Proof that A_5 is a Simple Group

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Definitions and Useful Theorems

- A normal subgroup H of G is defined as a subgroup which is closed under conjugation action $\forall g \in G \leftrightarrow$ a subgroup formed by combining its conjugacy classes
- A simple group is a group that does not contain any normal subgroups other than the trivial group and the group itself

Theorem 1. *Two permutations θ and θ' are in the same orbit under conjugation action iff they have the same partition type.*

Theorem 2. *(Lagrange's theorem) If H is a subgroup of G then $|H|$ divides $|G|$*

1 Proof

According to Theorem 1, all the same partition types in A_5 belong to the same conjugacy class. We can use a combinatorial approach to list the number of elements in each conjugacy class.

Let $\mathcal{P}$ denote the set of partition types of a permutation group. For A_5, $\mathcal{P}$ is $\mathcal{P} = \{(abcde), (abc), (ab)(cd), id\}$

So the number of elements of each conjugacy class are

$\mathcal{P}$	No. of elements in orbit
$(abcde)$	$4! = 24$
(abc)	$\binom{5}{2}.2! = 20$
$(ab)(cd)$	$\binom{5}{2}.\binom{3}{2}.\frac{1}{2}$=15
id	1

Since a normal subgroup is formed by combining conjugacy classes of the group, let us try to form a subgroup via these conjugacy classes and count the number of elements a possible subgroup can have. $|A_5| = 60$ so its subgroup can be of order 2,3,4,5,12,15,20 and 30 according to Theorem 2. Since it is a subgroup, it must contain the *id* element. Combining the remaining three conjugacy classes with the *id* class, we get 25,21 and 16, none dividing 60. Combining more conjugacy classes is useless, as they will be greater than 30 and not divide 60.

2 Conclusion

As shown above, there does not exist a subgroup that is a union of the conjugacy classes of A_5. Thus, we conclude that A_5 is a simple group.