§1. A First course.

§1.1. Conditional Expectation.

Given a probability space $(\Omega, \mathcal{F}_0, \mathbb{P})$, a σ-field $\mathcal{F} \subset \mathcal{F}_0$, and a r.v. $X \in \mathcal{F}_0$ with $\mathbb{E}|X| < \infty$.

Def 1.1. We define the conditional expectation of X given $\mathcal{F}$ ($\mathbb{E}[X|\mathcal{F}]$) to be any r.v. Y s.t.

(i). $Y \in \mathcal{F}$, i.e. $\mathcal{F}$ measurable

(ii). $\int_A Y \, d\mathbb{P} = \int_A X \, d\mathbb{P}$, $\forall A \in \mathcal{F}$.

Rmk: Any Y satisfying (i) and (ii) is said to be a version of $\mathbb{E}[X|\mathcal{F}]$, i.e. the representative element of the equivalence class in the a.e. sense.

Example 1.2. Given $(\Omega, \mathcal{F}_0, \mathbb{P})$, let $\Omega_1, \Omega_2, \Omega_n$ is a finite partition of Ω into disjoint sets and $\mathbb{P}(\Omega_i) > 0$ $(i = 1, 2 \cdots n)$. let $\mathcal{F} = \sigma(\Omega_1, \Omega_2, \cdots \Omega_n)$, then

$$\mathbb{E}[X|\mathcal{F}] = \frac{\int_{\Omega_i} X \, d\mathbb{P}}{\mathbb{P}(\Omega_i)} \quad \text{on } \Omega_i.$$

A degenerate but important special case is $\mathcal{F} = \{\phi, \Omega\}$, the trivial σ-field. In this case, $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}X$.

To start the existence of C.E, we recall ν is said to be absolutely continuous w.r.t μ ($\nu \ll \mu$) if $\mu(A) = 0$ implies $\nu(A) = 0$.

Radon-Nikodym Theorem. Let μ and ν be σ-finite measure on $(\Omega, \mathcal{F})$. If $\nu \ll \mu$, there is a function $f \in \mathcal{F}$ s.t. for all $A \in \mathcal{F}$.

$$\int_A f \, d\mu = \nu(A)$$

Proof. See Appendix A.4. in 《Probability: Theory and Examples》 — Durrett.

扫描全能王
3亿人都在用的扫描App

" ; . . Let $\mu = \mathbb{P}$ and

$\nu(A) = \int_A X \, d\mathbb{P}$. $X \in \mathcal{F}_0$

Since $\mathbb{E}|X| < \infty$; ν is a measure, and $\nu \ll \mu$.

R-N thm implies $\frac{d\nu}{d\mu} \in \mathcal{F}$ and,

$$\int_A X \, d\mathbb{P} = \nu(A) = \int_A \frac{d\nu}{d\mu} \, d\mathbb{P}.$$

Thm.1.3. Assume $\mathbb{E}|X|, \mathbb{E}|Y| < \infty$. $X, Y \subset \mathcal{F}_0$

(a). $\mathbb{E}[aX + Y | \mathcal{F}] = a\,\mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}]$. a.s

(b). If $X \leq Y$, $\mathbb{E}[X|\mathcal{F}] \leq \mathbb{E}[Y|\mathcal{F}]$. a.s.

(c). If X is $\mathcal{F}$-measurable. then $\mathbb{E}[X|\mathcal{F}] = X$ a.s.

(d). If $X \subset \mathcal{F}$ and $\mathbb{E}[|XY|] < \infty$., $\mathbb{E}[XY|\mathcal{F}] = X\mathbb{E}[Y|\mathcal{F}]$.

(e). If X is independend of $\mathcal{F}$. $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

(f). If $\mathcal{F}_1 \subset \mathcal{F}_2$. then

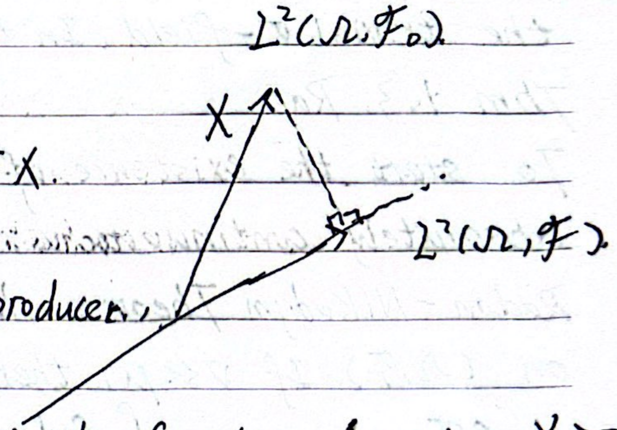
$$\mathbb{E}[\,\mathbb{E}[X|\mathcal{F}_1] \,|\, \mathcal{F}_2] = \mathbb{E}[X|\mathcal{F}_1].$$

$$\mathbb{E}[\,\mathbb{E}[X|\mathcal{F}_2] \,|\, \mathcal{F}_1] = \mathbb{E}[X|\mathcal{F}_1].$$

Proof. See «Durrett» or «Evans».

Thm1.4. Suppose $\mathbb{E}X^2 < \infty$. $\mathbb{E}[X|\mathcal{F}]$ is the variable $Y \in \mathcal{F}$ that minimizes the "error" $\mathbb{E}|X-Y|^2$.

Proof. See «Durrett».



Thm.1.5. If $X_n \geq 0$, and $\mathbb{E}X < \infty$., $X_n \uparrow X$.
then. $\mathbb{E}[X_n|\mathcal{F}] \uparrow \mathbb{E}[X|\mathcal{F}]$.

it provides a basical approximation producer,

take Thm1.3 (d) for example.

we only need to check. X is simple function. (w.o.l.g $X \geq 0$).

then take $X_n \uparrow X$ where X_n is simple function.

$\mathbb{E}[X_n Y|\mathcal{F}] = X_n \mathbb{E}[Y|\mathcal{F}] \Longrightarrow \mathbb{E}[XY|\mathcal{F}] = X\,\mathbb{E}[Y|\mathcal{F}]$. a.e.

and this prop is the importantingredient in the proof of thm 1.5.

since $\mathbb{E}[Y(X - \mathbb{E}[X|\mathcal{F}])] = 0 \quad \forall\, Y \subset \mathcal{F}$.

YUPIN

扫描全能王
3亿人都在用的扫描App

No: Date:

Another view.

$\mathcal{F} \subset \mathcal{F}_0$, and $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a closed subspace of $L^2(\Omega, \mathcal{F}_0, \mathbb{P})$. we can define a projection mapping i from $L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ to $L^2(\Omega, \mathcal{F}, \mathbb{P})$. For $X \subset \mathcal{F}_0$, $i(X) \subset \mathcal{F}$, and

$$\mathbb{E}[(X - i(X))Y] = 0, \quad \forall Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

hence $\int_A X \, d\mathbb{P} = \int_A i(X) \, d\mathbb{P}$, $\quad i(X) = \mathbb{E}[X | \mathcal{F}]$.

while $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is dense in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and i is a continuous map, hence i can be extended to $L^1(\Omega, \mathcal{F}_0, \mathbb{P})$ and the corresponding properties hold on but to be checked.

Thm 1.6. If φ is convex and $\mathbb{E}|X|, \mathbb{E}|\varphi(X)| < \infty$,

$$\varphi(\mathbb{E}[X|\mathcal{F}]) \le \mathbb{E}[\varphi(X)|\mathcal{F}].$$

Proof. see 《Durrett》 thm 4.1.10.

§ 1.2. Martingale and Brownian Motion

Def 1. Let $\mathcal{F}_n$ be a filtration, i.e. an increasing sequence of σ-field. If $\{X_n\}_{n=1}$ satisfies

(i). $\mathbb{E}[|X_n|] < \infty$ (ii) $X_n \subset \mathcal{F}_n$ (iii) $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n \ \forall n$.

$\{X_n\}$ is said to be a martingale, if "$=$" is replaced by "$\ge$" or "$\le$", $\{X_n\}$ is said to be submartingale or supermartingale.

Similarly, for a stochastic process $\{X_t\}_{t \ge 0}$, $\mathcal{F}_t := \sigma\{X_s, s \le t\}$ $\{X_t\}$ is a continuous martingale if $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$ $(t > s)$.

No: Date:

Def 1.8 A real-stochastic process $B(\cdot)$ is called Brownian motion (or Wiener process) if.

(i). $B(0)=0$ a.s

(ii). $B(t)-B(s) \sim N(0,t-s)$ $\forall t \geq s \geq 0$

(iii). $\forall 0<t_1<t_2<\cdots<t_n$, the r.v. $W(t_1)$, $W(t_2)-W(t_1)$, $W(t_n)-W(t_{n-1})$ are independent.

(iv). for a.e. $w\in\Omega$, the sample path $t\mapsto W(t,w)$ is continuous

Rmk: the construction of B.M. can be found in <<Evans>>

Def 1.9 We say f is uniformly Hölder continuous with exponent $1\geq r>0$ if there exists a constant K s.t.

$$|f(t)-f(s)|\leq K|t-s|^r \quad \forall s,t\in[0,T]$$

Thm 1.10 For a.e. $w\in\Omega$, $T>0$ the sample path $t\mapsto B(t,w)$ is uniformly Hölder continuous on $[0,T]$ for each $0<r<\frac{1}{2}$. which implies the path is a.s. continuous, however a.e. nowhere differentiable.

Proof. see <<Evans>>

Example 1.11. Check B.M. is a Martingale and $\mathbb{E}[B_sB_t]=s\wedge t$.

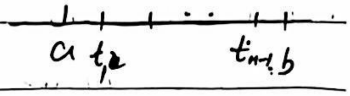
W.O.L.G. let $t>s$. $\mathbb{E}[B_t|\mathcal{F}_s]=\mathbb{E}[B_s+B_t-B_s|\mathcal{F}_s]$

$=B_s+\mathbb{E}[B_t-B_s|\mathcal{F}_s]=B_s$.

$\mathbb{E}[B_sB_t]=\mathbb{E}[\mathbb{E}[B_sB_t|\mathcal{F}_s]]=\mathbb{E}[B_s\mathbb{E}[B_t|B_s]]=\mathbb{E}[B_s\cdot B_s]=s=s\wedge t$.

扫描全能王
3亿人都在用的扫描App

Prop 1.12. Suppose $\mathbb{P}^n := \{a = t_1 < t_2 < \cdots < t_n = b\}$ and the mesh of $\mathbb{P}^n \longrightarrow 0$ $(n \to +\infty)$, then



$$\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \xrightarrow{L^2(\Omega)} b-a$$

Proof.

Notice that if $X \sim N(0,\sigma^2)$; $var(X^2) = \mathbb{E}[X^4] - \sigma^4 = 2\sigma^4$.

$X_i := (B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)$.

$\mathbb{E}[\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - (b-a)]^2 = \mathbb{E}[\sum_{i=0}^{n-1} [(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)]]^2$

$= \mathbb{E}[\sum_{i=0}^{n-1} X_i]^2 = \sum_{i=0}^{n-1} \mathbb{E}[X_i^2]$.

(for $i > j$, $\mathbb{E}[X_i X_j] = \mathbb{E}[\mathbb{E}[X_i X_j | \mathcal{F}_j]] = \mathbb{E}[X_j \mathbb{E}[X_i | \mathcal{F}_j]] = 0$)

$LHS = \sum_{i=0}^{n-1} 2(t_{i+1} - t_i)^2 \le \sum_{i=1}^{n-1} 2\|P\| (t_{i+1} - t_i) = 2(b-a)\|P\| \longrightarrow 0$.

Rmk: this assertion partly justifies $dW \approx (dt)^{\frac{1}{2}}$.

§1.3. Itô integral.

Def 1.13. A process is said to be progressively measurable if for $\forall t > 0$, the mapping:

$$(w, s) \longmapsto X_s(w)$$

defined on $\Omega \times [0,t]$ is measurable for the σ-field $\mathcal{F}_t \otimes \mathcal{B}([0,t])$

Rmk: Specially, a right or left continuous adapted process is progressively measurable.

Def 1.14. We denote by $\mathcal{L}^2(0,T)$ the space of P. measurable s.p. $G(\cdot)$ s.t. $\mathbb{E}(\int_0^T G^2 dt) < \infty$, denote $L^2(\Omega)$. $\mathbb{E}|X|^2 < \infty$. correspondingly $\mathcal{L}^1(0,T)$.

Def 1.15. A process $G \in \mathcal{L}^2(0,T)$ is called a step process if

$$G(t,w) = \sum_{i=1}^{n-1} e_i(w) \mathbb{1}_{(t_i, t_{i+1}]}(t) + e_0(w) \mathbb{1}_{\{0\}}(t). \quad n \in \mathbb{N}.$$

where $e_i \in \mathcal{F}_{t_i}$, $\mathcal{F}_{t_i} = \sigma(B_s, 0 \le s \le t_i)$.

we define Itô integral for step process

$$\int_0^T G(t,w)\, dB_t = \sum_{i=1}^{n-1} e_i(w) (B_{t_{i+1}} - B_{t_i})$$

CS 扫描全能王
3亿人都在用的扫描App

Lemma 1.16. (Itô Isometry for a simple process).

For simple process $G(t,w)$, we have.

$$\mathbb{E}\left|\int_0^T G(t,w)\,dB_t\right|^2 = \mathbb{E}\int_0^T |G(t,w)|^2\,dt$$

Proof.

$$LHS = \mathbb{E}\left|\sum_{i=1}^{n-1} e_i(w)[B_{t_{i+1}}-B_{t_i}]\right|^2 = \mathbb{E}\left|\sum_{i=1}^{n-1} e_i^2(w)(B_{t_{i+1}}-B_{t_i})^2\right|$$

$$= \sum_{i=1}^{n-1} \mathbb{E}\left|(B_{t_{i+1}}-B_{t_i})^2 e_i^2(w)\right| = \sum_{i=1}^{n-1} \mathbb{E}\left[\mathbb{E}[(B_{t_{i+1}}-B_{t_i})^2 e_i^2 \mid \mathcal{F}_{t_i}]\right]$$

$$= \sum_{i=1}^{n-1} \mathbb{E}[e_i^2(t_{i+1}-t_i)] = \mathbb{E}\int_0^T |G(t,w)|^2\,dt.$$

Lemma 1.17. If $G \in \mathcal{L}^2(0,T)$, there exists a sequence of bonnded step processes $G^n \in \mathcal{L}^2(0,T)$, s.t.

$$\mathbb{E}\left(\int_0^T |G(t,w)-G^n(t,w)|^2\,dt\right) \longrightarrow 0$$

Also we define a isometry map i from the dense subspace of $\mathcal{L}^2(0,T)$ to $L^2(\Omega)$, for $\forall\, G \in \mathcal{L}^2(0,T)$.

define $\int_0^T G(t,w)\,dB_t = \lim_{n\to\infty} \int_0^T G^n(t,w)\,dB_t$ in $L^2(\Omega)$

Thm 1.18. For $a, b \in \mathbb{R}$, $\forall\, G, H \in \mathcal{L}^2(0,T)$, we have.

(i). $\int_0^T aG + bH\,dB_t = a\int_0^T G\,dB_t + b\int_0^T H\,dB_t$

(ii). $\mathbb{E}\left(\int_0^T G(t,w)\,dB_t\right) = 0$

(iii). $\mathbb{E}\left(\left(\int_0^T G\,dB_t\right)^2\right) = \mathbb{E}\int_0^T G^2(t,w)\,dB_t$.

(iv). $\int_0^t G(t,w)\,dB_t$ is a martingale.

CS 扫描全能王
3亿人都在用的扫描App