Rob 501 - Mathematics for Robotics
Recitation #09

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1 Singular Value Decomposition

1. **Theorem**: Any matrix $A \in \mathbb{R}^{m\times n}$ can be factored as $A = U\Sigma V^\top$, where

   - $U \in \mathbb{R}^{m\times m}$ is an orthogonal matrix, and its columns are eigenvectors of $AA^\top$
   - $V \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. and its columns are eigenvectors of $A^\top A$
   - $\Sigma \in \mathbb{R}^{m\times n}$ is a rectangular matrix and its diagonal elements σ_i are singular values of A, i.e., σ_i^2 are eigenvalues of $AA^\top$ or $A^\top A$.

   This is called (full) singular value decomposition (SVD) of A. Moreover,

   - When $m \neq n$ and $k = \min\{m, n\}$, SVD can be reduced to thin SVD where $U \in \mathbb{R}^{m\times k}$, $\Sigma \in \mathbb{R}^{k\times k}$, $V \in \mathbb{R}^{n\times k}$.
   - When the number of non-zero singular values is p and $p < \min\{m, n\}$, SVD can be further reduced to compact SVD where $U \in \mathbb{R}^{m\times p}$, $\Sigma \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{n\times p}$.

2. **Remarks**: SVD has the following properties:

   - $\forall i, Av^i = \sigma_i u^i$, where v^i and u^i are the i-th column of V and U, respectively.
     v^i-s and u^i-s are called right singular vectors and left singular vectors of A, respectively.
   - SVD might not be unique. (See Ex.(b))
   - For general square matrix, SVD and eigen-decomposition have no relationship.(See Ex.(d))
     For symmetric positive definite matrix, SVD and eigen-decomposition are the same. (See Ex.(e))

3. **SVD and Rank:**
   Let $\{\sigma_1, \sigma_2, \cdots, \sigma_p\}$ be the singular values of A. $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_p \geq 0$, where p =min(m, n).

   $$A = U\Sigma V^\top = \sum_{i=1}^{p} \sigma_i u_i v_i^\top$$

   What is the rank of $u_i v_i^\top$?

   $$A - \sigma_p u_p v_p^\top = \sum_{i=1}^{p-1} \sigma_i u_i v_i^\top$$

   σ_p is the distance of A from the nearest singular matrix.

Fact: Suppose that rank $(A) = r$, so that σ_r is the smallest non-zero singular value. Then

(i) if an $n \times m$ matrix E satisfies $\|E\| \le \sigma_r$, then rank $(A+E) = r$

(ii) $\exists\, E$ with $\|E\| = \sigma_r$, s.t. rank $(A+E) < r$

4. Ex:

(a) $A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(a)

$$A^\top A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \quad AA^\top = \begin{bmatrix} 5 & -4 & 0 \\ -4 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det(\lambda I - A^\top A) = (\lambda - 5)^2 - 16 \quad \Rightarrow \lambda_1 = 9, \lambda_2 = 1, v^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1} = 3,\ \sigma_2 = \sqrt{\lambda_2} = 1,\ u^1 = \frac{1}{\sigma_1} A v^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, u^2 = \frac{1}{\sigma_2} A v^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$u^3 \perp \{u^1, u^2\}, \text{ set } u^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^\top}_{\text{full SVD}} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^\top}_{\text{thin SVD}}$$

(b)

$$A^\top A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad AA^\top = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det(\lambda I - A^\top A) = (\lambda - 2)^2 \quad \Rightarrow \lambda_1 = \lambda_2 = 2, \sigma_1 = \sigma_2 = \sqrt{2}.$$

There are infinite choice of v^1 and v^2.

- Option 1:

$$\text{choose } v^1 = \begin{bmatrix}1\\0\end{bmatrix}, v^2 = \begin{bmatrix}0\\1\end{bmatrix}, u^1 = \frac{1}{\sigma_1}Av^1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, u^2 = \frac{1}{\sigma_2}Av^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, u^3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}^\top}_{\text{full SVD}} = \underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}^\top}_{\text{thin SVD}}$$

- Option 2:

$$\text{choose } v^1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, v^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, u^1 = \frac{1}{\sigma_1}Av^1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, u^2 = \frac{1}{\sigma_2}Av^2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, u^3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^\top}_{\text{full SVD}} = \underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^\top}_{\text{thin SVD}}$$

(c)

$$A^\top A = \begin{bmatrix}0 & 1 & 0\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & -1\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}, \quad AA^\top = \begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\det(\lambda I - A^\top A) = \lambda(\lambda - 2) \quad \Rightarrow \lambda_1 = 2, \lambda_2 = 0, v^1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, v^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{2},\ \sigma_2 = \sqrt{\lambda_2} = 0,\ u^1 = \frac{1}{\sigma_1}Av^1 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\{u^2, u^3\} \perp u^1,\ \text{set } u^2 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ u^3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^\top}_{\text{full SVD}} = \underbrace{\begin{bmatrix}0 & 1\\ 1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^\top}_{\text{thin SVD}} = \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}\sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}^\top}_{\text{compact SVD}}$$

(d)

$$A^\top A = \begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix},$$

$$\det(\lambda I - A^\top A) = (\lambda - 5)^2 - 16 \quad \Rightarrow \lambda_1(A^\top A) = 9, \lambda_2(A^\top A) = 1,$$

$$\det(\lambda I - A) = \lambda^2 - 3 \quad \Rightarrow \lambda_1(A) = \sqrt{3}, \lambda_2(A) = -\sqrt{3}.$$

(e) • SVD

$$A^\top A = \begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix},$$

$$\det(\lambda I - A^\top A) = (\lambda-5)^2 - 16 \quad \Rightarrow \lambda_1 = 9, \lambda_2 = 1, v^1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, v^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1} = 3,\ \sigma_2 = \sqrt{\lambda_2} = 1,\ u^1 = \frac{1}{\sigma_1}Av^1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, u^2 = \frac{1}{\sigma_2}Av^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$A = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^\top$$

- Eigen-decomposition

$$\det(\lambda I - A) = (\lambda-2)^2 - 1 \quad \Rightarrow \lambda_1 = 3, \lambda_2 = 1, v^1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, v^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$A = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^\top$$

5. **Summary**: How to find SVD of $A \in \mathbb{R}^{m\times n}$?

   (a) Calculate $A^\top A$, find its eigenvalues λ_i-s and corresponding eigenvectors v^i-s.

   (b) Singular values of A are ordered set $\{\sigma_i = \sqrt{\lambda_i}\,|\, i = 1,\dots,n\}$.
   The right singular vectors of A are the eigenvectors of $A^\top A$, i.e., v^i-s.

   (c) Suppose there are p non-zero singular values, i.e., for $1 \le i \le p$, $\sigma_i > 0$, and for $p < j \le n$, $\sigma_j = 0$.
   For nonzero singular values σ_i-s, find the left singular vectors u^i based on $Av^i = \sigma_i u^i$.
   Corollary: $\operatorname{span}\{u^1,\dots,u^p\} = \operatorname{span}\{A^1,\dots,A^n\}$.

   (d) If $p < m$, complete $\{u^1,\dots,u^p\}$ to an orthonormal basis for $\mathbb{R}^m$ in the following way.
   $\operatorname{span}\{u^{p+1},\dots,u^m\} \perp \operatorname{span}\{u^1,\dots,u^p\}$ and $\operatorname{span}\{u^1,\dots,u^p\} = \operatorname{span}\{A^1,\dots,A^n\}$ implies that $\forall p < j \le m, u^j \perp \operatorname{span}\{A^1,\dots,A^n\}$, i.e.,

$$\begin{cases} (A^1)^\top u^j = 0 \\ \quad\vdots \\ (A^n)^\top u^j = 0 \end{cases} \implies \begin{bmatrix}(A^1)^\top\\ \vdots \\ (A^n)^\top\end{bmatrix} u^j = 0 \quad \implies A^\top u^j = 0$$

   Find all the u^j-s satisfying the equation above, and apply Gram-Schmidt process to get an orthonormal set $\{u^{p+1},\dots,u^m\}$.

   (e) For $m > n$, the resulting full SVD is

$$A = \underbrace{\left[\begin{array}{ccc|ccc} u^1 & \cdots & u^p & u^{p+1} & \cdots & u^m \end{array}\right]}_{\mathbb{R}^{m\times m}} \underbrace{\left[\begin{array}{ccccc} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_p & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & 0 \\ \hline 0 & & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & & 0 \end{array}\right]}_{\mathbb{R}^{m\times n}} \underbrace{\left[\begin{array}{ccc|ccc} v^1 & \cdots & v^p & v^{p+1} & \cdots & v^n \end{array}\right]^\top}_{\mathbb{R}^{n\times n}},$$

the resulting thin SVD is

$$A = \underbrace{\begin{bmatrix} u^1 & \cdots & u^n \end{bmatrix}}_{\mathbb{R}^{m\times n}} \underbrace{\begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_p & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}}_{\mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} v^1 & \cdots & v^p \mid v^{p+1} & \cdots & v^n \end{bmatrix}}_{\mathbb{R}^{n\times n}}^{\top},$$

and the resulting compact SVD is

$$A = \underbrace{\begin{bmatrix} u^1 & \cdots & u^p \end{bmatrix}}_{\mathbb{R}^{m\times p}} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_p \end{bmatrix}}_{\mathbb{R}^{p\times p}} \underbrace{\begin{bmatrix} v^1 & \cdots & v^p \end{bmatrix}}_{\mathbb{R}^{p\times n}}^{\top}.$$

For $m < n$, one can obtain full SVD, thin SVD, and compact SVD in a similar way.

2 QR Factorization

(a) **Theorem**: Given $A \in \mathbb{R}^{m\times n}$, $m \geq n, \text{rank}\{A\} = n$. Then there exists a matrix Q with orthonormal columns and an upper triangular matrix R such that $A = QR$. Moreover,

- If $Q \in \mathbb{R}^{m\times n}$, $R \in \mathbb{R}^{n\times n}$, $A = QR$ is called reduced QR decomposition.
- If $Q \in \mathbb{R}^{m\times m}$, $R \in \mathbb{R}^{m\times n}$, $A = QR$ is called full QR decomposition.

(b) How to compute Q and R ?

(c) Ex:

i. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

ii. $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$

i.

$$v^1 = A^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; q^1 = \frac{v^1}{\|v^1\|_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v^2 = A^2 - < A^2, q^1 > q^1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \frac{5}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \; q^2 = \frac{v^2}{\|v^2\|_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$r_{11} =< A^1, q^1 >= \sqrt{2}, \; r_{12} =< A^2, q^1 >= \frac{5}{\sqrt{2}}, \; r_{22} =< A^2, q^2 >= \frac{1}{\sqrt{2}},$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{5}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

ii.

$$v^1 = A^1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; q^1 = \frac{v^1}{\|v^1\|_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$v^2 = A^2 - < A^2, q^1 > q^1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, \; q^2 = \frac{v^2}{\|v^2\|_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$r_{11} = < A^1, q^1 > = \sqrt{2}, \; r_{12} = < A^2, q^1 > = \frac{1}{\sqrt{2}}, \; r_{22} = < A^2, q^2 > = \frac{3}{\sqrt{6}},$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$$

Find q^3 such that $\{q^1, q^2, q^3\}$ forms an orthonormal basis for $\mathbb{R}^3$, i.e., $q^3 \perp \operatorname{span}\{q^1, q^2\}$. Notice that $\operatorname{span}\{q^1, q^2\} = \operatorname{span}\{A^1, A^2\}$, thus $q^3 \perp \operatorname{span}\{A^1, A^2\}$,

$$\begin{cases} (A^1)^\top q^3 = 0 \\ (A^2)^\top q^3 = 0 \end{cases} \implies \begin{bmatrix} (A^1)^\top \\ (A^2)^\top \end{bmatrix} q^3 = 0 \implies A^\top q^3 = 0 \implies q^3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Thus, full QR factorization of A is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \\ 0 & 0 \end{bmatrix}$$