Rob 501 - Mathematics for Robotics
Recitation #2

Nils Smit-Anseeuw (Courtsey:Wubing Qin)

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1 Truth Tables

P	Q	$\sim$ P	$\sim$ Q	P $\wedge$ Q	P $\vee$ Q	$P \implies Q$	$\sim$ P $\vee$ Q	P $\wedge \sim$ Q
T	T	F	F	T	T	T	T	F
T	F	F	T	F	T	F	F	T
F	T	T	F	F	T	T	T	F
F	F	T	T	F	F	T	T	F

2 Negation of statements

1. Simple negation

   Ex:

   - $x > 2$ $x \leq 2$
   - at least 3 elements at most 2 elements
   - $p \wedge q$ $\sim p \vee \sim q$
   - $p \vee q$ $\sim p \wedge \sim q$
   - $x \in \mathbb{R}, x \neq 0$ $x \in \mathbb{R}, x = 0$
   - x satisfying $f(x) = 0$ is unique. x satisfying $f(x) = 0$ is not unique.

2. Statement with quantifiers Ex:

   - $\forall x \in X : P(x) < 0$ $\exists x \in X : P(x) \geq 0$
   - $\forall\, (x \in \mathbb{R}^n,\ x \neq 0) : x^T A x \geq 0$ $\exists\, (x \in \mathbb{R}^n,\ x \neq 0) : x^T A x < 0$
   - $\forall \epsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \geq N, |x_n - x^*| < \epsilon$ $\exists \epsilon > 0,\ \forall N \in \mathbb{N} :\ \exists n \geq N, |x_n - x^*| \geq \epsilon$

3. Statement with implications

   - $p \implies q$ $p \wedge \sim q$
   - $\forall \epsilon > 0, \exists \delta > 0 : \forall x,\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$
     $\exists \epsilon > 0, \forall \delta > 0 : \exists x,\ |x - x_0| < \delta \wedge |f(x) - f(x_0)| \geq \epsilon$

3 Proofs

For all integers $n \in \mathbb{N}$, Prove:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$P(1) = \dfrac{1 \times 2}{2} = 1$, Valid
Assume it is true for k, $P(k) = \dfrac{k(k+2)}{2}$
$P(k+1) = P(k) + (k+1) = \dfrac{k(k+1)}{2} + (k+1) = \dfrac{k^2+k+2k+2}{2} = \dfrac{k^2+3k+2}{2} = \dfrac{(k+1)(k+2)}{2}$

4 Subspace

Definition: Let $(\mathcal{X}, \mathcal{F})$ be a vector space and let $\mathcal{Y} \subset \mathcal{X}$. Then $(\mathcal{Y}, \mathcal{F})$ is a *subspace* of $(\mathcal{X}, \mathcal{F})$ if $(\mathcal{Y}, \mathcal{F})$ is a vector space when you use the rules of vector addition and scalar times vector multiplication defined on $(\mathcal{X}, \mathcal{F})$.

Proposition: $(\mathcal{X}, \mathcal{F})$ is a vector space and $\mathcal{Y} \subset \mathcal{X}$. The following are equivalent (TFAE):

- $(\mathcal{Y}, \mathcal{F})$ is a subspace
- a) $\forall y_1, y_2 \in \mathcal{Y}, \ y_1 + y_2 \in \mathcal{Y}$

  b) $\forall y \in \mathcal{Y}, \ \forall \alpha \in \mathcal{F}, \ \alpha y \in \mathcal{Y}$
- $\forall y_1, y_2 \in \mathcal{Y}, \ \forall \alpha \in \mathcal{F}, \ y_1 + \alpha y_2 \in \mathcal{Y}$
- $\forall y_1, y_2 \in \mathcal{Y}, \ \forall \alpha_1, \alpha_2 \in \mathcal{F}, \ \alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{Y}$

Which of the following are subspaces?:

1. $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^3, \mathbb{R}), \ \mathcal{Y} = \{x \in \mathbb{R}^3 : \ Cx = b;\ C, b$ are given constants$\}$
   If $b = 0$, $(\mathcal{Y}, \mathcal{F})$ is a subspace of $(\mathcal{X}, \mathcal{F})$; but if $b \neq 0$, $(\mathcal{Y}, \mathcal{F})$ is not a subspace of $(\mathcal{X}, \mathcal{F})$.
   $\forall y_1, y_2 \in \mathcal{Y}, \ \forall \alpha \in \mathcal{F}, \ C(y_1 + \alpha y_2) = Cy_1 + \alpha C y_2 = b + \alpha b$.
   If $b = 0$, $C(y_1 + \alpha y_2) = 0 = b$, i.e., $y_1 + \alpha y_2 \in \mathcal{Y}$.
   If $b \neq 0$, $C(y_1 + \alpha y_2) = (1+\alpha)b \neq b$, i.e., $y_1 + \alpha y_2 \notin \mathcal{Y}$.
2. $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^{n \times n}, \mathbb{R}), \ \mathcal{Y} = \{A \in \mathbb{R}^{n \times n} : \ A = A^\top\}$
   Yes. $\forall A_1, A_2 \in \mathcal{Y}, \ \forall \alpha \in \mathcal{F}, \ (A_1 + \alpha A_2)^\top = A_1^\top + \alpha A_2^\top = A_1 + \alpha A_2$, i.e., $A_1 + \alpha A_2 \in \mathcal{Y}$.
3. $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^{n \times n}, \mathbb{R}), \ \mathcal{Y} = \{A \in \mathbb{R}^{n \times n} : \ A$ is not invertible$\}$
   No. $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{Y}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{Y}$, $A_1 + A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible, so $A_1 + A_2 \notin \mathcal{Y}$.
4. $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^{n \times n}, \mathbb{R}), \ \mathcal{Y} = \{P \in \mathbb{R}^{n \times n} : \ A^2 = A\}$
   No. $A_1, A_2 \in \mathcal{Y}$, $(A_1 + A_2)^2 = A_1^2 + A_1A_2 + A_2A_1 + A_2^2 = A_1 + A_2 + A_1A_2 + A_2A_1 \neq A_1 + A_2$ if $A_1A_2 + A_2A_1 \neq 0$.