Exam Number:________

Date is Monday, December 17, 2018, 1:30 PM–3:20 PM

Rooms (First letter of last name)
DOW 1013 (A-P)   DOW 1014 (R-Z)

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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DO NOT OPEN THE EXAM BOOKLET UNTIL TOLD TO DO SO.

RULES:

1. CLOSED TEXTBOOK
2. CLOSED CLASS NOTES
3. CLOSED HOMEWORK
4. CLOSED HANDOUTS
5. 3 SHEETS OF NOTE PAPER (Front and Back), US Letter Size.
6. NO CALCULATORS, CELL PHONES, HEADSETS, nor DIGITAL DEVICES of any KIND.

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

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Anything written here will not be graded.
(You can use it for scratch paper.)

Problems 1 - 5 (30 points: 5 $\times$ 6)

Instructions. For each problem, there is at least one answer that is true (i.e., **T**) and one answer that is false (i.e., **F**). *You will receive no credit for your responses if you either mark all of the answers as true or all as false*, **because then we assume you are guessing**. Otherwise, each part of a question is worth 1.5 points.

1. Let $A = U\Sigma V^\top$ be the SVD of a real square matrix A, with $\Sigma = \text{diag}(20, \sigma_2, 14, \sigma_4)$. For $1 \leq i \leq 4$, let U_i and V_i denote the columns of U and V respectively. **Circle True or False as appropriate for the following statements:**

T **F** (a) The columns of U are e-vectors of $A^\top A$.

T **F** (b) The SVD is very useful for checking the numerical rank of a matrix.

T **F** (c) The rank of the matrix $B := U_1V_1^\top + U_2V_2^\top + U_3V_3^\top$ equals two. (This part of the question is **not using** the numerical rank.)

T **F** (d) $\lambda_{\max}(A^\top A) = 20$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue.

2. This problem looks at various minimum norm and minimum variance estimation problems associated with the equation

$$y = Cx + e,$$

where C is an $m \times n$ matrix and both m and n are greater than or equal to one. From the size of the matrix C, you easily deduce the dimensions of y, x and e. **Circle True or False as appropriate for the following statements:**

T **F** (a) Suppose that $e = 0$, x is deterministic, the rows of C are linearly independent, and the norm on $\mathbb{R}^n$ is $||x|| = \sqrt{x^\top Sx}$, with $S > 0$. Then $\widehat{x} = (C^\top SC)^{-1}SC^\top y$ satisfies

$$\widehat{x} = \arg\min_{y=Cx} ||x||^2.$$

T **F** (b) Assume x and e are deterministic, the columns of C are linearly independent, and the norm on $\mathbb{R}^n$ is $||x|| = \sqrt{x^\top Sx}$, with $S > 0$. Then $\widehat{x} = (C^\top SC)^{-1}SC^\top y$ satisfies $||y - C\widehat{x}|| = \inf_{x\in\mathbb{R}^n} ||y - Cx||$.

T **F** (c) Assume x is deterministic, e is a zero-mean random vector with covariance $\mathcal{E}\{ee^\top\} = Q > 0$, and the columns of C are linearly independent. Then $\widehat{x} = \underbrace{(C^\top Q^{-1}C)^{-1}C^\top Q^{-1}}_{K}\, y$ satisfies $KC = I$

T **F** (d) Assume both x and e are zero-mean random vectors with covariances $\mathcal{E}\{xx^\top\} = P > 0$, $\mathcal{E}\{ee^\top\} = Q > 0$, and $\mathcal{E}\{xe^\top\} = 0$. If the columns of C are linearly independent, then the minimum variance estimator (MVE) converges to the best linear unbiased estimator (BLUE) as the uncertainty in x tends to infinity, in other words: MVE $\to$ BLUE as $P \to \infty I$.

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3. Let $(\mathcal{X}, ||\cdot||)$ be a possibly infinite-dimensional (real) normed space. To eliminate trivial cases, you can assume that any subsets in the following questions are non-empty. **Circle True or False as appropriate for the following statements:**

T **F** (a) Suppose that $f : \mathcal{X} \to \mathbb{R}$ is continuous at each point of $\mathcal{X}$ and the subset $C \subset \mathcal{X}$ is compact. Then there exists $x_* \in \mathcal{X}$ such that

$$f(x_*) = \inf_{x \in C} f(x).$$

T **F** (b) If the sequence (x_n) is Cauchy, then $\exists\, x^* \in \mathcal{X}$ such that $x_n \to x^*$.

T **F** (c) Suppose that $S \subset \mathcal{X}$ is a subset and (x_n) is a sequence in S (that is, for all $n \geq 1$, $x_n \in S$). If there exists $x^* \in \mathcal{X}$ such that $x_n \to x^*$, then $x^* \in S$.

T **F** (d) Suppose that $(\mathcal{X}, ||\cdot||)$ is finite-dimensional, $S \subset \mathcal{X}$ is an open and <u>bounded</u> set, and (x_n) is a sequence in S (that is, for all $n \geq 1$, $x_n \in S$). Then (x_n) has a subsequence that is Cauchy, that is, there is a subsequence (x_{n_i}) of (x_n) and (x_{n_i}) is Cauchy.

4. Let $(\mathcal{X}, ||\cdot||)$ be a possibly infinite-dimensional (real) normed space. To eliminate trivial cases, you can assume that any subsets in the following questions are non-empty. **Circle True or False as appropriate for the following statements:**

T **F** (a) $S \subset \mathcal{X}$ is open if, and only if, $\forall x \notin S$, $d(x, S) > 0$.

T **F** (b) $S \subset \mathcal{X}$ is closed if, and only if, $x \notin S \implies d(x, S) > 0$.

T **F** (c) Suppose that $C \subset \mathcal{X}$ is convex. Then C is closed.

T **F** (d) Let $B_1(0) := \{x \in \mathcal{X} \mid ||x|| < 1\}$. Suppose that $f : B_1(0) \to \mathbb{R}$ is convex and there exists $y \in B_1(0)$ and $\delta > 0$ such that $B_\delta(y) \subset B_1(0)$ and $\forall\, z \in B_\delta(y)$, $f(y) \leq f(z)$. Then

$$f(y) = \inf_{x \in B_1(0)} f(x)$$

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5. Below is the model for a system:

$$x_{k+1} = A_k x_k + G_k w_k$$
$$y_k = C_k x_k + v_k$$

The notation used here is similar to the handout:

- The random vectors x_0 , and, $k \geq 0, w_k, v_k$ are all independent Gaussian random vectors
- $\forall\ k \geq 0, \forall\ l \geq 0, x_0, w_k, v_l$ are jointly Gaussian.
- $x \in \mathbb{R}^n, w \in \mathbb{R}^m, y \in \mathbb{R}^p, v \in \mathbb{R}^p$
- w_k is a 0-mean white noise process: $\mathcal{E}\{w_k\} = 0$ and $\operatorname{cov}(w_k, w_l) = R_k \delta_{kl}$, where $R_k > 0$
- v_k is a 0-mean white noise process: $\mathcal{E}\{v_k\} = 0$ and $\operatorname{cov}(v_k, v_l) = Q_k \delta_{kl}$, where $Q_k > 0$

We estimate the state at each time step using a Kalman filter, where:

- $K_k = P_{k|k-1} C_k^\top \left(C_k P_{k|k-1} C_k^\top + Q_k\right)^{-1}$ is the Kalman gain
- $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left(y_k - C_k \hat{x}_{k|k-1}\right)$ is the state estimate given $y_1, \ldots, y_k$.
- $P_{k|k} = P_{k|k-1} - K_k C_k P_{k|k-1}$ is the variance of $\hat{x}_{k|k}$.
- $\hat{x}_{k+1|k} = A_k \hat{x}_{k|k}$ is the state estimate at time $k+1$ given measurements $y_1, \ldots, y_k$.
- $P_{k+1|k} = A_k P_{k|k} A_k^\top + G_k R_k G_k^\top$ is the variance of $\hat{x}_{k+1|k}$

Circle True or False as appropriate for the following statements:

T **F** (a) $P_{k|k} \leq P_{k|k-1}$ (i.e. $P_{k|k-1} - P_{k|k}$ is positive semi-definite).

T **F** (b) $P_{k+1|k} \leq P_{k|k}$.

T **F** (c) Suppose $Q_k = rI$ where $r \in \mathbb{R}$ is a very large positive number. Then $\hat{x}_{k|k} \approx \hat{x}_{k|k-1}$ and $P_{k|k} \approx P_{k|k-1}$.

T **F** (d) Suppose $Q_k = \varepsilon I$ where $\varepsilon \in \mathbb{R}$ is a very small positive number. Then $C_k \hat{x}_{k|k} \approx y_k$.

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. <u>If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.</u> For example, we proved that real symmetric matrices have real e-values. So if you need this fact, simply state it and use it.

6. (20 points) Various things we have learned this term. **Part (c) is on the next page.**

(a) (6 points) Let $(\mathcal{X}, \mathbb{R})$ be a vector space. Recall from lecture that a function $f : \mathcal{X} \to \mathbb{R}$ is convex if it satisfies the following property:

$$\forall\ x, y \in \mathcal{X},\ \forall \lambda \in [0,1] : \quad f\big(\lambda x + (1-\lambda) y\big) \leq \lambda f(x) + (1-\lambda) f(y)$$

What is the condition for a function to be not convex? (i.e. give the negation of this definition).

(No need to show work for this one)

(b) (7 points) Suppose that A has QR factorization $A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Find x satisfying $Ax = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

$x =$ (Must show work below)

Extra space for part (b)

(c) (7 points) Suppose that $A \in \mathbb{R}^{2\times 2}$ has an SVD given by $A = U\Sigma V^\top$, where

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}, V = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

Give a matrix ΔA with $||\Delta A|| = 0.1$ such that $\text{rank}(A - \Delta A) = 1$; here $||\cdot||$ is the induced matrix 2-norm $||M|| = \sqrt{\lambda_{\max}(M^\top M)}$.

$\Delta A =$

(Showing your reasons is optional on this part.)

Note: If you are confident of your answer, you can just write it down. You do not even have to multiply out terms. You can leave your answer as a product of several terms. **If you are not confident**, then provide reasons to help with partial credit.

7. (15 points) Consider three jointly normal random variables X_1, X_2, and X_3 with

- Mean vector $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
- covariance matrix $\Sigma = \begin{bmatrix} 6 & 2 & 4 \\ 2 & 3 & 1 \\ 4 & 1 & 3 \end{bmatrix}$
- We only give the mean and covariance because these are necessary and sufficient to specify normal random variables and vectors. In your answers, you also specify densities by giving means and variances or covariances, as needed.

There are three parts to the question. Part (c) is on the next page.

(a) (6 points) Find the density of the random variable $Y = 2X_1 + 3X_2$.

$\mu_Y =$ and $\Sigma_Y =$

(Show your calculations below)

(b) (6 points) Find the density of the random vector $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ conditioned on $X_3 = 4$.

$\mu_{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Big\vert \{X_3=4\}} =$ and $\Sigma_{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Big\vert \{X_3=4\}} =$

(Show work below)

Extra space for part (b)

(c) (3 points) Find the conditional mean of $Y|\{X_3 = 4\}$, where $Y = 2X_1 + 3X_2$.

$\mu_{Y|\{X_3=4\}} =$	(Show work below)

8. (15 points) There are three (3) short answer questions. **Part (c) is on the next page.**

(a) **(5 Points)** Let $(\mathcal{X}, \mathbb{R}, < \bullet, \bullet, >)$ be an n-dimensional inner product space. Suppose we are given a point $x \in \mathcal{X}$, and a set of $1 \leq k < n$ linearly independent vectors $\left\{y^1, \ldots, y^k\right\}$ with $y^i \in \mathcal{X}$, and let $M = \text{span}\{y^1, \ldots, y^k\}$. Let $\left\{v^1, \ldots, v^k\right\}$ be an ortho**normal** basis for M. Define two points $\widehat{x}$ and x^* by

$$\begin{aligned} \widehat{x} &= \arg\min_{m \in M} < x - m, x - m > \\ x^* &= < x, v^1 > v^1 + \cdots + < x, v^k > v^k. \end{aligned}$$

Then $\hat{x} = x^*$

Circle **T or F**. You are not supposed to give a proof; only give a few short reasons why this is TRUE or FALSE.

Supporting Reasons:

(b) **(5 Points)** Consider the inner product space $(\mathcal{X}, \mathbb{R}, < \bullet, \bullet >)$, where $\mathcal{X} = \text{span}\{1, t, t^2, \sin(\pi t)\}$ viewed as functions from $[-1, 1]$ to $\mathbb{R}$. Define the inner product to be $< f, g >= \int_{-1}^{1} f(t)g(t)dt$. **Find the vector of minimum norm that satisfies:** $< f, t^2 >= 4$ and $< f, \sin(\pi t) >= \pi$.

Hint: You can use the following integrals: $\int_{-1}^{1} \sin(\pi t)dt = 0$, $\int_{-1}^{1} t\sin(\pi t)dt = \frac{2}{\pi}$, $\int_{-1}^{1} t^2 \sin(\pi t)dt = 0$, $\int_{-1}^{1} [\sin(\pi t)]^2 dt = 1$.

Answer: $f =$

Provide supporting calculations

Extra space for part (b)

(c) **(5 Points)** Starting from $x_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, perform one iteration of the Newton-Raphson Algorithm to find a "better" solution to the nonlinear equation $y = h(x)$, where $\underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} \exp(x_1) + x_2 \\ x_1 + (x_2)^2 + 2 \end{bmatrix}}_{h(x_1, x_2)}$.

Hint: $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$$\boxed{\text{Answer: First iterate} = \begin{bmatrix} \\ \end{bmatrix}}$$ **Provide supporting calculations**

9. (5 points) $\mathbf{A^+}$ Problem: Points earned here will go toward deciding who goes from an A to an A^+ at the end of the term.

The space of bounded infinite sequences $(\ell_\infty, ||\cdot||_\infty)$ can be made into an infinite dimensional normed space. An element $x \in \ell_\infty$ has the form

$$x = (x_1, x_2, x_3, \cdots)$$

where for $\forall\ i \geq 1, \quad x_i \in \mathbb{R}$ and where the norm

$$||x||_\infty := \sup_i |x_i| < \infty.$$

[Yes, each element of ℓ_∞ corresponds to a sequence in $\mathbb{R}$.]

Show that in this normed space, the closed unit ball

$$\overline{B}_1(0) = \{x \in \ell^\infty \quad | \quad ||x||_\infty \leq 1\}$$

is not compact. That is, find a sequence $s = (s_n)$ with $s_n \in \overline{B}_1(0)$ which does not have a Cauchy subsequence. [Yes, each element of the sequence (s_n) is itself a sequence in $\mathbb{R}$.]

Hint: Think about ℓ_∞ as if it were " $\mathbb{R}^\infty$ ", that is, an infinite-dimensional version of $\mathbb{R}^n$.

Extra space for question 9.

(Scratch Paper)

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(If you write anything here, be sure to indicate to which problem it applies.)