1. **Binomial** (n, p), $\quad (x = 0, 1, \ldots, n)$ R:x`binom`

$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

(a) $\boxed{F_{\text{Bin}(n,p)}(x) = F_{\text{Beta}(n-\lfloor x \rfloor, \lfloor x \rfloor + 1)}(1-p)}$

Check the R function below:

```
set.seed(0)
n = 1 + round(100*runif(1))
p = runif(1)
x = min(50*runif(1), n)
c(x, n, p)

# Compare the following two:
c(pbinom(x, size=n, prob=p), pbeta(1-p, n-floor(x), floor(x)+1) )
```

```
# Type your own value for n, p, and k.
# Assume that X is from Bin(n,p)
n = 4L
p = 1/2
k = 2
# p = runif(1)
# k   = sample(1L:(n-1L), size=1L)

# True value of P[X <= k]
pbinom(k, size=n, prob=p)

# Simulated value of  P[X <= k]
set.seed(1)
ITER = 100000L
countA = countB = 0

for ( i in seq_len(ITER) ) {
     U = runif(n)
     if ( sum(U <= p) <= k ) countA = countA + 1

     Y = sort(U)
     if( Y[k+1L] > p ) countB = countB + 1
}
countA / ITER
countB / ITER
```

Let U_i for $i = 1, 2, \ldots, n$ be a random sample from the uniform between zero and one. Their order statistics are given by $U_{(1)} \le U_{(2)} \le \cdots \le U_{(n)}$. Then we have

$$P(X \le k) = P(U_{(k+1)} > p).$$

Let $V_i = 1 - U_i$. Then V_i are also a random sample from the uniform between zero and one. Then we have $U_{(i)} = 1 - V_{(n-i+1)}$ so that $U_{(k+1)} = 1 - V_{(n-k)}$. Thus, we have

$$P(X \le k) = P(1 - V_{(n-k)} > p) = P(V_{(n-k)} < 1 - p).$$

For notational convenience, we let $m = n - k$. Then the pdf of $V_{(m)}$ is given by

$$f_m(v) = P(V_{(m)} = v) = \frac{n!}{(m-1)!(n-m)!} v^{m-1}(1-v)^{n-m},$$

that is,

$$f_{n-k}(v) = P(V_{(m)} = n - k) = \frac{n!}{(n-k-1)!k!} v^{n-k-1}(1-v)^k.$$

Thus, we have

$$\begin{aligned}
P(V_{(n-k)} < 1 - p) &= \int_0^{1-p} f_{n-k}(v)dv \\
&= \int_0^{1-p} \frac{n!}{(n-k-1)!k!} v^{n-k-1}(1-v)^k dv \\
&= \int_0^{1-p} \frac{\Gamma(n+1)}{\Gamma(n-k)\Gamma(k+1)} v^{(n-k)-1}(1-v)^{(k+1)-1} dv \\
&= \int_0^{1-p} f_{\text{Beta}(n-k,k+1)}(1-p)dv \\
&= F_{\text{Beta}(n-k,k+1)}(1-p).
\end{aligned}$$

(b) $\boxed{F_{\text{Bin}(n,p)}(r-1) = 1 - F_{X,\text{NB}(r,p)}(n-r)}$

Check the R function below:

```
# Binomial and Negative binomial
n = 2L
r = 2L
p = 1/4
c(pbinom(r-1, n,p), 1-pnbinom(n-r,r,p) )

# NOTE: wiki (Cumulative distribution function)
# https://en.wikipedia.org/wiki/Negative_binomial_distribution
# It works with p=1/2.
k = 2
c(pnbinom(k, r, p), pbinom(k, k+r,p) )
```

Let X be a binomial random variable with n and p. Let Y be a negative binomial random variable with pmf $f(y) = \binom{y-1}{r-1}p^r(1-p)^{y-r}$ for $y = r, r+1, \ldots$. That is, X is the number of successes in n trials and Y is the number of trials required to get r successes. Thus, we have

$$\begin{aligned}
P(X \geq r) &= P(\text{at least } r \text{ successes in } n \text{ trials}) \\
&= P(r\text{th success in } n\text{th or earlier trial}) \\
&= P(n \text{ or fewer trials required to get } r \text{ successes}) \\
&= P(Y \leq n).
\end{aligned}$$

Then we have $P(X \geq r) = 1 - P(X < r) = 1 - P(X \leq r-1)$ which results in $1 - P(X \leq r-1) = P(Y \leq n)$. Then we have

$$P(X \leq r-1) = F_{\text{Bin}(n,p)}(r-1) = 1 - P(Y \leq n) = 1 - F_{Y,\text{NB}(r,p)}(n),$$

that is, :

$$F_{\text{Bin}(n,p)}(r-1) = 1 - F_{Y,\text{NB}(r,p)}(n).$$

Next, we let X be a negative binomial random variable with pmf $f(x) = \binom{r+x-1}{x}p^r(1-p)^x$ for $x = 0, 1, \ldots$. That is, X is the number of failures required to get r successes and $X = Y - r$. Then it is immediate from $F_{\text{Bin}(n,p)}(r-1) = 1 - P(Y \leq n)$ that we have

$$F_{\text{Bin}(n,p)}(r-1) = 1 - P(Y \leq n) = 1 - P(X + r \leq n) = 1 - P(X \leq n-r)$$

Thus, we have

$$F_{\text{Bin}(n,p)}(r-1) = 1 - F_{X,\text{NB}(r,p)}(n-r).$$