Report For Discontinuous Galerkin Method

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1 Problem Statement

Using the third-order DG scheme to solve the Burgers' equation

$$\begin{cases} u_t + uu_x = 0, \\ u(x,0) = g(x) = 1/2 + \sin(x), \quad 0 \le x \le 2\pi, \end{cases} \tag{1}$$

with periodic boundary condition. This report will contain the following contents:

- Derive the exact solution of Burgers equation (1).
- Algorithm design and some related properties.
- Numerical results.
- Discussions.

2 Burgers' equation

In this section, we will introduce some theoretical results for Burgers' equation.

Choose the **flux** $f(u) := \frac{u^2}{2}$, Burgers' equation is a special type of the 1D conservation law:

$$\begin{cases} u_t + (f(u))_x = 0, \\ u(x,0) = g(x). \end{cases} \tag{2}$$

Consider the curve $y = \eta(t; x_0)$ satisfying:

$$\begin{aligned} \eta'(t; x_0) &= u(\eta(t; x_0), t), \\ \eta(0) &= x_0, \end{aligned} \tag{3}$$

along the curve $\eta(t; x_0)$, equation (1) is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(\eta(t; x_0), t) = 0, \\ u(x_0, 0) = g(x_0). \end{cases} \tag{4}$$

It means that $u(x,t)$ remains constant along the curve $\eta(t; x_0)$, and the curve $y = \eta(t; x_0)$ is the **characteristic line** for equation (1).

So, we find the solution $u(x,t)$ by the following two steps:

- Derive a characteristic line $y = \eta(t; \xi)$ from equation (3), which passes through the point (x, t).
- Set $u(x, t) := g(\xi)$.

By (3), $\eta(t; \xi) = \xi + g(\xi)t$, i.e.

$$x = \xi + ut.$$

Then we can derive the solution as an implicit form:

$$u(x, t) = g(x - u(x, t)t). \tag{5}$$

If the characteristic lines don't intersect for $t \in [0, T]$, the equation (1) is well-posed. Unfortunately, it isn't still true for each $T > 0$. Here is the characteristic lines if we choose $T = 1.2$.

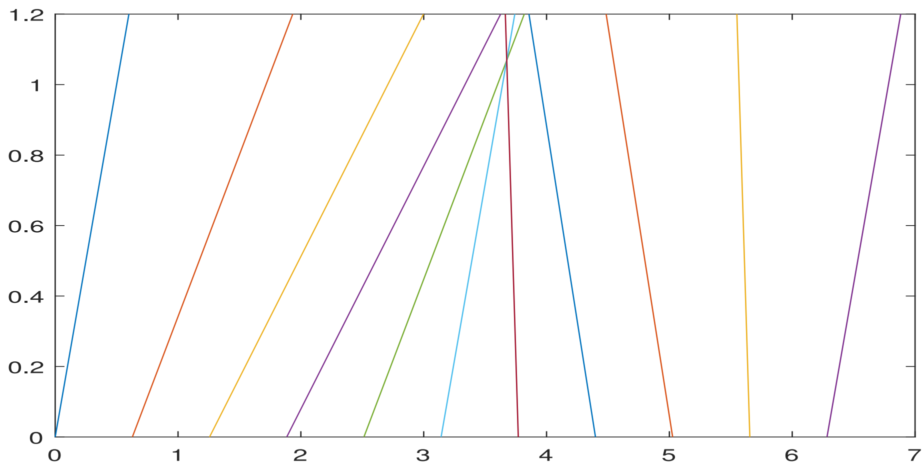


Figure 1: Characteristic lines.

If the characteristic lines intersect with each other, it means that equation (1) has no strong solutions, and its weak solutions are **discontinuous**. By [1, Exercise 3.3], the solution will break at time

$$T_b = \frac{-1}{\min g'(x)} = 1. \tag{6}$$

And the discontinuous point will move forward as well, which forms a **shock wave**. The entropy weak solution is a strong solution on both sides of the shock wave, and discontinuous on the shock wave points. In this problem, the shock wave originates from $x_0 = \pi$ with the speed

$$s = \frac{g(x_0^-) + g(x_0^+)}{2} = 0.5, \tag{7}$$

see [1, (3.26)].

3 Algorithm

In this section, we introduce the DG scheme briefly with some tricks for implementation.

3.1 Mesh and finite element space

In this assignment, we choose equidistant grids. Assume there are N elements on the interval $[0, 2\pi]$, mark $h := \frac{2\pi}{N}$, the $j-th$ element is

$$I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] = \left[\frac{2\pi(j-1)}{N}, \frac{2\pi j}{N}\right],$$

and the finite element space

$$V_h^2 := \{u \in L^2([0, 2\pi]) : u|_{I_j} \in P^2(I_j)\}.$$

$P_2(I_j)$ is the vector space of all polynomials on I_j with degree less than or equal to 2. We choose the Legendre-form basis for $P_2(I_j)$, i.e.

$$\begin{aligned} P_2(I_j) &= \mathrm{span}\{p_j^0, p_j^1, p_j^2\}, \\ p_j^0 &= 1, \quad p_j^1 = \frac{1}{h}\left(x - \frac{x_j + x_{j+1}}{2}\right), \\ p_j^2 &= \frac{3}{2}\left(\frac{2x - x_j - x_{j+1}}{h}\right)^2 - \frac{1}{2}. \end{aligned} \tag{8}$$

It's an orthogonal basis for $P_2(I_j)$.

3.2 Weak form and finite element approximation

Now derive the weak form for (2). Choose $v \in V_h^2$, integration by parts on I_j, we get:

$$\int_{I_j} u_t v \mathrm{d}x - \int_{I_j} f(u) v_x \mathrm{d}x + f(u)v|_{x^-_{j+\frac{1}{2}}} - f(u)v|_{x^+_{j-\frac{1}{2}}}, \forall u \in V_h^2. \tag{9}$$

Since u maybe discontinuous on $x_{j\pm\frac{1}{2}}$, we use **numerical flux** to substitute $f(u)|_{x_{j\pm\frac{1}{2}}}$. In this assignment, we choose **Lax-Friedriches flux**:

$$\hat{f}^{LF}(u^+, u^-) := \frac{1}{2}\left(f(u^-) + f(u^+) - \alpha(u^+ - u^-)\right), \quad \alpha = \max_u |f'(u)|. \tag{10}$$

Then, choose $u_j := c_j^0(t)p_j^0 + c_j^1(t)p_j^1 + c_j^2(t)p_j^2$, $v_{j,i} := p_j^i$, (9) and (10) derive a semi-discretize numerical scheme.

The Lax-Friedriches flux (10) is **consistent, Lipschitz continuous and monotone**, so this scheme satisfies discretize entropy inequality, see [2].

In fact, since $(p_i^2, p_j^2)_{L^2(I_j)} = k\delta_{ij}$, the mass matrix of (9) is just a diagonal matrix. So this scheme is efficient.

3.3 Time Integration

In this assignment, we use **TVD Runge-Kutta scheme** to solve the ODE system $U_t = \mathcal{L}(U)$:

$$\begin{aligned} U^{(1)} &= U^n + \Delta t \mathcal{L}(U^n), \\ U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}(U^{(1)} + \Delta t \mathcal{L}(U^{(1)})), \\ U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}(U^{(2)} + \Delta t \mathcal{L}(U^{(2)})). \end{aligned} \tag{11}$$

4 Numerical Result

In this section, we will give the numerical results.

4.1 Before break time

First, we test the L^1, $L^{\inf}$ error and convergence rates for this numerical scheme. We choose $T = 0.2$, $N = 20, 40, 80, 160, 320$ and $k = 0.1h$. Here are the results.

	20	rate	40	rate	80	rate	160	rate	320
L^1 norm	8.18e-4	2.82	1.16e-4	2.76	1.71e-5	2.75	2.53e-6	2.75	3.77e-7
L^∞ norm	1.10e-3	2.60	1.81e-4	2.62	2.95e-5	2.66	4.69e-6	2.67	7.38e-7

Table 1: The error and convergence rate

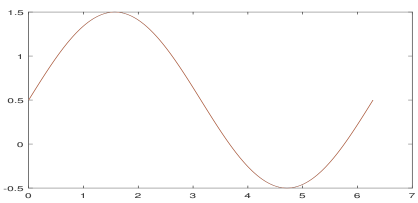
In this test, we choose $\alpha = 1.5$ for the Lax-Friedriches flux, and use Newton's method to derive the solution of Burgers' equation. We write the modified equation

$$w = \sin(x - wt - \frac{t}{2}), \quad u = w + \frac{1}{2},$$

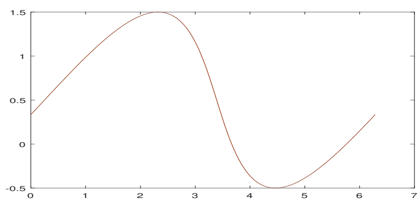
and set the initial value $w = 0.8$ beyond the shock wave, i.e. $x < \pi + 0.5t$, the initial value $w = -0.8$ after the shock wave, i.e. $x > \pi + 0.5t$.

4.2 After break time

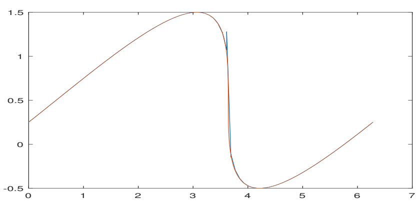
In this section, we give some figures for $T = 0, 0.5, 1, 1.5$ to show the continuous and discontinuous solutions.



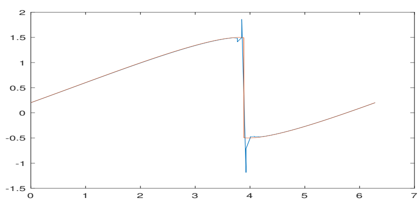
(a) $T = 0$



(b) $T = 0.5$



(c) $T = 1$

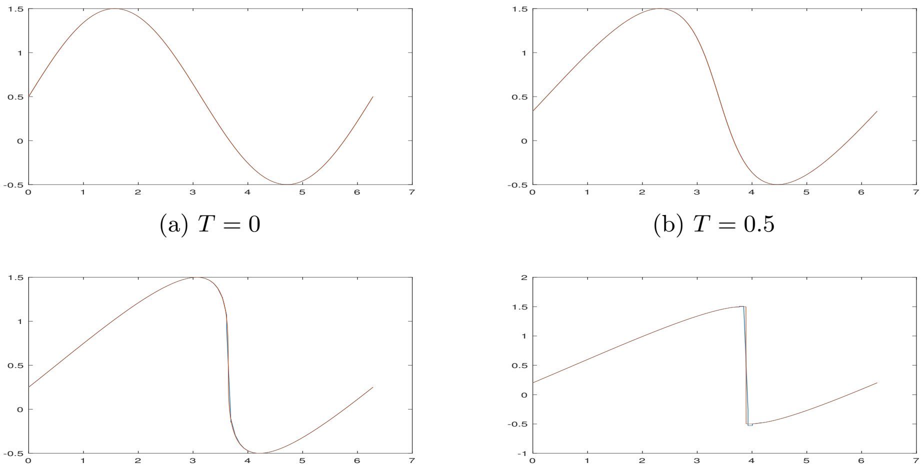


(d) $T = 1.5$

Figure 2: The numerical solution of Burgers equation

The red lines are the exact solutions, and the blue lines are the numerical solutions. Severe oscullations occur for $T > 1$, which means the P_2 element DG method isn't TV(Total Variation) stable.

Now, we try to operate the limiter introduced in [2, Section 3.2.2]. After we operate the limiter, we eliminate the oscullations. It shows:



(a) $T = 0$ (b) $T = 0.5$

(c) $T = 1$ (d) $T = 1.5$

Figure 3: The numerical solution of Burgers equation after limiter

These figures show that the numerical solutions fit the actual solution of Burgers' equation well.

5 Discussions

Now, we make some discussions for the numerical results.

- By (10), we should choose $\alpha = 1.5$. But for $\alpha = 0$ or $\alpha = 1$, the scheme also works, even gives better numerical result, why?
- How to make a prior estimation for the parameter α?
- In this assignment, we generate the initial value by interpolation method on $x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}, \frac{x_{j-\frac{1}{2}}+x_{j+\frac{1}{2}}}{2}$. If we try L^2-projection method to generate the initial value, things will different or not?
- In practise, we use the limiter for the final result, and the shock wave eliminates, why?
- The limiter operator limits $u(x^+_{j-\frac{1}{2}})$ and $u(x^-_{j+\frac{1}{2}})$, but I'm not sure how to get $\tilde{u}_{I_j}$ from this modified values. I set $\tilde{u}_{I_j}$ as the linear function, but the result seems not true. How to deal with this problem?

References

[1] Randall J.Leveque, Numerical Methods for Conservation Laws.

[2] Chi-Wang Shu, Discontinuous Galerkin Methods: General Approach and Stability.