Rob 501 - Mathematics for Robotics
Recitation #5

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1 Norms

1. Let $(\mathcal{X}, \mathbb{C})$ be a vector space. A function $||\cdot|| : \mathcal{X} \to \mathbb{R}$ is a norm if:

   - (**Non-negative**): $\forall\, x \in \mathcal{X}, ||x|| \geq 0$ and $||x|| = 0 \Leftrightarrow x = 0$.
   - (**Triangular inequality**): $\forall\, x,\, y \in \mathcal{X},\ ||x+y|| \leq ||x|| + ||y||$.
   - (**Scalability**): $\forall\, \alpha \in \mathbb{C},\ \forall\, x \in \mathcal{X},\ ||\alpha\, x|| = |\alpha|\, ||x||$.

   And, $(\mathcal{X},\, \mathbb{C},\, ||\cdot||)$ is called a normed space.

   Ex:

   (a) In $(\mathbb{R}^n,\, \mathbb{R})$, prove $||x||_1 = \sum_{i=1}^{n} |x_i|$ and $||x||_\infty = \max_{1 \leq i \leq n} |x_i|$ are norms.

   (b) In $(\mathbb{R}^2,\, \mathbb{R})$, plot the results of $||x||_1 = 1$, $||x||_2 = 1$, $||x||_\infty = 1$ on the $x_1 x_2$ plane. Then think about $||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$, $p \geq 1$.

   (c) In $(\mathbb{R}^2,\, \mathbb{R})$, given a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$ and define $f(x) = (x^\top A x)^{1/2}$. Is it a norm?

(d) Suppose $(\mathbb{R}^n, \mathbb{R}, ||\cdot||_V)$ is a normed space with some kind of norm $||\cdot||_V$ defined. In $(\mathbb{R}^{n\times n}, \mathbb{R})$, define $f_V(A) = \sup\limits_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \dfrac{||Ax||_V}{||x||_V}$. Is it a norm? Try to calculate $f_2(A)$.

2 Inner product

1. Let $(\mathcal{X}, \mathbb{C})$ be a vector space. A function $<\cdot, \cdot>: \mathcal{X}\times\mathcal{X}\to\mathbb{C}$ is an inner product if:
   - (**Hermitian symmetry**): $\forall x,\, y\in\mathbb{C}, <x,\, y> = \overline{<y,\, x>}$.
   - (**Linear in the first argument**): $\forall\alpha_1,\,\alpha_2\in\mathbb{C},\ x_1,\, x_2,\, y\in\mathcal{X},\ <\alpha_1x_1+\alpha_2x_2,\, y> = \alpha_1 <x_1,\, y> + \alpha_2 <x_2,\, y>$.
   - (**Non-negative**): $\forall x\in\mathcal{X}, <x, x>\geq 0$ and $<x,\, x> = 0 \Leftrightarrow x = 0$.

   And, $(\mathcal{X},\, \mathbb{C},\, <\cdot, \cdot>)$ is called an inner product space.

2. Let $(\mathcal{X},\, \mathbb{C},\, <\cdot, \cdot>)$ be an inner product space. Given two vectors $x,\, y\in\mathcal{X}$, x is orthogonal to y if $<x,\, y> = 0$.

   Ex:

   (a) In $(\mathbb{C}^n, \mathbb{C})$, define $<x,\, y> = x^\top\bar{y}$.

   (b) In $(\mathbb{R}^n, \mathbb{R})$, define $<x,\, y> = x^\top y$.

   (c) $\mathcal{X} = \{q(x)\,|\,\text{polynomials in } x \text{ with real coefficients of order } n,\ n\leq 3,\ x\in\mathbb{R}\}$, $\mathcal{F} = \mathbb{R}$.

   (i) Define $<f,\, g> := \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$. Is it an inner product?

   (ii) Given a set of vectors $\{1,\, x,\, x^2,\, x^3\}$, calculate their products defined in (i).

   (iii) Given another set of vectors $\left\{1,\, x,\, \frac{1}{2}(3x^2-1),\, \frac{1}{2}(5x^3-3x)\right\}$, calculate their products defined in (i).

(d) $\mathcal{X} = \{f \mid f : \mathbb{R} \to \mathbb{R},\ f(x+T) = f(x),\ T \text{ is a given constant}\}$, $\mathcal{F} = \mathbb{R}$.

(i) Define $< f, g > := \frac{1}{T}\int_0^T f(x)g(x)\,\mathrm{d}x$. Is it an inner product?

(ii) Given a set of vectors $\{1, \sin(\frac{2\pi}{T}x), \cos(\frac{2\pi}{T}x), \sin(\frac{4\pi}{T}x), \sin(\frac{4\pi}{T}x)\}$, calculate their products defined in (i).

3 Gram Schmidt process

1. In an inner product space $(\mathcal{X}, \mathcal{F}, < \cdot >)$, given $x, y \in \mathcal{X}$. *x is orthogonal to y* if $< x, y >= 0$, denoted as $x \perp y$.

2. A set S is *orthogonal* if $\forall\,(x, y \in S,\ x \neq y), x \perp y$.

3. A set S is *orthonormal* if S is orthogonal and $\forall\, x \in S$, $\|x\| = 1$.

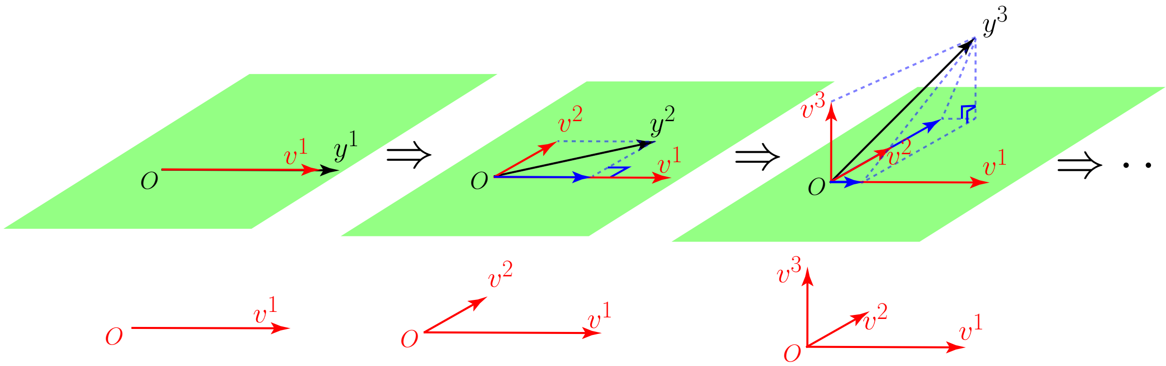
4. In a finite dimensional vector space, any set of linear independent vectors can be completed to a basis.

5. In an inner product space $(\mathcal{X}, \mathcal{F}, < \cdot >)$, given $S \subset \mathcal{X}$ a subset, the *orthogonal complement of S* is
$$S^{\perp} := \{x \in \mathcal{X} \mid\ < x, y >= 0,\ \forall\, y \in S\}.$$

6. In an inner product space $(\mathcal{X}, \mathcal{F}, < \cdot >)$, given a set of linear independent vectors $Y = \{y^1, y^2, \ldots, y^n\}$. There exists an orthogonal set $V = \{v^1, v^2, \ldots, v^n\}$ such that span $\{V\}$ = span $\{Y\}$. V can be otained by
$$\begin{aligned} k &= 1, \qquad v^1 = y^1, \\ k &\geq 2, \qquad v^k = y^k - \sum_{i=1}^{k-1} \frac{< y^k, v^i >}{< v^i, v^i >} v^i. \end{aligned}$$

7. Ex:

   - In $(\mathbb{R}^3, \mathbb{R})$, demonstrate how Gram Schmidt process works.

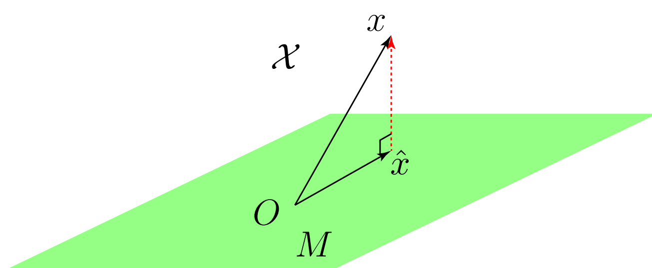


- In $(\mathbb{R}^3, \mathbb{R})$, given a set $Y = \left\{ y^1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, y^2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $< x, y >= x^\top y$, find an orthogonal set V such that span $\{V\}$ = span $\{Y\}$. Complete set V to a basis. What if you are asked to complete set V to an orthogonal basis.

- $\mathcal{X} = \{q(x) \,|\, \text{polynomials in } x \text{ with real coefficients of order } n,\ n \le 3,\ x \in \mathbb{R}\}$, $\mathcal{F} = \mathbb{R}$. The inner product is defined as $< f, g >:= \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$. Given a set of vectors $\{1, x, x^2\}$, apply Gram Schmidt process to find an orthogonal set V, and then find $V^\perp$.

4 Projection theorem

1. Projection Theorem: $(\mathcal{X}, \mathbb{R}, <\cdot, \cdot>)$ is a **finite dimensional inner product space**, and $M \subset \mathcal{X}$ is a subspace of $\mathcal{X}$. Then $\forall x \in \mathcal{X}$, there exists a unique $\hat{x} \in M$ such that $\|x - \hat{x}\| = \inf_{y \in M} \|x - y\|$. Moreover, $\hat{x}$ is characterized by $(x - \hat{x}) \perp M$.



2. Normal equation:
In a finite dimensional inner product space $(\mathcal{X}, \mathbb{R}, <\cdot>)$, $M = \text{span}\left\{y^1, y^2, \ldots, y^k\right\}$ is a subspace of $\mathcal{X}$ where $\{y^1, y^2, \ldots, y^k\}$ is a linear independent set. We need to find $\hat{x} = \arg\min_{m \in M} \|x - m\|$.
Suppose $\hat{x}$ takes the form
$$\hat{x} = \alpha_1 y^1 + \alpha_2 y^2 + \cdots + \alpha_k y^k,$$
then the coordinates of $\hat{x}$ expressed in the basis $\{y^1, y^2, \ldots, y^k\}$, i.e., $\alpha = \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_k\end{bmatrix}$ is given by the normal equation
$$\begin{bmatrix}< y^1, y^1 > & \cdots & < y^k, y^1 > \\ \vdots & \ddots & \vdots \\ < y^1, y^k > & \cdots & < y^k, y^k >\end{bmatrix} \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_k\end{bmatrix} = \begin{bmatrix}< x, y^1 > \\ \vdots \\ < x, y^k >\end{bmatrix}$$

3. A function $P : \mathcal{X} \to M$ is an orthogonal projection operator if $P(x) = \arg\min_{m \in M} \|x - m\|$.

4. Ex:

   - In $(\mathbb{R}^{2\times 2}, \mathbb{R})$, the inner product is defined as $< A, B >= \text{trace}(A^\top Q B)$, where $Q = \begin{bmatrix}2 & 1 \\ 1 & 1\end{bmatrix}$, and $M = \text{span}\left\{\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\right\}$. Given $x = \begin{bmatrix}1 & 2 \\ 3 & 4\end{bmatrix}$, find $\hat{x} = \arg\min_{y \in M} \|x - y\|$.

- $\mathcal{X} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$, $\mathcal{F} = \mathbb{R}$. Define inner product $< f,\, g >= \int_{-1}^{1} f(t)g(t)\mathrm{d}t$. $M = \operatorname{span}\left\{1,\, t,\, t^2\right\}$, $x = \mathrm{e}^t$, find $\hat{x} = \arg\min\limits_{y \in M} \|x - y\|$.

- In $(\mathbb{R}^n,\, \mathbb{R})$ with inner product defined in the standard way, i.e., $< x,\, y >= x^\top y$.
$M = \operatorname{span}\left\{v^1,\, v^2,\, \ldots,\, v^k\right\}$, $k \leq n$. Find the matrix representation of the othogonal projector that projects $x \in \mathcal{X}$ onto $\hat{x} \in M$. Use the standard basis $E = \{e^1,\, e^2,\, \ldots,\, e^n\}$ for $(\mathbb{R}^n,\, \mathbb{R})$, and basis $V = \{v^1,\, v^2,\, \ldots,\, v^k\}$ for subspace M. What if we also express the projection $\hat{x}$ in standard basis $E = \{e^1,\, e^2,\, \ldots,\, e^n\}$? What if V is an orthonormal set?