Rob 501 - Mathematics for Robotics
Recitation #3

Abhishek Venkataraman (Courtesy: Wubing Qin)

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1 Linear independence

1. Linear combination:
   Let $(\mathcal{X}, \mathcal{F})$ be a vector space. $v^1, v^2, \dots, v^k$ are vectors in $\mathcal{X}$, $\alpha_1, \alpha_2, \dots, \alpha_k$ are scalars in $\mathcal{F}$, *k is finite*. Then $\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k$ is a linear combination.

2. Linear independence:
   Let $(\mathcal{X}, \mathcal{F})$ be a vector space. $v^1, v^2, \dots, v^k$ are vectors in $\mathcal{X}$, $\alpha_1, \alpha_2, \dots, \alpha_k$ are scalars in $\mathcal{F}$, *k is finite*. These vectors are linearly independent if:
   $$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = 0 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

3. Span:
   Let $\mathcal{S}$ be a subset of a vector space $(\mathcal{X}, \mathcal{F})$. The span of $\mathcal{S}$ is the set of all linear combinations of elements in $\mathcal{S}$.

4. Basis:
   A basis for a vector space $(\mathcal{X}, \mathcal{F})$ is a set of linearly independent vectors whose span is the whole vector space. Note: A basis is not unique.

5. Dimension:
   Dimension is the largest number of linearly independent vectors (i.e., the number of vectors that form a basis).

   Ex:

   (a) In $(\mathbb{R}^{2\times 2}, \mathbb{R})$,

   (i) $v^1 = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $v^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $v^3 = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$, are they linearly independent?

   Take $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$,

   $$\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 = 0 \Longleftrightarrow \begin{bmatrix} 2\alpha_1 + \alpha_2 + 3\alpha_3 & \alpha_1 + 2\alpha_2 - \alpha_3 \\ -\alpha_1 - 2\alpha_2 + \alpha_3 & 3\alpha_1 + \alpha_2 + 2\alpha_3 \end{bmatrix} = 0 \Longleftrightarrow \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{x} = 0$$

   $\det(A) \neq 0 \implies x = 0 \implies$ Linearly independent.

(ii)$u^1 = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $u^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $u^3 = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$, are they linearly independent?
Take $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbb{R}$,

$$\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 = 0 \iff \begin{bmatrix} 2\alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + 2\alpha_2 - \alpha_3 \\ -\alpha_1 - 2\alpha_2 + \alpha_3 & 3\alpha_1 + \alpha_2 + 2\alpha_3 \end{bmatrix} = 0 \iff \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

One choice is $\alpha_1 = -1,\ \alpha_2 = 1,\ \alpha_3 = 1$. So they are linearly dependent.

(iii) What is the dimension? Can the sets in (i) or (ii) be bases?
From (i), we know dim≥ 3 since we already have 3 independent vectors. If we add another non-zero vector $v^4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, you can check that $v^1,\ v^2,\ v^3,\ v^4$ are still linear independent. If you try to add one more non-zero vectors to this set, they are always linear dependent.
Thus, dim $= 4$. Neither of them can be chosen as a basis.

(b) In $(\mathbb{C}^2,\ \mathbb{R})$,

(i) $v^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v^2 = \begin{bmatrix} j \\ 0 \end{bmatrix}$, $v^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v^4 = \begin{bmatrix} 0 \\ j \end{bmatrix}$, are they linearly independent?
Take $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \in \mathbb{R}$,

$$\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 + \alpha_4 v^4 = 0 \iff \begin{bmatrix} \alpha_1 + \alpha_2 j \\ \alpha_3 + \alpha_4 j \end{bmatrix} = 0 \quad \overset{\alpha_i \in \mathbb{R}}{\iff} \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

Thus, they are linearly independent.

(ii) $u^1 = \begin{bmatrix} 1+j \\ 0 \end{bmatrix}$, $u^2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $u^3 = \begin{bmatrix} 1 \\ 3+j \end{bmatrix}$, $u^4 = \begin{bmatrix} -1 \\ 3-j \end{bmatrix}$, are they linearly independent?

$$\alpha_1 u^1 + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4 = 0 \iff \begin{bmatrix} \alpha_1(1+j) + 2\alpha_2 + \alpha_3 - \alpha_4 \\ \alpha_3(3+j) + \alpha_4(3-j) \end{bmatrix} = 0 \overset{\alpha_i \in \mathbb{R}}{\iff} \begin{bmatrix} \alpha_1 + 2\alpha_2 + \alpha_3 - \alpha_4 \\ \alpha_1 \\ 3\alpha_3 + 3\alpha_4 \\ \alpha_3 - \alpha_4 \end{bmatrix} = 0,$$

implying $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Thus, they are linearly independent.

(iii) What is the dimension? Can the sets in (i) or (ii) be bases?
dim $=4$. Both of them can be chosen as a basis for $(\mathbb{C}^2,\ \mathbb{R})$.

(c) In $(\mathbb{C}^2,\ \mathbb{C})$,

(i) $v^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v^2 = \begin{bmatrix} j \\ 0 \end{bmatrix}$, $v^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v^4 = \begin{bmatrix} 0 \\ j \end{bmatrix}$, are they linearly independent?
Take $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \in \mathbb{R}$,

$$\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 + \alpha_4 v^4 = 0 \iff \begin{bmatrix} \alpha_1 + \alpha_2 j \\ \alpha_3 + \alpha_4 j \end{bmatrix} = 0 \quad \overset{\alpha_i \in \mathbb{C}}{\Longrightarrow} \quad \text{One option: } \alpha_1 = \alpha_3 = 1,\ \alpha_2 = \alpha_4 = j$$

Thus, they are linearly dependent.

(ii) $u^1 = \begin{bmatrix} 1+j \\ 0 \end{bmatrix}$, $u^2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $u^3 = \begin{bmatrix} 1 \\ 3+j \end{bmatrix}$, $u^4 = \begin{bmatrix} -1 \\ 3-j \end{bmatrix}$, are they linearly independent?
No, they are linearly dependent.

(iii) What is the dimension? Can the sets in (i) or (ii) be bases?
dim $=2$. Neither of them can be chosen as a basis.

(d) $\mathcal{X} = \{p(x) \,|\, \text{polynomials in } x \text{ of order } n,\, n \leq 3\}$, $\mathcal{F} = \mathbb{R}$,

(i) $v^1 = 1$, $v^2 = x$, $v^3 = x^2$, $v^4 = x^3$, are they linearly independent?
Yes, they are linearly independent.
Note: Taylor series is one way to project an infinitely differentiable function (function space is infinite dimensional) onto the subspace of polynomials with the basis $\{1, x, x^2, \ldots, x^k, \ldots\}$.

(ii) $u^1 = 1$, $u^2 = x$, $u^3 = \dfrac{1}{2}(3x^2 - 1)$, $u^4 = \dfrac{1}{2}(5x^3 - 3x)$, are they linearly independent?
Yes, they are linearly independent.
Note: The set above is the selected from Legendre polynomials, which is also widely used in engineering.

(iii) What is the dimension? Can the sets in (i) or (ii) be bases?
dim $=4$. Both of them can be chosen as a basis for $(\mathbb{C}^2, \mathbb{R})$.

2 Representation of vectors and Change of basis

1. Representation of vectors:
Given an n-dimensional vector space $(\mathcal{X}, \mathcal{F})$ with basis $V = \{v^1, v^2, \ldots, v^k\}$, any vector x can be written as

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n \quad \leftrightarrow \quad [x]_V = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

This is called the <u>representation of vector x</u> in the given basis V. The vector of coefficients $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathcal{F}^n$ is called the <u>coordinates of vector x</u> expressed in V, denoted as $[x]_V$.

2. Change of basis:
Let $V = \{v^1, v^2, \ldots, v^n\}$ and $U = \{u^1, u^2, \ldots, u^n\}$ be two bases for n-dimensional vector space $(\mathcal{X}, \mathcal{F})$, then there exists and $n \times n$ invertible matrix P such that $[x]_V = P[x]_U$ where the i-th column of P is the coordinates of vector u_i expressed in the basis V.

Ex:

(a) $\mathcal{X} = \mathbb{C}^2$, $\mathcal{F} = \mathbb{R}$, represent $x = \begin{bmatrix} 1+j \\ 1 \end{bmatrix}$ in the following basis.

$U = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} j \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ j \end{bmatrix} \right\}$, $V = \left\{ \begin{bmatrix} 1+j \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3+j \end{bmatrix}, \begin{bmatrix} -1 \\ 3-j \end{bmatrix} \right\}$, find $[x]_U$, $[x]_V$ and the matrix P.

Suppose $[x]_U = P\,[x]_V$, the i-th column of P is $P^i = [v^i]_U$.

$$[x]_U = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, [v^1]_U = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [v^2]_U = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [v^3]_U = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, [v^4]_U = \begin{bmatrix} -1 \\ 0 \\ 3 \\ -1 \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & -1/2 & 0 & -1/2 \\ 0 & 0 & 1/6 & 1/2 \\ 0 & 0 & 1/6 & -1/2 \end{bmatrix}, [x]_V = P^{-1}\,[x]_U = \begin{bmatrix} 1 \\ 0 \\ 1/6 \\ 1/6 \end{bmatrix}$$

(b) $\mathcal{X} = \{q(x)\,|\,\text{polynomials in } x \text{ of order } n,\, n \leq 3\}$, $\mathcal{F} = \mathbb{R}$, $q(x) = 2 + 3x - x^2$,
$U = \{1, x, x^2, x^3\}$, $V = \left\{1, x, \frac{1}{2}(3x^2 - 1), \frac{1}{2}(5x^3 - 3x)\right\}$. find $[q]_U$, $[q]_V$ and the matrix P.

Suppose $[q]_U = P\,[q]_V$, the i-th column of P is $P^i = [v^i]_U$.

$$[q]_U = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \end{bmatrix}, [v^1]_U = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [v^2]_U = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [v^3]_U = \begin{bmatrix} -1/2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}, [v^4]_U = \begin{bmatrix} 0 \\ -3/2 \\ 0 \\ 5/2 \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 5/2 \end{bmatrix},\ P^{-1} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix},\ [x]_V = P^{-1}\,[x]_U = \begin{bmatrix} 5/3 \\ 3 \\ -2/3 \\ 0 \end{bmatrix}$$