ENGN 0310 - Fall 2021

Final Exam

Instructions

- Your solutions must be submitted in the form of a single file to Canvas (pdf preferred), with the addition of a Mathematica file if you so choose.

- You can consult your handwritten class notes, the notes posted on the ENGN0310 course website, and the course textbook. Consulting any other sources is not allowed, which includes any non-ENGN0310 websites, and other people. Your exam should be entirely your own work and consulting other students is not allowed.

- Late submissions will not be accepted without a doctor's (or a similar) note. If problems arise during submission to Canvas you can email your solutions to Andrew_Bagnoli@brown.edu within the time limit of your exam for full credit.

- Once you begin your exam you have three hours to submit your work. Note that all exams are due by 11:59 pm on Wednesday, December 15$^{\text{th}}$ and cannot be submitted after that date, so if you start your exam after 8:59 pm you will not have the full time to complete your exam.

- In the interest of fairness to those taking the exam at times when Professor Kesari and/or TAs are not available questions cannot be answered during the exam, whether in person or by email.

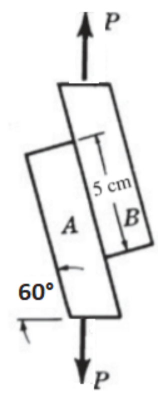

Figure 1: Figure for Problem 1.

Problem 1 (10 Points)

Consider the figure above, where two bars are connected by a weld of length 5 cm and depth 38 mm. The depth is in the direction into the plane of the paper. The ultimate tensile strength of the weld is 40 MPa and the ultimate shear strength is 15 MPa. Assuming that failure will always occur in the weld, how will the weld fail and what is the maximum allowable value of P?

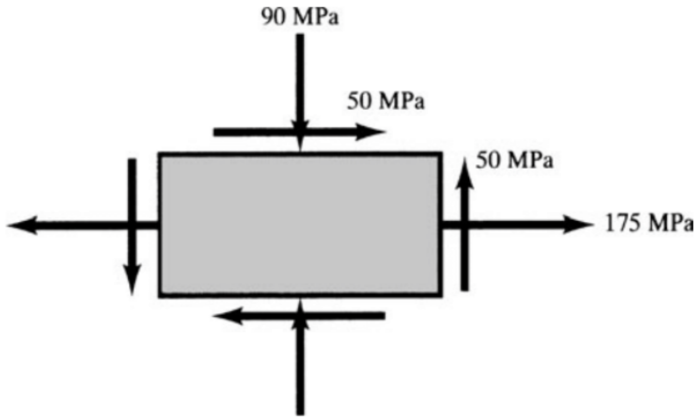

Figure 2: Figure for Problem 2.

Problem 2 (10 Points)

a) (2 pts) Given the loading in the figure above what is the stress tensor?

b) (4 pts) For this given stress tensor what are the maximum and minimum values of the scalar part of the normal component of the traction vector (i.e., of $\sigma(\phi)$), and what are the corresponding angles?

c) (4 pts) For this given stress tensor what are the maximum and minimum values of the magnitude of the shear component of the traction vector (i.e., of $|\tau(\phi)|$), and what are the corresponding angles?

d) (2 pts) For this given stress tensor draw the Mohr Circle.

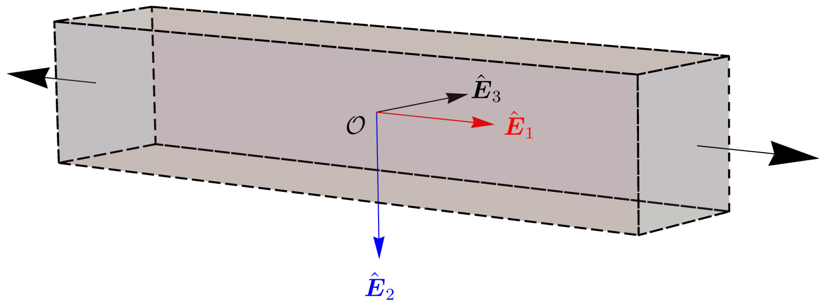

Figure 3: Figure for Problem 3

Problem 3 (25 Points)

The first topic we studied in the course was the tensile loading of bars. For a bar with its length in the $\hat{\boldsymbol{E}}_1$ direction we have been using the equation

$$E(X_1)A(X_1)u_1'(X_1) = F(X_1) \tag{1}$$

where $F(X_1)$ the axial force. It is defined as

$$F(X_1) = \boldsymbol{F}\left(X_1, \hat{\boldsymbol{E}}_1\right) \cdot \hat{\boldsymbol{E}}_1, \tag{2}$$

where $\boldsymbol{F}\left(X_1, \hat{\boldsymbol{E}}_1\right)$ is the net force acting on the internal surface $\Gamma\left(X_1, \hat{\boldsymbol{E}}_1\right)$. The quantity $E(X_1)$ is the Young's modulus at the bar's cross-section located at X_1, i.e, at $\Gamma(X_1)$. The quantity $A(X_1)$ is the area of $\Gamma(X_1)$. The quantity $u_1(X_1)$ is the displacement of $\Gamma(X_1)$ in the $\hat{\boldsymbol{E}}_1$ direction, and X_1 is the coordinate of a material particle in the direction of the bar's length.

In the following we will derive (1) for the case in which the Young's modulus and the cross-sectional area are constant along the length of the bar. For concreteness, we can take the bar to be of length $2l$ and have a square cross-section, with sides of length a. Two of the bar's lateral surfaces are perpendicular to $\pm\hat{\boldsymbol{E}}_2$ direction, while the other two lateral surfaces are perpendicular to $\pm\hat{\boldsymbol{E}}_3$ direction.

Take the deformation in the bar to be

$$\breve{x}_1(X_1, X_2, X_3) = \lambda_1 X_1 \tag{3}$$
$$\breve{x}_2(X_1, X_2, X_3) = \lambda_2 X_2 \tag{4}$$
$$\breve{x}_3(X_1, X_2, X_3) = \lambda_3 X_3 \tag{5}$$

1. (1pt) What is the displacement field? That is, what are the functions $\breve{u}_1(X_1, X_2, X_3)$, $\breve{u}_2(X_1, X_2, X_3)$, and $\breve{u}_3(X_1, X_2, X_3)$? Recall that

$$\breve{u}_i(X_1, X_2, X_3) = \breve{x}_i(X_1, X_2, X_3) - X_i,$$

where $i = 1,\ 2,\ 3.$

2. (3.5 pts) What is the strain field $\breve{\boldsymbol{\epsilon}}(X_1, X_2, X_3)$? Determining the strain field is equivalent to determining the components $\breve{\epsilon}_{ij}(X_1, X_2, X_3)$ where $i,\ j = 1,\ 2,\ 3.$ Recall the definition of the strain components:

$$\breve{\epsilon}_{ij}(X_1, X_2, X_3) = \frac{1}{2}\left(\frac{\partial \breve{u}_i(X_1, X_2, X_3)}{\partial X_j} + \frac{\partial \breve{u}_j(X_1, X_2, X_3)}{\partial X_i}\right), \tag{6}$$

where $i,\ j = 1,\ 2,\ 3$

3. (4.5 pts) Let us assume that the bar is composed of a homogeneous, linear elastic solid. What is the stress field $\breve{\boldsymbol{\sigma}}(X_1, X_2, X_3)$? Determining the stress field is equivalent to determining the components $\breve{\sigma}_{ij}(X_1, X_2, X_3)$ where $i,\ j = 1,\ 2,\ 3.$ Recall the definition of a homogeneous linear elastic solid. The stress and strain components in a homogeneous linear elastic solid are given by the Hooke's Law, which states that

$$\breve{\sigma}_{11}(X_1, X_2, X_3) = 2\mu\breve{\epsilon}_{11}(X_1, X_2, X_3) + \lambda \mathrm{Tr}(\breve{\boldsymbol{\epsilon}}(X_1, X_2, X_3)) \tag{7}$$
$$\breve{\sigma}_{22}(X_1, X_2, X_3) = 2\mu\breve{\epsilon}_{22}(X_1, X_2, X_3) + \lambda \mathrm{Tr}(\breve{\boldsymbol{\epsilon}}(X_1, X_2, X_3)) \tag{8}$$
$$\breve{\sigma}_{33}(X_1, X_2, X_3) = 2\mu\breve{\epsilon}_{33}(X_1, X_2, X_3) + \lambda \mathrm{Tr}(\breve{\boldsymbol{\epsilon}}(X_1, X_2, X_3)) \tag{9}$$
$$\breve{\sigma}_{23}(X_1, X_2, X_3) = 2\mu\breve{\epsilon}_{23}(X_1, X_2, X_3) \tag{10}$$
$$\breve{\sigma}_{13}(X_1, X_2, X_3) = 2\mu\breve{\epsilon}_{13}(X_1, X_2, X_3) \tag{11}$$
$$\breve{\sigma}_{12}(X_1, X_2, X_3) = 2\mu\breve{\epsilon}_{12}(X_1, X_2, X_3) \tag{12}$$
$$\breve{\sigma}_{32}(X_1, X_2, X_3) = \breve{\sigma}_{23}(X_1, X_2, X_3) \tag{13}$$
$$\breve{\sigma}_{31}(X_1, X_2, X_3) = \breve{\sigma}_{13}(X_1, X_2, X_3) \tag{14}$$
$$\breve{\sigma}_{21}(X_1, X_2, X_3) = \breve{\sigma}_{12}(X_1, X_2, X_3) \tag{15}$$

where

$$\mathrm{Tr}(\breve{\boldsymbol{\epsilon}}(X_1, X_2, X_3)) = \breve{\epsilon}_{11}(X_1, X_2, X_3) + \breve{\epsilon}_{22}(X_1, X_2, X_3) + \breve{\epsilon}_{33}(X_1, X_2, X_3)$$

and λ and μ are Lame's constants. The constant μ is also called the shear modulus.

4. (7 pts) Solve for $\lambda_2 - 1$ and $\lambda_3 - 1$ in terms of $\lambda_1 - 1$ by stipulating that the lateral surfaces of the bar are traction free. The vectors normal to the lateral surfaces are $\pm\hat{\boldsymbol{E}}_2$, and $\pm\hat{\boldsymbol{E}}_3$.

5. (3.5 pts) What is the traction vector on $\Gamma(X_1, \hat{\boldsymbol{E}}_1)$? That is determine $\breve{\boldsymbol{t}}(X_1, X_2, X_3, \hat{\boldsymbol{E}}_1)$.

6. (2 pts) What is the net force on $\Gamma(X_1, \hat{\boldsymbol{E}}_1)$? That is, what is $\boldsymbol{F}(X_1, \hat{\boldsymbol{E}}_1)$?

$$\boldsymbol{F}(X_1, \hat{\boldsymbol{E}}_1) = \int_{\Gamma(X_1)} \breve{\boldsymbol{t}}(X_1, X_2, X_3, \hat{\boldsymbol{E}}_1)\, d\Gamma \tag{16}$$

7. (3.5 pts) Take dot product with $\hat{\boldsymbol{E}}_1$ on both sides of the equation arrived at in the last part and simplify the resulting scalar equation using that

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{17}$$

$$\mu = \frac{E}{2(1+\nu)}, \tag{18}$$

$$\tag{19}$$

and $\lambda_1 - 1 = u_1'(X_1)$ to get (1).

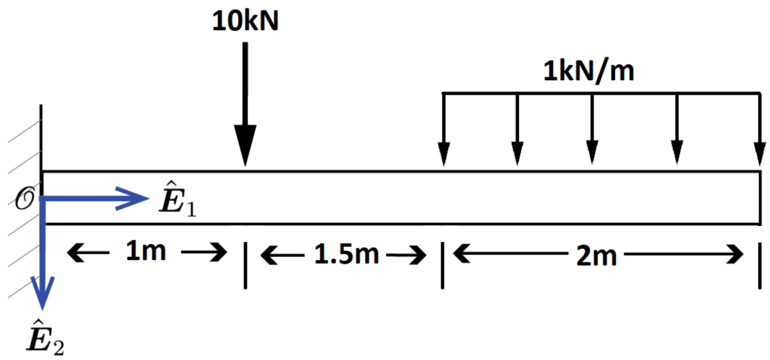

Figure 4: Figure for Problem 4.

Problem 4 (15 Points)

a) (10 pts) Draw the bending moment diagram for the beam in the figure above.

b) (5 pts) If the cross section of the beam is square, with side length 25 mm, what is the maximum value of $\sigma_{11}(X_1, X_2, X_3)$ in the bar?

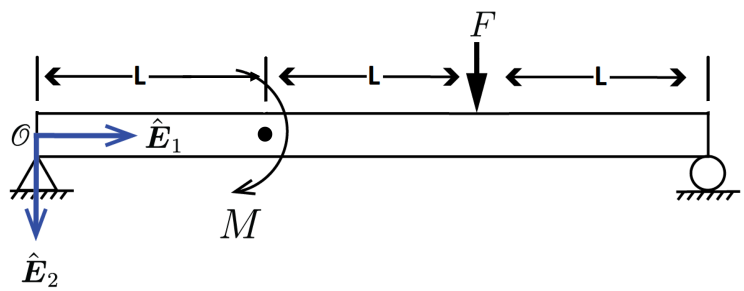

Figure 5: Figure for Problem 5.

Problem 5 (20 Points)

a) (2 pts) What are the reaction forces and/or moments, if any, for the beam in the figure above (in terms of F, M, and L).

b) (8 pts) What is the bending moment as a function of X_1 (The distance from the origin in the $\hat{\boldsymbol{E}}_1$ direction)? That is, what is $M(\cdot)$

c) (10 pts) What is the deflection of the neutral axis as a function of X_1 (i.e., what is $y(\cdot)$), assuming that the Young's Modulus E and the second moment of area I are constant throughout the bar?

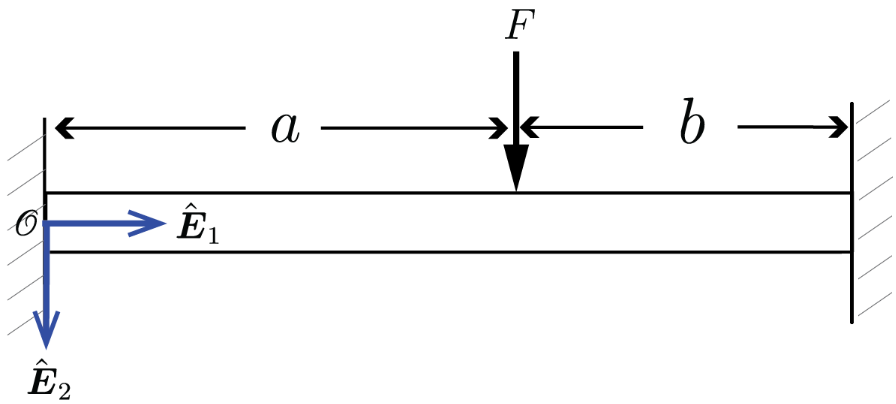

Figure 6: Figure for Problem 6

Problem 6 (20 Points)

a) (6 pts) What are the reaction forces and/or moments, if any, for the beam in the figure above (in terms of F, a, and b). (6 pts) What is the bending moment as a function of X_1. That is, what is $M(X_1)$? The only parameters in your function $M(\cdot)$ should be F, a, and b. (8 pts) Assuming that the Young's Modulus E and the second moment of area I are constant throughout the beam what is the neutral axis' deflection as a function of X_1, i.e., what is $y(X_1)$? The only parameters in your function $y(\cdot)$ should be F, a, b, E, and I.