① X_t – position V_t – velocity

$$\boxed{X_t = x_0 + \int_0^t \boxed{V_s}\, ds}$$

② X_t – wealth of a person

C_t – consumption rate at time t

$$X_t = x_0 + \int_0^t -\boxed{C_s}\, ds$$

One-dimensional ODE Given $F: [0,T] \times \mathbb{R} \to \mathbb{R}$

$$\begin{cases} x'(t) = F(t, x(t)) & t \in [0,T] \\ x(0) = x & x \in \mathbb{R} \end{cases} \qquad (1)$$

A classical solution to (1) is a differentiable function $X: [0,T] \to \mathbb{R}$ satisfying

(I) $X(0) = x$

(II) $X'(t) = F(t, X(t))$ for every $t \in [0,T]$

Theorem: Cauchy–Lipschitz, let $F \in C([0,T] \times \mathbb{R}; \mathbb{R})$ be Lipschitz continuous function i.e. $\exists L > 0$ s.t.

$$|F(t,y) - F(t,\bar{y})| \le L|y - \bar{y}| \qquad \forall t \in [0,T],\ y, \bar{y} \in \mathbb{R}$$

Then (1) admits a unique solution.

Formulation of optimal control problems.

A set $A \subseteq \mathbb{R}$, the set where control process take values.

The control process $\alpha: [0,T] \to A$

and the state process $X: [0,T] \to \mathbb{R}$

Admissible control process: $\mathcal{A} \in L^2([0,T]; A) = \{\alpha: [0,T] \to A \mid \int_0^T |\alpha(s)|^2 ds < \infty$

Dynamics of a control system

$$X_t = x_0 + \int_0^t f(s, X_s, \alpha_s)\, ds \qquad f: [0,T] \times \mathbb{R} \times A \to \mathbb{R}$$

$X_s = X_t + \int_t^s f\, ds$

Consider the following assumptions.

(H_1) $\exists L > 0$: $|f(t,y,a) - f(t,y',a')| \le L(|y-y'| + |a-a'|)$ $\forall t \in [0,T]$, $y', y \in \mathbb{R}$, $a', a \in A$

(H_2) $f(\cdot, y, a) \in L^1([0,T]; \mathbb{R})$ for $a \in A$, $y \in \mathbb{R}$

Under $(H_1)(H_2)$, the controlled dynamics admits a unique solution.

Optimization problem:

Define Reward/Loss function

$L: [0,T] \times \mathbb{R} \times A \to \mathbb{R}$ $\qquad g: \mathbb{R} \to \mathbb{R}$

Reward/Loss: $\boxed{J(t,x,\alpha)} = \int_t^T L(s, \boxed{X_s^{t,x,\alpha}}, \alpha_s)\, ds + \boxed{g(X_T^{t,x,\alpha})}$

$X(s)$ given $X(t) = x$

Then the optimization problem is

value function: $V(t,x) = \sup_{\alpha \in \mathcal{A}} J(t,x,\alpha)$ → Reward $\qquad$ Remark $\sup F = -\inf(-F)$

wellposedness of optimization problem ⟹ $J < \infty$

(H_3) $L(t,x,a) \le C_L(1 + |x|^2 + |a|^2)$

$g(x) \le C_g(1 + |x|^2)$

$(H_1)(H_2)(H_3)$ ⟶ well-defined optimization problem

Dynamical programming Principle (DPP)

$\forall \theta \in [t,T]$

$$V(t,x) = \sup_{\alpha \in \mathcal{A}} \left(\int_t^\theta L(s, X_s^{t,x,\alpha}, \alpha_s)\, ds + V(\theta, X_\theta^{t,x,\alpha}) \right)$$

Proof: $\forall \{\alpha\}_{s \in [t,T]} \in \mathcal{A}$

$$\sup J(t,x,\alpha) = \int_t^T L(t, X_s^{t,x,\alpha}, \alpha_s)\, ds + g(X_T^{t,x,\alpha})$$

$$= \int_t^\theta L\, ds + \underbrace{\int_\theta^T L(t, X_s^{t,x,\alpha}, \alpha_s)\, ds + \boxed{g(X_T^{t,x,\alpha})}}_{\le V(\theta, X_\theta^{t,x,\alpha})}$$

$= \sup \int_\theta^T L\, ds + g$

Since: $X_s^{t,x,\alpha} = X_s^{\theta, X_\theta^{t,x,\alpha}, \alpha}$ $\qquad$ uniqueness of ODE

$$\overset{\sup}{\le} \int_t^\theta L(s, X_s^{t,x,\alpha}, \alpha_s)\, ds + V(\theta, X_\theta^{t,x,\alpha})$$

$$V(t,x) = \sup_{\alpha \in \mathcal{A}} J(t,x,\alpha) \le \sup_{\alpha \in \mathcal{A}} \int_t^\theta L(s, X_s^{t,x,\alpha}, \alpha_s)\, ds + V(\theta, X_\theta^{t,x,\alpha}) \qquad ①$$

Converse direction: $\forall\, \alpha, \beta \in \mathcal{A}$

Define $\gamma_s = \mathbb{1}_{s \in [t,\theta]} \alpha_s + \mathbb{1}_{s \in [\theta,T]} \beta_s$

$$\boxed{\sup_\beta V(t,x)} \ge \int_t^T L(s, X_s^{t,x,\gamma}, \gamma_s)\, ds + g(X_T^{t,x,\gamma})$$

$$= \sup_\beta \int_t^\theta L(s, X_s^{t,x,\alpha}, \alpha_s)\, ds + \sup_\beta \boxed{\int_\theta^T L(s, X_s^{\theta, X_\theta^{t,x,\alpha}, \beta}, \beta_s)\, ds + g(X_T^{\theta, X_\theta^{t,x,\alpha}, \beta})}$$

$$\underbrace{\sup_\alpha V(t,x)}_{V(t,x)} \ge \sup_\alpha \int_t^\theta L(s, X_s^{t,x,\alpha}, \alpha_s)\, ds + V(\theta, X_\theta^{t,x,\alpha}) \qquad ②$$

combine ① and ② ⟹ DPP

The Hamilton-Jacobi-Bellman equation (HJB)

fix a control $\alpha \in \mathcal{A}$, $\forall h > 0$

By DPP $V(t,x) \ge \int_t^{t+h} L(s, X_s^{t,x,\alpha}, \alpha)\, ds + V(t+h, X_{t+h}^{t,x,\alpha})$

By assuming V is smooth enough ⟹ $V \in C^1([0,T] \times \mathbb{R})$

(L cts. f bounded)