Exam Number:________

ROB 501 Exam-I
[DO NOT OPEN UNTIL TOLD TO DO SO]

Tuesday, October 30, 2018, 6:00 PM–7:50 PM
TBD: Rooms (First letter of last name): (A-L) in EECS 1500; (M-We) in EECS 1311; and (Wu-Z) in EECS 1008.

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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SIGNATURE
(Sign **after** the exam is completed)

________________, ________________
LAST NAME (PRINTED), *FIRST NAME*

FILL IN YOUR NAME NOW. COPY THE HONOR CODE NOW. DO NOT COUNT PAGES.
DO NOT OPEN THE EXAM BOOKLET UNTIL TOLD TO DO SO.

RULES:

1. CLOSED TEXTBOOK
2. CLOSED CLASS NOTES
3. CLOSED HOMEWORK
4. CLOSED HANDOUTS
5. 2 SHEETS OF NOTE PAPER (Front and Back), US Letter Size.
6. NO CALCULATORS, CELL PHONES, HEADSETS, nor DIGITAL DEVICES of any KIND.

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

Problems 1 - 5 (30 points: 5×6)

Instructions. For each problem, there is at least one answer that is true (i.e., **T**) and one answer that is false (i.e., **F**). *You will receive no credit for your responses if you either mark all of the answers as true or all as false*, **because then we assume you are guessing**. Otherwise, each part of a question is worth 1.5 points.

1. (Questions on logic and proof methods) Recall that $\wedge$ is 'and', $\vee$ is 'or', and $\neg$ is 'not'. Recall also that the symbol $\Leftrightarrow$ and the written text, "if, and only if", "logically equivalent to", and "have the same truth table", all mean the same thing. For example, in HW, you verified that $\neg(p \wedge q)$ is "logically equivalent to" $(\neg p) \vee (\neg q)$ by proving "they have the same truth table". **Circle True or False as appropriate for the following statements:**

T **F** (a) There are statements about the natural numbers that can be proved with *Strong Induction* but cannot be proved with *Ordinary Induction*.

T **F** (b) Let n be a natural number. If n^2 is odd then so is n.

T **F** (c) $p \implies q$ is logically equivalent to $(\neg p) \vee q$

T **F** (d) The truth table given below is correct for $\neg q \implies p$

p	q	$\neg q \implies p$
1	1	1
1	0	1
0	1	0
0	0	1

2. (Facts about matrices) For $n, m \geq 1$, let A and B be an $n \times m$ real matrices. For any real matrix M, denote its i-th column by M_i and its ij-element by $[M]_{ij}$. **Circle True or False as appropriate for the following statements:**

T **F** (a) $\text{trace}(AA^\top) = \sum_{i=1}^{n} \left([A]_{ii}\right)^2$

T **F** (b) $[A^\top B]_{ij} = (A_i)^\top B_j$.

T **F** (c) $\text{span}\{A_1, A_2, \cdots, A_m\} = \{y \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^m, \text{ such that } y = Ax\}$.

T **F** (d) Suppose $n = m$ so that A is a square real matrix and let $x \in \mathbb{R}^n$, $x = [x_1, x_2, \ldots, x_n]^\top$. Then

$$x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = x^\top A.$$

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Anything written here will not be graded. You can use as scratch paper.

3. Let $(\mathcal{X}, \mathbb{R})$ be a finite-dimensional inner product space, let $S_1 \subset \mathcal{X}$ and $S_2 \subset \mathcal{X}$ be nonempty subsets (to be clear, they may or may not be subspaces). **Circle True or False as appropriate for the following statements:**

T **F** (a) If $\text{span}\{S_1\} \subset \text{span}\{S_2\}$, then $S_1 \subset S_2$.

T **F** (b) $\text{span}\{S_1\} \oplus S_1^{\perp} = \mathcal{X}$.

T **F** (c) $\text{span}\{S_1 \cup S_2\} = \text{span}\{S_1\} + \text{span}\{S_2\}$.

T **F** (d) $S_1^{\perp} \cap S_2^{\perp} = \left[\text{span}\{S_1\} \cap \text{span}\{S_2\}\right]^{\perp}$.

4. (Normed spaces, matrices, and inner products) **Circle True or False as appropriate for the following statements:**

T **F** (a) Let $(\mathcal{X}, \mathbb{R}, ||\cdot||)$ be an arbitrary normed space, let $M \subset \mathcal{X}$ be a subspace, and let $x \in \mathcal{X}$. Then there must exist a unique point $m_0 \in M$ satisfying $||x - m_0|| \leq ||x - m||, \forall m \in M$.

T **F** (b) Let P be an $n \times n$ real symmetric matrix. Suppose that $||x|| := \sqrt{x^\top P x}$ defines a norm on $(\mathbb{R}^n, \mathbb{R})$. Then P has all positive eigenvalues.

T **F** (c) The matrix $M = \begin{bmatrix} 6 & 2 & 4 \\ 2 & 3 & 1 \\ 4 & 1 & 1 \end{bmatrix}$ is positive definite.

T **F** (d) Let $(\mathcal{X}, \mathbb{R}, < \cdot, \cdot >)$ be an inner product space with basis $\{v^1, v^2, \ldots, v^n\}$. If $x \in \mathcal{X}$ satisfies $< x, v^i >= 0$, $\forall i \in \{1, \ldots, n\}$, then $x = 0$.

5. (Vector spaces, representations, and norms) Let $(\mathcal{X}, \mathbb{R}, ||\cdot||)$ be an n-dimensional normed space with $n \geq 4$ and let $L : \mathcal{X} \to \mathcal{X}$ be a linear operator. Let A be the matrix representation of $L : \mathcal{X} \to \mathcal{X}$ when the basis $\{u\} := \{u^1, \ldots, u^n\}$ is used on both copies of $\mathcal{X}$ (i.e., on the domain of L and its range (also called co-domain)). We define a second basis on $\mathcal{X}$ by scaling the first basis:

$$\{\bar{u}\} := \{\bar{u}^1 = u^1, \bar{u}^2 = 2u^2, \ldots, \bar{u}^k = ku^k, \ldots, \bar{u}^n = nu^n\}.$$

Circle True or False as appropriate for the following statements, where u^i and $\bar{u}^j$ always refer to elements of the given bases and the matrix A is as defined in the problem statement.

T **F** (a) The change of basis matrix from $\{u\}$ to $\{\bar{u}\}$ (i.e. $P \in \mathbb{R}^{n\times n}$ s.t. $[x]_{\bar{u}} = P[x]_u$) is $P = \mathrm{diag}\left([1, 2, \ldots, n]\right)$.

T **F** (b) $[L(u^3)]_{\{\bar{u}\}} = \frac{1}{3}A_3$, where A_3 is the third column of A.

T **F** (c) $[L(\bar{u}^3 + \bar{u}^4)]_{\{u\}} = 3A_3 + 4A_4$ where A_3 and A_4 are the corresponding columns of A.

T **F** (d) If the norm on $\mathcal{X}$ is strict, then $||u^1 + u^2|| < ||u^1|| + ||u^2||$ (Yes, here, u^1 and u^2 are the first two elements of the $\{u\}$ basis).

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Anything written here will not be graded. You can use as scratch paper.

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. <u>If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.</u> For example, we proved that the Projection Theorem. So if you need results associated with it, simply state them and use them.

6. (20 points) (Place your answers in the **boxes** and show your work below.) Let $(\mathcal{X}, \mathbb{R}, < \bullet, \bullet >)$ be the inner product space of continuous functions over $[-1, 1]$, i.e. $\mathcal{X} = \{f : [-1, 1] \to \mathbb{R}\}$, with the inner product defined as $< f, g >= \int_{-1}^{1} f(t)g(t)dt$.

(a) (12 points) Find an orthogonal basis $U = \{u^1, u^2, u^3, u^4\}$ for $M = \text{span}\{1, t, t^2, \sin(\pi t)\}$, where $u^i \in \mathcal{X}$ and $M \subset \mathcal{X}$

Hint: You can use the following integrals: $\int_{-1}^{1} \sin(\pi t)dt = 0$, $\int_{-1}^{1} t\sin(\pi t)dt = \frac{2}{\pi}$, $\int_{-1}^{1} t^2 \sin(\pi t)dt = 0$, $\int_{-1}^{1} [\sin(\pi t)]^2 dt = 1$, and $\int_{-1}^{1} t^k dt = 0$ when k is odd. Also, note that you are not asked to make them orthonormal.

(a)

$u^1 =$

$u^2 =$

$u^3 =$

$u^4 =$

(b) (8 points) Recall that the distance from a point $x \in \mathcal{X}$ to a subspace $M \subset \mathcal{X}$ is given by $d(x, M) = \inf_{m \in M} ||x - m||$ (using the norm induced by the inner product). Compute the distance $d_0 = d(\sin(\pi t), \text{span}\{t, t^2\})$, that is $d(x, M)$ for $x = \sin(\pi t)$ and $M = \text{span}\{t, t^2\}$. Hint: It will be a (non-negative) number and NOT a function.

(b) $d_0 =$

Show your calculations and reasoning below. No reasoning $\Longrightarrow$ no points.

Please show your work for question 6.

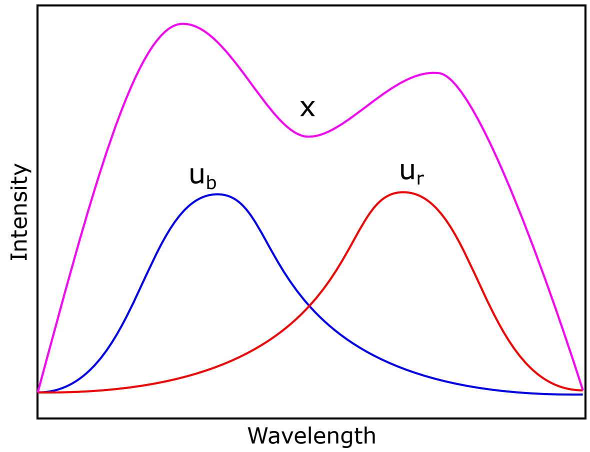

Figure 1: Diagram of our LED color profiles u_b, and u_r, and our target color x.

7. (15 points) The space of all color profiles can be specified by functions $I : [\Lambda_{min}, \Lambda_{max}] \to \mathbb{R}$, where $I(\lambda)$ is the intensity of the color at each wavelength λ. We can represent this space of colors as an inner product space $(\mathcal{X}, \mathbb{R}, < \cdot, \cdot >)$ where $\mathcal{X} := \{I : [\Lambda_{min}, \Lambda_{max}] \to \mathbb{R}\}$ is a finite-dimensional vector space of continuous functions and the inner product is defined as

$$< f, g > := \int_{\Lambda_{min}}^{\Lambda_{max}} f(\lambda)g(\lambda)d\lambda.$$

Suppose we have two LEDs, one red, one blue, with corresponding intensities $\{u_b, u_r\}$ that can be combined and scaled linearly to form different colors (see Figure. 1). Our goal is to best approximate a desired color with intensity function x using these two LEDs.

We are given the following properties:

$$< u_b, u_b >= 1 \qquad < u_b, u_r >= 0.5 \qquad < u_r, u_r >= 1 \qquad < x, u_b >= 2.5 \qquad < x, u_r >= 1.5$$

Find a linear combination of the colors blue and red that best approximates our desired color profile x. That is, find:

$$\alpha^* = \begin{bmatrix} \alpha_b^* \\ \alpha_r^* \end{bmatrix} = \arg \min_{\alpha_b, \alpha_r \in \mathbb{R}} ||x - (\alpha_b u_b + \alpha_r u_r)||^2$$

using the norm induced by the inner product. Record your answer in the box.

(a) (5 Points) Will your answer be unique? Briefly state why or why not.

(b) (10 points) $\alpha^* =$

Show your calculations below

Please show your work for question 7.

8. (15 points) (Proof Problem) Let $(\mathcal{X}, \mathcal{F})$ be a vector space and v^1, v^2, v^3 vectors in $\mathcal{X}$. Define the following two statements:

- P: each of the sets $\{v^1, v^2\}$, $\{v^2, v^3\}$, and $\{v^3, v^1\}$ is linearly independent.
- Q: the set $\{v^1, v^2, v^3\}$ is linearly independent.

For each of the following statements, decide if it is T or F and then support your conclusion with a proof or a counterexample:

(a) $P \implies Q$

(b) $Q \implies P$

Show your work below. You can use as true anything we have established in ROB 501 lecture or HW. I <u>cannot</u> answer any question of the form: **"do I have to prove this?"** or **"can I assume this?"** or **"have I shown enough?"**.

Remark: If the problem seems completely trivial, that is OK; please write down the few lines it takes to do a (complete) proof or to establish a counterexample. If the problem seems challenging, then maybe you need more than a few lines to work the problem. Both are possible.

(a) $P \implies Q$ is **[T F (circle one)]** and here is my supporting reasoning.

(b) $Q \implies P$ is **[T F (circle one)]** and here is my supporting reasoning.

Extra space for question 8.

9. (5 points) A$^+$ Problem: Points earned here will go toward deciding who goes from an A to an A^+ at the end of the term. Recall that for your GPA at Michigan, an A$^+$ counts the same as an A.

Def. Let $(\mathcal{X}, \mathbb{C})$ be a vector space over the complex numbers and $L : \mathcal{X} \to \mathcal{X}$ be a linear operator. $\lambda \in \mathbb{C}$ is an e-value of L if $\exists$ $(v \in \mathcal{X}, v \neq 0)$ such that $L(v) = \lambda v$; v is called an e-vector of L.

Prove this: If λ_1, λ_2, and λ_3 are *distinct* e-values of L, then a set of corresponding e-vectors $\{v^1, v^2, v^3\}$ is *linearly independent.*

Note: If you need to use a result from lecture or HW, clearly state the result and note that it is from ROB 501; in that case, you do not need to prove it. Otherwise, any other statements used in your proof should be justified here. **If your proof assumes that $(\mathcal{X}, \mathbb{C})$ is finite dimensional, you will earn at most three (3) points.**

(Scratch Paper)

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(If you write anything here, be sure to indicate to which problem it applies.)