Rob 501 - Mathematics for Robotics
Recitation #09

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1 Singular Value Decomposition

1. **Theorem**: Any matrix $A \in \mathbb{R}^{m\times n}$ can be factored as $A = U\Sigma V^\top$, where
   - $U \in \mathbb{R}^{m\times m}$ is an orthogonal matrix, and its columns are eigenvectors of $AA^\top$
   - $V \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. and its columns are eigenvectors of $A^\top A$
   - $\Sigma \in \mathbb{R}^{m\times n}$ is a rectangular matrix and its diagonal elements σ_i are singular values of A, i.e., σ_i^2 are eigenvalues of $AA^\top$ or $A^\top A$.

   This is called (full) singular value decomposition (SVD) of A. Moreover,
   - When $m \neq n$ and $k = \min\{m, n\}$, SVD can be reduced to thin SVD where $U \in \mathbb{R}^{m\times k}$, $\Sigma \in \mathbb{R}^{k\times k}$, $V \in \mathbb{R}^{n\times k}$.
   - When the number of non-zero singular values is p and $p < \min\{m, n\}$, SVD can be further reduced to compact SVD where $U \in \mathbb{R}^{m\times p}$, $\Sigma \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{n\times p}$.
2. **Remarks**: SVD has the following properties:
   - $\forall i, Av^i = \sigma_i u^i$, where v^i and u^i are the i-th column of V and U, respectively.
     v^i-s and u^i-s are called right singular vectors and left singular vectors of A, respectively.
   - SVD might not be unique. (See Ex.(b))
   - For general square matrix, SVD and eigen-decomposition are not necessarily the same.(See Ex.(d))
     For symmetric positive definite matrix, SVD and eigen-decomposition are the same. (See Ex.(e))
3. **SVD and Rank:**
   Let $\{\sigma_1, \sigma_2, \cdots, \sigma_p\}$ be the singular values of A. $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_p \geq 0$, where p =min(m, n).

$$A = U\Sigma V^\top = \sum_{i=1}^{p} \sigma_i u_i v_i^\top$$

   What is the rank of $u_i v_i^\top$?

$$A - \sigma_p u_p v_p^\top = \sum_{i=1}^{p-1} \sigma_i u_i v_i^\top$$

   σ_p is the distance of A from the nearest singular matrix.

Fact: Suppose that rank $(A) = r$, so that σ_r is the smallest non-zero singular value. Then

(i) if an $n \times m$ matrix E satisfies $\|E\| \leq \sigma_r$, then rank $(A+E) = r$

(ii) $\exists$ E with $\|E\| = \sigma_r$, s.t. rank $(A+E) < r$

4. Ex:

(a) $A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

2 QR Factorization

(a) **Theorem**: Given $A \in \mathbb{R}^{m \times n}$, $m \geq n, \text{rank}\{A\} = n$. Then there exists a matrix Q with orthonormal columns and an upper triangular matrix R such that $A = QR$. Moreover,

- If $Q \in \mathbb{R}^{m \times n}$, $R \in \mathbb{R}^{n \times n}$, $A = QR$ is called reduced QR decomposition.
- If $Q \in \mathbb{R}^{m \times m}$, $R \in \mathbb{R}^{m \times n}$, $A = QR$ is called full QR decomposition.

(b) How to compute Q and R ?

(c) Ex:

i. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

ii. $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$