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CENTRAL EXTENSIONS OF SIMPLICIAL GROUPS AND PRESHEAVES OF SIMPLICIAL GROUPS

By

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ABSTRACT

We show that path components of central extensions of a simplicial group G by a simplicial abelian group A are in bijection with the homotopy classes of maps between classifying spaces $[\overline{W}G, \overline{W}^2 A]$. Generalizing the classical correspondence for central extensions of a group G by an abelian group A. We then prove an analogous theorem for central extensions of presheaves of simplicial groups. Finally, we use the correspondence to show that the cup product factors through a Heisenberg central extension.

CHAPTER 1

INTRODUCTION

An *extension* of a group G by a group H is a group E such that we have a short exact sequence of groups

$$1 \to H \xrightarrow{i} E \xrightarrow{p} G \to 1$$

We say such an extension is *central* if $i(H)$ is contained in the center of E. Notice that this necessarily makes H abelian, so that a *central extension* of a group G by an abelian group A is a group E such that we have a short exact sequence of groups

$$0 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

where $i(A)$ is contained in the center of E. G acts trivially by conjugation on A in E, making A a trivial G-module.

Two central extensions of G by A, say E and E', are *isomorphic* if there is group homomorphism $f : E \to E'$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "f"] & G \arrow[d, equal] \\ A \arrow[r] & E' \arrow[r] & G \end{tikzcd}$$

This necessarily makes f an isomorphism and defines an equivalence relation. We denote these equivalence classes, which are called *isomorphism classes*, of central extensions of G by A by $\mathbf{Iso}(G, A)$.

It is a classical result that these isomorphism classes of central extensions are classified by degree 2 group cohomology

$$\mathbf{Iso}(G, A) \cong H^2(G, A),$$

where $H^2(G, A) = H^2_{\mathbf{Top}}(K(G, 1), A)$ and $K(G, 1)$ is an Eilenberg-MacLane space. An Eilenberg-MacLane space $K(A, n)$ is a topological space that has single non-trivial homotopy group in degree n

$$\pi_i K(A, n) = \begin{cases} A & i = n \\ 0 & i \neq n \end{cases}$$

It is well known that, by the representation theorem for cohomology,

$$H^2(G, A) \cong [K(G, 1), K(A, 2)].$$

Together these give a correspondence between isomorphism classes of central extensions and homotopy classes of maps between classifying spaces.

A pertinent example is given by the *Heisenberg group* $H_{A,B}$, defined for two abelian groups A, B as the set $A \times B \times (A \otimes B)$ with group operation defined by

$$(a, b, t)(a', b', t') = (a + a', b + b', t + t' + a \otimes b')$$

for $a, a' \in A$, $b, b' \in B$, and $t, t' \in A \otimes B$. This gives a *Heisenberg central extension* of groups

$$0 \to A \otimes B \to H_{A,B} \to A \times B \to 0.$$

The characteristic class of the Heisenberg central extension is given by $f(a, b, a', b') = a \otimes b'$ and according to above corresponds to a homotopy class

$$f \in [K(A \times B, 1), K(A \otimes B, 2)].$$

If central extensions are equipped with some additional structure then it is natural to ask if there is a similar correspondence. We know this is the case for Lie groups and topological groups [25], [26]. In this thesis we answer the question:

Is there a correspondence between equivalence classes of central extensions of simplicial groups (presheaves of simplicial groups) and homotopy classes of maps between classifying spaces?

This question was motivated by the work of Aldrovandi and Ramachandran in [1]. In this work they give a geometric interpretation of the cup product

$$H^1(X, A) \times H^1(X, B) \to H^2(X, A \otimes B)$$

for abelian sheaves A, B on an algebraic variety X, where the key step is showing that the Heisenberg sheaf "animates" the cup product. A crucial point is showing that the morphism between Eilenberg MacLane spaces

$$K(A \times B, 1) \to K(A \otimes B, 2)$$

given by the cup product corresponds to the characteristic class of the Heisenberg central extension of sheaves

$$0 \to A \otimes B \to H_{A,B} \to A \times B \to 0.$$

In this case it is possible to use central extensions of sheaves of groups, but in order to extend this to the cup product

$$H^i(X, A) \times H^j(X, B) \to H^{i+j}(X, A \otimes B)$$

for $i + j > 2$ one must show that the morphism between Eilenberg-MacLane spaces

$$K(A, i) \times K(B, j) \to K(A \otimes B, i + j)$$

given by the cup product corresponds to a Heisenberg central extension of simplicial sheaves of groups

$$0 \to K(A, i-1) \otimes K(B, j-1) \to H \to K(A, i-1) \times K(B, j-1) \to 0.$$

This is the case because $K(A, i-1)$ and $K(B, j-1)$ are homotopy types best represented by simplicial (pre)sheaves.

Thus one must consider central extensions of (pre)sheaves of simplicial groups. As the local homotopy theory for simplicial sheaves and simplicial presheaves are equivalent, we can work with presheaves of simplicial groups and this is the focus of our main theorems. We first do this for simplicial groups in chapter 4, which are presheaves of simplicial groups over a point, and then in chapter 6 we adapt the construction to presheaves of simplicial groups over any small site.

Let G be a simplicial group (presheaf of simplicial groups) and A a simplicial abelian group (presheaf of simplicial abelian groups). A central extension of G by A, definition 4.4.1 (definition 6.3.2) is a short exact sequence

$$0 \to A \to E \xrightarrow{p} G \to 1$$

of simplicial groups (presheaves of simplicial groups) where the natural morphism $\tilde{p} : E/A \to G$ is a weak equivalence (as opposed to an isomorphism).

As we only require the quotient to be a weak equivalence our notion of equivalent central extensions must also be up to weak equivalence. We say two central extensions of simplicial groups (presheaves of simplicial groups) are equivalent if there is an A-equivariant weak equivalence f making the following diagram commute:

$$
\begin{tikzcd}
E \arrow[dd, "f"', "\sim"] \arrow[dr, "p"] & \\
& G \\
E' \arrow[ur, "p'"'] &
\end{tikzcd}
$$

The equivalence classes are realized as the path components of the *category of central extensions* $\mathbf{C}(G,A)$, whose objects are central extensions and morphisms are the above commutative diagrams.

In general, homotopy classes of maps $[X,Y]_{\mathcal{M}}$ in a model category $\mathcal{M}$ are hard to compute and we use a result of Jardine [15]. Jardine shows that if a model category $\mathcal{M}$ satisfies certain (mild) conditions, then

$$[X,Y]_{\mathcal{M}} \cong \pi_0 H_{\mathcal{M}}(X,Y),$$

$H_{\mathcal{M}}(X,Y)$ is the *category of cocycles* from X to Y in $\mathcal{M}$. Its objects are *cocycles*:

$$
\begin{tikzcd}
& Z \arrow[dl, "f"', "\sim"] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}
$$

where the only requirement on the morphisms is that f be a weak equivalence. Morphisms between cocycles are commutative diagrams

$$
\begin{tikzcd}
& Z \arrow[dl, "f"', "\sim"] \arrow[dr, "g"] \arrow[dd, "w"] & \\
X & & Y \\
& Z' \arrow[ul, "f'", "\sim"'] \arrow[ur, "g'"'] &
\end{tikzcd},
$$

where w is (necessarily) a weak equivalence. $\pi_0 H_{\mathcal{M}}(X,Y)$ denotes the set of path components.

Our main theorems establish correspondences

$$\pi_0(\mathbf{C}(G,A)) \cong \pi_0(H_{\mathbf{sGrp}}(G,\overline{W}A))$$

for central extensions of simplicial groups (theorem 4.4.8), and

$$\pi_0(\mathbf{C}(G,A)) \cong \pi_0(H_{\mathbf{Pre(sGrpd)}_*}(G,\overline{W}A))$$

for central extensions of presheaves of simplicial groups (theorem 6.3.11). Then using the aforementioned result of Jardine, and applying a classifying space functor $\overline{W}$, we obtain the desired correspondences

$$\pi_0(\mathbf{C}(G,A)) \cong [\overline{W}G, \overline{W}^2 A]_{\mathbf{sSet}}$$

for central extensions of simplicial groups (corollary 4.4.9) and

$$\pi_0(\mathbf{C}(G,A)) \cong [\overline{W}G, \overline{W}^2 A]_{\mathbf{Pre(sSet)}_*}$$

for central extensions of presheaves of simplicial groups (corollary 6.3.12). Here $\overline{W}$ is a choice of classifying space such that, if G reduces to a single group, then $\overline{W}G$ reduces to an Eilenberg-MacLane space, $K(G,1)$.

We then show that

$$\mathbf{Iso}(G,A) \cong \pi_0(\mathbf{C}(G,A))$$

which together with our correspondence gives

$$\mathbf{Iso}(G,A) \cong [K(G,1), K(A,2)]$$

recovering the classical correspondence. The classical construction takes a central extension of groups

$$0 \to A \to E \xrightarrow{p} G \to 1$$

and chooses a based set theoretic cross section of p. That is a function $\sigma : G \to E$ such that $\sigma(1_G) = 1_E$ and $p \circ \sigma = \mathbf{id}_G$. It then defines the simplicial set morphism $f : \overline{W}G \to \overline{W}^2 A$ by defining a function $f : G \times G \to A$ by $f(g,h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$. We relate this to our construction by using bicategories to construct morphisms of simplicial sets from cross sections, which we use to show that both constructions give the same correspondence.

We end by showing that our correspondence factors the cup product through a Heisenberg central extension. The Heisenberg group $H_{A,B}$ naturally extends to a Heisenberg simplicial group (presheaf of simplicial groups) which gives a central extension of simplicial groups (presheaf of simplicial groups)

$$0 \to A \otimes B \to H_{A,B} \to A \times B \to 0.$$

These *Heisenberg central extensions of simplicial groups (presheaves of simplicial groups)* are extremely nice central extensions. Namely they are sectionwise and levelwise central extensions of groups that have simplicial set (simplicial presheaf) cross sections, which we use to show the cup product factors through a Heisenberg central extension, corollary 7.4.3 (corollary 7.6.3).

CHAPTER 2

GENERAL BACKGROUND

2.1 Simplicial Categories

Definition 2.1.1. The *ordinal number category*, Δ, has objects given by the ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$ for all $n \in \mathbb{N}$ and morphisms $\phi : [n] \to [m]$ are given by non-decreasing functions. Those functions such that if $i \leq j$ then $\phi(i) \leq \phi(j)$.

In particular there are two special kinds of morphisms called *coface maps* and *codegeneracy maps*. The coface maps are functions $d^i : [n] \to [n+1]$, $0 \leq i \leq n+1$ defined by:

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

So d^i is the function that skips i. The codegeneracy maps are functions $s^k : [n+1] \to [n]$, $0 \leq k \leq n$, defined by:

$$s^k(j) = \begin{cases} j & \text{if } j \leq k \\ j-1 & \text{if } j > k \end{cases}$$

So s^k is the function that doubles up on k. These two special kinds of morphisms in fact determine all the morphisms in Δ.

Theorem 2.1.2. *Every morphism $\phi : [n] \to [m]$ in Δ has a unique epi-monic factorization $\phi = d \circ s$, where the monic d is uniquely a composition of coface maps*

$$d = d^{i_1} \cdots d^{i_k} \text{ with } 0 \leq i_k < \cdots < i_1 \leq m$$

and the epi s is uniquely a composition of codegeneracy maps

$$s = s^{j_1} \cdots s^{j_t} \text{ with } 0 \leq j_1 < \cdots < j_t \leq n$$

Moreover these morphisms satisfy the following identities:

$$d^j d^i = d^i d^{j-1} \text{ if } i < j$$

$$s^j s^i = s^i s^{j+1} \text{ if } i \leq j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \text{ or } i = j+1 \\ d^{i-1} s^j & \text{if } i > j+1 \end{cases}$$

Definition 2.1.3. A *simplicial object* in a category, $\mathcal{C}$, is defined to be a contravariant functor from Δ to $\mathcal{C}$ i.e. a functor $F : \Delta^{op} \to \mathcal{C}$.

The *category of simplicial objects* in a category $\mathcal{C}$, denoted $s\mathcal{C}$, has as objects the simplicial objects in $\mathcal{C}$ and as morphisms the natural transformations between them. If X is an object of $s\mathcal{C}$ we will denote $X([n])$ by X_n. Also we will denote $X(d^i)$ by d_i, which we will call a face map, and $X(s^i)$ by s_i, that we will call a degeneracy map. More generally if $\phi : [n] \to [m]$ is a morphism in Δ, then we will denote $X(\phi) : X_m \to X_n$ by ϕ^*. Because the morphisms of Δ have a unique factorization in terms of coface and codegeneracy maps we have an equivalent definition of simplicial object. Namely a simplicial object X, in $\mathcal{C}$, is a collection of objects of $\mathcal{C}$, $\{X_n\}_{n \in \mathbb{N}}$, together with *face maps* $d_i : X_{n+1} \to X_n$ for $0 \leq i \leq n+1$ and *degeneracy maps* $s_j : X_n \to X_{n+1}$ for $0 \leq j \leq n$ satisfying the following *simplicial identities*:

$$d_i d_j = d_{j-1} d_i \text{ if } i < j$$

$$s_i s_j = s_{j+1} s_i \text{ if } i \leq j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$$

Using this definition a natural transformation from X to Y amounts to a collection of morphisms in $\mathcal{C}$, $\{f_n : X_n \to Y_n\}_{n \in \mathbb{N}}$, such that

$$d_i f_n = f_{n-1} d_i \qquad 0 \leq i \leq n$$

$$s_j f_n = f_{n+1} s_j \qquad 0 \leq j \leq n$$

Visually we like to write simplicial object as a diagram:

$$\cdots \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} X_2 \substack{\longleftarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} X_1 \substack{\longleftarrow \\ \longrightarrow \\ \longrightarrow} X_0$$

where the arrows represent the face and degeneracy maps. Often we omit drawing the degeneracy arrows for convenience. We call X_n the n-th level of the simplicial object X and say that a property p holds *levelwise* if it holds for every X_n and it is compatible with the simplicial identities. So that a natural transformation from $f : X \to Y$ amounts to levelwise morphisms $f_n : X_n \to Y_n$ such that the following diagram commutes when the same face or degeneracy map is used.

$$
\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_2 \arrow[l, bend right=20] \arrow[l, bend right=30] \arrow[l, bend right=40] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "f_2"] & X_1 \arrow[l, bend right=20] \arrow[l, bend right=30] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, "f_1"] & X_0 \arrow[l, bend right=30] \arrow[d, "f_0"] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & Y_2 \arrow[l, bend right=20] \arrow[l, bend right=30] \arrow[l, bend right=40] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Y_1 \arrow[l, bend right=20] \arrow[l, bend right=30] \arrow[r, shift left=1] \arrow[r, shift right=1] & Y_0 \arrow[l, bend right=30]
\end{tikzcd}
$$

When a category, $\mathcal{C}$, has products and tensor products, then the product and tensor product of simplicial objects, in $\mathcal{C}$, is defined levelwise.

Definition 2.1.4. The *Cartesian product* of two simplicial objects X and Y in a category with products $\mathcal{C}$, denoted $X \times Y$, is defined levelwise by:

$$(X \times Y)_n = X_n \times Y_n$$

Where the face and degeneracy maps are defined by:

$$d_i(x, y) = (d_i(x), d_i(y))$$
$$s_i(x, y) = (s_i(x), s_i(y))$$

More generally:

Definition 2.1.5. The *tensor product* of two simplicial objects X and Y in a monoidal category (a category with a tensor product) $\mathcal{C}$, denoted $X \otimes Y$, is defined levelwise by:

$$(X \otimes Y)_n = X_n \otimes Y_n$$

Where the face and degeneracy maps are defined by:

$$d_i(x \otimes y) = d_i(x) \otimes d_i(y)$$
$$s_i(x \otimes y) = s_i(x) \otimes s_i(y)$$

We will now give two examples of simplicial categories that we will use consistently throughout this paper.

Example 2.1.6. We call the category of simplicial objects in the category of sets the *category of simplicial sets* and denote it by **sSet**. A *simplicial set* is a functor $X : \Delta^{op} \to \mathbf{Set}$ or equivalently a collection of sets $\{X_n\}_{n\in\mathbb{N}}$ together with functions $d_i : X_{n+1} \to X_n$ for face maps and functions $s_j : X_n \to X_{n+1}$ for degeneracy maps that satisfy the simplicial identities. A morphism between to simplicial sets X and Y is a natural transformation between them, equivalently it is a collection of functions $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ such that $d_i f_n = f_{n-1} d_i$ for all $0 \le i \le n$ and $s_j f_n = f_{n+1} s_j$ for all $0 \le j \le n$.

Example 2.1.7. Similarly we call the category of simplicial objects in the category of groups the *category of simplicial groups* and denote it by **sGrp**. A *simplicial group* is a functor $G : \Delta^{op} \to \mathbf{Grp}$ or equivalently a collection of groups $\{G_n\}_{n\in\mathbb{N}}$ together with group homomorphisms $d_i : G_{n+1} \to G_n$ for face maps and group homomorphisms $s_j : G_n \to G_{n+1}$ for degeneracy maps that satisfy the simplicial identities. A morphism between to simplicial groups G and H is a natural transformation between them, equivalently it is a collection of group homomorphisms $\{f_n : G_n \to H_n\}_{n\in\mathbb{N}}$ such that $d_i f_n = f_{n-1} d_i$ for all $0 \le i \le n$ and $s_j f_n = f_{n+1} s_j$ for all $0 \le j \le n$.

An important construction that constructs a simplicial set from a category is the *nerve*.

Definition 2.1.8. The *nerve* of a category $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$ defined as follows $N(\mathcal{C})_k$ is the set of k-composable arrows of $\mathcal{C}$. That is a diagram of the form

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k$$

of objects and morphism in $\mathcal{C}$. The face maps are defined, for $0 < i < n$

$$d_0(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k) = X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k$$

$$d_i(X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k) = X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{f_{i+1}\circ f_i} X_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_k} X_k$$

$$d_k(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k) = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} X_{k-1}$$

The degeneracy maps are defined for $0 \le j \le k$

$$s_j(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k) = X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_j} X_j \xrightarrow{\mathbf{id_{X_j}}} X_j \xrightarrow{f_{j+1}} \cdots \xrightarrow{f_n} X_n$$

This defines a functor $N : \mathbf{Cat} \to \mathbf{sSet}$ and it can be proved that it is fully faithful.

2.2 Simplicial Sets

The category of simplicial sets has a homotopical structure, defined in Example 3.1.3, that is equivalent to the ordinary homotopy theory of topological spaces. This gives an algebraic, combinatorial model for standard homotopy theory. To see how they talk to each other we need the *realization functor* $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$, which we will now define. We will outline the steps given in [11](Section I.2). First, we need the standard n-simplex:

Definition 2.2.1. The *standard simplicial n-simplex*, Δ^n, is the simplicial set defined by

$$\Delta^n = hom_{\Delta}(-, [n])$$

In other words Δ^n is the contravariant functor on Δ represented by the set of homomorphisms in Δ to $[n]$. The Yoneda lemma then implies that $hom_{\mathbf{sSet}}(\Delta^n, X) \cong X_n$, so that the set of n-simplicies, X_n, correspond to natural transformations from Δ^n to X.

Definition 2.2.2. Let $n \geq 0$, $0 \leq i \leq n$, and σ be the unique non-degenerate n-simplex of Δ^n. The i^{th} *face* of Δ^n is $d_i(\sigma)$ and the (n,i) *horn* is the union of all faces of Δ^n except the i^{th} face.

A quick way to construct the realization functor uses the *simplex category*, $\Delta \downarrow X$ of a simplicial set X. The objects of $\Delta \downarrow X$ are natural transformations $\sigma : \Delta^n \to X$ (or simplicies of X). A morphism of $\Delta \downarrow X$ is given by the following commutative diagram in $\mathbf{sSet}$:

$$\begin{tikzcd}
\Delta^n \arrow[d, "\phi_*"'] \arrow[dr, "\sigma"] & \\
\Delta^m \arrow[r, "\gamma"'] & X
\end{tikzcd},$$

where ϕ_* is the unique map induced by the ordinal number morphism $\phi : [m] \to [n]$. A general result from category theory [10] implies there is an isomorphism

$$X \cong \varinjlim_{\Delta^n \to X \in \Delta \downarrow X} \Delta^n.$$

Therefore one can define the realization $|X|$ of a simplicial set X to be the colimit:

$$|X| = \varinjlim_{\Delta^n \to X \in \Delta \downarrow X} |\Delta^n|$$

in the category of topological spaces, where $|\Delta^n|$ denotes the standard topological n-simplex. Note that we use this notation as the realization of the simplical n-simplex Δ^n is in fact the topological n-simplex.

The realization functor is left adjoint to the *singular functor*, $Sing : \mathbf{Top} \to \mathbf{sSet}$. If T is a topological space, then $Sing(T)$ is defined levelwise by

$$Sing(T)_n = hom_{\mathbf{Top}}(|\Delta^n|, T).$$

The simplicial structure maps are given by if $\phi : [m] \to [n]$, then $\phi^* : Sing(T)_n \to Sing(T)_m$ is given by precomposition with the induced map $\phi_* : |\Delta^m| \to |\Delta^n|$.

Remark 2.2.3. The realization functor does not preserve products unless we restrict to the category of compactly generated Hausdorff spaces. Namely, in general $|X \times Y|$ need not be homeomorphic to $|X| \times |Y|$, but one does have

$$|X \times Y| \cong |X| \times_K |Y|$$

where $\times_k$ denotes the *Kelly space product*, which is the product in the category of compactly generated Hausdorff spaces. It is harmless, and not crucial in what follows, to restrict ourselves to compactly generated Hausdorff spaces and interpret **Top** to mean **CGHaus**.

2.3 Bisimplicial Sets

Definition 2.3.1. A *bisimplicial object* in a category $\mathcal{C}$ is a functor $\Delta^{op} \times \Delta^{op} \to \mathcal{C}$. Equivalently, it is a simplicial object in the category of simplicial objects in $\mathcal{C}$.

Definition 2.3.2. The *category of bisimplicial objects* in $\mathcal{C}$, denoted $s^2\mathcal{C}$, is the category that has as its objects the bisimplicial objects in $\mathcal{C}$ and as morphisms the natural transformations between them.

Remark 2.3.3. X being a bisimplicial object in a category $\mathcal{C}$ amounts to a collection of objects of $\mathcal{C}$, $\{X_{m,n} | m, n \in \mathbb{N}\}$, together with horizontal and vertical face and degeneracy maps. That is, for $0 \le i \le m$ and $0 \le j \le n$, we have morphisms of $\mathcal{C}$

$$X_{m+1,n} \xleftarrow{s_i^h} X_{m,n} \xrightarrow{d_i^h} X_{m-1,n}$$

and

$$X_{m,n+1} \xleftarrow{s_j^v} X_{m,n} \xrightarrow{d_j^v} X_{m,n-1}$$

such that for all m and n, both $X_{m,-}$ and $X_{-,n}$, are simplicial objects and the horizontal and vertical face and degeneracy maps commute. For all $0 \leq i \leq m$ and $0 \leq j \leq n$

$$d_i^h d_j^v = d_j^v d_i^h$$
$$d_i^h s_j^v = s_j^v d_i^h$$
$$s_i^h d_j^v = d_j^v s_i^h$$
$$s_i^h s_j^v = s_j^v s_i^h$$

Remark 2.3.4. We visualize a bisimplicial object as a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] & \vdots \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] & \vdots \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_{2,2} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & X_{1,2} \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & X_{0,2} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_{2,1} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left=1] \arrow[d, shift right=1] & X_{1,1} \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, shift left=1] \arrow[d, shift right=1] & X_{0,1} \arrow[d, shift left=1] \arrow[d, shift right=1] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_{2,0} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_{1,0} \arrow[r, shift left=1] \arrow[r, shift right=1] & X_{0,0}
\end{tikzcd}$$

where the horizontal arrows represent the horizontal face maps and the vertical arrows represent the vertical face maps. The degeneracy maps are omitted for convenience. The squares commute if the same horizontal and same vertical face maps are used. Moreover, each column $X_{m,-}$ is a simplicial object of $\mathcal{C}$ and each row $X_{-,n}$ is also a simplical object of $\mathcal{C}$.

Definition 2.3.5. If $F : s\mathcal{C} \to \mathcal{D}$ is a functor, we can obtain a functor $F : s^2\mathcal{C} \to s\mathcal{D}$ by either applying the functor to each row or to each column. We say that we apply F *row-wise* if we apply F to each row to obtain the functor $F : s^2\mathcal{C} \to s\mathcal{D}$ defined by $F(X_{m,n}) = F(X_{-,n})_m$. We say that we apply F *column-wise* if we apply F to each column to obtain the functor $F : s^2\mathcal{C} \to s\mathcal{D}$ defined by $F(X_{m,n}) = F(X_{m,-})_n$.

Definition 2.3.6. The diagonal of a bisimplicial object X in $\mathcal{C}$, denoted $diag(X)$, is the composite

$$\Delta^{op} \xrightarrow{\Delta} \Delta^{op} \times \Delta^{op} \xrightarrow{X} \mathcal{C}.$$

Δ is the diagonal functor $[n] \mapsto ([n], [n])$.

One uses the diagonal to obtain a simplicial set from a bisimplicial set. There is another functor that also forms a simplicial set from a bisimplicial set.

Definition 2.3.7. For X a bisimplicial set, the *total simplicial set* TX is defined levelwise by the equalizer

$$(TX)_n \to \prod_{i=0}^{n} X_{i,n-i} \rightrightarrows \prod_{i=0}^{n-1} X_{i,n-i-1}$$

Let $\pi_j : \prod_{i=0}^{n} X_{i,n-i} \to X_{j,n-j}$ be the projection, then the components of the two morphisms on the right are

$$\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{\pi_i} X_{i,n-i} \xrightarrow{d_0^v} X_{i,n-i-1}$$

and

$$\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{\pi_{i+1}} X_{i+1,n-i-1} \xrightarrow{d_{i+1}^h} X_{i,n-i-1}$$

Where the face maps $d_i : (TX)_n \to (TX)_{n-1}$ are given by

$$d_i = (d_i^v \pi_0, d_{i-1}^v \pi_1, \cdots, d_1^v \pi_{i-1}, d_i^h \pi_{i+1}, d_i^h \pi_{i+2}, \cdots, d_i^h \pi_n)$$

and the degeneracy maps $s_i : (TX)_n \to (TX)_{n+1}$ are given by

$$s_i = (s_i^v \pi_0, s_{i-1}^v \pi_1, \cdots, s_0^v \pi_i, s_i^h \pi_i, s_i^h \pi_{i+1}, \cdots, s_i^h \pi_n)$$

Remark 2.3.8. There is also an explicit description for the total simplicial set [2]. Let X be a bisimplicial set then

$$(TX)_n = \{(x_0, \cdots, x_n) \in \prod_{i=0}^{n} X_{i,n-i} : d_0^v x_i = d_{i+1}^h x_{i+1} \text{ for all } 0 \leq i \leq n\}$$

With face and degeneracy maps given by

$$d_i(x_0, \cdots, x_n) = (d_i^v x_0, d_{i-1}^v x_1, \cdots, d_1^v x_{i-1}, d_i^h x_{i+1}, d_i^h x_{i+2}, \cdots, d_i^h x_n)$$

$$s_i(x_0, \cdots, x_n) = (s_i^v x_0, s_{i-1}^v x_1, \cdots, s_0^v x_i, s_i^h x_i, s_i^h x_{i+1}, \cdots, s_i^h x_n)$$

Remark 2.3.9. This also gives a functor $T : \mathbf{s^2Grp} \to \mathbf{sGrp}$ and $T : \mathbf{s^2Ab} \to \mathbf{sAb}$ as one can quickly check that the simplicial structure maps behave with multiplication.

A result of fundamental importance is the following:

Theorem 2.3.10. *Let X be a bisimplicial set. Then there is a natural weak equivalence of simplicial sets*

$$diag(X) \xrightarrow{\sim} T(X)$$

Proof. The first published proof was given in [7], with more modern proofs given in [27], [29], and [16]. $\square$

2.4 Simplicial Groupoids

Throughout this paper we will also encounter simplicial objects in the category of groupoids, called *simplicial groupoids*. Many references use simplicial groupoid to mean simplicially enriched groupoid. In this paper we encounter both objects and will now define them both.

Definition 2.4.1. A *groupoid* is a category in which every morphism is an isomorphism. The category of groupoids, **Grpd**, has as objects groupoids and as morphisms functors between them.

Definition 2.4.2. A *simplicial groupoid* is a simplicial object in the category of groupoids, that is a contravariant functor $\Delta^{op} \to \mathbf{Grpd}$.

If $\mathcal{K}$ a monoidal category, then a $\mathcal{K}$-enriched category is a category whose morphisms form an object of $\mathcal{K}$. Similarly, a $\mathcal{K}$-enriched groupoid is a category whose morphisms for an object of $\mathcal{K}$ and moreover every morphism is an isomorphism. We do not define this notion explicitly, as the enriched categories we are interested in have a simpler equivalent definition, but the curious reader can refer to [24]. For example, a category is a **Set** enriched category and a groupoid is a **Set** enriched groupoid. Now, a *simplicially enriched groupoid* is a groupoid enriched in **sSet** which is equivalent to the following statement that we take as a definition.

Definition 2.4.3. A *simplicially enriched groupoid* is a simplicial groupoid with constant objects. That is, a simplicial groupoid G such that $Ob(G_n) = Ob(G_m)$ for all $m, n \geq 0$.

2.5 Presheaves on a Site

Definition 2.5.1. A *Grothendieck topology* $\mathcal{T}$ on a category $\mathcal{C}$ consists of a collection of subfunctors

$$R \subset hom(-, U), U \in \mathcal{C}$$

called covering sieves, such that the following axioms hold:

- (Base Change) If $R \subset hom(-,U)$ is covering and $\phi : V \to U$ is a morphism of $\mathcal{C}$, then the subfunctor $\phi^{-1}(R) = \{\gamma : W \to V | \phi \circ \gamma \in R\}$ is covering for V.

- (Local Character) Suppose that $R, R' \subset hom(-,U)$ are subfunctors and R is covering. If $\phi^{-1}(R')$ is covering for all $\phi : V \to U$, then R' is covering.

- (Identity) $hom(-,U)$ is covering for all $U \in \mathcal{C}$.

Typically, Grothendieck topologies arise from covering families in sites $\mathcal{C}$ having pullbacks. Covering families are sets of functors that generate covering sieves. More explicitly, suppose that $\mathcal{C}$ has pullbacks, then a Grothendieck pretopology (or basis for Grothendieck topology) $\mathcal{T}$ on $\mathcal{C}$ consists of families of sets of morphisms

$$\{\phi_\alpha : U_\alpha \to U\},\ U \in \mathcal{C}$$

called covering families, such that the following axioms hold:

- Suppose that $\{\phi_\alpha : U_\alpha \to U\}$ is a covering family and that $\psi : V \to U$ is a morphism of $\mathcal{C}$. Then the collection of morphisms $V \times_U U_\alpha \to V$ is a covering family for V.

- If $\{\phi_\alpha : U_\alpha \to U\}$ is a covering family and $\{\sigma_{\alpha,\beta} : W_{\alpha,\beta} \to U_\alpha\}$ is a covering family of U_α for all α. Then the family of composites
$$W_{\alpha,\beta} \xrightarrow{\sigma_{\alpha,\beta}} U_\alpha \xrightarrow{\phi_\alpha} U$$
is a covering family of U.

- The family $\{\mathbf{1} : U \to U\}$ is a covering family of U for all U in $\mathcal{C}$.

Definition 2.5.2. A *Grothendieck site* is a small category $\mathcal{C}$ that is equipped with a Grothendieck topology.

Example 2.5.3. Let X be a topological space. The site $op|_X$ is the category of open subsets $U \subset X$. Its objects are open subsets of X and morphisms are inclusions. A covering family for an open subset U is given by an open cover V_α of U.

Example 2.5.4. Suppose that $\mathcal{C}$ is a site and U is an object of $\mathcal{C}$. The over category $\mathcal{C}/U$ is the category whose objects are morphisms $V \to U$ in $\mathcal{C}$ and whose objects are commutative triangles

$$\begin{tikzcd} V \arrow[dd] \arrow[dr] & \\ & U \\ V' \arrow[ur] & \end{tikzcd}$$

The category $\mathcal{C}/U$ inherits a Grothendieck topology from $\mathcal{C}$, where a collection of maps $V_\alpha \to V \to U$ is a covering family if and only if the family of maps $V_\alpha \to V$ is a covering family of V.

Definition 2.5.5. A presheaf in a category $\mathcal{E}$ on a site $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathcal{E}$

The category of presheaves in a category $\mathcal{E}$ over a site $\mathcal{C}$, denoted $\mathbf{Pre}(\mathcal{C},\mathcal{E})$, has as objects functors $\mathcal{C}^{op} \to \mathcal{E}$ and as morphisms natural transformations between them. When a particular site $\mathcal{C}$ is fixed we will denote the category of presheaves in $\mathcal{E}$ over $\mathcal{C}$ by $\mathbf{Pre}(\mathcal{E})$.

Example 2.5.6. The simplex category Δ is a site, so that **sSet** is the category of presheaves in **Set** on Δ.

$$\mathbf{sSet} = \mathbf{Pre}(\Delta, \mathbf{Set})$$

2.6 Monoidal Categories

Definition 2.6.1. A *monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that is associative up to natural isomorphism and an object I that is both a left and right unit for $\otimes$ up to natural isomorphism.

Example 2.6.2. If A is an abelian group and E is a group with a free A-action. Then the action groupoid $E/\!\!/A$ is a monoidal category where all the required natural isomorphisms are identities.

Definition 2.6.3. A *lax monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories, equipped with a morphism $\eta : I_{\mathcal{D}} \to F(I_{\mathcal{C}})$ and a natural transformation μ between the functors given by the composites:

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

and

$$\mathcal{C} \times \mathcal{C} \xrightarrow{F \times F} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes_{\mathcal{D}}} \mathcal{D}$$

Every monoidal category gives rise to a bicategory. We refer the reader to [5] for the definitions of bicategories, 2-categories, and lax functors.

Definition 2.6.4. The *delooping bicategory* of a monoidal category $\mathcal{M}$ is the bicategory $\Omega^{-1}\mathcal{M}$ that has single object, the objects of $\mathcal{M}$ as morphisms, and the morphisms of $\mathcal{M}$ as deformations.

More precisely, $\Omega^{-1}\mathcal{M}(*,*) = \mathcal{M}$ and horizontal composition of morphisms and deformations is given by the tensor functor, $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. The identity at the object is the unit of the monoidal category, $1_* = I$. Moreover the associativity, left unit, and right unit constraints for $\Omega^{-1}\mathcal{M}$ are those of the monoidal category.

Example 2.6.5. If A is an abelian group and E is a group with a free A-action. Then $\Omega^{-1}(E/\!\!/A)$ is a 2-category because, the associativity, left unit, and right unit constraints are all identities.

2.7 Geometric Nerve

We now show how to obtain a simplicial set from a bicategory via the geometric nerve. Several notions of nerve are available for bicategories. Up to weak equivalence, they all define the same homotopy type. We use the geometric nerve as it directly relates to the $\overline{W}$ bar construction.

Definition 2.7.1. The *geometric nerve* of a bicategory $\mathcal{C}$ is the simplicial set

$$\Delta\mathcal{C} : \Delta^{op} \to \mathbf{Set}$$

defined levelwise to be the set of 2-functors

$$[p] \to \mathbf{LaxFunc}([p], \mathcal{C}),$$

where $[p]$ is regarded as a 2-category with objects $\{0, 1, \cdots, p\}$, exactly one arrow from $j \to i$ if $i \leq j$, and the only 2-morphisms are just identities. Then the simplicial structure maps are induced by, if $\alpha : [n] \to [m]$ in Δ (Note that being a morphism of Δ gives a 2-functor between the associated 2-categories), then $\alpha_* : \Delta\mathcal{C}_m \to \Delta\mathcal{C}_n$ is defined by sending a lax functor $F : [m] \rightsquigarrow \mathcal{C}$ to the composite $F \circ \alpha : [n] \rightsquigarrow \mathcal{C}$.

2.8 Grothendieck Construction for Lax Diagrams of Bicategories

We now would like to define the *Grothendieck Construction* for a lax diagram of bicategories. The Grothendieck construction is useful to “rectify” lax functors.

Definition 2.8.1. Let I be a small category. A *lax I-diagram of bicategories* is a lax functor $I^{op} \to \mathbf{BiCat}$, where $\mathbf{BiCat}$ is the tricategory of bicategories.

Definition 2.8.2. Let $\mathcal{F}, \mathcal{F}'$ be lax I-diagrams of bicategories, where I is a small category. A lax I-homomorphism $F = (F, \theta, \Pi, \Gamma) : \mathcal{F} \to \mathcal{F}'$ is given by:

For each object $i \in I$ there is a homomorphism of bicategories $F_i : \mathcal{F}_i \to \mathcal{F}'_i$. For each morphisms $a : j \to i$ in I a pseudo natural transformation

$$\begin{tikzcd} \mathcal{F}_i \arrow[r, "F_i"] \arrow[d, "a^*"'] & \mathcal{F}'_i \arrow[d, "a^*"] \\ \mathcal{F}_j \arrow[r, "F_j"'] \arrow[ur, phantom, "\overset{\theta}{\nearrow}"] & \mathcal{F}'_j \end{tikzcd}$$

Then for composable arrows there should be invertible modifications satisfying commutative diagrams which can be seen in [5]. We omit these diagrams as our modifications will be identities.

Definition 2.8.3. Let I be a small category. The *Grothendieck construction* on a lax I-diagram of bicategories $\mathcal{F} : I^{op} \to \mathbf{Bicat}$ is the large bicategory $\int_I \mathcal{F}$ defined as follows.

The objects are pairs (x, i), where i is an object of I and x is an object of the bicategory $\mathcal{F}(i)$. In other words:

$$Ob(\int_I \mathcal{F}) = \coprod_{i \in Ob(I)} Ob(\mathcal{F}(i))$$

The hom categories are then defined as

$$hom_{\int_I \mathcal{F}}((y, j), (x, i)) = \coprod_{j \xrightarrow{a} i} \mathcal{F}(j)(y, a^* x),$$

where the disjoint union is over all arrow $j \xrightarrow{a} i$ in I. This means that a 1-morphism $(u, a) : (y, j) \to (x, i)$ is given by a pair of morphisms, where $a : j \to i$ is a morphism of I and $u : y \to a^* x$ is a 1-morphism of $\mathcal{F}(j)$. For a pair of morphisms $(u, a), (u', a') : (y, j) \to (x, i)$, a 2-cell between (u, a) and (u', a') requires $a = a'$ and it is given by a 2-cell $\alpha : u \implies v$ in $\mathcal{F}(j)$.

For explicit definition of horizontal composition see [5].

Theorem 2.8.4. *[5] The Grothendieck construction carries a lax I-homomorphism $F : \mathcal{F} \to \mathcal{F}'$ to a morphism of bicategories $\int_I F : \int_I \mathcal{F} \to \int_I \mathcal{F}'$.*

CHAPTER 3

HOMOTOPY THEORY BACKGROUND

3.1 Model Categories

Homotopy theory in algebraic topology has proven to be a very useful theory. It is natural to desire a generalization that applies to other fields. To generalize this to other categories one should of course look at the category of topological spaces, **Top**. Using classical homotopy theory two spaces that are homotopy equivalent are thought of as being isomorphic, but they are not isomorphic in **Top**. Thus for a homotopy theory one would like a category, which one would call the *homotopy category*, in which these spaces are isomorphic. An easy way to achieve this would be to localize **Top** with respect to the class of homotopy equivalences. That is, to formally add in inverses for every homotopy equivalence in order to change all homotopy equivalences to isomorphisms. While such a construction is simple it lacks tools to do computations within the resulting category and is difficult to characterize. One of the main concerns is that after localizing the resulting category may no longer be locally small, which is necessary for most considerations. One would also like to consider weaker notions of homotopy such as weak homotopy equivalences. It is then natural to ask what is the additional structure needed on a category to provide such tools and a well defined homotopy category. Quillen's *model categories*, originally formulated in [22], are categories with the necessary additional structure to define a homotopy theory with respect to a chosen class of weak equivalences and a well defined homotopy category. In fact, the homotopy category constructed using model categories is equivalent to the homotopy category obtained by localizing with respect to the class of weak equivalences while also being well equipped for calculations. Thus model categories give the best of both worlds, where their cost is in showing their existence.

Definition 3.1.1. Let $\mathcal{C}$ be a category. A *model structure* on $\mathcal{C}$ is given by three distinguished classes of maps called *fibrations*, *cofibrations*, and *weak equivalences* — subject to the following axioms:

1. The category $\mathcal{C}$ is closed under limits and colimits.

2. Each of the three distinguished classes of maps are closed under retracts.

3. (2 out of 3) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that any two of f, g, or $g \circ f$ are weak equivalences, then so is the third.

4. (Lifting Axiom) Every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where j is a cofibration and q is a fibration has a solution so that both diagrams commute if either j or q is also a weak equivalence. We call a morphism that is both a fibration and a weak equivalence an *acyclic fibration* and similarly a morphism that is both a cofibration and a weak equivalence an *acyclic cofibration*. If there is a solution to the lifting problem we say that j has the *left lifting property* with respect to q and that q has the *right lifting property* with respect to j.

5. Any morphism $X \xrightarrow{f} Y$ can be functorially factored two ways:

   (a) $X \xrightarrow{i} Z \xrightarrow{q} Y$, where i is a cofibration and q is a weak equivalence as well as a fibration.

   (b) $X \xrightarrow{j} Z \xrightarrow{p} Y$, where j is a weak equivalence as well as a cofibration and p is a fibration.

Example 3.1.2. The prototypical example of a model category is the *category of topological spaces*, **Top**, equipped with the *classical model structure*. Its objects are topological spaces and morphisms are continuous maps. The model structure is given by the following:

1. Weak equivalences are the *weak homotopy equivalences*, continuous maps that induce isomorphisms on all homotopy groups.

2. Fibrations are the *Serre fibrations*, continuous maps that have the right lifting property with respect to all inclusions, $D^n \xhookrightarrow{i_0} D^n \times I$ that include the n-disk as $D^n \times \{0\}$.

3. Cofibrations are the continuous maps that have the left lifting property with respect to the continuous maps that are both weak homotopy equivalences and Serre fibrations.

Example 3.1.3. The category of simplicial sets, **sSet**, is a model category with model structure given by the following:

1. Weak equivalences are the *weak homotopy equivalences*, simplicial maps whose geometric realization is a weak homotopy equivalence of topological spaces.

2. Cofibrations are simplicial maps $X \xrightarrow{f} Y$ such that levelwise $X_n \xrightarrow{f_n} Y_n$ are injections for all n.

3. Fibrations are Kan fibrations. A *Kan fibration* is a morphism of simplicial sets that has the left lifting property with respect to all horn inclusions $\Lambda^k[n] \longrightarrow \Delta^n$ (definition 2.2.2). One can show that these are the morphisms that have the right lifting property with respect to those morphisms that are weak homotopy equivalences and levelwise injections [11].

Theorem 3.1.4. *[12] Let $\mathcal{M}$ be a model category. Then*

- *The class of cofibrations is closed under pushout.*
- *The class of morphisms that are cofibrations and weak equivalences is closed under pushout.*
- *The class of fibrations is closed under pullback.*
- *The class of morphisms that are fibrations and weak equivalences is closed under pullback.*

Definition 3.1.5. A model category $\mathcal{M}$ is *right proper* if weak equivalences are preserved under pullbacks along fibrations. That is, if for every weak equivalence $A \xrightarrow{g} Y$ and every fibration $X \xrightarrow{f} Y$ the morphism $X \times_Y A \longrightarrow X$ in the pullback diagram

$$\begin{tikzcd} X \times_Y A \arrow[r] \arrow[d] & A \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a weak equivalence.

Example 3.1.6. For example the model structures above for **Top** and **sSet** are right proper.

Every model structure discussed in this paper will be right proper. However, there are useful model structures that are not right proper. The following example from [23] is used in rational homotopy theory and is not right proper:

Example 3.1.7. Consider the alternative model structure on **sSet** given by:

- Weak Equivalences are *rational homology isomorphisms*. Those morphisms of simplicial sets $f : X \to Y$ that induce isomorphisms $H_n(X, \mathbb{Q}) \cong H_n(Y, \mathbb{Q})$ for all $n \geq 0$.

- Cofibrations are those morphisms of simplicial sets that are levelwise injections.
- Fibrations are called *rational fibrations* and are the simplicial maps that have the right lifting property with respect to simplicial maps that are weak homotopy equivalences and levelwise injections.

First, let $K(A,n)$ denote the Eilenberg-MacLane space, that is, a simplicial set such that $\pi_i(K(A,n)) \cong 0$ for all $i \neq n$ and $\pi_n(K(A,n)) \cong A$. Form the pullback square

$$\begin{tikzcd} K(\mathbb{Q}/\mathbb{Z},0) \arrow[r, "g"] \arrow[d] & C \arrow[d, "p"] \\ K(\mathbb{Z},1) \arrow[r, "f"] & K(\mathbb{Q},1) \end{tikzcd}$$

in which p is a rational fibration from a contractible space C and f is a rational homology isomorphism. However, g is not a rational homology isomorphism, showing that the structure is not right proper.

3.2 Homotopies

We will now define homotopy in a model category. To do this we need to generalize objects used in the homotopy theory of topological spaces, namely *path objects* and *cylinder objects*. Once these are defined, the notions of *left homotopy* and *right homotopy* are natural. This gives a nice geometric interpretation that allows one to do explicit calculations, which is one of the main advantages of using model categories.

Definition 3.2.1. An object X of a model category $\mathcal{M}$ is *fibrant* if the unique morphism to the terminal object $X \to *$ is a fibration. It is called *cofibrant* if the unique morphism from the initial object $\emptyset \to X$ is a cofibration.

Definition 3.2.2. Let $\mathcal{M}$ by a model category and X be an object of $\mathcal{M}$. The *diagonal map* is the morphism $diag : X \to X \prod X$ in the commutative diagram:

$$\begin{tikzcd} & & X \\ X \arrow[r, "diag"] \arrow[urr, bend left, "\mathbf{1}"] \arrow[drr, bend right, "\mathbf{1}"'] & X \prod X \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & X \end{tikzcd}$$

given by the universal property of products. This morphism always exists in a model category as a model category is closed under limits.

Definition 3.2.3. Let $\mathcal{M}$ be a model category and X be an object of $\mathcal{M}$. A *path object* of X, denoted $P(X)$, is an object of $\mathcal{M}$ fitting into the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "diag"'] & P(X) \arrow[r, "p"] & X \prod X \end{tikzcd}$$

where i is a weak equivalence.

A path object is not unique and does not have to literally be a path space in the category, but the path space in **Top** is a path object.

Example 3.2.4. Let **Top** have the classical model structure and X be a topological space. The path space, X^I, fits into the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "diag"'] & X^I \arrow[r, "p"] & X \prod X \end{tikzcd}$$

where i is the continuous function that sends each point of X to the constant path at that point and p is the continuous function that sends each path to its endpoints. This clearly factors the diagonal and i is a weak homotopy equivalence, so X^I is a path object in the classical model structure on **Top**.

Definition 3.2.5. Let $\mathcal{M}$ be a model category and X be an object of $\mathcal{M}$. The *folding map* is the morphism $fold : X \coprod X \to X$ in the commutative diagram:

$$\begin{tikzcd} X \arrow[dr, "i_1"] \arrow[drr, bend left, "\mathbf{1}"] & & \\ & X \coprod X \arrow[r, "fold"] & X \\ X \arrow[ur, "i_2"] \arrow[urr, bend right, "\mathbf{1}"'] & & \end{tikzcd}$$

given by the universal property of coproduct. Similarly to the diagonal map this morphism exists in any model category as a model category is closed under colimits.

Definition 3.2.6. Let $\mathcal{M}$ be a model category and X be an object of $\mathcal{M}$. A *cylinder object* of X, denoted $Cyl(X)$, is an object of $\mathcal{M}$ fitting into the commutative diagram:

$$\begin{tikzcd} X \coprod X \arrow[r, "i"] \arrow[rr, bend right, "fold"'] & Cyl(X) \arrow[r, "p"] & X \end{tikzcd}$$

where p is a weak equivalence.

Similar to path space objects, cylinder objects are not unique and do not have to be a cylinder in the category. However, the cylinder in **Top** is a cylinder object in the classical model structure.

Example 3.2.7. Let **Top** have the classical model structure and X be a topological space. Then the cylinder, $X \times I$, fits into the commutative diagram:

$$\begin{tikzcd} X \coprod X \arrow[r, "i"] \arrow[rr, bend right, "fold"'] & X \times I \arrow[r, "p"] & X \end{tikzcd}$$

where i sends one X in the disjoint union to $X \times 0$ and the other to $X \times 1$. The map p is the projection of $X \times I \to X$ which is a weak homotopy equivalence and clearly this factors the folding map, so $X \times I$ is a cylinder object in the classical model structure on **Top**.

With path and cylinder objects in hand it is natural to define left and right homotopies.

Definition 3.2.8. Let $\mathcal{M}$ be a model category and $f, g \in hom_{\mathcal{M}}(X, Y)$. The morphisms f and g are *right homotopic*, denoted $f \simeq_r g$, if for some path object $P(Y)$ of Y, there exists a morphism $h : X \to P(Y)$ of $\mathcal{M}$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ Y & P(Y) \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

Definition 3.2.9. Let $\mathcal{M}$ be a model category and $f, g \in hom_{\mathcal{M}}(X, Y)$. The morphisms f and g are *left homotopic*, denoted $f \simeq_l g$, if for some cylinder object $Cyl(X)$ of X, there exists a morphism $h : Cyl(X) \to Y$ of $\mathcal{M}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "f"'] & Cyl(X) \arrow[d, "h"] & X \arrow[l, "i_2"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

Definition 3.2.10. Let $\mathcal{M}$ be a model category and $f, g \in hom_{\mathcal{M}}(X, Y)$. The morphisms f and g are *homotopic*, denoted $f \simeq g$, if $f \simeq_r g$ and $f \simeq_l g$.

Definition 3.2.11. Let $\mathcal{M}$ be a model category and $f \in hom_{\mathcal{M}}(X, Y)$, then f is a *homotopy equivalence* if there exists a $g \in hom_{\mathcal{M}}(Y, X)$ such that $f \circ g \simeq \mathbf{1}_Y$ and $g \circ f \simeq \mathbf{1}_X$.

Now that we have homotopies defined, we would like for them to give an equivalence relation. Unfortunately, in general this is not an equivalence relation, so we need the following replacements.

Definition 3.2.12. Let X be an object of a model category $\mathcal{M}$. A *cofibrant replacement* of X is a cofibrant object X_c fitting in the diagram

$$\emptyset \xrightarrow{i} X_c \xrightarrow{q} X \ ,$$

where i is a cofibration and q is an acyclic fibration. This is guaranteed to exist by the factorization axiom. Similarly, a *fibrant replacement* of X is a fibrant object X_f fitting in the diagram

$$X \xrightarrow{j} X_f \xrightarrow{p} * \ ,$$

where j is an acyclic cofibration and p is a fibration. Again, this is guaranteed to exist by the factorization axiom.

Now, since these factorizations are functorial, they give rise to functors $Q : \mathcal{M} \to \mathcal{M}_c$ and $R : \mathcal{M} \to \mathcal{M}_f$, where $\mathcal{M}_c$ is the full subcategory of cofibrant objects, and $\mathcal{M}_f$ is the full subcategory of fibrant objects. Combining these operations gives a functor $QR : \mathcal{M} \to \mathcal{M}_{cf}$, where $\mathcal{M}_{cf}$ is the full subcategory of fibrant-cofibrant objects.

Theorem 3.2.13. *[12] Let $\mathcal{M}$ be a model category with X a cofibrant object of $\mathcal{M}$ and Y a fibrant object of $\mathcal{M}$. Then $\simeq, \simeq_r, \simeq_l$ are all equivalence relations on $hom_{\mathcal{M}}(X, Y)$. (In fact $\simeq_l$ is an equivalence relation if only X is cofibrant and $\simeq_r$ is an equivalence relation if only Y is fibrant.) Moreover there are bijections of sets*

$$\begin{tikzcd}
hom_{\mathcal{M}}(X,Y)/\simeq_r \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & hom_{\mathcal{M}}(X,Y)/\simeq_l \arrow[dl, "\cong"] \\
& hom_{\mathcal{M}}(X,Y)/\simeq &
\end{tikzcd}$$

Now with all of this machinery we have a concrete way to define the homotopy category. The homotopy category should have the same objects as $\mathcal{M}$ and the morphisms should by homotopy classes of morphisms. As homotopy is not an equivalence relation for any hom-set, we must take the homotopy class of maps between fibrant-cofibrant replacements.

Definition 3.2.14. Let $\mathcal{M}$ be a model category, then the *homotopy category*, $Ho(\mathcal{M})$, is the category that has the same objects as $\mathcal{M}$ and whose set of morphisms are defined as

$$hom_{Ho(\mathcal{M})}(X, Y) := hom_{\mathcal{M}}(QRX, QRY)/\simeq$$

for any pair of objects $X, Y \in \mathcal{M}$.

We now will formally define the localization of a category and reference the results showing that the homotopy category constructed is equivalent to the category obtained by localizing with respect to the class of weak equivalences.

Definition 3.2.15. Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a class of morphisms of $\mathcal{C}$. A *localization* of $\mathcal{C}$ with respect to $\mathcal{W}$ is the datum of a (large) category $\mathcal{W}^{-1}\mathcal{C}$ and a functor $L : \mathcal{C} \to \mathcal{W}^{-1}\mathcal{C}$ satisfying:

- $L(w)$ is an isomorphism for every $w \in \mathcal{W}$.
- For any (large) category $\mathcal{D}$ and any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(w)$ is an isomorphism for every $w \in \mathcal{W}$, there exists a functor $U : \mathcal{W}^{-1}\mathcal{C} \to \mathcal{D}$ such that the diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "L"] & \mathcal{D} \\ \mathcal{W}^{-1}\mathcal{C} \arrow[ur, "U"] & \end{tikzcd}$$

  commutes up to natural isomorphism.
- If U, V are two objects of the category of functors from $\mathcal{W}^{-1}\mathcal{C}$ to $\mathcal{D}$, denoted $\mathcal{D}^{\mathcal{W}^{-1}\mathcal{C}}$, then the natural map

$$hom_{\mathcal{D}^{\mathcal{W}^{-1}\mathcal{C}}}(U, V) \to hom_{\mathcal{D}^{\mathcal{C}}}(U \circ L, V \circ L)$$

  is a bijection.

Thus the homotopy category from this point of view would be a model category localized at its class of weak equivalences. If model categories are to be useful then these two notions of homotopy category should be equivalent.

Theorem 3.2.16. *[13] Let $\mathcal{M}$ be a model category with X a cofibrant object and Y a fibrant object, then there is a natural isomorphism*

$$hom_{\mathcal{W}^{-1}\mathcal{M}}(X,Y) \cong hom_{\mathcal{M}}(X,Y)/\simeq$$

Theorem 3.2.17. *[12] Let $\mathcal{M}$ be a model category, then the natural functor $i_{\mathcal{M}} : \mathcal{M} \to Ho(\mathcal{M})$ is a localization of $\mathcal{M}$ with respect to the class of weak equivalences.*

The universality of localizations then gives the desired equivalence between the two constructions.

Corollary 3.2.18. *Let $\mathcal{M}$ be a model category and $\mathcal{W}$ be its class of weak equivalences, then there is an equivalence of categories*

$$Ho(\mathcal{M}) \cong \mathcal{W}^{-1}\mathcal{M}$$

So, the localization of a model category is a locally small category. Since both notions of homotopy category are equivalent we will now use $Ho(\mathcal{M})$ to denote either of the constructions. Also, if X, Y are objects in a model category $\mathcal{M}$, then $[X,Y]_{\mathcal{M}}$ denotes the set of morphisms from X to Y in $Ho(\mathcal{M})$.

3.3 Quillen Functors

One would like an appropriate notion of morphism between two model categories. Such a morphism should be a functor between the model categories that should preserve the model structures in some way. In particular it should preserve path objects, cylinder objects, and homotopies. Two model categories should be "equivalent" if their homotopy categories are equivalent. We will discuss *derived functors* as these will be the induced functors on the homotopy category and then we will define an appropriate morphism between model categories called a *Quillen functor*.

Definition 3.3.1. Let $\mathcal{M}$ be a model category and $F : \mathcal{M} \to \mathcal{C}$ be a functor.

- A *left derived functor* of F is a pair (LF, l), where LF is a functor and l is a natural transformation such that

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "i_{\mathcal{M}}"'] & \arrow[d, phantom, "l\Uparrow"] & \mathcal{C} \\ & Ho(\mathcal{M}) \arrow[ur, "LF"'] & \end{tikzcd}$$

commutes, and for any other such pair (F', l') there exists a natural transformation $\alpha : F' \to LF$ such that $l \circ (\alpha \circ \mathbf{1}_{i_\mathcal{M}}) = l'$.

- A *right derived functor* of F is a pair (RF, r), where RF is a functor and r is a natural transformation such that

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "i_\mathcal{M}"'] & \arrow[d, phantom, "\Downarrow r"] & \mathcal{C} \\ & Ho(\mathcal{M}) \arrow[ur, "RF"'] & \end{tikzcd}$$

  commutes, and for any other such pair (F', r') there exists a natural transformation $\beta : RF \to F'$ such that $(\beta \circ \mathbf{1}_{i_\mathcal{M}}) \circ r = r'$.

The universal properties imply that the left and right derived functors are unique up to unique isomorphism. We will now refer to them as "the" left and right derived functors.

The derived functors give us a map from the homotopy category, but if $\mathcal{C}$ is a model category, one would like a morphism between their homotopy categories. This brings us to the *total derived functors.*

Definition 3.3.2. Let $\mathcal{M}, \mathcal{N}$ be model categories and $F : \mathcal{M} \to \mathcal{N}$ be a functor.

- The *total left derived functor*, $(\mathbb{L}F, l)$, is the left derived functor of the composite $i_\mathcal{N} \circ F : \mathcal{M} \to Ho(\mathcal{N})$.
- The *total right derived functor*, $(\mathbb{R}F, r)$, is the right derived functor of the composite $i_\mathcal{N} \circ F : \mathcal{M} \to Ho(\mathcal{N})$.

The natural question is whether derived functors exist:

Theorem 3.3.3. *[12] Let $\mathcal{M}$ be a model category and $F : \mathcal{M} \to \mathcal{C}$ be a functor of categories.*

- *If F maps acyclic cofibrations between cofibrant objects to isomorphisms, then the left derived functor (LF, l) exists. Moreover, if X is cofibrant, then l_X is an isomorphism.*
- *If F maps acyclic fibrations between fibrant objects to isomorphisms, then the right derived functor (RF, r) exists. Moreover, if X is fibrant, then r_X is an isomorphism.*

Corollary 3.3.4. *[12] Let $\mathcal{M}, \mathcal{N}$ be a model categories and $F : \mathcal{M} \to \mathcal{C}$ be a functor.*

- *If $i_\mathcal{M} \circ F$ maps acyclic cofibrations between cofibrant objects to isomorphisms, then the total left derived functor $(\mathbb{L}F, l)$ exists.*

- *If $i_{\mathcal{M}} \circ F$ maps acyclic fibrations between fibrant objects to isomorphisms, then the total right derived functor $(\mathbb{R}F, r)$ exists.*

Now that we know when the derived functors exist we can finally define a morphism between model categories. First we must recall the definition of an adjunction.

Definition 3.3.5. An *adjunction* between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $R : \mathcal{D} \to \mathcal{C}$ and $L : \mathcal{C} \to \mathcal{D}$ such that there is a natural isomorphism of the *hom*-functors

$$hom_{\mathcal{D}}(L(-), -) \cong hom_{\mathcal{C}}(-, R(-))$$

That is, for all objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ there is a natural bijection between the *hom*-sets

$$hom_{\mathcal{D}}(L(c), d) \cong hom_{\mathcal{C}}(c, R(d))$$

(L, R) are called an *adjoint pair* with L called the *left adjoint* and R called the *right adjoint*.

Definition 3.3.6. Let $\mathcal{M}, \mathcal{N}$ be model categories and $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ be an adjunction, then L is a *left Quillen functor* if it preserves cofibrations and weak equivalences between cofibrant objects. R is a *right Quillen functor* if it preserves fibrations and weak equivalences between fibrant objects. If L is a left Quillen functor and R is a right Quillen functor, then the adjunction is called a *Quillen adjunction*.

From corollary 3.3.4 total derived functors of L and R exist, so that there are induced functors between their homotopy categories. Moreover, one can refer to [12] to see that they preserve path objects, cylinder objects, and homotopies. Now that we have morphisms we can define when they give an "equivalence".

Definition 3.3.7. Let $\mathcal{M}, \mathcal{N}$ be model categories and $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ be a Quillen adjunction. L is a *left Quillen equivalence* and R is a *right Quillen equivalence* if, for every cofibrant object X of $\mathcal{M}$ and every fibrant object Y of $\mathcal{N}$, $L(X) \to Y$ is a weak equivalence if and only if its adjoint morphism $X \to R(Y)$ is a weak equivalence. The entire adjunction is called a *Quillen equivalence*.

For this to be an "equivalence" we should have that they induce an equivalence between homotopy categories.

Theorem 3.3.8. *[12] Let $\mathcal{M}, \mathcal{N}$ be model categories and $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ be a Quillen equivalence, then the induced adjoint pair or total derived functors*

$$\mathbb{L}L : Ho(\mathcal{M}) \leftrightarrows Ho(\mathcal{N}) : \mathbb{R}R$$

is an equivalence of categories.

3.4 Cocycle Categories

In this section we will cover some of the necessary background from Jardine's work [15]. Jardine proves that path components of a suitably defined *cocycle category* correspond to homotopy classes of maps. We will give an explicit example of this in the case of topological principal bundles. An advantage of this correspondence is that it allows you to represent an element of the homotopy category as a single roof, as opposed to a long zig-zag of morphisms.

Definition 3.4.1. A *cocycle* from X to Y in a model category $\mathcal{M}$ is pair of morphisms of $\mathcal{M}$

$$\begin{tikzcd} & Z \arrow[dl, "f"', "\sim"] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

where f is a weak equivalence.

Definition 3.4.2. The *category of cocycles* from X to Y in a model category $\mathcal{M}$, $H(X,Y)_{\mathcal{M}}$, is the category whose objects are the cocycles, and whose morphisms are commutative diagrams

$$\begin{tikzcd} & Z \arrow[dl, "f"', "\sim"] \arrow[dr, "g"] \arrow[dd, "\alpha", "\sim"'] & \\ X & & Y \\ & Z' \arrow[ul, "f'", "\sim"'] \arrow[ur, "g'"'] & \end{tikzcd}$$

Notice that by the 2 out of 3 property α is necessarily a weak equivalence.

Definition 3.4.3. Let $\mathcal{C}$ be a category and $\mathcal{K}$ be its class of morphisms. The class $\pi_0(\mathcal{C})$ of path components of $\mathcal{C}$ is obtained by declaring two objects x, y of $\mathcal{C}$ to be equivalent if there exists a zig-zag of morphisms of $\mathcal{C}$ of the form:

$$x \leftarrow x_1 \rightarrow x_2 \leftarrow \cdots \rightarrow x_n \leftarrow y$$

Equivalently, it is the equivalence class of objects of $\mathcal{C}$ where two objects are identified if there is a morphism between them in $\mathcal{K}^{-1}\mathcal{C}$.

Theorem 3.4.4. *(Jardine) [15] If $\mathcal{M}$ is a right proper model category for which the class of weak equivalences is closed under finite products, then there is a bijection*

$$\pi_0 H(X,Y) \cong [X,Y]_{\mathcal{M}} \text{ for all objects } X, Y \in \mathcal{M}$$

Proof. The bijection is given by sending the path component of a cocycle $X \xleftarrow{f} Z \xrightarrow{g} Y$ to the morphisms $g \circ f^{-1} \in [X,Y]_{\mathcal{M}}$. □

Remark 3.4.5. All model structures considered in this thesis are both right proper and closed under finite products.

Example 3.4.6. An example to motivate the use of such cocycles is given by principal fiber bundles in Topology. Let G be a topological group, then a *G-principal fiber bundle* is a continuous locally trivial fibration $\pi : P \to X$ equipped with a free and transitive continuous G action on P, $\rho : G \times P \to P$. (An equivalent condition is to say that the canonical map $G \times P \to P \times_X P$ defined by $(g,u) \mapsto (\rho(g,u), u)$ is an isomorphism.)

Consider a cover of X by open sets U such that there exists G-equivariant homeomorphisms $\phi_U : p^{-1}(U) \to U \times G$ making the following diagram commute

$$\begin{tikzcd} p^{-1}U \arrow[r, "\phi_U"] \arrow[d, "p"] & U \times G \arrow[dl, "\pi_U"] \\ U & \end{tikzcd}$$

where $U \times G$ has the right G-action $(u,g)h = (u,gh)$. This data corresponds to a choice of a cross section $\sigma_U : U \to p^{-1}U$, namely a continuous function such that $p \circ \sigma = \mathbf{1}$.

Any two sections σ_U, σ_V are uniquely related by

$$\sigma_{_U} = g_{_{UV}} \sigma_{_V}, \qquad g_{_{UV}} : U \cap V \to G$$

which results in the well known cocycle condition:

$$g_{_{UV}} g_{_{VW}} = g_{_{UW}}, \qquad \text{over } U \cap V \cap W$$

This gives a morphism between topological categories

$$\mathcal{U}_X \xrightarrow{p} G,$$

where G is a topological group thought of as a topological category with single object and $\mathcal{U}_X$ is the topological category whose objects are the open sets U of the open cover and a morphism $U \to V$ is given by $U \cap V$. p is the morphism that sends an object U of $\mathcal{U}_X$ to a point and a morphism $U \cap V$ of $\mathcal{U}_X$ to g_{UV}. After applying the nerve and geometric realization we obtain the cocycle [15]

$$X \xleftarrow{\simeq} |N\mathcal{U}_X| \xrightarrow{|N(p)|} |NG|$$

where $|NG| = BG$ is the classifying space of G.

CHAPTER 4

CENTRAL EXTENSIONS OF SIMPLICIAL GROUPS

The goal of this chapter is to define central extensions of simplicial groups and show that using an appropriate notion of equivalence, equivalence classes of these extensions correspond to homotopy classes of morphisms between classifying spaces. We start with some background on the model structures involved and we also recall the classical correspondence for isomorphism classes of central extensions of (ordinary) groups. We then discuss how to extend the notion of group to other categories and use this to define central extensions of simplicial groups. Then we construct two functors

$$\overline{F} : \mathbf{C}(\mathbf{G}, \mathbf{A})_{\mathbf{sGrp}} \to H(G, \overline{W}A)_{\mathbf{sGrp}}$$

and

$$F : H(G, \overline{W}A)_{\mathbf{sGrp}} \to \mathbf{C}(\mathbf{G}, \mathbf{A})_{\mathbf{sGrp}}$$

that we prove induce a bijection at the level of path components, which gives the desired correspondence as a corollary.

4.1 Model Structure for Simplicial Groups and Reduced Simplicial Sets

We will now present some model structures used throughout this paper as well as some results pertaining them. We will also define the $\overline{W}$ construction which is our choice of classifying space. This construction defines a functor that is part of a Quillen equivalence between the model categories of simplicial groups and that of reduced simplicial sets. Finally, we prove that the notion of homotopy between reduced simplicial sets in both the model structure for reduced simplicial sets and the model structure for simplicial sets coincide. We do this in order to be able to say that for two reduced simplicial sets X, Y there is an isomorphism $[X, Y]_{\mathbf{sSet}_0} \cong [X, Y]_{\mathbf{sSet}}$.

Definition 4.1.1. The category of simplicial groups, **sGrp**, has a model structure where

- Weak Equivalences are morphisms of simplicial groups such that the underlying morphism of simplicial sets is a weak equivalence.
- Fibrations are morphisms of simplicial groups such that the underlying morphism of simplicial sets is a fibration.
- Cofibrations are morphisms of simplicial groups that have the left lifting property with respect to all morphisms that are weak equivalences and fibrations.

Remark 4.1.2. There are many more characterizations of these classes of morphisms in **sGrp** which can be found in [22]. Every simplicial group is fibrant in this structure, as simplicial groups viewed as a simplicial set by forgetting the group structure are fibrant as simplicial sets [20].

Lemma 4.1.3. *The model structure of definition 4.1.1 on simplicial groups is right proper.*

Proof. Every object is fibrant by the above remark and every pullback of a weak equivalence of fibrant objects along a fibration is again a weak equivalence (see proposition 13.1.2 [12]). □

Definition 4.1.4. The category of *reduced simplicial sets*, $\mathbf{sSet}_0$, has as objects those simplicial sets with single vertex $X_0 = *$. The morphisms are just the morphisms of simplicial sets.

Definition 4.1.5. The category of reduced simplicial sets, $\mathbf{sSet}_0$, has a model structure where

- Weak equivalences are those that are weak equivalences of simplicial sets.
- Cofibrations are those that are cofibrations of simplicial sets.
- Fibrations are the morphisms that have the left lifting property with respect to all maps that are weak equivalences and cofibrations.

These two model categories are related via the $\overline{W}$ construction [11], which we will now define.

Definition 4.1.6. The functor $W : \mathbf{sGrp} \to \mathbf{sSet}$ is defined by

$$(WG)_n = G_n \times G_{n-1} \times \cdots \times G_0$$

with face and degeneracy maps defined by:

$$\begin{aligned} d_i(g_n, \cdots, g_0) &= (d_i g_n, d_{i-1} g_{n-1}, \cdots, (d_0 g_{n-i}) g_{n-i-1}, g_{n-i-2}, \cdots, g_0) && i < n \\ d_n(g_n, \cdots, g_0) &= (d_n g_n, d_{n-1} g_{n-1}, \cdots, d_1 g_1) && i = n \\ s_i(g_n, \cdots, g_0) &= (s_i g_n, s_{i-1} g_{n-1}, \cdots, s_0 g_{n-i}, 1, g_{n-i-1}, \cdots, g_0) && i \leq n+1 \end{aligned}$$

Notice that WG has a free G action $G \times WG \to WG$ defined levelwise by:

$$(h, (g_n, \cdots, g_0)) = (hg_n, g_{n-1}, \cdots, g_0)$$

Definition 4.1.7. $\overline{W}G$ is defined to be WG/G.

Remark 4.1.8. An equivalent characterization is the following, if G is a simplicial group, then $\overline{W}G$ is the reduced simplicial set defined levelwise by $(\overline{W}G)_n = G_{n-1} \times \cdots \times G_0$ for $n > 0$ and $(\overline{W}G)_0 = *$. The face and degeneracy maps are:

$$\begin{aligned}
d_0(g_{n-1}, \cdots, g_0) &= (g_{n-2}, \cdots, g_0) && i = 0 \\
d_i(g_{n-1}, \cdots, g_0) &= (d_{i-1}g_{n-1}, \cdots, (d_0 g_{n-i})g_{n-i-1}, g_{n-i-2}, \cdots, g_0) && 0 < i < n \\
d_n(g_{n-1}, \cdots, g_0) &= (d_{n-1}g_{n-1}, \cdots, d_1 g_1) && i = n \\
s_i(g_{n-1}, \cdots, g_0) &= (s_{i-1}g_{n-1}, \cdots, s_0 g_{n-i}, 1, g_{n-i-1}, \cdots, g_0) && i \leq n+1
\end{aligned}$$

One important fact is that WG is contractible and the quotient map $WG \to \overline{W}G$ is a fibration.

Lemma 4.1.9. *$\overline{W}$ is the right adjoint in an adjoint pair of functors*

$$G : \boldsymbol{sSet}_0 \leftrightarrows \boldsymbol{sGrp} : \overline{W}$$

where G is Kan's loop group functor [17]. Moreover these define a Quillen equivalence (see [11] Corollary 6.4).

$\overline{W}$ may seem mysterious at first glance, but it is a choice of classifying space and in fact is the classical Eilenberg-MacLane construction when applied to a constant simplicial group. Namely, if G is a constant simplicial group we have $\pi_i(\overline{W}G) = 1$ for $i \neq 1$ and $\pi_1(\overline{W}G) \cong G$. Thus its geometric realization is an Eilenberg-MacLane space

$$|\overline{W}G| \cong K(G, 1).$$

In order to be a classifying space we should have that $\overline{W}$ classifies principal fibrations, which we now define.

Definition 4.1.10. A *principal G-fibration* over B is a G-equivariant fibration $f : E \to B$ such that

- E has a free G-action
- B has a trivial G-action
- The induced map $E/G \to B$ is an isomorphism.

Two principal fibrations $f_1 : E_1 \to B$ and $f_2 : E_2 \to B$ are *isomorphic* if there is an equivariant isomorphism $g : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & E_2 \arrow[dl, "f_2"] \\ & B & \end{tikzcd}$$

Let $PF_G(B)$ denote the isomorphism class of principal G-fibrations over B. Then the following theorem shows that $\overline{W}$ is a classifying space.

Theorem 4.1.11. *[11] For all simplicial sets X, the morphism*

$$\Phi : [X, \overline{W}G] \to PF_G(B)$$

sending $[f] \in [X, \overline{W}G]$ to the pullback of $WG \to \overline{W}G$ along f is a bijection.

Notice $\mathbf{sSet}_0$ is a full subcategory of $\mathbf{sSet}$, so that if $f, g \in hom_{\mathbf{sSet}_0}(X, Y) = hom_{\mathbf{sSet}}(X, Y)$, we can talk about their being homotopic in $\mathbf{sSet}_0$ as well as in $\mathbf{sSet}$. The next lemma shows that these notions of homotopy coincide.

Lemma 4.1.12. *Let X, Y be reduced simplicial sets with X cofibrant and Y fibrant in $\mathbf{sSet}_0$ and let $f, g : X \to Y$ be two simplicial morphisms between them. Then $f \simeq g$ in $\mathbf{sSet}$ if and only if $f \simeq g$ in $\mathbf{sSet}_0$.*

Proof. First notice X is cofibrant in $\mathbf{sSet}$ as all objects are cofibrant. Also Y is fibrant in $\mathbf{sSet}_0$ so that it is fibrant in $\mathbf{sSet}$ ([11] Corollary 6.8).

For the forward direction, let h be a homotopy from f to g in $\mathbf{sSet}$ so that we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "f"'] & Cyl(X) \arrow[d, "h"] & X \arrow[l, "i_2"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

where $Cyl(X)$ is a cylinder object of **sSet**. Now let $c_1 : Cyl(X) \to Cyl(X)_0$ be the simplicial morphism that identifies all 0-cells of $Cyl(X)$. Also let $c_2 : X \coprod X \to X \coprod_0 X$ be the simplicial morphism that identifies the two 0-cells of $X \coprod X$. Notice that $X \coprod_0 X$ is the coproduct in $\mathbf{sSet}_0$ and there is the following commutative diagram:

$$\begin{tikzcd}
X \coprod X \arrow[r, "i"] \arrow[d, "c_1"] \arrow[rr, bend left, "fold"] & Cyl(X) \arrow[r, "p"] \arrow[d, "c_2"] & X \arrow[d, equal] \\
X \coprod_0 X \arrow[r, "\tilde{i}"] \arrow[rr, bend right, "fold_0"'] & Cyl(X)_0 \arrow[r, "\tilde{p}"] & X
\end{tikzcd}$$

so $Cyl(X)_0$ is a cylinder object in $\mathbf{sSet}_0$ moreover the homotopy h factors through the reduced cylinder

$$\begin{tikzcd}
Cyl(X) \arrow[rr, "h"] \arrow[dr, "c_1"'] & & Y \\
& Cyl(X)_0 \arrow[ur, "h_0"'] &
\end{tikzcd}$$

thus we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, equal] & Cyl(X) \arrow[d, "c_2"] & X \arrow[l, "i_2"'] \arrow[d, equal] \\
X \arrow[r, "\tilde{i}_1"] \arrow[dr, "f"'] & Cyl(X)_0 \arrow[d, "h_0"] & X \arrow[l, "\tilde{i}_2"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

which shows that $f \simeq g$ in $\mathbf{sSet}_0$.

For the other direction, let h be a homotopy from f to g in $\mathbf{sSet}_0$.

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[dr, "f"'] & Cyl(X)_0 \arrow[d, "h"] & X \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

where $Cyl(X)_0$ is a cylinder object in $\mathbf{sSet}_0$. $Cyl(X)_0$ is also a cylinder object of **sSet** because we have

$$\begin{tikzcd}
X \coprod X \arrow[r, "c_2"] \arrow[rrr, bend left, "fold"] & X \coprod_0 X \arrow[r, "i"] & Cyl(X)_0 \arrow[r, "p"] & X
\end{tikzcd}$$

so $f \simeq g$ in $\mathbf{sSet}$. $\square$

Corollary 4.1.13. *Let G, H be simplicial groups. Then*

$$[\overline{W}G, \overline{W}H]_{\mathbf{sSet}_0} \cong [\overline{W}G, \overline{W}H]_{\mathbf{sSet}}$$

Proof. Note that $\overline{W}G$ and $\overline{W}H$ are fibrant-cofibrant as simplicial sets and reduced simplicial sets so by theorem 3.2.16

$$[\overline{W}G, \overline{W}H]_{\mathbf{sSet}_0} \cong hom_{\mathbf{sSet}_0}(\overline{W}G, \overline{W}H)/\simeq \quad \text{and} \quad [\overline{W}G, \overline{W}H]_{\mathbf{sSet}} \cong hom_{\mathbf{sSet}}(\overline{W}G, \overline{W}H)/\simeq$$

The result then follows from lemma 4.1.12. $\square$

4.2 Central Extensions of Groups

We now recall the classical correspondence for central extensions of groups.

Definition 4.2.1. An *extension* of a group G by a group H is a group E such that we have a diagram of groups

$$1 \to H \xrightarrow{i} E \xrightarrow{p} G \to 1$$

where $ker(p) = Im(i)$ and $H \trianglelefteq E$. We say such an extension is *central* if $i(H)$ is contained in the center of E. Notice that this necessarily makes H abelian, so that a *central extension* of a group G by an abelian group A is a group E such that we have a short exact sequence of groups

$$0 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

where $i(A)$ is contained in the center of E.

Definition 4.2.2. A morphism of two central extensions of a group G by and abelian group A is a group homomorphism $f : E \to E'$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "f"] & G \arrow[d, equal] \\ A \arrow[r] & E' \arrow[r] & G \end{tikzcd}$$

This necessarily makes f an isomorphism and so it gives an equivalence relation. The equivalence classes of central extensions are called *isomorphism classes*. Let $\mathbf{Iso}(G, A)$ denote the set of isomorphism classes of central extensions of G by A.

Theorem 4.2.3. *There is a correspondence*

$$\mathbf{Iso}(G,A) \cong [\overline{W}G, \overline{W}^2 A]$$

For one direction of the correspondence, consider a central extension of groups $A \to E \xrightarrow{p} G$ and choose a based set theoretic cross section $\sigma : G \to E$, that is, a function (not necessarily a group homomorphism!) such that $p \circ \sigma = \mathbf{id}_G$ and such that $\sigma(1_G) = 1_E$. How σ fails to be a group homomorphism determines a function $f : G \times G \to A$ defined by $f(g,h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$. One can then use the fact that group multiplication in E is associative to see this function satisfies the cocycle condition

$$f(g,hk) + f(h,k) = f(gh,k) - f(g,h) = 0 \text{ for all } g,h,k \in G \tag{4.1}$$

One can show this is all that is required to extend $f : G \times G \to A$ to a simplicial set morphism $f : \overline{W}G \to \overline{W}^2 A$.

The other direction of the correspondence is constructed as follows. If $f \in [\overline{W}G, \overline{W}^2 A]$, then f is represented by a simplicial morphism $\tilde{f} : \overline{W}G \to \overline{W}^2 A$ as both $\overline{W}G$ and $\overline{W}^2 A$ are both fibrant and cofibrant in the model structure for simplicial sets. $\tilde{f}$ is entirely determined by $\tilde{f}_2 : G \times G \to A$, moreover, the simplicial identities ensure that $\tilde{f}_2$ satisfies the identity 4.1. Define the group H_f to be $G \times A$ as a set, with group law defined by

$$(g,a)(g',a') = (gg', a + a' + \tilde{f}_2(g,g'))$$

We send f to the central extension

$$0 \to A \to H_f \to G \to 1$$

It is also relatively easily verified that the two constructions are inverses of one another.

4.3 Simplicial Group Object Actions

We now need to generalize the notion of group to other categories other than the category of sets. Such an object is called a *group object* of the category. As soon as one has a group one would naturally like a group action, so we will also use the new generalized definition to generalize group actions. Then we look at the group object actions in the category of simplicial sets and simplicial groups to define the *simplicial action groupoid*. Finally, we will prove and state some technical results involving their relationship with our choice of classifying space $\overline{W}$.

Definition 4.3.1. A *group object* in a category $\mathcal{C}$ with finite products and a terminal object $*$ is an object G of $\mathcal{C}$ together with morphisms of $\mathcal{C}$

$$\begin{aligned} &\mathbf{1} : * \to G \text{ called the } \textit{unit map} \\ &(-)^{-1} : G \to G \text{ called the } \textit{inverse map} \\ &m : G \times G \to G \text{ called the } \textit{multiplication map} \end{aligned}$$

such that the following diagrams commute

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathbf{id}_G \times m"] \arrow[d, "m \times \mathbf{id}_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \cong * \times G \arrow[r, "\mathbf{1} \times \mathbf{id}_G"] \arrow[d, "\mathbf{id}_G \times \mathbf{1}"] \arrow[dr, "\mathbf{id}_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "diag"] \arrow[d, "diag"] \arrow[dr] & G \times G \arrow[r, "(-)^{-1} \times \mathbf{id}"] & G \times G \arrow[dd, "m"] \\ G \times G \arrow[d, "\mathbf{id} \times (-)^{-1}"] & * \arrow[dr, "\mathbf{1}"] & \\ G \times G \arrow[rr, "m"] & & G \end{tikzcd}$$

The first diagram corresponds to the associativity of the multiplication map. The second diagram corresponds to the unit map picking an element that is a left and right identity. The third makes sure that the inverse map is an inverse.

Example 4.3.2. The group objects in the category of sets are groups with the obvious unit, inverse, and multiplication maps.

Example 4.3.3. The group objects in the category of groups are abelian groups with the obvious unit, inverse, and multiplication maps. They must be abelian because the inverse map must be a group homomorphism. Namely we must have

$$(g \cdot h)^{-1} = g^{-1} \cdot h^{-1}$$

so

$$h^{-1} \cdot g^{-1} = g^{-1} \cdot h^{-1}$$

so that the group must be abelian. This is an application of the *Eckmann-Hilton argument* [9].

Definition 4.3.4. Let $\mathcal{C}$ be a category with finite product and a terminal object $*$. Let G be a group object of $\mathcal{C}$ and X be an object of $\mathcal{C}$. A *group object action* of G on X, is a morphism $\rho : G \times X \to X$ of $\mathcal{C}$ satisfying the following commutative diagrams:

$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times \mathbf{id}_X"] \arrow[d, "\mathbf{id}_G \times \rho"] & G \times X \arrow[d, "\rho"] \\ G \times X \arrow[r, "\rho"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1_G \times \mathbf{id}_X"] \arrow[dr, "\mathbf{id_X}"] & G \times X \arrow[d, "\rho"] \\ & X \end{tikzcd}$$

where $m : G \times G \to G$ is multiplication of the group object G and $1_G : * \to G$ is the unit map.

Example 4.3.5. The prototypical example is of course group actions in the category of sets.

Example 4.3.6. Another example would be group object actions in the category of groups. Here the group objects are the abelian groups so that an action of an abelian group A on a group G is a group homomorphism $\rho : A \times G \to G$ satisfying the commutative diagrams in **Grp**.

In particular, if $A \xrightarrow{i} E \xrightarrow{p} G$ is a central extension of groups, then $\rho : A \times E \to E$ defined by $\rho(a, e) = i(a)e$ is an action in **Grp**. Namely, ρ is a group action and also a group homomorphism because if $a, a' \in A$ and $e, e' \in E$

$$\rho(a, e)\rho(a', e') = i(a)ei(a')e' = i(a)i(a')ee' = \rho(aa', ee')$$

as $i(A)$ is in the center of E and i is a group homomorphism.

We can now define one of the main objects used in this paper. We will make heavy use of group object actions in the category of simplicial groups. As the group objects in the category of groups are abelian groups, we have that the group objects in the category of simplicial groups are the simplicial abelian groups. Thus, by definition a simplicial abelian group action of A on a simplicial group G is a morphism of simplicial groups $\rho : A \times G \to G$ satisfying the commutative diagrams in **sGrp**:

$$\begin{tikzcd} A \times A \times G \arrow[r, "m \times \mathbf{id}_G"] \arrow[d, "\mathbf{id}_A \times \rho"] & A \times G \arrow[d, "\rho"] \\ A \times G \arrow[r, "\rho"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "1_A \times \mathbf{id}_G"] \arrow[dr, "\mathbf{id_G}"] & A \times G \arrow[d, "\rho"] \\ & G \end{tikzcd}$$

In other words, at each level n we have an action of A_n on G_n in the category of groups, and those actions are compatible with the simplicial structure maps. That is, at each level $\rho_n : A_n \times G_n \to G_n$ is a group homomorphism satisfying the commutative diagrams in **Grp**:

$$\begin{tikzcd}
A_n \times A_n \times G_n \arrow[r, "m\times \mathbf{id}_{G_n}"] \arrow[d, "\mathbf{id}_{A_n}\times \rho_n"'] & A_n \times G_n \arrow[d, "\rho_n"] \\
A_n \times G_n \arrow[r, "\rho_n"] & G_n
\end{tikzcd}
\qquad
\begin{tikzcd}
G_n \arrow[r, "1_{A_n}\times \mathbf{id}_{G_n}"] \arrow[dr, "\mathbf{id}_{\mathbf{G_n}}"] & A_n \times G_n \arrow[d, "\rho_n"] \\
 & G_n
\end{tikzcd}$$

such that $d_i \rho_n = \rho_{n-1} d_i$ for all $0 \leq i \leq n$ and $s_j \rho_n = \rho_{n+1} s_j$ for all $0 \leq j \leq n$. We say such an action is *free* if it is levelwise a free action of groups.

Example 4.3.7. More generally, presheaves of abelian groups are the group objects in the category of presheaves of groups. A group object action in the category of presheaves of groups is then a morphism of presheaves of groups $A \times G \to G$, where A is a presheaf of abelian groups and G is a presheaf of groups, satisfying the necessary commutative diagrams. In particular this means that $A(U) \times G(U) \to G(U)$ is a sectionwise group morphism and also a group action. We say such an action is *free* if sectionwise it is a free action of groups.

Definition 4.3.8. Given a group action $\rho : G \times X \to X$, the *action groupoid* $X/\!\!/G$ is the groupoid that has objects given by X and for every $x, y \in X$ there is a morphism from x to y if and only if there is an element of g such that $\rho(g, x) = y$.

If we have a simplicial group action on a simplicial set, then we can apply this levelwise to obtain the following.

Definition 4.3.9. Let $\rho : G \times X \to X$ be an action in the category of simplicial sets. Then the *action simplicial groupoid*, denoted $X/\!\!/G$, is the simplicial groupoid defined by $(X/\!\!/G)_n = X_n/\!\!/G_n$ with structure maps induced from those in G and X.

Remark 4.3.10. The action simplicial groupoid is also defined for an action in the category of simplicial groups. That is, for a simplicial abelian group A acting on a simplicial group G. In this case all the structure maps are group homomorphisms.

Remark 4.3.11. Note that the simplicial action groupoid is a simplicial object in the category of groupoids and it is not necessarily a simplicially enriched groupoid.

We can apply the nerve levelwise to get a functor $N : \mathbf{sGrpd} \to \mathbf{s^2Set}$. After this we can apply the *total simplicial set functor* $T : \mathbf{s^2Set} \to \mathbf{sSet}$ of [2], [27].

Lemma 4.3.12. *Let G be a simplicial group, then*

$$TN(*/\!\!/G) \cong \overline{W}G$$

Proof. This is originally due to Duskin and appears in [27]. The isomorphism is defined levelwise by sending

$$(*, g^1_{n-1}, (g^1_{n-2}, g^2_{n-2}), \cdots, (g^1_0, \cdots, g^n_0)) \in TN(*/\!\!/G)_n \mapsto (g^1_{n-1}, g^2_{n-2}, \cdots, g^n_0) \in \overline{W}G_n$$

$\square$

Remark 4.3.13. $*/\!\!/G$ is the simplicially enriched groupoid with single object and morphisms given by G.

Remark 4.3.14. For an action in the category of simplicial groups, the nerve applied to the action simplicial groupoid, $N(G/\!\!/A)$, is a bisimplicial group. To see this one only needs to check that the face and degeneracies coming from the nerve are group homomorphisms. An element of $N_n(G_m/\!\!/A_m)$ is given by $(g_m, a^1_m, \cdots, a^n_m)$ and then for $0 < i \leq n$, $g_m, h_m \in G_m$ and $a^1_m, \cdots, a^n_m, b^1_m, \cdots, b^n_m \in A_m$

$$\begin{aligned}
d_i(g_m h_m, a^1_m b^1_m, \cdots, a^n_m b^n_m) &= (g_m h_m, a^1_m b^1_m, \cdots, a^i_m b^i_m a^{i+1}_m b^{i+1}_m, a^{i+2}_m b^{i+1}_m, \cdots, a^n_m b^n_m) \\
&= (g_m h_m, a^1_m b^1_m, \cdots, a^i_m a^{i+1}_m b^i_m b^{i+1}_m, a^{i+2}_m b^{i+1}_m, \cdots, a^n_m b^n_m) \\
&= d_i(g_m, a^1_m, \cdots, a^n_m) d_i(h_m, b^1_m, \cdots, b^n_m) \\
d_0(g_m h_m, a^1_m b^1_m, \cdots, a^n_m b^n_m) &= (\rho(a^1_m b^1_m, g_m h_m), a^2_m b^2_m, \cdots, a^n_m b^n_m) \\
&= (\rho(a^1_m, g_m)\rho(b^1_m, h_m), a^2_m b^2_m, \cdots, a^n_m b^n_m) \\
&= d_0(g_m, a^1_m, \cdots, a^n_m) d_0(h_m, b^1_m, \cdots, b^n_m)
\end{aligned}$$

Lemma 4.3.15. *Let A be a simplicial abelian group and let $A/\!\!/A$ denote the action simplicial groupoid given by multiplication. Then, as simplicial groups,*

$$TN(A/\!\!/A) \cong WA$$

and

$$TN(*/\!\!/A) \cong \overline{W}A$$

Proof. The first isomorphism is given by sending

$$(a_n^1, (a_{n-1}^1, a_{n-1}^2), \cdots, (a_0^1, \cdots, a_0^{n+1})) \in TN(A/\!\!/A)_n \mapsto (a_n^1, a_{n-1}^2, \cdots, a_0^{n+1}) \in WA_n$$

The second isomorphism is given by the isomorphism of lemma 4.3.12. $\square$

Definition 4.3.16. Let X be a simplicial set and G a simplicial group such that X has an G-action in **sSet**. The *Borel construction* is the simplicial set $WG \times_G X$, which is the quotient of $WG \times X$ by the diagonal action with respect to the given G action on X and the G action on WG given by multiplication.

Similarly, if G is a simplicial group and A a simplicial abelian group such that G has an A-action in **sGrp**, the Borel construction $WA \times_A G$ is a simplicial group.

Lemma 4.3.17. *Let G be a simplicial group and X be a simplicial set with a G-action in* **sSet**. *Then, as simplicial sets,*

$$WG \times_G X \cong TN(X/\!\!/G)$$

Similarly, if A is a simplicial abelian group and G is a simplical group with an A-action in **sGrp**. *Then, as simplicial groups,*

$$WA \times_A G \cong TN(G/\!\!/A)$$

Proof. This is a straightforward computation following the same ideas as the previous lemmas. (See proposition 3.73 in [21].) $\square$

Lemma 4.3.18. *Let $\rho : G \times X \to X$ be a free action in* **sSet**. *Then we have the following weak equivalence of simplicial sets:*

$$WG \times_G X \xrightarrow{\sim} X/G$$

Similarly, there is a weak equivalence of simplicial groups

$$WA \times_A G \xrightarrow{\sim} G/A$$

if $\rho : A \times G \to G$ is a free action in **sGrp**.

Proof. We prove this for a free action in **sGrp**, as the other proof is the same after forgetting some of the group structures. Think of G/A as TN applied to the action simplicial groupoid of the trivial action by $*$. Then define a morphism f between the action simplicial groupoids levelwise by $f(g) = [g]$ on objects and $f(a, g) = [g]$ on morphisms. The action is free so that f is a levelwise an equivalence of groupoids. Thus $N(f)$ is a weak equivalence of bisimplicial groups so that $TN(f)$ is a weak equivalence of simplicial groups as T preserves weak equivalences (see [7] section 7). $\square$

Lemma 4.3.19. *Let $\rho : A \times G \to G$ be a free action in* **sGrp**. *Then we have the following weak equivalence of simplicial groups:*

$$WA \times_{\overline{W}A} (WA \times_A G) \xrightarrow{\sim} G$$

Proof. First take the pullback of the action simplicial groupoids in the category of simplicial groupoids. Notice objects of $A/\!\!/A \times_{*/\!\!/A} G/\!\!/A$ are tuples $(a, g) \in A \times G$ and morphisms are given by $((\bar{a}, \bar{a}'), (a', \bar{g})) \in (A \times A) \times (A \times G)$ such that $\bar{a} = a'$. Then define $f : A/\!\!/A \times_{*/\!\!/A} G/\!\!/A \to A/\!\!/A \times G$ on objects by $f(a, g) = (a, \rho(-a, g))$ and on morphisms $f((\bar{a}, \bar{a}'), (\bar{a}, \bar{g})) = (\bar{a}, \bar{a}', \rho(-\bar{a}, \bar{g}))$. It is easy to check that f is an isomorphism of simplicial groupoids. Then both T and N preserve limits as they are right adjoint and WA is contractible so we have the following:

$$WA \times_{\overline{W}A} (WA \times_A G) \cong WA \times G \xrightarrow{\sim} G$$

$\square$

4.4 Central Extensions of Simplicial Groups

We will now define central extensions of simplicial groups and define two functors that will induce a bijection

$$\pi_0(\mathbf{C}(\mathbf{G}, \mathbf{A})_{\mathbf{sGrp}}) \cong \pi_0(H(G, \overline{W}A)_{\mathbf{sGrp}})$$

which then gives a correspondence

$$\pi_0(\mathbf{C}(\mathbf{G}, \mathbf{A})_{\mathbf{sGrp}}) \cong [\overline{W}G, \overline{W}^2 A]_{\mathbf{sSet}}$$

Definition 4.4.1. Let A be a simplicial abelian group and E, G be simplicial groups. We say that E is a central extension of G by A if we have the following:

- E has a free A-action, $\phi : A \times E \to E$, in the category of simplicial groups.
- G has a trivial A-action in the category of simplicial groups.
- There is an A-equivariant morphism $p : E \to G$ such that the induced map $\tilde{p} : E/A \xrightarrow{\sim} G$ is a weak equivalence of simplicial groups.

We will denote a central extension by

$$A \xrightarrow{i} E \xrightarrow{p} G,$$

where p is the equivariant morphism given by the definition and i is the inclusion defined levelwise by $i_n(a) = \phi_n(a, 1_E)$.

Definition 4.4.2. The category $\mathbf{C(G, A)_{sGrp}}$ of central extensions of simplicial groups is the category whose objects are central extensions of simplicial groups, and whose morphisms are given by commutative diagrams of equivariant morphisms:

$$\begin{tikzcd} E \arrow[d, "f"', "\sim"] \arrow[dr, "p"] & \\ E' \arrow[r, "p'"'] & G \end{tikzcd},$$

where f is a weak equivalence of simplicial groups.

Example 4.4.3. If we have morphisms of simplicial groups, $i : A \to E$ and $p : E \to G$ such that levelwise we have central extensions of groups

$$0 \to A_n \xrightarrow{i_n} E_n \xrightarrow{p_n} G_n \to 1$$

then $A \xrightarrow{i} E \xrightarrow{p} G$ is a central extension of simplicial groups. This is because E has a free A-action in the category of simplicial groups given by

$$A \times E \xrightarrow{i \times \mathbf{id}_E} E \times E \xrightarrow{m} E$$

where m is the multiplication of E. Also G has a trivial A-action in the category of simplicial groups and $p : E \to G$ is an A-equivariant morphism such that $\tilde{p} : E/A \xrightarrow{\cong} G$ is an isomorphism.

We make use of the following cocycle and functor to show that our main construction in this section induces a bijection at the level of path components. This is Observation 3.94 in [21] and we use it in a similar way as they do.

Definition 4.4.4. Let $\mathbf{q} : H(G, \overline{W}A)_{\mathbf{sGrp}} \to H(G, \overline{W}A)_{\mathbf{sGrp}}$ be the functor given by post-composition with the cocycle

$$\overline{W}A \cong * \times_A WA \xleftarrow{\sim} WA \times_A WA \xrightarrow{\sim} WA \times_A * \cong \overline{W}A.$$

The functor $\mathbf{q}$ has a clear left and right adjoint $\overline{\mathbf{q}}$ which is given by post-composition with the cocycle

$$\overline{W}A \cong WA \times_A * \xleftarrow{\sim} WA \times_A WA \xrightarrow{\sim} * \times_A WA \cong \overline{W}A.$$

Therefore both $\mathbf{q}$ and $\overline{\mathbf{q}}$ induce bijections at the level of path components.

Let $G \xleftarrow{\sim} Z \to \overline{W}A$ be a cocycle in $H(G, \overline{W}A)_{\mathbf{sGrp}}$, and notice WA has a free A-action $A \times WA \to WA$ given levelwise by

$$\bar{a}_n \cdot (a_n, a_{n-1}, \cdots, a_0) = (-\bar{a}_n + a_n, a_{n-1}, \cdots, a_0).$$

Pullback along $WA \to \overline{W}A$:

$$\begin{tikzcd}
 & WA \times_{\overline{W}A} Z \arrow[r] \arrow[d] & WA \arrow[d] \\
G & Z \arrow[l, "f"', "\sim"] \arrow[r] & \overline{W}A
\end{tikzcd}$$

and give $WA \times_{\overline{W}A} Z$ the free A-action, $\phi : A \times WA \times_{\overline{W}A} Z \to WA \times_{\overline{W}A} Z$, induced by the action on WA above. G has a trivial A-action, and the composite $p : WA \times_{\overline{W}A} Z \to Z \xrightarrow{\sim} G$ is an A-equivariant morphism. Moreover $\tilde{p} = f : Z \xrightarrow{\sim} G$, so that

$$A \xrightarrow{i} WA \times_{\overline{W}A} Z \xrightarrow{p} G$$

is a central extension of G by A according to definition 4.4.1.

Definition 4.4.5. Let $F : H(G, \overline{W}A)_{\mathbf{sGrp}} \to \mathbf{C(G, A)}_{\mathbf{sGrp}}$ be the functor sending the cocycle

$$G \xleftarrow{\sim} Z \to \overline{W}A$$

to the central extension

$$A \xrightarrow{i} WA \times_{\overline{W}A} Z \xrightarrow{p} G.$$

The functor sends a morphism of cocycles

$$\begin{tikzcd}
 & Z \arrow[dl, "\sim"'] \arrow[dr] \arrow[dd, "\sim"', "\alpha"] & \\
G & & \overline{W}A \\
 & Z' \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd}$$

to the morphism of central extensions

$$\begin{tikzcd}
WA \times_{\overline{W}A} Z \arrow[dd, "f"', "\sim"] \arrow[dr, "p"] & \\
 & G \\
WA \times_{\overline{W}A} Z' \arrow[ur, "p'"] &
\end{tikzcd},$$

where $f : WA \times_{\overline{W}A} Z \to WA \times_{\overline{W}A} Z'$ is the morphism given by the universal property of $WA \times_{\overline{W}A} Z'$

$$\begin{tikzcd}
WA \times_{\overline{W}A} Z \arrow[dr, "f", "\sim"'] \arrow[drr, bend left] \arrow[dd] & & \\
 & WA \times_{\overline{W}A} Z' \arrow[r] \arrow[d] & WA \arrow[d] \\
Z \arrow[r, "\alpha", "\sim"'] & Z' \arrow[r] & \overline{W}A
\end{tikzcd}$$

Remark 4.4.6. f is a weak equivalence because **sGrp** is right proper and $WA \times_{\overline{W}A} Z' \to Z'$ is a fibration.

Now given a central extension $A \xrightarrow{i} E \xrightarrow{p} G$ in $\mathbf{C(G, A)_{sGrp}}$, we can form the cocycle

$$G \xleftarrow{\sim} E/A \xleftarrow{\sim} WA \times_A E \to \overline{W}A,$$

where $WA \times_A E \to \overline{W}A$ is given by the morphism $A /\!\!/ A \to A /\!\!/ *$ induced by the terminal morphism $A \to *$.

Definition 4.4.7. The functor $\overline{F} : \mathbf{C(G,A)_{sGrp}} \to \mathbf{H(G,\overline{W}A)_{sGrp}}$ sends the central extension $A \xrightarrow{i} E \xrightarrow{p} G$ in $\mathbf{C(G,A)_{sGrp}}$ to the cocycle

$$G \xleftarrow{\sim} E/A \xleftarrow{\sim} WA \times_A E \to \overline{W}A.$$

It sends a morphism of central extensions

$$\begin{tikzcd} E \arrow[dd, "g"', "\sim"] \arrow[dr, "p"] & \\ & G \\ E' \arrow[ur, "p'"] & \end{tikzcd}$$

to the morphism of cocycles

$$\begin{tikzcd} & WA \times_A E \arrow[dl, "\sim"'] \arrow[dd, "\sim"', "\mathbf{id}\times g"] \arrow[dr] & \\ G & & \overline{W}A \\ & WA \times_A E' \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd}$$

Now we can state the main result of this chapter:

Theorem 4.4.8. *F and $\overline{F}$ induce bijections on path components*

$$\overline{F} : \pi_0(\mathbf{C(G,A)_{sGrp}}) \cong \pi_0(H(G,\overline{W}A)_{\mathbf{sGrp}}) : F$$

Corollary 4.4.9.

$$\pi_0(\mathbf{C(G,A)_{sGrp}}) \cong [\overline{W}G, \overline{W}^2 A]_{\mathbf{sSet}}$$

Proof. The model structure on **sGrp** is right proper and weak equivalences are closed under finite products so we have that:

$$\pi_0(H(G,\overline{W}A)_{\mathbf{sGrp}}) \cong [G,\overline{W}A]_{\mathbf{sGrp}}$$

$\overline{W}$ is part of a Quillen equivalence between **sGrp** and $\mathbf{sSet}_0$ so that we have:

$$[G,\overline{W}A]_{\mathbf{sGrp}} \cong [\overline{W}G, \overline{W}^2 A]_{\mathbf{sSet}_0}$$

Together with theorem 4.4.8 and corollary 4.1.13 we have the desired result. □

4.5 Proof of Theorem 4.4.8

We will show it is straightforward to see that $\overline{F}$ is right inverse to F on path components. To see that $\overline{F}$ is left inverse to F on path components we use a similar proof to Theorem 3.95 in [21] by showing $\overline{\mathbf{q}} \circ \overline{F}$ is left inverse to F on path components. But left and right inverses must coincide so that $\overline{\mathbf{q}} \circ \overline{F}$ and $\overline{F}$ induce the same function on path components.

First we will show that $\overline{F}$ is right inverse to F on path components. Let $A \xrightarrow{i} E \xrightarrow{p} G$ be a central extension of simplicial groups. Then $F(\overline{F}(A \hookrightarrow E \xrightarrow{p} G))$ is given by:

$$A \xrightarrow{j} WA \times_{\overline{W}A} (WA \times_A E) \xrightarrow{t} G$$

t is the composition

$$WA \times_{\overline{W}A} (WA \times_A E) \xrightarrow{\pi} WA \times_A E \xrightarrow{\sim} E/A \xrightarrow{\sim} G,$$

and j is induced by the free A action on $WA \times_{\overline{W}A} (WA \times_A E)$ induced by the free A-action on WA given levelwise by $\bar{a}_n \cdot (a_n, a_{n-1}, \cdots, a_0) = (\bar{a}_n^{-1} a_n, a_{n-1}, \cdots, a_0)$.

From lemma 4.3.19 we have a weak equivalence $WA \times_{\overline{W}A} (WA \times_A E) \xrightarrow{\sim} E$ that one can check is an A-equivariant morphism. Notice that we also have the following commutative diagram of simplicial groupoids:

$$\begin{tikzcd}
A/\!\!/A \times_{*/\!\!/A} E/\!\!/A \arrow[r, "\cong"] \arrow[d] & A/\!\!/A \times E \arrow[r] & E \arrow[d] \arrow[dr, "p"] & \\
 & & & G \\
E/\!\!/A \arrow[rr] & & E/A \arrow[ur, "\tilde{p}"] &
\end{tikzcd}$$

After applying TN to the diagram above, we see that $A \xrightarrow{i} E \xrightarrow{p} G$ and $F(\overline{F}(A \hookrightarrow E \xrightarrow{p} G))$ are in the same path component of $\mathbf{C(G, A)_{sGrp}}$.

Now we will show that $\overline{\mathbf{q}} \circ \overline{F}$ is left inverse to F on path components. Let $G \xleftarrow{\sim} Z \xrightarrow{f} \overline{W}A$ be a cocycle in $H(G, \overline{W}A)_{\mathbf{sGrp}}$. Then $\overline{F}(F(G \xleftarrow{\sim} Z \xrightarrow{f} \overline{W}A))$ is given by

$$G \xleftarrow{\sim} Z \xleftarrow{\sim} WA \times_A (WA \times_{\overline{W}A} Z) \rightarrow \overline{W}A$$

Now consider the following diagram:

$$\begin{tikzcd}
 & WA \times_A (WA \times_{\overline{W}A} Z) \arrow[dl, "\sim"'] \arrow[dd, "\sim"] \arrow[dr] & \\
G & & \overline{W}A \\
 & Z \arrow[ul, "\sim"] \arrow[ur, dashed, "\mathbf{q}(f)"] &
\end{tikzcd}$$

The left side triangle is clearly commutative. To see the right side triangle is commutative, notice that $(WA \times_{\overline{W}A} Z)/\!\!/A \to */\!\!/A$ factors as $Z \times_{\overline{W}A} WA)/\!\!/A \to WA/\!\!/A \to */\!\!/A$, and we have the following commutative diagram of simplicial groupoids:

$$\begin{tikzcd}
(Z \times_{\overline{W}A} WA)/\!\!/A \arrow[r] \arrow[d, "\sim"] & WA/\!\!/A \arrow[r, "\sim"] \arrow[d, "\sim"] & */\!\!/A \\
Z \arrow[r, "f"] & \overline{W}A &
\end{tikzcd}$$

After applying TN to it we have this shows the desired right triangle is commutative. So we have the $\overline{F} \circ F$ and $\mathbf{q}$ induce the same function on path components. Then applying $\overline{\mathbf{q}}$ we see that $\overline{\mathbf{q}} \circ \overline{F}$ is left inverse to F on path components as $\overline{\mathbf{q}} \circ \mathbf{q}$ induces the identity function on path components.

CHAPTER 5

LEVELWISE EXTENSIONS

5.1 Relation Between the Geometric Nerve and $\overline{W}$

Lemma 5.1.1. *Let A be an abelian group and E be a group with a free A-action. Then there is an isomorphism of simplicial sets*

$$\overline{W}N(E/\!\!/A) \cong \Delta\Omega^{-1}(E/\!\!/A)$$

Proof. See paragraph after Theorem 1 in [4]. □

Definition 5.1.2. Let G be a bisimplicial group, then we can apply $\overline{W}$ row-wise to obtain a bisimplicial set which we will denote by $\overline{W}G$. (See definition 2.3.5.)

Lemma 5.1.3. *Let G be a bisimplicial group. Then there is an isomorphism of simplicial sets*

$$\psi : T\overline{W}G \cong \overline{W}TG$$

Proof. An element of $T\overline{W}G$ element is given by

$$(*, g^0_{n-1}, (g^1_{n-2}, g^0_{n-2}), \cdots, (g^{n-1}_0, g^{n-2}_0, \cdots, g^0_0)),$$

where each $g^j_i \in G_{j,i}$ and the following are satisfied for all $1 \leq i \leq n$:

$$d^v_0(g^0_{n-1}) = d^{\overline{W}}_2(g^1_{n-2}, g^0_{n-2}) = d^h_1(g^1_{n-2})$$

$$\cdots$$

$$\begin{aligned} d^v_0(g^{i-1}_{n-i}, \cdots, g^0_{n-i}) &= (d^v_0(g^{i-1}_{n-i}), \cdots, d^v_0(g^0_{n-i})) \\ = d^{\overline{W}}_{i+1}(g^i_{n-i-1}, \cdots, g^0_{n-i-1}) &= (d^h_i(g^i_{n-i-1}), \cdots, d^h_1(g^1_{n-i-1})) \end{aligned}$$

$$\cdots$$

Equivalently, we have that for all $1 \leq i \leq n$ and all $0 \leq j \leq i-1$

$$d^v_0(g^{i-1-j}_{n-i}) = d^h_{i-j}(g^{i-j}_{n-i-1})$$

Now, an element of $\overline{W}TG$ is given by

$$((g^0_{n-1}, g^1_{n-2}, \cdots, g^{n-1}_0), (g^0_{n-2}, g^1_{n-3}, \cdots, g^{n-2}_0), \cdots, g^0_0),$$

where $(g^0_{n-1}, g^1_{n-2}, \cdots, g^{n-1}_0)$, $(g^0_{n-2}, g^1_{n-3}, \cdots, g^{n-2}_0)$, $\cdots$, g^0_0 are in $T_{n-1}G$, $T_{n-2}G$, $\cdots$, T_0G respectively. That is, we have

$$\begin{aligned}
d^v_0(g^k_{n-1-k}) &= d^h_{k+1}(g^{k+1}_{n-2-k}) && \text{for all } 0 \leq k \leq n-2 \\
d^v_0(g^k_{n-2-k}) &= d^h_{k+1}(g^{k+1}_{n-3-k}) && \text{for all } 0 \leq k \leq n-3 \\
&\cdots
\end{aligned}$$

Define $\psi : T\overline{W}G \to \overline{W}TG$ by

$$\begin{aligned}
&\psi(*, g^0_{n-1}, (g^1_{n-2}, g^0_{n-2}), \cdots, (g^{n-1}_0, g^{n-2}_0, \cdots, g^0_0)) \\
&= ((g^0_{n-1}, g^1_{n-2}, \cdots, g^{n-1}_0), (g^0_{n-2}, g^1_{n-3}, \cdots, g^{n-2}_0), \cdots, g^0_0)
\end{aligned}$$

ψ is well defined, note by above that for all $1 \leq i \leq n$ and all $0 \leq j \leq i-1$ that

$$d^v_0(g^{i-1-j}_{n-i}) = d^h_{i-j}(g^{i-j}_{n-i-1})$$

So we have that if $i = k+1$ and $j = 0$

$$d^v_0(g^k_{n-1-k}) = d^h_{k+1}(g^{k+1}_{n-2-k}) \text{ for all } 0 \leq k \leq n-2$$

Then if $i = k+2$ and $j = 1$ we have

$$d^v_0(g^k_{n-2-k}) = d^h_{k+1}(g^{k+1}_{n-3-k}) \text{ for all } 0 \leq k \leq n-3$$

Continuing in this way we see that $((g^0_{n-1}, g^1_{n-2}, \cdots, g^{n-1}_0), (g^0_{n-2}, g^1_{n-3}, \cdots, g^{n-2}_0), \cdots, g^0_0)$ is an element of $\overline{W}TG$. One must also check the compatibility with the simplicial structure maps. We will show this for d_i — the other structure maps work similarly.

d_i of the image is given by

$$\begin{aligned}
&d_i((g^0_{n-1}, g^1_{n-2}, \cdots, g^{n-1}_0), (g^0_{n-2}, g^1_{n-3}, \cdots, g^{n-2}_0), \cdots, g^0_0) \\
&= (d_{i-1}(g^0_{n-1}, g^1_{n-2}, \cdots, g^{n-1}_0), \cdots, d_1(g^0_{n-i+1}, \cdots, g^{n-i+1}_0), \\
&d_0(g^0_{n-i}, \cdots, g^{n-i}_0) \cdot (g^0_{n-i-1}, \cdots, g^{n-i-1}_0), \cdots, g^0_0) \\
&= ((d^v_{i-1}(g^0_{n-1}), \cdots, d^v_1(g^{i-2}_{n-i+1}), d^h_{i-1}(g^i_{n-i-1}), \cdots, d^h_{i-1}(g^{n-1}_0)), \\
&\cdots, (d^h_0(g^1_{n-i-1})g^0_{n-i-1}, \cdots, d^h_0(g^{n-i}_0)g^{n-i-1}_0), \cdots)
\end{aligned}$$

Then we have

$$
\begin{aligned}
&d_i(*, g^0_{n-1}, (g^1_{n-2}, g^0_{n-2}), \cdots, (g^{n-1}_0, g^{n-2}_0, \cdots, g^0_0)) \\
&= (*, d^v_{i-1}(g^0_{n-1}), \cdots, d^v_1(g^{i-2}_{n-i+1}, \cdots, g^0_{n-i+1}), d^h_i(g^i_{n-i-1}, \cdots, g^0_{n-i}), \cdots) \\
&= (*, d^v_{i-1}(g^0_{n-1}), \cdots, (d^v_1(g^{i-2}_{n-i+1}), \cdots, d^v_1(g^0_{n-i+1})), (d^h_{i-1}(g^i_{n-i-1}), \cdots, d^h_0(g^1_{n-i-1})g^0_{n-i-1}), \cdots)
\end{aligned}
$$

and after applying ψ we see that they are the same.

ψ has a clear inverse

$$
\psi^{-1}((g^0_{n-1}, g^1_{n-2}, \cdots, g^{n-1}_0), (g^0_{n-2}, g^1_{n-3}, \cdots, g^{n-2}_0), \cdots, g^0_0)
$$

$$
= (*, g^0_{n-1}, (g^1_{n-2}, g^0_{n-2}), \cdots, (g^{n-1}_0, g^{n-2}_0, \cdots, g^0_0))
$$

It is a similar calculation to see that this is well defined and behaves with the simplicial structure maps. $\square$

Definition 5.1.4. Let M be a simplicial monoidal category. Apply the delooping levelwise to obtain a simplicial 2-category, $\Omega^{-1}M$, where $(\Omega^{-1}M)_n = \Omega^{-1}(M_n)$. Then apply the geometric nerve functor levelwise to obtain the bisimplicial set $\Delta\Omega^{-1}M$, via $(\Delta\Omega^{-1}M)_{m,n} = (\Delta\Omega^{-1}M_n)_m$

Lemma 5.1.5. *Let A be a simplicial abelian group and E be a simplicial group with a free A-action. Then there is an isomorphism of simplicial sets*

$$
\overline{W}TN(E/\!\!/A) \cong T\Delta\Omega^{-1}(E/\!\!/A)
$$

Proof. Notice $\overline{W}TN(E/\!\!/A) \cong T\overline{W}N(E/\!\!/A)$ by lemma 5.1.3 as $N(E/\!\!/A)$ is a bisimplicial group. Then we have the isomorphism of bisimplicial sets $\overline{W}N(E/\!\!/A) \cong \Delta\Omega^{-1}(E/\!\!/A)$ where for each row the isomorphism $\overline{W}N(E_n/\!\!/A_n) \cong \Delta\Omega^{-1}(E_n/\!\!/A_n)$ is the one in lemma 5.1.1. $\square$

5.2 Levelwise Central Extensions

A diagram of simplicial groups

$$
A \xrightarrow{i} E \xrightarrow{p} G
$$

that is levelwise

$$
A_n \xrightarrow{i_n} E_n \xrightarrow{p_n} G_n
$$

a central extensions of groups is a central extension of simplicial groups according to our definition. There certainly are based set-theoretic cross sections $\sigma_n : G_n \to E_n$, but in general these do not define a simplicial morphism (they only define one up to homotopy). As recalled, a based set-theoretic cross section allows to find a 2-cocycle representing the characteristic class of an ordinary central extension. For diagrams of central extensions, this classical construction, applied levelwise, only provides lax diagrams $\Delta^{op} \to \mathbf{sSet}$, not true (bi)simplicial sets. We can rectify these diagrams, if we exhibit them as nerves of suitable bicategories, by an application of the Grothendieck construction. Moreover, to begin with, we will assume there is a simplicial set cross section. This simplifying assumption is actually justified is the notable case of the Heisenberg central extension, discussed later in this thesis. By assuming that σ is a simplicial set cross section we can obtain genuine diagrams.

Let:

$$A \xrightarrow{i} E \xrightarrow{p} G$$

be a diagram of simplicial groups which are levelwise a central extensions of groups, and assume there is a simplicial set cross section σ. σ gives a simplicial lax monoidal functor (see Definition 2.6.3) $G \to E/\!\!/A$ defined levelwise by σ_n on objects and sending $\mathbf{id}_g$ to $\mathbf{id}_{\sigma(g)}$ for any $g \in G_n$. The morphism $\eta : I_G \to F(I_{E/\!\!/A})$ is obvious and the natural transformation μ is defined for two objects $g, g' \in G_n$ by

$$\sigma_n(gg')\sigma_n(g')^{-1}\sigma_n(g)^{-1},$$

which is a morphism in $E/\!\!/A$ because $p(\sigma_n(gg')\sigma_n(g')^{-1}\sigma_n(g)^{-1}) = 1$.

We can then apply the delooping levelwise to get a simplicial lax functor $\sigma : \Omega^{-1}G \to \Omega^{-1}(E/\!\!/A)$. Since the geometric nerve is functorial, we can apply it levelwise to get a morphism of bisimplicial sets $\Delta(\sigma) : \Delta\Omega^{-1}G \to \Delta\Omega^{-1}(E/\!\!/A)$ and then one of simplicial sets $T\Delta(\sigma) : T\Delta\Omega^{-1}G \to T\Delta\Omega^{-1}(E/\!\!/A)$. Post composing with the morphism $T\Delta\Omega^{-1}(E/\!\!/A) \to T\Delta\Omega^{-1}(*/\!\!/A)$ induced by $E/\!\!/A \to */\!\!/A$, we obtain a morphism of simplicial sets, $\phi : \overline{W}G \to \overline{W}^2 A$ as the composite:

$$\phi : \overline{W}G \cong \Delta\Omega^{-1}G \to T\Delta\Omega^{-1}(E/\!\!/A) \to T\Delta\Omega^{-1}(*/\!\!/A) \cong \overline{W}^2 A$$

Remark 5.2.1. To see how this relates to the classical correspondence, let $A \to E \xrightarrow{p} G$ be a central extension of groups and let σ be a based cross section. Then $A \to E \to G$ can be thought of as a central extension of constant simplicial groups which has a simplicial set cross section defined levelwise by σ. Then one can calculate that the simplicial morphism

$$\phi : \overline{W}G \to \overline{W}^2 A$$

defined above is defined at level 2 by

$$\phi_2(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$$

So that this chooses the same morphism as the classical correspondence.

Theorem 5.2.2. *Let*

$$A \xrightarrow{i} E \xrightarrow{p} G$$

be a diagram of simplicial groups that is levelwise a central extension of groups that has a simplicial set cross section σ. Corollary 4.4.9 picks the homotopy class of

$$\phi \in [\overline{W}G, \overline{W}^2 A]_{\mathbf{sSet}}$$

Proof. There is the following commutative diagram of simplicial sets:

$$\begin{tikzcd}
\overline{W}G \arrow[d, "\cong"] & \overline{W}TN(E/\!\!/A) \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\cong"] & \overline{W}^2 A \arrow[d, "\cong"] \\
T\Delta\Omega^{-1}G & T\Delta\Omega^{-1}(E/\!\!/A) \arrow[l, "\sim"'] \arrow[r] & T\Delta\Omega^{-1}(*/\!\!/A) \\
 & T\Delta\Omega^{-1}G \arrow[ul, equal] \arrow[u, "T\Delta(\sigma)"] &
\end{tikzcd}$$

□

For the general case we proceed as follows. Let

$$A \xrightarrow{i} E \xrightarrow{p} G$$

be a diagram of central extensions and choose a set-theoretic cross section $\sigma_n : G_n \to E_n$ such that $\sigma_n(\mathbf{1}_{G_n}) = \mathbf{1}_{E_n}$ at each level. In general, $\{\sigma_n\}$ do not define a simplicial morphism, but we can use the Grothendieck construction to still construct a simplicial set morphism.

The levelwise sections define lax monoidal functors $\sigma_n : G_n \to E_n/\!\!/A_n$ and thus a homomorphism of bicategories $\sigma_n : \Omega^{-1}G_n \to \Omega^{-1}(E_n/\!\!/A_n)$. These combine to give a lax Δ-homomorphism (see [5] for definition)

$$F = (\sigma, \theta, \mathbf{id}, \mathbf{id}) : \Omega^{-1}G \to \Omega^{-1}(E/\!\!/A).$$

σ is defined levelwise by σ_n, and for any $\tau : [m] \to [n]$ in Δ, θ is the function $\theta_\tau : G_n \to A_m$ defined by $\theta_\tau(g) = \tau^*\sigma_n(g)(\sigma_m\tau^*(g))^{-1}$, for $g \in G_n$. $\theta_\tau(g)$ is an element of A_m for any $g \in G_n$, as we have

$$p_m(\tau^*\sigma_n(g)(\sigma_m\tau^*(g))^{-1}) = \tau^*p_n\sigma_n(g)(p_m\sigma_m\tau^*(g))^{-1} = \tau^*(g)(\tau^*(g))^{-1} = \mathbf{1}_G,$$

so that it is in the kernel of p_m and thus is an element of A_m. The necessary modifications are given by identities. One modification is trivial to see the other follows because we have the following commutative diagram for any pair of composable morphisms $[k] \xrightarrow{\psi} [m] \xrightarrow{\tau} [n]$ in Δ

$$\begin{tikzcd} F_k(\tau\psi)^* \arrow[r, "\theta_{\tau\psi}"] \arrow[d, "\theta_\psi\tau^*"'] & (\tau\psi)^*F_i \\ \psi^*F_j\tau^* \arrow[ur, "\psi^*\theta_\tau"'] & \end{tikzcd}$$

It is commutative because for all $g \in G_n$

$$\begin{aligned}
&\psi^*\phi_\tau(g)\phi_\psi\tau^*(g)\sigma_k(\tau\psi)^*(g)\\
&= \psi^*\phi_\tau(g)\psi^*\sigma_m\tau^*(g)\\
&= \psi^*(\phi_\tau(g)\sigma_m\tau^*(g))\\
&= (\tau\psi)^*\sigma_n(g)\\
&= \phi_{\tau\sigma}(g)\sigma_k(\tau\psi)^*(g)
\end{aligned}$$

After multiplying both sides by $(\sigma_k(\tau\psi)^*(g))^{-1}$ we have the desired identity.

Theorem 5.2.3. *Let*

$$A \xrightarrow{i} E \xrightarrow{p} G$$

be a diagram of simplicial groups that is levelwise a central extension of groups. The homotopy class of the central extension under Corollary 4.4.9 corresponds to the homotopy class of the composite

$$\Delta\int_\Delta \Omega^{-1}G \xrightarrow{\Delta\int_\Delta F} \Delta\int_\Delta \Omega^{-1}E/\!\!/A \to \Delta\int_\Delta \Omega^{-1}(*/\!\!/A)$$

in $[\Delta\int_\Delta \Omega^{-1}G, \Delta\int_\Delta \Omega^{-1}(*/\!\!/A)]_{\mathbf{sSet}}$.

Proof. F is built from levelwise cross sections of p, we have the following commutative diagram of lax Δ-homomorphisms

$$\begin{tikzcd} \Omega^{-1}G & \Omega^{-1}(E/\!\!/A) \arrow[l, "p"'] \arrow[r] & \Omega^{-1}(*/\!\!/A) \\ & \Omega^{-1}G \arrow[ul, equal] \arrow[u, "F"] \arrow[ur] & \end{tikzcd}.$$

Applying the Grothendieck construction, we get a diagram of homomorphisms of bicategories

$$\begin{tikzcd} \int_\Delta \Omega^{-1}G & \int_\Delta \Omega^{-1}(E/\!\!/A) \arrow[l, "\int_\Delta p"'] \arrow[r] & \int_\Delta \Omega^{-1}(*/\!\!/A) \\ & \int_\Delta \Omega^{-1}G \arrow[ul, equal] \arrow[u, "\int_\Delta F"] \arrow[ur] & \end{tikzcd}$$

By applying the geometric nerve we get a commutative diagram of simplicial sets

$$\begin{tikzcd} \Delta\int_\Delta \Omega^{-1}G & \Delta\int_\Delta \Omega^{-1}(E/\!\!/A) \arrow[l, "\Delta\int_\Delta p"', "\sim"] \arrow[r] & \Delta\int_\Delta \Omega^{-1}(*/\!\!/A) \\ & \Delta\int_\Delta \Omega^{-1}G \arrow[ul, equal] \arrow[u, "\Delta\int_\Delta F", "\sim"'] \arrow[ur] & \end{tikzcd}$$

Thus from the levelwise set theoretic cross sections we obtain a cocycle of simplicial sets in the same path component as the cocycle

$$\Delta\int_\Delta \Omega^{-1}G \xleftarrow[\sim]{\Delta\int_\Delta p} \Delta\int_\Delta \Omega^{-1}(E/\!\!/A) \longrightarrow \Delta\int_\Delta \Omega^{-1}(*/\!\!/A)$$

We need to express this cocycle in yet one more way. The following theorem is a generalization to bicategories of a celebrated result by Thomason.

Theorem 5.2.4. *[6] Suppose a category I is given. To every functor $\mathcal{C} : I^{op} \to$ **Bicat** there exists a natural weak equivalence of simplicial sets*

$$\eta : hocolim_I \Delta\mathcal{C} \xrightarrow{\sim} \Delta \int_I \mathcal{C}$$

Applying this we obtain commutative diagram

$$\begin{tikzcd} hocolim_\Delta \Delta\Omega^{-1}G \arrow[d, "\sim"] & hocolim_\Delta \Delta\Omega^{-1}(E/\!\!/A) \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & hocolim_\Delta \Delta\Omega^{-1}(*/\!\!/A) \arrow[d, "\sim"] \\ \Delta\int_\Delta \Omega^{-1}G & \Delta\int_\Delta \Omega^{-1}(E/\!\!/A) \arrow[l, "\Delta\int_\Delta p", "\sim"'] \arrow[r] & \Delta\int_\Delta \Omega^{-1}(*/\!\!/A) \end{tikzcd}$$

Without dwelling on the explicit nature of the homotopy colimit, we note that for any simplicial category $\mathcal{C}$ there is a natural weak equivalence of simplicial sets [3]

$$hocolim_{\Delta}\Delta\mathcal{C} \xrightarrow{\sim} diag\Delta\mathcal{C},$$

Therefore, together with the weak equivalence of theorem 2.3.10, and the isomorphism of lemma 5.1.5, we obtain the following diagram of simplicial sets

$$\begin{tikzcd}
hocolim_{\Delta}\Delta\Omega^{-1}G \arrow[d, "\sim"] & hocolim_{\Delta}\Delta\Omega^{-1}(E/\!\!/A) \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & hocolim_{\Delta}\Delta\Omega^{-1}(*/\!\!/A) \arrow[d, "\sim"] \\
\overline{W}G & \overline{W}TN(E/\!\!/A) \arrow[l, "\sim"'] \arrow[r] & \overline{W}^2 A
\end{tikzcd}$$

□

5.3 Classical Correspondence as Special Case of Corollary 4.4.9

Every central extension of groups gives rise to a central extension of constant simplicial groups. Now central extensions of groups are identified if they are isomorphic and central extensions of constant simplicial groups are identified if they are in the same path component. Therefore it is natural to ask how are isomorphism classes of central extensions related to weak path components of their corresponding constant simplicial group extensions. We will show, that as expected, there is a natural bijection between them, and that it allows one to recover the classic correspondence as a special case of Corollary 4.4.9.

Proposition 5.3.1. *Let A be an abelian group and G be a group. Then*

$$\mathbf{Iso}(G, A) \cong \pi_0\mathbf{C}(\mathbf{G}, \mathbf{A})_{\mathbf{sGrp}}$$

Proof. Define a function $\eta : \mathbf{Iso}(G, A) \to \pi_0\mathbf{C}(\mathbf{G}, \mathbf{A})_{\mathbf{sGrp}}$ by sending the isomorphism class of central extension of groups to the path component of its corresponding extension of constant simplicial groups. Clearly this is well defined and we will now construct its inverse function.

Given a central extension of the constant simplicial group G by the constant simplicial abelian group A, $A \to E \to G$, we look at its corresponding cocycle under Theorem 4.4.8 and then pullback along the universal bundle.

$$\begin{tikzcd}
 & WA \times_{\overline{W}A} (WA \times_A E) \arrow[r] \arrow[d] & WA \arrow[d] \\
G & WA \times_A E \arrow[l, "\sim"'] \arrow[r] & \overline{W}A
\end{tikzcd}$$

There is a free A-action on WA, $A \times WA \to A$ given levelwise by

$$\bar{a}_n \cdot (a_n, a_{n-1}, \cdots, a_0) = (-\bar{a}_n + a_n, a_{n-1}, \cdots, a_0)$$

This induces a free A-action on $WA \times_{\overline{W}A} (WA \times_A E)$, which in turn induces a free A-action on $\pi_0(WA \times_{\overline{W}A} (WA \times_A E))$.

To see that the induced action on $\pi_0(WA \times_{\overline{W}A} (WA \times_A E))$ is free, assume $\overline{a} \cdot [e, a] = [e, a]$, where $[-]$ denotes the path component. Then $[e, a - \overline{a}] = [e, a]$, meaning $(e, a - \overline{a})$ and (e, a) are in the same path component; therefore there exists an element $(e_1, a_1, a_1') \in (WA \times_{\overline{W}A} (WA \times_A E))_1$ such that $(a_1' \cdot d_0(e_1), a_1 + a_1') = (e, a - \overline{a})$ and $(d_1(e_1), a_1) = (e, a)$. Thus we have that $a_1 = a$, $a_1' = -\overline{a}$, and $(e_1, -\overline{a})$ is a representative of an equivalence class of $\pi_1(WA \times_A E, e)$. However, $\pi_1(WA \times_A E, e) \cong 0$ so that the induced map $\pi_1(WA \times_A E, e) \to \pi_1(\overline{W}A, *)$ must be the trivial morphism. Thus $-\overline{a}$ must be homotopic to 0 in $(\overline{W}A)_1$, so that $\overline{a} = 0$, as the only element homotopic to 0 in $(\overline{W}A)_1$ is itself.

Moreover, $\pi_0(WA \times_{\overline{W}A} (WA \times_A E))/A \cong \pi_0(WA \times_A E) \cong G$, so that we have

$$A \to \pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \to G$$

is a central extension of groups.

The inverse function $\eta^{-1} : \pi_0 \mathbf{C}(\mathbf{G}, \mathbf{A})_{\mathbf{sGrp}} \to \mathbf{Iso}(G, A)$ is defined by sending the path component of $A \to E \to G$ to the isomorphism class of

$$0 \to A \to \pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \to G \to 1$$

To see this is well defined, let $A \to E \to G$ and $A \to E' \to G$ be in the same path component. Then there is a zig-zag of A-equivariant weak equivalences between E and E', inducing an isomorphism $\pi_0(E) \cong \pi_0(E')$, which gives an isomorphism $\pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \cong \pi_0(WA \times_{\overline{W}A} (WA \times_A E'))$ fitting into the commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & \pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \arrow[r] \arrow[d, "\cong"] & G \arrow[d, equal] \\
A \arrow[r] & \pi_0(WA \times_{\overline{W}A} (WA \times_A E')) \arrow[r] & G
\end{tikzcd} .$$

To see they are inverses of each other, let $0 \to A \to E \to G \to 1$ be a central extension of groups. Then

$$\eta^{-1}\eta(0 \to A \to E \to G \to 1) = 0 \to A \to \pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \to G \to 1,$$

which we will now show is in the same isomorphism class as $0 \to A \to E \to G \to 1$.

We have already shown, in the proof of Theorem 4.4.8, that $A \to WA \times_{\overline{W}A} (WA \times_A E) \to G$ and $A \to E \to G$ are in the same path component. In particular, this gives an A-equivariant weak equivalence of simplicial groups

$$WA \times_{\overline{W}A} (WA \times_A E) \xrightarrow{\sim} E,$$

inducing an isomorphism

$$\pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \cong E,$$

which fits into the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & \pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \arrow[r] \arrow[d, "\cong"] & G \arrow[d, equal] \\ A \arrow[r] & E \arrow[r] & G \end{tikzcd}.$$

For the other direction, let $A \to E \to G$ be a central extension of a constant simplicial group G by a constant simplicial abelian group A. Then

$$\eta\eta^{-1}(A \to E \to G) = A \to \pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \to G,$$

which we will now show is in the same path component as $A \to E \to G$.

We have already shown in the proof of Theorem 4.4.8 that $A \to WA \times_{\overline{W}A} (WA \times_A E) \to G$ and $A \to E \to G$ are in the same path component. If we look at the long fibration sequence of

$$A \to E \to E/A$$

we see that, for all $n > 0$, $\pi_n(E)$ is trivial. Therefore, because $A \to WA \times_{\overline{W}A} (WA \times_A E) \to G$ and $A \to E \to G$ are in the same path component, $\pi_i(WA \times_{\overline{W}A} (WA \times_A E))$ is trivial for all $i > 0$. Thus the natural map

$$WA \times_{\overline{W}A} (WA \times_A E) \xrightarrow{\sim} \pi_0(WA \times_{\overline{W}A} (WA \times_A E))$$

is a weak equivalence. This shows $A \to \pi_0(WA \times_{\overline{W}A} (WA \times_A E)) \to G$ is in the same path component as $A \to WA \times_{\overline{W}A} (WA \times_A E) \to G$, and thus is in the same path component as $A \to E \to G$. $\square$

Proposition 5.3.1 together with Corollary 4.4.9 gives a correspondence

$$\mathbf{Iso}(G, A) \cong [\overline{W}G, \overline{W}^2 A]_{\mathbf{sSet}}$$

which we can compare with the classic correspondence using theorem 5.2.2. Then by Remark 5.2.1 we see that the classic correspondence for isomorphism classes of central extensions of groups is a special case of our correspondence when viewed as central extensions of constant simplicial groups.

CHAPTER 6

CENTRAL EXTENSIONS OF PRESHEAVES OF SIMPLICIAL GROUPS

We now will extend our results for central extensions of simplicial groups to central extensions of simplicial presheaves over a small site $\mathcal{C}$.

6.1 Simplicial Presheaves and Presheaves of Simplicially Enriched Groupoids

For the rest of this paper let $\mathcal{C}$ be a fixed small site. We will now discuss in some detail the categories of simplicial presheaves and presheaves of simplicially enriched groupoids. (It should be noted that often in the literature they are just called simplicial groupoids, but we explicitly call them simplicially enriched groupoids, because we also deal with presheaves of simplicial groupoids that do not have constant objects.) The category of simplicial presheaves, denoted **Pre**(**sSet**) is the category of presheaves in **sSet** on $\mathcal{C}$. The category of simplicial presheaves has a closed model structure given by Jardine [14]:

- Cofibrations are those morphisms of simplicial presheaves that are sectionwise monomorphisms.
- Weak equivalences are topological weak equivalences. If X is a simplicial presheaf on $\mathcal{C}$ and $x \in X(U)_0$, there is a sheaf $\tilde{\pi}_n(X|_U, x)$ on $\mathcal{C}/U$ defined as the sheaf associated to the presheaf

  $$(\phi : V \to U) \mapsto \pi_n(|X(V)|, x_V),$$

  where $|X(V)|$ is the realization of the simplicial set $X(V)$, and $\pi_n(|X(V)|, x_V)$ is the n-th homotopy group of the space $|X(V)|$, based at $x_v = \phi^*(x)$. The sheaf of path components $\tilde{\pi}_0(X)$ is defined similarly. Then a morphism $f : X \to Y$ of simplicial presheaves is a topological weak equivalence if it induces isomorphisms of sheaves:

  $$f_* : \tilde{\pi}_n(X|_U, x) \to \tilde{\pi}_n(Y|_U, f(x)), U \in \mathcal{C}, x \in X(U)_0$$

  $$f_* : \tilde{\pi}_0(X) \to \tilde{\pi}_0(Y)$$

- Fibrations are morphisms that have the right lifting property with respect to maps that are both cofibrations and topological weak equivalences. Sometimes they are called *global fibrations.*

For a simplicially enriched groupoid G, there is a simplicial set $\overline{W}G$ which is defined by analogy and extends the corresponding object for simplicial groups. Similarly, one can define the loop groupoid of a simplicial set X, which is left adjoint to $\overline{W}$. We will not define the loop groupoid, as we will only make use of the formal properties of the adjunction. For more details see [8].

Definition 6.1.1. [8] Let G be a simplicially enriched groupoid. $\overline{W}G$ is the simplicial set which has as 0-simplicies the objects of G. It has as n-simplicies ($n > 0$) the sequences of maps in G

$$X_n \xrightarrow{g_{n-1}} X_{n-1} \xrightarrow{g_{n-2}} \cdots \xrightarrow{g_1} X_1 \xrightarrow{g_0} X_0, \text{ where } g_i \in hom_i(X_{i+1}, X_i) \text{ for all } i,$$

with face and degeneracy maps given by

$$d_i(g_0, \cdots, g_{n-1}) = (g_0, \cdots, g_{n-i-1}, g_{n-i} \cdot d_0 g_{n-i+1}, d_1 g_{n-i+2}, \cdots, d_{i-1} g_{n-1})$$

$$s_i(g_0, \cdots, g_{n-1}) = (g_0, \cdots, g_{n-i}, \mathbf{id}, s_0 g_{n-i+1}, s_1 g_{n-i+2}, \cdots, s_{i-1} g_{n-1})$$

Remark 6.1.2. If a simplicially enriched groupoid G has one object, then this definition clearly reduces to that for simplicial groups. See the remark after Definition 4.1.7.

The functor $\overline{W}$ applied sectionwise, allows to transfer the model structure on $\mathbf{Pre}(\mathbf{sSet})$ to one on the category of presheaves of simplicially enriched groupoids, $\mathbf{Pre}(\mathbf{sGrpd})$ [18].

- Fibrations are morphisms of presheaves of simplicially enriched groupoids $f : G \to H$ such that $\overline{W}(f) : \overline{W}G \to \overline{W}H$ is a global fibration of simplicial presheaves.
- Weak equivalences are morphisms of presheaves of simplicially enriched groupoids $f : G \to H$ such that $\overline{W}(f) : \overline{W}G \to \overline{W}H$ is a topological weak equivalence of simplicial presheaves.
- Cofibrations are morphisms that have the left lifting property with respect to morphisms that are both fibrations and weak equivalences.

In fact the authors of [18] give an explicit description of the weak equivalences in this model structure. For any presheaf of simplicially enriched groupoids G on $\mathcal{C}$, any object $U \in \mathcal{C}$, and any $x \in Ob(G(U))$ we have that $G(U)(x,x)$ is a simplicial group. Therefore we have a presheaf $\pi_n^{simp}(G|_U, x, *)$ of simplicial homotopy groups based at $*$ on the over category $\mathcal{C}/U$. Defined by:

$$(\mathcal{C}/U)^{op} \to \mathbf{Grp}$$

$$(\phi : V \to U) \mapsto \pi_n(G(V)(x_V, x_V), *_V),$$

where x_V and $*_V$ are the images of x and $*$ in $X(V)$ respectively under the induced map $\phi^* : X(U) \to X(V)$. The simplicial homotopy groups exist because a simplicial group is a Kan Complex (see Lemma I.3.4 [11]). One defines $\pi_0^{simp}(G|_U, x)$ and $\pi_0^{simp}(G)$ similarly.

A map $f : G \to H$ of presheaves of simplicially enriched groupoids is a weak equivalence if it induces isomorphisms of sheaves

$$f_* : \tilde{\pi}_n^{simp}(G|_U, x, *) \to \tilde{\pi}_n^{simp}(H|_U, f(x), f(*)), n \geq 1, U \in \mathcal{C}, x \in Ob(X(U)), * \in X(U)(x,x)_0$$

$$f_* : \tilde{\pi}_0^{simp}(G|_U, x) \to \tilde{\pi}_0^{simp}(H|_U, f(x)), U \in \mathcal{C}, x \in Ob(X(U))$$

$$f_* : \tilde{\pi}_0^{simp}(G) \to \tilde{\pi}_0^{simp}(H)$$

Remark 6.1.3. By [14, Proposition 1.18] topological weak equivalences are the same as combinatorial weak equivalences, hence there is no mention of the realization in this description.

Theorem 6.1.4. *[18, Corollary 2.14] The adjunction $G : \mathbf{Pre(sSet)} \rightleftarrows \mathbf{Pre(sGrpd)} : \overline{W}$ is a Quillen equivalence.*

Corollary 6.1.5. *The adjunction $G : \mathbf{Pre(sSet)}_* \rightleftarrows \mathbf{Pre(sGrpd)}_* : \overline{W}$ is a Quillen equivalence between the induced model structure for pointed simplicial presheaves and pointed presheaves of simplicially enriched groupoids.*

Lemma 6.1.6. *The induced model structure on $\mathbf{Pre(sGrpd)}_*$ is a right proper.*

Proof. By [18, Theorem 3.15] $\mathbf{Pre(sGrpd)}$ is right proper. $\square$

6.2 Presheaves of Simplicial Group Object Actions

We now discuss group object actions in the category of presheaves of simplicial groups and prove some technical results involving their relationship with $\overline{W}$.

Definition 6.2.1. Let $\rho : G \times X \to X$ be an action in the category of simplicial presheaves. Then the *action presheaf simplicial groupoid*, denoted $X /\!\!/ G$, is the presheaf of simplicial groupoids defined by $(X/\!\!/G)_n = X_n /\!\!/ G_n$ with structure maps induced from those in G and X.

Remark 6.2.2. The action presheaf simplicial groupoid is also defined for an action in the category of presheaves of simplicial groups. That is, for a presheaf of simplicial abelian groups A acting on a presheaf of simplicial groups G. In this case all the structure maps are group homomorphisms.

Remark 6.2.3. Note the action presheaf simplical groupoid is a presheaf of simplicial objects in the category of groupoids, it is not necessarily a presheaf of simplicially enriched groupoids.

Remark 6.2.4. Many of the previously discussed functors extend to functors in the various presheaf categories when applied sectionwise. In this case, we use the same symbol. For example, if X is a presheaf of bisimplicial simplicial sets, then $T(X)$ is the presheaf of simplicial sets defined sectionwise by $T(X(U))$.

Lemma 6.2.5. *Let G be a presheaf of simplicial groups, then we have the following isomorphism of presheaves of simplicial groups:*

$$TN(*/\!\!/G) \cong \overline{W}G.$$

Proof. The isomorphism is given sectionwise by Lemma 4.3.12. $\square$

Lemma 6.2.6. *Let A be a presheaf of simplicial abelian groups and let $A/\!\!/A$ denote the action presheaf simplicial groupoid given by multiplication. Then, as presheaves of simplicial groups,*

$$TN(A/\!\!/A) \cong WA,$$

and

$$TN(*/\!\!/A) \cong \overline{W}A$$

Proof. The isomorphism are given sectionwise by Lemma 4.3.15 $\square$

Definition 6.2.7. Let X be a presheaf of simplicial sets and G a presheaf of simplicial groups such that X has a G-action. The *Borel construction* $WG \times_G X$ is the quotient of $WG \times X$ by the diagonal action.

Similarly, if G is a presheaf of simplicial groups and A a presheaf of simplicial abelian groups such that G has an A-action in $\mathbf{Pre}(\mathbf{sGrp})$, the Borel constuction $WA \times_A G$ is a presheaf of simplicial groups.

Lemma 6.2.8. *Let G be a presheaf of simplicial groups, and let X be a simplicial presheaf with a G-action. Then, as simplicial presheaves,*

$$WG \times_G X \cong TN(X /\!\!/ G).$$

Similarly, if A is a presheaf of simplicial abelian groups and G is a presheaf of simplical groups with an A-action in $\mathbf{Pre}(\mathbf{sGrp})$, then, as simplicial groups

$$WA \times_A G \cong TN(G /\!\!/ A).$$

Proof. The isomorphism is given sectionwise by that of Lemma 4.3.17 $\square$

Lemma 6.2.9. *Let $\rho : G \times X \to X$ be a free action in $\mathbf{Pre}(\mathbf{sSet})$. Then we have the following weak equivalence of simplicial presheaves:*

$$WG \times_G X \xrightarrow{\sim} X/G.$$

Similarly, there is a weak equivalence of presheaves of simplicial groups

$$WA \times_A G \xrightarrow{\sim} G/A,$$

if $\rho : A \times G \to G$ is a free action in $\mathbf{Pre}(\mathbf{sGrp})$.

Proof. We prove the statement for a free action in $\mathbf{Pre}(\mathbf{sGrp})$, as the other proof is the same after forgetting some of the group structures. Let $w : G /\!\!/ A \to G/A$ be defined sectionwise by setting $w(U) : (G /\!\!/ A)(U) \to (G/A)(U)$ to be the weak equivalence of Lemma 4.3.18. Each $w(U)$ is a weak equivalence of simplicial groups, so that if we treat these as simplicially enriched groupoids with single object, it is a weak equivalence of simplicially enriched groupoids. Sectionwise w is a weak equivalence of simplicially enriched groupoids and $\overline{W}$ is applied sectionwise and preserves weak equivalences, so that $\overline{W}(w)$ is sectionwise a weak equivalence of simplicial sets. Sectionwise weak equivalences are also weak equivalences (see Lemma 4.1 [16]) of simplicial presheaves so that $\overline{W}(w)$ is a weak equivalence in $\mathbf{sPre}$. Thus, w is a weak equivalence of presheaves of simplicially enriched groupoids, hence a weak equivalence of presheaves of simplical groups $\square$

Lemma 6.2.10. *Let $\rho : A \times G \to G$ be a free action in $\mathbf{Pre}(\mathbf{sGrp})$. Then we have the following weak equivalence of presheaves of simplicial groups:*

$$WA \times_{\overline{W}A} (WA \times_A G) \xrightarrow{\sim} G$$

Proof. The weak equivalence is defined sectionwise by the weak equivalence of Lemma 4.3.19, and sectionwise weak equivalences are weak equivalence of presheaves of simplicial groups (see Lemma 4.1 [16]). $\square$

6.3 Central Extensions of Presheaves of Simplicial Groups

We now define central extensions of presheaves of simplicial groups and define two functors

$$F : H(G, \overline{W}A)_{\mathbf{Pre(sGrpd)}_*} \to \mathbf{C(G, A)}_{\mathbf{Pre(sGrp)}}$$

and

$$\overline{F} : \mathbf{C(G, A)}_{\mathbf{Pre(sGrp)}} \to H(G, \overline{W}A)_{\mathbf{Pre(sGrpd)}_*}$$

that we prove induce a bijection on path components,

$$\pi_0(\mathbf{C(G, A)}_{\mathbf{Pre(sGrp)}}) \cong \pi_0(H(G, \overline{W}A)_{\mathbf{Pre(sGrpd)}_*}).$$

Definition 6.3.1. A morphism of presheaves of simplicial groups $f : G \to H$ is a weak equivalence if the induced map $f : \overline{W}G \to \overline{W}H$ is a weak equivalence of simplicial presheaves.

Definition 6.3.2. Let $A \in \mathbf{Pre}(\mathbf{sAb})$ and $E, G \in \mathbf{Pre}(\mathbf{sGrp})$. We say that E is a central extension of G by A if we have the following:

- E has a free A-action $\Phi : A \times E \to E$ in $\mathbf{Pre}(\mathbf{sGrp})$.
- G has a trivial A-action.
- There is an A-equivariant morphism $p : E \to A$ such that the induced map $\tilde{p} : E/A \xrightarrow{\sim} G$ is a weak equivalence of presheaves of simplicial groups.

We will denote a central extension by

$$A \xrightarrow{i} E \xrightarrow{p} G,$$

just as we did for simplicial groups. p is the morphisms given in the definition and i is the inclusion given by the composite

$$A \xrightarrow{\mathbf{id}\times 1_E} A \times E \xrightarrow{\Phi} E,$$

where 1_E is defined by $1_E(U)$ is the identity element of $E(U)$.

Definition 6.3.3. The category $\mathbf{C(G,A)_{Pre(sGrp)}}$ of central extensions of presheaves of simplicial groups is the category whose objects are central extensions of simplicial groups, and whose morphisms are given by commutative diagrams of equivariant morphisms:

$$\begin{tikzcd} E \arrow[dd, "f"', "\sim"] \arrow[dr, "p"] & \\ & G \\ E' \arrow[ur, "p'"] & \end{tikzcd},$$

where f is a weak equivalence of presheaves of simplicial groups.

Example 6.3.4. Let $A \xrightarrow{i} E \xrightarrow{p} G$ be a diagram of presheaves of simplicial groups such that sectionwise

$$A_n(U) \xrightarrow{i_n(U)} E_n(U) \xrightarrow{p_n(U)} G_n(U)$$

is a central extension of groups. Then $A \xrightarrow{i} E \xrightarrow{p} G$ is a central extension of presheaves of simplicial groups. This is because E has a free A-action in the category of presheaves of simplicial groups given by $A \times E \xrightarrow{i\times \mathbf{id}_E} E \times E \xrightarrow{m} E$ where m is the multiplication map of E. Clearly G has a trivial A action and $p : E \to G$ is an A-equivariant morphism such that $\tilde{p} : E/A \to G$ is an isomorphism.

Lemma 6.3.5. *Let G be a presheaf of simplicial groups and A be a presheaf of simplicial abelian groups. A cocycle of pointed presheaves of simplicially enriched groupoids*

$$\begin{tikzcd} & Z_* \arrow[dl, "f"', "\sim"] \arrow[dr, "g"] & \\ G_* & & \overline{W}A_* \end{tikzcd} \tag{6.1}$$

where G_* and A_* are pointed presheaves of simplicially enriched groupoids with single object. Let $Z_*(*,*)$ be defined sectionwise to be the simplical group of morphisms $Z_*(U)(*,*)$. Then we have:

$$\begin{tikzcd} & Z_* \arrow[dl, "f"', "\sim"] \arrow[dr, "g"] & \\ G & & \overline{W}A \\ & Z_*(*,*) \arrow[ul, "\tilde{f}", "\sim"'] \arrow[ur, "\tilde{g}"'] \arrow[uu, "i"', "\sim"] & \end{tikzcd}$$

Thus a cocycle such as Equation 6.1 in the Lemma is always equivalent to one formed from presheaves of simplicial groups.

Proof. We need only check that the natural inclusion $i : Z_*(*,*) \to Z_*$ is a weak equivalence of presheaves of simplicially enriched groupoids. Using the explicit description for weak equivalences of presheaves of simplicially enriched groupoids, we see that the only thing to check is that it induces an isomorphism $\tilde{\pi}_0^{simp}(Z_*(*,*)) \cong \tilde{\pi}_0^{simp}(Z_*)$, because i induces equalities $\pi_n^{simp}(Z_*(*,*)|_U,*,v) = \pi_n^{simp}(Z_*|_U,*,v)$ and $\pi_0^{simp}(Z_*(*,*)|_U,*) = \pi_0^{simp}(Z_*|_U,*)$. f is a weak equivalence so it induces an isomorphism $\tilde{\pi}_0^{simp}(Z_*) \cong \tilde{\pi}_0^{simp}(G) \cong *$ (Here $*$ represents the constant sheaf). Thus $\tilde{\pi}_0^{simp}(Z_*(*,*)) \cong \tilde{\pi}_0^{simp}(Z_*)$. □

Remark 6.3.6. We have the following cocycle of pointed presheaves of simplicially enriched groupoids:

$$\begin{tikzcd} \overline{W}A & * \times_A WA \arrow[l, "\cong"'] & WA \times_A WA \arrow[l, "\sim"'] \arrow[ll, bend left, "w"] \arrow[r, "\sim"] \arrow[rr, bend right, "v"'] & WA \times_A * \arrow[r, "\cong"] & \overline{W}A \end{tikzcd}$$

Unlike the cocycle in Definition 4.4.4, these are not necessarily fibrations in the model structure on presheaves of simplicially enriched groupoids, so one can not simple post-compose by pullback. Instead, as we are only concerned with path components, we can compose path components by using their correspondence with homotopy classes of maps (See Theorem 3.4.4).

Definition 6.3.7. Let $\mathbf{q} : \pi_0 H(G, \overline{W}A)_{\mathbf{Pre(sGrpd)}_*} \to \pi_0 H(G, \overline{W}A)_{\mathbf{Pre(sGrpd)}_*}$ be the function defined by sending the path component of a cocycle

$$\begin{tikzcd} & Z \arrow[dl, "f"', "\sim"] \arrow[dr, "g"] & \\ G & & \overline{W}A \end{tikzcd}$$

to $vw^{-}1gf^{-1} \in [G, \overline{WA}]$ and then to the corresponding cocycle under the bijection of Theorem 3.4.4.

The function $\mathbf{q}$ has a clear inverse function $\overline{\mathbf{q}}$ which is given by sending the path component of a cocycle to $wv^{-1}gf^{-1} \in [G, \overline{W}A]$ and then to the corresponding cocycle under the bijection of Theorem 3.4.4.

Let $G \xleftarrow{\sim} Z_* \to \overline{W}A$ be a cocycle in $H(G, \overline{W}A)_{\mathbf{Pre(sGrpd)}_*}$, and then replace it with the cocycle $G \xleftarrow{\sim} Z_*(*,*) \to \overline{W}A$ of Lemma 6.3.5. Notice WA has a free A-action $A \times WA \to WA$ given sectionwise and levelwise by

$$\bar{a}_n \cdot (a_n, a_{n-1}, \cdots, a_0) = (-\bar{a}_n + a_n, a_{n-1}, \cdots, a_0)$$

Pullback along $WA \to \overline{W}A$:

$$\begin{tikzcd}
 & WA \times_{\overline{W}A} Z_*(*,*) \arrow[r] \arrow[d] & WA \arrow[d] \\
G & Z_*(*,*) \arrow[l, "f"', "\sim"] \arrow[r] & \overline{W}A
\end{tikzcd}$$

and give $WA \times_{\overline{W}A} Z_*(*,*)$ the free A-action, $\phi : A \times (WA \times_{\overline{W}A} Z) \to WA \times_{\overline{W}A} Z$, induced by the action on WA above. G has a trivial A-action, and the composite $p : WA \times_{\overline{W}A} Z_*(*,*) \to Z_*(*,*) \xrightarrow{\sim} G$ is an A-equivariant morphism. Moreover $\tilde{p} = f : Z_*(*,*) \xrightarrow{\sim} G$ is a weak equivalence, so that

$$A \to WA \times_{\overline{W}A} Z_*(*,*) \xrightarrow{p} G$$

is a central extension of G by A according to definition 6.3.2.

Definition 6.3.8. Let $F : H(G, \overline{W}A)_{\mathbf{Pre(sGrpd)}_*} \to \mathbf{C(G,A)}_{\mathbf{Pre(sGrp)}}$ be the functor sending a cocycle

$$G \xleftarrow{\sim} Z_* \to \overline{W}A$$

to the central extension

$$A \to WA \times_{\overline{W}A} Z_*(*,*) \xrightarrow{p} G.$$

The functor sends a morphism of cocycles

$$\begin{tikzcd} & Z_* \arrow[dl, "\sim"'] \arrow[dd, "\sim"', "\alpha"] \arrow[dr] & \\ G & & \overline{W}A \\ & Z'_* \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd}$$

to the morphism of central extensions

$$\begin{tikzcd} WA \times_{\overline{W}A} Z_*(*,*) \arrow[dd, "f"', "\sim"] \arrow[dr, "p"] & \\ & G \\ WA \times_{\overline{W}A} Z'_*(*,*) \arrow[ur, "p'"] & \end{tikzcd},$$

where $f : WA \times_{\overline{W}A} Z_*(*,*) \to WA \times_{\overline{W}A} Z'_*(*,*)$ is the morphism given by the universal property of $WA \times_{\overline{W}A} Z'_*(*,*)$

$$\begin{tikzcd} WA \times_{\overline{W}A} Z_*(*,*) \arrow[dr, "f", "\sim"'] \arrow[drr, bend left] \arrow[dd] & & \\ & WA \times_{\overline{W}A} Z'_*(*,*) \arrow[r] \arrow[d] & WA \arrow[d] \\ Z_*(*,*) \arrow[r, "\alpha", "\sim"'] & Z'_*(*,*) \arrow[r] & \overline{W}A \end{tikzcd}$$

Remark 6.3.9. f is a weak equivalence, because $WA \times_{\overline{W}A} Z'_*(*,*) \to Z'_*(*,*)$ is a sectionwise fibration (hence a local fibration) and weak equivalences are preserved along pullbacks by local fibrations (lemma 4.37 [16]).

Definition 6.3.10. The functor $\overline{F} : \mathbf{C(G,A)}_{\mathbf{Pre(sGrp)}} \to H(G, \overline{W}A)_{\mathbf{Pre(sGrpd)}_*}$ sends the central extension $A \xrightarrow{i} E \xrightarrow{p} G$ in $\mathbf{C(G,A)}_{\mathbf{sGrp}}$ to the cocycle

$$G \xleftarrow{\simeq} E/A \xleftarrow{\simeq} WA \times_A E \to \overline{W}A,$$

thought of as a cocycle of pointed presheaves of simplical groupoids with single object. It sends a morphism of central extensions

$$
\begin{tikzcd}
E \arrow[dd, "g"', "\sim"] \arrow[dr, "p"] & \\
& G \\
E' \arrow[ur, "p'"] &
\end{tikzcd}
$$

to the morphism of cocycles

$$
\begin{tikzcd}
& WA \times_A E \arrow[dl, "\sim"'] \arrow[dd, "\sim"', "\mathbf{id}\times g"] \arrow[dr] & \\
G & & \overline{W}A \\
& WA \times_A E' \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd}
$$

Now we can state the main result of this chapter:

Theorem 6.3.11. *F and $\overline{F}$ induce bijections on path components*

$$\overline{F} : \pi_0(\mathbf{C}(\mathbf{G},\mathbf{A})_{\mathbf{Pre(sGrp)}}) \cong \pi_0(H(G,\overline{W}A)_{\mathbf{Pre(sGrpd)}_*}) : F$$

Corollary 6.3.12.

$$\pi_0(\mathbf{C}(\mathbf{G},\mathbf{A})_{\mathbf{Pre(sGrp)}}) \cong [\overline{W}G, \overline{W}^2 A]_{\mathbf{Pre(sSet)}_*}$$

Proof. The model structure on $\mathbf{Pre}(\mathbf{sGrp})$ is right proper, and weak equivalences are closed under finite products, so we have that:

$$\pi_0(H(G,\overline{W}A)_{\mathbf{Pre(sGrpd)}_*}) \cong [G,\overline{W}A]_{\mathbf{Pre(sGrpd)}_*}.$$

$\overline{W}$ is part of a Quillen equivalence between $\mathbf{Pre}(\mathbf{sGrpd})_*$ and $\mathbf{Pre}(\mathbf{sSet})_*$ therefore there is an isomorphism:

$$[G,\overline{W}A]_{\mathbf{Pre(sGrpd)}_*} \cong [\overline{W}G, \overline{W}^2 A]_{\mathbf{Pre(sSet)}_*}.$$

Together with Theorem 6.3.11 we obtain the desired bijection. $\square$

6.4 Proof of Theorem 6.3.11

The proof is similar to that of Theorem 4.4.8. We will show that $\overline{F}$ is right inverse to F on path components and then show that $\overline{\mathbf{q}} \circ \overline{F}$ is left inverse to F on path components. Then, since left and right inverses must be equal, we have that $\overline{\mathbf{q}} \circ \overline{F}$ and $\overline{F}$ induce the same function on path components.

First we will show that $\overline{F}$ is right inverse to F on path components. Notice $F(\overline{F}(A \hookrightarrow E \xrightarrow{p} G))$ is given by:

$$A \xrightarrow{j} WA \times_{\overline{W}A} (WA \times_A E) \xrightarrow{t} G,$$

where t is the composition

$$WA \times_{\overline{W}A} (WA \times_A E) \xrightarrow{\pi} WA \times_A E \xrightarrow{\sim} E/A \xrightarrow{\sim} G,$$

and j is induced by the free A action on $WA \times_{\overline{W}A} (WA \times_A E)$ induced by the free A-action on WA given sectionwise and levelwise by $\bar{a}_n \cdot (a_n, a_{n-1}, \cdots, a_0) = (\bar{a}_n^{-1} a_n, a_{n-1}, \cdots, a_0)$.

From Lemma 6.2.10 we have a weak equivalence $WA \times_{\overline{W}A} (WA \times_A E) \xrightarrow{\sim} E$ that one can check is an A-equivariant morphism. We also have the following commutative diagram of presheaves of simplicial groupoids:

$$\begin{tikzcd}
A/\!\!/A \times_{*/\!\!/A} E/\!\!/A \arrow[r, "\cong"] \arrow[d] & A/\!\!/A \times E \arrow[r] & E \arrow[d] \arrow[dr, "p"] & \\
 & & & G \\
E/\!\!/A \arrow[rr] & & E/A \arrow[ur, "\tilde{p}"] &
\end{tikzcd}$$

After applying TN to the diagram above, we see that $A \xrightarrow{i} E \xrightarrow{p} G$ and $F(\overline{F}(A \xrightarrow{i} E \xrightarrow{p} G))$ are in the same path component of $\mathbf{C(G, A)_{Pre(sGrp)}}$.

Now we will show that $\overline{\mathbf{q}} \circ \overline{F}$ is left inverse to F on path components. Let $G \xleftarrow{\sim} Z_* \xrightarrow{f} \overline{W}A$ be a cocycle in $\mathbf{H}(\mathbf{G}, \overline{\mathbf{W}}\mathbf{A})_{\mathbf{Pre(sGrpd)_*}}$. Then $\overline{F}(F(G \xleftarrow{\sim} Z_* \xrightarrow{f} \overline{W}A))$ is given by

$$G \xleftarrow{\sim} Z_*(*,*) \xleftarrow{\sim} WA \times_A (WA \times_{\overline{W}A} Z_*(*,*)) \to \overline{W}A$$

Notice that $(WA \times_{\overline{W}A} Z_*(*,*))/\!\!/A \to */\!\!/A$ factors as $(Z_*(*,*) \times_{\overline{W}A} WA)/\!\!/A \to WA/\!\!/A \to */\!\!/A$ and we have the following commutative diagram of presheaves of simplicial groupoids.

$$\begin{tikzcd}
(Z_*(*,*) \times_{\overline{W}A} WA)/\!\!/A \arrow[r] \arrow[d, "\sim"] & WA/\!\!/A \arrow[r, "\sim"] \arrow[d, "\sim"] & */\!\!/A \\
Z_*(*,*) \arrow[r, "f"] & \overline{W}A &
\end{tikzcd}$$

After applying TN sectionwise to the diagram we see that $\overline{F}(F(G \xleftarrow{\sim} Z_* \xrightarrow{f} \overline{W}A))$ is in the same path component as $\mathbf{q}(G \xleftarrow{\sim} Z_*(*,*) \xrightarrow{f} \overline{W}A)$, as they correspond to the same morphism in $[G, \overline{W}A]_{\mathbf{Pre(sGrpd)}_*}$. Thus $\overline{F}(F(G \xleftarrow{\sim} Z_* \xrightarrow{f} \overline{W}A))$ is in the same path component as $\mathbf{q}(G \xleftarrow{\sim} Z_* \xrightarrow{f} \overline{W}A))$ by Lemma 6.3.5. Applying $\overline{\mathbf{q}}$ to both cocycles we then see that $G \xleftarrow{\sim} Z_* \xrightarrow{f} \overline{W}A$ is in the same path component as $\overline{\mathbf{q}}(\overline{F}(F(G \xleftarrow{\sim} Z_* \xrightarrow{f} \overline{W}A)))$, proving $\overline{\mathbf{q}} \circ \overline{F}$ is left inverse to F.

CHAPTER 7

APPLICATION TO THE CUP PRODUCT

7.1 Abelian Categories

Definition 7.1.1. An *Ab-category* is a category $\mathcal{A}$, such that for every pair of objects $x, y \in \mathcal{A}$, each $hom(x, y)$ has the structure of an abelian group, and composition distributes over addition. In particular, given a diagram in $\mathcal{A}$ of the form

$$A \xrightarrow{f} B \underset{g}{\overset{g'}{\rightrightarrows}} C \xrightarrow{h} D$$

we have $h \circ (g + g') \circ f = h \circ g \circ f + h \circ g' \circ f$.

Definition 7.1.2. An *additive category* is an Ab-category $\mathcal{A}$ that has a zero object (an object that is both initial and terminal), and has all finite products.

Definition 7.1.3. An *abelian category* is an additive category $\mathcal{A}$ such that

- Every morphism in $\mathcal{A}$ has a kernel and cokernel.
- Every monic in $\mathcal{A}$ is the kernel of its cokernel.
- Every epic in $\mathcal{A}$ is the cokernel of its kernel.

Example 7.1.4. The prototypical examples are the categories of R-modules and abelian groups.

Definition 7.1.5. A chain complex C, in an abelian category $\mathcal{A}$, is a collection of objects of $\mathcal{A}$ $\{C_i\}_{i\in\mathbb{Z}}$, together with morphisms of $\mathcal{A}$ $\{\delta_i : C_i \to C_{i-1}\}_{i\in\mathbb{Z}}$, such that $\delta^2 = \delta_{i-1} \circ \delta_i = 0$ for all $i \in \mathbb{Z}$.

$$\cdots \xrightarrow{\delta_{i+2}} C_{i+1} \xrightarrow{\delta_{i+1}} C_i \xrightarrow{\delta_i} C_{i-1} \xrightarrow{\delta_{i-1}} \cdots$$

δ_i are called the *differentials* of C. A *non-negatively graded chain complex* is a chain complex such that $C_i = 0$ for all $i < 0$. The kernel of δ_i is the *i-cycles* of C, denoted $Z_i = Z_i(C)$. The image of δ_{i+1} is the *i-boundaries* of C, denoted $B_i = B_i(C)$. Because $\delta^2 = 0$ we have that

$$0 \subseteq B_i \subseteq Z_i \subseteq C_i$$

for all i. The i^{th} *homology* of C is the (sub)quotient $H_i(C) = Z_i/B_i$.

Definition 7.1.6. A morphism of chain complexes C, D in an abelian category $\mathcal{A}$ is a collection of morphisms $\{f_i : C_i \to D_i\}_{i\in\mathbb{Z}}$ such that $\delta_i f_i = f_{i-1}\delta_i$ for all i. In particular, they give a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta^C_{i+2}} & C_{i+1} & \xrightarrow{\delta^C_{i+1}} & C_i & \xrightarrow{\delta^C_i} & C_{i-1} & \xrightarrow{\delta^C_{i-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \xrightarrow{\delta^D_{i+2}} & D_{i+1} & \xrightarrow{\delta^D_{i+1}} & D_i & \xrightarrow{\delta^D_i} & D_{i-1} & \xrightarrow{\delta^D_{i-1}} & \cdots
\end{array}$$

Definition 7.1.7. The *category of chain complexes* in an abelian category $\mathcal{A}$, $Ch(\mathcal{A})$, has as objects the chain complexes in $\mathcal{A}$, and as morphisms chain complex morphisms. The *category of non-negatively graded chain complexes*, $Ch_+(\mathcal{A})$, is defined similarly.

We can iterate this construction and talk about chain complexes of chain complexes; these are usually called *double complexes.*

Definition 7.1.8. A *double complex* (or *bicomplex*) in an abelian category $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$, together with maps

$$\delta^h : C_{p,q} \to C_{p-1.q} \text{ and } \delta^v : C_{p,q} \to C_{p,q-1},$$

such that $\delta^h \circ \delta^h = \delta^v \circ \delta^v = \delta^v \circ \delta^h - \delta^h \circ \delta^v = 0$. It is useful to picture a double complex as a lattice

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{p+1,q+1} & \xrightarrow{\delta^h} & C_{p,q+1} & \xrightarrow{\delta^h} & C_{p-1,q+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \delta^v} & & \downarrow{\scriptstyle \delta^v} & & \downarrow{\scriptstyle \delta^v} & & \\
\cdots & \longrightarrow & C_{p+1,q} & \xrightarrow{\delta^h} & C_{p,q} & \xrightarrow{\delta^h} & C_{p-1,q} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \delta^v} & & \downarrow{\scriptstyle \delta^v} & & \downarrow{\scriptstyle \delta^v} & & \\
\cdots & \longrightarrow & C_{p+1,q-1} & \xrightarrow{\delta^h} & C_{p,q-1} & \xrightarrow{\delta^h} & C_{p-1,q-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \delta^v} & & \downarrow{\scriptstyle \delta^v} & & \downarrow{\scriptstyle \delta^v} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

in which the maps δ^h go horizontally, δ^v go vertically, and each commutes. Each row and each column is a chain complex.

Definition 7.1.9. The *total complex* of a double complex C, $\mathrm{Tot}(C)$, is the chain complex where

$$\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

with differential $\delta = \delta^v + (-1)^{\text{vertical degree}}\delta^h$.

Definition 7.1.10. The *tensor product* of two chain complexes C, D in an abelian category $\mathcal{A}$ is the chain complex

$$C \otimes D := \mathrm{Tot}(C \boxtimes D)$$

Where $C \boxtimes D$ is the double complex defined by $(C \boxtimes D)_{p,q} = C_p \otimes D_q$ with differentials $\delta^h = \delta \otimes \mathbf{id}_D$ and $\delta^v = \mathbf{id}_C \otimes \delta$.

Definition 7.1.11. Two chain complex morphisms $f, g : C \to D$ are *homotopic* (or *chain homotopic*) if there exists a collection of morphisms $\{s_n : C_n \to D_{n+1}\}_{n \in \mathbb{N}}$ such that

$$f_n - g_n = \delta^D_{n+1} \circ s_n + s_{n-1}\delta^C_n \text{ for all } n \in \mathbb{N}.$$

Remark 7.1.12. Equivalently, one can define chain homotopy using an interval object. We will do this for chain complexes of abelian groups for simplicity. The *standard interval object* for chain complexes of abelian groups is the chain complex

$$I = \cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{\mathbf{id} \oplus -\mathbf{id}} \mathbb{Z} \oplus \mathbb{Z} \to 0 \to 0 \to \cdots,$$

where $\mathbb{Z} \oplus \mathbb{Z}$ is in degree 0. A chain homotopy, between chain maps $f, g : C \to D$, is a morphism $h : C \otimes I \to D$ fitting in the following commutative diagram:

$$\begin{tikzcd}
C \arrow[d] \arrow[dr, "f"] & \\
C \otimes I \arrow[r, "h"] & D \\
C \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

7.2 Dold-Kan Correspondence

Definition 7.2.1. Let $\mathcal{A}$ be an abelian category. The *normalized chain complex functor*, $N : s\mathcal{A} \to Ch_+(\mathcal{A})$, sends a simplicial object in $\mathcal{A}$ to the non-negatively graded chain complex in $\mathcal{A}$. It is defined by

$$N_n(\mathcal{A}) = \bigcap_{i=0}^{n-1} ker(d_i : \mathcal{A}_n \to \mathcal{A}_{n-1})$$

with differential $\delta_i = (-1)^n d_n$.

Definition 7.2.2. Let $\mathcal{A}$ be an abelian category. Define a functor $\Gamma : Ch_+(\mathcal{A}) \to s\mathcal{A}$ as follows. Let C be a non-negatively graded chain complex in $\mathcal{A}$. Then

$$\Gamma_n(C) = \bigoplus_{p \leq n} \bigoplus_{\tau} C_p(\tau)$$

where τ ranges over all surjections $[n] \to [p]$ in Δ, and $C_p(\tau)$ denotes the copy of C_p indexed by τ. The simplicial structure maps are defined over the restrictions $\alpha_*(\tau)$ for every index τ. Let $\alpha : [m] \to [n]$ in Δ, then $\alpha_* : \Gamma_m(C) \to \Gamma_n(C)$ is defined as follows. For each surjection $\tau : [n] \to [m]$, find the epi-monic factorization (Theorem 2.1.2) of $\tau\alpha$:

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \tau} \\ [q] & \xrightarrow{s} & [p] \end{array}$$

If $p = q$ (in which case $\tau\alpha = d$) we take $\alpha_*(\tau)$ to be the natural identification of $C_p(\tau)$ with the summand $C_p(d)$ of $\Gamma_m(C)$. If $p = q+1$ and $d = d^p$ (in which case the image of $\tau\alpha$ is $\{0, 1, \cdots, p-1\}$ of $[p]$) we take $\alpha_*(\tau)$ to be the map

$$C_p(\tau) = C_p \xrightarrow{\delta} C_{p-1} = C_{p-1}(s).$$

Otherwise define the map to be 0.

Theorem 7.2.3. *(Dold-Kan Correspondence) Let $\mathcal{A}$ be an abelian category. The functors N and Γ define an equivalence of categories*

$$\Gamma : s\mathcal{A} \cong Ch_+(\mathcal{A}) : N$$

Moreover, under this correspondence simplicial homotopic maps correspond to chain homotopic maps.

Proof. The proof is a direct computation which can be found in Section 8.4 of [28]. $\square$

Example 7.2.4. In particular this gives an equivalence of categories

$$\mathbf{sAb} \cong Ch_+(\mathbf{Ab})$$

where N sends simplicial homotopic maps to chain homotopic maps and Γ sends chain homotopic maps to simplicial homotopic maps.

Example 7.2.5. $\mathbf{Pre}(\mathbf{Ab})$ is an abelian category as well so we have the following equivalences of categories:

$$\mathbf{Pre}(s\mathbf{Ab}) \cong s(\mathbf{Pre}(\mathbf{Ab})) \cong Ch_+(\mathbf{Pre}(\mathbf{Ab})) \cong \mathbf{Pre}(Ch_+(\mathbf{Ab}))$$

Moreover, the composite equivalence $\mathbf{Pre}(s\mathbf{Ab}) \cong \mathbf{Pre}(Ch_+(\mathbf{Ab}))$ is obtained by applying the Dold-Kan correspondence between $\mathbf{sAb}$ and $Ch_+(\mathbf{Ab})$ sectionwise.

Definition 7.2.6. Let A, B be two simplicial abelian groups. The *Alexander Whitney map* is the morphism of chain complexes

$$\Delta_{A,B} : C(A \otimes B) \to C(A) \otimes C(B),$$

defined for two n-simplicies $a \in A_n$ and $b \in B_n$ by

$$\Delta_{A,B}(a \otimes b) = \bigoplus_{p+q=n} (d_{p+1} \cdots d_n a) \otimes (d_0^p b),$$

where d_0^p denotes applying d_0 p-times. This map can then be restricted to normalized chain complexes to give a natural transformation

$$\Delta_{A,B} : N(A \otimes B) \to N(A) \otimes N(B).$$

Definition 7.2.7. Let A, B be two simplicial abelian groups. The *Shuffle map* is the morphism of chain complexes

$$\nabla_{A,B} : C(A) \otimes C(B) \to C(A \otimes B),$$

defined by for $a \in A_p$ and $b \in B_q$ such that $p + q = n$

$$\nabla_{A,B}(a \otimes b) = \sum_{(\mu,\nu)} \mathbf{sign}(\mu, \nu)(s_\nu(a)) \otimes (s_\mu(b)),$$

where the sum ranges over all p, q-shuffles (μ, ν). That is, permutations $(\mu_1, \cdots, \mu_p, \nu_1, \cdots, \nu_q)$ of $(1, \cdots, n)$ such that $\mu_1 < \mu_2 < \cdots < \mu_p$ and $\nu_1 < \nu_2 < \cdots < \nu_q$. The corresponding degeneracy maps are defined by

$$s_\mu = s_{\mu_p - 1} \circ \cdots \circ s_{\mu_2 - 1} \circ s_{\mu_1 - 1}$$

and

$$s_\nu = s_{\nu_q - 1} \circ \cdots \circ s_{\nu_2 - 1} \circ s_{\nu_1 - 1}.$$

This map can then be restricted to normalized chain complexes to give a natural transformation

$$\nabla_{A,B} : N(A) \otimes N(B) \to N(A \otimes B).$$

Proposition 7.2.8. *On normalized complexes, $\Delta_{A,B}$ is a homotopy equivalence, with homotopy inverse given by $\nabla_{A,B}$.*

Proof. See Section 29 [19]. □

7.3 Simplicial Cup Product

Definition 7.3.1. Let A be a bisimplicial object in an abelian category $\mathcal{A}$. Then the *normalized bicomplex* of A, $\overline{N}(A)$, is defined by

$$\overline{N}_{p,q}(A) = \left(\bigcap_{i=0}^{p-1} ker(d_i^h : A_{p,q} \to A_{p-1,q}) \right) \cap \left(\bigcap_{i=0}^{q-1} ker(d_i^v : A_{p,q} \to A_{p,q-1}) \right),$$

with horizontal differential $\delta^h = d_p^h$ and vertical differential $\delta^v = (-1)^p d_q^v$.

Definition 7.3.2. Let G be a simplicial group. The bisimplicial group $\overline{K}(G, i)$ (G must be abelian for $i > 1$) is defined by

$$\overline{K}(G, i)_{m,n} = \Gamma(G_n[i])_m,$$

where $G_n[i]$ is the chain complex of groups that has the group G_n at level i and is trivial everywhere else. The vertical face and degeneracy maps are those induced by the face and degeneracy maps of G, and the horizontal face and degeneracy maps are induced by the face and degeneracy maps of $\Gamma(G_n[i])$.

Lemma 7.3.3. *(Cegarra-Remedios) Let A be a bisimplicial abelian group, then $NTA \cong Tot\overline{N}A$*

Proof. See section 5 of [7]. ◻

Definition 7.3.4. Let C be a chain complex of abelian groups. Let $C[n]$ be the chain complex C shifted up n, that is, $C[n]_k = C_{n+k}$.

Lemma 7.3.5. *Let A be a simplicial abelian group. Then $NT\overline{K}(A,n) \cong (NA)[n]$ for all $n \in \mathbb{N}$.*

Proof. $NT\overline{K}(A,n) \cong \mathrm{Tot}\overline{NK}(A,n)$ by Lemma 7.3.3. $\overline{NK}(A,n)$ is the bicomplex that is all zero except for the n^{th} column which is $N(A)$, therefore after taking the total complex $\mathrm{Tot}\overline{NK}(A,n) \cong (NA)[n]$. ◻

Lemma 7.3.6. *Let G be a simplicial group. Then there is an isomorphism of simplicial sets*

$$T\overline{K}(G,1) \cong \overline{W}G.$$

Which is an isomorphism of simplicial (abelian) groups if G is a simplical abelian group.

Proof. This is originally due to Duskin. The isomorphism is defined levelwise by sending an element $(g_{n-1}, \cdots, g_0)$ of $\overline{W}G$ to the element

$$(1, g_{n-1}, (d_0(g_{n-1}), g_{n-2}), (d_0^2(g_{n-1}), d_0(g_{n-2}), g_{n-3}), \cdots, (d_0^{n-1}(g_{n-1}), d_0^{n-2}(g_{n-2}, \cdots, d_0(g_1), g_0))$$

of $T\overline{K}(G,1)$. ◻

Lemma 7.3.7. *Let A be a simplicial abelian group. Then there is an isomorphism of simplicial abelian groups*

$$T\overline{K}(A,n) \cong \overline{W}^n(A).$$

Proof. We will prove this by strong induction on n. The base case for $n = 1$ is Lemma 7.3.6.

Assume that $T\overline{K}(A,n) \cong \overline{W}^n(A)$ for all $1 \leq n \leq k$ and notice that by Lemma 7.3.5 we have

$$NT\overline{K}(A,n) \cong (NA)[n], \text{ for all } 1 \leq n \leq k.$$

Γ is part of an equivalence of categories, so that

$$T\overline{K}(A,n) \cong \Gamma NT\overline{K}(A,n) \cong \Gamma((NA)[n]), \text{ for all } 1 \leq n \leq k. \tag{7.1}$$

By the inductive hypothesis

$$\overline{W}^{k+1}A = \overline{W}(\overline{W}^k A) \cong T\overline{K}(\overline{W}^k A, 1),$$

and since N is part of an equivalence of categories, by Lemma 7.3.5

$$N\overline{W}^{k+1}A \cong NT\overline{K}(\overline{W}^k A, 1) \cong (N\overline{W}^k A)[1]. \tag{7.2}$$

By the inductive hypothesis and equation 7.1

$$\overline{W}^k A \cong T\overline{K}(A, k) \cong \Gamma((NA)[k]).$$

Applying N we obtain

$$N\overline{W}^k A \cong N\Gamma((NA)[k]) \cong (NA)[k],$$

which together with equation 7.2 yields

$$N\overline{W}^{k+1}A \cong (N\overline{W}^k A)[1] \cong ((NA)[k])[1] \cong (NA)[k+1].$$

Applying Γ and equation 7.1 we obtain the desired result

$$\overline{W}^{k+1}A \cong \Gamma N\overline{W}^{k+1}A \cong \Gamma((NA)[k+1]) \cong T\overline{K}(A, k+1).$$

□

Definition 7.3.8. Let $X \in \mathbf{sSet}$ and $C \in Ch_+(\mathbf{Ab})$. Then the *i-th hypercohomolgy* of X is

$$\mathbf{H}^i(X, C) = [X, \overline{W}^i\Gamma(C)]_{\mathbf{sSet}}.$$

Definition 7.3.9. The *cup product*

$$\cup : \mathbf{H}^i(X, C) \times \mathbf{H}^j(X, D) \to \mathbf{H}^{i+j}(X, C \otimes D)$$

is defined as follows. Let $c \in \mathbf{H}^i(X, C)$ and $d \in \mathbf{H}^j(X, D)$. Then the cup product $c \cup d \in \mathbf{H}^{i+j}(X, C \otimes D)$ is given by the composite

$$X \xrightarrow{diag} X \times X \xrightarrow{(c,d)} \overline{W}^i(\Gamma(C)) \times \overline{W}^j(\Gamma(D)) \xrightarrow{\phi_{C,D}} \overline{W}^{i+j}(C \otimes D),$$

where $\phi_{C,D}$ is given by:

$$\overline{W}^i(\Gamma(C)) \times \overline{W}^j(\Gamma(D)) \cong \Gamma(C[i]) \times \Gamma(D[j]) \xrightarrow{\otimes} \Gamma(C[i]) \otimes \Gamma(D[j]) \xrightarrow{\psi} \overline{W}^{i+j}(C \otimes D),$$

and ψ is the homotopy equivalence given by:

$$\Gamma(C[i]) \otimes \Gamma(D[j]) \cong \Gamma N(\Gamma(C[i]) \otimes \Gamma(D[j])) \xrightarrow{\Gamma(\Delta_{\Gamma(C[i]),\Gamma(D[j])})} \Gamma(N\Gamma(C[i]) \otimes N\Gamma(D[j])) \cong$$

$$\Gamma(C[i] \otimes D[j]) \cong \Gamma(C \otimes D[i+j]) \cong \overline{W}^{i+j}(C \otimes D).$$

Remark 7.3.10. ψ is a homotopy equivalence because the Dold-Kan correspondence sends chain homotopies to simplicial homotopies (hence chain homotopy equivalences to simplicial homotopy equivalences) making $\Gamma(\Delta_{A,B})$ a homotopy equivalence. Therefore, the homotopy class of

$$\phi_{C,D} \in [\overline{W}^i(\Gamma(C)) \times \overline{W}^j(\Gamma(D)), \overline{W}^{i+j}(C \otimes D)]$$

corresponds to the homotopy class of

$$\otimes \in [\Gamma(C[i]) \times \Gamma(D[j]), \Gamma(C[i]) \otimes \Gamma(D[j])].$$

7.4 Simplicial Heisenberg Extensions

Definition 7.4.1. Given two simplicial abelian groups A, B consider the central extension of simplicial groups

$$A \otimes B \to H_{A,B} \to A \times B,$$

where $H_{A,B}$ is the simplicial group which as a simplicial set is $A \times B \times A \otimes B$, and whose group law is defined levelwise by

$$(a_n, b_n, t_n)(a'_n, b'_n, t'_n) = (a_n + a'_n, b_n + b'_n, t_n + t'_n + a_n \otimes b'_n),$$

for $a_n, a'_n \in A_n$, $b_n, b'_n \in B_n$, and $t_n, t'_n \in (A \otimes B)_n$. We call this extension a *Heisenberg central extension of simplicial groups* and $H_{A,B}$ a *simplicial Heisenberg group.*

Proposition 7.4.2. *Let $C, D \in Ch_+(\mathbf{Ab})$ and $i, j > 1$. Then the homotopy class of $\phi_{C,D}$ in the cup product (Definition 7.3.9) corresponds to the path component of the Heisenberg central extension of simplicial groups*

$$\Gamma(C[i-1]) \otimes \Gamma(D[j-1]) \to H_{\Gamma(C[i-1]),\Gamma(D[j-1])} \to \Gamma(C[i-1]) \times \Gamma(D[j-1])$$

Proof. Let $A = \Gamma(C[i-1])$ and $B = \Gamma(D[j-1])$.

$$\Gamma(C[i-1]) \otimes \Gamma(D[j-1]) \to H_{\Gamma(C[i-1]),\Gamma(D[j-1])} \to \Gamma(C[i-1]) \times \Gamma(D[j-1])$$

has a simplicial set cross section $\sigma : A \times B \to H_{A,B}$ defined levelwise by $\sigma(a_n, b_n) = (a_n, b_n, 0)$, so that by Theorem 5.2.2 it corresponds to

$$\phi : \overline{W}(A \times B) \cong T\Delta\Omega^{-1}(A \times B) \xrightarrow{T\Delta(\sigma)} T\Delta\Omega^{-1}(H_{A,B}/\!\!/A \otimes B) \to T\Delta\Omega^{-1}(*/\!\!/A \otimes B) \cong \overline{W}^2(A \otimes B).$$

Let α be the composite bisimplicial group morphism

$$\alpha = \Delta\Omega^{-1}(A \times B) \xrightarrow{\Delta(\sigma)} \Delta\Omega^{-1}(H_{A,B}/\!\!/A \otimes B) \to \Delta\Omega^{-1}(*/\!\!/A \otimes B),$$

which fits into the following commutative diagram

$$\begin{tikzcd} \Delta\Omega^{-1}(A \times B) \arrow[r, "\alpha"] \arrow[d, "\cong"] & \Delta\Omega^{-1}(*/\!\!/A \otimes B) \\ \Delta\Omega^{-1}A \times \Delta\Omega^{-1}B \arrow[r, "\otimes"] & \Delta\Omega^{-1}A \otimes \Delta\Omega^{-1}B \arrow[u, "f"] \end{tikzcd} .$$

f is the bisimplicial group morphism defined levelwise by the simplicial group morphism

$$f_n : \Delta\Omega^{-1}A_n \otimes \Delta\Omega^{-1}B_n \to \Delta\Omega^{-1}(*/\!\!/A_n \otimes B_n),$$

which is given by

$$f_{n,2}(a_n, a'_n, b_n, b'_n) = a_n \otimes b'_n.$$

This defines a simplicial map because level 0 and 1 must be trivial and any level above 2 is completely determined by the simplicial identities. The diagram is commutative because levelwise we have the commutative diagram:

$$\begin{tikzcd} \Delta\Omega^{-1}(A_n \times B_n) \arrow[r, "\alpha_n"] \arrow[d, "\cong"] & \Delta\Omega^{-1}(*/\!\!/A_n \otimes B_n) \\ \Delta\Omega^{-1}A_n \times \Delta\Omega^{-1}B_n \arrow[r, "\otimes"] & \Delta\Omega^{-1}A_n \otimes \Delta\Omega^{-1}B_n \arrow[u, "f_n"] \end{tikzcd} \tag{7.3}$$

f is levelwise a homotopy equivalence because

$$\Gamma(A_n[1]) \otimes \Gamma(B_n[1]) \cong \Delta\Omega^{-1}A_n \otimes \Delta\Omega^{-1}B_n \xrightarrow{f} \Delta\Omega^{-1}(*/\!\!/A_n \otimes B_n) \cong \Gamma((A_n \otimes B_n)[2]),$$

factors as

$$\Gamma(A_n[1]) \otimes \Gamma(B_n[1]) \cong \Gamma N(\Gamma(A_n[1]) \otimes \Gamma(B_n[1])) \xrightarrow{\Gamma(\Delta_{\Gamma(A_n[1]),\Gamma(B_n[1])})} \Gamma(N\Gamma(A_n[1]) \otimes N\Gamma(B_n[1])) \cong$$

$$\Gamma(A_n[1] \otimes B_n[1]) \cong \Gamma((A_n \otimes B_n)[2]).$$

Moreover we have the following commutative diagram:

$$\begin{tikzcd}
T(\Delta\Omega^{-1}A \times \Delta\Omega^{-1}B) \arrow[r, "T(\otimes)"] \arrow[d, "\cong"] & T(\Delta\Omega^{-1}A \otimes \Delta\Omega^{-1}B) \arrow[d, "\cong"] \\
T\Delta\Omega^{-1}A \times T\Delta\Omega^{-1}B \arrow[r, "\otimes"] \arrow[d, "\cong"] & T\Delta\Omega^{-1}A \otimes T\Delta\Omega^{-1}B \arrow[d, "\cong"] \\
\Gamma(C[i]) \times \Gamma(D[j]) \arrow[r, "\otimes"] & \Gamma(C[i]) \otimes \Gamma(D[j])
\end{tikzcd}$$

which we can then paste with the diagram obtained by applying T to equation 7.3 to obtain:

$$\begin{tikzcd}
T\Delta\Omega^{-1}(A \times B) \arrow[r, "T(\alpha)"] \arrow[d, "\cong"] & T\Delta\Omega^{-1}(*/\!\!/(A \otimes B) \\
T(\Delta\Omega^{-1}A \times \Delta\Omega^{-1}B) \arrow[r, "T(\otimes)"] \arrow[d, "\cong"] & T(\Delta\Omega^{-1}A \otimes \Delta\Omega^{-1}B) \arrow[u, "T(f)"'] \arrow[d, "\cong"] \\
\Gamma(C[i]) \times \Gamma(D[j]) \arrow[r, "\otimes"] & \Gamma(C[i]) \otimes \Gamma(D[j])
\end{tikzcd}$$

f is levelwise a homotopy equivalence, therefore $T(f)$ is a homotopy equivalence. The diagram above shows ϕ factors as the tensor product pre-composed with an isomorphism and post-composed with a homotopy equivalence. Thus, the class of

$$\phi \in [\overline{W}(A \times B), \overline{W}^2(A \otimes B)]$$

corresponds to the class of

$$\otimes \in [\Gamma(C[i]) \times \Gamma(D[j]), \Gamma(C[i]) \otimes \Gamma(D[j])].$$

Finally by Remark 7.3.10 we obtain the desired result.

□

Corollary 7.4.3. *The* cup product *(Definition 7.3.9)*

$$\cup : \mathbf{H}^i(X, C) \times \mathbf{H}^j(X, D) \to \mathbf{H}^{i+j}(X, C \otimes D)$$

factors through $X \to \Gamma(C[i]) \times \Gamma(D[j])$ and the Heisenberg central extension of simplicial groups of Proposition 7.4.2.

7.5 Simplicial Presheaf Cup Product

All presheaves will be over a fixed small site $\mathcal{C}$.

Definition 7.5.1. Let G be a presheaf of simplicial groups. The presheaf of bisimplicial groups $\overline{K}(G,i)$, is given sectionwise by

$$\overline{K}(G,i)(U) = \overline{K}(G(U),i)$$

In particular, sectionwise it is the bisimplicial group of Definition 7.3.2.

Lemma 7.5.2. *Let G be a presheaf of simplicial groups. Then there is an isomorphism of simplicial presheaves*

$$T\overline{K}(G,1) \cong \overline{W}G,$$

which is an isomorphism of presheaves of simplicial groups if G is a presheaf of simplicial abelian groups.

Proof. The isomorphism is given sectionwise by Lemma 7.3.6. $\square$

Lemma 7.5.3. *Let A be a presheaf of simplicial abelian groups. Then there is an isomorphism of presheaves of simplicial abelian groups*

$$T\overline{K}(A,n) \cong \overline{W}^n A.$$

Proof. The isomorphism is given sectionwise by Lemma 7.3.7. $\square$

Definition 7.5.4. Let $X \in \mathbf{Pre}(\mathbf{sSet})$ and $C \in \mathbf{Pre}(Ch_+(\mathbf{Ab}))$. Then the *i-th hypercohomolgy* of X is

$$\mathbf{H}^i(X,C) = [X, \overline{W}^i\Gamma(C)]_{\mathbf{Pre(sSet)}}.$$

Remark 7.5.5. Similar to the simplical case, if $C \in \mathbf{Pre}(Ch_+(\mathbf{Ab}))$, then $C[n]$ denotes the presheaf that is defined sectionwise to be $C(U)[n]$ (Definition 7.3.4).

Definition 7.5.6. The *cup product*

$$\cup : \mathbf{H}^i(X,C) \times \mathbf{H}^j(X,D) \to \mathbf{H}^{i+j}(X,C \otimes D)$$

is defined as follows. Let $c \in \mathbf{H}^i(X,C)$ and $d \in \mathbf{H}^j(X,D)$. Then the cup product $c \cup d \in \mathbf{H}^{i+j}(X, C \otimes D)$ is given by the composite

$$X \xrightarrow{diag} X \times X \xrightarrow{(c,d)} \overline{W}^i(\Gamma(C)) \times \overline{W}^j(\Gamma(D)) \xrightarrow{\phi_{C,D}} \overline{W}^{i+j}(C \otimes D),$$

where $\phi_{C,D}$ is given by:

$$\overline{W}^i(\Gamma(C)) \times \overline{W}^j(\Gamma(D)) \cong \Gamma(C[i]) \times \Gamma(D[j]) \xrightarrow{\otimes} \Gamma(C[i]) \otimes \Gamma(D[j]) \xrightarrow{\psi} \overline{W}^{i+j}(C \otimes D),$$

and ψ is the homotopy equivalence given by

$$\Gamma(C[i]) \otimes \Gamma(D[j]) \cong \Gamma N(\Gamma(C[i]) \otimes \Gamma(D[j])) \xrightarrow{\Gamma(\Delta_{\Gamma(C[i]),\Gamma(D[j])})} \Gamma(N\Gamma(C[i]) \otimes N\Gamma(D[j])) \cong$$

$$\Gamma(C[i] \otimes D[j]) \cong \Gamma(C \otimes D[i+j]) \cong \overline{W}^{i+j}(C \otimes D)$$

Remark 7.5.7. $\phi_{C,D}$ is the tensor product $\otimes$ precomposed with an isomorphism and post composed with a homotopy equivalence so that

$$\phi_{C,D} \in [\overline{W}^i(\Gamma(C)) \times \overline{W}^j(\Gamma(D)), \overline{W}^{i+j}(C \otimes D)]$$

corresponds to the homotopy class of

$$\otimes \in [\Gamma(C[i]) \times \Gamma(D[j]), \Gamma(C[i]) \otimes \Gamma(D[j])].$$

7.6 Simplicial Presheaf Heisenberg Extensions

Definition 7.6.1. Given two presheaves of simplicial abelian groups A, B consider the central extension of presheaves of simplicial groups

$$A \otimes B \to H_{A,B} \to A \times B,$$

where $H_{A,B}$ is the presheaf of simplicial groups defined sectionwise to be $H_{A(U),B(U)}$. In particular, sectionwise this is a Heisenberg central extension of simplicial groups. We call this extension the *Heisenberg central extension of presheaves of simplicial groups* and $H_{A,B}$ the *simplicial presheaf Heisenberg group.*

Proposition 7.6.2. *Let $C, D \in \mathbf{Pre}(Ch_+(\mathbf{Ab}))$ and $i, j > 1$. Then the homotopy class of the morphism $\phi_{C,D}$ in the cup product (Definition 7.3.9) corresponds to the path component of the Heisenberg central extension of simplicial groups*

$$\Gamma(C[i-1]) \otimes \Gamma(D[j-1]) \to H_{\Gamma(C[i-1]),\Gamma(D[j-1])} \to \Gamma(C[i-1]) \times \Gamma(D[j-1])$$

Proof. Apply the proof of Proposition 7.4.2 sectionwise. □

Corollary 7.6.3. *The* cup product *(Definition 7.5.6)*

$$\cup : \mathbf{H}^i(X, C) \times \mathbf{H}^j(X, D) \to \mathbf{H}^{i+j}(X, C \otimes D)$$

factors through $X \to \Gamma(C[i]) \times \Gamma(D[j])$ and the Heisenberg central extension of presheaves of simplicial groups of Proposition 7.6.2.

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BIOGRAPHICAL SKETCH

Michael Niemeier was born in Omaha, Nebraska. He obtained his B.S. in Mathematics from the University of Nebraska Lincoln. After completing his PhD at Florida State University he will be a mathematics teacher at the Indiana Academy for Science, Mathematics, and Humanities.