SIAM J. MATH. ANAL.
Vol. 56, No. 6, pp. 7074–7107

ALMOST-SURE STABILITY OF THE SINGLE MODE SOLUTION OF A NOISY NONLINEAR AUTOPARAMETRIC SYSTEM*

PETER H. BAXENDALE† AND N. SRI NAMACHCHIVAYA‡

Abstract. For a pendulum suspended below a vibrating block with white noise forcing, the solution in which the pendulum remains vertical is called the single mode solution. When this solution becomes unstable there is energy transfer from the block to the pendulum, helping to absorb the vibrations of the block. We study the Lyapunov exponent λ governing the almost-sure stability of the process linearized along the single mode solution. The linearized equation is excited by a combination of white and colored noise processes, which makes the evaluation of λ nontrivial. Depending on the relative sizes of the noise intensity and the damping coefficients, λ may take either negative or positive values. For white noise forcing of intensity $\varepsilon\nu$ we prove $\lambda(\varepsilon) = \lambda_0 + \varepsilon^2\lambda_2 + O(\varepsilon^4)$ as $\varepsilon \to 0$, where λ_0 and λ_2 are given explicitly in terms of the parameters of the system.

Key words. autoparametric system, Lyapunov exponent, stability boundary, resonance

MSC codes. 37H15, 60H10, 70K20

DOI. 10.1137/23M1556502

1. Introduction.

Autoparametric systems are vibrating systems that consist of two subsystems. The primary system is an oscillator that is directly excited by some external forcing, and the secondary system is coupled nonlinearly to the oscillator in such a way that it can be at rest while the oscillator is vibrating.

In this paper we consider the two-degree-of-freedom system shown in Figure 1. The primary system consists of a block of mass m_1 attached by a linear spring of stiffness k_1 and a viscous dashpot with a damping coefficient c_1. The block is excited directly by (downward) forcing $\Xi(t)$. The secondary system comprises a simple pendulum of length ℓ attached to the block. The mass of the pendulum bob is m_2 and the pendulum motion is damped by a viscous damper of coefficient c_2. This system appears in Hatwal, Mallik, and Ghosh [15]. See also [4, 6] and a similar model with a nonlinear spring and a compound pendulum in [35]. A closely related system with a vertically mounted cantilever beam with a tip mass appears in Haxton and Barr [16].

It is a characteristic property of an autoparametric system that there is a solution involving forced motion of the primary system while the secondary system remains at rest. Here it corresponds to the "single mode" solution in which the pendulum remains directly below the vibrating block at all times. As parameters in the system are varied, it may happen that the single mode solution becomes unstable and the pendulum begins to move. In this case the motion of the primary system causes excitation of the secondary system, and there is energy transfer from the vibrating block to the pendulum. In effect, the pendulum acts as a vibration absorber. Our primary concern is the almost-sure stability analysis of the single mode solution when the forcing function $\Xi(t)$ is a multiple of white noise.

*Received by the editors March 6, 2023; accepted for publication (in revised form) July 3, 2024; published electronically November 1, 2024.
https://doi.org/10.1137/23M1556502
Funding: The work of the second author was partially supported by the National Sciences and Engineering Research Council of Canada Discovery grant RGPIN-2018-03735.
†Department of Mathematics, University of Southern California, Los Angeles, CA 90089 USA (baxendal@usc.edu).
‡Department of Applied Mathematics, University of Waterloo, Waterloo, ON N2L3G1, Canada (navam@uwaterloo.ca).

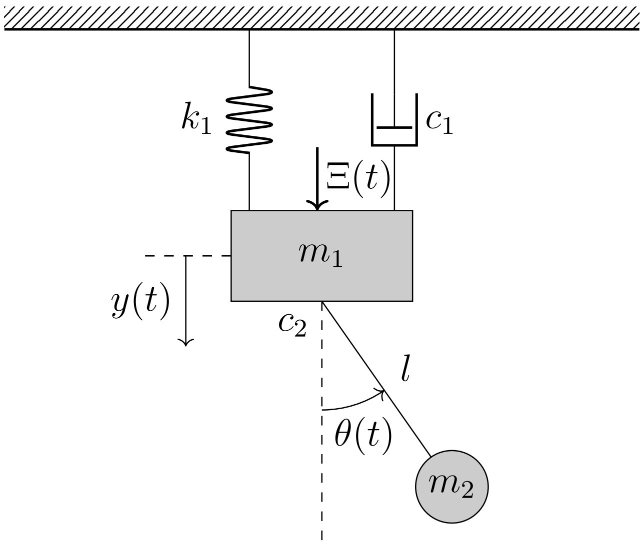

FIG. 1. *Block and pendulum.*

In addition to the block and pendulum and block and beam examples mentioned above, autoparametric systems are used to model the dynamics of structural and mechanical systems such as the vibration of an initially deflected shallow arch [32], inplane and out-of-plane motions of suspended cables [10], and the pitching and rolling motions of a ship [27, 29]. A larger collection of examples is given in the monograph [33].

Periodically excited autoparametric systems have been studied extensively; see, for example, [16, 28, 15, 11, 4, 6]. These are deterministic systems, and the most interesting situations occur when the natural frequencies of the primary (excited mode) and the secondary (unexcited) systems are in 2:1 resonance. Issues of bifurcation theory arise, and numerical computations frequently show chaotic behavior.

Autoparametric systems with stochastic forcing have also been studied. Gaussian and non-Gaussian closure techniques were used in [18, 30, 17, 12, 23] to approximate the moments of solutions of the associated Fokker–Planck equation. Sufficient conditions for almost-sure stability of the single mode solution were obtained in [1]. Stochastic averaging techniques were used in [26].

The equations of motion for the block and pendulum system shown in Figure 1 are

$$\begin{aligned}(m_1+m_2)\ddot{y}(t)+c_1\dot{y}(t)+k_1y(t)-m_1\ell(\ddot{\theta}(t)\sin\theta(t)+\dot{\theta}^2(t)\cos\theta(t))&=\Xi(t),\\ m_2\ell^2\ddot{\theta}(t)+c_2\dot{\theta}(t)+m_2\ell(g-\ddot{y}(t))\sin\theta(t)&=0,\end{aligned}\tag{1.1}$$

where y represents the downwards displacement of the block relative to its rest (unforced) position, and θ is the angle of the pendulum; see [15]. We specialize to the case of white noise forcing, that is, $\Xi(t)=\hat{\nu}\dot{W}(t)$, where W is a standard Wiener process and $\hat{\nu}$ represents the noise intensity.

Letting $\eta=y/\ell$ represent the dimensionless position of the block, the equations of motion (1.1) can be rewritten as

$$\begin{aligned}\ddot{\eta}(t)+2\zeta_1\dot{\eta}(t)+\chi^2\eta(t)-R\big(\ddot{\theta}(t)\sin\theta(t)+\dot{\theta}^2(t)\cos\theta(t)\big)&=\nu\dot{W}(t),\\ \ddot{\theta}(t)+2\zeta_2\dot{\theta}(t)+\big(\kappa^2-\ddot{\eta}(t)\big)\sin\theta(t)&=0,\end{aligned}\tag{1.2}$$

where the constants ζ_1 and ζ_2 are scaled damping coefficients, $\chi^2 = k_1/(m_1+m_2)$, and $\kappa = \sqrt{g/\ell}$ is the frequency of the undamped pendulum. The parameter $R = m_2/(m_1+m_2)$, $0<R<1$, represents the mass ratio, and $\nu = \hat{\nu}/[\ell(m_1+m_2)]$ is the scaled noise intensity.

The rigorous interpretation of (1.2) as a 4-dimensional stochastic differential equation (SDE) is carried out in section 2. See Theorem 2.1, which guarantees existence and uniqueness and exponential moments for solutions with arbitrary initial conditions. The single mode solution of the system (1.2) is of the form

$$\big(\eta(t),\dot{\eta}(t)\big) = \big(\bar{\eta}(t),\dot{\bar{\eta}}(t)\big) \quad \text{and} \quad \big(\theta(t),\dot{\theta}(t)\big) = \big(0,0\big), \tag{1.3}$$

where the motion $\{(\bar{\eta}(t),\dot{\bar{\eta}}(t)) : t \ge 0\}$ of the block is described by

$$\ddot{\bar{\eta}}(t) + 2\zeta_1\dot{\bar{\eta}}(t) + \chi^2\bar{\eta}(t) = \nu\dot{W}(t). \tag{1.4}$$

The single mode (1.4) has a stationary solution $\{(\bar{\eta}(t),\dot{\bar{\eta}}(t)) : t \ge 0\}$ with Gaussian stationary probability measure with mean $(0,0)$ and covariance matrix $R = \frac{\nu^2}{4\zeta_1}\left(\begin{smallmatrix} 1/\chi^2 & 0 \\ 0 & 1\end{smallmatrix}\right)$.

The stability or instability of the single mode solution is governed by the large time behavior of the process obtained by linearizing the 4-dimensional SDE along the single mode solution $(\bar{\eta}(t),\dot{\bar{\eta}}(t),0,0)$. In particular it is determined by the sign of the Lyapunov exponent

$$\lambda := \lim_{t\to\infty}\frac{1}{t}\log\|\,(\varphi(t),\dot{\varphi}(t))\,\| \tag{1.5}$$

associated with the linear SDE

$$\ddot{\varphi}(t) + 2\zeta_2\dot{\varphi}(t) + \big(\kappa^2 + 2\zeta_1\dot{\bar{\eta}}(t) + \chi^2\bar{\eta}(t)\big)\varphi(t) = \nu\varphi(t)\dot{W}(t), \tag{1.6}$$

where $(\bar{\eta}(t),\dot{\bar{\eta}}(t))$ is given by (1.4); see section 2.1.

The novelty of the linear SDE (1.6) is that it is parametrically excited by the colored noise processes $\bar{\eta}(t)$ and $\dot{\bar{\eta}}(t)$ as well as the white noise (generalized) process $\dot{W}(t)$ which is driving them in (1.4). The techniques and results developed in this paper apply to a larger class of excitations of (1.6) consisting of combinations of Gaussian colored noise and white noise processes. This larger class is described in section 3.

It is shown in section 4 that the right side of (1.5) exists as an almost-sure limit and is given by Khas'minskii's integral formula (4.8) for all initial values $(\bar{\eta},\dot{\bar{\eta}},\varphi,\dot{\varphi})$ with $(\varphi,\dot{\varphi}) \ne (0,0)$. Although the integrand $Q(v,\psi)$ in (4.8) is given explicitly, the measure m is characterized only as the unique invariant probability measure for an associated diffusion process, and it is hard to evaluate the integral and obtain an exact formula for the Lyapunov exponent. However, assuming the pendulum is underdamped, so that the damped frequency $\kappa_d = \sqrt{\kappa^2-\zeta_2^2}$ is real and positive, a simple bound on Q gives the following upper bound on the Lyapunov exponent:

$$\lambda \le -\zeta_2 + \frac{\nu}{2\kappa_d}\sqrt{\frac{\chi^2+4\zeta_1^2}{4\zeta_1} + \frac{\nu^2}{2\kappa_d^2}}. \tag{1.7}$$

See Proposition 4.2. This gives almost-sure stability for sufficiently small noise intensity ν.

The main results in the paper are given in sections 5 and 6. Theorem 5.1 and Propositions 5.1 and 5.2 give rigorous asymptotic estimates for the Lyapunov exponent

$\lambda(\varepsilon)$ for the system obtained when the noise intensity is rescaled $\nu \rightsquigarrow \varepsilon\nu$. For the block and pendulum system we show

$$\lambda(\varepsilon) = -\zeta_2 + \varepsilon^2 \frac{2\kappa_d^2\nu^2}{(\chi^2 - 4\kappa_d^2)^2 + 16\zeta_1^2\kappa_d^2} + O(\varepsilon^4) \tag{1.8}$$

as $\varepsilon \to 0$. The asymptotic expansion in Theorem 5.1 for the case $\nu \rightsquigarrow \varepsilon\nu$ remains valid as the pendulum damping $\zeta_2 \to 0$, and this gives the results in section 6 for the case in which the noise intensity and the pendulum damping are both rescaled: $\nu \rightsquigarrow \varepsilon\nu$ and $\zeta_2 \rightsquigarrow \varepsilon^2\zeta_2$. Using $\widehat{\lambda}(\varepsilon)$ in place of $\lambda(\varepsilon)$ to denote the different scaling, we have

$$\widehat{\lambda}(\varepsilon) = \varepsilon^2 \left(-\zeta_2 + \frac{2\kappa^2\nu^2}{(\chi^2 - 4\kappa^2)^2 + 16\zeta_1^2\kappa^2} \right) + O(\varepsilon^4) \tag{1.9}$$

as $\varepsilon \to 0$. This shows clearly the critical noise to damping ratio ν^2/ζ_2 separating unstable regions with $\widehat{\lambda}(\varepsilon) > 0$ from stable regions with $\widehat{\lambda}(\varepsilon) < 0$. See Figure 2. Recalling that κ is the frequency of the undamped pendulum and that χ is the frequency of the undamped block and pendulum system when the pendulum is at rest directly below the block, we give a brief discussion of the similarities and differences in the 2:1 resonance effects for white noise and periodic forcing. The final three sections contain the proofs.

2. Well-posedness for the nonlinear system. Since $0 < R < 1$, the equations (1.2) can be rearranged to give

(2.1)

$$\begin{aligned}
&\ddot{\eta}(t) + \frac{2\zeta_1\dot{\eta}(t) + \chi^2\eta(t) - R\dot{\theta}^2(t)\cos\theta(t) + 2R\zeta_2\dot{\theta}(t)\sin\theta(t) + \kappa^2 R\sin^2\theta(t)}{(1 - R\sin^2\theta(t))} \\
&\quad = \frac{\nu}{(1 - R\sin^2\theta(t))}\dot{W}(t), \\
&\ddot{\theta}(t) + \frac{2\zeta_2\dot{\theta}(t) + \kappa^2\sin\theta(t) + 2\zeta_1\dot{\eta}(t)\sin\theta(t) + \chi^2\eta\sin\theta(t) - R\dot{\theta}^2(t)\sin\theta(t)\cos\theta(t)}{(1 - R\sin^2\theta(t))} \\
&\quad = \frac{\nu\sin\theta(t)}{(1 - R\sin^2\theta(t))}\dot{W}(t).
\end{aligned}$$

This second order system for (η, θ) can now be turned into a first order system of stochastic differential equations for the process $(\eta, \dot{\eta}, \theta, \dot{\theta})$ taking values in the space $N = \mathbb{R}^2 \times \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}$. Write

$$(v_1, v_2, u_1, u_2) = (\eta, \dot{\eta}, \theta, \dot{\theta}).$$

Then (2.1) gives

$$\begin{aligned}
dv_1 &= v_2\,dt, \\
dv_2 &= \frac{-2\zeta_1 v_2 - \chi^2 v_1 + Ru_2^2\cos u_1 - 2R\zeta_2 u_2\sin u_1 - R\kappa^2\sin^2 u_1}{(1 - R\sin^2 u_1)}dt \\
&\quad + \frac{\nu}{(1 - R\sin^2 u_1)}dW(t), \\
du_1 &= u_2\,dt, \\
du_2 &= \frac{-2\zeta_2 u_2 - \kappa^2\sin u_1 - 2\zeta_1 v_2\sin u_1 - \chi^2 v_1\sin u_1 + Ru_2^2\sin u_1\cos u_1}{(1 - R\sin^2 u_1)}dt \\
&\quad + \frac{\nu\sin u_1}{(1 - R\sin^2 u_1)}dW(t).
\end{aligned} \tag{2.2}$$

We see that $du_1(t)dW(t)=0$ and so the Itô and Stratonovich interpretations of (2.2) are identical.

For ease of notation write $U(t) = (v_1(t), v_2(t), u_1(t), u_2(t)) \in N$. Noting that $|u_1(t)| \le 1$ we abuse notation by writing $\|U(t)\| = \sqrt{v_1^2(t)+v_2^2(t)+u_2^2(t)}$.

THEOREM 2.1.

(i) *For every initial condition $U(0)\in N$ the system* (2.2) *has a unique solution $\{U(t): t\ge 0\}$ valid for all $t\ge 0$.*

(ii) *There exists $b>0$ such that for every initial $U(0)$ the process $\{U(t): t\ge 0\}$ satisfies the condition $\|U(t)\|\le b$ for infinitely many t as $t\to\infty$.*

(iii) *There exist $\beta>0$ and K such that every invariant probability measure σ for* (2.2) *satisfies*

$$\int_N \exp\bigl(\beta\|U\|^2\bigr)d\sigma(U)\le K.$$

(iv) *There exist $0<\beta\le\beta_1$ and K_1 and for every $\gamma>0$ there exists K_2 such that*

$$\mathbb{E}\exp\bigl(\beta\|U(t)\|^2\bigr)\le K_1 e^{-\gamma t}\exp\left(\beta_1\|U(0)\|^2\right)+K_2$$

for all initial conditions $U(0)\in N$ and all $t\ge 0$.

The proof of Theorem 2.1 is given in section 7.

Remark 2.1. The corresponding calculation for white noise forcing of the beam model discussed in [18, 30, 23] fails because it involves division by a term of the form $1-R\theta^2$ rather than $1-R\sin^2\theta$. There is no guarantee that the full nonlinear system, for example [18, eqns. (4), (5)], has a solution valid for all time $t\ge 0$.

Remark 2.2. Suppose that the simple pendulum is replaced by a compound pendulum with mass m_2, moment of inertia I about the pivot point, and distance d of the center of mass from the pivot point. The equations of motion are now

$$\begin{aligned}(m_1+m_2)\ddot{y}(t)+c_1\dot{y}(t)+k_1y(t)-m_2d\bigl(\ddot{\theta}(t)\sin\theta(t)+\dot{\theta}^2(t)\cos\theta(t)\bigr)&=\Xi(t),\\ I\ddot{\theta}(t)+c_2\dot{\theta}(t)+m_2d(g-\ddot{y}(t))\sin\theta(t)&=0.\end{aligned}\tag{2.3}$$

Letting $\eta=y/L$, where $L=I/(m_2d)$ is the effective length of the compound pendulum, the equations of motion (2.3) can be written in the form (1.2), where now $\kappa=\sqrt{g/L}$ is the frequency of the compound pendulum, and $R:=(d/L)\cdot m_2/(m_1+m_2)<1$ because $L\ge d$.

There exists at least one stationary solution, the single mode solution described in (1.3), (1.4) above. In addition there is an "unstable single mode" solution in which the pendulum is at rest directly above the block. It is described by (1.4) together with $(\theta(t),\dot{\theta}(t))=(\pi,0)$. There may or may not be other stationary solutions. We conjecture the existence of other "nonlinear" solutions depends on the stability of the single mode solution; see section 2.2.

2.1. Stability of the single mode solution. In order to determine the stability of the single mode solution we study the long time behavior of the linearization of the system along the single mode solution (1.3), (1.4). Writing $\eta(t)=\bar{\eta}(t)+\delta x(t)$ and $\theta(t)=0+\delta\varphi(t)$ in (1.2) and letting $\delta\to 0$ gives the set of variational equations

$$\begin{aligned}\ddot{x}(t)+2\zeta_1\dot{x}(t)+\chi^2x(t)&=0,\\ \ddot{\varphi}(t)+2\zeta_2\dot{\varphi}(t)+\bigl(\kappa^2-\ddot{\bar{\eta}}(t)\bigr)\varphi(t)&=0,\end{aligned}\tag{2.4}$$

where $\bar{\eta}_t$ is given by (1.4). More rigorously, linearizing the system (2.2) along the solution $(\bar{\eta}, \dot{\bar{\eta}}, 0, 0)$ gives a first order system for $(x, \dot{x}, \varphi, \dot{\varphi})$ which is equivalent to

$$\begin{gathered} \ddot{x}(t) + 2\zeta_1 \dot{x}(t) + \chi^2 x(t) = 0, \\ \ddot{\varphi}(t) + 2\zeta_2 \dot{\varphi}(t) + \big(\kappa^2 + 2\zeta_1 \dot{\bar{\eta}}(t) + \chi^2 \bar{\eta}(t)\big)\varphi(t) = \nu \varphi(t) \dot{W}(t). \end{gathered} \tag{2.5}$$

We see that the second line in (2.4) together with the information about $\bar{\eta}(t)$ in (1.4) is rigorously interpreted as the second line in (2.5).

The stability of the single mode solution is determined by the long time behavior of the linearized process $\{(x(t), \dot{x}(t), \varphi(t), \dot{\varphi}(t)) : t \geq 0\}$. The top equation in (2.5) is damped and unforced, so $(x(t), \dot{x}(t)) \to (0, 0)$ as $t \to \infty$. Therefore it is enough to consider the parametric excitation of $\{(\varphi(t), \dot{\varphi}(t)) : t \geq 0\}$ caused by the single mode vibration $\bar{\eta}$. For simplicity of notation we replace $\bar{\eta}$ with η and consider the long term growth or decay rate for the process $\{(\varphi(t), \dot{\varphi}(t)) : t \geq 0\}$ given by

$$\ddot{\eta}(t) + 2\zeta_1 \dot{\eta}(t) + \chi^2 \eta(t) = \nu \dot{W}(t), \tag{2.6}$$

$$\ddot{\varphi}(t) + 2\zeta_2 \dot{\varphi}(t) + \big(\kappa^2 + 2\zeta_1 \dot{\eta}(t) + \chi^2 \eta(t)\big)\varphi(t) = \nu \varphi(t) \dot{W}(t). \tag{2.7}$$

The stability of the single mode solution is determined by the almost-sure exponential growth rate of (2.7), expressed precisely as the Lyapunov exponent

$$\lambda := \lim_{t \to \infty} \frac{1}{t} \log \| (\varphi(t), \dot{\varphi}(t)) \|. \tag{2.8}$$

We will see in Corollary 4.1 that the almost-sure limit in (2.8) exists and takes the same value for all initial conditions $(\eta(0), \dot{\eta}(0), \varphi(0), \dot{\varphi}(0))$ with $(\varphi(0), \dot{\varphi}(0)) \neq (0, 0)$.

2.2. Invariant measures. The single mode solution takes values in the invariant subset $H_0 = \{(v_1, v_2, 0, 0) : v_1, v_2 \in \mathbb{R}\} \subset N$, and the unstable single mode solution takes values in the invariant subset $H_\pi = \{(v_1, v_2, \pi, 0) : v_1, v_2 \in \mathbb{R}\} \subset N$. Let σ_0 and σ_π denote the corresponding invariant measures for the diffusion process $\{U(t) : t \geq 0\}$ on H_0 and H_π respectively. If $U(0) \in H_0$ then $U(t)$ converges to σ_0 in distribution, and similarly if $U(0) \in H_\pi$ then $U(t)$ converges to σ_π in distribution.

Now consider the process $\{U(t) : t \geq 0\}$ with $U(0) \in N \setminus (H_0 \cup H_\pi)$. From Theorem 2.1 we know that $U(t)$ exists for all time and that $\|U(t)\|$ does not tend to infinity. In order to discover whether $\{U(t) : t \geq 0\}$ is recurrent or transient on $N \setminus (H_0 \cup H_\pi)$, and thus determine whether or not there is an invariant measure for $\{U(t) : t \geq 0\}$ on $N \setminus (H_0 \cup H_\pi)$, it is necessary to determine the behavior of the process when it is close to H_0 or H_π. Does it tend to get even closer, or does it tend to move away again? The behavior of $\|(\varphi(t), \dot{\varphi}(t))\|$ for the linearization (2.6), (2.7) along H_0 acts as an approximation to the distance $\|(u_1(t), u_2(t))\|$ of $U(t) = (v_1(t), v_2(t), u_1(t), u_2(t))$ from H_0 when $U(t)$ is close to H_0. The sign of λ has implications for the almost sure growth or decay of $\|(\varphi(t), \dot{\varphi}(t))\|$, and so it should have implications for the distance of $U(t)$ from H_0 whenever this distance is small. If $\lambda < 0$ we expect this distance to converge to 0, so that the process $\{U(t) : t \geq 0\}$ is transient on $N \setminus (H_0 \cup H_\pi)$ and there is no invariant measure on $N \setminus (H_0 \cup H_\pi)$. Conversely if $\lambda > 0$ we expect that $U(t)$ will tend to move away from H_0, allowing the possibility of recurrent behavior of $N \setminus (H_0 \cup H_\pi)$.

The creation of a new invariant measure as λ passes from negative to positive is shown by Baxendale [8] for the case of a stochastic differential equation in $\mathbb{R}^n$ with a fixed point at 0. That is, with $N \setminus (H_0 \cup H_\pi)$ replaced by $\mathbb{R}^n \setminus \{0\}$. This result uses techniques developed by Baxendale and Stroock [7] in the setting of $M \times M \setminus \{\text{diagonal}\}$ for the 2-point motion of a stochastic differential equation on a compact manifold M. In both cases the linearization is taken along a compact submanifold ($\{0\}$ and the diagonal, respectively), and eigenfunctions for the associated moment Lyapunov exponent can be used to construct suitable Lyapunov functions to describe the behavior of the nonlinear system near the submanifold. But these methods do not apply when the linearization is along a noncompact submanifold, and in that case suitable Lyapunov functions have to be constructed on a case by case basis. This is carried out, for example, in Coti Zelati and Hairer [13] with $\mathbb{R}^3 \setminus (\{(0,0)\} \times \mathbb{R})$. In the absence of an explicit construction, we make the following conjecture.

CONJECTURE 2.1.

(i) *If $\lambda < 0$, then σ_0 and σ_π are the only invariant measures for $\{U(t) : t \geq 0\}$ on $N = \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}$, and $U(t)$ converges in distribution to the single mode invariant measure σ_0 for all $U(0) \notin H_\pi$.*

(ii) *If $\lambda > 0$, there is also an invariant measure, $\widehat{\sigma}$ say, on $N \setminus (H_0 \cup H_\pi)$ such that $U(t)$ converges to $\widehat{\sigma}$ in distribution for all $U(0) \notin (H_0 \cup H_\pi)$.*

3. A more general linear system. Much of our analysis and computation of the Lyapunov exponent (2.8) is valid in a more general setting. Notice that (2.7) can be rewritten as

$$\ddot{\varphi}(t) + 2\zeta_2 \dot{\varphi}(t) + \big(\kappa^2 - \ddot{\eta}(t)\big)\varphi(t) = 0 \tag{3.1}$$

with $\eta(t)$ given by (2.6). We will replace the generalized process $\ddot{\eta}(t)$ by a more general $\xi(t)$.

Let $\{v(t) : t \geq 0\}$ be the $\mathbb{R}^d$ valued Ornstein–Uhlenbeck process given by

$$dv(t) = Av(t)dt + BdW(t), \tag{3.2}$$

where A is a $d \times d$ matrix, B is a $d \times m$ matrix, and $W(t)$ is a standard m-dimensional Brownian motion. See, for example, Gardiner [14, sect 4.4] or Liberzon and Brockett [24]. We shall assume throughout that the eigenvalues of A have strictly negative real parts, and that (A, B) is a controllable pair, that is,

$$\operatorname{rank}\{B, AB, \ldots, A^{d-1}B\} = d. \tag{3.3}$$

Since the eigenvalues of A have strictly negative real parts, the equation (3.2) has a stationary solution

$$v(t) = \int_{-\infty}^{t} e^{(t-s)A} B dW(s)$$

with a stationary probability measure μ, say, which is mean-zero Gaussian with $d \times d$ covariance matrix R given by

$$R_{jk} = \sum_{\ell=1}^{m} \int_{-\infty}^{0} \big(e^{-sA}Be_\ell\big)_j \big(e^{-sA}Be_\ell\big)_k ds = \int_0^\infty \big(e^{tA}BB^* e^{tA^*}\big)_{jk} dt. \tag{3.4}$$

The controllable pair assumption is then equivalent to the assumption that R is invertible, and this in turn is equivalent to the assumption that $\operatorname{supp}(\mu) = \mathbb{R}^d$.

Choose $a \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}^m$ and define

$$\xi(t) = \sum_{j=1}^{d} a_j v_j(t) + \sum_{\ell=1}^{m} \gamma_\ell \dot{W}_\ell(t) = \langle a, v(t) \rangle + \langle \gamma, \dot{W}(t) \rangle. \tag{3.5}$$

At the cost of replacing m with $m+1$ and adjoining a column of zeros to the matrix B, we can include the situation where $\xi(t)$ includes some white noise which is independent of the white noise driving $v(t)$. With this choice of $\xi(t)$ the equation

$$\ddot{\varphi}(t)+2\zeta_2\dot{\varphi}(t)+\big(\kappa^2-\xi(t)\big)\varphi(t)=0 \tag{3.6}$$

has a well-defined meaning as the second order SDE

$$\ddot{\varphi}(t)+2\zeta_2\dot{\varphi}(t)+\big(\kappa^2-\langle a,v(t)\rangle\big)\varphi(t)=\sum_{\ell=1}^{m}\gamma_\ell\varphi(t)\dot{W}_\ell(t). \tag{3.7}$$

We will consider the Lyapunov exponent

$$\lambda:=\lim_{t\to\infty}\frac{1}{t}\log\|\,(\varphi(t),\dot{\varphi}(t))\,\| \tag{3.8}$$

for the more general system given by (3.2), (3.7).

Since $v(t)=\binom{\eta(t)}{\dot{\eta}(t)}$ satisfies

$$dv(t)=\begin{pmatrix}0 & 1\\ -\chi^2 & -2\zeta_1\end{pmatrix}v(t)dt+\begin{pmatrix}0\\ \nu\end{pmatrix}dW(t) \tag{3.9}$$

and

$$\ddot{\eta}(t)=-\chi^2\eta(t)-2\zeta_1\dot{\eta}(t)+\nu\dot{W}(t)$$

we see that the system (2.6), (2.7) is a special case of (3.2), (3.7).

4. Khas'minskii's integral formula. Define $u(t)=\binom{\varphi(t)}{\dot{\varphi}(t)}$; then (3.7) can be written as the 2-dimensional linear SDE

$$du(t)=\begin{pmatrix}0 & 1\\ -\kappa^2+\langle a,v(t)\rangle & -2\zeta_2\end{pmatrix}u(t)dt+\sum_{\ell=1}^{m}\begin{pmatrix}0 & 0\\ \gamma_\ell & 0\end{pmatrix}u(t)dW_\ell(t). \tag{4.1}$$

We note that the Stratonovich and Itô interpretations of (4.1) are the same. Following Khas'minskii [21], write $u(t)=\|u(t)\|\binom{\cos\psi(t)}{\sin\psi(t)}$. Applying Itô's formula to (4.1) gives

$$d(\log\|u(t)\|)=Q(v(t),\psi(t))dt+\sum_{\ell=1}^{m}\gamma_\ell\sin\psi(t)\cos\psi(t)dW_\ell(t)$$

and

$$d\psi(t)=h(v(t),\psi(t))dt+\sum_{\ell=1}^{m}\gamma_\ell\cos^2\psi(t)dW_\ell(t), \tag{4.2}$$

where

$$Q(v,\psi)=\big(1-\kappa^2+\langle a,v\rangle\big)\sin\psi\cos\psi-2\zeta_2\sin^2\psi+\frac{\|\gamma\|^2}{2}\cos^2\psi\cos 2\psi \tag{4.3}$$

and

$$h(v,\psi)=-1+\big(1-\kappa^2+\langle a,v\rangle\big)\cos^2\psi-2\zeta_2\sin\psi\cos\psi-\|\gamma\|^2\sin\psi\cos^3\psi.$$

For $(\varphi(0),\dot\varphi(0))\neq(0,0)$ we have

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log\sqrt{\varphi^2(t)+\dot\varphi^2(t)}&=\lim_{t\to\infty}\frac{1}{t}\log\|u(t)\|\\ &=\lim_{t\to\infty}\frac{1}{t}\left[\int_0^t Q(v(s),\psi(s))ds+\sum_{\ell=1}^m\int_0^t\frac{\gamma_\ell}{2}\sin 2\psi(s)\,dW_\ell(s)\right]\end{aligned}\tag{4.4}$$

in the sense that the almost-sure existence and value of the limit on the left under initial conditions $(v,\varphi,\dot\varphi)$ with $(\varphi,\dot\varphi)\neq(0,0)$ are equivalent to the almost-sure existence and value of the limit on the right under initial conditions (v,ψ).

Since

$$N(t)=\sum_{\ell=1}^m\int_0^t\frac{\gamma_\ell}{2}\sin 2\psi(s)\,dW_\ell(s)$$

is a continuous martingale with quadratic variation $\langle N\rangle_t\leq\|\gamma\|^2t/4$, it follows that

$$\mathbb{P}^{(v,\psi)}\left(\frac{1}{t}N(t)\to 0\text{ as }t\to\infty\right)=1\tag{4.5}$$

for all $(v,\psi)\in M$.

Proposition 4.1. *Assume that the eigenvalues of A have strictly negative real parts, and that (A,B) is a controllable pair. Assume also that the coefficients $a\in\mathbb{R}^d$ and $\gamma\in\mathbb{R}^m$ in* (3.5) *are not both zero. Then the diffusion $\{(v(t),\psi(t)):t\geq 0\}$ on $M:=\mathbb{R}^d\times\mathbb{R}/(2\pi\mathbb{Z})$ given by* (3.2), (4.2) *has a unique stationary probability measure m, say, with smooth density $\rho(v,\psi)$. Moreover, if $F:M\to\mathbb{R}$ is m integrable, then*

$$\mathbb{P}^{(v,\psi)}\left(\frac{1}{t}\int_0^t F(v(s),\psi(s))ds\to\int_M F\,dm\ \text{ as }t\to\infty\right)=1\tag{4.6}$$

for every $(v,\psi)\in M$.

The proof of Proposition 4.1 is given in section 8. The v marginal of m is the stationary probability measure μ on $\mathbb{R}^d$ which is mean-zero Gaussian with covariance matrix R given by (3.4). There exist constants k_0 and k_1 such that $|Q(v,\psi)|\leq k_0+k_1\|v\|$ and so

$$\int_M|Q(v,\psi)|dm(v,\psi)\leq\int_{\mathbb{R}^d}\big(k_0+k_1\|v\|\big)d\mu(v)<\infty.$$

By (4.4) and (4.5) and using Proposition 4.1 we obtain the following corollary.

Corollary 4.1.

(i) *Under the assumptions of Proposition* 4.1 *we have*

$$\mathbb{P}^{(v,\varphi,\dot\varphi)}\left(\frac{1}{t}\log\|(\varphi(t),\dot\varphi(t))\|\to\int_M Q\,dm\ \text{ as }t\to\infty\right)=1\tag{4.7}$$

for every $(v,\varphi,\dot\varphi)$ with $(\varphi,\dot\varphi)\neq(0,0)$.

(ii) *The Lyapunov exponent λ defined in* (3.8) *exists as an almost-sure limit for all initial conditions $(v,\varphi,\dot\varphi)$ with $(\varphi,\dot\varphi)\neq(0,0)$ and*

$$\lambda=\int_M Q(v,\psi)\,dm(v,\psi).\tag{4.8}$$

Direct evaluation of (4.8) is hard, because we need to first solve $\mathcal{A}^* m = 0$, where $\mathcal{A}$ denotes the generator of the diffusion $\{(v(t), \psi(t)) : t \geq 0\}$ on M given by (3.2), (4.2). There is a simple case when $a = 0$, because then $v(t)$ disappears from (4.1) and the problem reduces to a constant coefficient linear SDE in $\mathbb{R}^2$. In this case the exact formula of Imkeller and Lederer [20] will apply. But this excludes the original block-pendulum model. The block-pendulum case is interesting because of the different sorts of noise in (2.7): white noise $\dot{W}(t)$ as well as colored noise $\eta(t)$ and $\dot{\eta}(t)$.

Before proceeding to the small noise estimates, we give a simple upper bound for the Lyapunov exponent λ.

PROPOSITION 4.2. *Assume $\zeta_2 < \kappa$ and let $\kappa_d = \sqrt{\kappa^2 - \zeta_2^2}$ denote the damped frequency of the pendulum. Then*

$$\lambda \leq -\zeta_2 + \frac{\sqrt{\langle a, Ra\rangle}}{2\kappa_d} + \frac{\|\gamma\|^2}{2\kappa_d^2}, \tag{4.9}$$

where R is the covariance matrix (3.4). *For the original block and pendulum setting*

$$\lambda \leq -\zeta_2 + \frac{\nu}{2\kappa_d}\sqrt{\frac{\chi^2 + 4\zeta_1^2}{4\zeta_1}} + \frac{\nu^2}{2\kappa_d^2}. \tag{4.10}$$

The proof of Proposition 4.2 uses a slightly different version of the function $Q(v, \psi)$ and is given in section 9.1. The result is motivated by an estimate in Ariaratnam [1].

5. Small forcing. In this section we consider the effect of small forcing of the block. Precisely, we replace $\nu \rightsquigarrow \varepsilon\nu$ for small ε. The linearized system (2.6), (2.7) becomes

$$\ddot{\eta}(t) + 2\zeta_1\dot{\eta}(t) + \chi^2\eta(t) = \varepsilon\nu\dot{W}(t),$$
$$\ddot{\varphi}(t) + 2\zeta_2\dot{\varphi}(t) + \big(\kappa^2 + 2\zeta_1\dot{\eta}(t) + \chi^2\eta(t)\big)\varphi(t) = \varepsilon\nu\varphi(t)\dot{W}(t).$$

Equivalently, writing $\eta(t) = \varepsilon\tilde{\eta}(t)$ and then dropping the tilde, we obtain

$$\ddot{\eta}(t) + 2\zeta_1\dot{\eta}(t) + \chi^2\eta(t) = \nu\dot{W}(t), \tag{5.1}$$

$$\ddot{\varphi}(t) + 2\zeta_2\dot{\varphi}(t) + \big(\kappa^2 + 2\varepsilon\zeta_1\dot{\eta}(t) + \varepsilon\chi^2\eta(t)\big)\varphi(t) = \varepsilon\nu\varphi(t)\dot{W}(t). \tag{5.2}$$

For the more general problem in section 3 we replace $\xi(t) \rightsquigarrow \varepsilon\xi(t)$ in (3.6), that is, $a \rightsquigarrow \varepsilon a$ and $\gamma \rightsquigarrow \varepsilon\gamma$, giving the equation

$$\ddot{\varphi}(t) + 2\zeta_2\dot{\varphi}(t) + \big(\kappa^2 - \varepsilon\langle a, v(t)\rangle\big)\varphi(t) = \sum_{\ell=1}^{m} \varepsilon\gamma_\ell\varphi(t)\dot{W}_\ell(t), \tag{5.3}$$

where $\{v(t) : t \geq 0\}$ is still given by (3.2). We now define the Lyapunov exponent

$$\lambda(\varepsilon) := \lim_{t\to\infty} \frac{1}{t} \log \|\,(\varphi(t), \dot{\varphi}(t))\,\|, \tag{5.4}$$

where $\{\varphi(t) : t \geq 0\}$ is given by (5.3).

Define the matrix valued cosine transform of A by

$$\widehat{S}_A(\omega) = \frac{1}{\pi}\int_0^\infty e^{tA}\cos\omega t\, dt, \tag{5.5}$$

and recall that R denotes the covariance matrix of the stationary probability measure μ for $\{v(t) : t \geq 0\}$.

THEOREM 5.1.

(i) *Assume that the eigenvalues of A have strictly negative real parts, and that (A,B) is a controllable pair. Assume $\zeta_2 < \kappa$ and let $\kappa_d = \sqrt{\kappa^2 - \zeta_2^2}$ denote the damped frequency of the pendulum. For $\{\varphi(t) : t \geq 0\}$ given by (5.3) we have*

$$\lambda(\varepsilon) = -\zeta_2 + \varepsilon^2 \lambda_2(2\kappa_d) + O(\varepsilon^4) \qquad \text{as } \varepsilon \to 0, \tag{5.6}$$

where

$$\lambda_2(\omega) = \frac{\pi}{\omega^2}\left(\langle a, \widehat{S}_A(\omega) R a\rangle + \langle a, \widehat{S}_A(\omega) B\gamma\rangle + \frac{\|\gamma\|^2}{2\pi}\right). \tag{5.7}$$

(ii) *Moreover, given matrices A and B and vectors a and γ the asymptotic (5.6) is uniform for κ_d bounded away from 0 and ∞. That is, given A, B, a, γ, and $0 < c_1 < c_2 < \infty$, there exists K such that*

$$\left|\lambda(\varepsilon) + \zeta_2 - \varepsilon^2 \lambda_2(2\kappa_d)\right| \leq K\varepsilon^4 \tag{5.8}$$

whenever $0 < \varepsilon \leq 1$ and $c_1 \leq \kappa_d \leq c_2$.

The proof for Theorem 5.1 is given in section 9.

Observe that the destabilizing effect of the noise is strongest when the damped frequency κ_d maximizes the function $\lambda_2(2\omega)$. We will expand on this observation in section 6.

In the special case when $a = 0$ we have $\lambda_2(\omega) = \|\gamma\|^2/(2\omega^2)$ and we recover the result of Auslender and Milstein [3] that

$$\lambda(\varepsilon) = -\zeta_2 + \frac{\varepsilon^2 \|\gamma\|^2}{8\kappa_d^2} + O(\varepsilon^4) \qquad \text{as } \varepsilon \to 0.$$

But when $a \neq 0$ we have to do some work to simplify the formula (5.7) for $\lambda_2(\omega)$. The covariance matrix R is determined by (3.4). The cosine transform $\widehat{S}_A(\omega)$ defined in (5.5) satisfies

$$\widehat{S}_A(\omega) = \frac{1}{\pi}\Re\left(\int_0^\infty e^{tA} e^{-i\omega t}\, dt\right) = -\frac{1}{\pi}\Re\Big((A - i\omega I_d)^{-1}\Big). \tag{5.9}$$

Hence, given the matrices A and B and the vectors a and γ, all the terms in (5.7) can be calculated.

We will give formulas for $\lambda_2(\omega)$ in terms of power spectral density. Recall that for any L^2 stationary mean zero scalar process $X(t)$ the power spectral density function S_X is given by

$$S_X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{E}[X(t)X(0)] e^{-i\omega t}\, dt = \frac{1}{\pi}\int_0^\infty \mathbb{E}[X(t)X(0)] \cos \omega t\, dt. \tag{5.10}$$

We note that many authors omit the factor $1/2\pi$.

The block and pendulum setting has $\xi(t) = \ddot{\eta}(t)$ where $\dot{\eta}(t)$ is a well-defined L^2 stationary process. The first result extends this case to the general system of section 3.

Fix $\alpha \in \mathbb{R}^d$ and define $\xi(t) = \dot{V}(t)$ where $V(t) = \langle \alpha, v(t)\rangle$. Since

$$\xi(t) = \langle \alpha, \dot{v}(t)\rangle = \langle \alpha, Av(t)\rangle + \langle \alpha, B\dot{W}(t)\rangle = \langle A^*\alpha, v(t)\rangle + \langle B^*\alpha, \dot{W}(t)\rangle, \tag{5.11}$$

we have (3.5) with $a = A^*\alpha$ and $\gamma = B^*\alpha$. Note that $V(t)$ is a well-defined L^2 stationary process, so that it has a well-defined autocovariance function $\mathbb{E}[V(t)V(0)]$ and hence a well-defined power spectral density $S_V(\omega)$ using the formula (5.10).

PROPOSITION 5.1. *For ξ given by* (5.11) *we have*

$$\lambda_2(\omega) = \pi S_V(\omega). \tag{5.12}$$

Specializing to the original block and pendulum case, we have $\xi(t) = \ddot{\eta}(t) = \dot{v_2}(t)$ where $v(t) = \binom{\eta(t)}{\dot{\eta}(t)}$ as in (3.9). Therefore we may apply Proposition 5.1 with $V(t) = v_2(t) = \dot{\eta}(t)$. For the system (3.9) the power spectral density $S_{\dot{\eta}}(w)$ is well known; see, for example, [14, sect. 4.4].

COROLLARY 5.1. *For $\xi = \ddot{\eta}$ given by* (2.6) *we have*

$$\lambda_2(\omega) = \pi S_{\dot{\eta}}(\omega) = \frac{\omega^2 \nu^2}{2[(\chi^2 - \omega)^2 + 4\zeta_1^2 \omega^2]}. \tag{5.13}$$

In the general case (3.5) when $\xi(t) = \langle a, v(t)\rangle + \langle \gamma, \dot{W}(t)\rangle$ with $\gamma \neq 0$ then ξ is not a well-defined L^2 process. It does not have an autocovariance function, and the formula (5.10) for the power spectral density does not apply. However we can consider the power spectral density $S_{\xi_\delta}(\omega)$ for a mollified version ξ_δ of ξ.

Suppose $\psi : \mathbb{R} \to [0, \infty)$ is piecewise continuous with support in $[-1, 1]$ and $\int_{-\infty}^{\infty} \psi(t)dt = 1$. Define $\psi_\delta(t) = (1/\delta)\psi(t/\delta)$. For any continuous function f we have

$$f_\delta(t) := f * \psi_\delta(t) = \int_{-\infty}^{\infty} f(s)\psi_\delta(t-s)ds = \int_{-\infty}^{\infty} f(t - \delta u)\psi(u)du \to f(t) \tag{5.14}$$

as $\delta \to 0$, so that the functions $\psi_\delta(t - \cdot)$ converge to the Dirac delta distribution at t. Then the process ξ_δ defined informally as $\xi * \psi_\delta$ and precisely by

$$\xi_\delta(t) = \int_{-\infty}^{\infty} \langle a, v(s)\rangle \psi_\delta(t-s)ds + \left\langle \gamma, \int_{-\infty}^{\infty} \psi_\delta(t-s)dW(s)\right\rangle \tag{5.15}$$

is a well-defined L^2 process, and it converges in a weak sense to the generalized process ξ.

PROPOSITION 5.2. *For $\xi(t) = \langle a, v(t)\rangle + \langle \gamma, \dot{W}(t)\rangle$ and $\xi_\delta(t)$ given by* (5.15) *the limit*

$$\lim_{\delta\to 0} S_{\xi_\delta}(\omega) = \langle a, \widehat{S}_A(\omega)Ra\rangle + \langle a, \widehat{S}_A(\omega)B\gamma\rangle + \frac{\|\gamma\|^2}{2\pi} \tag{5.16}$$

exists and does not depend on the choice of ψ. Defining $S_\xi(\omega) := \lim_{\delta\to 0} S_{\xi_\delta}(\omega)$ we have

$$\lambda_2(\omega) = \frac{\pi}{\omega^2} S_\xi(\omega). \tag{5.17}$$

The proofs of Propositions 5.1 and 5.2 are given in section 9.

6. Small forcing and small pendulum damping. In this section we consider the combined effect of small forcing of the block together with small damping of the pendulum. Precisely, we replace

$$\nu \rightsquigarrow \varepsilon\nu \quad \text{and} \quad \zeta_2 \rightsquigarrow \varepsilon^2\zeta_2$$

for small ε. The equation for φ is now

$$\ddot{\varphi}(t) + 2\varepsilon^2\zeta_2\dot{\varphi}(t) + \big(\kappa^2 + 2\varepsilon\zeta_1\dot{\eta}(t) + \varepsilon\chi^2\eta(t)\big)\varphi(t) = \varepsilon\nu\varphi(t)\dot{W}(t) \tag{6.1}$$

for the original block and pendulum setting, and

$$\ddot{\varphi}(t) + 2\varepsilon^2\zeta_2\dot{\varphi}(t) + \big(\kappa^2 - \varepsilon\langle a, v(t)\rangle\big)\varphi(t) = \sum_{\ell=1}^{m} \varepsilon\gamma_\ell\varphi(t)\dot{W}_\ell(t) \tag{6.2}$$

in the general setting. In order to distinguish this scaling from the previous one in section 5 we write

$$\widehat{\lambda}(\varepsilon) := \lim_{t\to\infty} \frac{1}{t}\log\|\,(\varphi(t),\dot{\varphi}(t))\,\| \tag{6.3}$$

for the Lyapunov exponent where $\{\varphi(t) : t \geq 0\}$ is given by (6.2).

THEOREM 6.1. *Assume that the eigenvalues of A have strictly negative real parts, and that (A, B) is a controllable pair. For the process $\big\{\big(\varphi(t),\dot{\varphi}(t)\big) : t \geq 0\big\}$ given by* (6.2) *we have*

$$\widehat{\lambda}(\varepsilon) = \varepsilon^2\big(-\zeta_2 + \lambda_2(2\kappa)\big) + O(\varepsilon^4) \qquad \text{as } \varepsilon \to 0, \tag{6.4}$$

where $\lambda_2(\omega)$ is given by (5.7).

Proof. The damped frequency for (6.2) is $\kappa_d = \sqrt{\kappa^2 - \varepsilon^4\zeta_2^4} = \kappa + O(\varepsilon^4)$, and so $\lambda_2(2\kappa_d) = \lambda_2(2\kappa) + O(\varepsilon^4)$. The result now follows directly from Theorem 5.1(ii). ∎

Returning to the original block and pendulum case and using (5.13) we have

$$\widehat{\lambda}(\varepsilon) = \varepsilon^2\left(-\zeta_2 + \frac{2\nu^2\kappa^2}{(\chi^2 - 4\kappa^2)^2 + 16\zeta_1^2\kappa^2}\right) + O(\varepsilon^4) \qquad \text{as } \varepsilon \to 0. \tag{6.5}$$

Recall that the long term exponential growth or decay rate for the process $\{(\varphi(t), \dot{\varphi}(t)) : t \geq 0\}$ is given by the Lyapunov exponent $\widehat{\lambda}(\varepsilon)$. Putting $\widehat{\lambda}(\varepsilon) = 0$ in (6.5) provides the almost-sure *stability boundary* in terms of the excitation intensity ν. Hence the second order approximation of the almost-sure stability boundary is given by

$$\nu_c^2 = \frac{\zeta_2\big[(\chi^2 - 4\kappa^2)^2 + 16\zeta_1^2\kappa^2\big]}{2\kappa^2} = \zeta_2\left(\frac{(\chi^2 - 4\kappa^2)^2}{2\kappa^2} + 8\zeta_1^2\right). \tag{6.6}$$

It is clear that the dissipation in both the primary (ζ_1) and secondary (ζ_2) systems has a stabilizing effect on the single mode solution. Although no particular attention was given to the $1:2$ resonance in the analysis of the linearized system, the stability boundary (6.6) shows the significance of internal resonance, $\chi \approx 2\kappa$, in determining the instability region in the (κ, ν) parameter space, which is of significance in applications. For fixed χ, ζ_1, and ζ_2 the critical noise intensity ν_c, as a function of κ, has a minimum value $\sqrt{8\zeta_1^2\zeta_2}$ attained when $\kappa = \chi/2$. Figure 2 shows ν_c as a function of κ with $\chi = 1$ and $\zeta_2 = 1$ and $\zeta_1 = 0.02$, 0.2, 0.4, and 0.7.

The behavior displayed in Figure 2 mimics that of the instability tongues and transition curves in the stability chart for Mathieu's equation with linear viscous damping and cosine periodic forcing. More specifically the equation

$$\ddot{\varphi}(t) + 2\varepsilon\zeta_2\dot{\varphi}(t) + (\kappa^2 - \varepsilon\nu\cos\omega t)\varphi(t) = 0 \tag{6.7}$$

with small periodic forcing $\varepsilon\nu$ and small dissipation $\varepsilon\zeta_2$ has first order approximation (as $\varepsilon \to 0$) of the stability boundary given by

$$\varepsilon^2(\nu_c^2 - 16\zeta_2^2) = \omega^2\left(\frac{4\kappa^2}{\omega^2} - 1\right)^2. \tag{6.8}$$

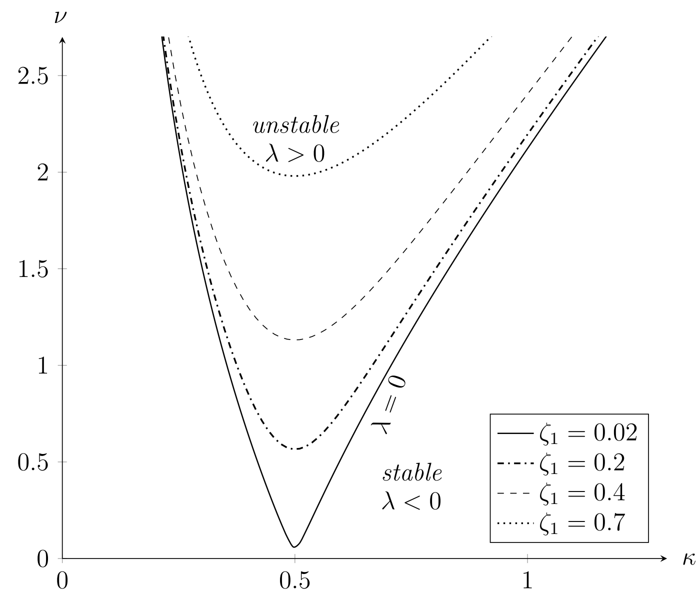

FIG. 2. *Almost-sure stability boundaries with* $\chi = 1$ *and* $\zeta_2 = 1$ *and* $\zeta_1 = 0.02$, 0.2, 0.4, *and* 0.7.

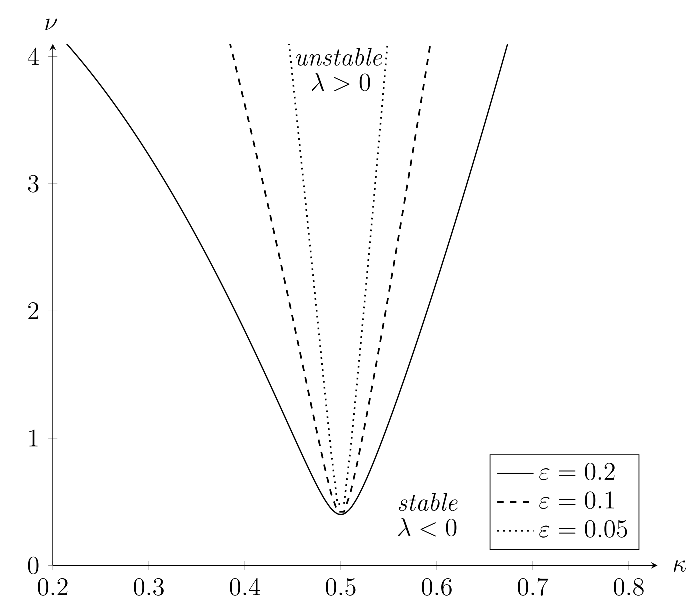

FIG. 3. *Almost-sure stability boundaries for the Mathieu equation* (6.7) *with* $\omega = 1$, $\zeta_2 = 0.1$, *and* $\varepsilon = 0.2$, 0.1, *and* 0.05.

See Verhulst [34, page 241] for the case $\omega = 2$. Note that in the deterministic setting the damping is of the same order ε as the forcing. Figure 3 shows ν_c as a function of κ with $\omega = 1$, $\zeta_2 = 0.1$, and $\varepsilon = 0.2$, 0.1, and 0.05. Notice that in this model the "width" of the instability region decreases as ε decreases.

A more directly relevant comparison is seen when we replace white noise forcing in the autoparametric system with periodic (deterministic) forcing. Consider the system

$$\ddot\eta(t)+2\zeta_1\dot\eta(t)+\chi^2\eta(t)-R\big(\ddot\theta(t)\sin\theta(t)+\dot\theta^2(t)\cos\theta(t)\big)=\varepsilon\nu\cos\omega t,$$
$$\ddot\theta(t)+2\varepsilon\zeta_2\dot\theta(t)+\big(\kappa^2-\ddot\eta(t)\big)\sin\theta(t)=0, \tag{6.9}$$

with forcing of some fixed frequency ω and small intensity $\varepsilon\nu$, and small pendulum damping $\varepsilon\zeta_2$. Linearizing along the single mode solution $\theta(t)\equiv 0$ we get

$$\ddot\eta(t)+2\zeta_1\dot\eta(t)+\chi^2\eta(t)=\varepsilon\nu\cos\omega t, \tag{6.10}$$
$$\ddot\varphi(t)+2\varepsilon\zeta_2\dot\varphi(t)+\big(\kappa^2-\ddot\eta(t)\big)\varphi(t)=0. \tag{6.11}$$

The stationary solution of (6.10) is

$$\eta(t)=\frac{\varepsilon\nu}{\sqrt{(\chi^2-\omega^2)^2+4\zeta_1^2\omega^2}}\cos(\omega t+\alpha),$$

where $\alpha=\arg(\chi^2-\omega^2+2i\zeta_1\omega)$, and so

$$\ddot\eta(t)=\frac{\varepsilon\omega^2\nu}{\sqrt{(\chi^2-\omega^2)^2+4\zeta_1^2\omega^2}}\cos(\omega t+\alpha+\pi).$$

Therefore, (6.11) has the form of Mathieu's equation (6.7) with ν replaced by $\frac{\omega^2\nu}{\sqrt{(\chi^2-\omega^2)^2+4\zeta_1^2\omega^2}}$. The phase change $\pi+\alpha$ has no effect on the stability, and the (first order) stability boundary is

$$\varepsilon^2\left(\frac{\omega^4\nu_c^2}{(\chi^2-\omega^2)^2+4\zeta_1^2\omega^2}-16\zeta_2^2\right)=\omega^2\left(\frac{4\kappa^2}{\omega^2}-1\right)^2. \tag{6.12}$$

A multiplicative change in the vertical coordinate will convert the stability regions for the Mathieu equation (6.7) shown in Figure 3 into the corresponding regions for the periodically forced autoparametric system (6.10), (6.11).

Remark 6.1. For the periodically forced system (6.10), (6.11) the functions $\eta(t)$ and $\ddot\eta(t)$ are related by a simple multiplicative factor. It makes little theoretical difference whether $\nu\cos\omega t$ is applied as forcing on the block or is assumed to describe the motion of the pivot point. The situation with stochastic forcing is very different. With white noise forcing $\{\eta(t):t\ge 0\}$ is an L^2 process with continuous sample paths, whereas $\{\ddot\eta(t):t\ge 0\}$ exists only as a generalized process and has to be interpreted in terms of stochastic integrals.

7. Proof of Theorem 2.1.

Let $\mathcal{L}$ denote the generator of the diffusion $\{(v_1(t), v_2(t),u_1(t),u_2(t)):t\ge 0\}$ on $N=\mathbb{R}^2\times\mathbb{R}/(2\pi\mathbb{Z})\times\mathbb{R}$ given by (2.2). Then

$$\begin{aligned}\mathcal{L}=v_2\frac{\partial}{\partial v_1}&+\left(\frac{-2\zeta_1v_2-\chi^2v_1+Ru_2^2\cos u_1-2R\zeta_2u_2\sin u_1-R\kappa^2\sin^2u_1}{(1-R\sin^2u_1)}\right)\frac{\partial}{\partial v_2}\\&+u_2\frac{\partial}{\partial u_1}\\&+\left(\frac{-2\zeta_2u_2-\kappa^2\sin u_1-2\zeta_1v_2\sin u_1-\chi^2v_1\sin u_1+Ru_2^2\sin u_1\cos u_1}{(1-R\sin^2u_1)}\right)\frac{\partial}{\partial u_2}\\&+\frac{\nu^2}{2(1-R\sin^2u_1)^2}\left(\frac{\partial^2}{\partial v_2^2}+2\sin u_1\frac{\partial^2}{\partial v_2\partial u_2}+\sin^2u_1\frac{\partial^2}{\partial u_2^2}\right).\end{aligned}$$

7.1. Construction of a Lyapunov function.

Up to normalization, the energy of the block and pendulum system is given by

$$E(v_1,v_1,u_1,u_2)=\frac12v_2^2+\frac12Ru_2^2-Rv_2u_2\sin u_1+\frac12\chi^2v_1^2+R\kappa^2(1-\cos u_1). \tag{7.1}$$

LEMMA 7.1.

$$\mathcal{L}E(v_1, v_1, u_1, u_2) = -2\zeta_1 v_2^2 - 2R\zeta_2 u_2^2 + \frac{\nu^2}{2(1 - R\sin^2 u_1)}.$$
$$\mathcal{L}\Big(v_1(v_2 - Ru_2 \sin u_1)\Big) = v_2^2 - Rv_2 u_2 \sin u_1 - 2\zeta_1 v_1 v_2 - \chi^2 v_1^2.$$

Proof. This is direct calculation. ∎

For α to be chosen later, define

$$F(v_1, v_2, u_1, u_2) = E(v_1, v_1, u_1, u_2) + \alpha\Big(v_1(v_2 - Ru_2 \sin u_1)\Big). \tag{7.2}$$

PROPOSITION 7.1. (i) *There exist $\alpha > 0$ and positive constants $c_1, \ldots, c_5$ such that*

$$c_1 \|(v_1, v_2, u_2)\|^2 \le F(v_1, v_2, u_1, u_2) \le c_2 + c_3 \|(v_1, v_2, u_2)\|^2 \tag{7.3}$$

and

$$\mathcal{L}F(v_1, v_2, u_1, u_2) \le c_4 - c_5 \|(v_1, v_2, u_2)\|^2. \tag{7.4}$$

(ii) *For $\alpha > 0$ as in* (i), *there exists $\beta_0 > 0$ such that for all $\gamma > 0$ there exists c_6 such that*

$$\mathcal{L}(e^{\beta_0 F(v_1, v_2, u_1, u_2)}) \le -\gamma e^{\beta_0 F(v_1, v_2, u_1, u_2)} + c_6. \tag{7.5}$$

Proof. (i) Several applications of the Cauchy–Schwarz inequality give

$$\begin{aligned}
&\frac{1-\sqrt{R}}{2}(v_2^2 + Ru_2^2) + \frac{1}{2}\chi^2 v_1^2 - \alpha\Big(v_1^2 + \frac{1}{2}v_2^2 + \frac{1}{2}R^2 u_2^2\Big) \\
&\qquad \le F(v_1, v_2, u_1, u_2) \\
&\qquad \le R\kappa^2 + \frac{1+\sqrt{R}}{2}(v_2^2 + Ru_2^2) + \frac{1}{2}\chi^2 v_1^2 + \alpha\Big(v_1^2 + \frac{1}{2}v_2^2 + \frac{1}{2}R^2 u_2^2\Big).
\end{aligned}$$

Thus the upper and lower bounds on F are satisfied whenever $0 < \alpha < \min(1 - \sqrt{R}, \chi^2/2)$.

Similarly, using Lemma 7.1,

$$\begin{aligned}
&\mathcal{L}F(v_1, v_2, u_1, u_2) \\
&\quad \le \frac{\nu^2}{2(1-R)} - 2\zeta_1 v_2^2 - 2R\zeta_2 u_2^2 + \alpha\Big(v_2^2 + R|v_2 u_2| + 2\zeta_1 |v_1 v_2| - \chi^2 v_1^2\Big) \\
&\quad \le \frac{\nu^2}{2(1-R)} - 2\zeta_1 v_2^2 - 2R\zeta_2 u_2^2 + \frac{\alpha}{2}\Big((2 + R + \frac{4\zeta_1^2}{\chi^2})v_2^2 + Ru_2^2 - \chi^2 v_1^2\Big).
\end{aligned}$$

The upper bound on $\mathcal{L}F$ now follows by choosing α sufficiently small.

(ii) Let Γ denote the carré du champ operator associated with $\mathcal{L}$; see, for example, [5]. Since $\partial F/dv_2$ and $\partial F/du_2$ both grow at most linearly with $\|(v_1, v_2, u_2)\|$ and the coefficients of $dW(t)$ in (2.2) are bounded, there exists c_7 such that $\Gamma(F,F) \le c_7 \|(v_1, v_2, u_2)\|^2$. Then

$$\begin{aligned}
\mathcal{L}\left(e^{\beta_0 F}\right) &= \Big(\beta_0 \mathcal{L}F + \beta_0^2 \Gamma(F,F)\Big) e^{\beta_0 F} \\
&\le \Big(\beta_0(c_4 - c_5\|(v_1, v_2, u_2)\|^2) + \beta_0^2 c_7 \|(v_1, v_2, u_2)\|^2\Big) e^{\beta_0 F} \\
&= \Big(\beta_0 c_4 - \beta_0(c_5 - \beta_0 c_7)\|(v_1, v_2, u_2)\|^2\Big) e^{\beta_0 F}.
\end{aligned}$$

Choose $\beta_0 > 0$ so that $\beta_0 c_7 < c_5$. For each $\gamma > 0$ there a radius r such that $\mathcal{L}(e^{\beta_0 F}) \leq -\gamma e^{\beta_0 F}$ if $\|(v_1, v_2, u_2)\| \geq r$, and of course $\mathcal{L}(e^{\beta_0 F})$ is bounded in the set $\|(v_1, v_2, u_2)\| \leq r$. ◻

7.2. Consequences of the Lyapunov function. Recall the notation $(v_1, v_2, u_1, u_2) = U \in N$ and the abuse of notation $\|(v_1, v_2, u_2)\| = \|U\|$. Fix α and β so that the results of Proposition 7.1 are valid for the function F.

Proof of Theorem 2.1(i). The coefficients of (2.2) are locally Lipschitz, so there is a well-defined local solution. Proposition 7.1 implies $F(U) \to \infty$ as $\|U\| \to \infty$, and there exists c such that $\mathcal{L}(1+F)(U) \leq c(1+F)(U)$ for all U. The result of Khas'minskii [22, Thm 3.5] implies that the well-defined local solution exists for all time and is a Feller process. ◻

Proof of Theorem 2.1(ii). Notice that if $0 < c < c_5$ and $d = c_4 + c$ and $b^2 = c_4/(c_5 - c)$, then

$$\mathcal{L}F(U) \leq c_4 - c_5\|U\|^2 \leq -c(1 + \|U\|^2) + d1_{\|U\| \leq b}.$$

This gives condition (CD2) of Meyn and Tweedie [25]. It now follows from [25, Thm. 4.3(i)] that $\mathbb{P}(\|U_t\| \leq b \text{ for some } t \geq \delta) = 1$ for all initial conditions $U(0)$ and all $\delta > 0$. ◻

Proof of Theorem 2.1(iii). The inequality (7.5) implies that $e^{\beta_0 F}$ satisfies condition (CD2) of [25]. Taking $\beta = c_1\beta_0$ and using (7.3) for the first inequality, and then [25, Thm. 4.3(ii)] for the second inequality gives

$$\int_N \exp\bigl(\beta\|U\|^2\bigr) d\pi(U) \leq \int_N \exp\bigl(\beta_0 F(U)\bigr) d\pi(U) \leq \frac{c_6}{\gamma}$$

for every invariant probability measure π for (2.2). ◻

Proof of Theorem 2.1(iv). The function $e^{\beta_0 F}$ also satisfies condition (CD3) of [25], so by the calculation in the proof of [25, Thm. 6.1] we have

$$\mathbb{E}[e^{\beta_0 F(U_t)}] \leq e^{-\gamma t} e^{\beta_0 F(U)} + \frac{c_6}{\gamma}.$$

It remains only to use the upper and lower bounds on $F(U)$ given in (7.3). We have $\beta = c_1\beta_0$ and $\beta_1 = c_3\beta_0$ and $K_1 = e^{c_2\beta_0}$. ◻

8. Proof of Proposition 4.1. The generator $\mathcal{A}$ of the process $\{(v(t), \psi(t)) : t \in \mathbb{R}\}$ on $M := \mathbb{R}^d \times R/(2\pi\mathbb{Z})$ given by (3.2), (4.2) can be written in the Hörmander form

$$\mathcal{A} = \frac{1}{2}\sum_{\ell=1}^{m} V_\ell^2 + V_0,$$

where the vector fields $V_0, V_1, \ldots V_m$ are given by

$$V_0(v,\psi) = \begin{pmatrix} Av \\ -1 + (1 - \kappa^2 + \langle a, v\rangle)\cos^2\psi - \zeta_2 \sin 2\psi \end{pmatrix} \quad \text{and} \quad V_\ell(v,\psi) = \begin{pmatrix} Be_\ell \\ \gamma_\ell \cos^2\psi \end{pmatrix}$$

for $1 \leq \ell \leq m$.

8.1. Hypoellipticity. Let $L = L(V_0, V_1, \ldots, V_m)$ be the Lie algebra generated by the vector fields $V_0, V_1, \ldots, V_m$ and let $L_0 = L_0(V_0; V_1, \ldots, V_m)$ be the ideal in L generated by $V_1, \ldots, V_m$.

For any smooth vector field V on M, let $\mathcal{M}(V)$ denote the Lie bracket

$$\mathcal{M}(V) = [V, V_0],$$

so that $-\mathcal{M}$ is the operation of taking the Lie derivative of a vector field with respect to V_0.

LEMMA 8.1.
(i) *For all* $k \geq 0$ *and* $1 \leq \ell \leq m$

$$\mathcal{M}^k(V_\ell)(v,\psi) = \begin{pmatrix} A^k B e_\ell \\ f_{kl}(v,\psi) \end{pmatrix}$$

for some smooth function $f_{kl} : M \to \mathbb{R}$.
(ii) *For a vector field of the form* $V(v,\psi) = \binom{v_0}{0}$ *we have*

$$\mathcal{M}(V)(v,\psi) = \begin{pmatrix} A v_0 \\ \langle a, v_0 \rangle \cos^2 \psi \end{pmatrix}.$$

(iii) *For a vector field of the form* $V(v,\psi) = \binom{v_0}{\alpha \cos^2 \psi}$ *we have*

$$[\mathcal{M}(V), V](v,\psi) = \begin{pmatrix} 0 \\ 2\alpha^2 \cos^2 \psi \end{pmatrix} \quad \text{and} \quad [\mathcal{M}^2(V), \mathcal{M}(V)](v, \pm\pi/2) = \begin{pmatrix} 0 \\ 4\alpha^2 \end{pmatrix}.$$

Proof. The calculations for (i) and (ii) are elementary and direct. The calculations for (iii) are longer, but still elementary and direct. We omit the details. ∎

PROPOSITION 8.1. *Assume* $a \in \mathbb{R}^d$ *and* $\gamma \in \mathbb{R}^m$ *are not both* 0. *Assume also that* (A, B) *is a controllable pair. Then* $L_0(v,\psi) = T_{(v,\psi)}M$ *for all* $(v,\psi) \in M$.

Proof. If $\gamma \neq 0$ choose ℓ_0 such that $\gamma_{\ell_0} \neq 0$ and consider the finite subset

$$\{\mathcal{M}^k(V_\ell) : 0 \leq k \leq d-1, 1 \leq \ell \leq m\} \cup \{[\mathcal{M}(V_{\ell_0}), V_{\ell_0}], [\mathcal{M}^2(V_{\ell_0}), \mathcal{M}(V_{\ell_0})]\} \equiv N_1 \cup N_2,$$

say, of L_0. By Lemma 8.1(i) at each (v,ψ) the v components of the vector fields in N_1 form the set $\{A^k B e_\ell : 0 \leq k \leq d-1, 1 \leq \ell \leq m\}$ and this set spans $\mathbb{R}^d$ because of the controllable pair condition. By Lemma 8.1(iii) with $V = V_{\ell_0}$, at each (v,ψ) the vector fields in N_2 span $\{0\} \times \mathbb{R}$. Together the set $(N_1 \cup N_2)(v,\psi)$ spans $\mathbb{R}^d \times \mathbb{R} = T_{(v,\psi)}M$.

If $\gamma = 0$ and $a \neq 0$ the controllable pair condition implies there exist $k_0 \geq 0$ and $1 \leq \ell_0 \leq m$ such that $\langle a, A^k B e_{\ell_0} \rangle = 0$ for $0 \leq k < k_0$ and $\langle a, A^{k_0} B e_{\ell_0} \rangle \neq 0$. By Lemma 8.1(ii) we have $\mathcal{M}^k(V_{\ell_0})(v,\psi) = \binom{A^k B e_{\ell_0}}{0}$ for $k \leq k_0$ and

$$\mathcal{M}^{k_0+1}(V_{\ell_0})(v,\psi) = \begin{pmatrix} A^{k_0+1} B e_{\ell_0} \\ \langle a, A^{k_0} B e_{\ell_0} \rangle \cos^2 \psi \end{pmatrix}.$$

Since $\langle a, A^{k_0} B e_{\ell_0} \rangle \neq 0$ we can replace V_{ℓ_0} with $\mathcal{M}^{k_0+1}(V_{\ell_0})$ in N_2 above. We can apply Lemma 8.1(iii) with $V = \mathcal{M}^{k_0+1}(V_{\ell_0})$, so that the vector fields in the new N_2 span $\{0\} \times \mathbb{R}$ and the proof is completed as before. ∎

8.2. Controllability.

LEMMA 8.2. *Fix $t>0$ and piecewise continuous $c:[0,t]\to\mathbb{R}$ with c not identically zero. For all ψ_0 and ψ_1 there exists $\alpha\in\mathbb{R}$ such that the path $\psi:[0,t]\to\mathbb{R}$ given by*

$$\frac{d\psi(s)}{ds}=-1+\big(1-\kappa^2+\alpha c(s)\big)\cos^2\psi(s)-\zeta_2\sin 2\psi(s),\quad 0\le s\le t, \tag{8.1}$$

with $\psi(0)=\psi_0$ has $\psi(t)=\psi_1 \bmod 2\pi$.

Proof. At the cost of changing the signs of c and α, we may assume there exist a subinterval $[t_0,t_1]\subset[0,t]$ and $\delta>0$ such that $c(s)\le-\delta$ for $t_0\le s\le t_1$. If $\beta>\zeta_2^2-1$ then the right side of

$$\frac{d\tilde\psi(s)}{ds}=-1-\beta\cos^2\tilde\psi(s)-\zeta_2\sin 2\tilde\psi(s),\quad t_0\le s\le t_1, \tag{8.2}$$

is never zero. Separating variables and integrating gives

$$\arctan\left(\frac{\tan\tilde\psi(s)+\zeta_2}{\sqrt{\beta+1-\zeta_2^2}}\right)=-\left(\sqrt{\beta+1-\zeta_2^2}\right)s+\text{ constant.}$$

A transition from $\tilde\psi=\pi/2$ to $\tilde\psi=-\pi/2$ (or vice-versa) changes the left side by π, and so the time taken for $\tilde\psi$ to move distance π is $\pi/\sqrt{\beta+1-\zeta_2^2}$. Therefore $\tilde\psi(t_1)-\tilde\psi(t_0)\to-\infty$ as $\beta\to\infty$, and then by comparison $\psi(t_1)-\psi(t_0)\to-\infty$ as $\alpha\to\infty$.

Since the right side of (8.1) is negative whenever $\psi(s)\equiv\pi/2 \bmod \pi$, on any subinterval $[t_2,t_3]$ we have $\psi(t_3)-\psi(t_2)\le\pi$. Therefore

$$\begin{aligned}\psi(t)&=\psi_0+\Big(\psi(t_0)-\psi(0)\Big)+\Big(\psi(t_1)-\psi(t_0)\Big)+\Big(\psi(t)-\psi(t_1)\Big)\\&\le\psi_0+2\pi+\Big(\psi(t_1)-\psi(t_0)\Big)\to-\infty\end{aligned}$$

as $\alpha\to\infty$. Since $\psi(t)$ depends continuously on α, it follows by the intermediate value theorem that $\psi(t)$ takes values of the form $2\pi n+\psi_1$ for infinitely many α as $\alpha\to\infty$, and we are done. ∎

PROPOSITION 8.2. *Assume $a\in\mathbb{R}^d$ and $\gamma\in\mathbb{R}^m$ are not both 0. Assume also that (A,B) is a controllable pair. Given (v_0,ψ_0) and (v_1,ψ_1) in M and $t>0$ there exists a piecewise continuous function $b:[0,t]\to\mathbb{R}^m$ such that the path $p:[0,t]\to M$ defined by*

$$\frac{dp(s)}{ds}=V_0(p(s))+\sum_{\ell=1}^m b_\ell(s)V_\ell(p(s)),\quad 0\le s\le t, \tag{8.3}$$

with $p(0)=(v_0,\psi_0)$ has $p(t)=(v_1,\psi_1)$.

Proof. The controllable pair condition allows us to control the v component of a path. There is a control path p_1, say, which sends v_0 to 0 in the time interval $[0,t/3]$. It sends (v_0,ψ_0) to $(0,\psi_2)$ for some $\psi_2\in\mathbb{R}/(2\pi\mathbb{Z})$. There is also a control path p_2, say, which sends 0 to v_1 in the time interval $[2t/3,t]$. It induces an invertible linear mapping of $\mathbb{R}^2$, and so there is $\psi_3\in\mathbb{R}/(2\pi\mathbb{Z})$ so that p_2 sends $(0,\psi_3)$ to (v_1,ψ_1). It only remains to find a path during the middle interval $[t/3,2t/3]$ which sends $(0,\psi_2)$ to $(0,\psi_3)$. Equivalently, we see that it is sufficient to prove Proposition 8.2 in the special case $v_0=v_1=0$.

Assume first $\gamma = 0$ and $a \neq 0$. The controllable pair condition implies there exists $1 \le \ell \le m$ and $0 < t_0 < t$ such that $\langle a, e^{sA} Be_\ell \rangle \neq 0$ for $0 < s \le t_0$. Choose control function $s \mapsto b(s)e_\ell$ for a piecewise continuous $b : [0,t] \to \mathbb{R}$. The v component of (8.3) is

$$\frac{dv(s)}{ds} = Av(s) + b(s)Be_\ell, \quad v(0) = 0,$$

and this has solution

$$v(t) = \int_0^t b(s)\left(e^{(t-s)A}Be_\ell\right) ds.$$

In order to satisfy $v(t) = 0$ we require

$$\int_0^t b(s)\left(e^{(t-s)A}Be_\ell\right)_i ds = 0, \quad i = 1, 2, \ldots, d. \tag{8.4}$$

The ψ component of (8.3) is

$$\frac{d\psi(s)}{ds} = -1 + (1 - \kappa^2 + \langle a, v(s)\rangle)\cos^2\psi(s) - \zeta_2 \sin 2\psi(s).$$

Since

$$\langle a, v(s)\rangle = \int_0^s b(r)\langle a, e^{(s-r)A}Be_\ell\rangle dr$$

and $\langle a, e^{sA}Be_\ell\rangle \neq 0$ for $0 < s \le t_0$, there exists a piecewise continuous $b : [0, t_0] \to \mathbb{R}$ $\langle a, v(s)\rangle \neq 0$. Now extend b to the full interval $[0,t]$ in such a way that (8.4) is satisfied, and apply Lemma 8.2. It suffices to replace the function b by αb for suitably chosen α. This completes the proof in the case that $\gamma = 0$ and $a \neq 0$.

Now suppose instead that $\gamma \neq 0$. There exists a linear mapping $T : \mathbb{R}^m \to \mathbb{R}^d$ such that $T\gamma = a$. Define functions $f_\ell(v) = \langle Te_\ell, v\rangle$. Then

$$\sum_{\ell=1}^m f_\ell(v)V_\ell(v,\psi) = \begin{pmatrix} BT^*v \\ \langle a, v\rangle \cos^2\psi \end{pmatrix},$$

and so

$$V_0(v,\psi) - \sum_{\ell=1}^m f_\ell(v)V_\ell(v,\psi) = \begin{pmatrix} (A - BT^*)v \\ -1 + (1-\kappa^2)\cos^2\psi - \zeta_2 \sin 2\psi \end{pmatrix}.$$

Fix $1 \le k \le m$ such that $\gamma_k \neq 0$. Consider a piecewise continuous function $b : [0,t] \to \mathbb{R}$ to be chosen later, and consider the controlled path along the time dependent vector field

$$V_0(v,\psi) - \sum_{\ell=1}^m f_\ell(v)V_\ell(v,\psi) + b(s)V_k(v,\psi).$$

The v component of (8.3) is now

$$\frac{dv(s)}{ds} = (A - BT^*)v(s) + b(s)Be_k, \quad v(0) = 0,$$

and this has solution

$$v(t) = \int_0^t b(s)\Big(e^{(t-s)(A-BT^*)}Be_k\Big)ds.$$

In order to satisfy $v(t) = 0$ we require

$$\int_0^t b(s)\Big(e^{(t-s)(A-BT^*)}Be_k\Big)_i ds = 0, \quad i = 1, 2, \ldots, d. \tag{8.5}$$

The ψ component of (8.3) is now

$$\frac{d\psi(s)}{ds} = -1 + (1 - \kappa^2 + b(s)\gamma_k)\cos^2\psi(s) - \zeta_2 \sin 2\psi(s).$$

In order to apply Lemma 8.2 it suffices to choose a nonidentically zero function b satisfying (8.5), and the proof is completed as before. ∎

8.3. Transition probabilities and invariant measures. We use the results of the previous two sections to obtain results about the diffusion process $\{(v(t), \psi(t)) : t \geq 0\}$ on $M = \mathbb{R}^d \times (\mathbb{R}/(2\pi\mathbb{Z}))$.

Proof of Proposition 4.1. For ease of notation write $(v, \psi) = x$, and let $P_t(x, U) = \mathbb{P}^x(x(t) \in U)$ denote the transition probability for the diffusion. The Lie algebra result in Proposition 8.1 implies condition (E) of Ichihara and Kunita [19]. By [19, Theorem 3] there exists smooth $p : (0, \infty) \times M \times M \to [0, \infty)$ such that $P_t(x, U) = \int_U p(t, x, y)dy$. The stability (eigenvalue condition) for A implies the existence of the stationary probability μ for $\{v(t) : t \geq 0\}$, and hence the existence of at least one stationary probability m for $\mathcal{A}$. Then [19, Theorem 4] implies m has a smooth density $\rho(x)$.

Fix an open set $U \subset M$, and let $t > 0$. The stationarity of m implies

$$m(U) = \int_M P_t(x, U)\rho(x)dx.$$

The support theorem of Stroock and Varadhan [31] together with Proposition 8.2 implies $P_t(x, U) > 0$ for all $x \in M$, and hence $m(U) > 0$. This implies that m is unique and that $\mathrm{supp}(m) = M$. It follows (see [19, Prop. 5.1]) that $P_t(x, \cdot)$ is absolutely continuous with respect to m for all $t > 0$ and all $x \in M$.

Birkhoff's ergodic theorem implies

$$\mathbb{P}^x\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t F(x(s))ds = \int_M F dm\right) = 1$$

for m-almost all $x \in M$. Finally for fixed $x \in M$, by conditioning on behavior at time 1 we get

$$\begin{aligned}
&\mathbb{P}^x\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t F(x(s))ds = \int_M F dm\right) \\
&= \int \mathbb{P}^y\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t F(x(s))ds = \int_M F dm\right) P_1(x, dy) \\
&= 1
\end{aligned}$$

because $P_1(x, \cdot)$ is absolutely continuous with respect to m and

$$\mathbb{P}^y\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t F(x(s))ds = \int_M F dm\right) = 1$$

for m-almost all $y \in M$. ∎

9. Proofs for section 5. In the proofs we will assume that $a \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}^m$ are not both zero. The case when a and γ are both zero is equivalent to setting $\varepsilon = 0$. Then (5.3) has nonrandom constant coefficients and an elementary eigenvalue calculation gives $\lambda(\varepsilon) \equiv -\zeta_2$.

9.1. Khas'minskii's formula revisited. Recall that $\kappa_d = \sqrt{\kappa^2 - \zeta_2^2}$ denotes the damped frequency of the pendulum. For the asymptotic analysis as $\varepsilon \to 0$ it is convenient to replace $u(t) = \binom{\varphi(t)}{\dot\varphi(t)}$ with $\tilde u(t) = \binom{\kappa_d \varphi(t)}{\zeta_2\varphi(t)+\dot\varphi(t)}$, so that $\varphi(t) = \tilde u_1(t)/\kappa_d$ and $\dot\varphi(t) = \tilde u_2(t) - (\zeta_2/\kappa_d \tilde u_1(t)$. Then we get the 2-dimensional linear SDE

$$d\tilde u(t) = \begin{pmatrix} -\zeta_2 & \kappa_d \\ -\kappa_d + \frac{\varepsilon}{\kappa_d}\langle a, v(t)\rangle & -\zeta_2 \end{pmatrix} \tilde u(t)dt + \frac{\varepsilon}{\kappa_d}\sum_{\ell=1}^m \begin{pmatrix} 0 & 0 \\ \gamma_\ell & 0\end{pmatrix} \tilde u(t) dW_\ell(t). \tag{9.1}$$

Write $\tilde u(t) = \|\tilde u(t)\| \binom{\cos\tilde\psi(t)}{\sin\tilde\psi(t)}$. The transformation $\psi(t) \mapsto \tilde\psi(t)$ is given by a diffeomorphism of $\mathbb{R}/(2\pi\mathbb{Z})$, so the ergodicity result Proposition 4.1 applies equally well to the diffusion $\{(v(t),\tilde\psi(t)) : t \ge 0\}$. Also $\big|\log\|u(t)\| - \log\|\tilde u(t)\|\big|$ is bounded. Thus the method used to obtain formula (4.8) for λ in Corollary 4.1 is equally valid when applied to the process $\{\tilde u(t) : t \ge 0\}$.

Applying Itô's formula to (9.1) gives

$$d\log\|\tilde u(t)\| = \widetilde{Q}^\varepsilon(v(t),\tilde\psi(t))dt + \sum_{\ell=1}^m \frac{\varepsilon\gamma_\ell}{\kappa_d}\sin\tilde\psi(t)\cos\tilde\psi(t)\, dW_\ell(t)$$

and

$$d\tilde\psi(t) = \tilde h^\varepsilon(v(t),\tilde\psi(t))dt + \sum_{\ell=1}^m \frac{\varepsilon\gamma_\ell}{\kappa_d}\cos^2\tilde\psi(t)\, dW_\ell(t) \tag{9.2}$$

where

$$\widetilde{Q}^\varepsilon(v,\psi) = -\zeta_2 + \frac{\varepsilon}{\kappa_d}\langle a, v\rangle \sin\psi\cos\psi + \frac{\varepsilon^2\|\gamma\|^2}{2\kappa_d^2}\cos^2\psi\cos 2\psi$$

and

$$\tilde h^\varepsilon(v,\psi) = -\kappa_d + \frac{\varepsilon}{\kappa_d}\langle a, v\rangle \cos^2\psi - \frac{\varepsilon^2\|\gamma\|^2}{\kappa_d^2}\sin\psi\cos^3\psi.$$

Repeating the arguments in section 4 leading up to Corollary 4.1 we get

$$\lambda(\varepsilon) = \int_M \widetilde{Q}^\varepsilon(v,\psi)d\tilde m^\varepsilon(v,\psi), \tag{9.3}$$

where $\tilde m^\varepsilon$ is the unique invariant probability measure for the diffusion $\{(v(t),\tilde\psi(t)) : t \in \mathbb{R}\}$ on M with generator $\widetilde{\mathcal{A}}^\varepsilon$ given by (3.2), (9.2). For convenience of notation we drop tildes for the rest of this section.

Proof of Proposition 4.2. For this proof we take $\varepsilon = 1$. We have

$$\begin{aligned} Q(v,\psi) &= -\zeta_2 + \frac{1}{\kappa_d}\langle a, v\rangle \sin\psi\cos\psi + \frac{\|\gamma\|^2}{2\kappa_d^2}\cos^2\psi\cos 2\psi \\ &\le -\zeta_2 + \frac{1}{2\kappa_d}|\langle a, v\rangle| + \frac{\|\gamma\|^2}{2\kappa_d^2} \end{aligned}$$

and so

$$\begin{aligned}\lambda = \int_M Q(v,\psi)dm(v,\psi) &\leq -\zeta_2 + \frac{1}{2\kappa_d}\int_{\mathbb{R}^d} |\langle a,v\rangle| d\mu(v) + \frac{\|\gamma\|^2}{2\kappa_d^2} \\ &\leq -\zeta_2 + \frac{1}{2\kappa_d}\left(\int_{\mathbb{R}^d}\langle a,v\rangle^2 d\mu(v)\right)^{1/2} + \frac{\|\gamma\|^2}{2\kappa_d^2} \\ &= -\zeta_2 + \frac{\sqrt{\langle a, Ra\rangle}}{2\kappa_d} + \frac{\|\gamma\|^2}{2\kappa_d^2}.\end{aligned}$$

For the block and pendulum we have $R = \frac{\nu^2}{4\zeta_1}\left(\begin{smallmatrix} 1/\chi^2 & 0 \\ 0 & 1\end{smallmatrix}\right)$, $a = \left(\begin{smallmatrix} -\chi^2 \\ -2\zeta_1\end{smallmatrix}\right)$, and $\gamma = \nu$. ◻

9.2. Adjoint expansion. Instead of evaluating the right side of (9.3) directly, we will consider the adjoint equation $\mathcal{A}^\varepsilon F^\varepsilon = Q^\varepsilon - \lambda(\varepsilon)$ using the asymptotic expansion method originated by Arnold, Papanicolaou, and Wihstutz [2]. The integrand Q^ε in (9.3) can be written $Q^\varepsilon = -\zeta_2 + \varepsilon Q_1 + \varepsilon^2 Q_2$ where

$$Q_1(v,\psi) = \frac{1}{\kappa_d}\langle a,v\rangle \sin\psi\cos\psi \qquad \text{and} \qquad Q_2(v,\psi) = \frac{\|\gamma\|^2}{2\kappa_d^2}\cos^2\psi\cos 2\psi.$$

The diffusion process $\{v(t) : t\in\mathbb{R}\}$ given by (3.2) has generator $\mathcal{G}$ given by

$$\mathcal{G}f(v) = \sum_{j=1}^d (Av)_j \frac{\partial f}{\partial v_j}(v) + \frac{1}{2}\sum_{j,k=1}^d (BB^*)_{jk}\frac{\partial^2 f}{\partial v_j \partial v_k}. \tag{9.4}$$

For $j = 1,\dots,d$ we have

$$dv_j(t)d\psi(t) = \sum_{\ell=1}^m (Be_\ell)_j\left(\frac{\varepsilon\gamma_\ell}{\kappa_d}\cos^2\psi(t)\right)dt = \frac{\varepsilon}{\kappa_d}\cos^2\psi(t)(B\gamma)_j dt. \tag{9.5}$$

Therefore, the generator $\mathcal{A}^\varepsilon$ of the diffusion $\{(v(t),\psi(t)) : t\in\mathbb{R}\}$ can be written as $\mathcal{A}^\varepsilon = \mathcal{A}_0 + \varepsilon\mathcal{A}_1 + \varepsilon^2\mathcal{A}_2$, where

$$\mathcal{A}_0 = \mathcal{G} - \kappa_d\frac{\partial}{\partial\psi}$$

and

$$\mathcal{A}_1 = \frac{1}{\kappa_d}\cos^2\psi\left(\langle a,v\rangle\frac{\partial}{\partial\psi} + \sum_{j=1}^d (B\gamma)_j\frac{\partial^2}{\partial\psi\partial v_j}\right)$$

and

$$\mathcal{A}_2 = \frac{\|\gamma\|^2}{\kappa_d^2}\left(-\sin\psi\cos^3\psi\frac{\partial}{\partial\psi} + \cos^4\psi\frac{\partial^2}{\partial\psi^2}\right).$$

Expanding the adjoint equation $\mathcal{A}^\varepsilon F^\varepsilon = Q^\varepsilon - \lambda(\varepsilon)$ as

$$(\mathcal{A}_0 + \varepsilon\mathcal{A}_1 + \varepsilon^2\mathcal{A}_2)(\varepsilon F_1 + \varepsilon^2 F_2 + \cdots) = (-\zeta_2 + \varepsilon Q_1 + \varepsilon^2 Q_2) - (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots)$$

and equating powers of ε we get $\lambda_0 = -\zeta_2$ and

$$\mathcal{A}_0 F_1 = Q_1 - \lambda_1, \tag{9.6}$$

$$\mathcal{A}_0 F_2 + \mathcal{A}_1 F_1 = Q_2 - \lambda_2, \tag{9.7}$$

$$\mathcal{A}_0 F_3 + \mathcal{A}_1 F_2 + \mathcal{A}_2 F_1 = -\lambda_3, \tag{9.8}$$

and so on. In the next three sections we will solve these equations, finding explicit values for λ_1, λ_2, and λ_3. We will find an explicit formula for F_1 and characterizations of the functions F_2 and F_3 which are sufficiently precise so as to enable a rigorous asymptotic estimate to be made in section 9.3.

Remark 9.1. Solving the system recursively, all the equations will be of the form

$$\mathcal{A}_0 F(v,\psi) = G(v,\psi) - \lambda,$$

where G is given and λ and F are to be found. For an exponentially ergodic Markov process with generator $\mathcal{B}$, semigroup $\{P_t : t \geq 0\}$, and invariant probability m, then $P_t G \to \int G dm$ exponentially fast as $t \to \infty$. Then taking $\lambda = \int G dm$ and

$$F = -\int_0^\infty (P_t G - \lambda) dt \tag{9.9}$$

solves $\mathcal{B}F = G - \lambda$. But this is not the case for $\mathcal{A}_0$; for example $G(v,\psi) = \cos\psi$ has $\lambda = \int G dm = 0$ but $P_t G(v,\psi) = \cos(\psi - \kappa_d t)$ so that the integral for F does not converge. This causes some extra work, especially when solving the second and third equations. The construction of D_n in Lemmas 9.1 and 9.2 is related to, but distinctly different from, the construction in (9.9).

9.2.1. The first equation. We need to find λ_1 and $F_1(v,\psi)$ so that

$$\mathcal{A}_0 F_1(v,\psi) = Q_1(v,\psi) - \lambda_1$$

where

$$Q_1(v,\psi) = \frac{1}{2\kappa_d}\langle a, v\rangle \sin 2\psi = -\frac{1}{2\kappa_d}\Re\Big(i\langle a, v\rangle e^{i2\psi}\Big).$$

Define

$$F_1(v,\psi) = \frac{1}{2\kappa_d}\Re\bigg(ie^{i2\psi}\int_0^\infty \langle a, e^{tA}v\rangle e^{-i2\kappa_d t} dt\bigg). \tag{9.10}$$

Here $\Re$ denotes the real part of the complex valued expression in parentheses. Since the eigenvalues of A have negative real parts, the integrand decays exponentially quickly with t so that $F_1(v,\psi)$ is well defined. Moreover it is linear in v and we may differentiate with respect to v inside the integral. We get

$$\begin{aligned}
\mathcal{A}_0 F_1(v,\psi) &= \frac{1}{2\kappa_d}\Re\bigg(ie^{i2\psi}\int_0^\infty \Big(\langle a, e^{tA}Av\rangle - i2\kappa_d\langle a, e^{tA}v\rangle\Big)e^{-i2\kappa_d t}dt\bigg)\\
&= \frac{1}{2\kappa_d}\Re\bigg(ie^{i2\psi}\int_0^\infty \frac{\partial}{\partial t}\Big(\langle a, e^{tA}v\rangle e^{-i2\kappa_d t}\Big)dt\bigg)\\
&= -\frac{1}{2\kappa_d}\Re\Big(ie^{i2\psi}\langle a, v\rangle\Big)\\
&= Q_1(v,\psi).
\end{aligned} \tag{9.11}$$

Therefore $\lambda_1 = 0$. For ease of notation we write

$$b = \bigg(\int_0^\infty e^{tA^*}e^{-i2\kappa_d t}dt\bigg)a \in \mathbb{C}^d \tag{9.12}$$

so that

$$F_1(v,\psi) = \frac{1}{2\kappa_d}\Re\left(ie^{i2\psi}\int_0^\infty \langle e^{tA^*}a, v\rangle e^{-i2\kappa_d t}dt\right) = \frac{1}{2\kappa_d}\Re\left(ie^{i2\psi}\langle b, v\rangle\right). \tag{9.13}$$

9.2.2. The second equation. We need to find λ_2 and $F_2(v,\psi)$ such that

$$\mathcal{A}_0 F_2(v,\psi) = -\mathcal{A}_1 F_1(v,\psi) + Q_2(v,\psi) - \lambda_2.$$

Using (9.13) we have

$$\begin{aligned}
\mathcal{A}_1 F_1(v,\psi) &= \frac{1}{\kappa_d}\cos^2\psi\left(\langle a, v\rangle + \sum_{j=1}^d (B\gamma)_j \frac{\partial}{\partial v_j}\right)\frac{\partial F_1}{\partial \psi} \\
&= -\frac{1}{\kappa_d^2}\cos^2\psi\,\Re\Big(e^{i2\psi}\big(\langle a, v\rangle\langle b, v\rangle + \langle b, B\gamma\rangle\big)\Big) \\
&= -\frac{1}{4\kappa_d^2}\Re\Big(\big(1 + 2e^{i2\psi} + e^{i4\psi}\big)\big(\langle a, v\rangle\langle b, v\rangle + \langle b, B\gamma\rangle\big)\Big).
\end{aligned}$$

Also

$$Q_2(v,\psi) = \frac{\|\gamma\|^2}{2\kappa_d^2}\cos^2\psi\cos 2\psi = \frac{\|\gamma\|^2}{8\kappa_d^2}\Re\big(1 + 2e^{i2\psi} + e^{i4\psi}\big).$$

So we need to solve

$$\mathcal{A}_0 F_2(v,\psi) = \frac{1}{4\kappa_d^2}\Re\Big(\big(1 + 2e^{i2\psi} + e^{i4\psi}\big)\big(\langle a, v\rangle\langle b, v\rangle + \langle b, B\gamma\rangle + \frac{\|\gamma\|^2}{2}\big)\Big) - \lambda_2. \tag{9.14}$$

Let $\mathcal{F}_2$ denote the space of functions $M \to \mathbb{C}$ consisting of complex linear combinations of functions of the form $e^{in\psi}C(v,v)$ and $e^{in\psi}$ where C is bilinear and $n = 0, 2, 4$. The formulas above show that $-\mathcal{A}_1 F_1(v,\psi) + Q_2(v,\psi) \in \mathcal{F}_2$. Then we can break down the problem of solving (9.14) into a collection of (complex valued) problems of the form

$$\mathcal{A}_0 F(v,\psi) = e^{in\psi}C(v,v) - \lambda \tag{9.15}$$

and

$$\mathcal{A}_0 F(v,\psi) = e^{in\psi} - \lambda. \tag{9.16}$$

The problem (9.16) is trivial: if $n \neq 0$ take $F(v,\psi) = (i/n\kappa_d)e^{in\psi}$ and $\lambda = 0$, and if $n = 0$ take $F(v,\psi) = 0$ and $\lambda = 1$. The problem (9.15) is more interesting and the following lemma, generalizing the calculation in (9.11) to the bilinear setting, will be useful.

LEMMA 9.1. *Suppose $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is bilinear. For $n \geq 0$ define*

$$D_n(v^{(1)}, v^{(2)}) = -\int_0^\infty C(e^{tA}v^{(1)}, e^{tA}v^{(2)})e^{-in\kappa_d t}dt. \tag{9.17}$$

Then D_n is well defined and bilinear and there exists $E_n \in \mathbb{C}$ such that

$$\mathcal{A}_0\big(e^{in\psi}D_n(v,v)\big) = e^{in\psi}C(v,v) + e^{in\psi}E_n.$$

Moreover

(9.18)

$$E_0 = -\sum_{\ell=1}^m \int_0^\infty C(e^{tA}Be_\ell, e^{tA}Be_\ell)dt = -\sum_{j,k=1}^d C(e_j, e_k)R_{jk} = -\int_{\mathbb{R}^d} C(v,v)d\mu(v)$$

where R is the covariance matrix for the invariant probability measure μ.

Proof. D_n is well defined because the eigenvalues of A have negative real parts. We will break down the calculation of $\mathcal{A}_0\big(e^{in\psi}D_n(v,v)\big)$ into first and second order derivatives. We have

$$\begin{aligned}
&\left(\sum_{j=1}^{d}(Av)_j\frac{\partial}{\partial v_j}-\kappa_d\frac{\partial}{\partial\psi}\right)\big(e^{in\psi}D_n(v,v)\big)\\
&\quad=e^{in\psi}\Big(D_n(Av,v)+D_n(v,Av)-in\kappa_dD_n(v,v)\Big)\\
&\quad=-e^{in\psi}\int_0^\infty\Big(C(Ae^{tA}v,e^{tA}v)+C(e^{tA}v,Ae^{tA}v)-in\kappa_dC(e^{tA}v,e^{tA}v)\Big)e^{-in\kappa_dt}dt\\
&\quad=-e^{in\psi}\int_0^\infty\frac{\partial}{\partial t}\Big(C(e^{tA}v,e^{tA}v)e^{-in\kappa_dt}\Big)dt\\
&\quad=e^{in\psi}C(v,v),
\end{aligned}$$

and

$$\frac{1}{2}\sum_{j,k=1}^{d}(BB^*)_{jk}\frac{\partial^2D_n}{\partial v_j\partial v_k}(v,v)=\frac{1}{2}\sum_{j,k=1}^{d}(BB^*)_{jk}\Big(D_n(e_j,e_k)+D_n(e_k,e_j)\Big):=E_n$$

where e_j and e_k denote the jth and kth standard unit vectors in $\mathbb{R}^d$. Together we have

$$\begin{aligned}
&\mathcal{A}_0\big(e^{in\psi}D_n(v,v)\big)\\
&\quad=\left(\sum_{j=1}^{d}(Av)_j\frac{\partial}{\partial v_j}-\kappa_d\frac{\partial}{\partial\psi}\right)\big(e^{in\psi}D_n(v,v)\big)+\frac{e^{in\psi}}{2}\sum_{j,k=1}^{d}(BB^*)_{jk}\frac{\partial^2D_n}{\partial v_j\partial v_k}(v,v)\\
&\quad=e^{in\psi}C(v,v)+e^{in\psi}E_n.
\end{aligned}$$

Finally

$$\begin{aligned}
E_0&=-\frac{1}{2}\sum_{j,k=1}^{d}(BB^*)_{jk}\int_0^\infty\Big(C(e^{tA}e_j,e^{tA}e_k)+C(e^{tA}e_k,e^{tA}e_j)\Big)dt\\
&=-\sum_{\ell=1}^{m}\int_0^\infty C(e^{tA}Be_\ell,e^{tA}Be_\ell)dt\\
&=-\sum_{j,k=1}^{d}C(e_j,e_k)\sum_{\ell=1}^{m}\int_0^\infty\big(e^{tA}Be_\ell\big)_j\big(e^{tA}Be_\ell\big)_kdt,
\end{aligned}$$

and the rest of (9.18) follows immediately from (3.4). ∎

The lemma implies that at the cost of translating the function F by an element in $\mathcal{F}_2$ we can reduce a problem of the form (9.15) to a problem of the form (9.16). For $n\neq0$ we can solve (9.15) with $F\in\mathcal{F}_2$ and $\lambda=0$. The remaining case of interest is (9.15) with $n=0$, which has a solution with $F\in\mathcal{F}_2$ and $\lambda=-E_0$. Since the right side of (9.14) contains a bilinear term with $C(v,v)=\langle a,v\rangle\langle b,v\rangle$ we note that the corresponding value of $-E_0$ from (9.18) is

$$-E_0=\sum_{j,k=1}^{d}\langle a,e_j\rangle\langle b,e_j\rangle R_{jk}=\sum_{j,k=1}^{d}a_jb_kR_{jk}=\langle b,Ra\rangle.$$

Putting all the calculations together, and going through the right side of (9.14) term by term, we obtain the following result.

PROPOSITION 9.1. *The equation* (9.14) *has a solution with* $F_2 = \Re(\widehat{F}_2)$ *for some* $\widehat{F}_2 \in \mathcal{F}_2$ *and*

$$\lambda_2 = \frac{1}{4\kappa_d^2}\Big(\langle \Re(b), Ra\rangle + \langle \Re(b), B\gamma\rangle + \frac{\|\gamma\|^2}{2}\Big). \tag{9.19}$$

Finally, using (9.12) and (5.5) we have

$$\langle \Re(b), Ra\rangle = \Re\left\langle a, \int_0^\infty e^{tA}e^{-i2\kappa_d t}Ra\right\rangle = \pi\langle a, \widehat{S}_A(2\kappa_d)Ra\rangle$$

and similarly

$$\langle \Re(b), B\gamma\rangle = \pi\langle a, \widehat{S}_A(2\kappa_d)B\gamma\rangle,$$

so that

$$\lambda_2 = \frac{\pi}{4\kappa_d^2}\left(\langle a, \widehat{S}_A(2\kappa_d)Ra\rangle + \langle a, \widehat{S}_A(2\kappa_d)B\gamma\rangle + \frac{\|\gamma\|^2}{2\pi}\right). \tag{9.20}$$

This is the formula for $\lambda_2(2\kappa_d)$ in Theorem 5.1, but the asymptotic behavior of $\lambda(\varepsilon)$ has not yet been proved.

9.2.3. The third equation. Next we consider

$$\mathcal{A}_0 F_3 = -\mathcal{A}_1 F_2 - \mathcal{A}_2 F_1 - \lambda_3. \tag{9.21}$$

Let $\mathcal{F}_3$ denote the space of functions $M \to \mathbb{C}$ consisting of complex linear combinations of functions of the form $e^{in\psi}C(v,v,v)$ and $e^{in\psi}D(v)$ where C is trilinear and D is linear and $n = 0, 2, 4, 6$. The exact formula (9.13) for F_1 and the characterization of F_2 in Proposition 9.1 together imply that $-\mathcal{A}_1 F_2 - \mathcal{A}_2 F_1$ is the real part of a function in $\mathcal{F}_3$.

LEMMA 9.2. *Suppose* $C : (\mathbb{R}^d)^3 \to \mathbb{C}$ *is trilinear. For* $n \geq 0$ *define*

$$D_n(v^{(1)}, v^{(2)}, v^{(3)}) = -\int_0^\infty C(e^{tA}v^{(1)}, e^{tA}v^{(2)}, e^{tA}v^{(3)})e^{-in\kappa_d t}dt.$$

Then D_n *is well defined and trilinear and there exists a linear mapping* $E_n : \mathbb{R}^d \to \mathbb{C}$ *such that*

$$\mathcal{A}_0\big(e^{in\psi}D_n(v,v,v)\big) = e^{in\psi}C(v,v,v) + e^{in\psi}E_n(v).$$

Proof. The method of proof is the same as for Lemma 9.1, and we omit the details. ∎

The lemma implies that at the cost of translating the function F by an element of $\mathcal{F}_3$ we can reduce a problem of the form

$$\mathcal{A}_0 F(v,\psi) = e^{in\psi}C(v,v,v) \tag{9.22}$$

for $n = 0, 2, 4, 6$ into a problem of the form

$$\mathcal{A}_0 F(v,\psi) = e^{in\psi}E_n(v). \tag{9.23}$$

The method used in section 9.2.1 to obtain an explicit solution of (9.6) can be equally applied here to obtain an explicit solution for (9.23) with $F \in \mathcal{F}_3$. Repeating the term by term approach used in the previous section we get the next result.

PROPOSITION 9.2. *The equation* (9.21) *has a solution with* $F_3 = \Re(\widehat{F}_3)$ *for some* $\widehat{F}_3 \in \mathcal{F}_3$ *and* $\lambda_3 = 0$.

9.3. Proof of Theorem 5.1. When applying the adjoint method on a noncompact space such as M the following result of Baxendale and Goukasian [9, Prop. 3] is important.

PROPOSITION 9.3. *Let* $\{X(t) : \ t \geq 0\}$ *be a diffusion process on a* σ*-compact manifold* M *with invariant probability measure* m. *Let* $\mathcal{B}$ *be an operator acting on* $C^2(M)$ *functions that agrees with the generator of* $\{X(t) : t \geq 0\}$ *on* C^2 *functions with compact support. Let* $f \in C^2(M)$, *and assume* f *and* $\mathcal{B}f$ *are* m*-integrable. Suppose there exists a positive* $G \in C^2(M)$ *and* $k < \infty$ *satisfying* $\mathcal{B}G(x) \leq kG(x)$ *and* $\mathcal{B}f(x) \leq G(x)$ *for all* $x \in M$, *and* $f(x)/G(x) \to 0$ *as* $x \to \infty$. *Then* $\int_M \mathcal{B}f(x)dm(x) = 0$.

Remark 9.2. For an example as to why we need something like Proposition 9.3 consider

$$dX_t = \left(-aX_t^3 + \frac{\nu^2}{2X_t}\right)dt + \nu dW_t$$

on $M = (0,\infty)$ with $f(x) = \log x$. We get $\mathcal{B}f(x) = -ax^2$ but clearly $\int_0^\infty (-ax^2)dm(x) \neq 0$.

Proof of Theorem 5.1(i). So far we have shown the existence of functions F_1, F_2, and F_3 satisfying (9.6), (9.7), (9.8) with $\lambda_1 = \lambda_3 = 0$ and λ_2 given by the formula (9.20). Since F_1, F_2, and F_3 are smooth functions of (v,ψ), we have

$$\begin{aligned}
\mathcal{A}^\varepsilon(\varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3) &= (\mathcal{A}_0 + \varepsilon\mathcal{A}_1 + \varepsilon^2\mathcal{A}_2)(\varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3) \\
&= \varepsilon\mathcal{A}_0 F_1 + \varepsilon^2(\mathcal{A}_0 F_2 + \mathcal{A}_1 F_1) + \varepsilon^3(\mathcal{A}_0 F_3 + \mathcal{A}_1 F_2 + \mathcal{A}_2 F_1) \\
&\quad + \varepsilon^4(\mathcal{A}_1 F_3 + \mathcal{A}_2 F_2) + \varepsilon^5 \mathcal{A}_2 F_3 \\
&= \varepsilon Q_1 + \varepsilon^2(Q_2 - \lambda_2) + \varepsilon^4(\mathcal{A}_1 F_3 + \mathcal{A}_2 F_2) + \varepsilon^5\mathcal{A}_2 F_3 \\
&= Q^\varepsilon + \zeta_2 - \varepsilon^2\lambda_2 + \varepsilon^4(\mathcal{A}_1 F_3 + \mathcal{A}_2 F_2) + \varepsilon^5 \mathcal{A}_2 F_3. \qquad (9.24)
\end{aligned}$$

Fix $\varepsilon > 0$ for the moment. From the explicit formula (9.10) for F_1 and the characterizations of F_2 and F_3 in Propositions 9.1 and 9.2 we know that $\varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3$ grows at most like $\|v\|^3$, and that $\mathcal{A}_1 F_3$ grows at most like $\|v\|^4$, and that $\mathcal{A}_2 F_2$ grows at most like $\|v\|^2$, and that $\mathcal{A}_2 F_3$ grows at most like $\|v\|^3$. We also know that the v marginal of m^ε is the Gaussian measure μ with mean 0 and covariance matrix R, so that

$$\int_M \|v\|^k dm^\varepsilon(v,\psi) = \int_{\mathbb{R}^d} \|v\|^k d\mu(v) < \infty \qquad (9.25)$$

for all $k \geq 0$. Since

$$\mathcal{A}^\varepsilon(\|v\|^4) = \mathcal{G}(\|v\|^4) = 4\langle Av, v\rangle\|v\|^2 + 2\mathrm{tr}(BB^*)\|v\|^2 + 4\|B^*v\|^2$$

we can apply Proposition 9.3 with $\mathcal{B} = \mathcal{A}^\varepsilon$ and $f = \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3$ and $G = c_1 + c_2\|v\|^4$ for suitable positive constants c_1 and c_2. Integrating (9.24) with respect to the invariant probability m^ε gives

$$0 = \int Q^\varepsilon dm^\varepsilon + \zeta_2 - \varepsilon^2\lambda_2 + \varepsilon^4 \int_M (\mathcal{A}_1 F_3 + \mathcal{A}_2 F_2)dm^\varepsilon + \varepsilon^5 \int_M \mathcal{A}_2 F_3 dm^\varepsilon. \qquad (9.26)$$

Using Khas'minskii's formula (9.3), this gives

$$\lambda(\varepsilon) = -\zeta_2 + \varepsilon^2\lambda_2 - \varepsilon^4 \int_M (\mathcal{A}_1 F_3 + \mathcal{A}_2 F_2)dm^\varepsilon - \varepsilon^5 \int_M \mathcal{A}_2 F_3 dm^\varepsilon. \qquad (9.27)$$

Finally the growth estimates above on $\mathcal{A}_1 F_3$ and $\mathcal{A}_2 F_2$ and $\mathcal{A}_2 F_3$ together with (9.25) imply that the integrals

$$\int_M (\mathcal{A}_1 F_3 + \mathcal{A}_2 F_2) dm^\varepsilon \quad \text{and} \quad \int_M \mathcal{A}_2 F_3 dm^\varepsilon$$

are bounded uniformly in ε. At this point we let $\varepsilon \to 0$ in (9.27) and obtain

$$\lambda(\varepsilon) = \varepsilon^2 \lambda_2 + O(\varepsilon^4)$$

as $\varepsilon \to 0$. This completes the proof of Theorem 5.1. □

Proof of Theorem 5.1(ii). From (9.14) we see that $\mathcal{A}_0 F_2$ is a finite sum of terms of the form $\Re\left(e^{in\psi} c\right)$ or $\Re\left(e^{in\psi} C(v,v)\right)$ where the constants c and the coefficients in the bilinear mappings C are continuous functions of κ_d for $0 < \kappa_d < \infty$. The construction in section 9.2.2 implies that the same is true for F_2, and then also for $\mathcal{A}_2 F_2$. So given $0 < c_1 < c_2 < \infty$ there exists K_1 such that $|\mathcal{A}_2 F_2(\psi, v)| \le K_1(1 + \|v\|^2)$ whenever $c_1 \le \kappa_d \le c_2$.

Similarly from (9.21) we see that $\mathcal{A}_0 F_3$ is a finite sum of terms of the form $\Re\left(e^{in\psi} D(v)\right)$ or $\Re\left(e^{in\psi} C(v,v,v)\right)$ where the coefficients in the linear mappings D and trilinear mappings C are continuous functions of κ_d for $0 < \kappa_d < \infty$. The construction in section 9.2.3 implies that the same is true of F_2. So given $0 < c_1 < c_2 < \infty$ there exists K_2 and K_3 such that $|\mathcal{A}_1 F_3(\psi, v)| \le K_2(1 + \|v\|^4)$ and $|\mathcal{A}_2 F_3(\psi, v)| \le K_3(1 + \|v\|^3)$ whenever $c_1 \le \kappa_d \le c_2$.

These estimates can now be used in (9.27) to give the uniform estimate (5.8). □

9.4. Proof of Proposition 5.1. Taking $a = A^*\alpha$ and $\gamma = B^*\alpha$ in (5.7) we get

$$\begin{aligned}\lambda_2(\omega) &= \frac{\pi}{\omega^2}\left(\langle A^*\alpha, \widehat{S}_A(\omega) R A^*\alpha\rangle + \langle A^*\alpha, \widehat{S}_A(\omega) B B^*\alpha\rangle + \frac{\|B^*\alpha\|^2}{2\pi}\right) \\ &= \frac{\pi}{\omega^2}\left(\langle \alpha, A\widehat{S}_A(\omega) R A^*\alpha\rangle + \langle \alpha, A\widehat{S}_A(\omega) B B^*\alpha\rangle + \frac{\langle \alpha, BB^*\alpha\rangle}{2\pi}\right).\end{aligned}$$

The covariance matrix R in (3.4) satisfies

$$AR + RA^* = -BB^*; \tag{9.28}$$

see Gardiner [14, sect. 4.4]. Substituting for BB^* and noting $\langle \alpha, BB^*\alpha\rangle = -\langle \alpha, AR\alpha\rangle - \langle \alpha, RA^*\alpha\rangle = -2\langle \alpha, AR\alpha\rangle$ we get

$$\lambda_2(\omega) = \frac{\pi}{\omega^2}\left(-\langle \alpha, A\widehat{S}_A(\omega) AR\alpha\rangle - \frac{\langle \alpha, AR\alpha\rangle}{\pi}\right). \tag{9.29}$$

Integrating by parts twice gives

$$A\widehat{S}_A(\omega) A = \frac{1}{\pi}\int_0^\infty A e^{tA} A \cos\omega t \, dt = -\frac{1}{\pi} A - \frac{\omega^2}{\pi}\int_0^\infty e^{tA}\cos\omega t \, dt = -\frac{1}{\pi} A - \omega^2 \widehat{S}_A(\omega).$$

Substituting for $A\widehat{S}_A(\omega) A$ in (9.29) gives

$$\lambda_2(\omega) = \pi \langle \alpha, \widehat{S}_A(\omega) R\alpha\rangle. \tag{9.30}$$

Let $R(t)$ denote the autocovariance matrix for $\{v(t) : t \in \mathbb{R}\}$. Then $R(0)$ is the covariance matrix R of the stationary version, and $R(t) = e^{tA}R$ for $t \ge 0$. The autocovariance function for V is

$$R_V(t) = \mathbb{E}[V(t)V(0)] = \langle \alpha, R(t)\alpha\rangle = \langle \alpha, e^{tA} R\alpha\rangle$$

for $t \geq 0$. The power spectral density of V is

$$S_V(\omega) = \frac{1}{\pi}\int_0^\infty R_V(t)\cos\omega t\,dt = \langle \alpha, \widehat{S}_A(\omega)R\alpha\rangle,$$

and so

$$\lambda_2(\omega) = \pi\langle \alpha, \widehat{S}_A(\omega)R\alpha\rangle = \pi S_V(\omega),$$

as required.

9.5. Proof of Proposition 5.2. We use the stationary version of v given by

$$v(t) = \int_{-\infty}^t e^{(t-s)A}BdW(s) = \sum_{\ell=1}^m \int_{-\infty}^t e^{(t-s)A}Be_\ell dW_\ell(s),$$

where $e_1,\dots,e_m$ denote the standard basis vectors in $\mathbb{R}^m$. Then

$$\begin{aligned}\int_{-\infty}^\infty \langle a, v(u)\rangle\psi_\delta(t-u)du &= \sum_{\ell=1}^m \int_{-\infty}^\infty \left(\int_{-\infty}^u \left\langle a, e^{(u-s)A}Be_\ell\right\rangle dW_\ell(s)\right)\psi_\delta(t-u)du\\ &= \sum_{\ell=1}^m \int_{-\infty}^\infty \left(\int_s^\infty \left\langle a, e^{(u-s)A}Be_\ell\right\rangle \psi_\delta(t-u)du\right)dW_\ell(s),\end{aligned}$$

and so

$$\xi_\delta(t) = \sum_{\ell=1}^m \int_{-\infty}^\infty \left(\int_s^\infty \langle a, e^{(u-s)A}Be_\ell\rangle\,\psi_\delta(t-u)du + \gamma_\ell\psi_\delta(t-s)\right)dW_\ell(s) \equiv \sum_{\ell=1}^m \xi_{\delta,\ell}(t),$$

say. Since the $\{W_\ell : 1 \leq \ell \leq m\}$ are independent, the $\{\xi_{\delta,\ell} : 1 \leq \ell \leq m\}$ are independent, and the autocovariance function for ξ_δ is the sum of the autocovariance functions for each of the $\xi_{\delta,\ell}$. The computation of each $E[\xi_{\delta,\ell}(t)\xi_{\delta,\ell}(0)]$ can be broken down into 4 terms. Therefore

$$\begin{aligned}&\mathbb{E}[\xi_\delta(t)\xi_\delta(0)]\\ &\quad= \sum_{\ell=1}^\infty \mathbb{E}[\xi_{\delta,\ell}(t)\xi_{\delta,\ell}(0)]\\ &\quad= \sum_{\ell=1}^m \int_{-\infty}^\infty \left(\int_s^\infty \langle a, e^{(u-s)A}Be_\ell\rangle\,\psi_\delta(t-u)du\right)\left(\int_s^\infty \langle a, e^{(w-s)A}Be_\ell\rangle\,\psi_\delta(-w)dw\right)ds\\ &\qquad+ \sum_{\ell=1}^m \int_{-\infty}^\infty \left(\int_s^\infty \langle a, e^{(u-s)A}Be_\ell\rangle\,\psi_\delta(t-u)du\right)\gamma_\ell\psi_\delta(-s)ds\\ &\qquad+ \sum_{\ell=1}^m \int_{-\infty}^\infty \left(\int_s^\infty \langle a, e^{(u-s)A}Be_\ell\rangle\,\psi_\delta(-u)du\right)\gamma_\ell\psi_\delta(t-s)ds\\ &\qquad+ \sum_{\ell=1}^m \gamma_\ell^2 \int_{-\infty}^\infty \psi_\delta(t-s)\psi_\delta(-s)\,ds\\ &\quad= \int_{-\infty}^\infty \left(\int_s^\infty\int_s^\infty \langle a, e^{(u-s)A}BB^*e^{(w-s)A^*}a\rangle\psi_\delta(t-u)\psi_\delta(-w)dudw\right)ds\\ &\qquad+ \int_{-\infty}^\infty \left(\int_s^\infty \langle a, e^{(u-s)A}B\gamma\rangle\,\psi_\delta(t-u)du\right)\psi_\delta(-s)ds\end{aligned}$$

$$+\int_{-\infty}^{\infty}\left(\int_{s}^{\infty}\langle a, e^{(u-s)A}B\gamma\rangle\,\psi_\delta(-u)du\right)\psi_\delta(t-s)ds$$
$$+\int_{-\infty}^{\infty}\|\gamma\|^2\psi_\delta(t-s)\psi_\delta(-s)\,ds$$
$$=\int_{-\infty}^{\infty}\left(I_\delta^{(1)}(s,t)+I_\delta^{(2)}(s,t)+I_\delta^{(3)}(s,t)+I_\delta^{(4)}(s,t)\right)ds,$$

say. Then

$$S_{\xi_\delta}(\omega)=\frac{1}{\pi}\int_0^\infty \mathbb{E}[\xi_\delta(t)\xi_\delta(0)]\cos\omega t\,dt=\sum_{i=1}^4\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^\infty I_\delta^{(i)}(s,t)\,ds\right)\cos\omega t\,dt.$$

We consider the limit as $\delta\to 0$ for each of the four terms separately.

***i* = 1.**

$$I_\delta^{(1)}(s,t)=\int_s^\infty\int_s^\infty\left\langle a, e^{(u-s)A}BB^*e^{(w-s)A^*}a\right\rangle\psi_\delta(t-u)\psi_\delta(-w)\,dudw.$$

Recalling $t\geq 0$ we have

$$I_\delta^{(1)}(s,t)\to\begin{cases}\langle a, e^{(t-s)A}BB^*e^{-sA^*}a\rangle & \text{if } s<0,\\ 0 & \text{if } s>0\end{cases}$$

as $\delta\to 0$. Also $I_\delta^{(1)}(s,t)=0$ if $s>\delta$ and

$$\left|I_\delta^{(1)}(s,t)\right|\leq\sup_{-\delta\leq r_1,r_2\leq\delta}\left|\left\langle a, e^{(t+r_1-s)A}BB^*e^{(r_2-s)A^*}a\right\rangle\right|$$
$$\leq\left(\sup_{|r|\leq\delta}\|e^{rA^*}a\|\right)^2\|e^{(t-s)A}BB^*e^{-sA^*}\|.$$

Therefore by the dominated convergence theorem we have

$$\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^\infty I_\delta^{(1)}(s,t)\,ds\right)\cos\omega t\,dt$$
$$\to\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^0\langle a, e^{(t-s)A}BB^*e^{-sA^*}a\rangle\,ds\right)\cos\omega t\,dt$$

as $\delta\to 0$. Since

$$\int_{-\infty}^0\langle a, e^{(t-s)A}BB^*e^{-sA^*}a\rangle\,ds=\left\langle a, e^{tA}\left(\int_{-\infty}^0 e^{-sA}BB^*e^{-sA^*}\,ds\right)a\right\rangle=\langle a, e^{tA}Ra\rangle$$

where R is the covariance matrix for the invariant probability measure for $v(t)$ (see (3.4)), we have

$$\lim_{\delta\to 0}\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^\infty I_\delta^{(1)}(s,t)\,ds\right)\cos\omega t\,dt=\int_0^\infty\langle a, e^{tA}Ra\rangle\cos\omega t\,dt=\langle a,\widehat{S}_A(\omega)Ra\rangle.$$

***i* = 2.**

$$\int_{-\infty}^\infty I_\delta^{(2)}(s,t)\,ds=\int_{-\infty}^\infty\left(\int_s^\infty\langle a, e^{(u-s)A}B\gamma\rangle\psi(t-u)\,du\right)\psi(-s)\,ds.$$

For $t > 0$ we have

$$\int_{-\infty}^{\infty} I_\delta^{(2)}(s,t)\,ds \to \langle a, e^{tA}B\gamma\rangle$$

as $\delta \to 0$. Also

$$\left|\int_{-\infty}^{\infty} I_\delta^{(2)}(s,t)\,ds\right| \le \sup_{|r|\le 2\delta}\left|\langle a, e^{(t+r)A}B\gamma\rangle\right| \le \left(\sup_{|r|\le 2\delta}\|e^{rA^*}a\|\right)\|e^{tA}B\gamma\|.$$

Therefore by the dominated convergence theorem we have

$$\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} I_\delta^{(2)}(s,t)\,ds\right)\cos\omega t\,dt \to \frac{1}{\pi}\int_0^{\infty}\langle a, e^{tA}B\gamma\rangle\cos\omega t\,dt = \langle a, \widehat{S}_A(\omega)B\gamma\rangle$$

as $\delta \to 0$.

$\boldsymbol{i = 3.}$

$$\int_{-\infty}^{\infty} I_\delta^{(3)}(s,t)\,ds = \int_{-\infty}^{\infty}\left(\int_s^{\infty}\langle a, e^{(u-s)A}B\gamma\rangle\,\psi_\delta(-u)du\right)\psi_\delta(t-s)ds.$$

Note that $|\int_{-\infty}^{\infty} I_\delta^{(3)}(s,t)\,ds| \le \sup_{r\ge 0}|\langle a, e^{rA}B\gamma\rangle| < \infty$. Also, if $0 < 2\delta < t$, then $\int_{-\infty}^{\infty} I_\delta^{(3)}(s,t)\,ds = 0$. Therefore by the dominated convergence theorem we have

$$\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} I_\delta^{(3)}(s,t)\,ds\right)\cos\omega t\,dt \to 0$$

as $\delta \to 0$.

$\boldsymbol{i = 4.}$ The substitution $u = s - t$ gives

$$\int_{-\infty}^{\infty} I_\delta^{(4)}(s,t)\,ds = \|\gamma\|^2\int_{-\infty}^{\infty}\gamma_\delta(t-s)\psi_\delta(-s)ds = \int_{-\infty}^{\infty} I_\delta^{(4)}(s,-t)\,ds,$$

and so

$$\begin{aligned}\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} I_\delta^{(4)}(s,t)\,ds\right)\cos\omega t\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} I_\delta^{(4)}(s,t)\,ds\right)\cos\omega t\,dt\\ &= \frac{\|\gamma\|^2}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_\delta(t-s)\psi_\delta(-s)\cos\omega t\,dsdt.\end{aligned}$$

Notice that $\int_{-\infty}^{\infty}\psi_\delta(t-s)\psi_\delta(-s)ds = 0$ if $|t| > 2\delta$. Also

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_\delta(t-s)\psi_\delta(-s)\,dsdt &= \int_{-\infty}^{\infty}\psi_\delta(-s)\left(\int_{-\infty}^{\infty}\psi_\delta(t-s)\,dt\right)ds\\ &= \int_{-\infty}^{\infty}\psi_\delta(-s)\,ds = 1.\end{aligned}$$

Therefore

$$\begin{aligned}0 &\le \frac{\|\gamma\|^2}{2\pi} - \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} I_\delta^{(4)}(s,t)\,ds\right)\cos\omega t\,dt\\ &= \frac{\|\gamma\|^2}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_\delta(t-s)\psi_\delta(-s)\big(1-\cos\omega t\big)\,dsdt\\ &\le \frac{\|\gamma\|^2}{2\pi}\sup_{|t|\le 2\delta}\big|1-\cos\omega t\big|\end{aligned}$$

and so

$$\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^\infty I_\delta^{(4)}(s,t)\,ds\right)\cos\omega t\,dt\to\frac{\|\gamma\|^2}{2\pi}$$

as $\delta\to 0$.

Together these four limits give (5.16), and the rest of Proposition 5.2 follows directly from the definition (5.7).

REFERENCES

[1] S. T. Ariaratnam, *Stability of modes at rest in stochastically forced nonlinear oscillators*, Internat. J. Non-Linear Mech., 26 (1991), pp. 819–825.

[2] L. Arnold, G. Papanicolaou, and V. Wihstutz, *Asymptotic analysis of the Lyapunov exponent and rotation number of the random oscillator and applications*, SIAM J. Appl. Math., 46 (1986), pp. 427–450, https://doi.org/10.1137/0146030.

[3] E. I. Auslender and G. N. Mil'shtein, *Asymptotic expansions of the Liapunov index for linear stochastic systems with small noise*, J. Appl. Math. Mech., 46 (1982), pp. 277–283.

[4] A. K. Bajaj, S. I. Chang, and J. M. Johnson, *Amplitude modulated dynamics of a resonantly excited autoparametric two degree-of-freedom system*, Nonlinear Dyn., 5 (1994), pp. 433–457.

[5] D. Bakry and M. Émery, *Diffusions hypercontractives*, in Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math. 1123, Springer, Berlin, 1985, pp. 177–206.

[6] B. Banerjee, A. K. Bajaj, and P. Davies, *Resonant dynamics of an autoparametric system: A study using higher-order averaging*, Internat. J. Non-Linear Mech., 31 (1996), pp. 21–39.

[7] P. H. Baxendale and D. W. Stroock, *Large deviations and stochastic flows of diffeomorphisms*, Probab. Theory Related Fields, 80 (1988), pp. 169–215.

[8] P. H. Baxendale, *Invariant measures for nonlinear stochastic differential equations*, in Lyapunov Exponents (Oberwolfach, 1990), Lecture Notes in Math. 1486, Springer, Berlin, 1991, pp. 123–140.

[9] P. H. Baxendale and L. Goukasian, *Lyapunov exponents of nilpotent Itô systems with random coefficients*, Stochastic Process. Appl., 95 (2001), pp. 219–233.

[10] F. Benedettini, G. Rega, and F. Vestroni, *Modal coupling in the free nonplanar finite motion of an elastic cable*, Meccanica, 21 (1986), pp. 38–46.

[11] M. P. Cartmell and J. W. Roberts, *Simultaneous combination resonances in an autoparametrically resonant system*, J. Sound Vib., 123 (1988), pp. 81–101.

[12] W. K. Chang and R. A. Ibrahim, *Multiple internal resonance in suspended cables under random in-plane loading*, Nonlinear Dyn., 12 (1997), pp. 275–303.

[13] M. Coti Zelati and M. Hairer, *A noise-induced transition in the Lorenz system*, Comm. Math. Phys., 383 (2021), pp. 2243–2274.

[14] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, 3rd ed., Springer Series in Synergetics 13, Springer-Verlag, Berlin, 2004.

[15] H. Hatwal, A. K. Mallik, and A. Ghosh, *Forced nonlinear oscillations of an autoparametric system. Part 1: Periodic responses, Part 2: Chaotic responses*, Trans. ASME J. Appl. Mech., 50 (1983), pp. 657–668.

[16] R. S. Haxton and A. D. S. Barr, *The autoparametric vibration absorber*, Trans. ASME J. Engrg. Ind., 94 (1972), pp. 119–125.

[17] R. A. Ibrahim and H. Heo, *Autoparametric vibration of coupled beams under random support motion*, ASME J. Vib. Acoustics Stress Reliability, 108 (1986), pp. 421–426.

[18] R. A. Ibrahim and J. W. Roberts, *Stochastic stability of the stationary response of a system with autoparametric coupling*, Z. Angew. Math. Mech., 57 (1977), pp. 643–649.

[19] K. Ichihara and H. Kunita, *A classification of the second order degenerate elliptic operators and its probabilistic characterization*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 30 (1974), pp. 235–254.

[20] P. Imkeller and C. Lederer, *An explicit description of the Lyapunov exponents of the noisy damped harmonic oscillator*, Dynam. Stability Systems, 14 (1999), pp. 385–405.

[21] R. Z. Khas'minskii, *Necessary and sufficient conditions for asymptotic stability of linear stochastic systems*, Theory Probab. Appl., 12 (1967), pp. 144–147.

[22] R. Khasminskii, *Stochastic Stability of Differential Equations*, 2nd ed., Stoch. Model. Appl. Probab. 66, Springer, Heidelberg, 2012.

[23] W. K. Lee and D. S. Cho, *Damping effect of a randomly excited autoparametric system*, J. Sound Vib., 236 (2000), pp. 23–31.

[24] D. Liberzon and R. W. Brockett, *Spectral analysis of Fokker–Planck and related operators arising from linear stochastic differential equations*, SIAM J. Control Optim., 38 (2000), pp. 1453–1467, https://doi.org/10.1137/S0363012998338193.

[25] S. P. Meyn and R. L. Tweedie, *Stability of Markovian processes.* III. *Foster-Lyapunov criteria for continuous-time processes*, Adv. Appl. Probab., 25 (1993), pp. 518–548.

[26] N. S. Namachchivaya, D. Kok, and S. T. Ariaratnam, *Stability of noisy nonlinear autoparametric systems*, Nonlinear Dyn., 47 (2007), pp. 143–165.

[27] A. H. Nayfeh, D. T. Mook, and L. R. Marshall, *Nonlinear coupling of pitch and roll modes in ship motions*, J. Hydronautics, 7 (1973), pp. 145–152.

[28] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1979.

[29] M. A. S. Neves and C. A. Rodríguez, *A coupled non-linear mathematical model of parametric resonance of ships in head seas*, Appl. Math. Model., 33 (2009), pp. 2630–2645.

[30] J. W. Roberts, *Random excitation of a vibratory system with autoparametric interaction*, J. Sound Vib., 69 (1980), pp. 101–116.

[31] D. W. Stroock and S. R. S. Varadhan, *On the support of diffusion processes with applications to the strong maximum principle*, in Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, CA, 1970/1971), Vol. III: Probability Theory, 1972, pp. 333–359.

[32] W-N. Tien, N. S. Namachchivaya, and A. K. Bajaj, *Non-linear dynamics of a shallow arch under periodic excitation* I. $1:2$ *internal resonance*, Int. J. Nonlinear Mech., 29 (1994), pp. 349–366.

[33] A. Tondl, T. Ruijgrok, F. Verhulst, and R. Nabergoj, *Autoparametric Resonance in Mechanical Systems*, Cambridge University Press, Cambridge, 2000.

[34] F. Verhulst, *Parametric and autoparametric resonance*, Acta Appl. Math., 70 (2002), pp. 231–264.

[35] J. Warminski and K. Kecik, *Autoparametric vibrations of a nonlinear system with pendulum*, Math. Probl. Eng. (Nonlinear dynamics and their applications to engineering sciences), (2006), 80705.