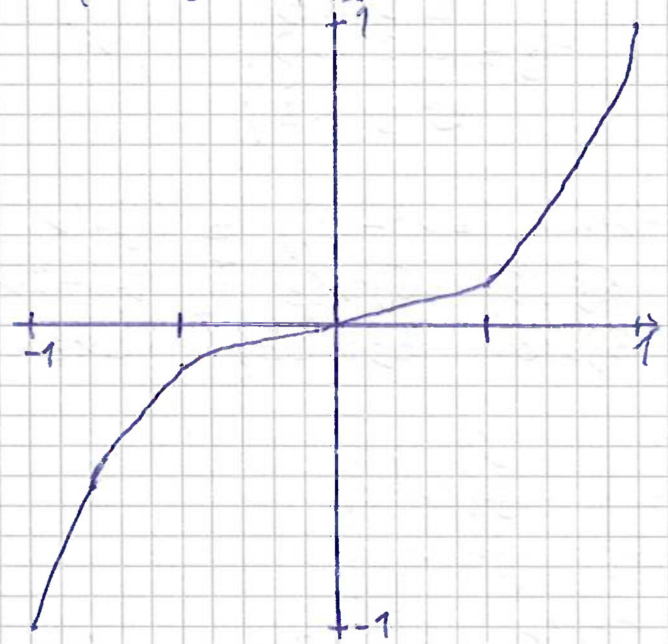
Theory Background Exercises

Functions and Domain Sketches

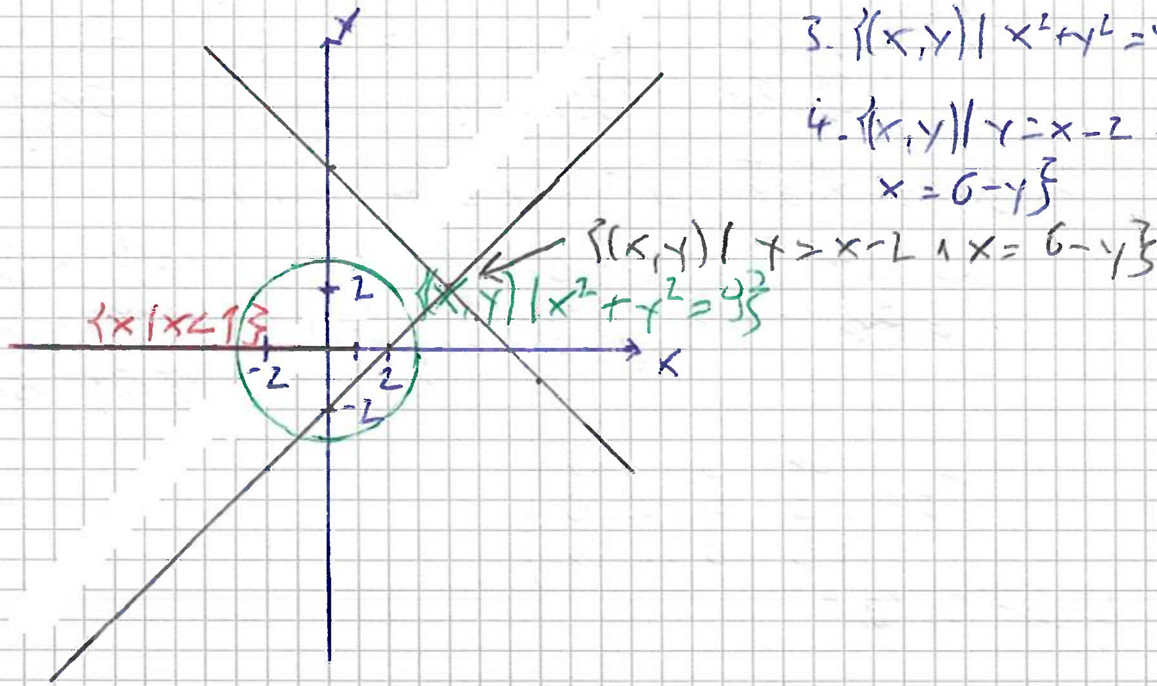
1. $f(x) = x^3$, $x \in [-1, 1]$



2. $\{x \mid x < 1\}$

3. $\{(x,y) \mid x^2 + y^2 = 4\}$

4. $\{(x,y) \mid y = x - 2 \wedge x = 6 - y\}$



Multivariate Functions and their Derivatives

1. $f(x,y) = x^2 + y^2$

$$\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix} , \quad Hf = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

2. $f(x,y,z) = 2xy + x^2 + y^2 + z(x^2 - y^2)$

$$\nabla f = \begin{pmatrix} 2y + 2x + 2xz \\ 2x + 2y + 2yz \\ x^2 - y^2 \end{pmatrix} \quad Hf = \begin{pmatrix} 2 + 2z & 2 & 2x \\ 2 & 2 + 2z & 2y \\ 2x & 2y & 0 \end{pmatrix}$$

3. $f(x) = x^T x$, with $x \in \mathbb{R}^n$, $n > 2$

$$x^T \cdot x = (x_1 \dots x_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i^2$$

$$\nabla f = \begin{pmatrix} 2x_1 \\ \vdots \\ 2x_n \end{pmatrix}, \quad Hf = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 2 \end{pmatrix}$$, with Hf an $n \times n$ matrix.

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1. $f(x,y,z) = (2xy + x^2 + y^2,\ z(x^2 - y^2)) = (f_1(x,y,z),\ f_2(x,y,z))$

$$Jf = \begin{pmatrix} \delta f_1/\delta x & \delta f_1/\delta y & \delta f_1/\delta z \\ \delta f_2/\delta x & \delta f_2/\delta y & \delta f_2/\delta z \end{pmatrix}^T = \begin{pmatrix} 2y + 2x & 2x + 2y & 0 \\ 2xz & 2yz & x^2 - y^2 \end{pmatrix}$$

2. $$f(x,y) = \begin{pmatrix} x^2 y \\ x - y \\ 3xy^2 \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{pmatrix}$$

$$Jf = \begin{pmatrix} \delta f_1/\delta x & \delta f_1/\delta y \\ \delta f_2/\delta x & \delta f_2/\delta y \\ \delta f_3/\delta x & \delta f_3/\delta x \end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\ 1 & -1 \\ 3y^2 & 6xy \end{pmatrix}$$

3. $f(x) = x \odot x$, with $x \in \mathbb{R}^n$, $n > 2$

$$\mathrm{diag}(x) \cdot x = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ 0 & 0 & \dots & x_n \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

$$Jf = \begin{pmatrix} 2x_1 & 0 & \dots & 0 \\ 0 & 2x_2 & \dots & 0 \\ 0 & 0 & \dots & 2x_n \end{pmatrix}$$

Linear Systems

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & 10 \\ 1 & 2 & -3 & 5 \end{array}\right) \begin{array}{l} L_1' \\ L_2' = L_2 - L_1 \\ L_3' = L_3 - L_1 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 8 \\ 0 & 1 & -4 & 3 \end{array}\right) L_2' = -\tfrac{1}{2} L_2 \Leftrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -4 \\ 0 & 1 & -4 & 3 \end{array}\right) L_3' = L_3 - L_2 \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -5 & 7 \end{array}\right) L_3' = -\tfrac{1}{5} L_3 \Leftrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -7/5 \end{array}\right) \begin{array}{l} L_1' = L_1 - L_2 \\ L_2' = L_2 - L_3 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -13/5 \\ 0 & 0 & 1 & -7/5 \end{array}\right) \Leftrightarrow \begin{cases} x = 6 \\ y = -13/5 \\ z = -7/5 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 2 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right) \begin{array}{l} \\ L_2' = \frac{1}{2} L_2 \\ \\ L_4' = L_4 - L_2 \\ \\ \end{array} \Leftrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1/2 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right) \Leftrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1/2 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right) \begin{array}{l} \\ \\ \\ L_4' = L_4 + 2L_5 \\ \\ \end{array} \Leftrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1/2 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right) \begin{array}{l} \\ \\ L_3' = L_3 - 2L_4 + L_5 \\ \\ \\ \end{array} \Leftrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right) \begin{array}{l} \\ L_2' = L_2 - L_3 + L_5 \\ \\ \\ \\ \end{array} \Leftrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -31/2 \\ 0 & 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right) \begin{array}{l} L_1' = L_1 - 2L_3 + L_5 \\ \\ \\ \\ \\ \end{array} \Leftrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -29 \\ 0 & 1 & 0 & 0 & 0 & -31/2 \\ 0 & 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right) \Leftrightarrow \begin{cases} x_1 = -29 \\ x_2 = -31/2 \\ x_3 = 13 \\ x_4 = -7 \\ x_5 = -3 \end{cases}$$

Engineering example

1. $a(x) = \begin{pmatrix} e^x & \sin(2\pi/3\, x) & x^3 \end{pmatrix}, \quad X = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$

$$y(x) = a\,X = \begin{pmatrix} e^x & \sin(2\pi/3\, x) & x^3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

2. $$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} e^{x_1} & \sin(2\pi/3\, x_1) & x_1^3 \\ e^{x_2} & \sin(2\pi/3\, x_2) & x_2^3 \\ e^{x_3} & \sin(2\pi/3\, x_3) & x_3^3 \\ e^{x_4} & \sin(2\pi/3\, x_4) & x_4^3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_2 \\ \beta_3 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} -5 \\ 1 \\ 6 \\ 34 \end{pmatrix} = \begin{pmatrix} e^{-1} & -\sqrt{3}/2 & -1 \\ 1 & 0 & 0 \\ e & \sqrt{3}/2 & 1 \\ e^2 & -\sqrt{3}/2 & 8 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

3. $$Y - AX = \begin{pmatrix} -5 \\ 1 \\ 6 \\ 34 \end{pmatrix} - \begin{pmatrix} 1/e & -\sqrt{3}/2 & -1 \\ 1 & 0 & 0 \\ e & \sqrt{3}/2 & 1 \\ e^2 & -\sqrt{3}/2 & 8 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} =$$

$$\begin{pmatrix} 5 \\ 1 \\ 6 \\ 34 \end{pmatrix} - \begin{pmatrix} \beta_0/e - \sqrt{3}\beta_1/2 - \beta_2 \\ \beta_0 \\ e\beta_0 + \sqrt{3}\beta_1/2 + \beta_2 \\ e^2\beta_0 - \sqrt{3}\beta_1/2 + 8\beta_2 \end{pmatrix} =$$

$$\begin{pmatrix} 5 - \beta_0/e + \sqrt{3}\beta_1/2 + \beta_2 \\ 1 - \beta_0 \\ 6 - e\beta_0 - \sqrt{3}\beta_1/2 - \beta_2 \\ 34 - e^2\beta_0 + \sqrt{3}\beta_1/2 - 8\beta_2 \end{pmatrix}$$

$\|Y-AX\|_2^2 = (5-\beta_0/e+\sqrt{3}\beta_1/2+\beta_2)^2 + (1-\beta_0)^2 +$
$(6-e\beta_0-\sqrt{3}\beta_1/2-\beta_2)^2 + (34-e^2\beta_0+\sqrt{3}\beta_1/2-8\beta_2)^2$

By definition, we have:

$\|Y-AX\|_2^2 = (Y-AX)^T(Y-AX) = Y^TY - Y^TAX - (AX)^TY$
$+(AX)^TAX = Y^TY - 2Y^TAX + (AX)^TAX =$
$Y^TY - 2Y^TAX + X^TA^TAX$

Since we want to compute $\min \|Y-AX\|_2^2$, we want to find the vector X^* such that $\nabla_X \|Y-AX\|_2^2 = 0$ and $\|Y-AX^*\|_2^2 \leq \|Y-AX\|_2^2 \quad \forall X \in \mathbb{R}^3$. $\nabla_X \|Y-AX\|_2^2$ is equ. to:

$\nabla_X (Y^TY - 2Y^TAX + X^TA^TAX) = 0 - 2Y^TA + 2A^TAX$
$\Rightarrow 2A^TAX^* - 2Y^TA = 0 \iff$
$2A^TAX^* = 2Y^TA = 2A^TY \iff$
$X^* = \frac{1}{2}(A^TA)^{-1}2A^TY = (A^TA)^{-1}A^TY$

$\Rightarrow \underline{\underline{X^* = (A^TA)^{-1}A^TY}}$

4. $f(\beta_0,\beta_1,\beta_2) = (5-\beta_0/e+\sqrt{3}\beta_1/2+\beta_2)^2 + (1-\beta_0)^2 + (6-$
$e\beta_0-\sqrt{3}\beta_1/2-\beta_2)^2 + (34-e^2\beta_0+\sqrt{3}\beta_1/2-8\beta_2)^2$

$\frac{\delta f}{\delta \beta_0} = 2(5-\beta_0/e+\sqrt{3}\beta_1/2+\beta_2)\cdot(-1/e) + 2(1-\beta_0)(-1)$
$+ 2(6-e\beta_0-\sqrt{3}\beta_1/2-\beta_2)\cdot(-e) + 2(34-e^2\beta_0+\sqrt{3}\beta_1/2-8\beta_2)$
$\cdot(-e^2) = -10/e + 2\beta_0/e^2 - \sqrt{3}\beta_1/e - 2\beta_2/e - 2 + 2\beta_0 - 12$
$+ 2e^2\beta_0 + \sqrt{3}\beta_1 e + 2e\beta_2 - 68e^2 + e^4\beta_0 - \sqrt{3}e^2\beta_1 + 16e^2\beta_2 =$
$(2e^2+1/e^2+2)\beta_0 + (\sqrt{3}e - \sqrt{3}/e)\beta_1 + (16e - 2/e)\beta_2 - 12e-$

$\frac{\delta f}{\delta \beta_1} = (\sqrt{3}e - \sqrt{3}/e)\beta_0 + 3\beta_1 + 9\sqrt{3}\beta_2 - \sqrt{3}$

$\frac{\delta f}{\delta \beta_2} = (16e - 2/e)\beta_0 + 9\sqrt{3}\beta_1 + 130\beta_2 - 80$

$$\nabla f = \begin{pmatrix} (2e^2 + 1/e^2 + 2)\beta_0 + (\sqrt{3}e - \sqrt{3}/e)\beta_1 + (10e - 2/e)\beta_2 - 12e - 2 \\ (\sqrt{3}e - \sqrt{3}/e)\beta_0 + 3\beta_1 + 9\sqrt{3}\beta_2 - \sqrt{3} \\ (10e - 2/e)\beta_0 + 9\sqrt{3}\beta_1 + 130\beta_2 - 86 \end{pmatrix}$$

5.

$$Hf = \begin{pmatrix} \delta f^2/\delta\beta_0^2 & \delta f^2/\delta\beta_0\delta\beta_1 & \delta^2 f/\delta\beta_0\delta\beta_2 \\ \delta f^2/\delta\beta_0\delta\beta_1 & \delta f^2/\delta\beta_1^2 & \delta^2 f/\delta\beta_1\delta\beta_2 \\ \delta f^2/\delta\beta_0\delta\beta_2 & \delta f^2/\delta\beta_1\delta\beta_2 & \delta f^2/\delta\beta_2^2 \end{pmatrix} =$$

$$\begin{pmatrix} 2e^2 + 1/e^2 + 2 & \sqrt{3}e - \sqrt{3}/e & 10e - 2/e \\ \sqrt{3}e - \sqrt{3}/e & 3 & 9\sqrt{3} \\ 10e - 2/e & 9\sqrt{3} & 130 \end{pmatrix}$$

6. The Hessian matrix Hf is symmetric since $a_{ij} = a_{ji}$ $\forall i, j = 1, \dots, 3$.

A matrix A is positive definite $\Leftrightarrow \forall x \in \mathbb{R}^n$, $x^T A x > 0$.
$A \in M_{n \times n}$

To prove a symmetric positive definite matrix, we must satisfy the condition that all its eigenvalues are positive. To compute the eigen values $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we have to solve the following equation:

$$|A - \lambda I| = 0$$

$$|A - \lambda I| = \begin{vmatrix} 2e^2 + 1/e^2 + 2 - \lambda & \sqrt{3}e - \sqrt{3}/e & 10e - 2/e \\ \sqrt{3}e - \sqrt{3}/e & 3 - \lambda & 9\sqrt{3} \\ 10e - 2/e & 9\sqrt{3} & 130 - \lambda \end{vmatrix} =$$

$$(2e^2 + 1/e^2 + 2 - \lambda)(3 - \lambda)(130 - \lambda) + (\sqrt{3}e - \sqrt{3}/e)9\sqrt{3}(10e - 2/e)$$
$$+ (10e - 2/e)(\sqrt{3}e - \sqrt{3}/e)9\sqrt{3} - \left((10e - 2/e)(3 - \lambda)(10e - 2/e) + 243(2e^2 + 1/e^2 + 2 - \lambda) + (\sqrt{3}e - \sqrt{3}/e)^2(130 - \lambda)\right) =$$
$$-\lambda^3 + 342\lambda^2 + 378\lambda + \underbrace{340e^2 - 21/e^2 - 81 + 12\sqrt{3}/e - 432\sqrt{3}}_{C} = 0$$

To solve the eigenvalues, I used a solver online which gave me: $\lambda_1 \doteq 344.519$, $\lambda_{2,3} \doteq -1.20 \pm 3.75i$, Since the matrix has two complex eigen values, it is no positive semi-definite.