Interior Point Method And Algorithms Overview

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Problem Statement

Consider optimizing a large-scale problem with $n = 10^6$ variables using the following methods:

(a) Gradient Descent Method

(b) Newton's Method

(c) Augmented Lagrangian Method using Newton's Solver

(d) Infeasible Start Newton's Method

(e) Active Set Method using Newton's Solver

(f) Primal-Dual Interior Point Method using Newton's Solver with Hessian Modification

(g) Gauss-Newton Method

(h) L-BFGS Method

We evaluate the above methods for the following scenarios, aiming to find a local minimum and specifying the algorithm expected to provide the shortest runtime. Each problem starts from a feasible point.

Solutions

1. A quadratic problem which is strictly convex with linear equality constraints.

Algorithms that can find a solution:

- Newton's Method
- Primal-Dual Interior Point Method
- Augmented Lagrangian Method

Optimal Choice: Primal-Dual Interior Point Method

Justification: This problem is strictly convex, meaning a unique global minimum exists. Quadratic problems with equality constraints are efficiently solved using second-order methods. The Primal-Dual Interior Point Method is optimal here because:

- It directly handles equality constraints using a barrier formulation.
- It exploits the structure of quadratic problems to achieve fast convergence.
- Its polynomial-time complexity makes it highly efficient for large-scale problems.

Newton's Method is also suitable but less efficient for very large-scale problems due to higher memory requirements for storing the full Hessian.

2. A linear program with inequality constraints.

Algorithms that can find a solution:

- Primal-Dual Interior Point Method
- Active Set Method

Optimal Choice: Primal-Dual Interior Point Method

Justification: Linear programming problems with inequality constraints are a standard application of the Primal-Dual Interior Point Method. Key reasons for this choice:

- It efficiently handles inequality constraints using barrier functions.

- It scales well with problem size, making it ideal for large-scale problems with $n = 10^6$.
- Compared to the Active Set Method, the Interior Point Method avoids iterating over all constraints, providing faster convergence for large systems.

3. A convex quadratic problem with linear inequality constraints, solved via a sequence of convex quadratic subproblems with a small number of linear equality constraints.

Algorithms that can find a solution:

- Active Set Method
- Augmented Lagrangian Method

Optimal Choice: Active Set Method
Justification: The Active Set Method is particularly efficient for problems involving quadratic objectives and inequality constraints when the number of equality constraints is small. Detailed reasons:

- It iteratively identifies and solves the active set of constraints, which simplifies solving subproblems with equality constraints.
- The small number of equality constraints ensures that the method's overhead remains low.
- Compared to the Augmented Lagrangian Method, it typically converges faster for problems with a manageable number of constraints.

4. An unconstrained non-convex problem.

Algorithms that can find a solution:

- Gradient Descent Method
- Newton's Method
- L-BFGS Method

Optimal Choice: L-BFGS Method
Justification: For large-scale unconstrained optimization, L-BFGS (Limited-memory Broyden–Fletcher–Goldfarb–Shanno) is optimal because:

- It approximates the Hessian using a limited-memory approach, reducing computational and storage costs.
- It converges faster than Gradient Descent due to its use of second-order information, even without computing the full Hessian.
- Newton's Method, while effective for non-convex problems, is impractical here due to the large scale of $n = 10^6$, as computing and storing the Hessian is infeasible.

Gradient Descent is less efficient due to slower convergence rates, especially for non-convex problems.

5. A non-convex least-squares problem of the form $f(x) = \sum_{i=1}^{m} f_i(x)^2$ with $x \in \mathbb{R}^n$. The gradient of f_i is dense, and the Hessian is not available.

Algorithms that can find a solution:

- Gauss-Newton Method
- L-BFGS Method

Optimal Choice: Gauss-Newton Method
Justification: The Gauss-Newton Method is specifically designed for solving least-squares problems. Reasons for choosing this method:

- It avoids explicit computation of the full Hessian by approximating it using the Jacobian of $f_i(x)$, which is computationally efficient.
- The least-squares structure of the problem allows the method to exploit the sparsity or structure of the Jacobian.
- L-BFGS, while effective for non-convex problems, does not leverage the specific least-squares structure, making Gauss-Newton more efficient in this case.

Summary of Optimal Choices

1. Primal-Dual Interior Point Method
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3. Active Set Method

4. L-BFGS Method
5. Gauss-Newton Method