Rob 501 - Mathematics for Robotics
Recitation #6

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1 Matrices

1.
   - $A \in \mathbb{R}^{n\times n}$ is symmetric if $A^\top = A$.
   - $A \in \mathbb{R}^{n\times n}$ is skew symmetric if $A^\top = -A$.
   - $A \in \mathbb{R}^{n\times n}$ is orthogonal if $A^\top A = AA^\top = I$.
   - $A \in \mathbb{R}^{n\times n}$ is normal if $A^\top A = AA^\top$.
2. Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$.
   - A is positive definite if $\forall\, x \neq 0$, $x^\top Ax > 0$. We denote it as $A \succ 0$.
   - A is positive semi-definite or non-negative definite if $\forall\, x \neq 0$, $x^\top Ax \geq 0$. We denote it as $A \succeq 0$.
   - A is negative definite if $\forall\, x \neq 0$, $x^\top Ax < 0$. We denote it as $A \prec 0$.
   - A is negative semi-definite or non-positive definite if $\forall\, x \neq 0$, $x^\top Ax \leq 0$. We denote it as $A \preceq 0$.
3. Given a real matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, $\Delta_k = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$ is called the leading principal minor of order k.
4. We have proven the following statement in the lecture. Given a symmetric matrix $P \in \mathbb{R}^{n\times n}$.
$$P \succeq 0 \iff (\exists\, N \in \mathbb{R}^{n\times n} : P = NN^\top)$$
5. Given a symmetric matrix $A \in \mathbb{R}^{m\times m}$. The following are equivalent (TFAE):
   - A is positive definite.
   - All the eigenvalues of A are positive, or the minimum eigenvalue of A is positive.
   - All the leading principal minors of A are positive definite.
   - All the leading principal minors of A have positive determinants.
6. Theorem 1: For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $Q^\top AQ = \Lambda$, where $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix. In other words, any real symmetric matrix is orthogonally diagonalizable.
7. Matrix inversion lemma: Suppose the following matrix product are compatible and the inverses exists
$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$
This is also called Sherman-Morrison-Woodbury formula or Woodbury matrix identity.

8. Other useful matrix identities: Given A, P, $Q \in \mathbb{R}^{n\times n}$, then

   - $(I+A)^{-1} = I - (I+A)^{-1}A$,
   - $(I+PQ)^{-1}P = P(I+QP)^{-1}$,

   if the inverses exist.

   - $(I+A)^{-1} = (I+A)^{-1}(I+A-A) = (I+A)^{-1}(I+A) - (I+A)^{-1}A = I - (I+A)^{-1}A$,
   - $P+PQP = P(I+QP)$ and $P+PQP = (I+PQ)P$, thus $P(I+QP) = (I+PQ)P$, pre-multiplying both sides by $(I+PQ)^{-1}$ and then post-multiplying both sides by $(I+QP)^{-1}$ yield the result.

Ex:

(1) Which of the following matrices are orthogonally diagonalizable?

$$\text{a)}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \qquad \text{b)}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix} \qquad \text{c)}\begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}$$

(a) A is symmetric, so it is orthogonally diagonalizable.

(b)

$$A^*A = \begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$$
$$AA^* = \begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix} = \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} = A^*A \quad\Longrightarrow\quad A \text{ is normal.}$$
$$\det(\lambda I - A) = \det\begin{bmatrix}\lambda-2 & -1\\ 1 & \lambda-2\end{bmatrix} = (\lambda-2)^2+1 \quad\Longrightarrow\quad \lambda_{1,2} = 2 \pm i$$
$$\Longrightarrow\quad A \text{ is not orthogonally diagonalizable.}$$

(c)

$$A^*A = \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}5 & 1\\ 1 & 2\end{bmatrix}$$
$$AA^* = \begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}5 & -1\\ -1 & 2\end{bmatrix} \neq A^*A \quad\Longrightarrow\quad A \text{ is not normal.}$$

(2) For those matrices that are normal in (1) find Q and Λ.

(a)

$$\det(\lambda I - A) = \det\begin{bmatrix}\lambda-2 & -1\\ -1 & \lambda-2\end{bmatrix} = \lambda^2 - 4\lambda + 3 \quad\Longrightarrow\quad \lambda_1 = 3,\ \lambda_2 = 1$$
$$\lambda_1 I - A = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \quad\Longrightarrow\quad v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$
$$\lambda_2 I - A = \begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix} \quad\Longrightarrow\quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$$
$$\therefore \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

(3) Prove matrix inversion lemma.

$$
\begin{aligned}
(A+BCD)^{-1} &= \left((I+BCDA^{-1})A\right)^{-1} = A^{-1}(I+BCDA^{-1})^{-1} \\
&= A^{-1}\Big(I-(I+\underbrace{BC}_{P}\underbrace{DA^{-1}}_{Q})^{-1}\underbrace{BC}_{P}DA^{-1}\Big) \\
&= A^{-1}\Big(I-\underbrace{BC}_{P}(I+\underbrace{DA^{-1}}_{Q}\underbrace{BC}_{P})^{-1}DA^{-1}\Big) \\
&= A^{-1}-A^{-1}BC(I+DA^{-1}BC)^{-1}DA^{-1} \\
&= A^{-1}-A^{-1}B\Big((I+DA^{-1}BC)C^{-1}\Big)^{-1}DA^{-1} \\
&= A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}
\end{aligned}
$$

(4) Given $A=\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix}$, $B=\begin{bmatrix}1&2\\1&2\\1&2\\1&2\end{bmatrix}$, $C=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $D=B^\top$. Is $(A+BCD)$ invertible?
Notice that $(A+BCD)^{-1}=A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$ cannot be applied here since C is not invertible, but it is cumbersome to calculate the inverse of a 4-by-4 matrix. So we try $(A+BCD)^{-1}=A^{-1}-A^{-1}BC(I+DA^{-1}BC)^{-1}DA^{-1}$.

$$
DA^{-1}BC=\begin{bmatrix}1&1&1&1\\2&2&2&2\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1/2&0&0\\0&0&1/3&0\\0&0&0&1/4\end{bmatrix}\begin{bmatrix}1&2\\1&2\\1&2\\1&2\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}25/12&0\\25/6&0\end{bmatrix}
$$

$$
(I+DA^{-1}BC)^{-1}=\begin{bmatrix}37/12&0\\25/6&1\end{bmatrix}^{-1}=\frac{12}{37}\begin{bmatrix}1&0\\-25/6&37/12\end{bmatrix}
$$

Then we can compute $(A+BCD)^{-1}=A^{-1}-A^{-1}BC(I+DA^{-1}BC)^{-1}DA^{-1}$.

(5) Given two vectors u, $v\in\mathbb{R}^n$, when is the matrix $(I+uv^\top)$ invertible?
If $(I+uv^\top)$ is nonsingular, by applying matrix inversion lemma, we get

$$
(I+uv^\top)^{-1}=I-u(1+v^\top u)^{-1}v^\top=I-\frac{1}{1+v^\top u}uv^\top,
$$

When $v^\top u\neq -1$, $(I+uv^\top)^{-1}=I-\dfrac{1}{1+v^\top u}uv^\top$.

2 Block Matrices / Partitioned Matrices

1. What is a block matrix? Examples.

2. Given two block matrices $A=\begin{bmatrix}A_{11}&\cdots&A_{1q}\\\vdots&\ddots&\vdots\\A_{p1}&\cdots&A_{pq}\end{bmatrix}\in\mathbb{R}^{m\times n}$, $B=\begin{bmatrix}B_{11}&\cdots&B_{1r}\\\vdots&\ddots&\vdots\\B_{q1}&\cdots&B_{qr}\end{bmatrix}\in\mathbb{R}^{n\times l}$, where $A_{ij}\in\mathbb{R}^{m_i\times n_j}$ and $B_{jk}\in\mathbb{R}^{n_j\times l_k}$ are block matrices and $\sum_{i=1}^{p}m_i=m$, $\sum_{j=1}^{q}n_j=n$, $\sum_{k=1}^{r}l_k=l$.

- $A^\top = \begin{bmatrix} A_{11}^\top & \cdots & A_{p1}^\top \\ \vdots & \ddots & \vdots \\ A_{1q}^\top & \cdots & A_{pq}^\top \end{bmatrix}$.
- For $k \in \mathbb{R}$, $kA = \begin{bmatrix} kA_{11} & \cdots & kA_{1q} \\ \vdots & \ddots & \vdots \\ kA_{p1} & \cdots & kA_{pq} \end{bmatrix}$.
- Suppose $C = AB$, then C can be partitioned as $C = \begin{bmatrix} C_{11} & \cdots & C_{1r} \\ \vdots & \ddots & \vdots \\ C_{p1} & \cdots & C_{pr} \end{bmatrix} \in \mathbb{R}^{m\times l}$, where

$$C_{ik} = \sum_{j=1}^{q} A_{ij} B_{jk} \in \mathbb{R}^{m_i \times l_j}.$$

3. A block matrix of the form $A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_p \end{bmatrix}$ is called a block diagonal matrix.

   If $\forall\, 1 \le i \le p$, A_i is a square matrix, then $\det A = \prod_{i=1}^{p} \det A_i$, and the eigenvalues of A are the collection of the eigenvalues of A_i, i.e., $\{\lambda \mid Ax = \lambda x, x \neq 0\} = \{\lambda_i \mid \forall\, 1 \le i \le p,\ A_i y = \lambda_i y, y \neq 0\}$

4. Block LDU decomposition:
   Given a block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, D are square matrices, and D is invertible, then

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}.$$

   This is called block LDU decomposition of M. Also,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}.$$

   $D - CA^{-1}B$ is called Schur complement of A in M, and $A - BD^{-1}C$ is called Schur complement of D in M.

   Lemma:

   - If M is symmetric, i.e., $A = A^\top$, $D = D^\top$, $C = B^\top$, then the following are equivalent:
     - $M \succ 0$.
     - $A \succ 0$ and $D - B^\top A^{-1} B \succ 0$.
     - $D \succ 0$ and $A - BD^{-1}B^\top \succ 0$.
   - $\det(M) = \det(A)\det(D - CA^{-1}B) = \det(D)\det(A - BD^{-1}C)$
   - Suppose in a given matrix M, $A \in \mathbb{R}^{n\times n}$, $B = -u \in \mathbb{R}^n$, $C = v^\top \in \mathbb{R}^{1\times n}$, $D = 1$, we obtain the matrix determinant lemma using the first lemma here:

$$\det(A + uv^\top) = (1 + v^\top A^{-1} u)\det(A)$$

   - In general,

$$\det(A + UWV^\top) = \det(A)\det(W)\det(W^{-1} + V^\top A^{-1} U)$$

     if A, W are invertible square matrices.

- Note that even if A, B, C, D are square matrices of the same size, $\det(M) \neq \det(AD - BC)$ in general.

Ex: Given $M = \begin{bmatrix} 7 & 6 & 4 \\ 6 & 6 & 4 \\ 4 & 4 & 4 \end{bmatrix}$. Is M positive definite?

We can use any method given in the first part. However, to calculate the determinant or eigenvalues of a 3-by-3 matrix could be cumbersome. Here we can use Schur complement instead.

$$M = \left[\begin{array}{cc|c} 7 & 6 & 4 \\ 6 & 6 & 4 \\ \hline 4 & 4 & 4 \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

- D is positive definite since $4 > 0$.
- $A - BD^{-1}C = \begin{bmatrix} 7 & 6 \\ 6 & 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} 4^{-1} \begin{bmatrix} 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$.
  $A - BD^{-1}C$ is positive definite since $3 > 0$ and $2 - 2 \times \dfrac{1}{3} \times 2 > 0$.
- Thus, M is positive definite.