irrational number $x \in \mathbb{I}$, i.e.,
   $\forall\, x \in \mathbb{I}, \forall\, \epsilon > 0,\ \exists\, m, n \in \mathbb{Z},\ m$ and n are co-prime : $|x - \dfrac{m}{n}| < \epsilon$.
   Thus, for a given $x \in \mathbb{I}$, $\mathrm{d}(x, \mathbb{Q}) = 0$.

   (b) Are the sets below open or closed?

   - In normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$,

$P_1 = \{0\}$	closed
$P_2 = [0,\ 1]$	closed
$P_3 = (0,\ 1)$	open
$P_4 = [0,\ 1)$	neither open nor closed
$P_5 = \mathbb{R}$	both open and closed
$P_6 = \emptyset$	both open and closed

   - In normed space $(\mathbb{R}^2, \mathbb{R}, \|\cdot\|)$,

$P_1 = (0,\ 1) \times (0,\ 1)$	open
$P_2 = [0,\ 1] \times (0,\ 1)$	neither open nor closed
$P_3 = \{(x,\ y) \in \mathbb{R}^2 \,\|\, y = 2x + 1\}$	closed

   (c) Recall from recitation 2, we have shown that the set of rational numbers $\mathbb{Q}$ with standard $+$ and $\times$ operation is a field, and a field over itself is a vector space, so $(\mathbb{Q}, \mathbb{Q})$ is a vector space. If we define the norm in $(\mathbb{Q}, \mathbb{Q})$ as $\|x - y\| = |x - y|$ for all $x, y \in \mathbb{Q}$, then $(\mathbb{Q}, \mathbb{Q}, \|\cdot\|)$ is a normed space.

   - In normed space $(\mathbb{Q}, \mathbb{Q}, \|\cdot\|)$, $\overline{\mathbb{Q}} = ?$ $\mathring{\mathbb{Q}} = ?$ Is $\mathbb{Q}$ open or closed?
     $\overline{\mathbb{Q}} = \{x \in \mathbb{Q} \,|\, \mathrm{d}(x,\ \mathbb{Q}) = 0\} = \mathbb{Q}$
     $\mathring{\mathbb{Q}} = \{x \in \mathbb{Q} \,|\, \mathrm{d}(x,\ \sim\mathbb{Q}) > 0\} = \mathbb{Q}$
     $\mathbb{Q}$ is both open and closed in the normed space $(\mathbb{Q}, \mathbb{Q}, \|\cdot\|)$.
   - In normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$, $\overline{\mathbb{Q}} = ?$ $\mathring{\mathbb{Q}} = ?$ Is $\mathbb{Q}$ open or closed? $\overline{\mathbb{I}} = ?$ $\mathring{\mathbb{I}} = ?$ Is $\mathbb{I}$ open or closed?
     $\overline{\mathbb{Q}} = \{x \in \mathbb{R} \,|\, \mathrm{d}(x,\ \mathbb{Q}) = 0\} = \{x \in \mathbb{Q} \,|\, \mathrm{d}(x,\ \mathbb{Q}) = 0\} \bigcup \{x \in \mathbb{I} \,|\, \mathrm{d}(x,\ \mathbb{Q}) = 0\} = \mathbb{Q} \bigcup \mathbb{I} = \mathbb{R}$
     $\mathring{\mathbb{Q}} = \{x \in \mathbb{R} \,|\, \mathrm{d}(x,\ \sim\mathbb{Q}) > 0\} = \{x \in \mathbb{R} \,|\, \mathrm{d}(x,\ \mathbb{I}) > 0\}$
     $= \{x \in \mathbb{Q} \,|\, \mathrm{d}(x,\ \mathbb{I}) > 0\} \bigcup \{x \in \mathbb{I} \,|\, \mathrm{d}(x,\ \mathbb{I}) > 0\} = \emptyset$
     $\mathbb{Q}$ is neither open nor closed in the normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$.
     Similarly,
     $\overline{\mathbb{I}} = \{x \in \mathbb{R} \,|\, \mathrm{d}(x,\ \mathbb{I}) = 0\} = \mathbb{R}$
     $\mathring{\mathbb{I}} = \{x \in \mathbb{R} \,|\, \mathrm{d}(x,\ \sim\mathbb{I}) > 0\} = \{x \in \mathbb{R} \,|\, \mathrm{d}(x,\ \mathbb{Q}) > 0\} = \emptyset$
     $\mathbb{I}$ is neither open nor closed in the normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$.

   (d)

$$\bigcap_{n=1}^{\infty}\left(-1+\frac{1}{n},\ 1\right)=(0,\ \ 1), \qquad \bigcup_{n=1}^{\infty}\left[-1,\ \frac{1}{n}\right]=[-1,\ \ 1],$$

$$\bigcap_{n=1}^{\infty}\left(-1-\frac{1}{n},\ 1+\frac{1}{n}\right)=[-1,\ \ 1], \qquad \bigcup_{n=1}^{\infty}\left[-1+\frac{1}{n},\ 1-\frac{1}{n}\right]=(-1,\ \ 1),$$

$$\bigcap_{n=1}^{\infty}\left(-1-\frac{1}{n},\ 1\right)=[-1,\ \ 1), \qquad \bigcup_{n=1}^{\infty}\left[-1+\frac{1}{n},\ \frac{1}{n}\right]=(-1,\ \ 1].$$

2 Completeness and compactness

1. A sequence (x_n) in a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$ is a Cauchy sequence if

$$\forall \epsilon > 0, \exists N(\epsilon) < +\infty : \forall n, m \geq N, \|x_n - x_m\| < \epsilon.$$

2. A sequence (x_n) in a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$ is a convergent sequence if there exists an $x^* \in \mathcal{X}$ such that

$$\forall \epsilon > 0, \exists N(\epsilon) < +\infty : \forall n \geq N, \|x_n - x^*\| < \epsilon.$$

3. A normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$ is complete if every Cauchy sequence in $\mathcal{X}$ converges to a limit in $\mathcal{X}$. A complete normed space is called Banach space.

4. In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$, a subset $P \subset \mathcal{X}$ is a complete set if every Cauchy sequence in P converges to a limit in P.

5. In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$, a subset $P \subset \mathcal{X}$ is (sequentially) compact if every sequence in P has a convergent subsequence whose limit belongs to P.

6. **Theorem:**
   - In a finite dimensional normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$, a subset of $\mathcal{X}$ is compact if and only if it is closed and bounded.
   - In a finite dimensional normed space, every bounded sequence has a convergent subsequence.
   - In a normed space, any finite dimension subspace is complete.
   - Any closed subset of a complete set is complete.

7. Ex:
   (a) Consider the sequence (x_n) below.
   - $x_n = \left(1 + \frac{1}{n}\right)^n$
     Notice that $x_n \in \mathbb{Q}$, but $x_n \to e \notin \mathbb{Q}$
   - $x_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$. Considering Taylor series $\arctan x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} x^k$.
     Notice that $x_n \in \mathbb{Q}$, but$x_n \to \frac{\pi}{4} \notin \mathbb{Q}$

   (b) Is the normed space $(\mathbb{Q}, \mathbb{Q}, \|\cdot\|)$ complete? How about normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$?
   $(\mathbb{Q}, \mathbb{Q}, \|\cdot\|)$ is not complete since the sequence given in (a) doesn't have a limit in it.
   $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$ is complete.

   (c) In normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$, whether the following sets are compact and complete?
   - $[0, +\infty)$
     It is complete because it is a closed subset of a Banach space. However, it is not compact due to its unboundedness. Or, you can think about a sequence, such as $x_n = n$, which doesn't have a convergent subsequence, so it is not compact.
   - $[0, 1)$
     It is neither complete nor compact. You can consider a Cauchy sequence $x_n = 1 - \frac{1}{n}$. The limit doesn't belong to $[0, 1)$.
   - $[0, 1]$
     It is both complete and compact.