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Computing the Maximal Positive Invariant set for the constrained zonotopic case

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***Abstract*— The maximal positive invariant (MPI) set results from a finite set recurrence instantiated by the intersection of input and state bounds (e.g., the stage constraints of the linear model predictive control problem). When these constraints take the form of hyper-rectangles, zonotopes or constrained zonotopes, the resulting polyhedral MPI set may be succinctly described as a constrained zonotope, eliminating the need for explicit enumeration of its halfspaces. In this paper we discuss the various MPI computation algorithms (with both exact and sufficient stop conditions), recasted in the framework of constrained zonotopes. We analyze one of these variations over a dynamical system whose dimension can be arbitrarily increased in order to asses changes in computation time and storage requirements with respect to the polyhedral case (under the simplifying assumption of closed-loop invertibility of the state matrix).**

***Index Terms*— maximal positive invariant set; constrained zonotope; minimal representation; model predictive control**

I. Introduction

Model Predictive Control (MPC) is a popular approach in both academia and industry as it explicitly accounts for constraints, cost and model dynamics [1]. Arguably, the main difficulties with MPC originate in the calculations carried to ensure recursive feasibility [2], [3]. The computationally-expensive step is usually to compute the terminal set. The maximal positive invariant (MPI) set is the common choice for it but it comes at a steep price: the time required to obtain it, the memory required for storing, and even the additional computational effort of integrating it in the MPC algorithm [4]. It is worth mentioning that there are approaches which try to avoid these complexities, e.g., by working with implicit forms via a sequence of sets [5], by fixing the complexity [6] or, even, by prolonging the prediction horizon sufficiently such as to avoid altogether the need of a terminal set [7].

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While a large body of literature discusses the computation of MPI sets [4], [8], [9], to the best of the authors' knowledge, only polyhedral sets have been used. Arguably, exploiting the particularities of the MPI set, should lead to efficiencies in its computation. To this end, and the main impetus for this paper, we note that the set sequence spanning the MPI set:

- i) is a repetition of matrix multiplications and set intersections applied to 'simple' sets (hyper-rectangles, zonotopes or constrained zonotopes);
- ii) is spanned by a 'seed' set which is usually an intersection between two hyper-rectangles (those defining the state constraints and, respectively, the input constraints);
- iii) the stop condition that halts it reduces to testing (in exact or sufficient form) a set inclusion.

Since handling polyhedra quickly becomes intractable for large dimensions and number of constraints, several alternatives have been considered over the past few decades. Due to their resilience to the "dimensionality curse", zonotopes [10] and, more recently, constrained zonotopes [11] have been increasingly used, e.g., for reachability analysis [12] or fault diagnosis [13]. Both representations enable significant complexity reductions in standard set operations. In particular, constrained zonotopes can be used to approximate arbitrarily-well any convex set [11, Theorem 1] and are closed under intersection and set addition. These properties allow us to have the following contributions:

- i) construct the MPI set as a constrained zonotope with (partial) minimal representation, with the simplifying assumption of closed-loop invertibility of the state matrix;
- ii) propose linear programs which test set inclusion (either as a variation of the AH-polytope result from [14] or, directly, through the constrained zonotope description);
- iii) analyze the behavior and complexity of the MPI algorithm, adapted to exploit the underlying structure.

We analyze all the aforementioned MPI methods over a benchmark whose dimension can be varied arbitrarily [15].

The rest of the paper is organized as follows. Section II introduces the basic notions about constrained zonotopes theory. Section III details the equivalence between constrained zonotope and AH-polytope representations, and adapts the AH-polytope inclusion condition for constrained zonotopes. Section IV details the variants of constrained zonotope MPI construction. Section V analyzes the relative performance of the proposed schemes and Section VI draws the conclusions.

Notation: $O_{m\times n} \in \mathbb{R}^{m\times n}$ is the matrix with m rows and n columns whose entries are zero. Whenever $m = n$, we use the shorthand notation O_n. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. The symbol $\mathbf{1}$ represents the column vector of values of one. For an arbitrary matrix $G \in \mathbb{R}^{m\times n}$, G_i denotes its i-th column and $G_j^\top$ its j-th row. For two sets, X and Y, their Minkowski sum is defined as $X \oplus Y = \{x + y : \forall x \in X, \forall y \in Y\}$. $|A|$, with $A \in \mathbb{R}^{m\times n}$, represents the element-wise absolute value of A. For a vector $x \in \mathbb{R}^n$, its infinity norm is given as $\|x\|_\infty := \max(|x_1|, |x_2|, \ldots, |x_n|)$. $\mathbb{B}_\infty^n$ is the n-dimensional unit ball of the infinity-norm. 'blk$(X_1, \ldots, X_n)$' denotes the block-diagonal concatenation of matrices $X_1, \ldots, X_n$.

II. Preliminaries

Later in the paper we analyze the effect of a particular set representation (the constrained zonotope) in the MPI set construction. To arrive at it, we need first to introduce the notions of polytope and zonotope.

Any polytope (i.e. a bounded polyhedron) has a dual representation [16], as an intersection of half-spaces (*H-rep*):

$$P = \{x \in \mathbb{R}^d : Hx \leq h\}, \tag{1}$$

or, equivalently, as the convex hull of extreme points (*V-rep*):

$$P = \{x \in \mathbb{R}^d : x = V\alpha, \alpha^\top \mathbf{1} = 1, \alpha \geq 0\}. \tag{2}$$

Pair $(H, h) \in \mathbb{R}^{q\times d} \times \mathbb{R}^q$ describes the q constraints (the "half-spaces"), and $V \in \mathbb{R}^{d\times\nu}$ gathers the ν extreme vertices.

Zonotopes, a sub-class of polytopes, are endowed with a third representation (*G-rep*) due to their central symmetry:

$$Z = \langle c, G\rangle = \{x \in \mathbb{R}^d : x = c + G\lambda : \|\lambda\|_\infty \leq 1\}. \tag{3}$$

Pair $(c, G) \in \mathbb{R}^d \times \mathbb{R}^{d\times D}$ denotes the zonotope's center and, respectively, its D generators.

A. Constrained zonotopes

Constrained zonotopes provide an excellent compromise between numerical complexity and fidelity of representation.

Definition 1 (Constrained Generator (CG-rep), [11]): A convex set $Z \subset \mathbb{R}^d$ is a constrained zonotope if there exists $(c, G, F, \theta) \in \mathbb{R}^d \times \mathbb{R}^{d\times D} \times \mathbb{R}^{q\times D} \times \mathbb{R}^q$ such that

$$\begin{aligned} Z &= \langle c, G, F, \theta\rangle \\ &= \left\{x \in \mathbb{R}^d : x = c + G\lambda, \|\lambda\|_\infty \leq 1, F\lambda = \theta\right\}. \end{aligned} \tag{4}$$

Compared to (3), for *CG-rep*, central symmetry is no longer necessary (i.e., a polytope is a constrained zonotope).

Constrained zonotopes have useful properties [11], they:

i) are closed under affine transformations:

$$r + RZ_1 = \langle r + Rc_1, RG_1, F_1, \theta_1\rangle; \tag{5}$$

ii) are closed under Minkowski sum:

$$Z_1 \oplus Z_2 = \left\langle c_1 + c_2, \begin{bmatrix} G_1 & G_2\end{bmatrix}, \begin{bmatrix} F_1 & 0 \\ 0 & F_2\end{bmatrix}, \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}\right\rangle; \tag{6}$$

iii) are closed under set intersection:

$$Z_1 \cap Z_2 = \left\langle c_1, \begin{bmatrix} G_1 & 0\end{bmatrix}, \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \\ G_1 & -G_2\end{bmatrix}, \begin{bmatrix}\theta_1 \\ \theta_2 \\ c_2 - c_1\end{bmatrix}\right\rangle. \tag{7}$$

where $Z_1 := \langle c_1, G_1, F_1, \theta_1\rangle$ and $Z_2 := \langle c_2, G_2, F_2, \theta_2\rangle$.

Noteworthy, set operations (6) and (7) increase the representation complexity. However, there exists techniques [11], [17] which may remove redundancies.

B. Set inclusion conditions

Arguably, the main element in the MPI computation is checking a set inclusion to validate the algorithm's stop condition. Sufficient but efficient formulations are introduced in [14] which build upon the notion of *AH polytope*, defined as the affine transformation of a polytope. Based on this definition we recall the main result from [14] which provides sufficient conditions for set inclusion verification.

Proposition 1: [AH-polytope inclusion, [14, Thm. 1]] Let $\mathbb{U} = \{\bar{u}\} \oplus U\mathbb{P}_u \subset \mathbb{R}^d$, $\mathbb{V} = \{\bar{v}\} \oplus V\mathbb{P}_v \subset \mathbb{R}^d$, where $\mathbb{P}_u = \{u \in \mathbb{R}^{d_u} : H_u u \leq h_u\}$, $\mathbb{P}_v = \{v \in \mathbb{R}^{d_v} : H_v v \leq h_v\}$ and $(H_u, h_u) \in \mathbb{R}^{q_u\times d_u} \times \mathbb{R}^{q_u}$, $(H_v, h_v) \in \mathbb{R}^{q_v\times d_v} \times \mathbb{R}^{q_v}$, $(U, \bar{u}) \in \mathbb{R}^{d\times d_u} \times \mathbb{R}^d$, $(V, \bar{v}) \in \mathbb{R}^{d\times d_v} \times \mathbb{R}^d$.

Then, set inclusion $\mathbb{U} \subseteq \mathbb{V}$ holds if

$$\exists\, \Gamma \in \mathbb{R}^{d_v\times d_u},\ \beta \in \mathbb{R}^{d_v},\ \Lambda \in \mathbb{R}_+^{q_v\times q_u} \tag{8}$$

such that

$$U = V\Gamma, \qquad \bar{v} - \bar{u} = V\beta, \tag{9a}$$

$$\Lambda H_u = H_v\Gamma \qquad \Lambda h_u \leq h_v + H_v\beta. \tag{9b}$$

Zonotopes (the affine mappings of the unit ball of the infinity norm [18]) are a particular case of an AH-polytope:

$$\langle c, G\rangle := \{c\} \oplus G\mathbb{B}_\infty^D, \tag{10}$$

thus, Prop. 1 may be adapted, as done in [14, Cor. 4].

III. Set inclusions for constrained zonotopes

While any constrained zonotope is a polytope, meaning that standard set inclusion tests can be applied, these require the explicit description of the half-space (1), or, exponentially worse, of the vertex (2) formulations. To avoid both the effort spent in enumerating these elements and the complexity of the associated checks, we show here efficient formulations which exploit the constrained generator description from (4).

Lemma 1: Any constrained zonotope is an AH-polytope. □

Proof: Consider a constrained zonotope $Z = \langle c, G, F, \theta\rangle$, defined as in (4), and assume a partitioning $F = \begin{bmatrix} F_a & F_b\end{bmatrix}$, $\lambda = \begin{bmatrix}\lambda_a^\top & \lambda_b^\top\end{bmatrix}^\top$ such that F_a is square and invertible[1]. Then $F\lambda = \theta$ implies $\lambda_a = F_a^{-1}(\theta - F_b\lambda_b)$, which, introduced back in (4), gives

$$Z = \left\{(c + G\bar{\theta}) + G\bar{F}\lambda_b \mid \left\|\bar{F}\lambda_b + \bar{\theta}\right\|_\infty \leq 1\right\}, \tag{11}$$

$$\text{where } \bar{F} := \begin{bmatrix} -F_a^{-1}F_b \\ I\end{bmatrix}, \quad \bar{\theta} := \begin{bmatrix} F_a^{-1}\theta \\ 0\end{bmatrix}. \tag{12}$$

Noting that inequality $\left\|\bar{F}\lambda_b + \bar{\theta}\right\|_\infty \leq 1$ is equivalent with

$$-\mathbf{1} \leq \bar{F}\lambda_b + \bar{\theta} \leq \mathbf{1}, \quad -\mathbf{1} - \bar{\theta} \leq \bar{F}\lambda_b \leq \mathbf{1} - \bar{\theta},$$

shows that (11), after a re-arranging, may be put in the form

$$Z = \left\{(c + G\bar{\theta}) + G\bar{F}\lambda_b \mid \begin{bmatrix}\bar{F} \\ -\bar{F}\end{bmatrix}\lambda_b \leq \begin{bmatrix}\mathbf{1} - \bar{\theta} \\ \mathbf{1} + \bar{\theta}\end{bmatrix}\right\}.$$

Denoting

$$\{\bar{u}, U, H_u, h_u\} \leftarrow \left\{c + G\bar{\theta}, G\bar{F}, \begin{bmatrix}\bar{F} \\ -\bar{F}\end{bmatrix}, \begin{bmatrix}\mathbf{1} - \bar{\theta} \\ \mathbf{1} + \bar{\theta}\end{bmatrix}\right\}, \tag{13}$$

we arrive to the AH-polytope definition of Proposition 1. ■

[1] As long as F is full row rank, there will always exist such a decomposition, possibly after a permutation of its columns.

Theorem 4 of [17] states the same result but through the use of pseudo-inverse matrices. For further use we prefer the form exposed in Lemma 1. Consider two constrained zonotopes

$$Z_{1,2} = \langle c_{1,2}, G_{1,2}, F_{1,2}, \theta_{1,2} \rangle, \tag{14}$$

defined as in (4) and for which we wish to check inclusion $Z_1 \subseteq Z_2$. Lemma 1 leads to Proposition 1's corollary. For further use: $(c_i, G_i, F_i, \theta_i) \in \mathbb{R}^{d_i} \times \mathbb{R}^{d_i \times D_i} \times \mathbb{R}^{q_i \times D_i} \times \mathbb{R}^{q_i}$.

Corollary 1: For constrained zonotopes $Z_{1,2}$ defined as in (14), set inclusion $Z_1 \subseteq Z_2$ holds if

$$\exists\, \Gamma \in \mathbb{R}^{D_2 \times D_1},\ \beta \in \mathbb{R}^{D_2}, \Lambda \in \mathbb{R}_+^{2q_2 \times 2q_1} \tag{15}$$

such that

$$G_1 \bar{F}_1 = G_2 \bar{F}_2 \Gamma, \tag{16a}$$

$$(c_2 + G_2\bar{\theta}_2) - (c_1 + G_1\bar{\theta}_1) = G_2 \bar{F}_2 \beta, \tag{16b}$$

$$\Lambda \begin{bmatrix} \bar{F}_1 \\ -\bar{F}_1 \end{bmatrix} = \begin{bmatrix} \bar{F}_2 \\ -\bar{F}_2 \end{bmatrix} \Gamma, \tag{16c}$$

$$\Lambda \begin{bmatrix} \mathbf{1} - \bar{\theta}_1 \\ \mathbf{1} + \bar{\theta}_1 \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} - \bar{\theta}_2 \\ \mathbf{1} + \bar{\theta}_2 \end{bmatrix} + \begin{bmatrix} \bar{F}_2 \\ -\bar{F}_2 \end{bmatrix} \beta. \tag{16d}$$

Proof: As per Lemma 1, constrained zonotopes (14) may be put in the AH-polytope form [14]. Applying (13) in (8)-(9) directly leads to (15)–(16), concluding the proof. ■

Remark 1: The symmetries of (16c)–(16d) allow a compact reformulation. Denote $\Lambda = \begin{bmatrix} \Lambda' & \Lambda'' \\ \Lambda'' & \Lambda' \end{bmatrix}$ and group $\bar{\Lambda} := \Lambda' - \Lambda''$ and $\tilde{\Lambda} := \Lambda' + \Lambda''$ then (16c)–(16d) become $\bar{\Lambda}\bar{F}_1 = \bar{F}_2\Gamma$ and $|\bar{\Lambda}\bar{\theta}_1 - \bar{\theta}_2 + \bar{F}_2\beta| \leq \mathbf{1} - \tilde{\Lambda}\mathbf{1}$, respectively. Since, by construction, $\bar{\Lambda}\mathbf{1} \leq \tilde{\Lambda}\mathbf{1}$ we have that checking

$$\bar{\Lambda}\bar{F}_1 = \bar{F}_2\Gamma, \quad |\bar{\Lambda}\bar{\theta}_1 - \bar{\theta}_2 + \bar{F}_2\beta| \leq \mathbf{1} - \tilde{\Lambda}\mathbf{1} \tag{17}$$

implies (16c)–(16d). The advantage of (17) is both in problem size ($\bar{\Lambda}, \tilde{\Lambda}$ have half of Λ's entries) and in that $\bar{\Lambda}$ may have arbitrary, not elementwise positive, values. ♦

The question arises whether the same methodology used in Proposition 1 and [14, Cor. 4] may be applied to $Z_1 \subseteq Z_2$, without first passing through Lemma 1.

Proposition 2: For constrained zonotopes $Z_{1,2}$ defined as in (14), set inclusion $Z_1 \subseteq Z_2$ holds if

$$\exists\, \Gamma \in \mathbb{R}^{D_2 \times D_1},\ \beta \in \mathbb{R}^{D_2}, \Pi \in \mathbb{R}^{q_2 \times q_1} \tag{18}$$

such that

$$c_2 - c_1 = G_2\beta, \qquad G_1 = G_2\Gamma, \tag{19a}$$

$$\Pi F_1 = F_2\Gamma, \qquad \Pi\theta_1 = \theta_2 + F_2\beta, \tag{19b}$$

$$|\Gamma|\mathbf{1} + |\beta| \leq \mathbf{1}. \tag{19c}$$

Proof: The inclusion is verified if $x \in Z_1$ implies that $x \in Z_2$. Otherwise stated, implication

$$(\forall\, \lambda_1 \text{ s.t. } x = c_1 + G_1\lambda_1 \text{ and } F_1\lambda_1 = \theta_1, \|\lambda_1\|_\infty \leq 1)$$
$$\implies (\exists\, \lambda_2 \text{ s.t. } x = c_2 + G_2\lambda_2 \text{ and } F_2\lambda_2 = \theta_2, \|\lambda_2\|_\infty \leq 1)$$

has to hold. Using (19a) in $x = c_1 + G_1\lambda_1$ we arrive at

$$x = c_2 - G_2\beta + G_2\Gamma\lambda_1 = c_2 + G_2(\Gamma\lambda_1 - \beta), \tag{20}$$

where, by noting $\lambda_2 = \Gamma\lambda_1 - \beta$, it becomes clear that the set inclusion condition $Z_1 \subseteq Z_2$ holds if

$$F_2(\Gamma\lambda_1 - \beta) = \theta_2, \tag{21a}$$

$$\|\Gamma\lambda_1 - \beta\|_\infty \leq 1, \tag{21b}$$

are shown to hold. Left-multiplying with Π in the initial assumption $F_1\lambda_1 = \theta_1$ and applying (19b) leads directly to (21a). For (21b), we combine the norm properties and the initial assumption that $\|\lambda_1\|_\infty \leq 1$ to derive the following chain of implications: $(\|\Gamma\lambda_1 - \beta\|_\infty \leq 1) \Longleftarrow (\|\Gamma\lambda_1\|_\infty + \|\beta\|_\infty \leq 1) \Longleftarrow (|\Gamma|\mathbf{1} + |\beta| \leq \mathbf{1})$. Since the last term is (19c), (21b) holds, thus concluding the proof. ■

Remark 2: Corollary 1, its symmetry-exploiting modification in (17) and Proposition 2 propose alternative sufficient conditions for checking set inclusion $Z_1 \subseteq Z_2$. It remains an open question which of these methods is less conservative (or whether they are equivalent). While the computation time for solving an optimization problem in which these linear relations are embedded is affected by many factors, Table I may help in deciding which of these variants to use.

(eqs.)	# vars.	# eqs.	# ineqs.
(16)	$D_2(D_1+1) + 4q_2q_1$	$d(D_1+1) + 2q_2D_1$	$2q_2$
(17)	$D_2(D_1+1) + 2q_2q_1$	q_2D_1	q_2
(19)	$D_2(D_1+1) + q_2q_1$	$(1+D_1)(d+q_2)$	D_2

TABLE I: Complexity representation ♦

Remark 3: An ancillary topic is the size of the internal description for a constrained zonotope. The intersection operation (7) is particularly egregious, especially when the operation is repeated iteratively. The idea, exploited later in the paper, is to apply [17, Theorem 2] for a subset of equality indices (those that correspond to the second term in intersection (7)). This has the twin benefit of controlling the size of the representation and the number of redundancy checks. ♦

IV. Application to the MPI computation

Consider the standard linear-quadratic MPC problem:

$$\min_{\bar{u}_0,\ldots,\bar{u}_{N-1}} \sum_{k=0}^{N-1} \bar{x}_k^\top Q\bar{x}_k + \bar{u}_k^\top R\bar{u}_k + \bar{x}_N^\top P\bar{x}_N \tag{22a}$$

$$\text{s.t.} \quad \bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k,\ \forall k = 0,\ldots,N-1, \tag{22b}$$

$$\bar{u}_k \in \mathcal{U}, \quad \bar{x}_{k+1} \in \mathcal{X},\ \forall k = 0,\ldots,N-1, \tag{22c}$$

$$\bar{x}_N \in \Omega, \tag{22d}$$

with the components given in the usual way: linear dynamics (22b) with state $x_k \in \mathbb{R}^d$ and input $u_k \in \mathbb{R}^m$, used for prediction along the finite horizon of length N; stage constraints (22c) for cost and input ($\mathcal{X} \subset \mathbb{R}^d$, $\mathcal{U} \subset \mathbb{R}^m$); terminal constraint (22d) and cost (22a) defined by positive (semi)-definite stage (Q, R) and terminal cost (P) matrices.

At the end of the prediction horizon ($k \geq N$) in the MPC problem (22), the control action switches to a fixed feedback law $\bar{u}_k = K\bar{x}_k$ which has to ensure closed-loop stability and admissibility ($\bar{x}_{N+\ell} \in \mathcal{X}, \bar{u}_{N+\ell} \in \mathcal{U}$ for any $\ell \geq 0$ if $x_N \in \Omega$). To simplify the notation, we note $A_\circ := A + BK$ to obtain the LTI dynamics

$$\bar{x}_{k+1} = A_\circ\bar{x}_k, \tag{23}$$

with matrix $A_\circ \in \mathbb{R}^{d\times d}$ and a set $\overline{\mathcal{X}} \subset \mathbb{R}^d$ which bounds the state. Further, let us recall the standard recurrence for MPI (maximal positive invariant) set construction:

$$\Omega_0 = \overline{\mathcal{X}}, \quad \Omega_{k+1} = A_\circ^{-1}\Omega_k \cap \overline{\mathcal{X}}, \quad \text{with} \tag{24a}$$

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$$\overline{\mathcal{X}} := \mathcal{X} \cap \{x \in \mathbb{R}^d :\ Kx \in \mathcal{U}\}. \tag{24b}$$

Under mild and reasonable assumptions [5, Thm. 3], recurrence (24a) is guaranteed to stop by arriving at a fixed set point $\Omega := \Omega_{\bar{k}} = \Omega_{\bar{k}+1}$, for some finite index $\bar{k}$.

Since by construction (see (26) and further details in [8]) we have that $\Omega_{k+1} \subseteq \Omega_k$ for all k, it suffices to check that

$$\Omega_k \subseteq \Omega_{k+1}. \tag{25}$$

Equivalently, recurrence (24a) may be written as

$$\Omega_{k+1} = \bigcap_{j=0}^{k+1} \mathcal{Y}_j = \bigcap_{j=0}^{k+1} A_\circ^{-j}\overline{\mathcal{X}} = \Omega_k \cap A_\circ^{-(k+1)}\overline{\mathcal{X}}, \tag{26}$$

and, consequently, the stop condition (25) becomes[2]

$$\Omega_k \subseteq \mathcal{Y}_{k+1} \Leftrightarrow \Omega_k \subseteq A_\circ^{-(k+1)}\overline{\mathcal{X}}. \tag{27}$$

Often, conditions (25), (27) are replaced by the sufficient condition (by way of noting that $\Omega_k \subseteq \overline{\mathcal{X}}$):

$$\overline{\mathcal{X}} \subseteq A_\circ^{-(k+1)}\overline{\mathcal{X}}. \tag{28}$$

Assuming[3] that the state/input constraint sets are zonotopes

$$\mathcal{X} = \langle c_X, G_X\rangle, \quad \mathcal{U} = \langle c_U, G_U\rangle, \tag{29}$$

means that, via (7), $\overline{\mathcal{X}}$ is a constrained zonotope:

$$\begin{aligned}\bar{\mathcal{X}} &= \langle c_{\bar{X}}, G_{\bar{X}}, F_{\bar{X}}, \theta_{\bar{X}}\rangle \\ &= \left\langle c_X, \begin{bmatrix} G_X & 0\end{bmatrix}, \begin{bmatrix} KG_X & -G_U\end{bmatrix}, c_U - Kc_X\right\rangle,\end{aligned} \tag{30}$$

with $\left(\bar{c}, \bar{G}, \bar{F}, \bar{\theta}\right) \in \mathbb{R}^d \times \mathbb{R}^{d\times\bar{D}} \times \mathbb{R}^{\bar{q}\times\bar{D}} \times \mathbb{R}^{\bar{q}}$, with shorthand $\bar{q}, \bar{D}$ chosen accordingly.

Since constrained zonotopes are closed under affine transformations and intersections, all terms Ω_k in set recurrence (24a) are also constrained zonotopes defined by recurrence:

$$\Omega_{k+1} = \left\langle A_\circ^{-1}c_k, \begin{bmatrix} A_\circ^{-1}G_k & 0\end{bmatrix}, \begin{bmatrix} F_k & 0 \\ 0 & F_{\bar{X}} \\ A_\circ^{-1}G_k & -G_{\bar{X}}\end{bmatrix}, \begin{bmatrix}\theta_k \\ \theta_{\bar{X}} \\ c_{\bar{X}} - A_\circ^{-1}c_k\end{bmatrix}\right\rangle \tag{31a}$$

$$= \left\langle c_{\bar{X}}, \begin{bmatrix} G_{\bar{X}} & 0\end{bmatrix}, \begin{bmatrix} F_{\bar{X}} & 0 \\ 0 & F_k \\ G_{\bar{X}} & -A_\circ^{-1}G_k\end{bmatrix}, \begin{bmatrix}\theta_{\bar{X}} \\ \theta_k \\ A_\circ^{-1}c_k - c_{\bar{X}}\end{bmatrix}\right\rangle \tag{31b}$$

with $\Omega_k = \langle c_k, G_k, F_k, \theta_k\rangle$. The two forms, (31a) and (31b), are equivalent and appear from the order in which we consider the terms in intersection (24a) – a commutative operation. We may now introduce the following lemma.

Lemma 2: Set $\Omega_k = \langle c_k, G_k, F_k, \theta_k\rangle$, the k-th term of the recurrence (24a), iterated from starting set (30) is given by

$$c_k = c_{\bar{X}}, \qquad G_k = \begin{bmatrix} G_{\bar{X}} & \mathbf{0}_{d\times k\bar{D}}\end{bmatrix}, \tag{32a}$$

$$F_k = \begin{bmatrix} I_{k+1}\otimes F_{\bar{X}} \\ \Phi_k\cdot(I_k\otimes G_{\bar{X}})\end{bmatrix}, \qquad \theta_k = \begin{bmatrix}\mathbf{1}_{k+1}\otimes\theta_{\bar{X}} \\ \mathbf{1}_k\otimes\tilde{c}_{\bar{X}}\end{bmatrix}, \tag{32b}$$

with '$\otimes$' denoting the Kroenecker product and shorthands

$$\tilde{c}_{\bar{X}} = (A_\circ^{-1} - I)c_{\bar{X}}, \tag{33a}$$

$$\Phi_k = \begin{bmatrix} 0 & \dots & I & -A_\circ^{-1} \\ 0 & \dots & -A_\circ^{-1} & A_\circ^{-2} \\ \vdots & \ddots & \vdots & \vdots \\ I & \dots & (-A_\circ^{-1})^{k-1} & (-A_\circ^{-1})^k\end{bmatrix}. \tag{33b}$$

Proof: Iterating (31b) with shorthand (33) when starting from (30) leads to (32), thus concluding the proof. ■

The structure exposed by Lemma 2 allows to adapt Proposition 2 for set inclusions (25) and (27).

Corollary 2: Set inclusion (25) holds if

$$\begin{aligned}&\exists\,\Gamma \in \mathbb{R}^{(k+1)\tilde{D}\times(k+1)\tilde{D}},\ \Pi \in \mathbb{R}^{[(k+1)n_c+(k+1)d]\times[(k+1)\tilde{D}+kd]} \\ &\exists\,\Gamma_1 \in \mathbb{R}^{\tilde{D}\times\tilde{D}},\ \Gamma_{2\ell} \in \mathbb{R}^{\tilde{D}\times\tilde{D}},\ \forall l = 1:k \\ &\exists\,\Pi_1, \Pi_2 \in \mathbb{R}^{\tilde{q}\times\tilde{q}},\ \Pi_{2\ell} \in \mathbb{R}^{\tilde{D}\times\tilde{D}}, \Pi_{2\ell} \in \mathbb{R}^{d\times d}\end{aligned} \tag{34}$$

such that

$$G_{\bar{X}} = G_{\bar{X}}\Gamma_1, \tag{35a}$$

$$\Pi_1 F_{\bar{X}} = F_{\bar{X}}\Gamma_1, \quad \Pi_1\theta_{\bar{X}} = \theta_{\bar{X}}, \tag{35b}$$

$$\Pi_2 F_{\bar{X}} = F_{\bar{X}}\Gamma_{2\ell}, \quad \Pi_2\theta_{\bar{X}} = \theta_{\bar{X}}, \tag{35c}$$

$$\begin{aligned}&\Pi_{3\ell}\tilde{c}_{\bar{X}} = \tilde{c}_{\bar{X}}, \\ &\mathrm{blk}(\Pi_{3\ell})\Phi_k(I_k\otimes G_{\bar{X}}) = \Phi_k(I_k\otimes G_{\bar{X}})\mathrm{blk}(\Gamma_{2\ell}),\end{aligned} \tag{35d}$$

$$\begin{aligned}&\Pi_4(\mathbf{1}_k\otimes\tilde{c}_{\bar{X}}) = \tilde{c}_{\bar{X}}, \\ &\Pi_4\Phi_k(I_k\otimes G_{\bar{X}}) = G_{\bar{X}} + \phi^\top(I_k\otimes G_{\bar{X}})\mathrm{blk}(\Gamma_{2\ell}),\end{aligned} \tag{35e}$$

$$|\Gamma_1|\mathbf{1} \le 1, |\Gamma_{2\ell}|\mathbf{1} \le 1 \tag{35f}$$

with $\phi^\top = \begin{bmatrix} -A_\circ^{-1} & \dots & (-A_\circ^{-1})^{k+1}\end{bmatrix}$, a shorthand for the bottom-right corner of matrix Φ_{k+1}. □

Proof: Applying (19) for set inclusion (25) with notation (32) we arrive at:

$$c_{\bar{X}} - c_{\bar{X}} = \begin{bmatrix} G_{\bar{X}} & \mathbf{0}_{d\times(k+1)\bar{D}}\end{bmatrix}\beta, \tag{36a}$$

$$\begin{bmatrix} G_{\bar{X}} & \mathbf{0}_{d\times k\bar{D}}\end{bmatrix} = \begin{bmatrix} G_{\bar{X}} & \mathbf{0}_{d\times(k+1)\bar{D}}\end{bmatrix}\Gamma, \tag{36b}$$

$$\Pi\begin{bmatrix} I_{k+1}\otimes F_{\bar{X}} \\ \Phi_k\cdot(I_k\otimes G_{\bar{X}})\end{bmatrix} = \begin{bmatrix} I_{k+2}\otimes F_{\bar{X}} \\ \Phi_{k+1}\cdot(I_{k+1}\otimes G_{\bar{X}})\end{bmatrix}\Gamma, \tag{36c}$$

$$\Pi\begin{bmatrix}\mathbf{1}_{k+1}\otimes\theta_{\bar{X}} \\ \mathbf{1}_k\otimes(A_\circ^{-1} - I)c_{\bar{X}}\end{bmatrix} = \begin{bmatrix}\mathbf{1}_{k+2}\otimes\theta_{\bar{X}} \\ \mathbf{1}_{k+1}\otimes(A_\circ^{-1} - I)c_{\bar{X}}\end{bmatrix} + \begin{bmatrix} I_{k+2}\otimes F_{\bar{X}} \\ \Phi_{k+1}\cdot(I_{k+1}\otimes G_{\bar{X}})\end{bmatrix}\beta, \tag{36d}$$

$$|\Gamma|\mathbf{1} + |\beta| \le \mathbf{1}. \tag{36e}$$

From (36a) we have that[4] $\beta = 0$. Applying this and with notation ϕ, (36c)–(36d) become

$$\Pi\begin{bmatrix} I_{k+1}\otimes F_{\bar{X}} \\ \Phi_k\cdot(I_k\otimes G_{\bar{X}})\end{bmatrix} = \begin{bmatrix} F_{\bar{X}} & 0 \\ 0 & I_{k+1}\otimes F_{\bar{X}} \\ 0 & \Phi_k(I_k\otimes G_{\bar{X}}) \\ G_{\bar{X}} & \phi^\top(I_k\otimes G_{\bar{X}})\end{bmatrix}\Gamma, \tag{37a}$$

$$\Pi\begin{bmatrix}\mathbf{1}_{k+1}\otimes\theta_{\bar{X}} \\ \mathbf{1}_k\otimes(A_\circ^{-1} - I)c_{\bar{X}}\end{bmatrix} = \begin{bmatrix}\theta_{\bar{X}} \\ \mathbf{1}_{k+1}\otimes\theta_{\bar{X}} \\ \mathbf{1}_k\otimes(A_\circ^{-1} - I)c_{\bar{X}} \\ (A_\circ^{-1} - I)c_{\bar{X}}\end{bmatrix}. \tag{37b}$$

[2]For ease of computation, the closed-loop state matrix $A_\circ$ is assumed to be invertible. The general case may be handled, for example, through an SVD decomposition which isolates the matrix's kernel.

[3]A reasonable assumption since these sets almost always come from magnitude bounds.

[4]In fact, it can be any vector in the kernel of $\begin{bmatrix} G_{\bar{X}} & \mathbf{0}_{d\times(k+1)\bar{D}}\end{bmatrix}$ but in view of (36e), $\beta = 0$ is still the best choice.

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Take:

$$\Gamma = \begin{bmatrix} e_1 \otimes \Gamma_1 & \mathrm{blk}(\Gamma_{2\ell}) \end{bmatrix}^\top, \; \Pi = \begin{bmatrix} e_1^\top \otimes \Pi_1 & 0 \\ I_{k+1} \otimes \Pi_2 & 0 \\ 0 & \mathrm{blk}(\Pi_{3\ell}) \\ 0 & \Pi_4 \end{bmatrix}$$

in blocks of appropriate dimensions, with $1 \leq \ell \leq k$ and $e_1^\top$ the fist row of identity matrix I_{k+1}. Introducing this notation in (37) and using (35b)–(35e) we observe that (36c) and (36d) hold. Noting that (36b) and (36e) are reformulations of (35a) and, respectively (35f) concludes the proof. ■

Corollary 3: Set inclusion (27) holds if

$$\exists\, \Gamma \in \mathbb{R}^{\bar{D}\times(k+1)\bar{D}},\; \beta \in \mathbb{R}^{\bar{D}}, \Pi \in \mathbb{R}^{\bar{q}\times[(k+1)\bar{q}+kd]} \tag{38}$$

such that

$$A_\circ^{k+1} c_{\bar{X}} - c_{\bar{X}} = G_{\bar{X}}\beta, \tag{39a}$$

$$\begin{bmatrix} A_\circ^{k+1} G_{\bar{X}} & 0_{d\times k\bar{D}} \end{bmatrix} = G_{\bar{X}}\Gamma, \tag{39b}$$

$$\Pi \begin{bmatrix} I_{k+1} \otimes F_{\bar{X}} \\ \Phi_k \cdot (I_k \otimes G_{\bar{X}}) \end{bmatrix} = F_{\bar{X}}\Gamma, \tag{39c}$$

$$\Pi \begin{bmatrix} \mathbf{1}_{k+1} \otimes \theta_{\bar{X}} \\ \mathbf{1}_k \otimes (A_\circ^{-1} - I)c_{\bar{X}} \end{bmatrix} = \theta_{\bar{X}} + F_{\bar{X}}\beta, \tag{39d}$$

$$|\Gamma|\mathbf{1} + |\beta| \leq \mathbf{1}. \tag{39e}$$

Proof: Set inclusion (27) may be equivalently written as $A_\circ^{k+1}\Omega_k \subseteq \bar{\mathcal{X}}$. Then, using (5) and (19), with notation (32), we arrive at (39), thus concluding the proof. ■

Remark 4: Simply enumerating a list of constraints does not guarantee the feasibility of a problem. Case in point, in Corollary 3, there are $n_{\text{constr}} = (k+1)D^2 + D + (k+1)q^2 + kdq$ variables that have to verify the $n_{\text{eqs}} = d + (k+1)dD + (k+1)qD + q$ equalities that appear in (39a)–(39d). For the case later given in Sec. V, where $q = d$, we have that $n_{\text{constr}} = (k+1)(D^2 + d^2) + kd^2 + D > n_{\text{eqs}} = 2(k+1)dD + 2d$ is always true which means that there are enough degrees of freedom to tackle (39e). ♦

V. Illustrative Example

The question arises whether exploiting the constrained zonotope structure of the MPI set is actually more efficient than the standard polyhedral interpretation. The response is a qualified 'yes' as it strongly depends on choices made along the procedure. We have identified three distinct steps:

- Step 1) checking the set inclusion ((25), (27) or (28)) via one of the validations methods ((16), (17) or (19), possibly with the improvements of Cor. 2 and 3);
- Step 2) the application of a generator reduction procedure as discussed in Remark 3 (at each step of the set recurrence (24a) and/or, at its end);
- Step 3) the actual cost of embedding the MPI set into the larger MPC problem and solving it.

Since each of these steps may greatly influence the total computation time we analyze them separately, with various implementation choices. To do so, we consider the 'CSE' dynamics from the $COMPl_eib$ [15] whose size is proportional with a parameter (counting the coupled springs, dampers and masses) that can be changed arbitrarily. The mass positions and velocities define the state; its input are the two forces exerted at the ends of the coupled springs chain [15]. The continuous-time model is given by

$$\dot{x} = \underbrace{\begin{bmatrix} 0_{\ell\times\ell} & I_\ell \\ -k\mu^{-1}K_c & -\mu^{-1}\delta I_\ell \end{bmatrix}}_{\mathrm{A}\in\mathbb{R}^{2\ell\times 2\ell}} x + \underbrace{\begin{bmatrix} 0_{\ell\times 2} \\ \mu^{-1}D_c \end{bmatrix}}_{\mathrm{B}\in\mathbb{R}^{2\ell\times 2}} u, \tag{40}$$

and
$$K_c = \underbrace{\begin{bmatrix} 1 & -1 & \cdots & 0 & 0 \\ -1 & -2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -2 & -1 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}}_{\in\mathbb{R}^{\ell\times\ell}}, \; D_c = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\in\mathbb{R}^{\ell\times 2}}.$$

The numerical values considered are $\mu = 4,\ \delta = 1,\ k = 1$ and the resulted system is discretized with the forward Euler method for a sampling time of 1 sec. The bounding sets are $\mathcal{X} = \{x \in \mathbb{R}^{2\ell} : \|x\|_\infty \leq 1\}$, $\mathcal{U} = \{u \in \mathbb{R}^2 : \|u\|_\infty \leq 1\}$, which, put in formulation (30), gives $\overline{\mathcal{X}} = \langle 0_{2\ell}, \begin{bmatrix} I_{2\ell} & 0_{2\ell\times(2\ell+2)} \end{bmatrix}, \begin{bmatrix} K & -I_2 \end{bmatrix}, 0_2 \rangle$, with the static gain matrix K being the result of a pole placement procedure. For a parameter range of $\ell \in \{2, \ldots, 9\}$ we test now the steps identified earlier. First, we show in Fig. 1 the time spent checking

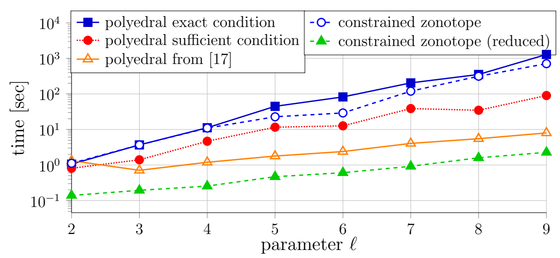

Fig. 1: Time to check the set inclusion condition.

the set inclusion (Step 1). We consider the exact and sufficient conditions (27) and (28) for the polyhedral representation (and , respectively) and for the constrained zonotope representation, implemented as in Corollary 3 (and , respectively). The 'online-redundancy check' method [9], , is introduced as comparison (the fastest, in our implementation from [9]).

We observe comparable testing times between the non-minimal constrained zonotope and the standard polyhedral implementation for (27). The difference becomes clear in favor of the former for $\ell \geq 5$. For a fair comparison, the same number of steps ($\bar{k}$, obtained from [9], is considered).

The reduction procedure (Step 2), as presented in [17, Theorem 2] requires the test of a linear program for each combination of equality and generator and is surprisingly costly[5]. We implemented a scheme where a limited reduction procedure (only for the newly introduced equalities) is carried at the intermediary steps and a costly, 'all cases', reduction is done at the end of the set recurrence. The results are illustrated in Fig. 2 where, in addition to the computation

[5]Worth mentioning, Algorithm 1 also given in [17] does not help as it outputs conservative estimations of the generator bounding interval.

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times for MPT3 and our own implementation from [9] we depict the time spent in the intermediary steps (), in the final step () and the number of LPs solved ($\times$ 100), to give an insight in the computational effort. The

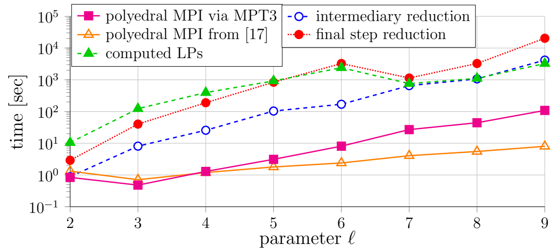

Fig. 2: Redundancy check in the constrained zonotope case versus polyhedral implementations.

result of these reductions steps is illustrated in Fig. 3 where we observe indeed a marked reduction in the representation's complexity. Lastly, problem size is not a perfect indicator of

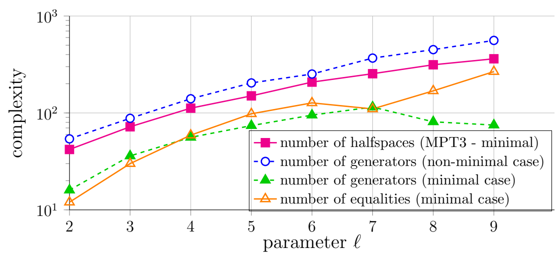

Fig. 3: Complexity of the set representation.

computation time since solver heuristics greatly influence it. We depict in Fig. 4 the total run time (Step 3) for solving the MPC problem (horizon $N = 10$, $Q = I_{2\ell}$, $R = I_2$, simulation horizon of 100 steps) in the nominal – without MPI (), with MPI as a polyhedral set in irredundant description () and MPI given as a constrained zonotope in redundant () and minimal () descriptions. We

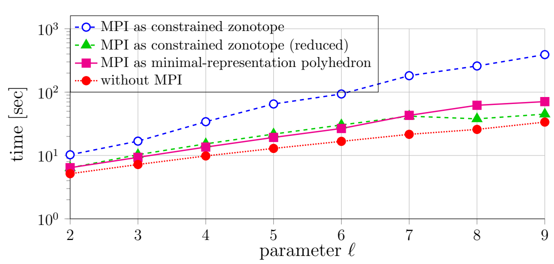

Fig. 4: Computation time for the MPC problem.

observe that the constrained zonotope representation reduces the computation time if put in its minimal form but it is less efficient than the standard polyhedral approach otherwise. The constrained zonotope variants (redundant and reduced forms) implement the exact stop condition (27) and the generator reduction procedure from Remark 3.

The optimizations are solved in MATLAB, using CasADi [19], on a computer with a 2.1GHz CPU with 12 cores and 32GB of RAM, while the zonotopic and polyhedral sets are implemented by using CORA [20] and MPT3 [21].

VI. Conclusions

We have adapted the maximal positive invariant (MPI) procedure to the constrained zonotope case. We have shown improvements in representation complexity and reduction in run time for the associated MPC problem. Future work focuses on exploiting the symmetry of the problem, better heuristics for the redundant generator elimination and embedding in the robust MPC framework.

References

[1] S. Sun, A. Romero, P. Foehn, E. Kaufmann, and D. Scaramuzza, "A Comparative Study of Nonlinear MPC and Differential-Flatness-Based Control for Quadrotor Agile Flight," *IEEE Transactions on Robotics*, vol. 38, no. 6, p. 3357–3373, Dec. 2022.

[2] H. Chen and F. Allgower, "A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability," *Automatica*, vol. 34, no. 10, pp. 1205–1217, 1998.

[3] N. T. Nguyen and I. Prodan, "Stabilizing a multicopter using an NMPC design with a relaxed terminal region," *IFAC-PapersOnLine*, vol. 54, no. 6, pp. 126–132, 2021.

[4] S. V. Raković and W. S. Levine, *Handbook of model predictive control*. Springer, 2018.

[5] S. V. Raković and S. Zhang, "The implicit maximal positively invariant set," *IEEE Transactions on Automatic Control*, vol. 68, no. 8, pp. 4738–4753, 2023.

[6] M. Lazar and M. Tetteroo, "Computation of terminal costs and sets for discrete–time nonlinear MPC," *IFAC-PapersOnLine*, vol. 51, no. 20, pp. 141–146, 2018.

[7] F. Rußwurm, W. Esterhuizen, K. Worthmann, and S. Streif, "On MPC without terminal conditions for dynamic non-holonomic robots," *IFAC-PapersOnLine*, vol. 54, no. 6, pp. 133–138, 2021.

[8] E. Gilbert and K. Tan, "Linear systems with state and control constraints: the theory and application of maximal output admissible sets," *IEEE Tran. on Automatic Control*, vol. 36, no. 9, pp. 1008–1020, 1991.

[9] B. Gheorghe, F. Stoican, and I. Prodan, "On the complexity of computing the maximal positive invariant set," in *Proceedings of the European Control Conference*, 2024, accepted, available at https://shorturl.at/uCFJX.

[10] K. Fukuda, "From the zonotope construction to the Minkowski addition of convex polytopes," *Journal of Symbolic Computation*, vol. 38, no. 4, pp. 1261–1272, 2004.

[11] J. K. Scott, D. M. Raimondo, G. R. Marseglia, and R. D. Braatz, "Constrained zonotopes: A new tool for set-based estimation and fault detection," *Automatica*, vol. 69, pp. 126–136, 2016.

[12] M. Althoff, O. Stursberg, and M. Buss, "Computing reachable sets of hybrid systems using a combination of zonotopes and polytopes," *Nonlinear analysis: hybrid systems*, vol. 4, no. 2, pp. 233–249, 2010.

[13] A. Ingimundarson, J. M. Bravo, V. Puig, J. Alamo, and P. Guerra, "Robust fault detection using zonotope-based set-membership consistency test," *International journal of adaptive control and signal processing*, vol. 23, no. 4, pp. 311–330, 2009.

[14] S. Sadraddini and R. Tedrake, "Linear encodings for polytope containment problems," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, p. 4367–4372.

[15] F. Leibfritz, "COMPleib: COnstrained Matrix–optimization Problem library – a collection of test examples for nonlinear semidefinite programs, control system design and related problems." Department of Mathematics, Univ. Trier, Germany, Tech. Rep., 2006.

[16] G. M. Ziegler, *Lectures on polytopes*. Springer Science & Business Media, 2012, vol. 152.

[17] V. Raghuraman and J. P. Koeln, "Set operations and order reductions for constrained zonotopes," *Automatica*, vol. 139, p. 110204, 2022.

[18] K. Fukuda, "Polyhedral computation," 2020, publisher: Department of Mathematics, Institute of Theoretical Computer Science ETH Zurich.

[19] J. A. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, "CasADi: a software framework for nonlinear optimization and optimal control," *Mathematical Programming Computation*, vol. 11, no. 1, pp. 1–36, 2019.

[20] M. Althoff, "An introduction to CORA 2015," in *Proc. of the 1st and 2nd Workshop on Applied Verification for Continuous and Hybrid Systems*, December 2015.

[21] M. Herceg, M. Kvasnica, C. Jones, and M. Morari, "Multi-Parametric Toolbox 3.0," in *Proc. of the European Control Conference*, Zürich, Switzerland, July 17–19 2013, pp. 502–510.