**Machine Learning** **(Due: 18th March 24:00)**

# Submission Assignment #1 Solution

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**Problem 1: Basic Vector Operations** ( points)

**(1)** $||a||_2 = \sqrt{1^2+2^2+3^2} = \sqrt{14}$ $||b||_2 = \sqrt{(-8)^2+1^2+2^2} = \sqrt{69}$

**(2)** $||a-b||_2 = \sqrt{[1-(-8)]^2+(2-1)^2+(3-2)^2} = \sqrt{83}$

**(3)** $\because \langle a,b\rangle = a^Tb = 1\times(-8)+2\times 1+3\times 2 = 0$

$\therefore a\perp b$

**Problem 2: Basic Matrix Operations** ( points)

**(1)** $\because \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 3 & -5 & 3 & 0 & 1 & 0 \\ 6 & -6 & 4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{3}{8} & -\frac{7}{8} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$ $\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{3}{8} & -\frac{7}{8} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$

$\det(A) = \begin{vmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{vmatrix} = 16$

**(2)** $rank\,(A) = 3$

**(3)** $trace\,(A) = 0$

**(4)** $A + A^T = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 9 \\ 0 & -10 & -3 \\ 9 & -3 & 8 \end{bmatrix}$

**(5)** $\because A^{-1} \neq A^T$ $\therefore A$ *is not an orthogonal matrix.*

**(6)** $|\lambda I - A| = \begin{vmatrix} \lambda-1 & 3 & -3 \\ -3 & \lambda+5 & -3 \\ -6 & 6 & \lambda-4 \end{vmatrix} = \lambda^3 - 12\lambda - 16 = (\lambda+2)^2(\lambda-4) = 0$

$\therefore \lambda_1 = \lambda_2 = -2, \lambda_3 = 4$

$if\ \lambda = 4, [4I - A] = \begin{bmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{bmatrix}, (4I-A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \begin{cases} x+y-z=0 \\ 4y-2z=0 \end{cases} \to \begin{cases} z=2y \\ x=y \end{cases} \begin{bmatrix} y \\ y \\ 2y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\therefore \eta_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$if\ \lambda = -2, [-2\lambda - A] = \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{bmatrix}, (-2\lambda - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0, x+y-z=0 \to x = y-z$

$\begin{bmatrix} y-z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\therefore \eta_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \eta_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \eta_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \eta_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \eta_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

**(7)** $Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, s.t. Q^{-1}AQ = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

**(8)** $\|A\|_{2,1} = \sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2} + \sqrt{a_{21}^2 + a_{22}^2 + a_{23}^2} + \sqrt{a_{31}^2 + a_{32}^2 + a_{33}^2} = \sqrt{19} + \sqrt{43} + 2\sqrt{22}$
$\|A\|_F = \sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2 + a_{33}^2} = 5\sqrt{6}$

**(9)** $Using\ SVD, A = U\Sigma V^T, where\ U = \begin{bmatrix} -0.3348 & -0.8018 & -0.4950 \\ -0.5399 & -0.2673 & 0.7982 \\ -0.7722 & 0.5345 & -0.3434 \end{bmatrix}, \Sigma = \begin{bmatrix} 12.0648 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0.6631 \end{bmatrix},$
$V = \begin{bmatrix} -0.5461 & 0.8018 & -0.2428 \\ 0.6911 & 0.2673 & -0.6716 \\ -0.4735 & -0.5345 & -0.7 \end{bmatrix}$
$\therefore \|A\|_* = trace(\Sigma) = 14.7279$
$spectral\ norm: \|A\|_2 = \max\{\sigma_A\} = 12.0648$

**Problem 3: Linear Equations** ( points)

**(1)** $\begin{cases} 2x_1 + 2x_2 + 3x_3 = 1\,(a) \\ x_1 - x_2 = -1\,(b) \\ -x_1 + 2x_2 + x_3 = 2\,(c) \end{cases} \rightarrow \begin{cases} (a) - (c) : 3x_1 + 2x_3 = -1 \\ (b) \times 2 + (c) : x_1 + x_3 = 0 \end{cases} \xrightarrow{solve} \begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$

**(2)** $\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

**(3)** $\because A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \xrightarrow[transformation]{elementary\ row} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$
$\therefore rank(A) = 3$

**(4)** $\because \begin{bmatrix} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -4 & -3 \\ 0 & 1 & 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & -1 & 6 & 4 \end{bmatrix}$
$\therefore A^{-1} = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}, \det(A) = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix} = -1$

**(5)** $x = A^{-1}b = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

**(6)** $\langle x, b \rangle = x^T b = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 1$

$x \otimes b = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

**(7)** $||b||_1 = 1 + 1 + 2 = 4$
$||b||_2 = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$
$||b||_\infty = \max\{1, 1, 2\} = 2$

**(8)**
$y^T Ay = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = 7y_1^2 + 2y_3^2 + 7y_1y_2 + 2y_2y_3 + 9y_1y_3$
$\frac{\partial y^T Ay}{\partial y_1} = 14y_1 + 7y_2 + 9y_3, \frac{\partial y^T Ay}{\partial y_2} = 7y_1 + 2y_3, \frac{\partial y^T Ay}{\partial y_3} = 4y_3 + 9y_1 + 2y_2$
$\nabla_y y^T Ay = \begin{bmatrix} 14y_1 + 7y_2 + 9y_3 & 7y_1 + 2y_3 & 9y_1 + 2y_2 + 4y_3 \end{bmatrix}$

**(9)** $\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}$

**(10)** $rank\,(A_1) = rank\,(A) = 3$

**(11)** $to\ solve\ the\ equation, A_1 x = b\ \leftrightarrow\ Ax = b, x = A^{-1}b, x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};$

$and\ x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\ is\ also\ the\ solution\ of\ A_1 x = b.$