On Computing the Minkowski Difference of Zonotopes

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Abstract

Zonotopes are becoming an increasingly popular set representation for formal verification techniques. This is mainly due to their efficient representation and their favorable computational complexity of important operations in high-dimensional spaces. In particular, zonotopes are closed under Minkowski addition and linear maps, which can be very efficiently implemented. Unfortunately, zonotopes are not closed under Minkowski difference for dimensions greater than two. However, we present an algorithm that efficiently computes a halfspace representation of the Minkowski difference of two zonotopes. In addition, we present an efficient algorithm that computes an approximation of the Minkowski difference in generator representation. The efficiency of the proposed solution is demonstrated by numerical experiments. These experiments show a reduced computation time in comparison to that when first the halfspace representation of zonotopes is obtained and the Minkowski difference is performed subsequently.

1 Introduction

Zonotopes have recently enjoyed a lot of popularity as a set representation for formal methods in engineering and computer science. One of the main reasons is that zonotopes can be efficiently stored in computer systems, especially when dealing with high-dimensional problems. Another important reason is that zonotopes are closed under linear maps and Minkowski addition as shown in Sec. 2. However, to the best knowledge of the author, there exists no published algorithm for computing the Minkowski difference of zonotopes. Such an algorithm would open up new possibilities for formal methods in engineering and computer science. We first review existing literature on zonotopes for formal verification and state estimation of continuous dynamic systems, formal verification of computer programs, and problems in computational geometry.

Today, zonotopes are widely used to compute the reachable set of continuous dynamic systems, i.e., the set of states that are reachable by the solution of a differential equation when the initial state, inputs, and parameters are uncertain within bounded sets. Early works on this problem used a variety of set representations, such as polytopes [13], ellipsoids [37], oriented hyper-rectangles [49], and level sets [41]. All the mentioned set representations are either not closed under important operations (e.g. ellipsoids and oriented hyper-rectangles are not closed under Minkowski addition) or are computationally inefficient in high-dimensional spaces (e.g. polytopes and level sets). For linear continuous systems in particular, zonotopes provide an excellent compromise between accuracy and efficiency as first demonstrated by Kühn [36]. Later, Girard [24] extended the approach to uncertain inputs. This work also developed a method for computing the reachable set for all times except isolated points in time. It has led to a wrapping-free algorithm, i.e., an algorithm for which over-approximations are not accumulating, in [25].

Further extensions of the previously mentioned work are the use for systems with uncertain parameters [6], nonlinear ordinary differential equations [7], and nonlinear differential-algebraic systems [5]. Reachable sets have been applied to many technical realizations, such as automated vehicles [4], human-robot collaboration [45], and smart grids [2].

Zonotopes are also used to rigorously estimate the states of dynamical systems as an alternative to observers that optimize with respect to the best estimate, such as Kalman filters. One of the first works that uses zonotopes for state-bounding observers is [14]. As with reachability analysis, this work has been extended to nonlinear systems in [1,15] and systems with uncertain parameters [39]. A further application of zonotopes for continuous dynamic systems is model-predictive control with guaranteed stability [11]. Advances in using zonotopes for control and guaranteed state estimation are summarized in [38].

In computer science, zonotopes are used as abstract domains in abstract interpretation for static program analysis [26], whose implementation details can be found in [21]. An extension of zonotopes with (sub-)polyhedric domains can be found in [22]. Further extensions of zonotopes exist, but for the brevity of the literature review, they are not presented. Zonotopes are also used in automated theorem provers as a set representation [31]. Finally, zonotopes are used as bounding volumes to facilitate fast collision detection algorithms [28].

Zonotopes are also an active research area in computational geometry. However, this paper mainly focuses on computational aspects on Minkowski difference targeted for applications in engineering and computer science. Thus, recent developments regarding combinatorics and relations to other mathematical problems are only briefly reviewed. This is not to say that this review is complete. The association of zonotopes with higher Bruhat orders is described in [17]. Coherence and enumeration of tilings of 3-zonotopes are addressed in [8]. Properties of zonotopes with large 2D-cuts are derived in [48] with examples provided by the *Ukrainian easter eggs*. In [12], an enclosure of zonoids by zonotopes is derived, which has the same support values for fixed directions. A bound for the number of generators with equal length of a zonotope required to enclose a ball up to a certain Hausdorff distance is obtained in [10]. In [18] it is shown that the problem of maximizing a quadratic form in n binary variables when the underlying (symmetric) matrix is positive semidefinite, can be reduced to searching the extreme points of a zonotope. The problem of listing all extreme points of a zonotope is addressed in [20].

As shown above, the applications of zonotopes are manifold. Most works use zonotopes to enclose other sets, resulting in over-approximations. However, for many applications the ability to compute the Minkowski difference¹ is essential. The use of the Minkowski difference can be exemplified through the computation of invariance sets of dynamic systems [40], reachability analysis [46], robust model predictive control [47], optimal control [33], robotic path planning [9], robust interval regression analysis [32], and cooperative games [16]. Providing an algorithm for computing the Minkowski difference of zonotopes would open up many new possibilities. Minkowski difference is well-known and rather straightforward to implement for polytopes for which open source implementations exist, see e.g. [30]. Since a zonotope is a special case of a polytope, one could use the algorithms for polytopes. However, this is less efficient compared to the novel computation for zonotopes as presented later.

The paper is organized as follows: We recall in Sec. 2 some preliminaries on zonotopes and provide the definition of the Minkowski difference. In Sec. 3, algorithms are presented for com-

 $^{^{1}}$ Note that the term $Pontryagin\ difference$ is often used as a synonym for Minkowski difference. However, we use the term $Minkowski\ difference$ throughout this paper.

puting the halfspace representation of the Minkowski difference of zonotopes. The resulting halfspace representation is used to obtain an approximation of the Minkowski difference in generator representation in Sec. 4. The performance of the approach is demonstrated by numerical experiments in Sec. 5.

2 Preliminaries and Problem Statement

We first recall the representation of a zonotope. Throughout this paper, we index elements of vectors and matrices by subscripts and enumerate vectors or matrices by superscripts in parentheses to avoid confusion with the exponentiation of a variable. For instance $A_{ij}^{(k)}$ is the element of the $i^{\rm th}$ row and $j^{\rm th}$ column of the $k^{\rm th}$ matrix $A^{(k)}$.

Definition 1 (Zonotope (G-Representation)). Zonotopes are parameterized by a center $c \in \mathbb{R}^n$ and generators $g^{(i)} \in \mathbb{R}^n$ and defined for $c \in \mathbb{R}^n$, $g^{(i)} \in \mathbb{R}^n$ as

$$\mathcal{Z} = \left\{ c + \sum_{i=1}^{p} \beta_i \, g^{(i)} \middle| \beta_i \in [-1, 1] \right\}. \tag{1}$$

The order of a zonotope is defined as $\varrho = \frac{p}{n}$.

We write in short $\mathcal{Z}=(c,g^{(1)},\ldots,g^{(p)})$. Zonotopes are a compact way of representing sets in high-dimensional spaces. More importantly, linear maps $M\otimes\mathcal{Z}:=\{Mz|z\in\mathcal{Z}\}\ (M\in\mathbb{R}^{q\times n})$ and Minkowski addition $\mathcal{Z}_1\oplus\mathcal{Z}_2:=\{z_1+z_2|z_1\in\mathcal{Z}_1,z_2\in\mathcal{Z}_2\}$, as required in many of the applications mentioned in Sec. 1, can be computed efficiently and exactly [35]. Given $\mathcal{Z}_1=(c,g^{(1)},\ldots,g^{(p_1)})$ and $\mathcal{Z}_2=(d,h^{(1)},\ldots,h^{(p_2)})$ one can efficiently compute

$$\mathcal{Z}_1 \oplus \mathcal{Z}_2 = (c + d, g^{(1)}, \dots, g^{(p_1)}, h^{(1)}, \dots, h^{(p_2)}),
M \otimes \mathcal{Z}_1 = (M c, M g^{(1)}, \dots, M g^{(p_1)}).$$
(2)

Note that in the remainder of this paper, the symbol for set-based multiplication is often omitted for simplicity of notation. Further, one or both operands of set-based operations can be singletons and set-based multiplication has precedence over Minkowski addition. A zonotope can be interpreted in three ways, see e.g. [29, p. 364] and [50, Sec. 7.3]. All interpretations are now introduced, as we use each one in this paper to show certain properties concisely.

Minkowski addition of line segments (first interpretation) A zonotope can be interpreted as the Minkowski addition of line segments $l^{(i)} = [-1,1] g^{(i)}$, which is visualized step-by-step for \mathbb{R}^2 in Fig. 1.

Projection of a hypercube (second interpretation) A zonotope can be interpreted as the projection of a p-dimensional unit hypercube $\mathcal{C} = [-1,1]^p$ onto the n-dimensional space by the matrix of generators $G = [g^{(1)}, \ldots, g^{(p)}]$, which is then translated to the center c: $\mathcal{Z} = c \oplus G \otimes \mathcal{C}$. We write in short $\mathcal{Z} = (c, G)$.

Polytopes whose faces are centrally symmetric (third interpretation) A zonotope can be interpreted as a polytope whose j-faces are centrally symmetric. As later shown in Sec. 3, the facets ((n-1)-faces) are obtained by choosing particular halfspaces $\{x \in \mathbb{R}^n | c^{(i)}^T x \leq d_i\}$, $c^{(i)} \in \mathbb{R}^n$, $d_i \in \mathbb{R}$ whose intersection forms the halfspace representation (H-representation) of a zonotope:

$$\mathcal{Z}_H = \left\{ x \in \mathbb{R}^n \middle| C \, x \le d \right\},\tag{3}$$

where $C = \begin{bmatrix} c^{(1)}, & \dots, & c^{(q)} \end{bmatrix}^T$ and $d = \begin{bmatrix} d_1, & \dots, & d_q \end{bmatrix}^T$.

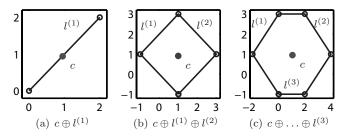


Figure 1: Step-by-step construction of a zonotope.

Related to the Minkowski addition is the Minkowski difference. Given the minuend \mathcal{Z}_m and the subtrahend \mathcal{Z}_s of equal dimension, the Minkowski difference is defined as (see [42])

$$\mathcal{Z}_m \ominus \mathcal{Z}_s = \{x \in \mathbb{R}^n | x \oplus \mathcal{Z}_s \subseteq \mathcal{Z}_m\},$$

such that $(\mathcal{Z}_m \ominus \mathcal{Z}_s) \oplus \mathcal{Z}_s \subseteq \mathcal{Z}_m$. We refer to $\mathcal{Z}_m \ominus \mathcal{Z}_s$ as the difference. When $\tilde{\mathcal{Z}}_m = \mathcal{A} \oplus \mathcal{Z}_s$, where \mathcal{A} is an arbitrary set, we have $(\tilde{\mathcal{Z}}_m \ominus \mathcal{Z}_s) \oplus \mathcal{Z}_s = \tilde{\mathcal{Z}}_m$. An alternative definition (see [23,42]) is

$$\mathcal{Z}_m \ominus \mathcal{Z}_s = \bigcap_{z_s \in \mathcal{Z}_s} (\mathcal{Z}_m - z_s). \tag{4}$$

The Minkowski difference can be obtained by translations along generators in Theorem 1, which is based on Lemma 1.

Lemma 1 (Minkowski Difference from Finitely Many Intersections). When the subtrahend \mathcal{Z}_s is convex, it suffices to compute the Minkowski difference from intersections of translations by the vertices $v^{(i)} \in \mathcal{V}$ of \mathcal{Z}_s :

$$\mathcal{Z}_m\ominus\mathcal{Z}_s=igcap_{v^{(i)}\in\mathcal{V}}(\mathcal{Z}_m-v^{(i)})$$

Proof. See [34, Theorem 2.1].

Theorem 1 (Minkowski Difference from Generators). When the sets \mathcal{Z}_m and \mathcal{Z}_s are zonotopes, it suffices to apply the following recursive procedure using only the generators $g^{(s,i)}$ of $\mathcal{Z}_s = (c^{(s)}, g^{(s,1)}, \ldots, g^{(s,p_s)})$ to obtain $\mathcal{Z}_m \ominus \mathcal{Z}_s$:

$$\mathcal{Z}_{int}^{(1)} = \mathcal{Z}_m - c^{(s)}$$

$$\forall i = 1 \dots p_s : \quad \mathcal{Z}_{int}^{(i+1)} = (\mathcal{Z}_{int}^{(i)} + g^{(s,i)}) \cap (\mathcal{Z}_{int}^{(i)} - g^{(s,i)})$$

$$\mathcal{Z}_m \ominus \mathcal{Z}_s = \mathcal{Z}_{int}^{(p_s + 1)}$$

Proof. As shown in [16, eq. (2)], it generally holds for sets \mathcal{A} , \mathcal{B} , and \mathcal{C} that

$$\mathcal{A} \ominus (\mathcal{B} \oplus \mathcal{C}) = (\mathcal{A} \ominus \mathcal{B}) \ominus \mathcal{C}. \tag{5}$$

Let us rewrite $\mathcal{Z}_m \ominus \mathcal{Z}_s = \mathcal{Z}_m \ominus (c^{(s)} \bigoplus_{i=1}^{p_s} [-1,1] \otimes g^{(s,i)})$. By recursively applying (5) we obtain

$$\mathcal{Z}_m \ominus \mathcal{Z}_s = \left(\left(\left(\mathcal{Z}_m - c^{(s)} \right) \ominus \left[-1, 1 \right] \otimes g^{(1)} \right) \ominus \dots \right) \ominus \left[-1, 1 \right] \otimes g^{(p_s)}. \tag{6}$$

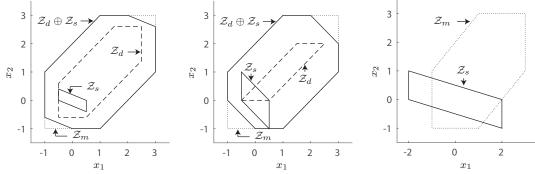
The Minkowski difference with $[-1,1] \otimes g^{(s,i)}$ can be further simplified according to Lemma 1 by only considering the extreme cases, such that for a set \mathcal{A} we have

$$\mathcal{A} \ominus [-1, 1] \otimes g^{(s,i)} = (\mathcal{A} + g^{(s,i)}) \cap (\mathcal{A} - g^{(s,i)}). \tag{7}$$

Inserting (7) into (6) results in the theorem to be proven.

To provide the reader with a better understanding of the Minkowski difference of zonotopes, we show three distinctive examples in Fig. 2: (a) the zonotope order of $\mathcal{Z}_d = \mathcal{Z}_m \ominus \mathcal{Z}_s$ equals the one of the minuend, (b) the order is reduced, and (c) the result is the empty set. We choose

$$\begin{split} & \mathcal{Z}_{m} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right), \\ & \mathcal{Z}_{s,1} = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \right), \mathcal{Z}_{s,2} = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \right), \mathcal{Z}_{s,3} = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \right). \end{split}$$



(a) $\mathcal{Z}_d = \mathcal{Z}_m \ominus \mathcal{Z}_{s,1}$; result has the (b) $\mathcal{Z}_d = \mathcal{Z}_m \ominus \mathcal{Z}_{s,2}$; result has a (c) $\mathcal{Z}_d = \mathcal{Z}_m \ominus \mathcal{Z}_{s,3}$; result is empty. same order as \mathcal{Z}_m .

Figure 2: Results of different Minkowski differences.

One can observe in Fig. 2(a) that all halfspaces of \mathcal{Z}_m remain for $\mathcal{Z}_m \ominus \mathcal{Z}_{s,1}$. The result of $\mathcal{Z}_m \ominus \mathcal{Z}_{s,2}$ in Fig. 2(b) does not require all halfspaces of \mathcal{Z}_m . Finally, Fig. 2(c) shows that $\mathcal{Z}_m \ominus \mathcal{Z}_{s,3} = \emptyset$. The halfspace representation of the Minkowski difference of zonotopes is addressed in the next section.

3 Halfspace Conversion of Zonotopes

As it is later shown in Sec. 4, zonotopes are not closed under Minkowski difference (unless in the two-dimensional case). One possibility to obtain the Minkowski addition, is to first

convert both zonotopes into halfspace representation and then use standard algorithms for obtaining the Minkowski difference. In this section, we propose a more efficient algorithm for obtaining the halfspace representation of the Minkowski difference. For that purpose, we require the n-dimensional cross product, which is an extension of the well-known cross product of two three-dimensional vectors. The resulting vector is orthogonal to all n-1 n-dimensional vectors. Definition 2 (n-dimensional cross product (see [43])). Given are n-1 vectors $h^{(i)} \in \mathbb{R}^n$ which are stored in a matrix $H = [h^{(1)}, \ldots, h^{(n-1)}] \in \mathbb{R}^{n \times n-1}$. We further denote by $H^{[i]} \in \mathbb{R}^{n-1 \times n-1}$ the matrix H, where the ith row is removed. The cross product nX(H) of the vectors stored in H is defined as

$$y = nX(H) = \left[\det(H^{[1]}), \dots, (-1)^{i+1} \det(H^{[i]}), \dots, (-1)^{n+1} \det(H^{[n]})\right]^T.$$

From now on, we assume that all generators of each zonotope are not aligned. If two aligned generators would exist, e.g. $\gamma g^{(i)} = g^{(j)}, \ \gamma \in \mathbb{R}$, one could easily adjust $g^{(i)} := (\gamma + 1)g^{(i)}$ and remove $g^{(j)}$. For a general zonotope, the generator matrix G is of dimension $n \times p$. The normal vector of each facet is obtained from the n-dimensional cross product of n-1 generators, which have to be selected from p generators for each non-parallel facet, so that one obtains $2\binom{p}{n-1}$ facets [27, Lemma 3.1]. This is always possible since we assume without loss of generality that all generators are not aligned. The generators that span a facet are obtained by canceling p-n+1 generators from the generator matrix G, which is denoted by $G^{(\gamma,\ldots,\eta)}$, where γ,\ldots,η are the p-n+1 indices of the generators that are taken out of G.

Theorem 2 (H-Representation of Zonotopes). The halfspace representation $Cx \leq d$ of a zonotope (c,G) with p independent generators is

$$C = \begin{bmatrix} C^+ \\ -C^+ \end{bmatrix}, \quad C^+ = \begin{bmatrix} C^+_1 \\ \vdots \\ C^+_{\nu} \end{bmatrix}, \quad C^+_i = \frac{nX(G^{\langle \gamma, \dots, \eta \rangle})^T}{\|nX(G^{\langle \gamma, \dots, \eta \rangle})\|_2},$$
$$d = \begin{bmatrix} d^+ \\ d^- \end{bmatrix} = \begin{bmatrix} C^+ c + \Delta d \\ -C^+ c + \Delta d \end{bmatrix}, \quad \Delta d = \sum_{\nu=1}^p |C^+ g^{(\nu)}|.$$

The index i varies from 1 to $\nu = \binom{p}{n-1}$.

Proof. The i^{th} facet is spanned by n-1 generators, which are obtained by canceling p-n+1 generators with indices γ, \ldots, η from G, which is denoted by $G^{\langle \gamma, \ldots, \eta \rangle}$. This is illustrated for a two-dimensional example in Fig. 3. As illustrated in Fig. 4, the normal vector of this facet is obtained by the normalized n-dimensional cross product (see Def. 2):

$$C_i^+ = nX(G^{\langle \gamma, \dots, \eta \rangle})^T / \|nX(G^{\langle \gamma, \dots, \eta \rangle})\|_2.$$

It is sufficient to compute ν normal vectors denoted by a superscript '+', as the remaining ν normal vectors denoted by a superscript '-' differ only in sign due to the central symmetry of zonotopes. A possible point $x^{(i)}$ on the i^{th} halfspace is obtained by adding generators in the direction of the normal vector to the center (see Fig. 3):

$$x^{(i)} = c + \sum_{\upsilon=1}^p \mathrm{sgn}(C_i^+ g^{(\upsilon)}) g^{(\upsilon)},$$

where sgn() is the sign function returning the sign of a value. Note that the generators spanning the i^{th} facet are not required to reach the halfspace. To keep the result simple, however, we

add them in the above formula, since they only translate $x^{(i)}$ on the facet. Since the elements d_i^+ are the scalar products of any point on the i^{th} halfspace with its normal vector, we obtain

$$d_i^+ = C_i^+ x^{(i)} = C_i^+ \Big(c + \sum_{v=1}^p \operatorname{sgn}(C_i^+ g^{(v)}) g^{(v)} \Big) = C_i^+ c + \sum_{v=1}^p |C_i^+ g^{(v)}| = C_i^+ c + \Delta d_i.$$

Analogously, the values for d_i^- are obtained.

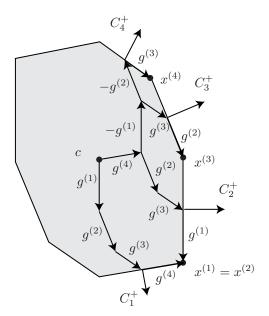


Figure 3: Various generator additions to reach corresponding facets in two dimensions. Only the generators not spanning the facet are required to reach the facet.

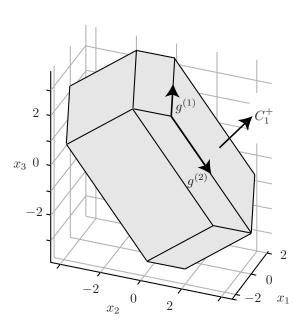


Figure 4: Normal vector of a facet of a threedimensional zonotope spanned by two generators.

Next, we directly obtain the halfspace representation of the intersection of two zonotopes, which are identical, except that one of them is translated by a vector 2h.

Lemma 2 (H-Representation of Intersection of Translated Zonotopes). Given are two zonotopes $\mathcal{Z}_o = (c - h, G)$ and $\mathcal{Z}_t = (c + h, G)$. The halfspace representation $C x \leq d$ of the intersection $\mathcal{Z}_o \cap \mathcal{Z}_t$ has an identical C matrix as presented in Theorem 2, but a changed d vector

$$d = \begin{bmatrix} d^+ \\ d^- \end{bmatrix} = \begin{bmatrix} C^+ c + \Delta d - \Delta d_{trans} \\ -C^+ c + \Delta d - \Delta d_{trans} \end{bmatrix}, \quad \Delta d = \sum_{v=1}^p |C^+ g^{(v)}|, \quad \Delta d_{trans} = |C^+ h|. \tag{8}$$

Proof. Since the translated zonotope \mathcal{Z}_t has the same generators as the original zonotope \mathcal{Z}_o , both halfspace representations of \mathcal{Z}_t and \mathcal{Z}_o will have the same normal vectors, see Theorem 2. For each normal vector C_i^+ , one of the corresponding halfspaces from \mathcal{Z}_o or \mathcal{Z}_t is a subset of the other one as shown in Fig. 5. This, of course, is analogous for C_i^- . Consequently, the number of required normal vectors for the intersection is unchanged compared to Theorem 2 as also illustrated in Fig. 5.

However, the intersection results in a subset of \mathcal{Z}_o , which is equivalent in reducing the values of d_i^+ and d_i^- by $\Delta d_{trans,i}$. From the duality of intersection of translated sets and Minkowski difference (see (4)) and after introducing 0_n as the *n*-dimensional vector of zeros, we have that

$$(0_n, h) \oplus (\mathcal{Z}_o \cap \mathcal{Z}_t) = (0_n, h) \oplus \left(\underbrace{(c - h, G) \cap (c + h, G)}_{(c, G) \ni (0_n, h)}\right) = (c, G). \tag{9}$$

The Minkowski addition of $(0_n, h)$ in (9) can be seen as an additional generator so that $\Delta d = \sum_{v=1}^{p} |C^+ g^{(v)}| + |C^+ h|$ in (8). To compensate for the change in Δd , we choose $\Delta d_{trans} = |C^+ h|$ in (8).

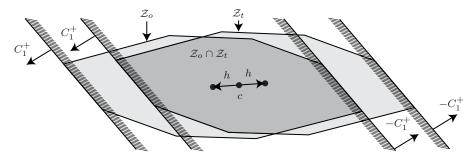


Figure 5: Either the halfspace belonging to C_1^+ or C_1^- is redundant for \mathcal{Z}_o when computing the interaction $\mathcal{Z}_o \cap \mathcal{Z}_t$. The same applies to \mathcal{Z}_t

The above lemma is used to compute the halfspace representation of $\mathcal{Z}_m \ominus \mathcal{Z}_s$.

Theorem 3 (H-Representation of Minkowski Difference). Given are the two zonotopes $\mathcal{Z}_m = (c^{(m)}, G^{(m)})$ and $\mathcal{Z}_s = (c^{(s)}, G^{(s)})$. A halfspace representation $C x \leq d$ of the Minkowski difference $\mathcal{Z}_m \ominus \mathcal{Z}_s$ has an identical C matrix as Theorem 2, but a changed d vector:

$$d = \begin{bmatrix} d^{+} \\ d^{-} \end{bmatrix} = \begin{bmatrix} C^{+} (c^{(m)} - c^{(s)}) + \Delta d - \Delta d_{trans} \\ -C^{+} (c^{(m)} - c^{(s)}) + \Delta d - \Delta d_{trans} \end{bmatrix},$$

$$\Delta d = \sum_{v=1}^{p_{m}} |C^{+} g^{(m,v)}|, \quad \Delta d_{trans} = \sum_{v=1}^{p_{s}} |C^{+} g^{(s,v)}|.$$
(10)

Proof. As shown in Theorem 1, the Minkowski difference can be computed from finitely many translations of \mathcal{Z}_m by the generators of \mathcal{Z}_s . The halfspace representation of intersecting \mathcal{Z}_m with one translated version of itself has been shown in Lemma 2. Repeated application of Lemma 2 results in the summation of values of Δd_{trans} to $\Delta d_{trans} = \sum_{v=1}^{p_s} |C^+ g^{(s,v)}|$, thus proving the theorem.

In the next section, we use the halfspace representation of the Minkowski difference to obtain an approximation in generator representation.

4 Generator Removal and Contraction

As shown in Sec. 2, it is possible that not all generators of the minuend \mathcal{Z}_m are preserved when performing a Minkowski difference. In the extreme case, the result is the empty set,

and all generators have been removed. We present how to detect the generators that can be removed, followed by an algorithm that contracts the remaining generators such that the result of the Minkowski difference is approximated. Unfortunately, we cannot exactly represent the Minkowski difference by a zonotope.

Proposition 1 (Zonotopes are not closed under Minkowski difference). Zonotopes are not closed under Minkowski difference, i.e., given the zonotopes \mathcal{Z}_m , \mathcal{Z}_s , the set $\mathcal{Z}_d = \mathcal{Z}_m \ominus \mathcal{Z}_s$ is not a zonotope.

Proof. We prove the proposition by a counterexample with $\mathcal{Z}_m = (\mathbf{0}, G_m)$ and $\mathcal{Z}_s = (\mathbf{0}, G_s)$, where $\mathbf{0}$ is a vector of zeros and

$$G_m = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad G_s = \frac{1}{3} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

As one can see in the plot of $\mathcal{Z}_m \ominus \mathcal{Z}_s$ in Fig. 6 obtained by CORA [3] and the MPT toolbox [30], not all faces are centrally symmetric as it has to be for zonotopes.

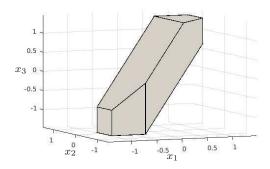


Figure 6: The Minkowski difference of two zonotopes that is not a zonotope in general.

4.1 Generator Removal

The overall algorithm for generator removal is shown in Alg. 1. First, the normal vectors of the H-representation of the minuend \mathcal{Z}_m are stored together with the indices $\alpha_i, \ldots, \kappa_i$ of generators $g^{(m,i)}$ $(i=1,\ldots,p_m)$, which span the corresponding facets. This is denoted for the i^{th} halfspace by $\langle C_i, [\alpha_i, \ldots, \kappa_i] \rangle$, where $\alpha_i, \ldots, \kappa_i \in \mathbb{N}^+$ are the indices of the generators. The information for all normal vectors in C^+ is stored in a set \mathcal{C}^{full} such that $\forall i: \langle C_i, [\alpha_i, \ldots, \kappa_i] \rangle \in \mathcal{C}^{full}$.

Next, the halfspace representation of the intersection is computed as presented in Theorem 3, and the minimum H-representation is obtained using linear programming [19, Sec. 2.21]. This results in the set of halfspaces after the intersection $\hat{C} \subseteq \mathcal{C}^{full}$, where $\mathcal{C}^{red} = \mathcal{C}^{full} \setminus \hat{C}$ is the set of redundant halfspaces. The generator indices $[\alpha_j, \ldots, \kappa_j]$ are stored in a vector $\operatorname{ind}^{(j)}$ for the j^{th} redundant halfspace C_j^{red} . The entries of $\operatorname{ind}^{(j)}$ are 1 if the generator contributes to the halfspace and 0 otherwise (see Alg. 1, line 3-5):

$$\forall i = 1, \dots, p_m : \quad \operatorname{ind}_i^{(j)} = \begin{cases} 1, & \text{if } i \in \{\alpha_j, \dots, \kappa_j\}, \\ 0, & \text{otherwise.} \end{cases}$$

The generators, which are no longer required, are selected based on $ind^{(j)}$ by the following proposition.

Proposition 2 (Generator removal). The i^{th} generator of the minuend \mathcal{Z}_m with p_m generators in n-dimensional space is no longer required for the difference $\mathcal{Z}_m \ominus \mathcal{Z}_s$ if and only if (see Alg. 1, line 8-13)

$$\mathtt{ind}_i = 2 inom{p_m-1}{n-2}, \quad \mathtt{ind} = \sum_{j=1}^{\varsigma} \mathtt{ind}^{(j)},$$

where ς is the number of redundant halfspaces. Please note that ind_i refers to the i^{th} entry of the vector ind .

Proof. Each facet is spanned by n-1 generators. By fixing one generator for a facet, one can still select n-2 generators from p_m-1 generators, such that each generator spans $2\binom{p_m-1}{n-2}$ facets. If a generator is removed, those $2\binom{p_m-1}{n-2}$ facets no longer exist, such that a generator can only be removed if and only if all $2\binom{p_m-1}{n-2}$ facets to which it contributes are redundant.

The indices removedGeneratorIndex from Alg. 1 are used to remove generators such that the matrix $\hat{G}^{(m)} = [g^{(m,ind_1)}, \dots, g^{(m,ind_v)}]$ of remaining generators is obtained, where $ind_1, \dots, ind_v \notin \texttt{removedGeneratorIndex}$ and $ind_i \in \{1, \dots, p_m\}$.

Algorithm 1 Generator Removal

```
Input: \varsigma redundant generators \langle C_k^{red}, [\alpha_k, \dots, \kappa_k] \rangle.
Output: removedGeneratorIndex
  1: ind = [0, \ldots, 0]
 2: for k = 1 ... \varsigma do
           for i = 1 \dots p_m do
               \operatorname{ind}_{i}^{(k)} = \begin{cases} 1, & \text{if } i \in \{\alpha_{k}, \dots, \kappa_{k}\}, \\ 0, & \text{otherwise.} \end{cases}
  4:
  5:
           ind = ind + ind^{(k)}
  7: end for
  8: removedGeneratorIndex \leftarrow \emptyset
 9: for i=1\dots p_m do
          if \operatorname{ind}_i = 2\binom{p_m-1}{n-2} then
                removedGeneratorIndex = removedGeneratorIndex \cup i
11:
           end if
12:
13: end for
```

4.2 Generator Contraction

So far we have removed the generators that are not required anymore. It remains to adjust the lengths of the remaining generators so that the spanned zonotope approximates the Minkowski difference $\mathcal{Z}_m \ominus \mathcal{Z}_s$. We introduce the vector of stretching factors $\mu = [\mu_1, \ldots, \mu_{p_m}]$. Since we have already removed the generators that do not span any of the remaining halfspaces in

Sec. 4.1, no stretching factor has length 0 so that $\forall i : 0 < \mu_i \leq 1$. To obtain the proposed approximation, let us first obtain the \hat{d}^+ vector of non-redundant halfspaces as

$$\hat{d}^{+} = \hat{C}^{+} \left(c^{(m)} - c^{(s)} \right) + \Delta \hat{d} - \Delta \hat{d}_{trans}$$

$$= \hat{C}^{+} \left(c^{(m)} - c^{(s)} \right) + \sum_{v=1}^{p_{m}} |\hat{C}^{+} g^{(m,v)}| - \sum_{v=1}^{p_{s}} |\hat{C}^{+} g^{(s,v)}|.$$
(11)

Only in special cases it is possible that the approximating zonotope has identical normal vectors of the constraining halfspaces than the exact result. The main idea is to compute the \hat{d}^+ values of the halfspace representation for a given stretching vector μ when assuming that the normal vectors are identical, which they are not:

$$\hat{d}^{+} = \hat{C}^{+} \left(c^{(m)} - c^{(s)} \right) + \sum_{\nu=1}^{\hat{p}_{m}} |\hat{C}^{+} \hat{g}^{(m,\nu)}| \mu_{\nu} = \hat{C}^{+} \left(c^{(m)} - c^{(s)} \right) + |\hat{C}^{+} \hat{G}^{(m)}| \mu. \tag{12}$$

Next, we demand that the \hat{d}^+ values of the minuend after stretching in (12) are identical to the ones of the Minkowski difference $\mathcal{Z}_m \ominus \mathcal{Z}_s$ in (11) resulting in

$$\hat{C}^{+}(c^{(m)} - c^{(s)}) + |\hat{C}^{+}\hat{G}^{(m)}|\mu = \hat{C}^{+}(c^{(m)} - c^{(s)}) + \sum_{v=1}^{p_{m}} |\hat{C}^{+}g^{(m,v)}| - \sum_{v=1}^{p_{s}} |\hat{C}^{+}g^{(s,v)}|$$

$$|\hat{C}^{+}\hat{G}^{(m)}|\mu = \sum_{v=1}^{p_{m}} |\hat{C}^{+}g^{(m,v)}| - \sum_{v=1}^{p_{s}} |\hat{C}^{+}g^{(s,v)}|$$

$$\mu = |\hat{C}^{+}\hat{G}^{(m)}|^{-1} (\sum_{v=1}^{p_{m}} |\hat{C}^{+}g^{(m,v)}| - \sum_{v=1}^{p_{s}} |\hat{C}^{+}g^{(s,v)}|).$$

We can now state the approximating generator representation of $\mathcal{Z}_m \ominus \mathcal{Z}_s$ as

$$(c^{(m)} - c^{(s)}, \mu_1 \,\hat{g}^{(m,1)}, \dots, \mu_{\hat{p}_m} \,\hat{g}^{(m,\hat{p}_m)})$$

where \hat{p}_m is the number of generators $\hat{g}^{(m,i)}$. Since the proposed approximation can be obtained from solving a set of linear equations, its implementation is computationally cheap. Determining the set of non-redundant halfspaces is the most time consuming process, as discussed in the next section, which presents the results of numerical experiments.

5 Numerical Experiments

In this section, the performance of computing the Minkowski difference of random zonotopes with various orders and for a different number of dimensions is assessed. The computation times are compared with those of polytopes, whose halfspace representation is obtained according to Theorem 2. We additionally assess the computation time when combining Minkowski difference with Minkowski addition, as it is required in many applications, see e.g. [46]. Please note that the results are not directly comparable, since the results when using zonotopes are approximative.

5.1 Random Generation of Zonotopes

To fairly assess the performance, random zonotopes are generated. Each scenario has the following parameters:

- The dimension n of the Euclidean space \mathbb{R}^n .
- The order ϱ_s of the subtrahend \mathcal{Z}_s .
- The order ϱ_m of the minuend \mathcal{Z}_m .
- The maximum length $l^{s,max}$ of the generators $||g^{(s,i)}||_2$ of \mathcal{Z}_s is selected as 1 without loss of generality.
- The maximum length $l^{m,max}$ of the generators $\|g^{(m,i)}\|_2$ of \mathcal{Z}_m is selected as $3\frac{\varrho_s}{\varrho_m}$. The fraction $\frac{\varrho_s}{\varrho_m}$ ensures that the size of the minuend and subtrahend are comparable when using different orders. The factor of 3 is selected to balance the three cases as shown in Fig. 2: (a) No order reduction of the minuend, (b) order reduction of the minuend, and (c) empty sets.

Given the above parameters, a simple method for obtaining random generators would be to first randomly generate each entry of a generator by uniformly sampling values within [-1,1]. The generator would then be stretched to a length that is uniformly distributed within $[0,l^{max}]$. However, the directions of the resulting generators would not be uniformly distributed. Thus, we first generate points that are uniformly distributed on a unit hypersphere according to [44]. Next, the generators are defined as the vector from the origin to the points on the hypersphere, which are stretched to achieve the desired length of the generators $l^{(i)} = ||g^{(i)}||_2$. This is uniformly distributed within $[0,l^{max}]$.

5.2 Comparison with Polytope Implementation

In this subsection, we compare the computation times of our own MATLAB implementation compared to those of the MATLAB toolbox Multi-Parametric Toolbox 3.0 with its default settings [30]. For each scenario specified by the dimension n and the zonotope orders ϱ_s and ϱ_m , we randomly generate 100 instances of the Minkowski difference $\mathcal{Z}_m \ominus \mathcal{Z}_s$, as well as the combined use of Minkowski difference and addition $(\mathcal{Z}_m \ominus \mathcal{Z}_s) \oplus \mathcal{Z}_s$. The latter expression is not meaningful, but the average computation times would not change if we add a set other than \mathcal{Z}_s with the same order and dimension. Tab. 1 lists the results on (i) the average computation time for each instance for $\mathcal{Z}_m \ominus \mathcal{Z}_s$ and $(\mathcal{Z}_m \ominus \mathcal{Z}_s) \oplus \mathcal{Z}_s$, (ii) the average order of the resulting zonotopes of $\mathcal{Z}_m \ominus \mathcal{Z}_s$, and (iii) on the percentage of empty results of $\mathcal{Z}_m \ominus \mathcal{Z}_s$. All computations are performed on a laptop with an Intel i7-3520M CPU with 2.90GHz and 4 cores. Parallelization of algorithms is not used in all evaluations.

We first discuss the results for the Minkowski difference in Tab. 1. It can be observed that the zonotope implementation is in all cases faster than the polytope implementation of the *Multi-Parametric Toolbox 3.0*, e.g. for 6 dimensions and an order of 2 for the minuend and the subtrahend, the computation with zonotopes is more than 40 times faster. The computation times for polytopes of larger dimensions have been declared as *did not finish* (dnf) since a single set computation takes more than two hours, which amounts to more than one week for 100 instances. Most of the computation time for the Minkowski difference of zonotopes is spent

Table 1: Average computational time of Minkowski difference and its combination with Minkowski addition for various scenarios. Each Scenario is run 100 times and we declare a scenario as *did not finish* (dnf) when a single set computation takes more than two hours, which amounts to more than one week for 100 instances.

				$\mathcal{Z}_m\ominus\mathcal{Z}_s$				$(\mathcal{Z}_m\ominus\mathcal{Z}_s)\oplus\mathcal{Z}_s$	
	Order		Computat	Computation Time		Resulting Sets		Computation Time	
	\mathcal{Z}_m	\mathcal{Z}_s	Zonotope	Polytope	Avg. Order	Empty Sets	Zonotope	Polytope	
2	2	2	$0.0107\mathrm{s}$	$0.0385\mathrm{s}$	1.6026	22%	$0.0108\mathrm{s}$	$0.0583\mathrm{s}$	
2	4	2	$0.0152\mathrm{s}$	$0.0599\mathrm{s}$	3.1141	8%	$0.0153\mathrm{s}$	$0.0872\mathrm{s}$	
2	2	4	$0.0098\mathrm{s}$	$0.0450\mathrm{s}$	1.6866	33%	$0.0099\mathrm{s}$	$0.0663\mathrm{s}$	
2	4	4	$0.0166\mathrm{s}$	$0.0683\mathrm{s}$	3.1398	7%	$0.0167\mathrm{s}$	$0.1022\mathrm{s}$	
4	2	2	$0.0825\mathrm{s}$	$0.7160\mathrm{s}$	1.6279	57%	$0.0826\mathrm{s}$	291.38 s	
4	4	2	$16.886\mathrm{s}$	$35.644\mathrm{s}$	3.1175	17%	$16.887\mathrm{s}$	dnf	
4	2	4	$0.0779\mathrm{s}$	$42.484\mathrm{s}$	1.6106	48%	$0.0816\mathrm{s}$	dnf	
4	4	4	$18.692\mathrm{s}$	$169.78\mathrm{s}$	3.1152	11%	$18.693\mathrm{s}$	dnf	
6	2	2	$12.957{\rm s}$	$549.64\mathrm{s}$	1.6742	78%	$12.958\mathrm{s}$	dnf	
6	2.5	2	$1115.2\mathrm{s}$	dnf	2.0873	58%	$1115.2\mathrm{s}$	dnf	
6	2	2.5	$26.234\mathrm{s}$	dnf	1.6667	76%	$26.235\mathrm{s}$	dnf	
6	2.5	2.5	$3235.2\mathrm{s}$	dnf	2.0799	52%	$3235.2\mathrm{s}$	dnf	
8	1.8	1.8	$235.25\mathrm{s}$	dnf	1.5000	91%	$235.25\mathrm{s}$	dnf	

computing the minimum halfspace representation [19, Sec. 2.21]. Linear programming is used to determine generators that need to be removed as shown in Sec. 4.1.

The difference in computation time is substantially different when combining Minkowski difference and Minkowski addition. For dimensions equal or larger than 4 we obtain the result *did not finish* (dnf), except when the order of the minuend and the subtrahend are 2 for 4 dimensions. In this case, however, the zonotope computation is already 3500 times faster. Thus, zonotopes are several orders of magnitude faster when combining Minkowski difference with Minkowski addition.

The approximation error has not been evaluated since obtaining the volume for higherdimensional polytopes is infeasible.

6 Conclusions

To the best knowledge of the author, this is the first work that presents an approach for approximating the Minkowski difference of zonotopes. Although the Minkowski difference can be computed exactly for the halfspace representation of zonotopes, doing so would result in the loss of the generator representation of a zonotope. It would require to perform subsequent computations, such as linear maps or Minkowski addition, using a halfspace representation. For such operations in particular, however, zonotopes in generator representation are very efficient

and easily outperform the halfspace representation by several orders of magnitude in highdimensional spaces. Even though the presented approximative Minkowski difference approach is computationally more expensive than linear maps and Minkowski addition for zonotopes, it is significantly faster than state-of-the-art algorithms for halfspace representations.

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