Rob 501 - Mathematics for Robotics
Recitation #4

Abhishek Venkataraman (Courtesy: Wubing Qin)

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1 Linear Operator and Matrix Representation

Let $\mathcal{X}$ and $\mathcal{Y}$ be two vector space over the same field $\mathcal{F}$. A mapping $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ is a linear operator if $\forall \alpha_1, \alpha_2 \in \mathcal{F}$ and $\forall x_1, x_2 \in \mathcal{X}$: $\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{L}(x_1) + \alpha_2 \mathcal{L}(x_2)$.

Note: A linear map must satisfy $\mathcal{L}(0) = 0$.

A matrix representation of a linear operator $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$, with a basis $U = \{u^1, u^2, \dots, u^m\}$ for the space $(\mathcal{X}, \mathcal{F})$ and a basis $V = \{v^1, v^2, \dots, v^n\}$ for the space $(\mathcal{Y}, \mathcal{F})$, is an $n \times m$ matrix A such that

$$[\mathcal{L}(x)]_V = A\,[x]_U,$$

where $[x]_U$ is the coordinates of x expressed in the basis U, $[\mathcal{L}(x)]_V$ is the coordinates of $\mathcal{L}(x)$ expressed in the basis V, and the i-th column of A is $[\mathcal{L}(u^i)]_V$.

Ex: Let $(\mathbb{P}^n, \mathbb{R})$ be the space of polynomials of degree less than or equal to n over the field $\mathbb{R}$.
Define the map $\mathcal{L} : \mathbb{P}^2 \to \mathbb{P}^3$ as:

$$(\mathcal{L}(p))(x) = \int_0^x p(s)\mathrm{d}s$$

1. Is $\mathcal{L}$ a linear operator?

   $\forall \alpha_1, \alpha_2 \in \mathcal{F}, \forall p_1(x), p_2(x) \in \mathbb{P}^2$:

   $$\big(\mathcal{L}(\alpha_1 p_1 + \alpha_2 p_2)\big)(x) = \int_0^x (\alpha_1 p_1 + \alpha_2 p_2)(s)\mathrm{d}s = \int_0^x \big(\alpha_1 p_1(s) + \alpha_2 p_2(s)\big)\mathrm{d}s = \alpha_1 \int_0^x p_1(s)\mathrm{d}s + \alpha_2 \int_0^x p_2(s)\mathrm{d}s$$
   $$= \alpha_1 \big(\mathcal{L}(p_1)\big)(x) + \alpha_2 \big(\mathcal{L}(p_2)\big)(x) = \big(\alpha_1 \mathcal{L}(p_1) + \alpha_2 \mathcal{L}(p_2)\big)(x)$$

   So it is a linear operator.

2. Find the matrix representation A_1 of $\mathcal{L}$ with respect to the bases $U_1 = \{1, x, x^2\}$ for $\mathbb{P}^2$ and $V_1 = \{1, x, x^2, x^3\}$ for $\mathbb{P}^3$.

   Suppose $[\mathcal{L}(p)]_{V_1} = A_1\,[p]_{U_1}$,

$$\begin{aligned}
\big(\mathcal{L}(u^1)\big)(x) &= \int_0^x 1 \cdot \mathrm{d}s = x \\
\big(\mathcal{L}(u^2)\big)(x) &= \int_0^x s \,\mathrm{d}s = \frac{1}{2}x^2 \\
\big(\mathcal{L}(u^3)\big)(x) &= \int_0^x s^2 \,\mathrm{d}s = \frac{1}{3}x^3
\end{aligned} \quad \Longrightarrow \quad \big[\mathcal{L}(u^1)\big]_{V_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \big[\mathcal{L}(u^2)\big]_{V_1} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix},\ \big[\mathcal{L}(u^3)\big]_{V_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}.$$

Thus,

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

3. Change the basis in $\mathbb{P}^2$ to $U_2 = \left\{1,\, x,\, \frac{1}{2}(3x^2-1)\right\}$. Find the new matrix representation A_2.

   Suppose $[p]_{U_1} = P\,[p]_{U_2}$. If we can find P, then $[\mathcal{L}(p)]_{V_1} = A_1\,[p]_{U_1} = A_1\,P\,[p]_{U_2}$, i.e., $A_2 = A_1\,P$. From change of basis, we know that the i-th column of P is the coordinates of u^i in U_2 expressed in the basis U_1, so

$$P = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{bmatrix},\ A_2 = A_1\,P = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

4. Keep the basis U_1 in $\mathbb{P}^2$, but change the basis in $\mathbb{P}^3$ to $V_2 = \left\{1,\, x,\, \frac{1}{2}(3x^2-1),\, \frac{1}{2}(5x^3-3x)\right\}$. Find the new matrix representation A_3.

   Suppose $\left[\mathcal{L}(p)\right]_{V_1} = Q\left[\mathcal{L}(p)\right]_{V_2}$. If we can find Q, then

$$[\mathcal{L}(p)]_{V_1} = A_1\,[p]_{U_1} \quad \Rightarrow \quad Q\left[\mathcal{L}(p)\right]_{V_2} = A_1\,[p]_{U_1} \quad \Rightarrow \quad \left[\mathcal{L}(p)\right]_{V_2} = Q^{-1}\,A_1\,[p]_{U_1}$$

   i.e., $A_3 = Q^{-1}\,A_1$.
   From change of basis, we know that the i-th column of Q is the coordinates of v^i in V_2 expressed in the basis V_1, so

$$Q = \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 5/2 \end{bmatrix},\ Q^{-1} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix},$$

$$A_3 = Q^{-1}\,A_1 = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 1/6 & 0 \\ 1 & 0 & 1/5 \\ 0 & 1/3 & 0 \\ 0 & 0 & 2/15 \end{bmatrix}.$$

2 Similarity Transform

1. Let $A, B \in \mathbb{C}^{n\times n}$. A and B are <u>similar</u> if there exists an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that $B = PAP^{-1}$.

2. If A is similar to a diagonal matrix, A is said to be <u>diagonalizable</u>.

   Ex: Check whether the following matrices are diagonalizable. If they are, calculate A^{100}.
   (1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (2) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

   (1) $\det(\lambda I - A) = \det \begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 \;\Rightarrow\; \lambda_1 = \lambda_2 = 1.$

   $(\lambda_1 I - A)v^1 = 0 \;\Rightarrow\; \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} v^1 = 0 \;\Rightarrow\; v^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   We cannot find another eigenvector v^2, so it is not diagonalizable.

(2) $\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -1 \\ -4 & \lambda - 1 \end{bmatrix} = (\lambda - 3)(\lambda + 1) \;\Rightarrow\; \lambda_1 = 3,\ \lambda_2 = -1.$

$(\lambda_1 I - A)v^1 = 0 \;\Rightarrow\; \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} v^1 = 0 \;\Rightarrow\; v^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$(\lambda_2 I - A)v^2 = 0 \;\Rightarrow\; \begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix} v^2 = 0 \;\Rightarrow\; v^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Thus, we have

$$A \underbrace{\begin{bmatrix} v^1 & v^2 \end{bmatrix}}_{V} = \begin{bmatrix} Av^1 & Av^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 v^1 & \lambda_2 v^2 \end{bmatrix} = \underbrace{\begin{bmatrix} v^1 & v^2 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda} \;\Rightarrow\; A = V\Lambda V^{-1}$$

$$A^{100} = (V\Lambda V^{-1})^{100} = (V\Lambda V^{-1})(V\Lambda V^{-1})(V\Lambda V^{-1})\cdots(V\Lambda V^{-1}) = V\Lambda^{100}V^{-1}$$

Application:

$$\begin{aligned} \mathrm{e}^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots \\ &= VV^{-1} + V\Lambda V^{-1} + \frac{1}{2!}V\Lambda^2V^{-1} + \frac{1}{3!}V\Lambda^3V^{-1} + \frac{1}{4!}V\Lambda^4V^{-1} + \cdots \\ &= V\left(I + \Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \frac{1}{4!}\Lambda^4 + \cdots\right)V^{-1} = V\mathrm{e}^{\Lambda}V^{-1} \end{aligned}$$

$$\Rightarrow \mathrm{e}^A = \mathrm{e}^{V\Lambda V^{-1}} = V\mathrm{e}^{\Lambda}V^{-1}$$