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Problem 1: Constraints Elimination Solution

We aim to minimize:

$$f(x) = e^{1^T x} + e^{-1^T x},$$

subject to the equality constraints:

$$Ax = b,$$

where:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Step 1: Express x in terms of free variables

Using *constraint elimination*, we rewrite x as:

$$x = x_p + Zy,$$

where:

- x_p is a particular solution to $Ax = b$,
- Z is a basis for the null space of A,
- $y \in \mathbb{R}$ is the free variable.

1.1 Compute a Particular Solution (x_p)

To find x_p, solve:

$$Ax_p = b \quad \Longrightarrow \quad \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

This expands to the system of equations:

$$x_1 - 2x_2 = 0,$$
$$2x_1 + x_2 = 5.$$

From the first equation:

$$x_1 = 2x_2.$$

Substitute $x_1 = 2x_2$ into the second equation:

$$2(2x_2) + x_2 = 5 \quad \Longrightarrow \quad 4x_2 + x_2 = 5 \quad \Longrightarrow \quad 5x_2 = 5.$$

Solve for x_2:

$$x_2 = 1.$$

Using $x_1 = 2x_2$:

$$x_1 = 2(1) = 2.$$

Since there is no restriction on x_3 from the equations, set:

$$x_3 = 0.$$

Thus, the particular solution is:

$$x_p = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

1.2 Compute the Null Space Basis (Z)

To find Z, solve $Ax = 0$:

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Expanding:

$$x_1 - 2x_2 = 0, \quad 2x_1 + x_2 = 0.$$

Solve for x_1 and x_2 in terms of x_3:

$$x_1 = 2x_2, \quad x_2 = 0 \implies x_1 = 0.$$

Let $x_3 = t$, where $t \in \mathbb{R}$. Then:

$$x = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the null space basis is:

$$Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Step 2: Substitute x into the Objective Function

The solution to $Ax = b$ is:

$$x = x_p + Zy = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ y \end{bmatrix}.$$

Substitute x into $f(x)$:

$$f(x) = e^{1^T x} + e^{-1^T x}.$$

Here:

$$1^T x = 2 + 1 + y = 3 + y.$$

Thus:

$$f(y) = e^{3+y} + e^{-(3+y)}.$$

Step 3: Minimize $f(y)$

3.1 Compute the Derivative

The derivative of $f(y)$ is:

$$f'(y) = \frac{d}{dy}\left(e^{3+y} + e^{-(3+y)}\right) = e^{3+y} - e^{-(3+y)}.$$

3.2 Solve for Critical Points

Set $f'(y) = 0$:

$$e^{3+y} = e^{-(3+y)}.$$

Simplify:

$$e^{2(3+y)} = 1 \quad \Longrightarrow \quad 2(3+y) = 0 \quad \Longrightarrow \quad y = -3.$$

3.3 Verify Minimum

The second derivative is:

$$f''(y) = e^{3+y} + e^{-(3+y)} > 0 \quad \text{(for all } y\text{)}.$$

Thus, $y = -3$ is a minimum.

Step 4: Compute the Minimum Value

Substitute $y = -3$ into $f(y)$:

$$f(-3) = e^{3-3} + e^{-(3-3)} = e^0 + e^0 = 1 + 1 = 2.$$

Step 5: Back-Substitute to Find x

The corresponding x is:

$$x = \begin{bmatrix} 2 \\ 1 \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

Final Answer

The minimum value of the objective function is:

$$\boxed{f(x) = 2 \quad \text{at } x = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}}.$$

Problem 2: Solving the KKT System

We aim to solve the convex quadratic optimization problem:

$$\text{Minimize } f(x) = c^T x + \frac{1}{2} x^T Q x,$$

subject to:

$$Ax = b.$$

Step 1: The Data we have

The problem parameters are:

$$c = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}, \quad Q = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Step 2: KKT Conditions

The KKT system is given by:

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}.$$

Substitute the given matrices:

$$\begin{bmatrix} 6 & 0 & 2 & 1 & 0 \\ 0 & 4 & 0 & 1 & 1 \\ 2 & 0 & 6 & 2 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -7 \\ 0 \\ 2 \end{bmatrix}.$$

This expands into the following system of equations:

$$\begin{aligned} 6x_1 + 0x_2 + 2x_3 + \lambda_1 &= 3, \\ 0x_1 + 4x_2 + 0x_3 + \lambda_1 + \lambda_2 &= 2, \\ 2x_1 + 0x_2 + 6x_3 + 2\lambda_1 + \lambda_2 &= -7, \\ x_1 + x_2 + 2x_3 &= 0, \\ x_2 + x_3 &= 2. \end{aligned}$$

Step 3: Solve the Equality Constraints ($Ax = b$)

The equality constraints are:

$$x_1 + x_2 + 2x_3 = 0, \quad x_2 + x_3 = 2.$$

From the second equation:

$$x_2 = 2 - x_3.$$

Substitute $x_2 = 2 - x_3$ into the first equation:

$$x_1 + (2 - x_3) + 2x_3 = 0 \quad \Longrightarrow \quad x_1 + 2 + x_3 = 0 \quad \Longrightarrow \quad x_1 = -2 - x_3.$$

Thus, the solution for x in terms of the free variable x_3 is:

$$x_1 = -2 - x_3, \quad x_2 = 2 - x_3, \quad x_3 = x_3.$$

Step 4: Back-Substitute into the KKT System

Substitute x_1, x_2, x_3 into the first three equations of the KKT system:

$$\begin{aligned} 6(-2-x_3)+0(2-x_3)+2x_3+\lambda_1&=3,\\ 0(-2-x_3)+4(2-x_3)+0x_3+\lambda_1+\lambda_2&=2,\\ 2(-2-x_3)+0(2-x_3)+6x_3+2\lambda_1+\lambda_2&=-7.\end{aligned}$$

To solve this system, I used the Python library `SymPy` to perform symbolic computation. The code I used is:

```
from sympy import symbols, Eq, solve
x3, lambda1, lambda2 = symbols('x3 lambda1 lambda2')
eq1 = Eq(6*(-2 - x3) + 2*x3 + lambda1, 3)
eq2 = Eq(4*(2 - x3) + lambda1 + lambda2, 2)
eq3 = Eq(2*(-2 - x3) + 6*x3 + 2*lambda1 + lambda2, -7)
solve([eq1, eq2, eq3], (x3, lambda1, lambda2))
```

This code solves the system of equations, yielding the values:

$$x_3=-1,\quad \lambda_1=11,\quad \lambda_2=-21.$$

Final Solution

After solving the above equations, the optimal values of x and λ are:

$$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\quad \lambda=\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}.$$

Where:

$$x_3=-1,\quad \lambda_1=11,\quad \lambda_2=-21.$$